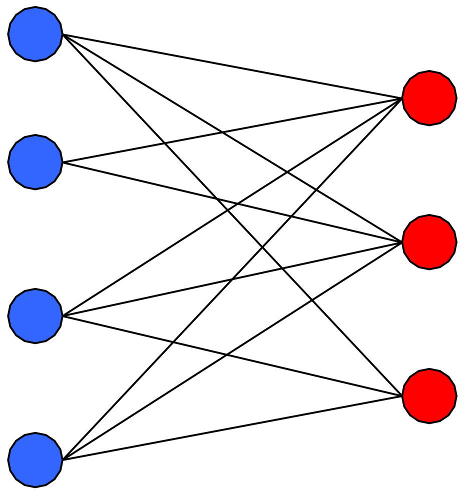
Introduction to Algorithms

Bipartiteness - DFS

Bipartite Graphs

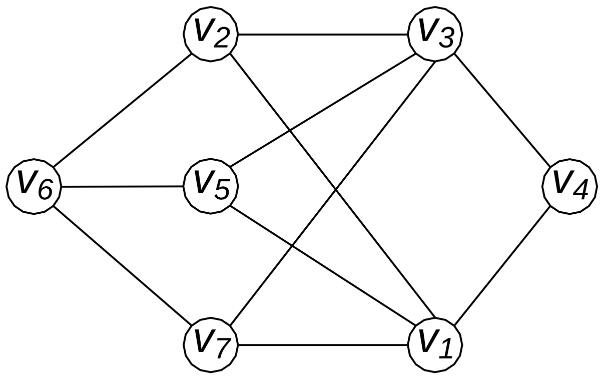
Definition: An undirected graph G=(V,E) is bipartite
  if you can partition the node set into 2 parts (say, blue/red or left/right) so that
  all edges join nodes in different parts
  i.e., no edge has both ends in the same part.

Application:
- Scheduling: machine=red, jobs=blue
- Stable Matching: men=blue, wom=red

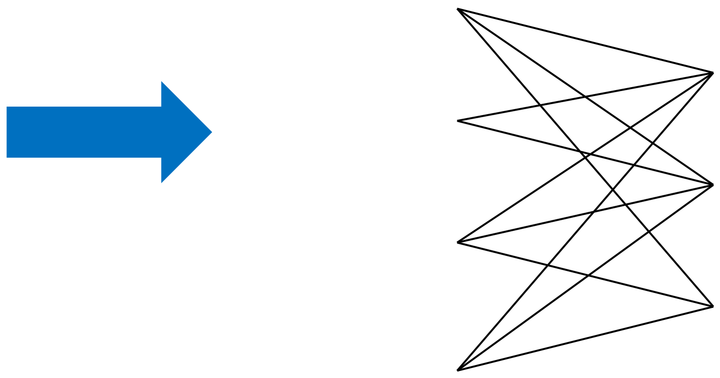

a bipartite graph

Testing Bipartiteness

Problem: Given a graph G, is it bipartite?



a bipartite graph G



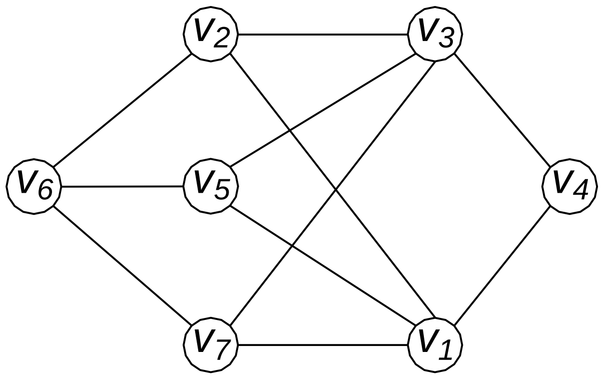
Testing Bipartiteness

Problem: Given a graph G, is it bipartite?

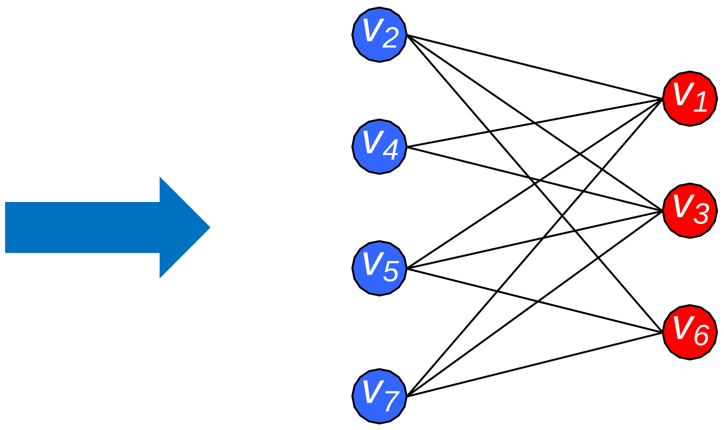
Many graph problems become:

- Easier if the underlying graph is bipartite (matching)
- Tractable if the underlying graph is bipartite (independent set)

Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

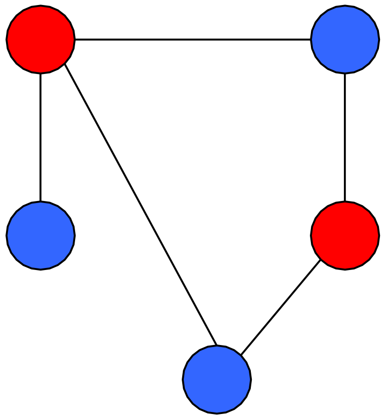


another drawing of G

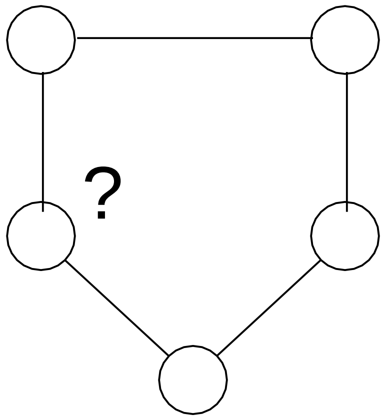
An Obstruction to Bipartiteness

Lemma: If G is bipartite, then it does not contain an odd length cycle.

Pf: We cannot 2-color an odd cycle, let alone G.



bipartite
(2-colorable)



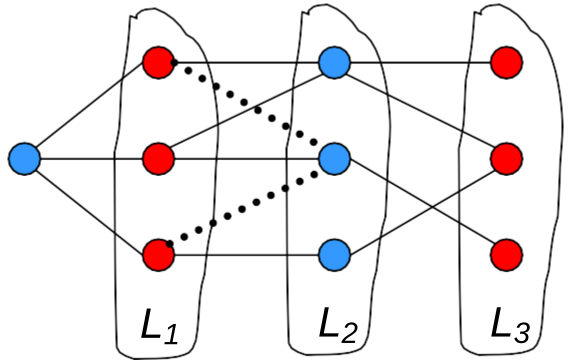
not bipartite
(not 2-colorable)

A Characterization of Bipartite Graphs

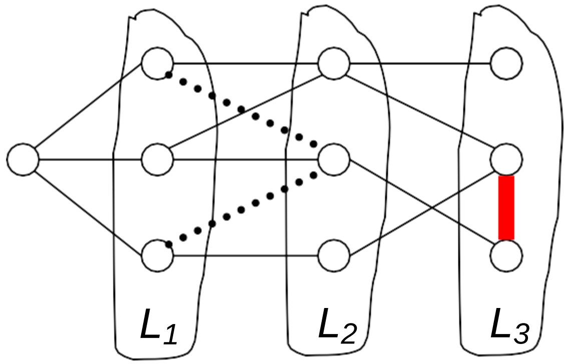
Lemma: Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(s). Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

A Characterization of Bipartite Graphs

Lemma: Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(s). Exactly one of the following holds.

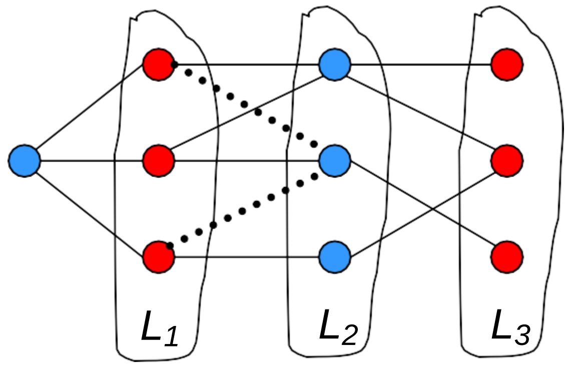
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

Suppose no edge joins two nodes in the same layer.

By previous lemma, all edges join nodes on adjacent levels.



Case (i)

Bipartition:
blue = nodes on odd levels,
red = nodes on even levels.

A Characterization of Bipartite Graphs

Lemma: Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS(s). Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

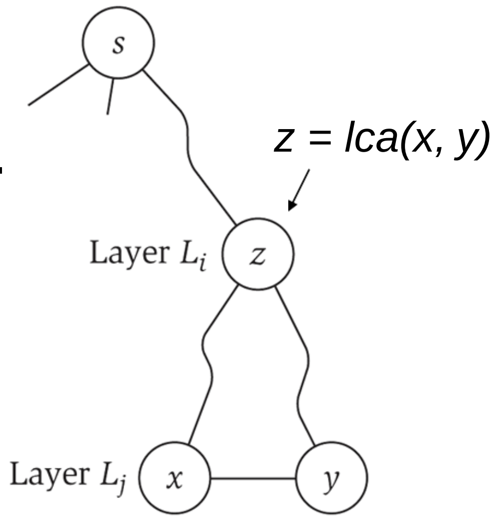
Suppose (x, y) is an edge & x, y in same level L_j.

Let z = their lowest common ancestor in BFS tree.

Let L_i be level containing z.

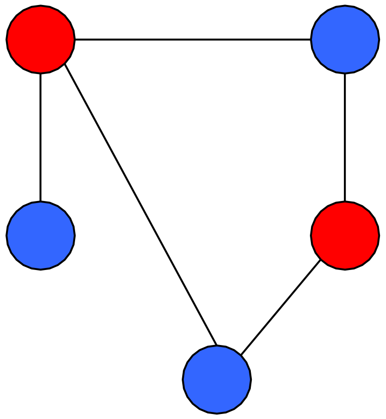
Consider cycle that takes edge from x to y, then tree from y to z, then tree from z to x.

Its length is 1 + (j-i) + (j-i), which is odd.

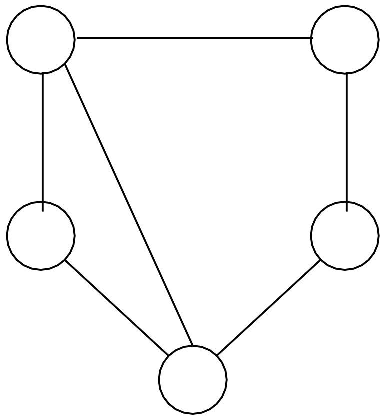


Obstruction to Bipartiteness

Cor: A graph G is bipartite iff it contains no odd length cycles.



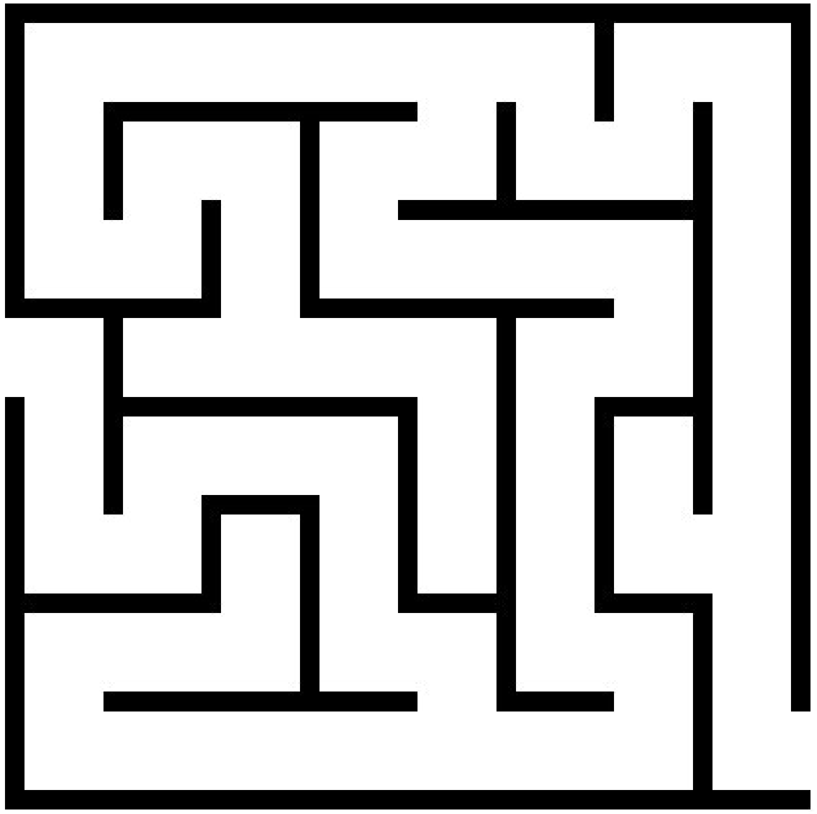
bipartite
(2-colorable)



not bipartite
(not 2-colorable)

Depth First Search

Follow the first path you find as far as you can go; back up to last unexplored edge when you reach a dead end, then go as far as you can



Naturally implemented using recursive calls or a stack

DFS(s) – Recursive version

Global Initialization: mark all vertices undiscovered

```
DFS(v)
    Mark v discovered

    for each edge {v,x}
        if (x is undiscovered)
            Mark x discovered
            DFS(x)

    Mark v full-discovered
```

DFS(A)

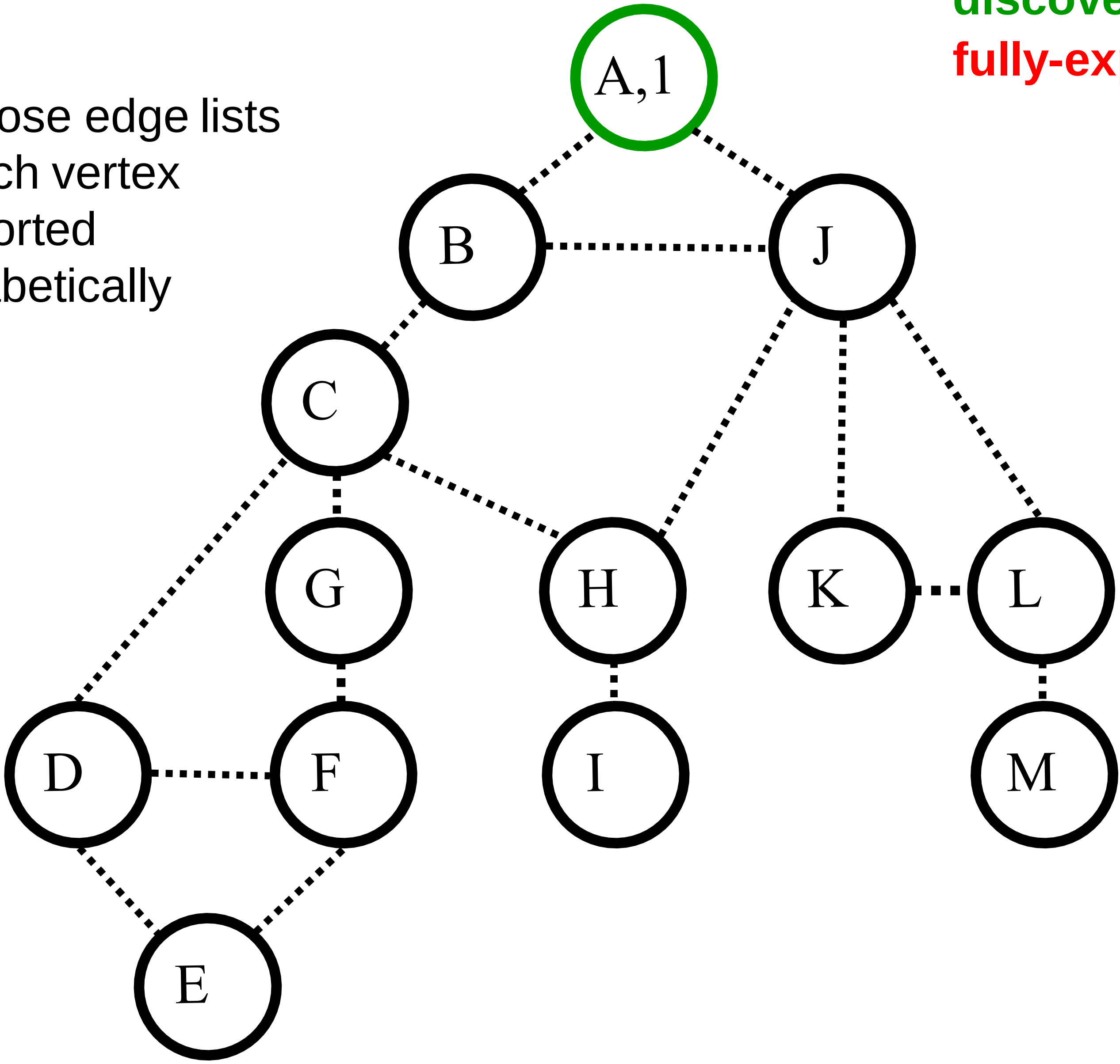
Color code:

undiscovered

discovered

fully-explored

Suppose edge lists at each vertex are sorted alphabetically



Call Stack (Edge list):

A (B,J)

st[] = {1}

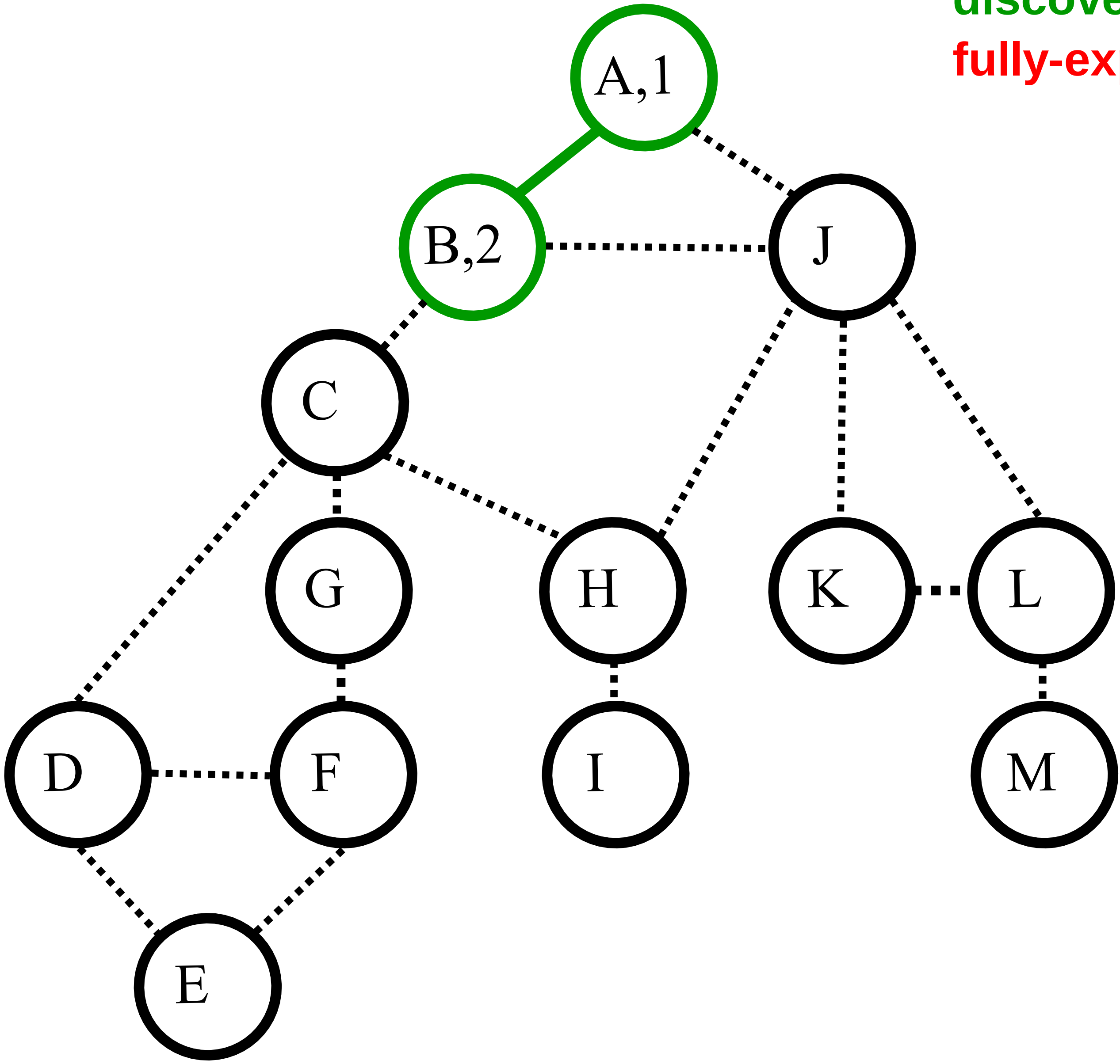
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (A,C,J)

st[] =
{1,2}

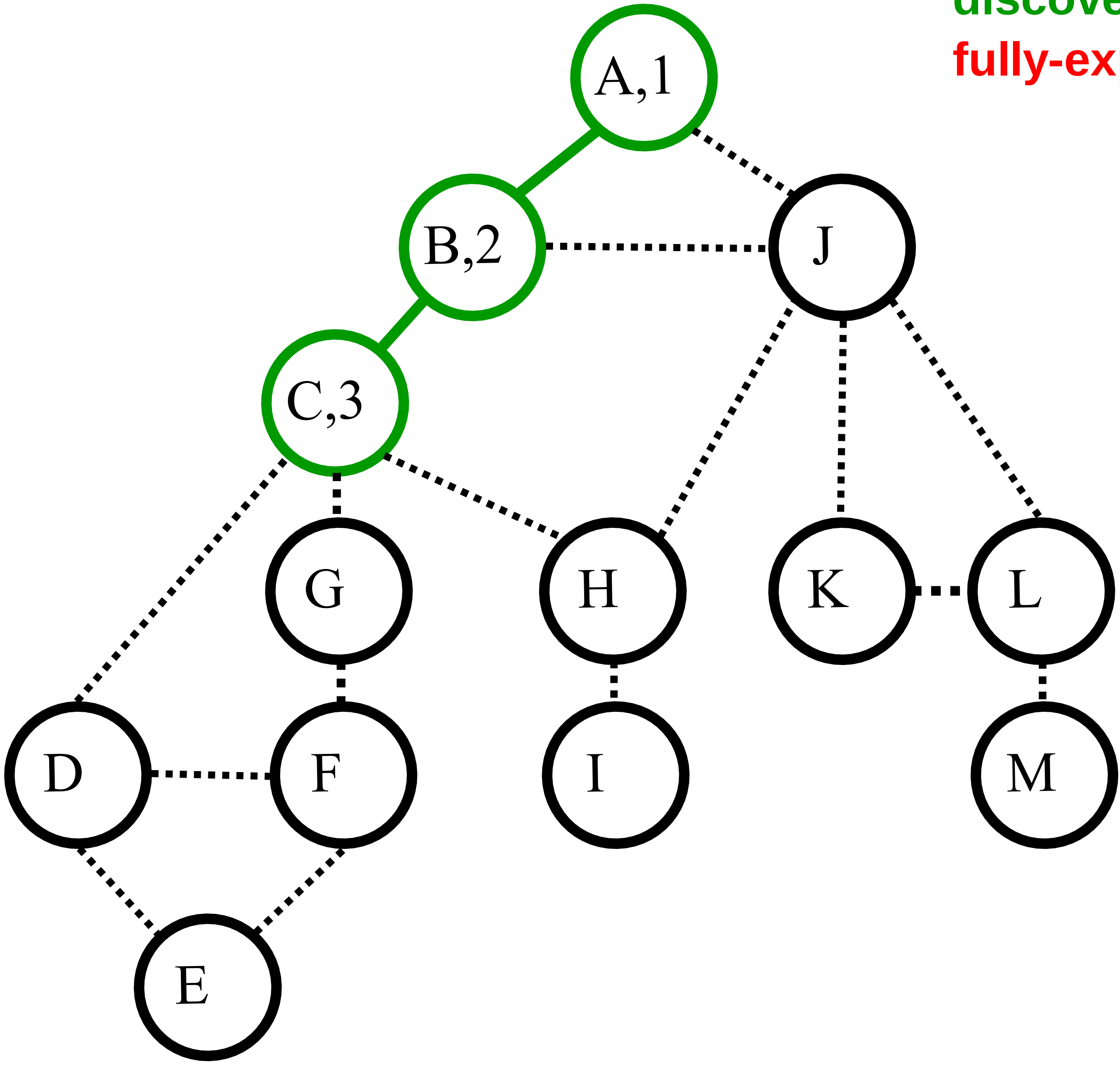
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (B,D,G,H)

st[] =
{1,2,3}

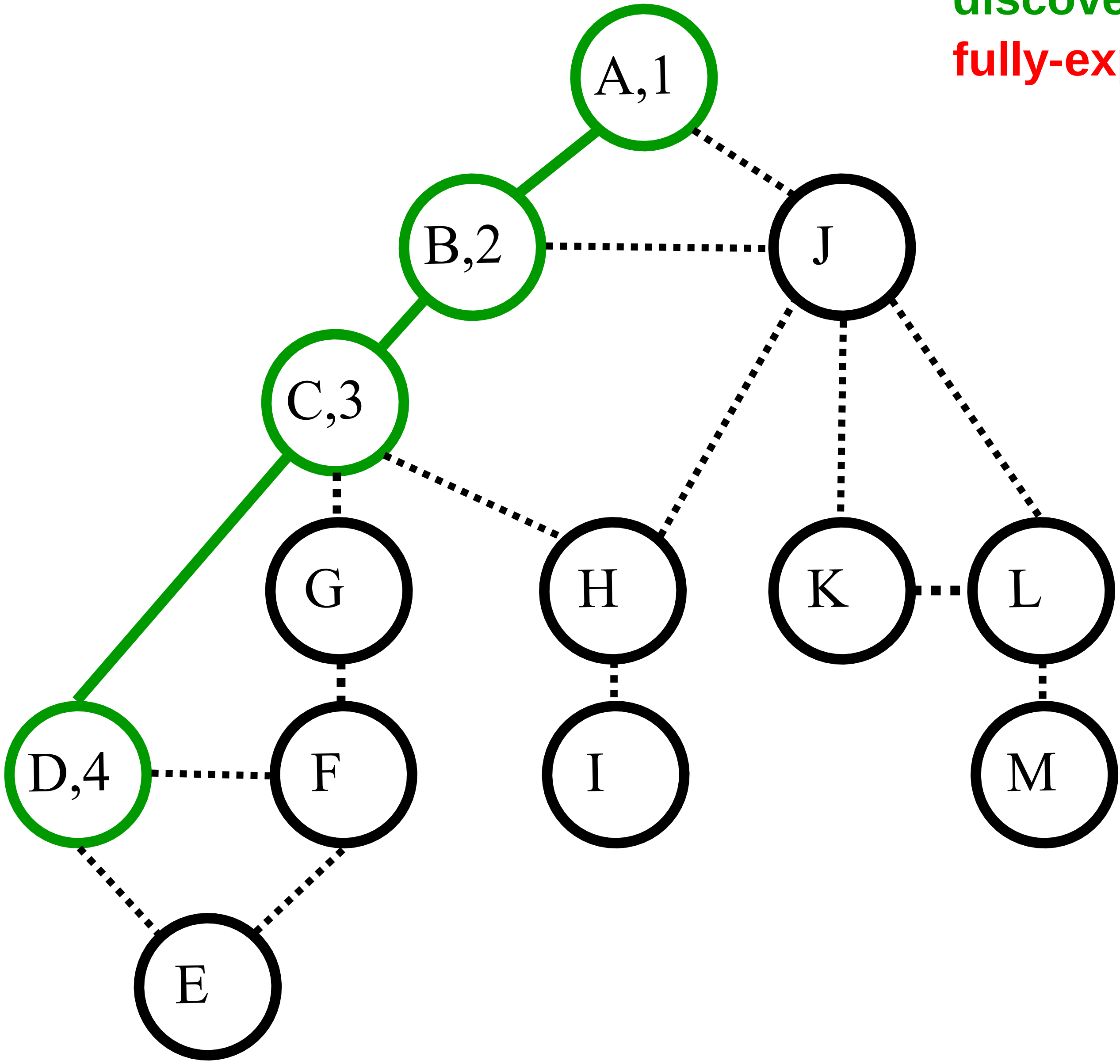
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (C,E,F)

st[] =
{1,2,3,4}

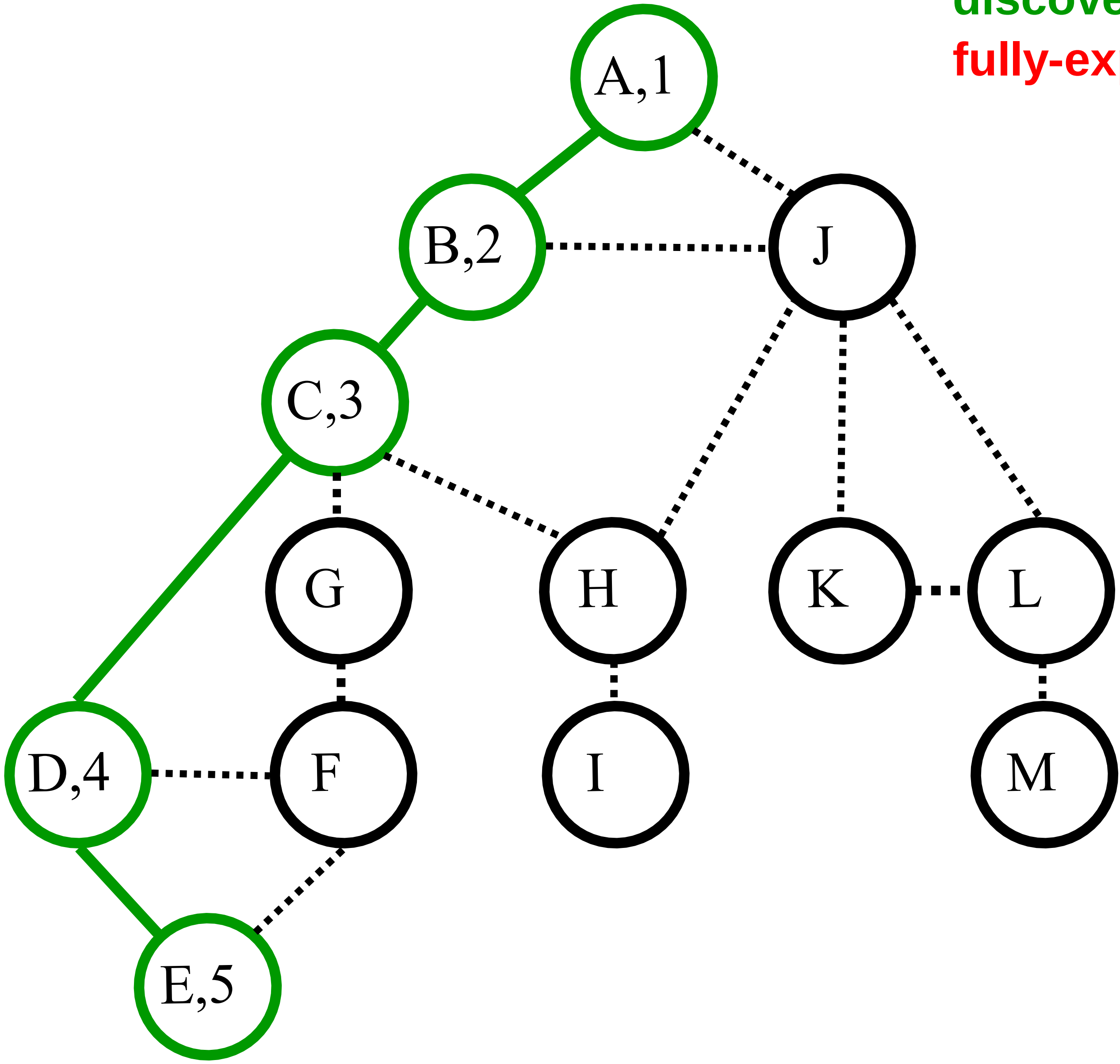
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (D,F)

st[] =
{1,2,3,4,5}

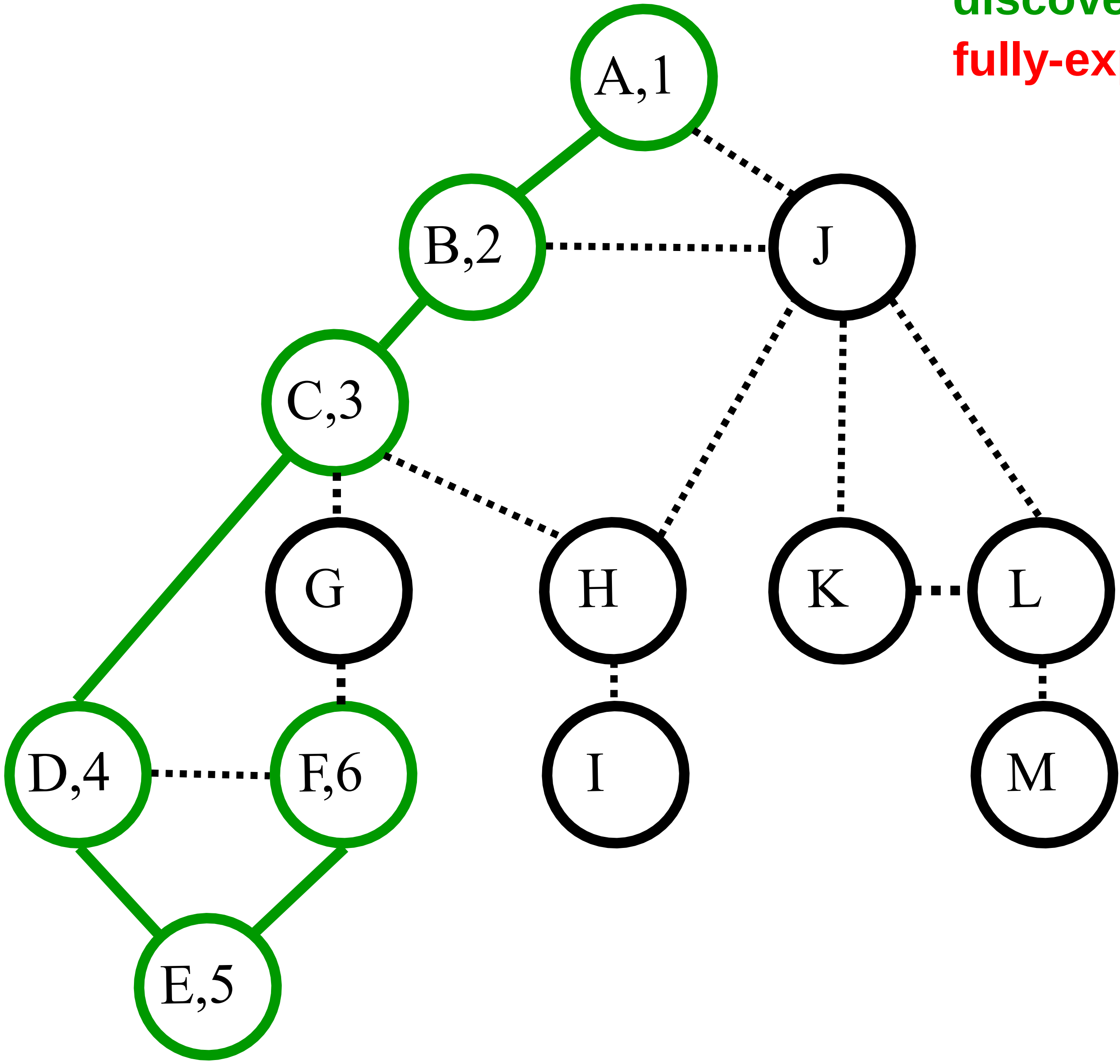
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)

st[] =
{1,2,3,4,5,
6}

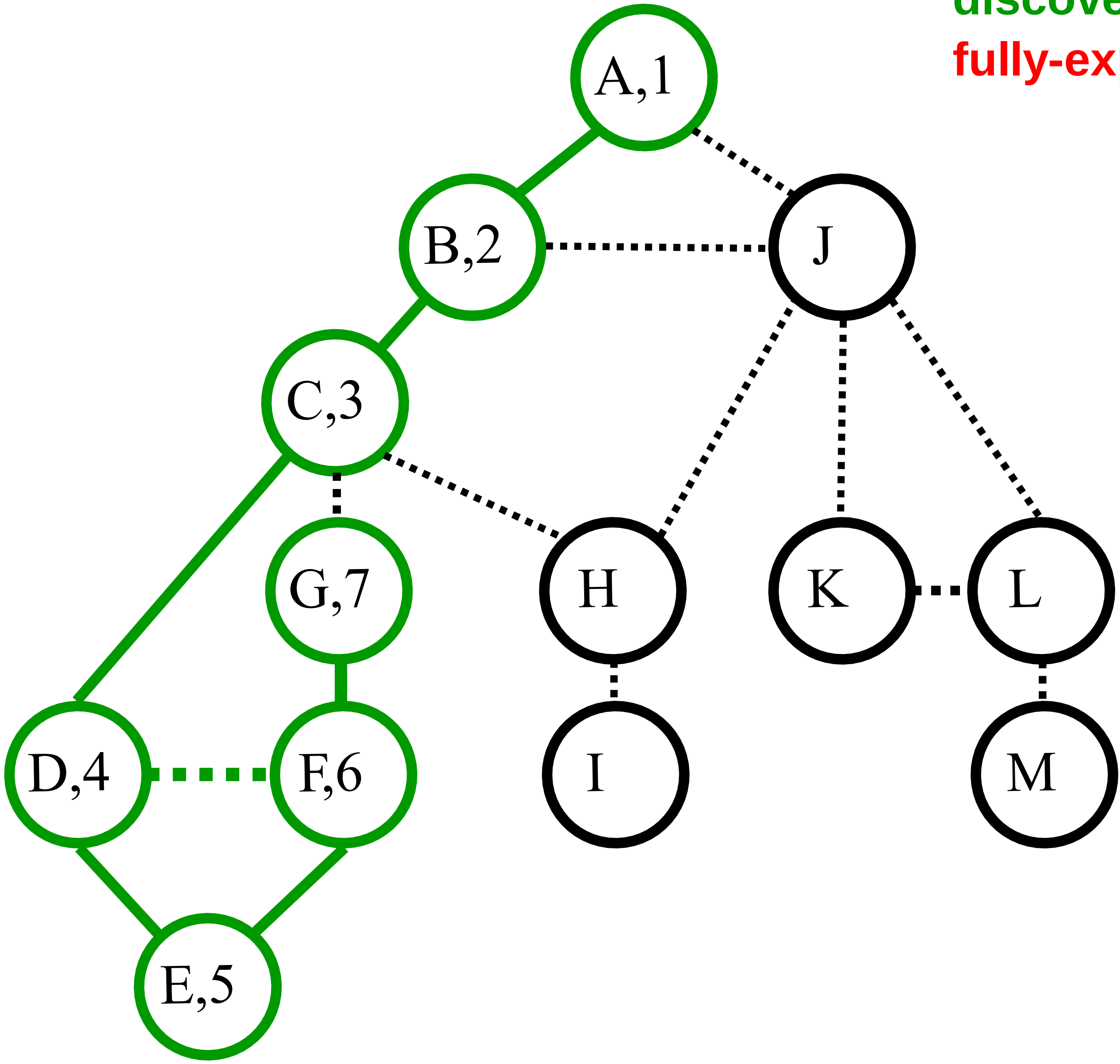
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)

st[] =
{1,2,3,4,5,
6,7}

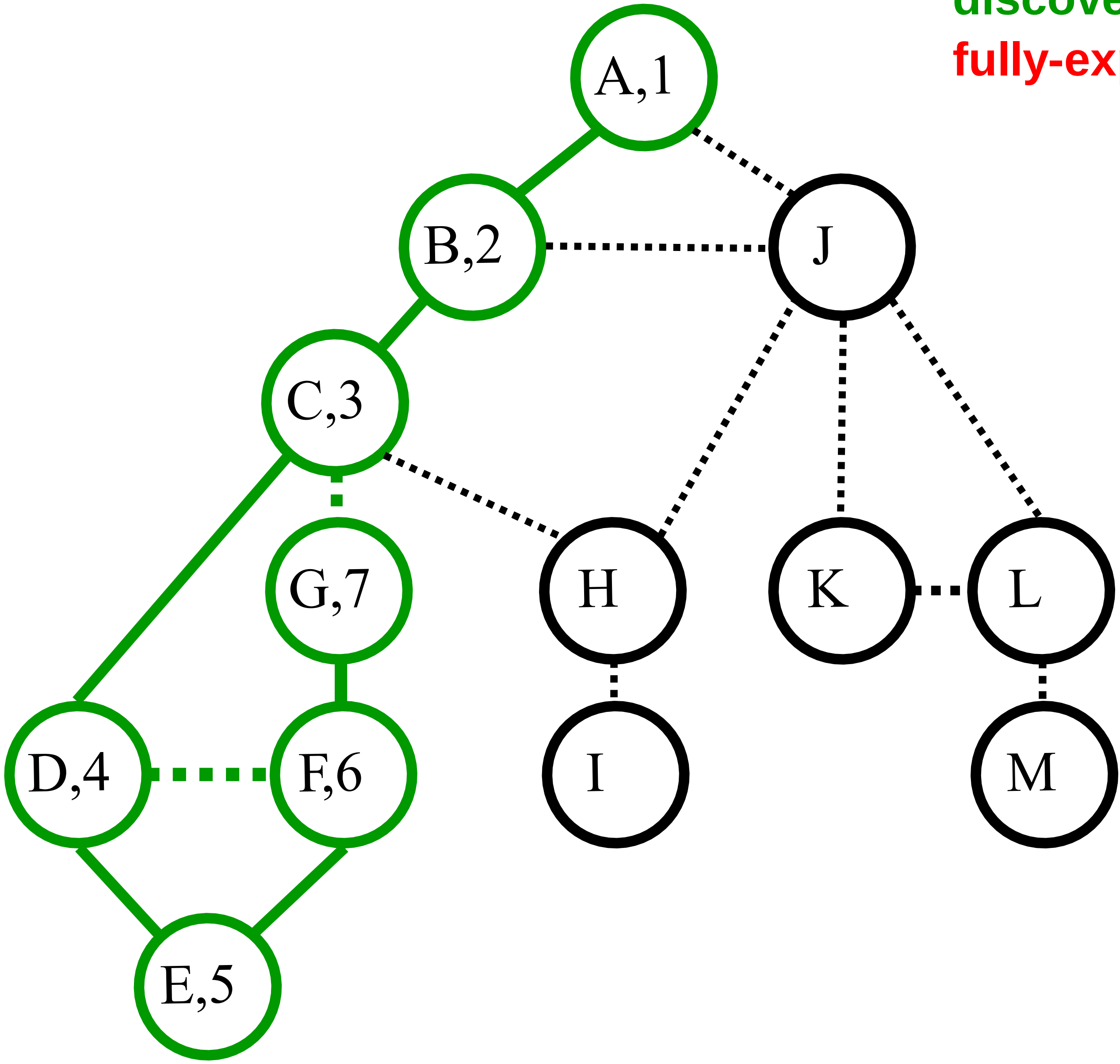
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

st[] =
{1,2,3,4,5,
6,7}

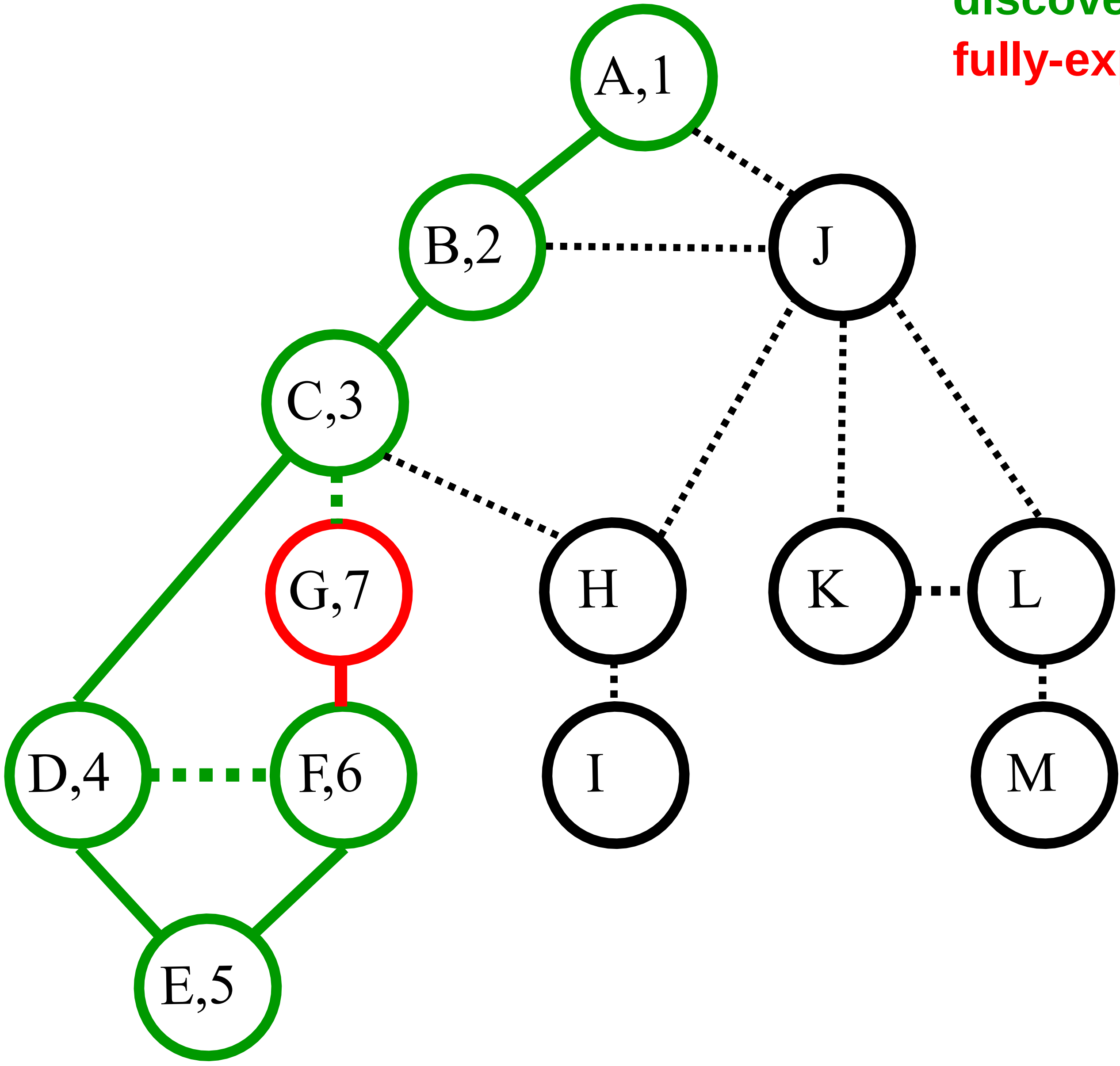
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

st[] =
{1,2,3,4,5,
6}

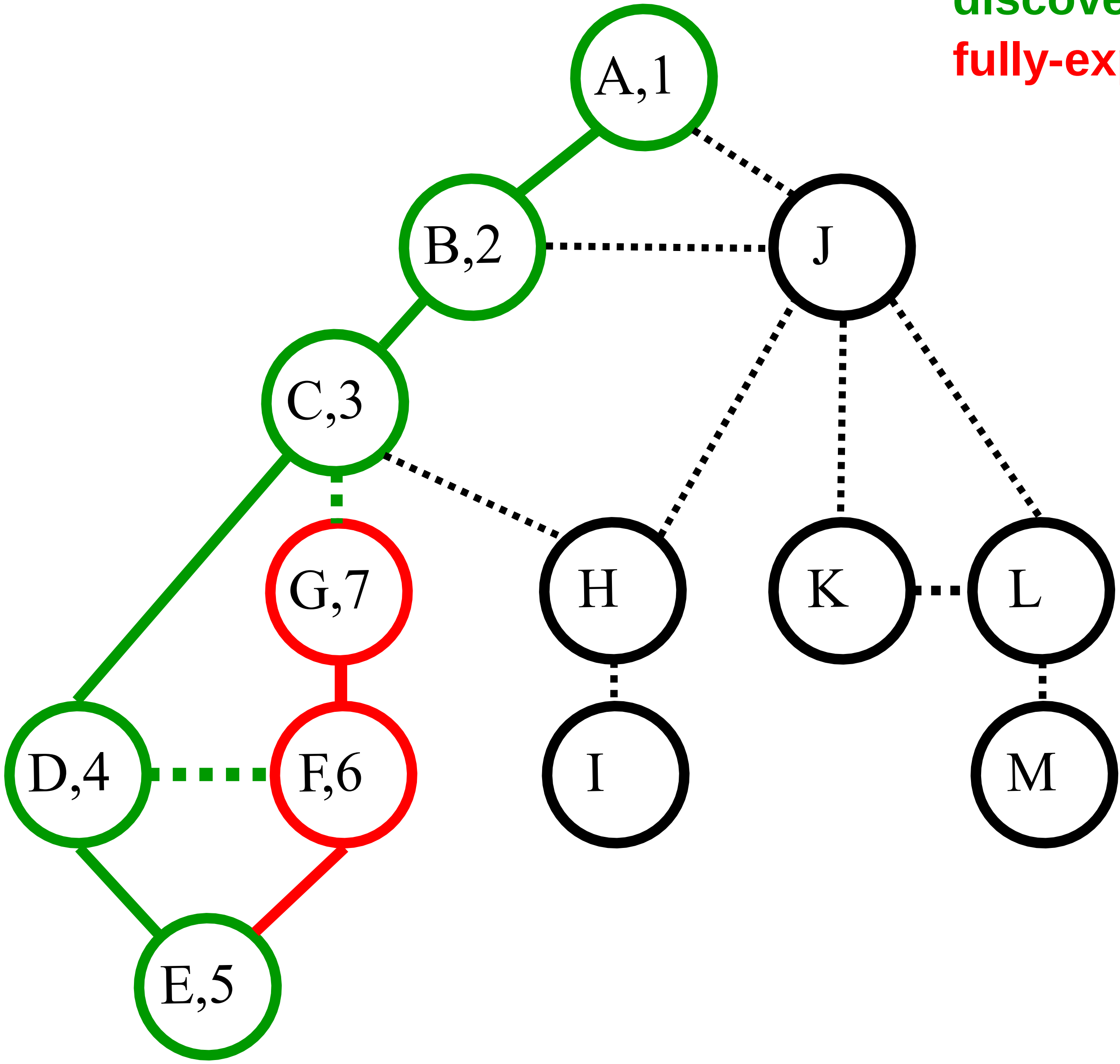
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

st[] =
{1,2,3,4,5}

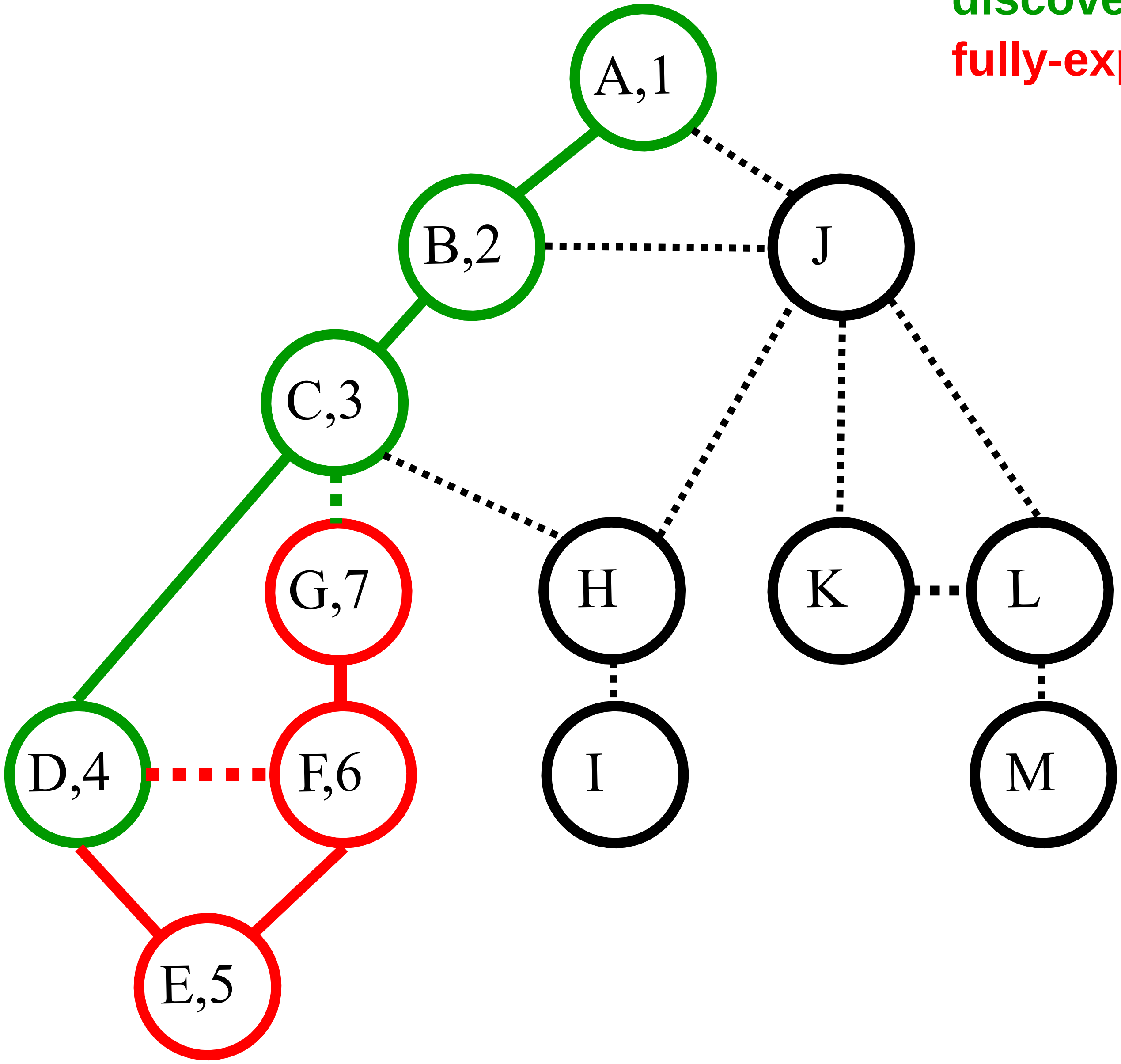
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)

st[] =
{1,2,3,4}

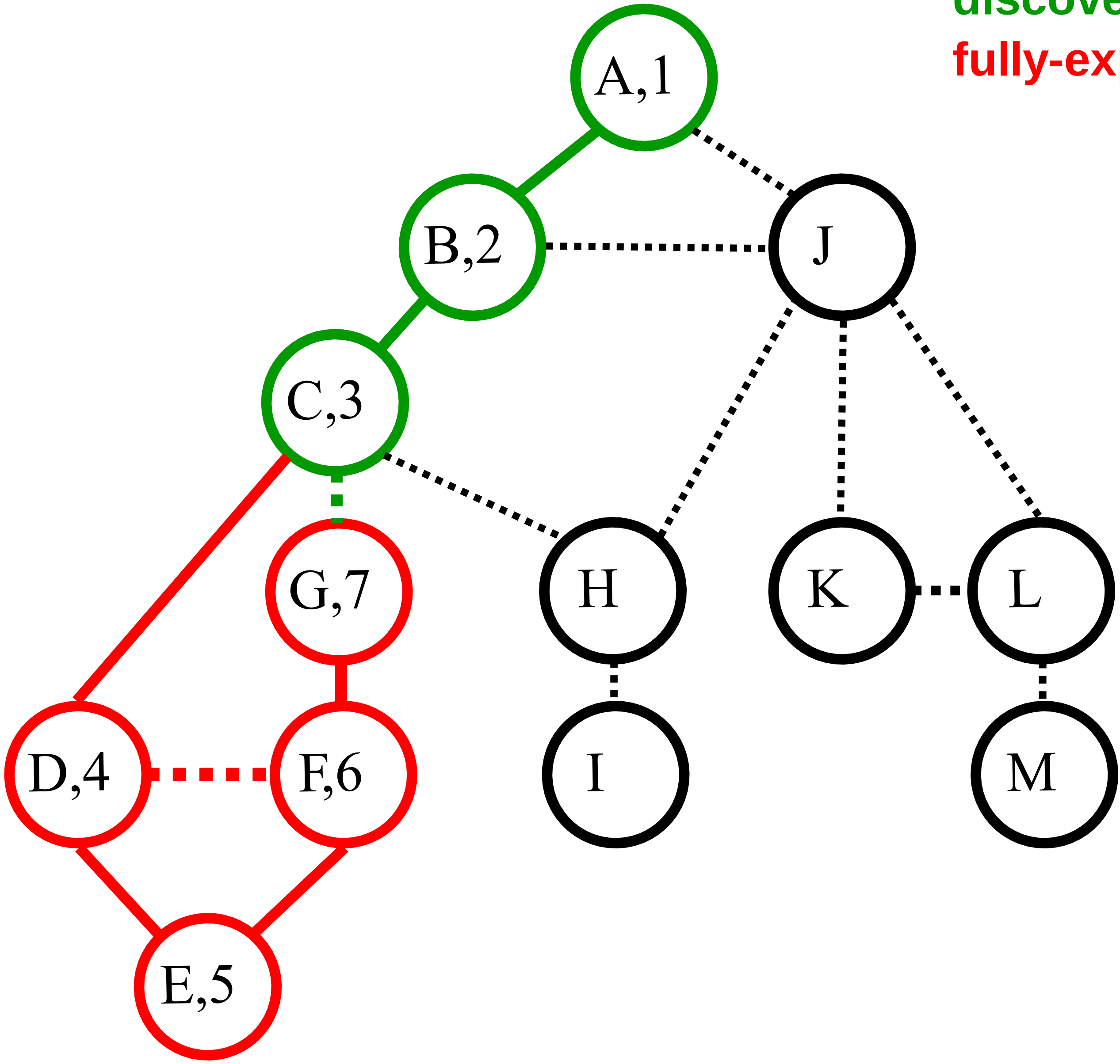
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)

st[] =
{1,2,3}

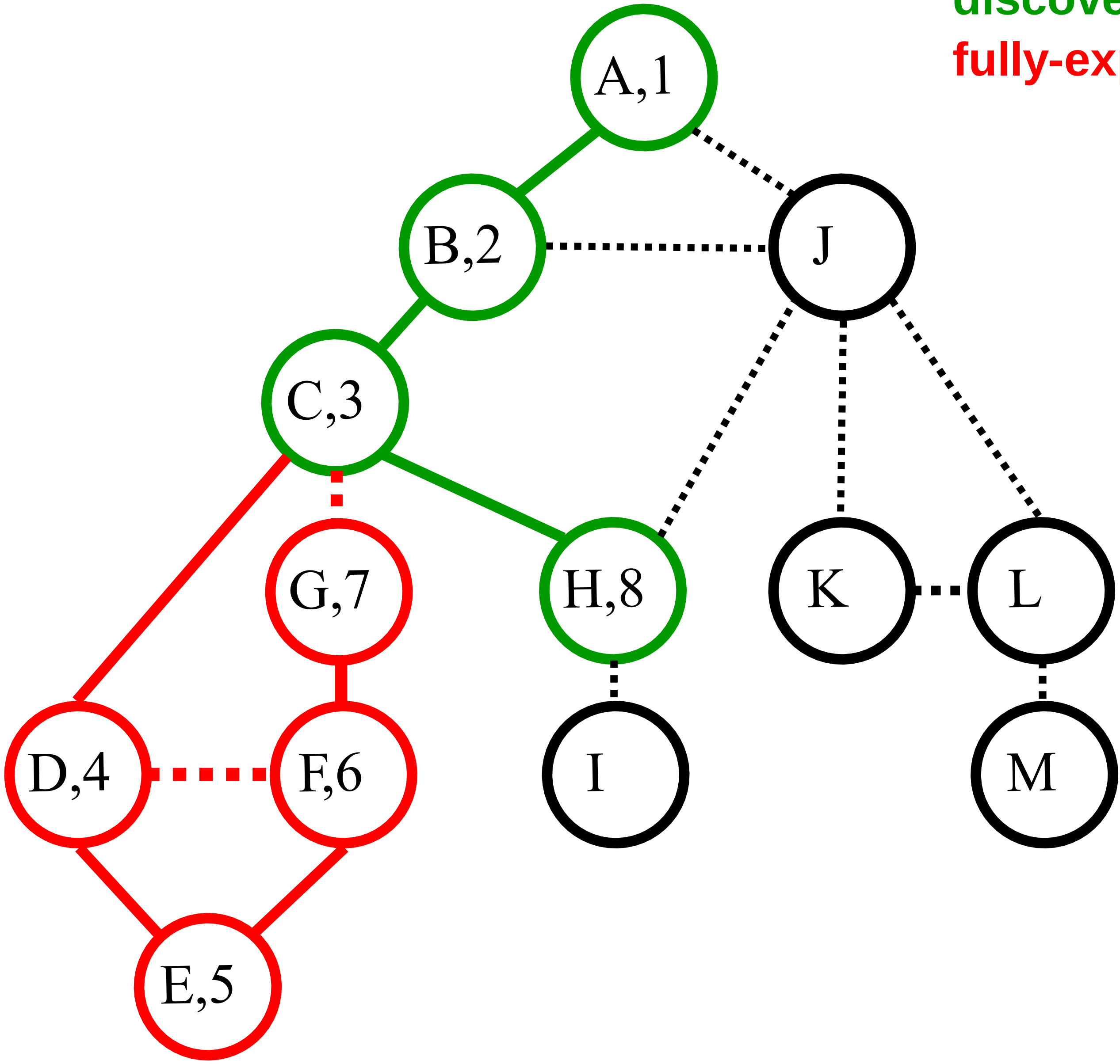
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (C,I,J)

st[] =
{1,2,3,8}

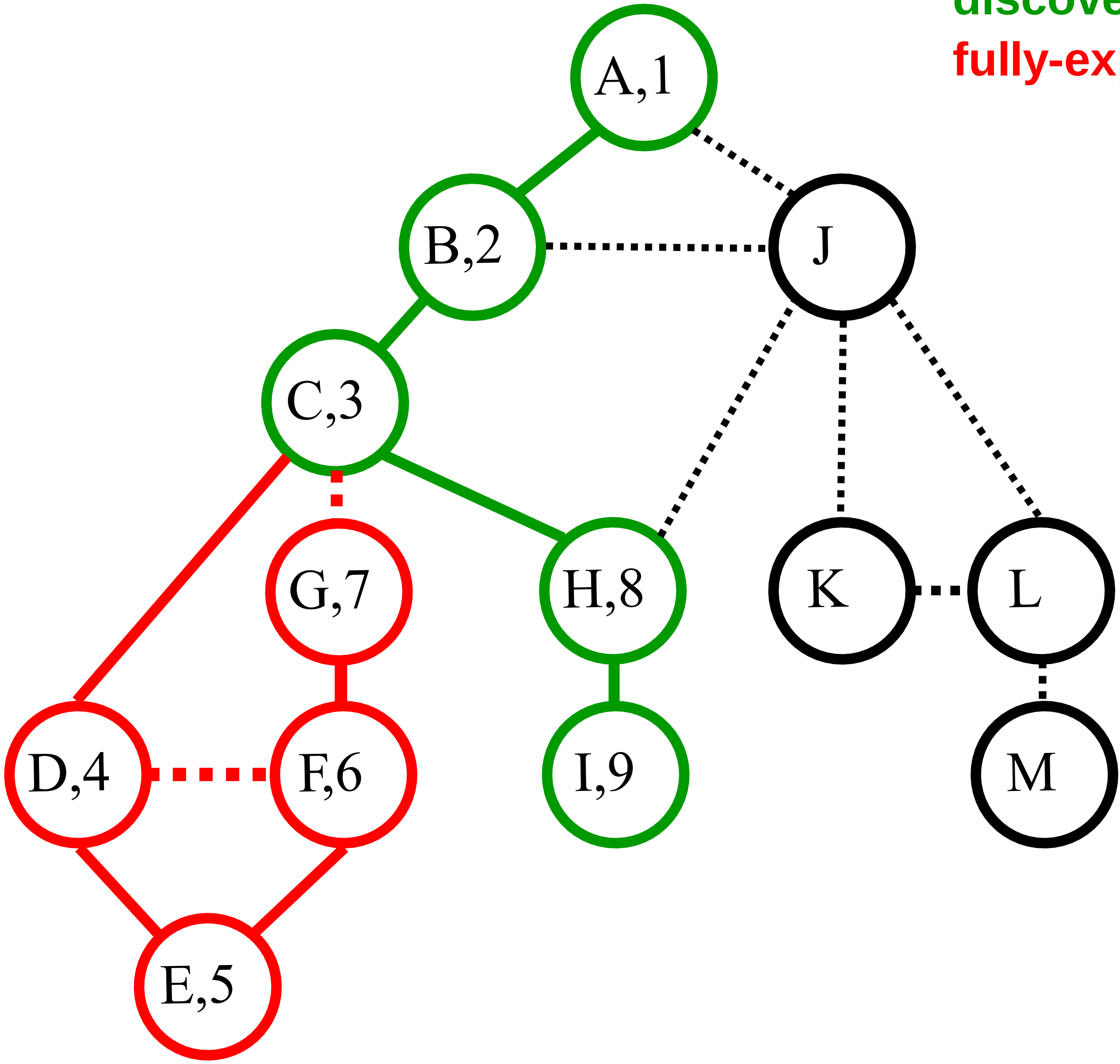
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

st[] =
{1,2,3,8,9}

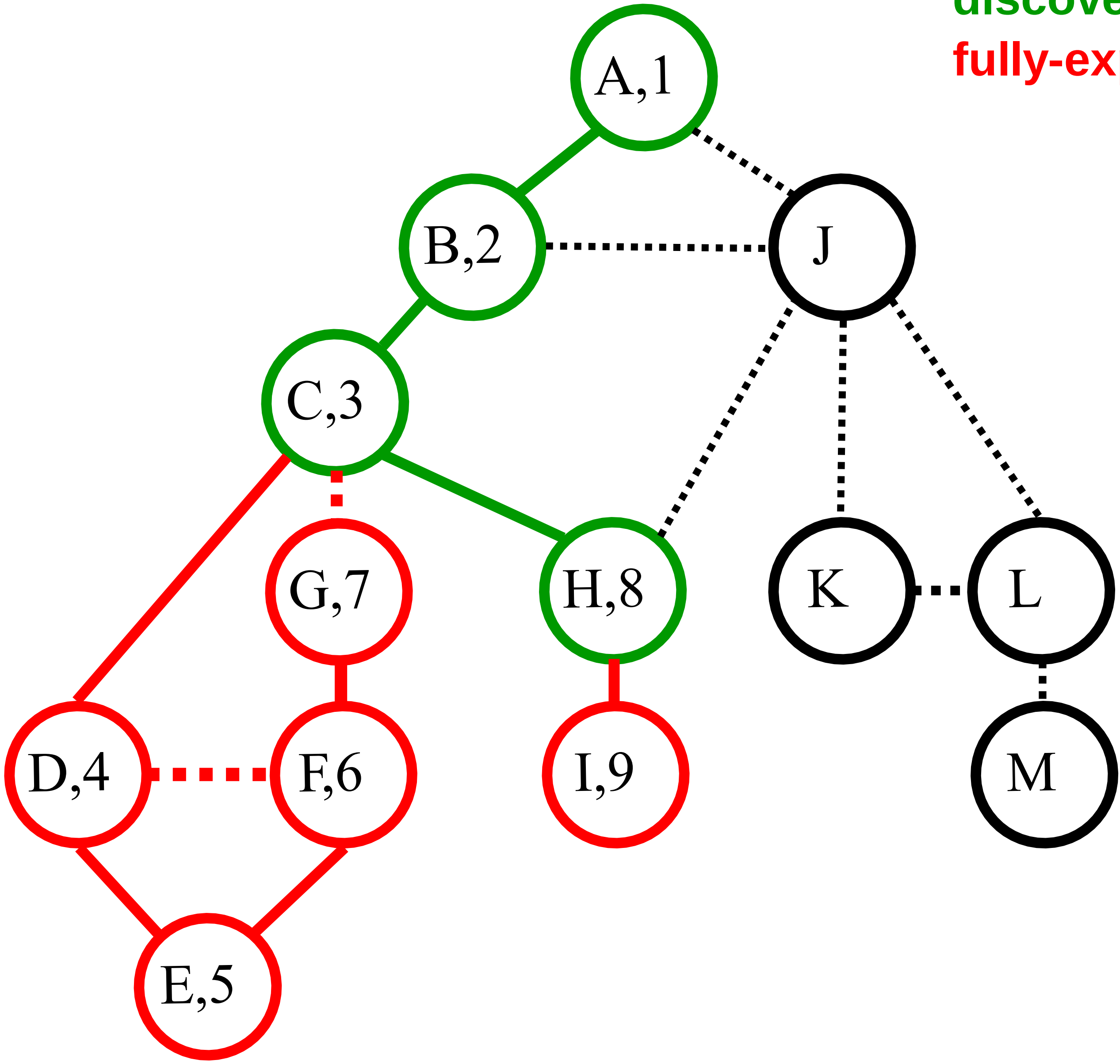
DFS(A)

Color code:

undiscovered

discovered

fully-explored



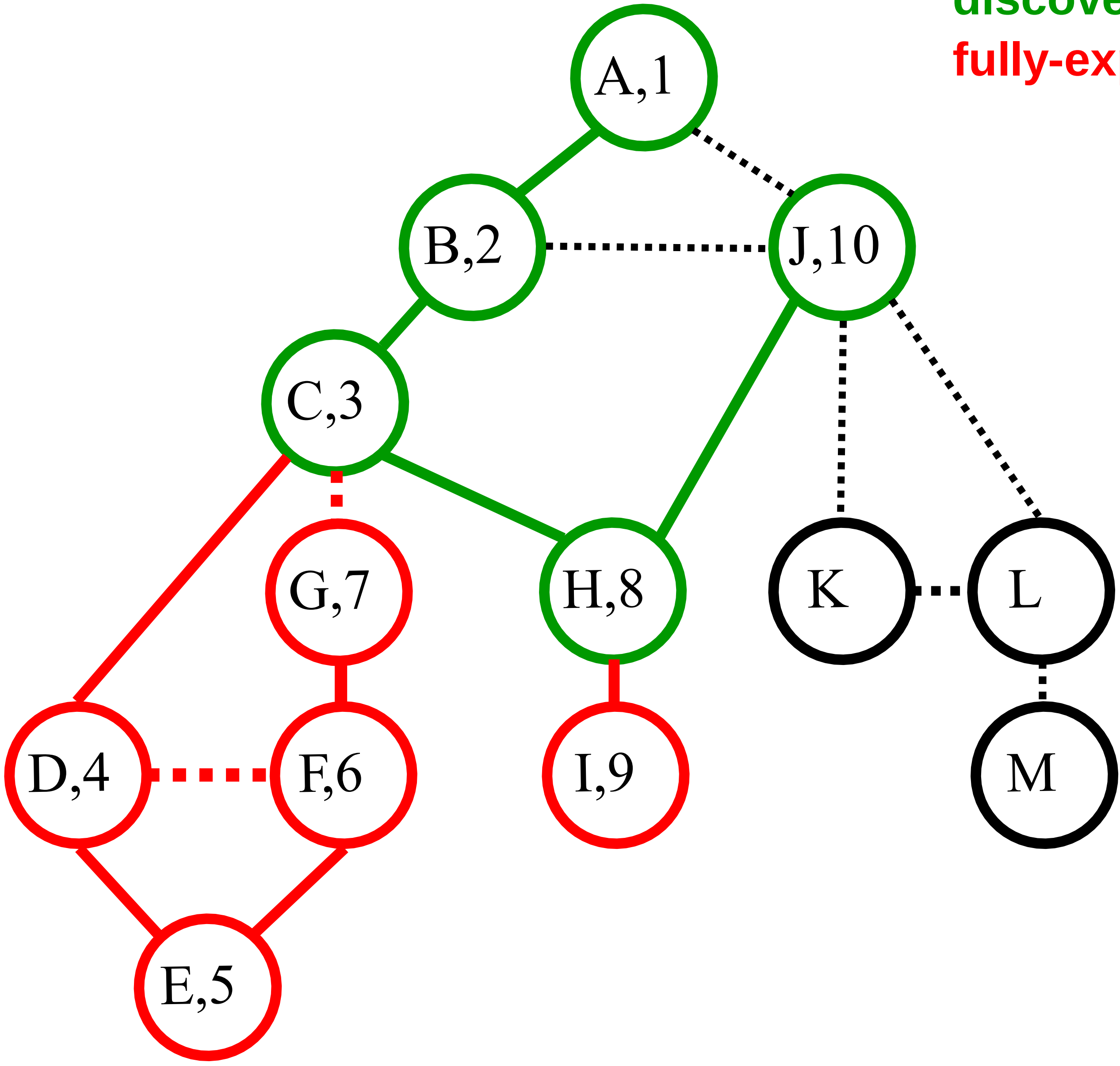
Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)

st[] =
{1,2,3,8}

DFS(A)

Color code:
undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

st[] =
{1,2,3,8,
10}

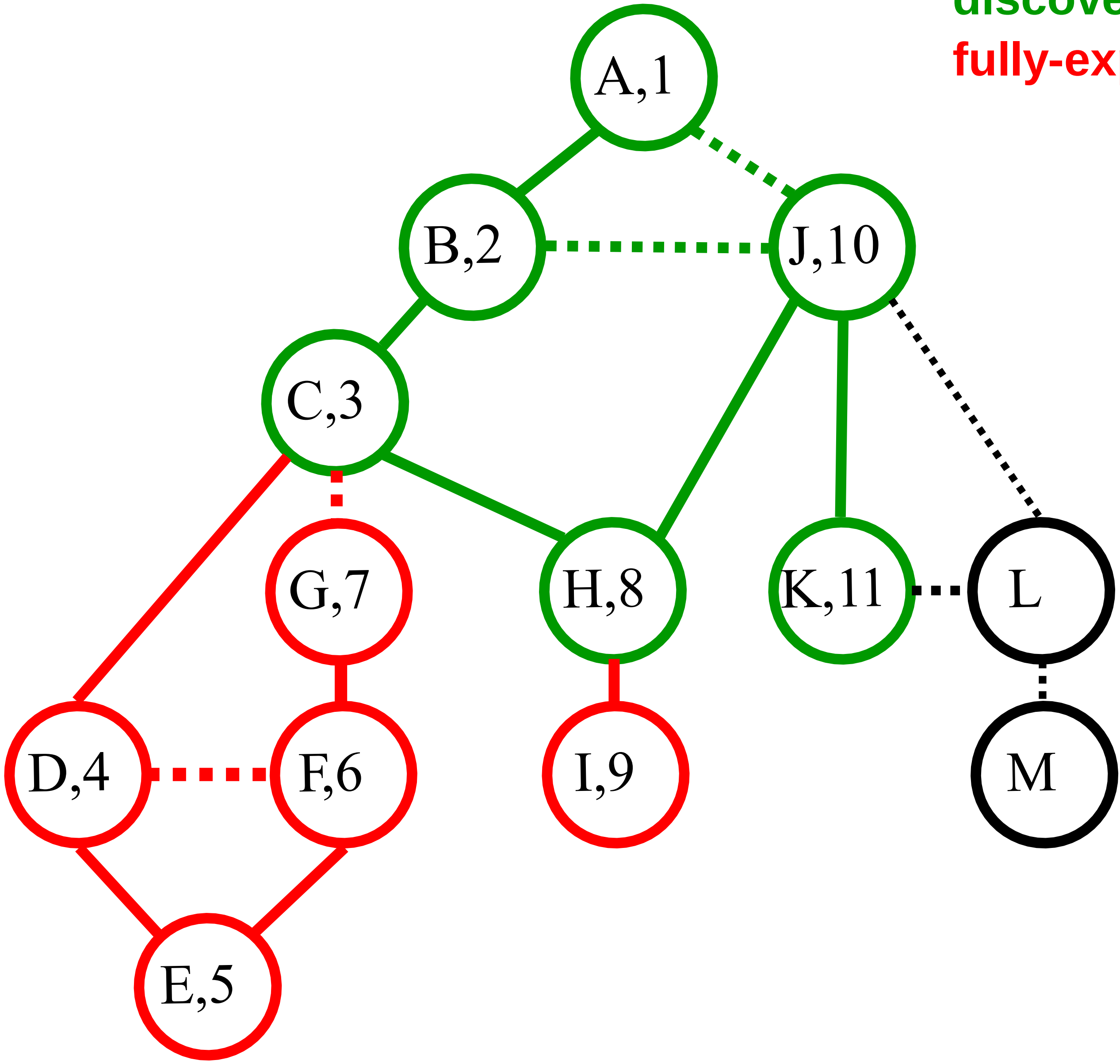
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (J,L)

st[] = {1,2,3,8,10,11}

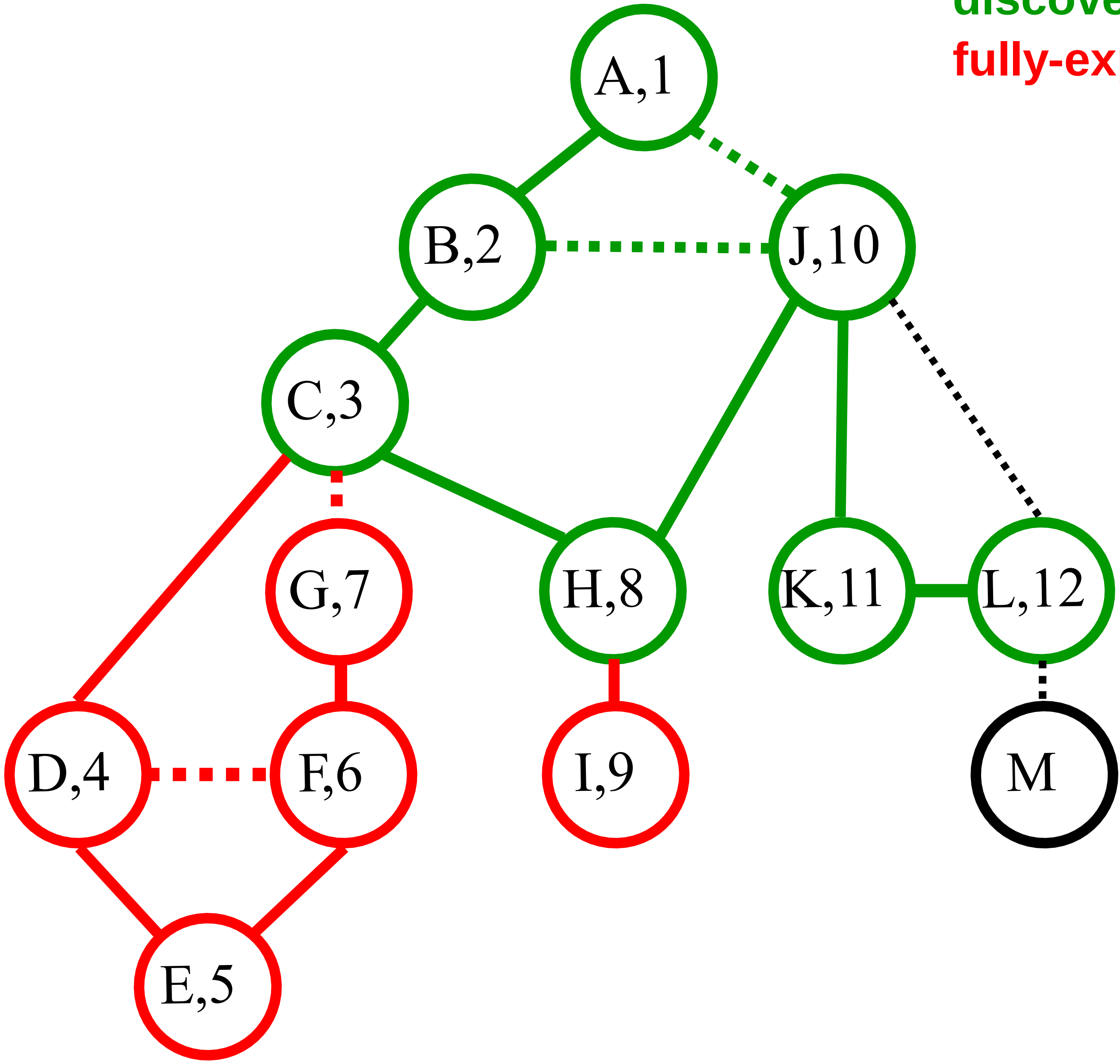
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (J,K,M)

st[] =
{1,2,3,8,10
,11,12}

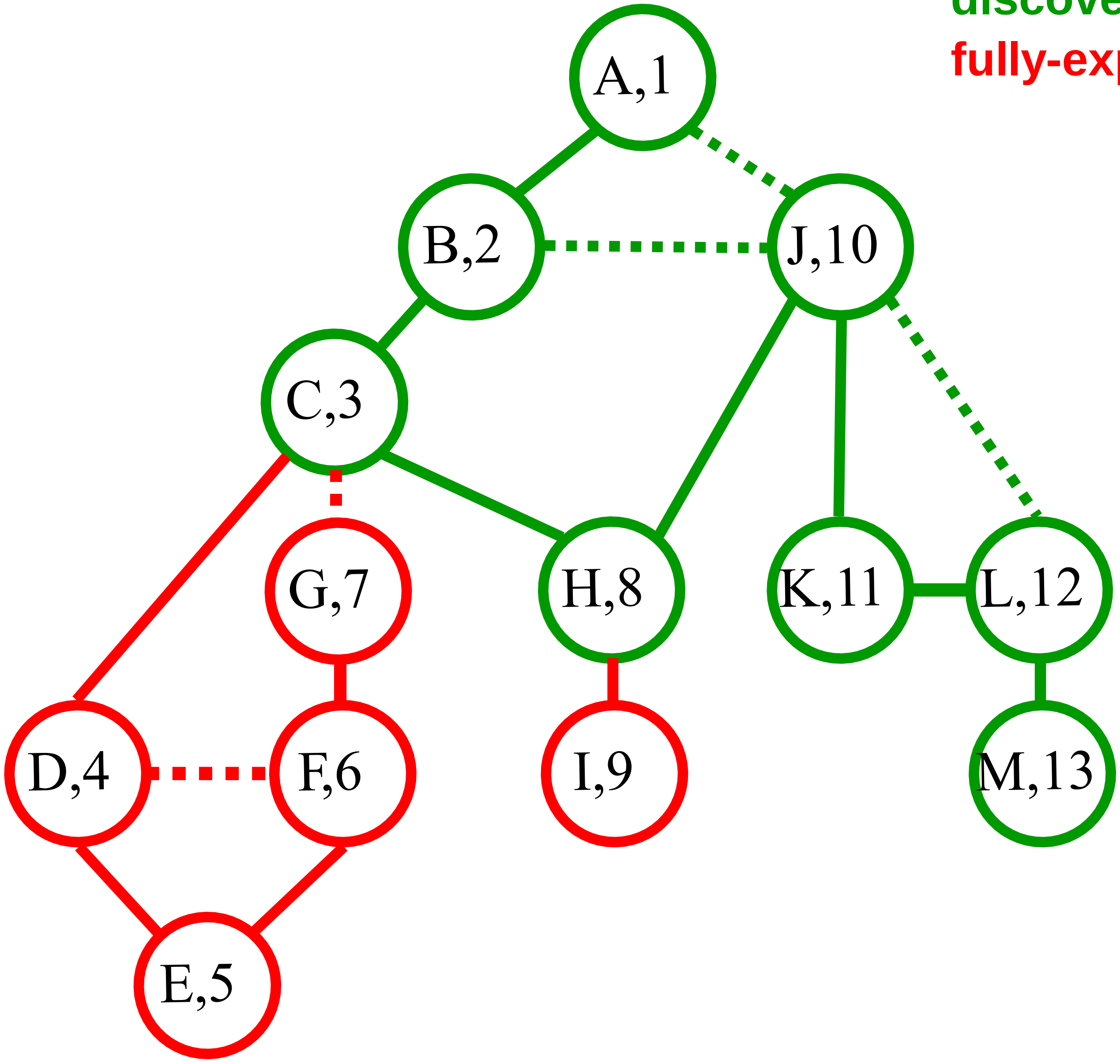
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)
M(L)

st[] =
{1,2,3,8,10
,11,12,13}

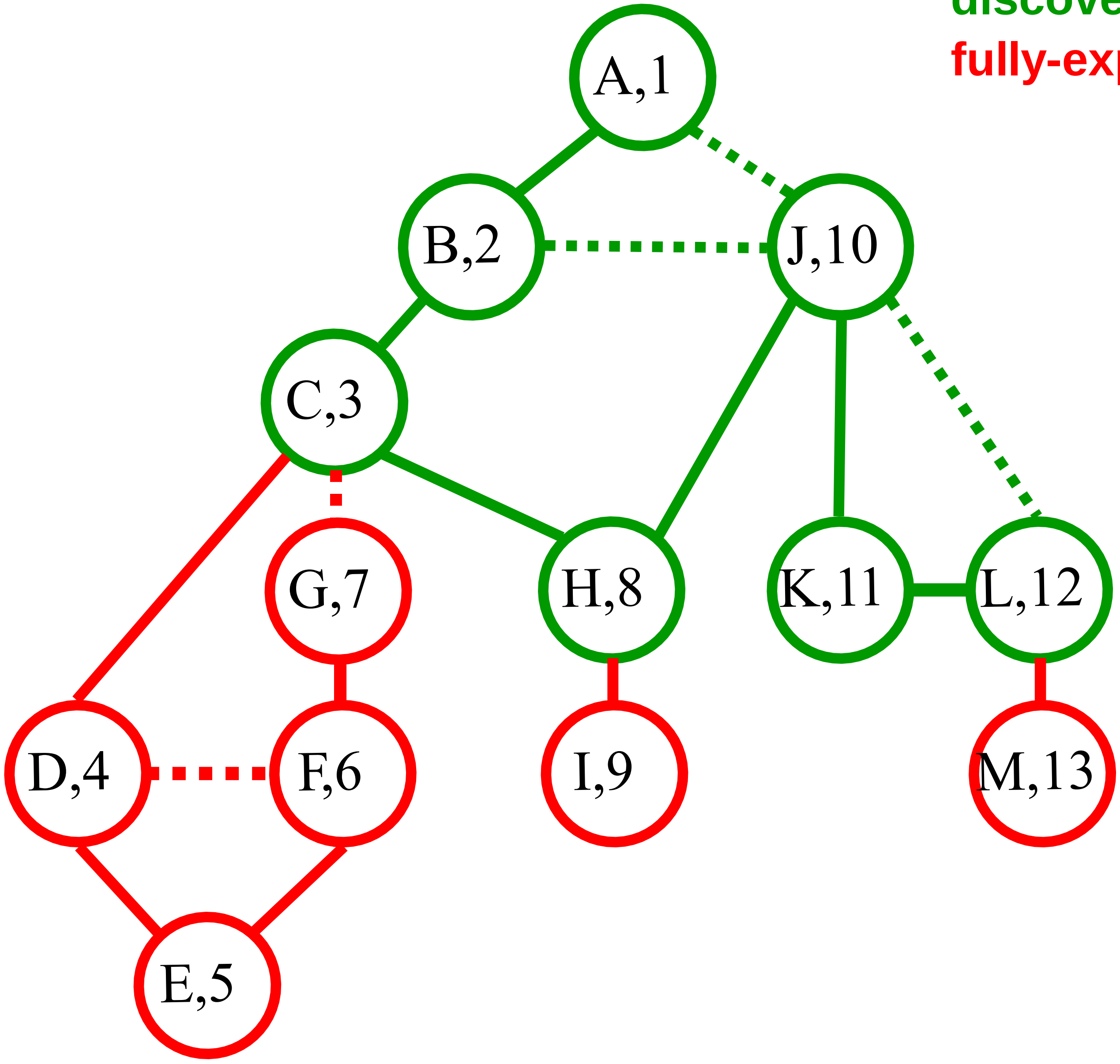
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)

st[] = {1,2,3,8,10,11,12}

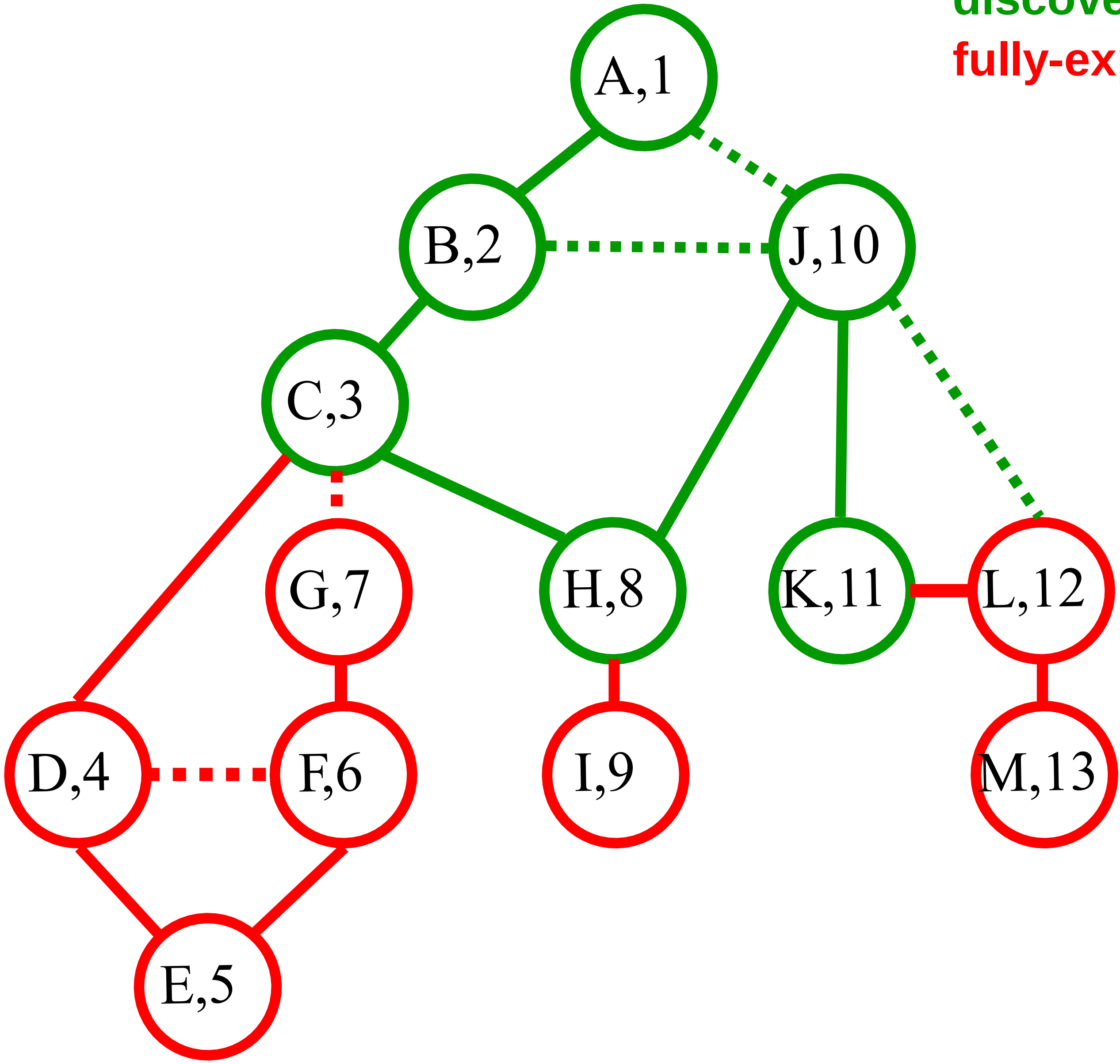
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)

st[] =
{1,2,3,8,10
,11}

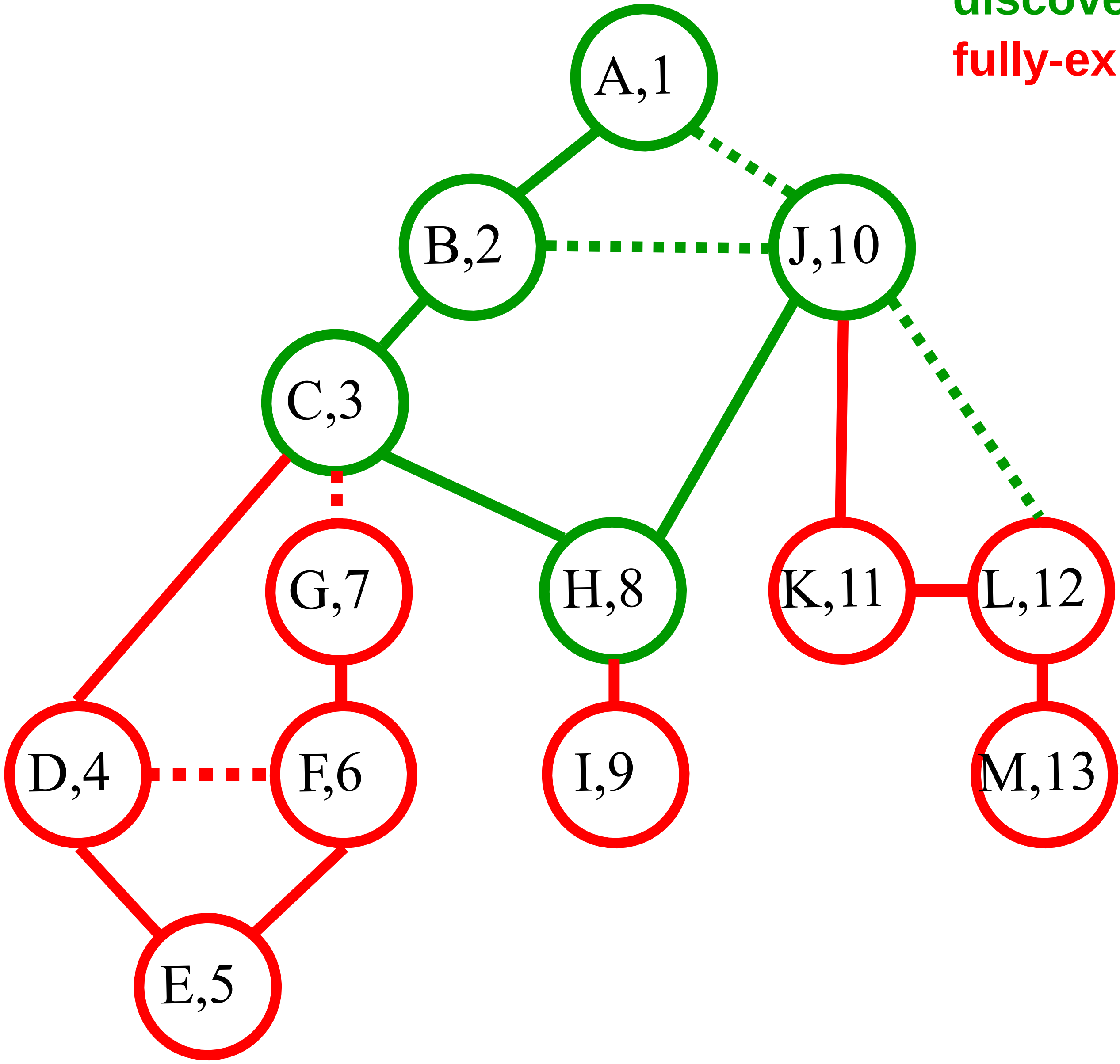
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)

st[] =
{1,2,3,8,
10}

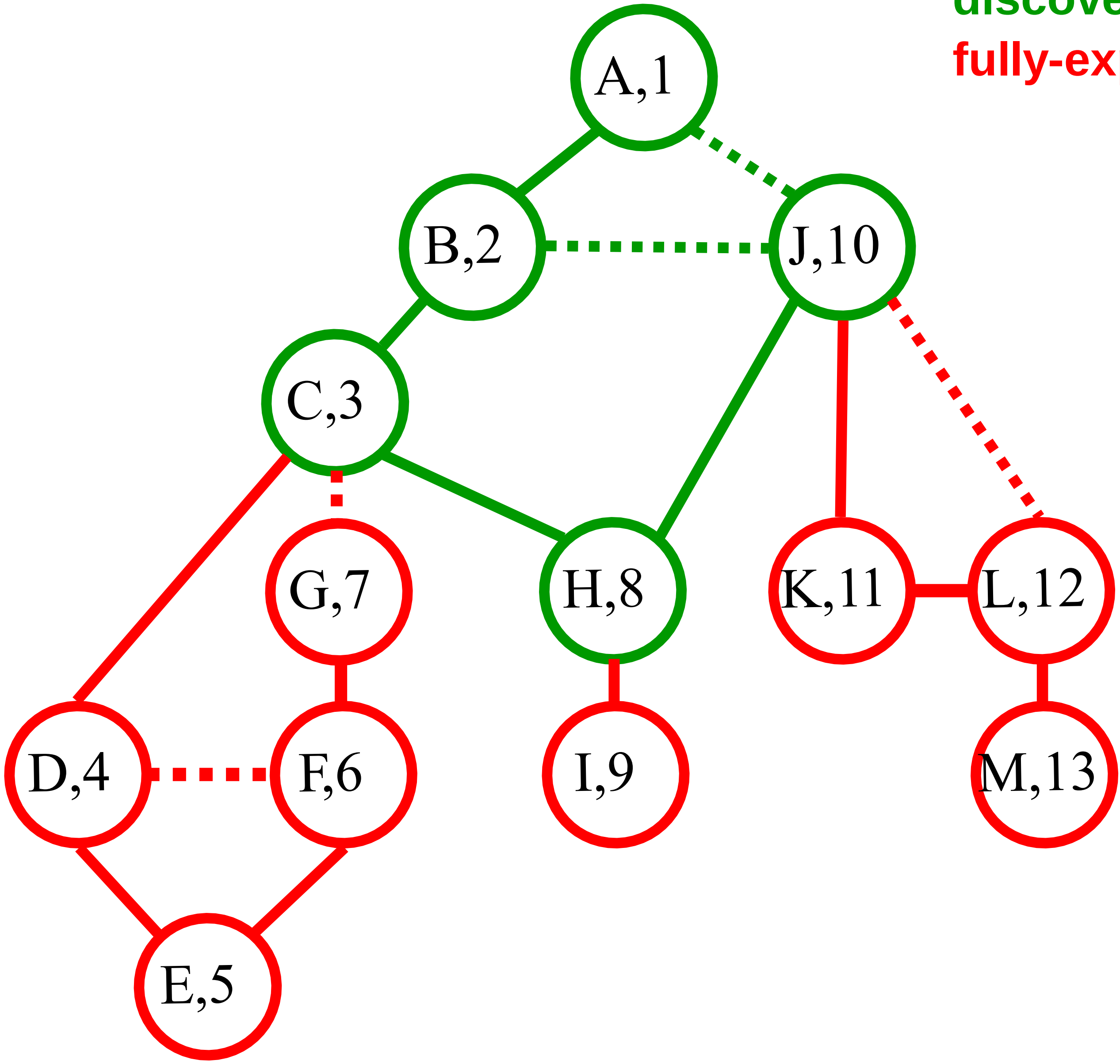
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,~~L~~)

st[] =
{1,2,3,8,
10}

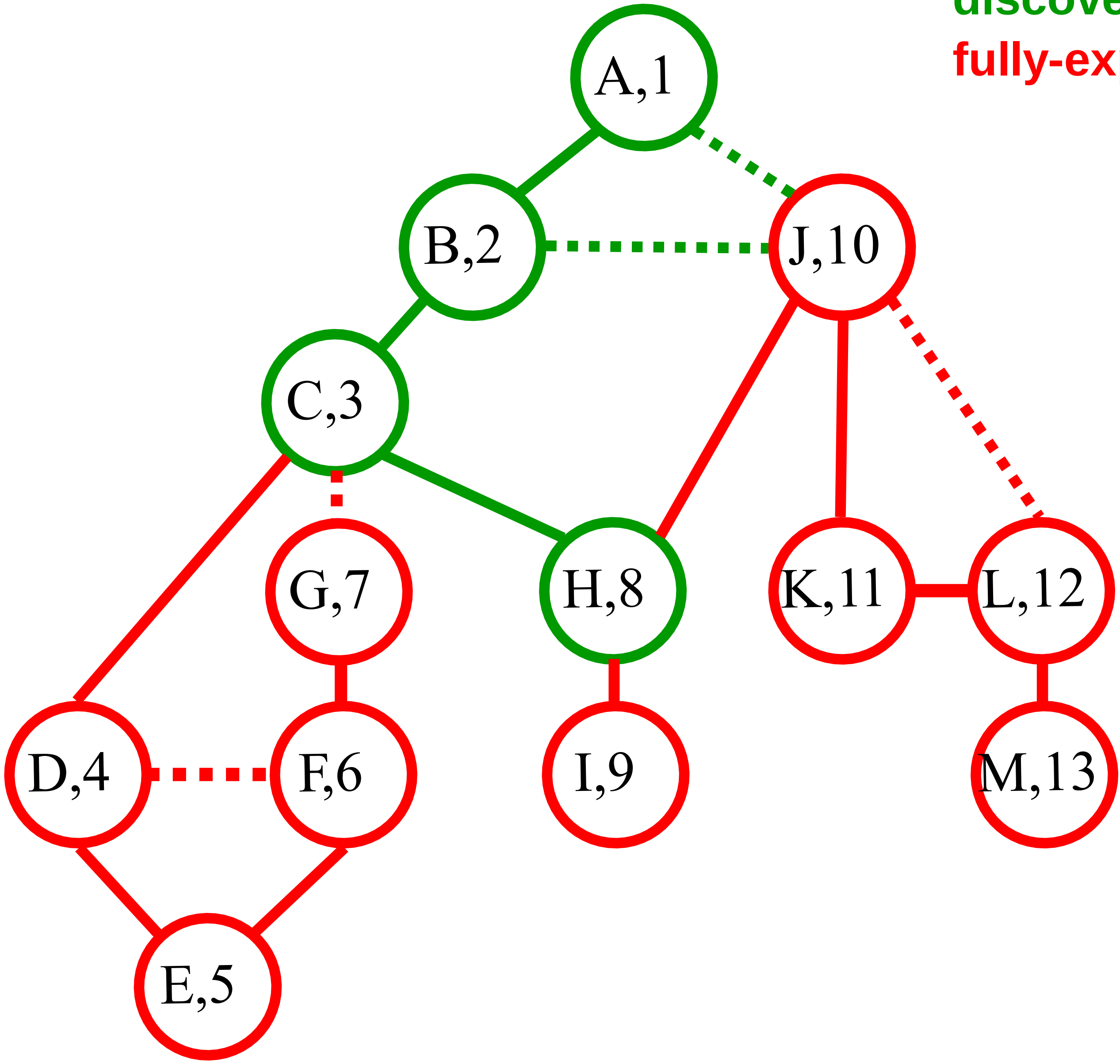
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)

st[] =
{1,2,3,8}

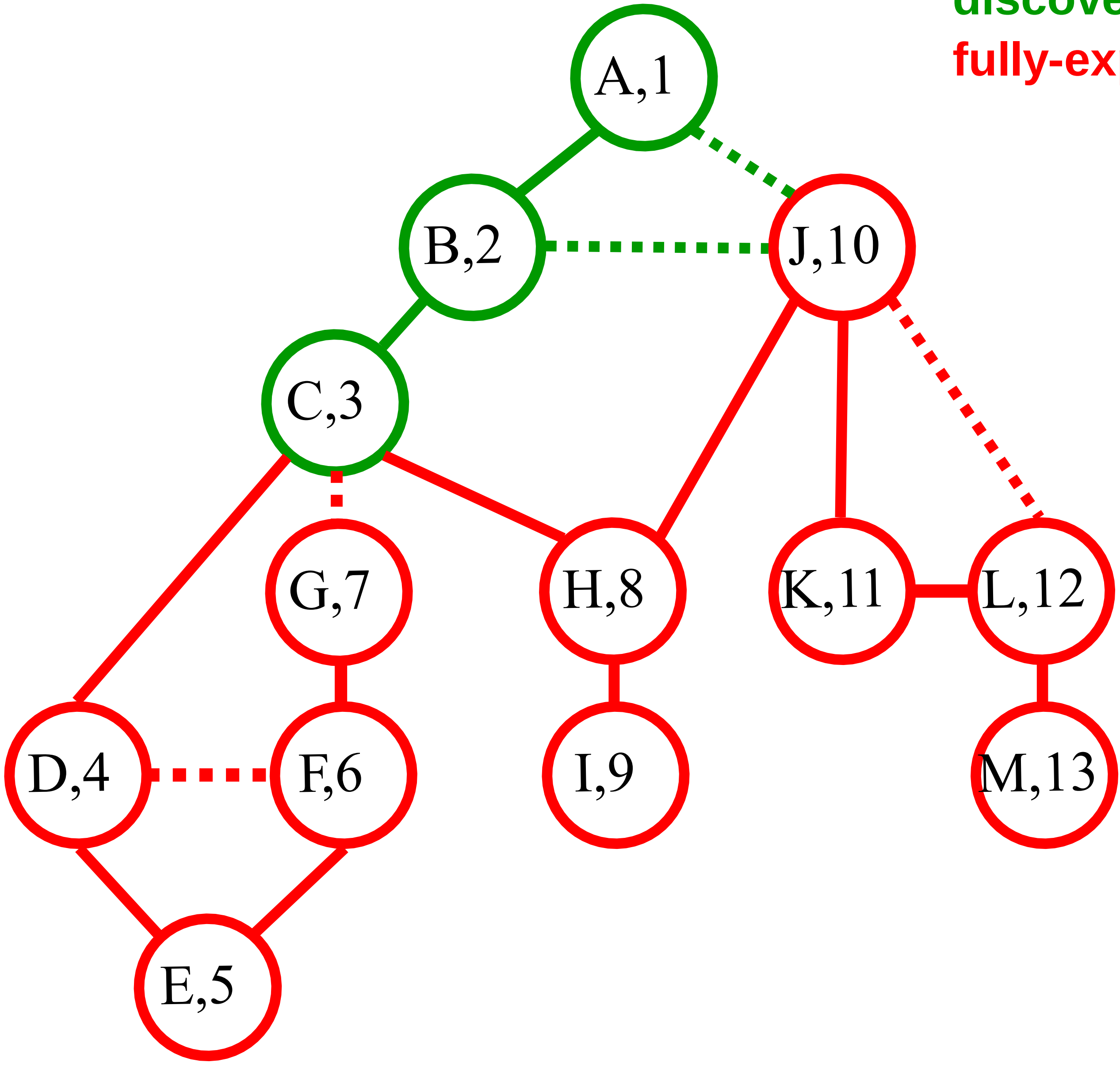
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)

st[] =
{1,2,3}

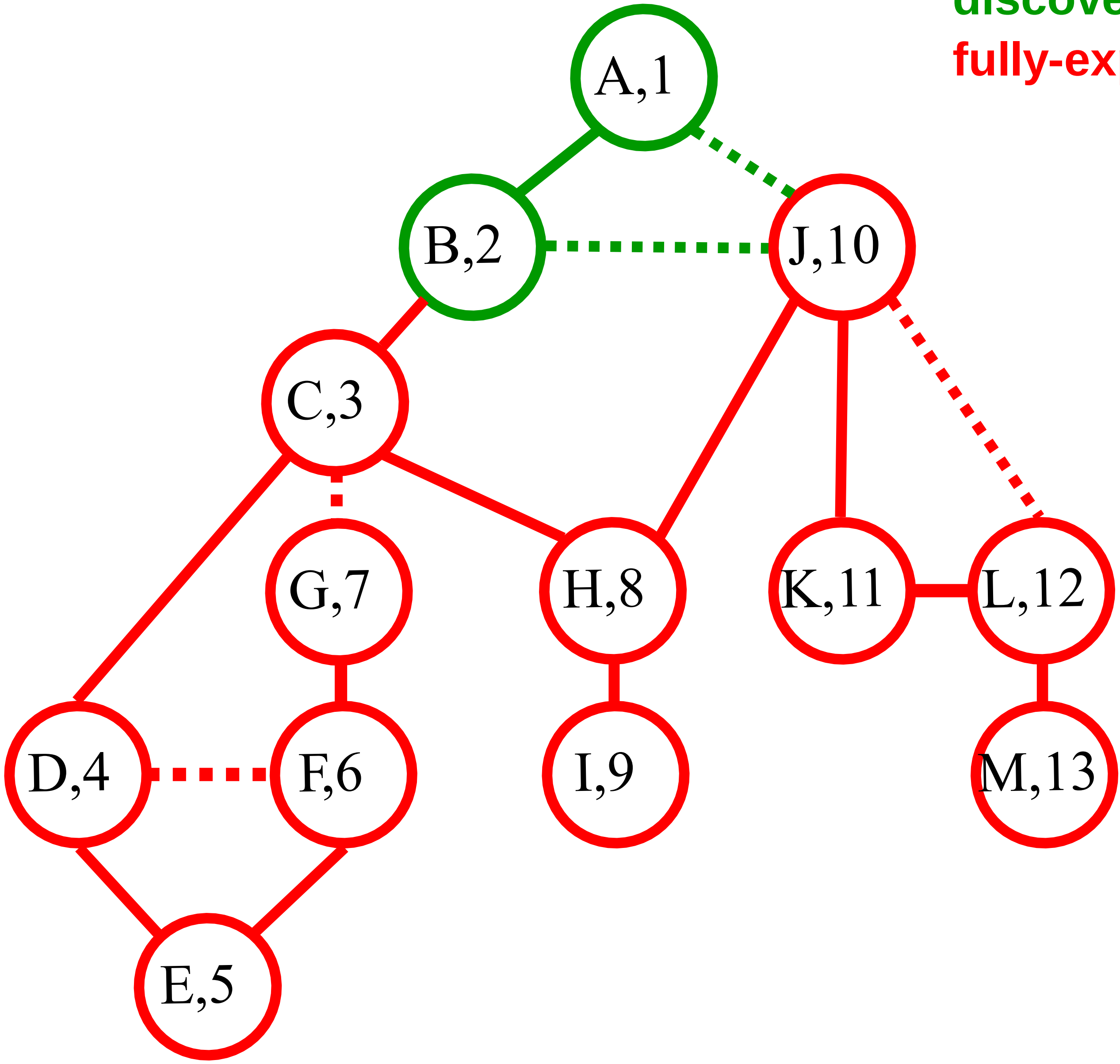
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

st[] =
{1,2}

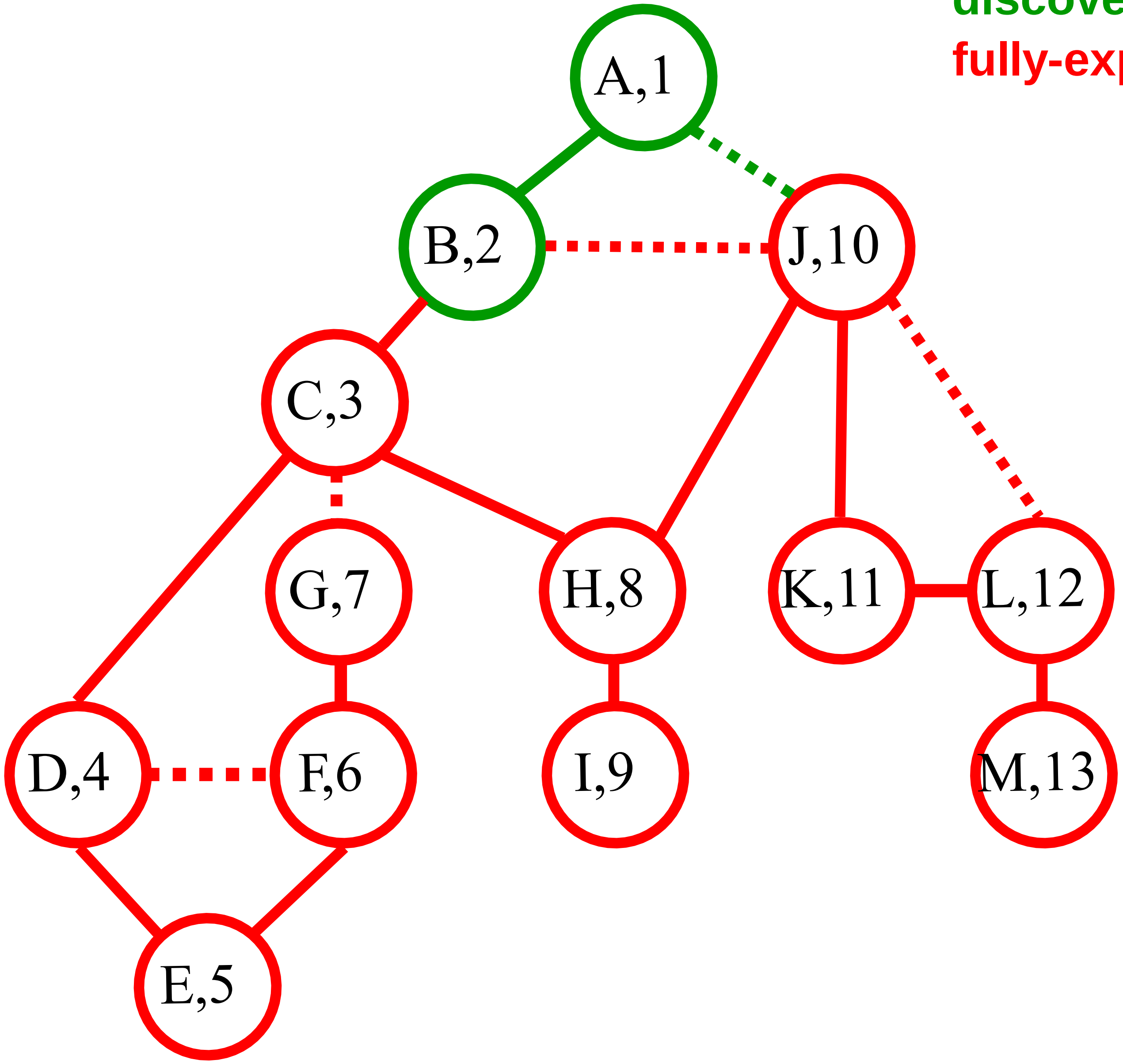
DFS(A)

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

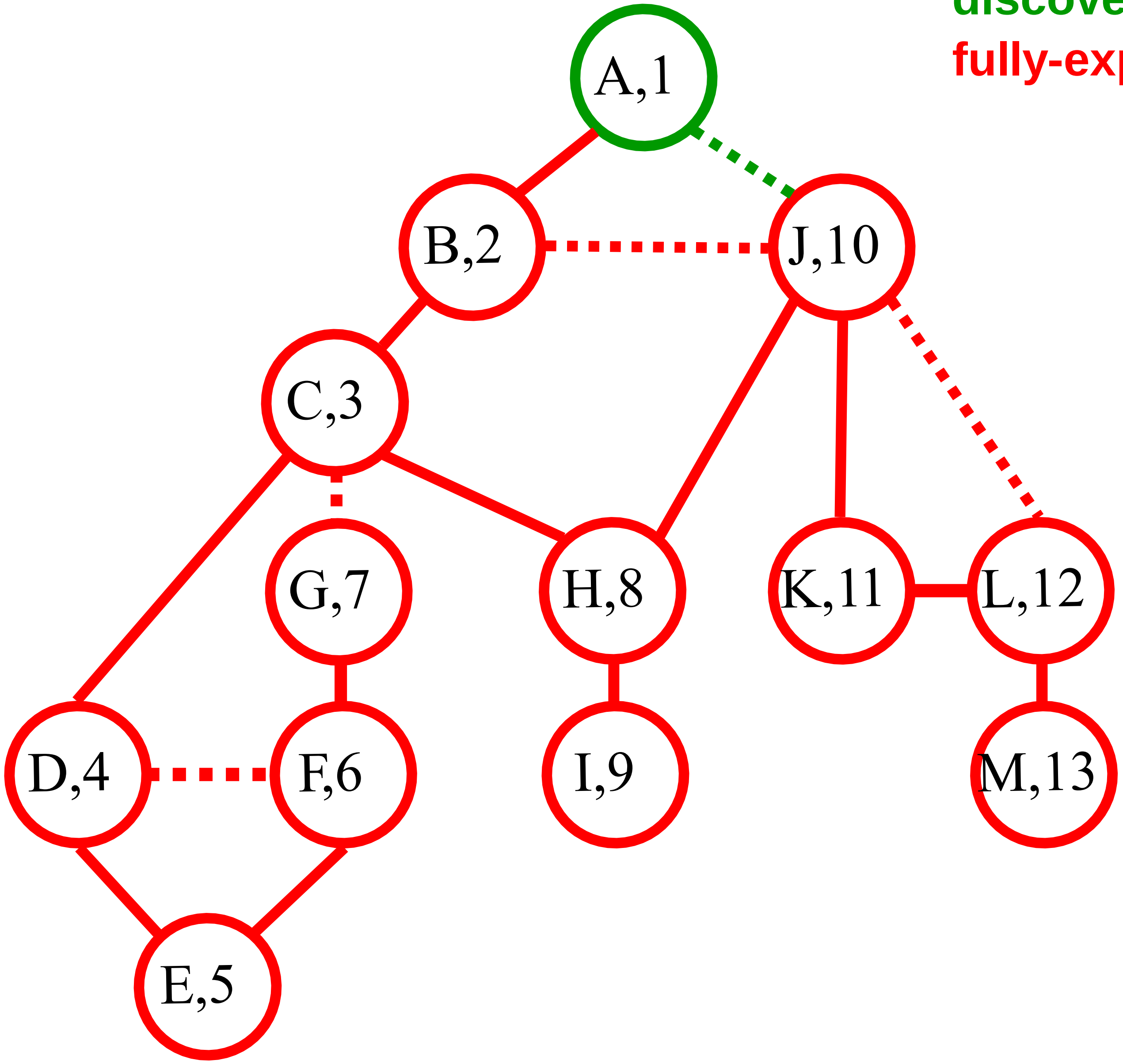
A ($\cancel{B}$,J)
B ($\cancel{A}$,$\cancel{C}$,$\cancel{J}$)

st[] =
{1,2}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



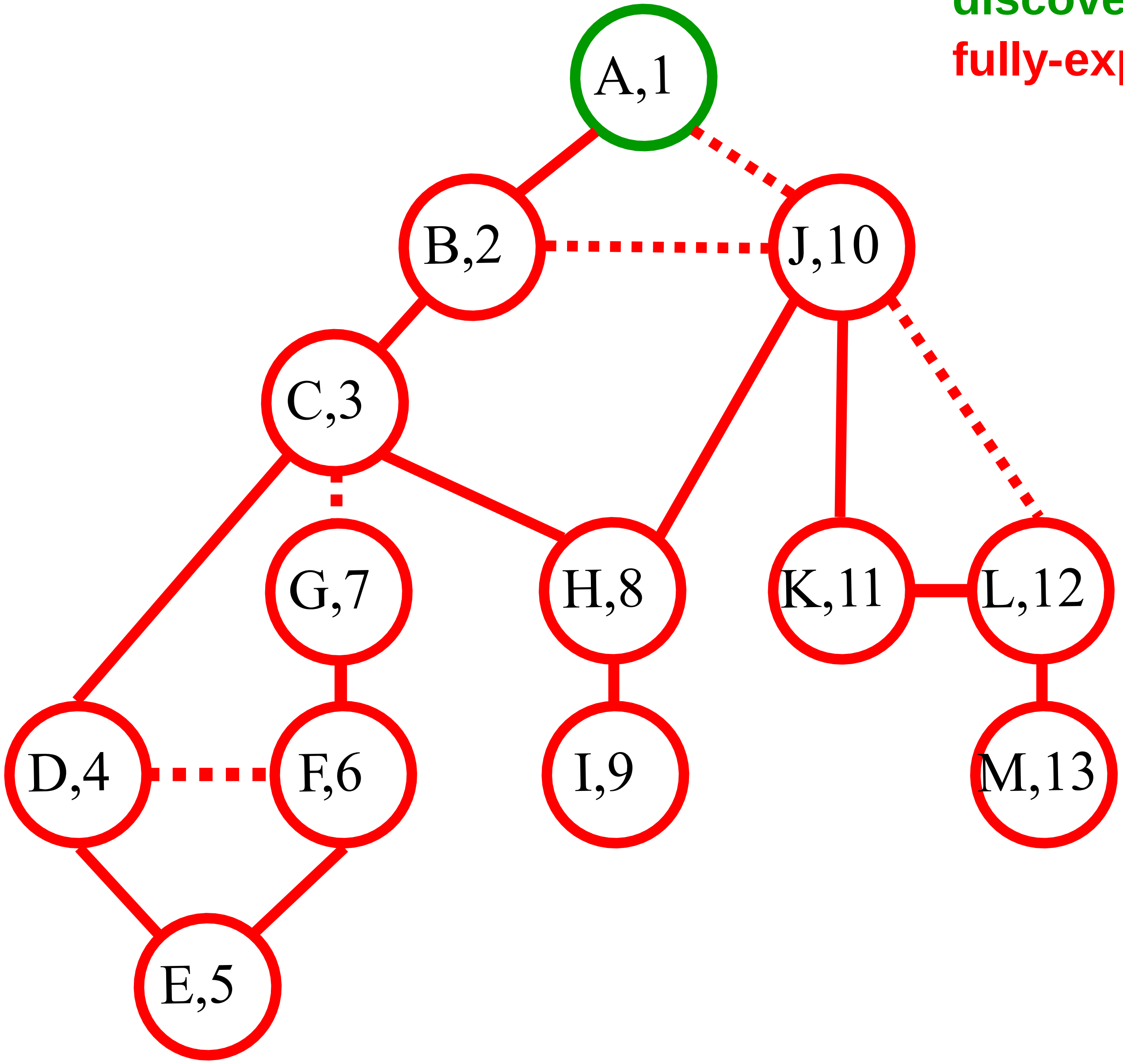
Call Stack:
(Edge list)

A (~~B~~,J)

st[] =
{1}

DFS(A)

Color code:
undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

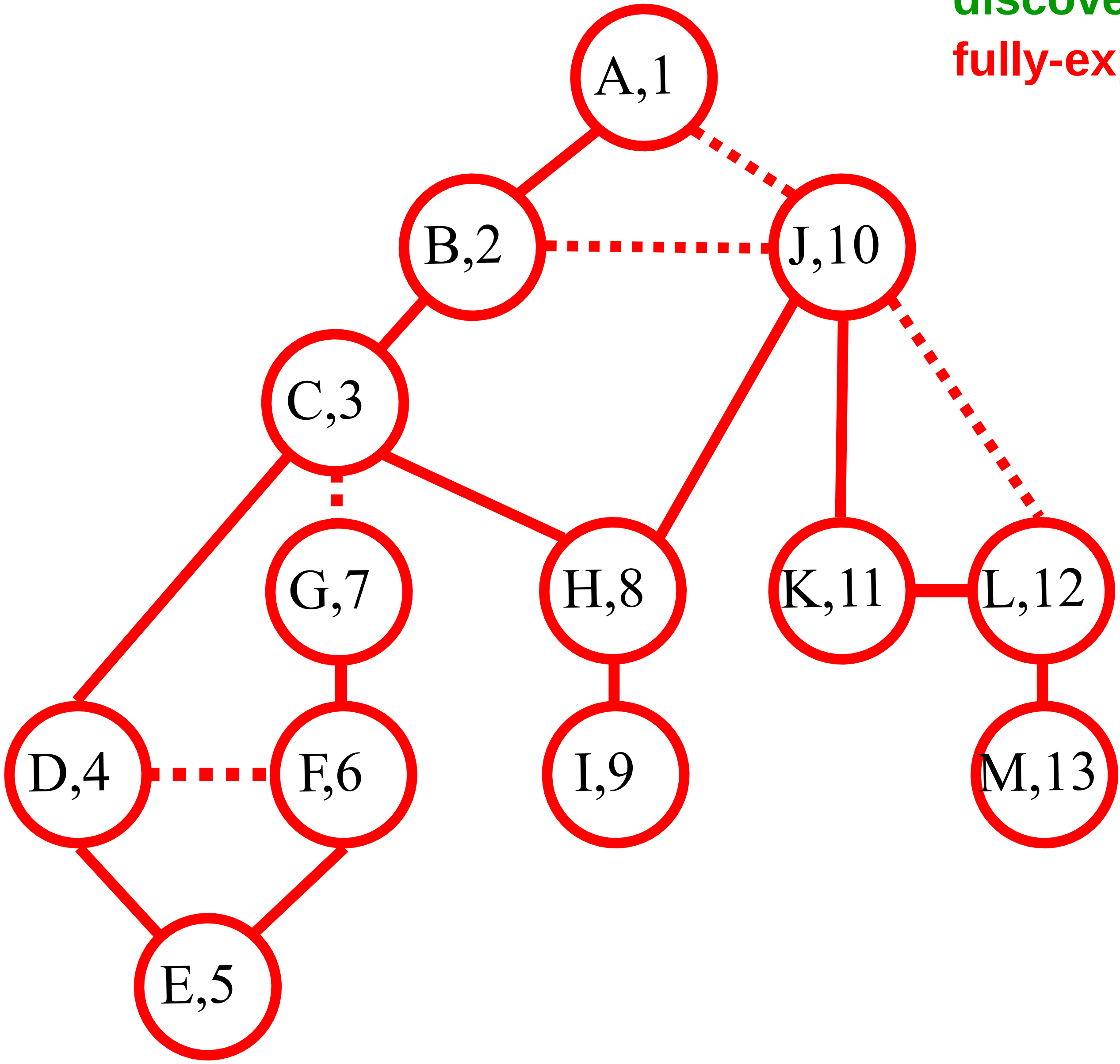
A (~~B~~,~~J~~)

st[] =
{1}

DFS(A)

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

TA-DA!!

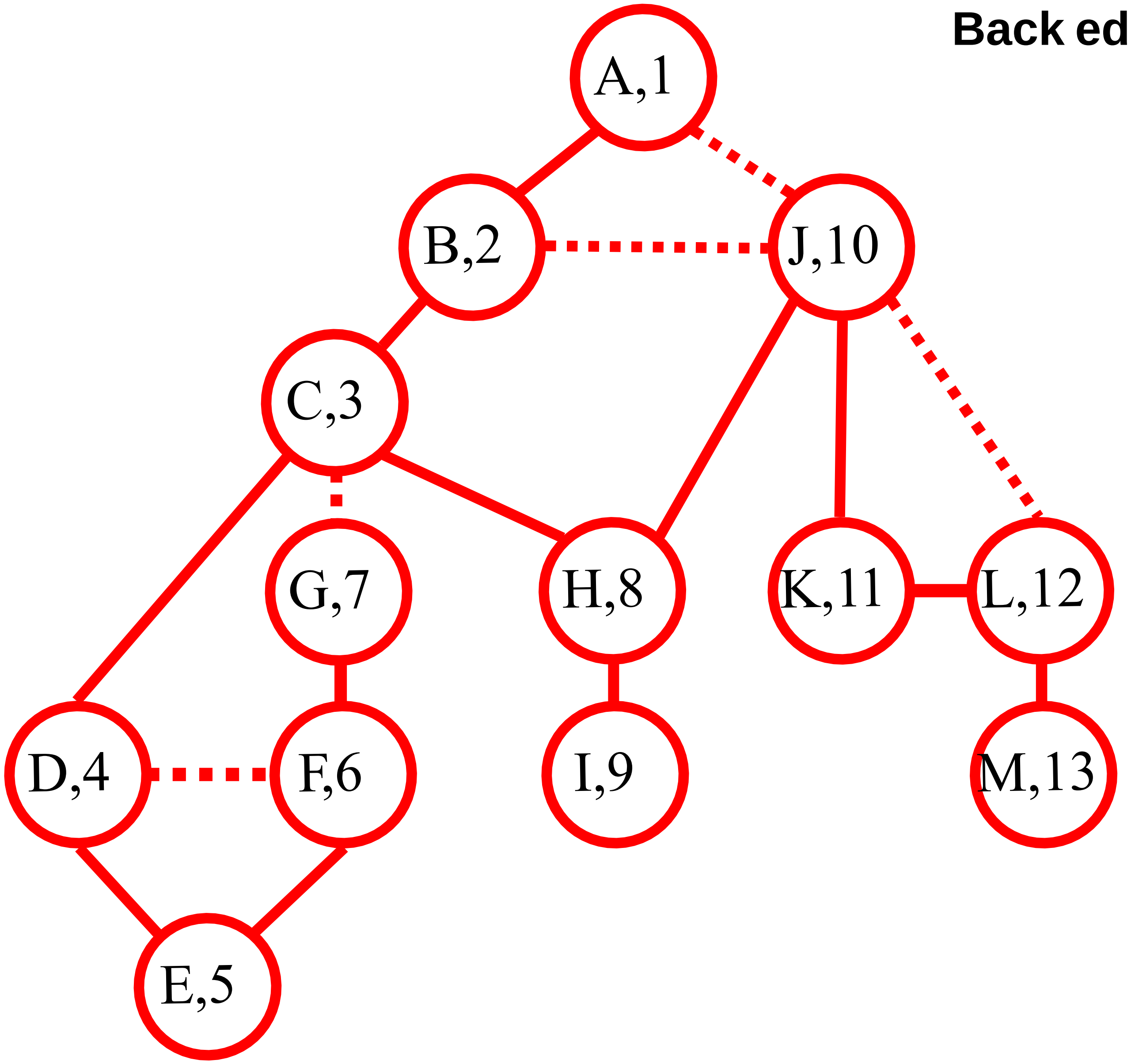
st[] = {}

DFS(A)

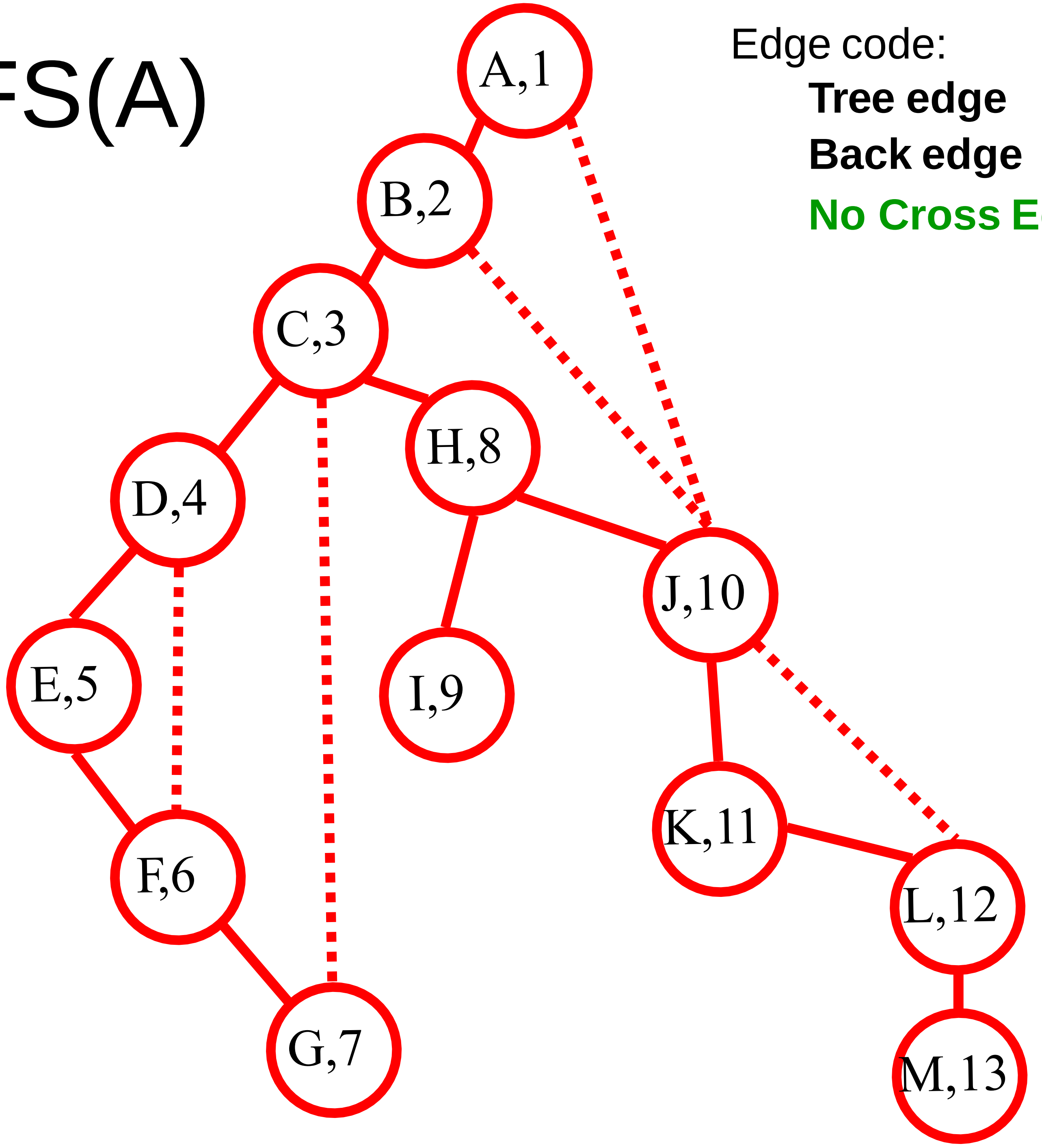
Edge code:

Tree edge —

Back edge ⋯



DFS(A)



Edge code:

Tree edge ——

Back edge ▪▪▪▪▪

No Cross Edges!

Properties of (undirected) DFS

Like BFS(s):

- DFS(s) visits x iff there is a path in G from s to x

  So, we can use DFS to find connected components

- Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G

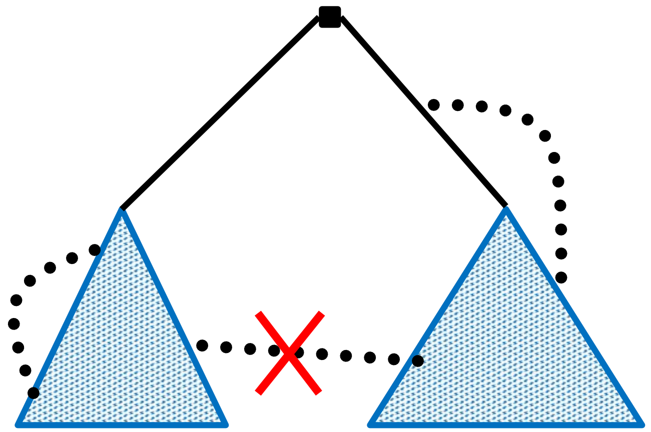
Unlike the BFS tree:

- The DF spanning tree isn't minimum depth
- Its levels don't reflect min distance from the root
- Non-tree edges never join vertices on the same or adjacent levels

Non-Tree Edges in DFS

All non-tree edges join a vertex and one of its descendants/ancestors in the DFS tree

BFS tree ≠ DFS tree, but, as with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple" – only descendant/ancestor



Non-Tree Edges in DFS

Obs: During DFS(x) every vertex marked visited is a descendant of x in the DFS tree

Lemma: For every edge $\{x, y\}$, if $\{x, y\}$ is not in DFS tree, then one of x or y is an ancestor of the other in the tree.

Proof:

One of x or y is visited first, suppose WLOG that x is visited first and therefore DFS(x) was called before DFS(y)

Since $\{x, y\}$ is not in DFS tree, y was visited when the edge {x,y} was examined during DFS(x)

Therefore y was visited during the call to DFS(x) so y is a descendant of x.

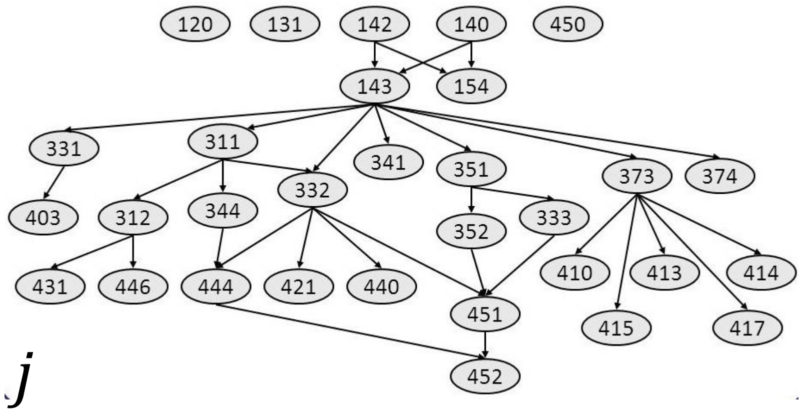
DAGs and Topological Ordering

Precedence Constraints

In a directed graph, an edge (i, j) means task i must occur before task j.

Applications

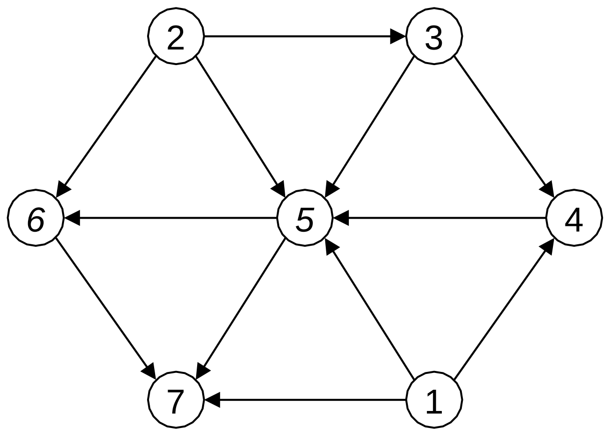
- Course prerequisite:
  course i must be taken before j
- Compilation:
  must compile module i before j
- Computing overflow:
  output of job i is part of input to job j
- Manufacturing or assembly:
  send it before paint it



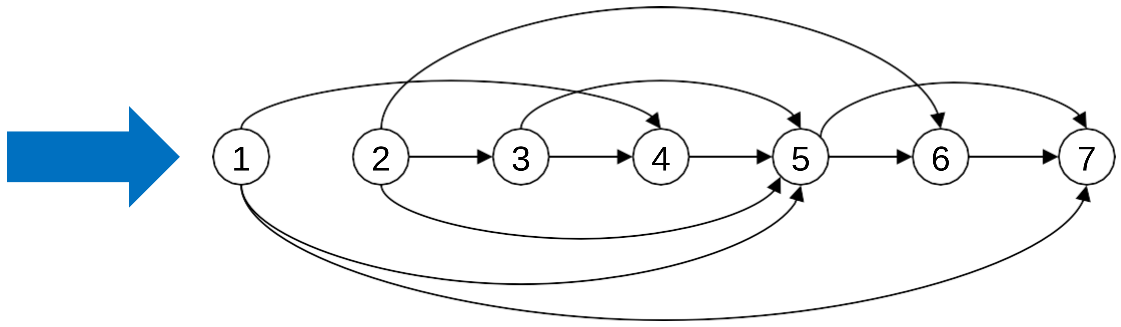
Directed Acyclic Graphs (DAG)

A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

Def: A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.



a DAG



a topological ordering of that DAG– all edges left-to-right

DAGs: A Sufficient Condition

Lemma: If G has a topological order, then G is a DAG.

Pf. (by contradiction)

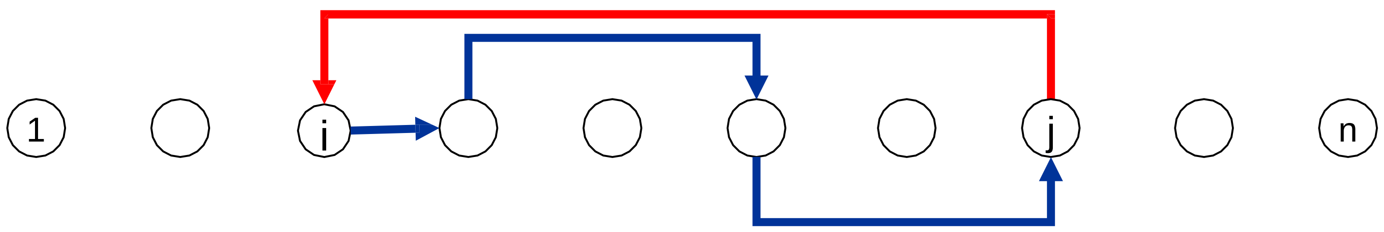
Suppose that G has a topological order 1,2, ... $, n$ and that G also has a directed cycle C.

Let i be the lowest-indexed node in C, and let j be the node just before i ; thus (j, i) is an (directed) edge.

By our choice of i , we have $i < j$.

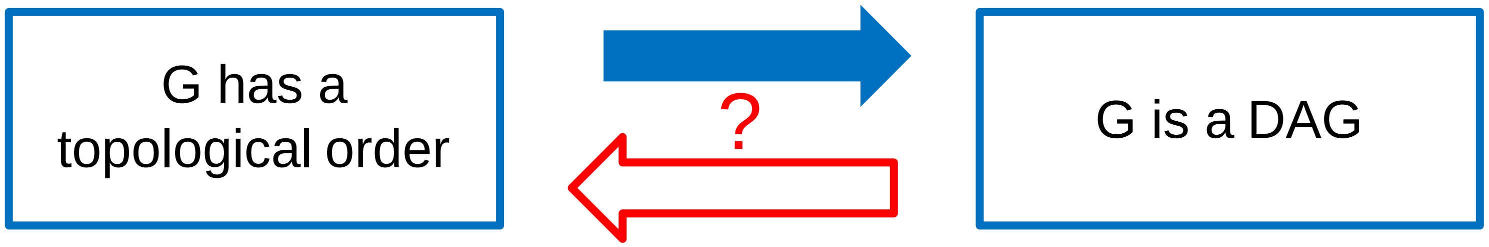
On the other hand, since (j, i) is an edge and 1, ... $, n$ is a topological order, we must have $j < i$, a contradiction

the directed cycle C



the supposed topological order: 1,2,…,n

DAGs: A Sufficient Condition



Every DAG has a source node

Lemma: If G is a DAG, then G has a node with no incoming edges (i.e., a source).

Pf. (by contradiction)

Suppose that G is a DAG and it has no source

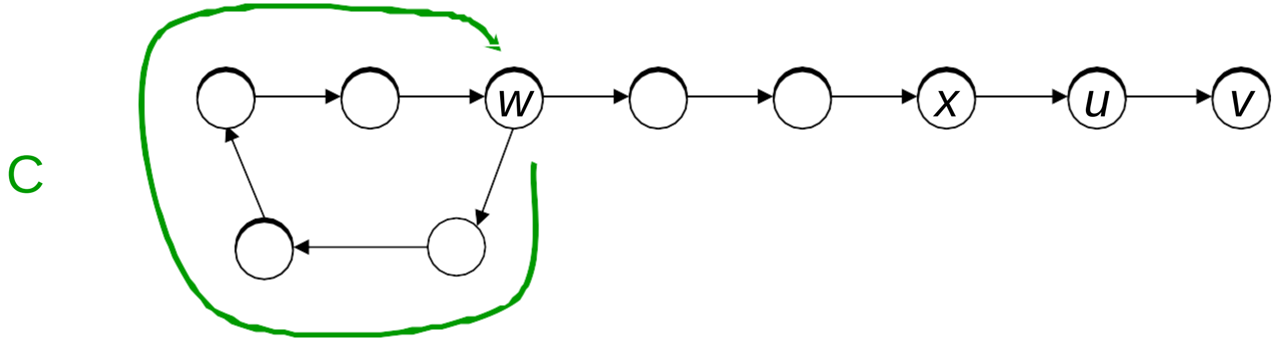
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Is this similar to a previous proof?

Repeat until we visit a node, say w, twice.

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.



DAG => Topological Order

Lemma: If G is a DAG, then G has a topological order

Pf. (by induction on n)
Base case: true if n = 1.

IH: Every DAG with n-1 vertices has a topological ordering.

IS: Given DAG with n nodes, find a source node v.

$G - \{ v \}$ is a DAG, since deleting v cannot create cycles.

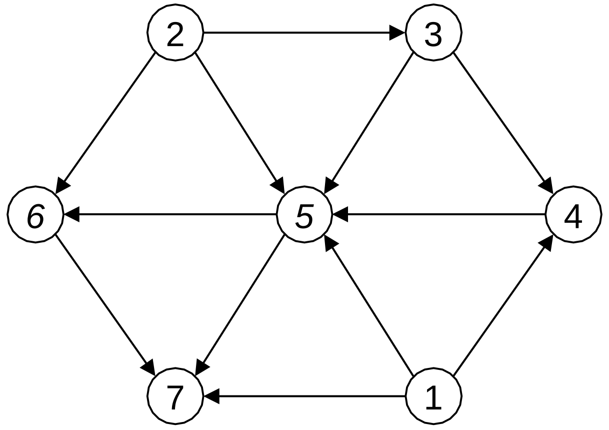
Reminder: Always remove vertices/edges to use IH

By IH, $G - \{ v \}$ has a topological ordering.
Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges.

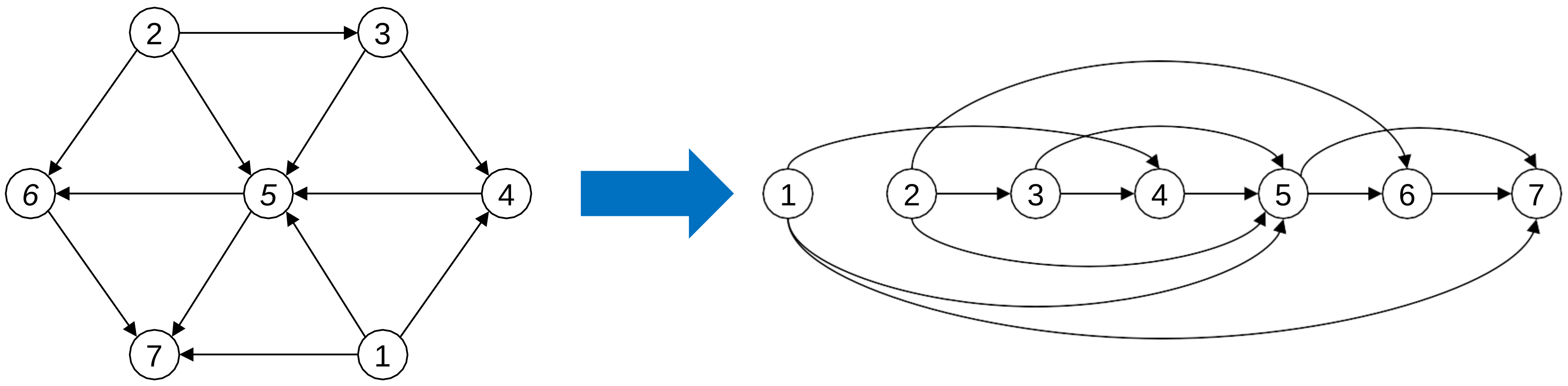
A Characterization of DAGs

$$\begin{tikzcd}
\boxed{\text{G has a topological order}} \arrow[r, shift left] & \boxed{\text{G is a DAG}} \arrow[l, shift left]
\end{tikzcd}$$

Topological Order Algorithm: Example



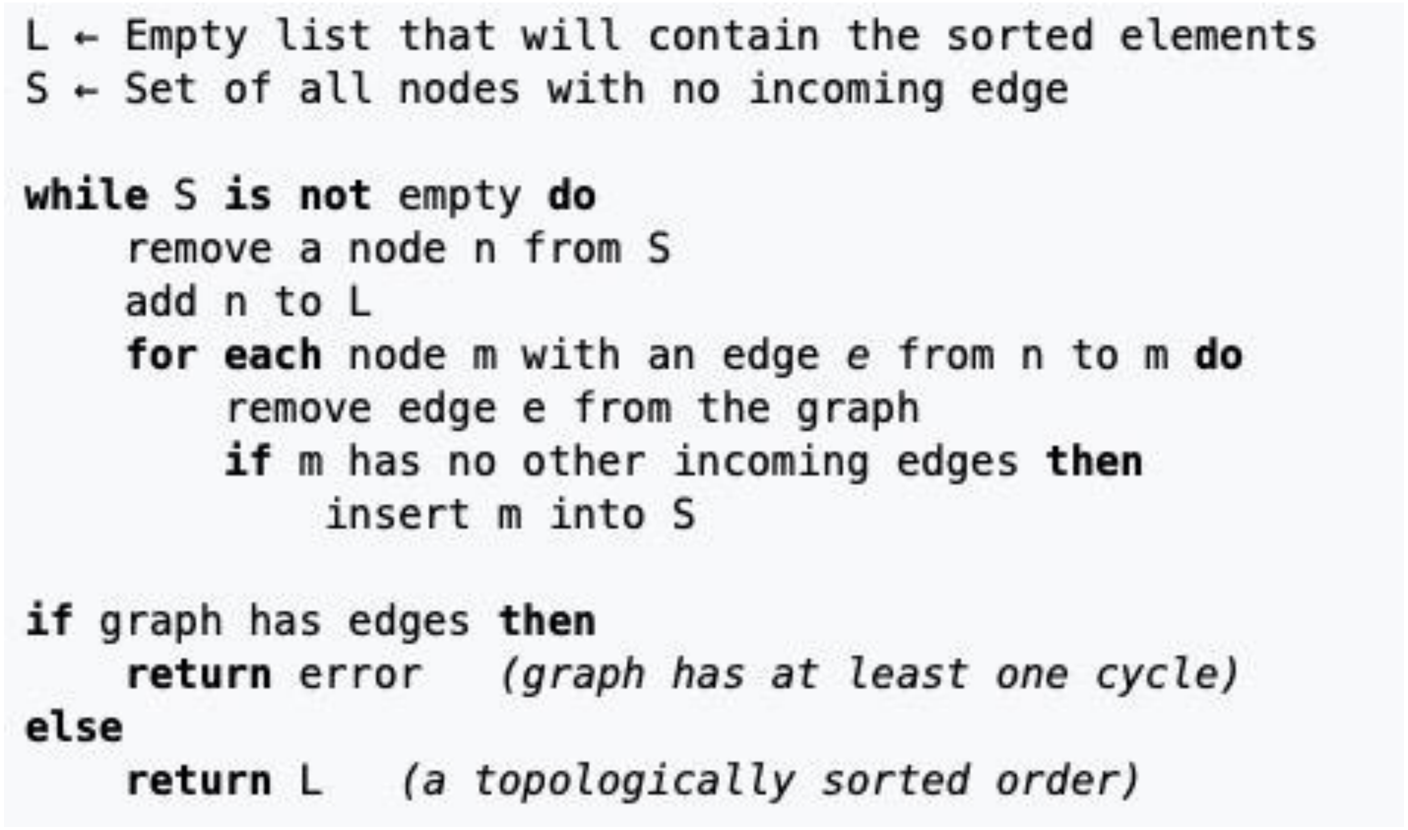
Topological Order Algorithm: Example



Topological order: 1, 2, 3, 4, 5, 6, 7

Topological Sorting Algorithm

Initialization:



```
L ← Empty list that will contain the sorted elements
S ← Set of all nodes with no incoming edge

while S is not empty do
    remove a node n from S
    add n to L
    for each node m with an edge e from n to m do
        remove edge e from the graph
        if m has no other incoming edges then
            insert m into S

if graph has edges then
    return error   (graph has at least one cycle)
else
    return L   (a topologically sorted order)
```

Time: O(m + n) (assuming edge-list representation of graph)

Summary

- Graphs: abstract relationships among pairs of objects
- Terminology: node/vertex/vertices, edges, paths, multi-edges, self-loops, connected
- Representation: Adjacency list, adjacency matrix
- Nodes vs Edges: $m = O(n^2)$, often less
- BFS: Layers, queue, shortest paths, all edges go to same or adjacent layer
- DFS: recursion/stack; all edges ancestor/descendant
- Algorithms: Connected Comp, bipartiteness, topological sort