04_Step_3

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1 12 steps to Navier-Stokes

You should have completed Steps 1 and 2 before continuing. This IPython notebook continues the presentation of the **12 steps to Navier-Stokes**, the practical module taught in the interactive CFD class of Prof. Lorena Barba.

1.1 Step 3: Diffusion Equation in 1-D

The one-dimensional diffusion equation is:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$$

The first thing you should notice is that —unlike the previous two simple equations we have studied— this equation has a second-order derivative. We first need to learn what to do with it!

1.1.1 Discretizing $\frac{\partial^2 u}{\partial x^2}$

The second-order derivative can be represented geometrically as the line tangent to the curve given by the first derivative. We will discretize the second-order derivative with a Central Difference scheme: a combination of Forward Difference and Backward Difference of the first derivative. Consider the Taylor expansion of u_{i+1} and u_{i-1} around u_i:

$$u_{i+1} = u_i + \Delta x \frac{\partial u}{\partial x}\bigg|_i + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_i + \frac{\Delta x^3}{3}\frac{\partial^3 u}{\partial x^3}\bigg|_i + O(\Delta x^4)$$

$$u_{i-1} = u_i - \Delta x \frac{\partial u}{\partial x}\bigg|_i + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2}\bigg|_i - \frac{\Delta x^3}{3}\frac{\partial^3 u}{\partial x^3}\bigg|_i + O(\Delta x^4)$$

If we add these two expansions, you can see that the odd-numbered derivative terms will cancel each other out. If we neglect any terms of $O(\Delta x^4)$ or higher (and really, those are very small), then we can rearrange the sum of these two expansions to solve for our second-derivative.

$$u_{i+1} + u_{i-1} = 2u_i + \Delta x^2 \frac{\partial^2 u}{\partial x^2}\bigg|_i + O(\Delta x^4)$$

Then rearrange to solve for $\frac{\partial^2 u}{\partial x^2}\Big|_i$ and the result is:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x^2)$$

1.1.2 Back to Step 3

We can now write the discretized version of the diffusion equation in 1D:

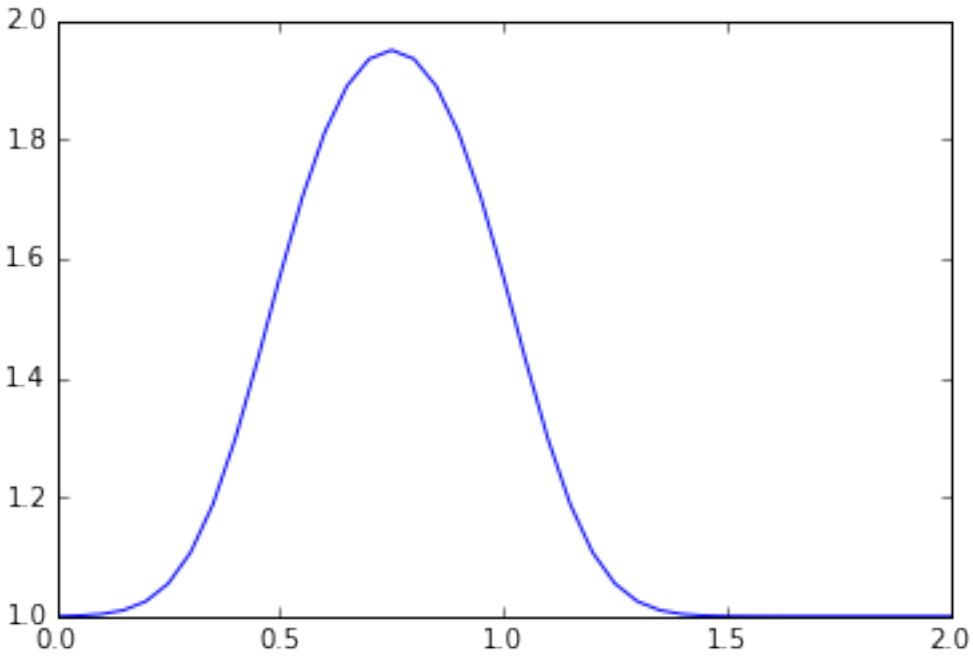
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \nu \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

As before, we notice that once we have an initial condition, the only unknown is u_i^{n+1}, so we re-arrange the equation solving for our unknown:

$$u_i^{n+1} = u_i^n + \frac{\nu \Delta t}{\Delta x^2}(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

The above discrete equation allows us to write a program to advance a solution in time. But we need an initial condition. Let's continue using our favorite: the hat function. So, at $t = 0$, $u = 2$ in the interval $0.5 \leq x \leq 1$ and $u = 1$ everywhere else. We are ready to number-crunch!

```
In [1]: import numpy as np                  #loading our favorite library
        import matplotlib.pyplot as plt     #and the useful plotting library
        %matplotlib inline

        nx = 41
        dx = 2./(nx-1)
        nt = 20    #the number of timesteps we want to calculate
        nu = 0.3   #the value of viscosity
        sigma = .2 #sigma is a parameter, we'll learn more about it later
        dt = sigma*dx**2/nu #dt is defined using sigma ... more later!


        u = np.ones(nx)      #a numpy array with nx elements all equal to 1.
        u[.5/dx : 1/dx+1]=2  #setting u = 2 between 0.5 and 1 as per our I.C.s

        un = np.ones(nx) #our placeholder array, un, to advance the solution in time

        for n in range(nt):  #iterate through time
            un = u.copy() ##copy the existing values of u into un
            for i in range(1,nx-1):
                u[i] = un[i] + nu*dt/dx**2*(un[i+1]-2*un[i]+un[i-1])

        plt.plot(np.linspace(0,2,nx), u);
```



1.2 Learn More

For a careful walk-through of the discretization of the diffusion equation with finite differences (and all steps from 1 to 4), watch **Video Lesson 4** by Prof. Barba on YouTube.

```
In [2]: from IPython.display import YouTubeVideo
        YouTubeVideo('y2WaK7_iMRI')
```

```
Out[2]: <IPython.lib.display.YouTubeVideo at 0x7f9d3c0fb350>
```

```
In [3]: from IPython.core.display import HTML
        def css_styling():
            styles = open("../styles/custom.css", "r").read()
            return HTML(styles)
        css_styling()
```

```
Out[3]: <IPython.core.display.HTML at 0x7f9d3e6d0a50>
```

(The cell above executes the style for this notebook.)