07_Step_5

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1 12 steps to Navier-Stokes

Up to now, all of our work has been in one spatial dimension (Steps 1 to 4). We can learn a lot in just 1D, but let's grow up to flatland: two dimensions.

In the following exercises, you will extend the first four steps to 2D. To extend the 1D finite-difference formulas to partial derivatives in 2D or 3D, just apply the definition: a partial derivative with respect to x is the variation in the x direction at constant y.

In 2D space, a rectangular (uniform) grid is defined by the points with coordinates:

$$x_i = x_0 + i\Delta x$$

$$y_i = y_0 + i\Delta y$$

Now, define $u_{i,j} = u(x_i, y_j)$ and apply the finite-difference formulas on either variable x, y acting separately on the i and j indices. All derivatives are based on the 2D Taylor expansion of a mesh point value around $u_{i,j}$.

Hence, for a first-order partial derivative in the x-direction, a finite-difference formula is:

$$\left.\frac{\partial u}{\partial x}\right|_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{\Delta x} + \mathcal{O}(\Delta x)$$

and similarly in the y direction. Thus, we can write backward-difference, forward-difference or central-difference formulas for Steps 5 to 12. Let's get started!

1.1 Step 5: 2-D Linear Convection

The PDE governing 2-D Linear Convection is written as

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} = 0$$

This is the exact same form as with 1-D Linear Convection, except that we now have two spatial dimensions to account for as we step forward in time.

Again, the timestep will be discretized as a forward difference and both spatial steps will be discretized as backward differences.

With 1-D implementations, we used i subscripts to denote movement in space (e.g. $u_i^n - u_{i-1}^n$). Now that we have two dimensions to account for, we need to add a second subscript, j, to account for all the information in the regime.

Here, we'll again use i as the index for our x values, and we'll add the j subscript to track our y values.

With that in mind, our discretization of the PDE should be relatively straightforward.

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + c\frac{u_{i,j}^n - u_{i-1,j}^n}{\Delta x} + c\frac{u_{i,j}^n - u_{i,j-1}^n}{\Delta y} = 0$$

As before, solve for the only unknown:

$$u_{i,j}^{n+1} = u_{i,j}^n - c\frac{\Delta t}{\Delta x}(u_{i,j}^n - u_{i-1,j}^n) - c\frac{\Delta t}{\Delta y}(u_{i,j}^n - u_{i,j-1}^n)$$

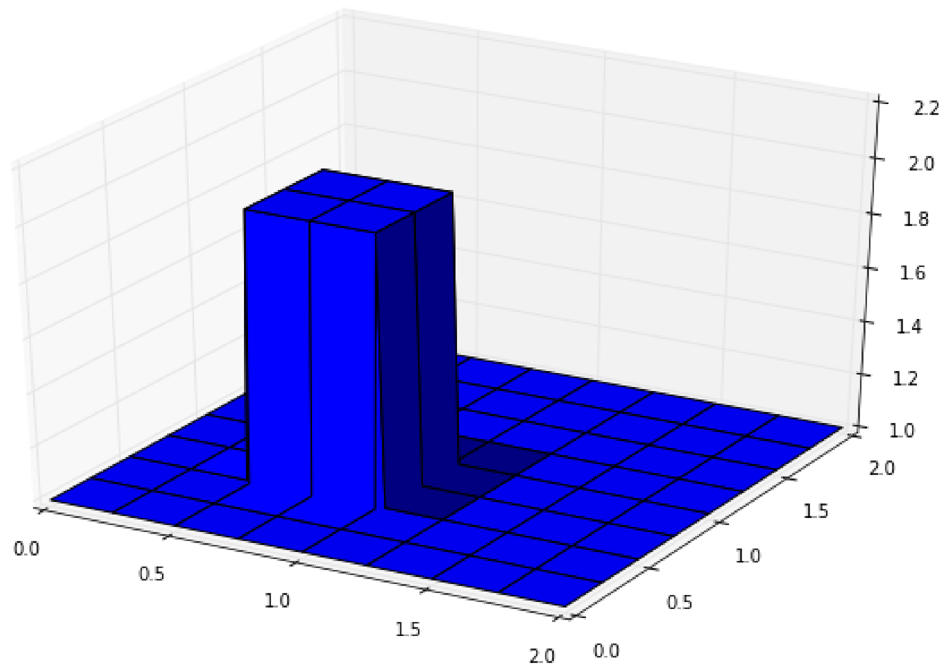
We will solve this equation with the following initial conditions:

$$u(x) = \begin{cases} 2 \text{ for} & 0.5 \le x \le 1 \\ 1 \text{ for} & \text{everywhere else} \end{cases}$$

and boundary conditions:

$$u = 1 \text{ for } \begin{cases} x = 0,\ 2 \\ y = 0,\ 2 \end{cases}$$

In [1]:
```python
from mpl_toolkits.mplot3d import Axes3D    ##New Library required for projected 3d plots

import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline

###variable declarations
nx = 81
ny = 81
nt = 100
c = 1
dx = 2.0/(nx-1)
dy = 2.0/(ny-1)
sigma = .2
dt = sigma*dx

x = np.linspace(0,2,nx)
y = np.linspace(0,2,ny)

u = np.ones((ny,nx)) ##create a 1xn vector of 1's
un = np.ones((ny,nx)) ##

###Assign initial conditions

u[.5/dy:1/dy+1,.5/dx:1/dx+1]=2 ##set hat function I.C. : u(.5<=x<=1 && .5<=y<=1 ) is 2

###Plot Initial Condition
fig = plt.figure(figsize=(11,7), dpi=100)          ##the figsize parameter can be used to produ
ax = fig.gca(projection='3d')
X, Y = np.meshgrid(x,y)
surf = ax.plot_surface(X,Y,u[:])
```



1.1.1 3D Plotting Notes

To plot a projected 3D result, make sure that you have added the Axes3D library.

```
from mpl_toolkits.mplot3d import Axes3D
```

The actual plotting commands are a little more involved than with simple 2d plots.

```
fig = plt.figure(figsize=(11,7), dpi=100)
ax = fig.gca(projection='3d')
surf2 = ax.plot_surface(X,Y,u[:])
```

The first line here is initializing a figure window. The **figsize** and **dpi** commands are optional and simply specify the size and resolution of the figure being produced. You may omit them, but you will still require the

```
fig = plt.figure()
```

The next line assigns the plot window the axes label 'ax' and also specifies that it will be a 3d projection plot. The final line uses the command

```
plot_surface()
```

which is equivalent to the regular plot command, but it takes a grid of X and Y values for the data point positions.

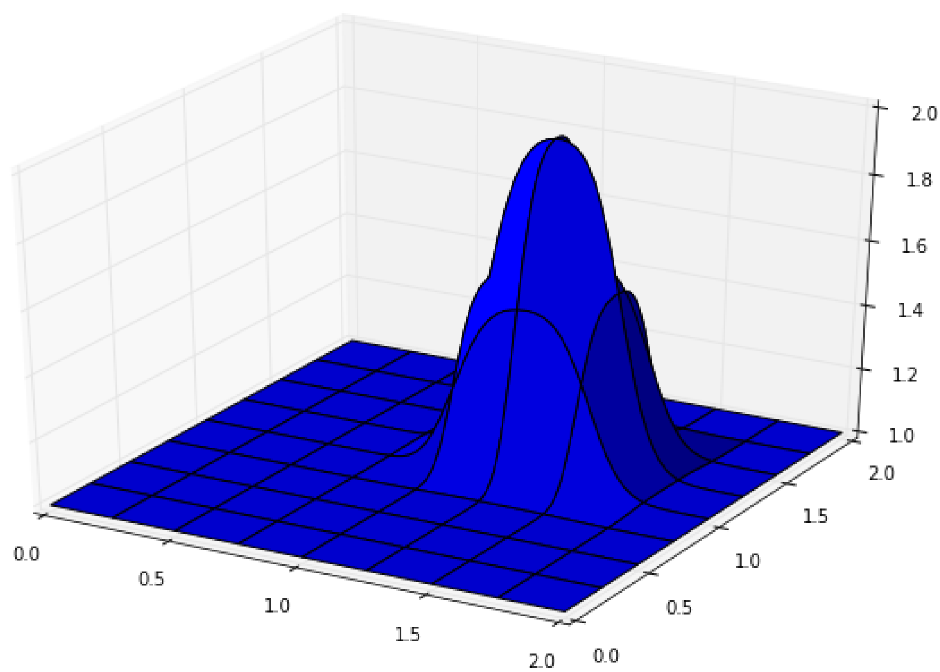
Note The `X` and `Y` values that you pass to `plot_surface` are not the 1-D vectors `x` and `y`. In order to use matplotlibs 3D plotting functions, you need to generate a grid of `x, y` values which correspond to each coordinate in the plotting frame. This coordinate grid is generated using the numpy function `meshgrid`.

```
X, Y = np.meshgrid(x,y)
```

1.1.2 Iterating in two dimensions

To evaluate the wave in two dimensions requires the use of several nested for-loops to cover all of the `i`'s and `j`'s. Since Python is not a compiled language there can be noticeable slowdowns in the execution of code with multiple for-loops. First try evaluating the 2D convection code and see what results it produces.

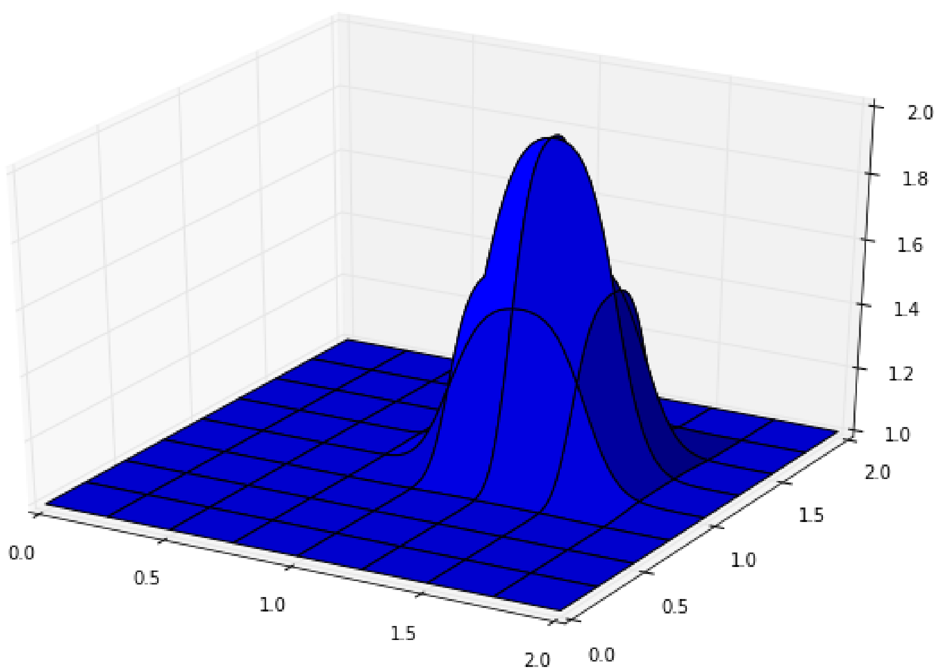
```
In [2]: u = np.ones((ny,nx))
        u[.5/dy:1/dy+1,.5/dx:1/dx+1]=2

        for n in range(nt+1): ##loop across number of time steps
            un = u.copy()
            row, col = u.shape
            for j in range(1, row):
                for i in range(1, col):
                    u[j,i] = un[j, i] - (c*dt/dx*(un[j,i] - un[j,i-1]))-(c*dt/dy*(un[j,i]-un[j-1,i]))
                    u[0,:] = 1
                    u[-1,:] = 1
                    u[:,0] = 1
                    u[:,-1] = 1

        fig = plt.figure(figsize=(11,7), dpi=100)
        ax = fig.gca(projection='3d')
        surf2 = ax.plot_surface(X,Y,u[:])
```



1.2 Array Operations

Here the same 2D convection code is implemented, but instead of using nested for-loops, the same calculations are evaluated using array operations.

In [3]:
```
u = np.ones((ny,nx))
u[.5/dy:1/dy+1,.5/dx:1/dx+1]=2

for n in range(nt+1): ##loop across number of time steps
    un = u.copy()
    u[1:,1:]=un[1:,1:]-(c*dt/dx*(un[1:,1:]-un[1:,:-1]))-(c*dt/dy*(un[1:,1:]-un[:-1,1:]))
    u[0,:] = 1
    u[-1,:] = 1
    u[:,0] = 1
    u[:,-1] = 1

fig = plt.figure(figsize=(11,7), dpi=100)
ax = fig.gca(projection='3d')
surf2 = ax.plot_surface(X,Y,u[:])
```



1.3 Learn More

The video lesson that walks you through the details for Step 5 (and onwards to Step 8) is **Video Lesson 6** on You Tube:

In [4]:
```
from IPython.display import YouTubeVideo
YouTubeVideo('tUg_dE3NXoY')
```

Out[4]: <IPython.lib.display.YouTubeVideo at 0x7f0240312cd0>

In [5]:
```
from IPython.core.display import HTML
def css_styling():
```

```
    styles = open("../styles/custom.css", "r").read()
    return HTML(styles)
css_styling()
```

Out[5]: <IPython.core.display.HTML at 0x7f0240312610>

(The cell above executes the style for this notebook.)