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Vector Analysis HW #5
Sections 15.1 & 15.2

1. Find the gradient vector field $\vec{\nabla} f$ of $f(x,y,z) = x\cos\frac{y}{z}$.

$$\vec{\nabla} f = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle$$
$$\boxed{\vec{\nabla} f = \left\langle \cos\frac{y}{z}, \frac{-x}{z}\sin\frac{y}{z}, \frac{xy}{z^2}\sin\frac{y}{z}\right\rangle}$$

2. Evaluate the line integral
$$\int_C xyz\,ds$$
where C is described by $\vec{r}(t) = \langle 2\sin t, t, -2\cos t\rangle, 0 \leq t \leq \pi$.
$$\frac{d\vec{r}}{dt} = \langle 2\cos t, 1, 2\sin t\rangle$$

$$
\begin{aligned}
\int_C xyz\,ds &= \int_a^b x(t)\cdot y(t)\cdot z(t)\cdot ||\vec{r}\,'(t)||\,dt \\
&= \int_0^\pi (2\sin t)\cdot(t)\cdot(-2\cos t)\cdot\sqrt{(2\cos t)^2+1^2+(2\sin t)^2}\,dt \\
&= \int_0^\pi -4t\cos t\sin t\cdot\sqrt{4\cos^2 t+1+4\sin^2 t}\,dt \\
&= \int_0^\pi -4t\cos t\sin t\cdot\sqrt{1+4(\cos^2 t+\sin^2 t)}\,dt \\
&= \int_0^\pi -4t\cos t\sin t\cdot\sqrt{5}\,dt \\
&= -4\sqrt{5}\int_0^\pi t\cos t\sin t\,dt \\
u = t \quad dv &= \cos t\sin t\,dt \\
du = dt \quad v &= \frac{1}{2}\sin^2 t \\
&= -4\sqrt{5}\left(\frac{1}{2}t\sin^2 t\Big|_0^\pi - \int_0^\pi \frac{1}{2}\sin^2 t\,dt\right) \\
&= -4\sqrt{5}\left(\left(\frac{1}{2}\pi\sin^2\pi\right)-\left(\frac{1}{2}0\sin^2 0\right)-\int_0^\pi \frac{1}{2}\cdot\frac{1}{2}(1-\cos(2t))\,dt\right) \\
&= -4\sqrt{5}\left((0)-(0)-\frac{1}{4}\int_0^\pi (1-\cos(2t))\,dt\right) \\
&= -4\sqrt{5}\cdot -\frac{1}{4}\left[t+\frac{1}{2}\sin(2t)\right]_0^\pi \\
&= \sqrt{5}\left(\left(\pi+\frac{1}{2}\sin(2\pi)\right)-\left(0+\frac{1}{2}\sin(0)\right)\right) \\
&= \sqrt{5}\,((\pi)-(0))
\end{aligned}
$$

$$\boxed{\int_C xyz\,ds = \pi\sqrt{5}}$$

3. A thin wire is bent into the shape of a semicircle $x^2+y^2=4, x\geq 0$. If the linear density function is a constant k, find the mass of the wire.

   Parametrization:

$$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle,\ -\frac{\pi}{2}\leq t\leq\frac{\pi}{2}$$

$$\frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t\rangle$$

Evaluating Line Integral:

$$\begin{aligned}
M = \int_C \delta(x,y,z)\,ds &= \int_a^b k \cdot ||\vec{r}\,'(t)||\,dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \cdot \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \cdot \sqrt{4\sin^2 t + 4\cos^2 t}\,dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \cdot \sqrt{4(\sin^2 t + \cos^2 t)}\,dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \cdot \sqrt{4}\,dt \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} k \cdot 2\,dt \\
&= 2k \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,dt \\
&= 2k \big[t\big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&= 2k\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) \\
&= 2k\left(\frac{\pi}{2} + \frac{\pi}{2}\right)
\end{aligned}$$

$$\boxed{M = \int_C \delta(x,y,z)\,ds = 2\pi k}$$

4. Use a line integral to find the area of the surface that extends upward from the semicircle $y = \sqrt{4 - x^2}$ in the xy-plane to the surface $z = x^2 y$. Parametrization:

$$\vec{r}(t) = \langle 2\cos t, 2\sin t\rangle,\ 0 \le t \le \pi$$

$$\frac{d\vec{r}}{dt} = \langle -2\sin t, 2\cos t\rangle$$

Evaluating Line Integral:

$$\begin{aligned}
\int_C z\,ds &= \int_a^b x(t)^2 \cdot y(t) \cdot ||\vec{r}\,'(t)||\,dt \\
&= \int_0^\pi (2\cos t)^2 \cdot (2\sin t) \cdot \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt \\
&= \int_0^\pi 4\cos^2 t \cdot 2\sin t \cdot \sqrt{4\sin^2 t + 4\cos^2 t}\,dt \\
&= \int_0^\pi 8\cos^2 t \sin t \cdot \sqrt{4(\sin^2 t + \cos^2 t)}\,dt \\
&= \int_0^\pi 16\cos^2 t \sin t \cdot\,dt
\end{aligned}$$

$u = \cos t \quad du = -\sin t\,dt$

$$\begin{aligned}
&= -16\int_1^{-1} u^2\,du \\
&= 16\int_{-1}^{1} u^2\,du \\
&= 16\left[\frac{1}{3}u^3\right]_{-1}^{1} \\
&= \frac{16}{3}((1) - (-1))
\end{aligned}$$

$$\boxed{\int_C z\,ds = \frac{32}{3}}$$

5. Consider the vector field $\vec{F}(x, y) = \langle x^2, -xy\rangle$ and C to be the portion of the circle $x^2 + y^2 = 1$ in the first quadrant traversed counterclockwise, graphed below (but like not really).

   (a) Would you guess that the line integral $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero? Explain.

   I would guess negative. Near the beginning and end of the curve, it looks almost orthogonal, but the middle of the curve's flow seems to completely be going against the grain in the opposite direction of the vector field.

   (b) Calculate the line integral to verify your educated guess in (a).

   Parametrization:

$$\vec{r}(t) = \langle \cos t, \sin t\rangle,\ 0 \le t \le \frac{\pi}{2}$$

$$\frac{d\vec{r}}{dt} = \langle -\sin t, \cos t\rangle$$

Evaluating Line Integral:

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \vec{F}(x(t),y(t))\cdot\vec{r}\,'\,dt \\
&= \int_0^{\frac{\pi}{2}} \langle(\cos t)^2, -(\cos t)(\sin t)\rangle\cdot\langle -\sin t,\cos t\rangle\,dt \\
&= \int_0^{\frac{\pi}{2}} (-\cos^2 t\sin t + -\cos^2 t\sin t)\,dt \\
&= 2\int_0^{\frac{\pi}{2}} -\cos^2 t\sin t\,dt
\end{aligned}$$

$u = \cos t \;\; du = -\sin t\,dt$

$$\begin{aligned}
&= 2\int_1^0 u^2\,du \\
&= -2\int_0^1 u^2\,du \\
&= -2\left[\frac{1}{3}u^3\right]_0^1
\end{aligned}$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = \frac{-2}{3} < 0}$$

6. Evaluate the line integral

$$\int_C \vec{F}\cdot d\vec{r}$$

where $\vec{F}(x,y,z) = \langle \sin x, \cos y, xz\rangle$ and C is given by $\vec{r}(t) = \langle t^3, -t^2, t\rangle$, $0 \le t \le 1$.

$$\frac{d\vec{r}}{dt} = \langle 3t^2, -2t, 1\rangle$$

Evaluating Line Integral:

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \vec{F}(x(t),y(t),z(t))\cdot \vec{r}\,'\,dt \\
&= \int_0^1 \langle \sin(t^3), \cos(-t^2), (t^3\cdot t)\rangle \cdot \langle 3t^2, -2t, 1\rangle\,dt \\
&= \int_0^1 \left(3t^2\sin(t^3) + -2t\cos(-t^2) + t^4\right)\,dt \\
&= \left[-\cos(t^3) + \sin(-t^2) + \frac{1}{5}t^5\right]_0^1 \\
&= \left(-\cos(1) + \sin(-1) + \frac{1}{5}\right) - (-\cos(0)) \\
&= -\cos(1) - \sin(1) + \frac{1}{5} + 1
\end{aligned}$$

$$\boxed{\int_C \vec{F}\cdot d\vec{r} = -\cos(1) - \sin(1) + \frac{6}{5}}$$

7. Show that a constant force field (i.e. $\vec{F}(x,y) = \langle F_1, F_2\rangle$ for some constants F_1 and F_2) does zero work on a particle that moves once, traversed counterclockwise, around the circle $x^2+y^2=1$. Is this also true for a force field $\vec{F}(x,y) = \langle kx, ky\rangle$, where k is a constant?

Parametrization:

$$\vec{r}(t) = \langle \cos t, \sin t\rangle,\ 0\le t\le 2\pi$$
$$\frac{d\vec{r}}{dt} = \langle -\sin t, \cos t\rangle$$

Evaluating Line Integral Where $\vec{F}(x,y) = \langle F_1, F_2\rangle$:

$$\begin{aligned}
W = \int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \langle F_1, F_2\rangle \cdot \langle -\sin t, \cos t\rangle\,dt \\
&= \int_0^{2\pi} \left(-F_1\sin t + F_2\cos t\right)\,dt \\
&= \left[F_1\cos t + F_2\sin t\right]_0^{2\pi} \\
&= (F_1\cos 2\pi + F_2\sin 2\pi) - (F_1\cos 0 + F_2\sin 0) \\
&= (F_1 + 0) - (F_1 + 0)
\end{aligned}$$

$$\boxed{W = \int_C \vec{F}\cdot d\vec{r} = 0}$$

Evaluating Line Integral Where $\vec{F}(x,y) = \langle kx, ky\rangle$:

$$\begin{aligned} W = \int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi} \langle k\cos t, k\sin t\rangle \cdot \langle -\sin t, \cos t\rangle \, dt \\ &= \int_0^{2\pi} (-k\sin t\cos t + k\sin t\cos t)\; dt \\ &= \int_0^{2\pi} 0\, dt \end{aligned}$$

$$\boxed{W = \int_C \vec{F}\cdot d\vec{r} = 0}$$

$$\boxed{\therefore \text{ the particle does zero work in both force fields.}}$$

8. Find the work done by the force field $\vec{F}(x,y,z) = \langle y+z, x+z, x+y\rangle$ on a particle that moves along the line segment from $(1,0,0)$ to $(3,4,2)$.

   Parametrization:

   $$\vec{r}(t) = (1-t)\langle 1,0,0\rangle + t\langle 3,4,2\rangle = \langle 1+2t, 4t, 2t\rangle,\ 0 \le t \le 1$$

   Evaluating Line Integral:

   $\vec{F}$ is a conservative vector field that has a potential function $\varphi(x,y,z) = xy + xz + yz$.

   $$\begin{aligned} W = \int_C \vec{F}\cdot d\vec{r} &= \varphi(3,4,2) - \varphi(1,0,0) \\ &= (12+8+6) - (0) \end{aligned}$$

   $$\boxed{W = \int_C \vec{F}\cdot d\vec{r} = 26}$$

9. Find the work done by the force field $\vec{F}(x,y,z) = \langle x+y, xy, -z^2\rangle$ on a particle that moves along the line segments from $(0,0,0)$ to $(1,3,1)$ to $(2,-1,4)$.

   Parametrization:

   $$\vec{a}(t) = \langle t, 3t, t\rangle,\ 0 \le t \le 1$$

   $$\frac{d\vec{a}}{dt} = \langle 1,3,1\rangle$$

   $$\vec{b}(t) = (1-t)\langle 1,3,1\rangle + t\langle 2,-1,4\rangle = \langle 1+t, 3-4t, 1+3t\rangle,\ 0 \le t \le 1$$

   $$\frac{d\vec{b}}{dt} = \langle 1,-4,3\rangle$$

Evaluating Line Integral:

$$
\begin{aligned}
W = \int_C \vec{F} \cdot d\vec{r} &= \int_{C_1} \vec{F} \cdot d\vec{a} + \int_{C_2} \vec{F} \cdot d\vec{b} \\
&= \int_0^1 \langle t + 3t, 3t \cdot t, -t^2 \rangle \cdot \langle 1, 3, 1 \rangle \, dt \\
&\quad + \int_0^1 \langle 1 + t + 3 - 4t, (1 + t)(3 - 4t), -(1 + 3t)^2 \rangle \cdot \langle 1, -4, 3 \rangle \, dt \\
&= \int_0^1 \langle 4t, 3t^2, -t^2 \rangle \cdot \langle 1, 3, 1 \rangle \, dt \\
&\quad + \int_0^1 \langle 4 - 3t, 3 - t - 4t^2, -1 - 6t - 9t^2 \rangle \cdot \langle 1, -4, 3 \rangle \, dt \\
&= \int_0^1 (4t + 9t^2 - t^2) \, dt \\
&\quad + \int_0^1 ((4 - 3t) + (-12 + 4t + 16t^2) + (-3 - 18t - 27t^2)) \, dt \\
&= \int_0^1 (4t + 8t^2) \, dt + \int_0^1 (-11 - 17t - 11t^2) \, dt \\
&= \int_0^1 (-11 - 13t - 3t^2) \, dt \\
&= \left[-11t - \frac{13}{2} t^2 - t^3 \right]_0^1
\end{aligned}
$$

$$\boxed{W = \int_C \vec{F} \cdot d\vec{r} = -\frac{37}{2}}$$

10. Confirm that $\varphi(x, y) = ye^x + x \sin y$ is a potential function for $\vec{F}(x, y) = \langle ye^x + \sin y, e^x + x \cos y \rangle$.

$$
\begin{aligned}
\vec{\nabla} \varphi(x, y) &= \langle \varphi_x(x, y), \varphi_y(x, y) \rangle \\
&= \langle ye^x + \sin y, e^x + x \cos y \rangle
\end{aligned}
$$

$$\boxed{\vec{\nabla} \varphi(x, y) = \vec{F}(x, y)}$$