15_Step_11

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1 12 steps to Navier-Stokes

The final two steps in this interactive module teaching beginning CFD with Python will both solve the Navier-Stokes equations in two dimensions, but with different boundary conditions.

The momentum equation in vector form for a velocity field $\vec{v}$ is:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

This represents three scalar equations, one for each velocity component (u, v, w). But we will solve it in two dimensions, so there will be two scalar equations.

Remember the continuity equation? This is where the Poisson equation for pressure comes in!

1.1 Step 11: Cavity Flow with Navier-Stokes

Here is the system of differential equations: two equations for the velocity components u, v and one equation for pressure:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right)$$

From the previous steps, we already know how to discretize all these terms. Only the last equation is a little unfamiliar. But with a little patience, it will not be hard!

1.1.1 Discretized equations

First, let's discretize the u-momentum equation, as follows:

$$\frac{u_{i,j}^{n+1} - u_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\frac{u_{i,j}^{n} - u_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\frac{u_{i,j}^{n} - u_{i,j-1}^{n}}{\Delta y} \tag{1}$$

$$= -\frac{1}{\rho}\frac{p_{i+1,j}^{n} - p_{i-1,j}^{n}}{2\Delta x} + \nu\left(\frac{u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}}{\Delta x^2} + \frac{u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n}}{\Delta y^2}\right) \tag{2}$$

Similarly for the v-momentum equation:

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta t} + u_{i,j}^{n}\frac{v_{i,j}^{n} - v_{i-1,j}^{n}}{\Delta x} + v_{i,j}^{n}\frac{v_{i,j}^{n} - v_{i,j-1}^{n}}{\Delta y} \tag{3}$$

$$= -\frac{1}{\rho}\frac{p_{i,j+1}^{n} - p_{i,j-1}^{n}}{2\Delta y} + \nu\left(\frac{v_{i+1,j}^{n} - 2v_{i,j}^{n} + v_{i-1,j}^{n}}{\Delta x^2} + \frac{v_{i,j+1}^{n} - 2v_{i,j}^{n} + v_{i,j-1}^{n}}{\Delta y^2}\right) \tag{4}$$

Finally, the discretized pressure-Poisson equation can be written thus:

$$\frac{p_{i+1,j}^{n} - 2p_{i,j}^{n} + p_{i-1,j}^{n}}{\Delta x^2} + \frac{p_{i,j+1}^{n} - 2 * p_{i,j}^{n} + p_{i,j-1}^{n}}{\Delta y^2} = \rho\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right)\right.$$

$$\left. - \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} - 2\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right]$$

You should write these equations down on your own notes, by hand, following each term mentally as you write it.

As before, let's rearrange the equations in the way that the iterations need to proceed in the code. First, the momentum equations for the velocity at the next time step.

The momentum equation in the u direction:

$$u_{i,j}^{n+1} = u_{i,j}^{n} - u_{i,j}^{n}\frac{\Delta t}{\Delta x}(u_{i,j}^{n} - u_{i-1,j}^{n}) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}(u_{i,j}^{n} - u_{i,j-1}^{n})$$

$$-\frac{\Delta t}{\rho 2\Delta x}(p_{i+1,j}^{n} - p_{i-1,j}^{n}) + \nu\left(\frac{\Delta t}{\Delta x^2}(u_{i+1,j}^{n} - 2u_{i,j}^{n} + u_{i-1,j}^{n}) + \frac{\Delta t}{\Delta y^2}(u_{i,j+1}^{n} - 2u_{i,j}^{n} + u_{i,j-1}^{n})\right)$$

The momentum equation in the v direction:

$$v_{i,j}^{n+1} = v_{i,j}^{n} - u_{i,j}^{n}\frac{\Delta t}{\Delta x}(v_{i,j}^{n} - v_{i-1,j}^{n}) - v_{i,j}^{n}\frac{\Delta t}{\Delta y}(v_{i,j}^{n} - v_{i,j-1}^{n})$$

$$-\frac{\Delta t}{\rho 2\Delta y}(p_{i,j+1}^{n} - p_{i,j-1}^{n}) + \nu\left(\frac{\Delta t}{\Delta x^2}(v_{i+1,j}^{n} - 2v_{i,j}^{n} + v_{i-1,j}^{n}) + \frac{\Delta t}{\Delta y^2}(v_{i,j+1}^{n} - 2v_{i,j}^{n} + v_{i,j-1}^{n})\right)$$

Almost there! Now, we rearrange the pressure-Poisson equation:

$$p_{i,j}^{n} = \frac{(p_{i+1,j}^{n} + p_{i-1,j}^{n})\Delta y^2 + (p_{i,j+1}^{n} + p_{i,j-1}^{n})\Delta x^2}{2(\Delta x^2 + \Delta y^2)} - \frac{\rho\Delta x^2\Delta y^2}{2(\Delta x^2 + \Delta y^2)}\times$$

$$\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) - \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right.$$

$$\left. - 2\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right]$$

The initial condition is $u, v, p = 0$ everywhere, and the boundary conditions are:

$u = 1$ at $y = 2$ (the "lid");
$u, v = 0$ on the other boundaries;
$\frac{\partial p}{\partial y} = 0$ at $y = 0$;
$p = 0$ at $y = 2$
$\frac{\partial p}{\partial x} = 0$ at $x = 0, 2$

1.2 Implementing Cavity Flow

```
In [1]: from mpl_toolkits.mplot3d import Axes3D
        from matplotlib import cm
        import matplotlib.pyplot as plt
        import numpy as np
        %matplotlib inline

        nx = 41
        ny = 41
        nt = 500
        nit=50
        c = 1
        dx = 2.0/(nx-1)
        dy = 2.0/(ny-1)
        x = np.linspace(0,2,nx)
        y = np.linspace(0,2,ny)
        X,Y = np.meshgrid(x,y)

        rho = 1
        nu = .1
        dt = .001

        u = np.zeros((ny, nx))
        v = np.zeros((ny, nx))
        p = np.zeros((ny, nx))
        b = np.zeros((ny, nx))
```

The pressure Poisson equation that's written above can be hard to write out without typos. The function `buildUpB` below represents the contents of the square brackets, so that the entirety of the PPE is slightly more manageable.

```
In [2]: def buildUpB(b, rho, dt, u, v, dx, dy):

            b[1:-1,1:-1]=rho*(1/dt*((u[1:-1,2:]-u[1:-1,0:-2])/(2*dx)+(v[2:,1:-1]-v[0:-2,1:-1])/(2*dy))-\
                              ((u[1:-1,2:]-u[1:-1,0:-2])/(2*dx))**2-\
                              2*((u[2:,1:-1]-u[0:-2,1:-1])/(2*dy)*(v[1:-1,2:]-v[1:-1,0:-2])/(2*dx))-\
                              ((v[2:,1:-1]-v[0:-2,1:-1])/(2*dy))**2)

            return b
```

The function `presPoisson` is also defined to help segregate the different rounds of calculations. Note the presence of the pseudo-time variable `nit`. This sub-iteration in the Poisson calculation helps ensure a divergence-free field.

```
In [3]: def presPoisson(p, dx, dy, b):
            pn = np.empty_like(p)
            pn = p.copy()
```

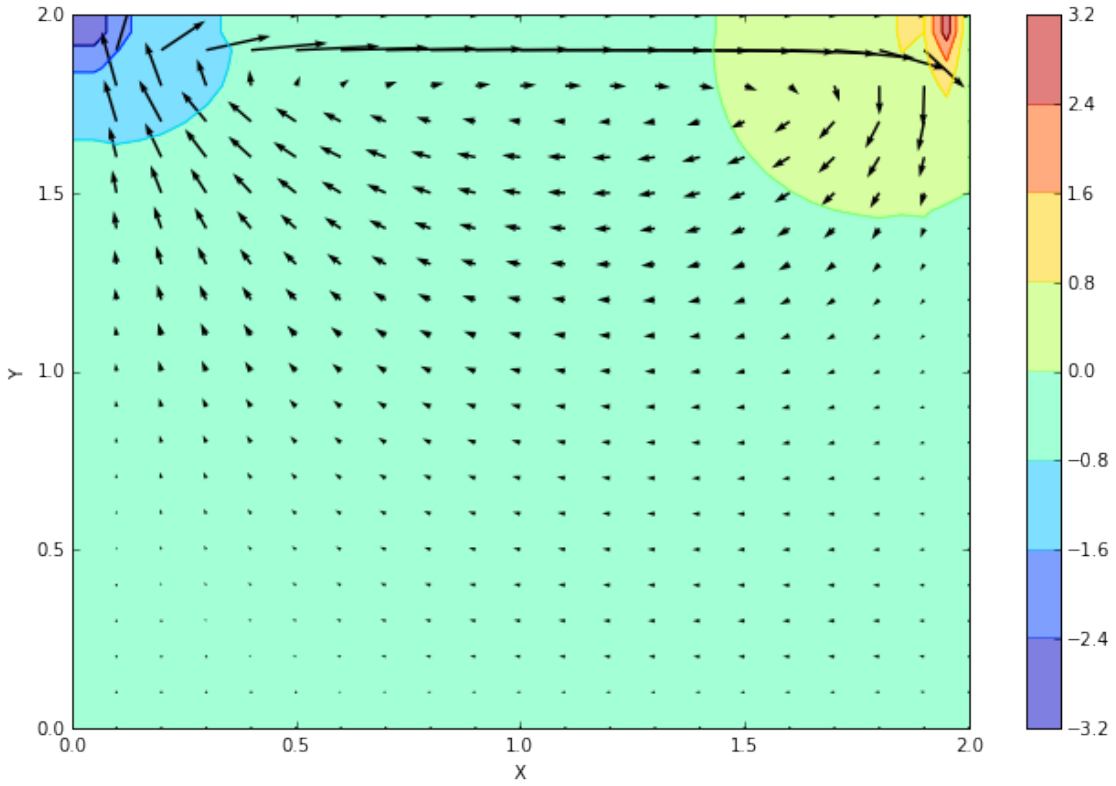
```
        for q in range(nit):
            pn = p.copy()
            p[1:-1,1:-1] = ((pn[1:-1,2:]+pn[1:-1,0:-2])*dy**2+(pn[2:,1:-1]+pn[0:-2,1:-1])*dx**2)/\
                           (2*(dx**2+dy**2)) -\
                           dx**2*dy**2/(2*(dx**2+dy**2))*b[1:-1,1:-1]

            p[-1,:] =p[-2,:] ##dp/dy = 0 at y = 2
            p[0,:] = p[1,:]  ##dp/dy = 0 at y = 0
            p[:,0]=p[:,1]    ##dp/dx = 0 at x = 0
            p[:,-1]=0        ##p = 0 at x = 2

        return p
```

Finally, the rest of the cavity flow equations are wrapped inside the function `cavityFlow`, allowing us to easily plot the results of the cavity flow solver for different lengths of time.

```
In [4]: def cavityFlow(nt, u, v, dt, dx, dy, p, rho, nu):
            un = np.empty_like(u)
            vn = np.empty_like(v)
            b = np.zeros((ny, nx))

            for n in range(nt):
                un = u.copy()
                vn = v.copy()

                b = buildUpB(b, rho, dt, u, v, dx, dy)
                p = presPoisson(p, dx, dy, b)

                u[1:-1,1:-1] = un[1:-1,1:-1]-\
                               un[1:-1,1:-1]*dt/dx*(un[1:-1,1:-1]-un[1:-1,0:-2])-\
                               vn[1:-1,1:-1]*dt/dy*(un[1:-1,1:-1]-un[0:-2,1:-1])-\
                               dt/(2*rho*dx)*(p[1:-1,2:]-p[1:-1,0:-2])+\
                               nu*(dt/dx**2*(un[1:-1,2:]-2*un[1:-1,1:-1]+un[1:-1,0:-2])+\
                               dt/dy**2*(un[2:,1:-1]-2*un[1:-1,1:-1]+un[0:-2,1:-1]))

                v[1:-1,1:-1] = vn[1:-1,1:-1]-\
                               un[1:-1,1:-1]*dt/dx*(vn[1:-1,1:-1]-vn[1:-1,0:-2])-\
                               vn[1:-1,1:-1]*dt/dy*(vn[1:-1,1:-1]-vn[0:-2,1:-1])-\
                               dt/(2*rho*dy)*(p[2:,1:-1]-p[0:-2,1:-1])+\
                               nu*(dt/dx**2*(vn[1:-1,2:]-2*vn[1:-1,1:-1]+vn[1:-1,0:-2])+\
                               (dt/dy**2*(vn[2:,1:-1]-2*vn[1:-1,1:-1]+vn[0:-2,1:-1])))

                u[0,:] = 0
                u[:,0] = 0
                u[:,-1] = 0
                u[-1,:] = 1    #set velocity on cavity lid equal to 1
                v[0,:] = 0
                v[-1,:]=0
                v[:,0] = 0
                v[:,-1] = 0


            return u, v, p
```

Let's start with `nt = 100` and see what the solver gives us:

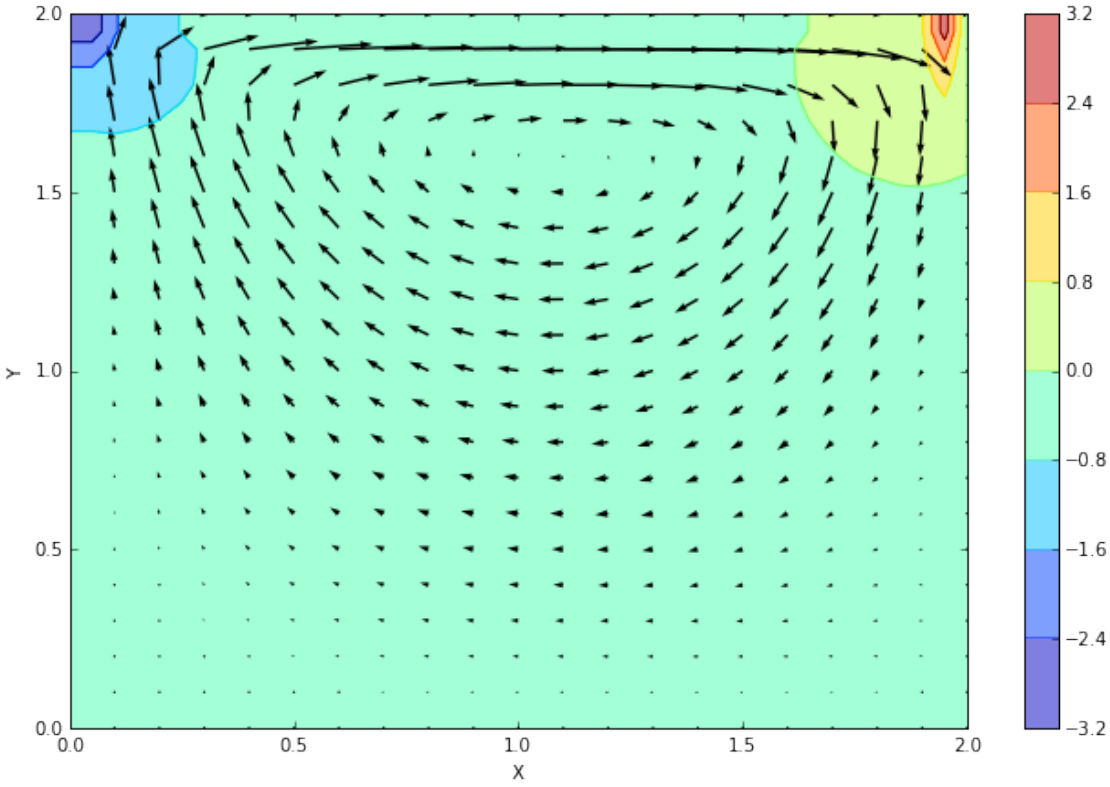
In [5]:
```
u = np.zeros((ny, nx))
v = np.zeros((ny, nx))
p = np.zeros((ny, nx))
b = np.zeros((ny, nx))
nt = 100
u, v, p = cavityFlow(nt, u, v, dt, dx, dy, p, rho, nu)
fig = plt.figure(figsize=(11,7), dpi=100)
plt.contourf(X,Y,p,alpha=0.5)    ###plnttong the pressure field as a contour
plt.colorbar()
plt.contour(X,Y,p)               ###plotting the pressure field outlines
plt.quiver(X[::2,::2],Y[::2,::2],u[::2,::2],v[::2,::2]) ##plotting velocity
plt.xlabel('X')
plt.ylabel('Y');
```



You can see that two distinct pressure zones are forming and that the spiral pattern expected from lid-driven cavity flow is beginning to form. Experiment with different values of `nt` to see how long the system takes to stabilize.

In [6]:
```
u = np.zeros((ny, nx))
v = np.zeros((ny, nx))
p = np.zeros((ny, nx))
b = np.zeros((ny, nx))
nt = 700
u, v, p = cavityFlow(nt, u, v, dt, dx, dy, p, rho, nu)
fig = plt.figure(figsize=(11,7), dpi=100)
plt.contourf(X,Y,p,alpha=0.5)
plt.colorbar()
```

```
plt.contour(X,Y,p)
plt.quiver(X[::2,::2],Y[::2,::2],u[::2,::2],v[::2,::2])
plt.xlabel('X')
plt.ylabel('Y');
```



1.3 Learn More

The interactive module **12 steps to Navier-Stokes** is one of several components of the Computational Fluid Dynamics class taught by Prof. Lorena A. Barba in Boston University between 2009 and 2013.

For a sample of what the othe components of this class are, you can explore the **Resources** section of the Spring 2013 version of the course's Piazza site.

```
In [7]: from IPython.core.display import HTML
        def css_styling():
            styles = open("../styles/custom.css", "r").read()
            return HTML(styles)
        css_styling()
```

```
Out[7]: <IPython.core.display.HTML at 0x7f38b46046d0>
```

(The cell above executes the style for this notebook.)