Uniformly random spanning trees in almost-linear time

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The weighted uniformly random spanning tree problem

Given an undirected graph G with weights (conductances) $\{c_e\}_{e \in E(G)}$ on its edges, sample a spanning tree T of G with probability proportional to $\prod_{e \in E(T)} c_e$.

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- Algorithmic applications of electrical flows

Matrix-based algorithms

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n: number of vertices
Idea: go through edges one by one and flip coins conditioned on prior choices

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Algorithm:

- Pick an arbitrary vertex $u \in V(G)$.
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- Only need first visits (at most n such visits) ☺

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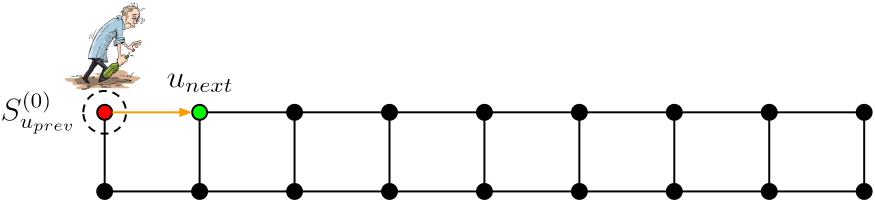
  - This work $\tilde{O}(m^{1+o(1)})$

Aldous-Broder remix

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$

- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(0)}$.
  - Replace u with the non-$S_u^{(0)}$ endpoint of this edge.
- Return the edges used to visit each vertex for the first time.

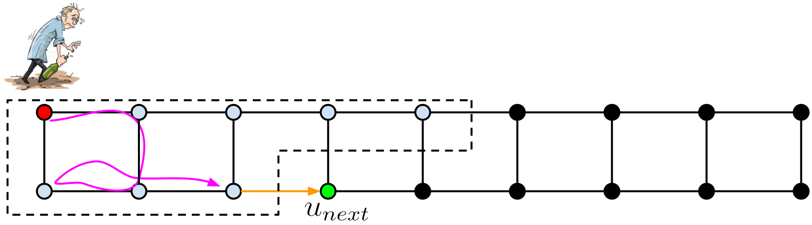


[0] laohblogger.com

Wishful thinking

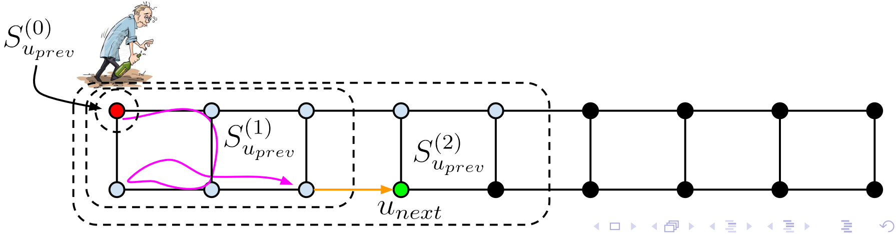
- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$

- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - Sample the edge that the random walk starting at u uses to exit the set of visited vertices.
  - Replace u with the unvisited endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Shortcutting meta-algorithm

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and **pick** shortcutters $\{S_v^{(i)}\}_{i=1}^{\sigma_0}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - Let $i^* \in \{0, 1, \ldots, \sigma_0\}$ be the maximum value of i for which all vertices in $S_u^{(i)}$ have been visited.
  - **Sample** the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the non-$S_u^{(i^*)}$ endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
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- While there is an unvisited vertex
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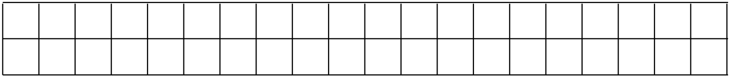
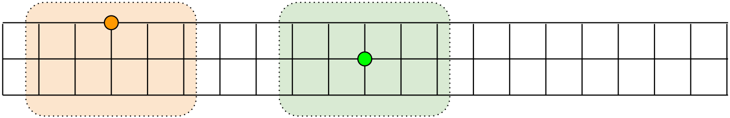
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and **pick** shortcutters $S_v^{(1)}$ to be the 2-neighborhood of v for all $v \in V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - Let $i^* \in \{0, 1, 2, 3\}$ be the maximum value of i for which all vertices in $S_u^{(i)}$ have been visited.
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and **pick** shortcutters $S_v^{(2)}$ to be the 4-neighborhood of v for all $v \in V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - Let $i^* \in \{0, 1, 2, 3\}$ be the maximum value of i for which all vertices in $S_u^{(i)}$ have been visited.
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
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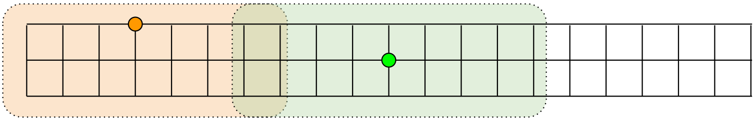

Example

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and **pick** shortcutters $S_v^{(3)}$ to be the 8-neighborhood of v for all $v \in V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - Let $i^* \in \{0, 1, 2, 3\}$ be the maximum value of i for which all vertices in $S_u^{(i)}$ have been visited.
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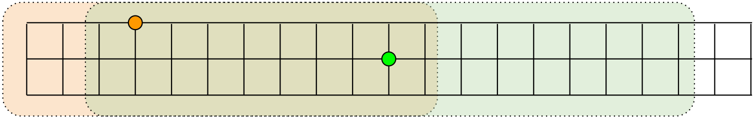
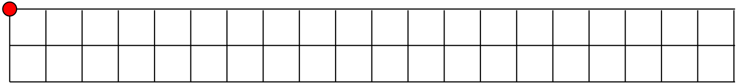


Example on walk step 1

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- **Pick** an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example on walk step 2

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
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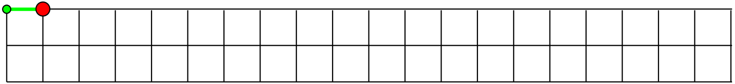

Example on walk step 3

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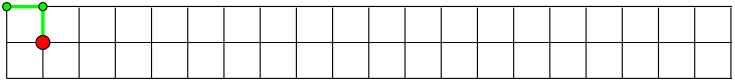

Example on walk step 4

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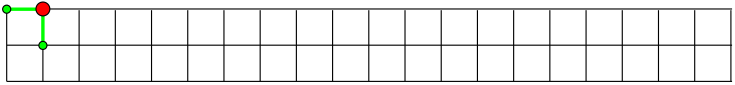

Example on walk step 5

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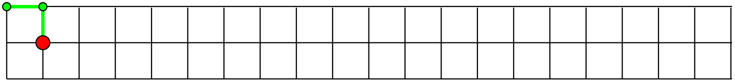

Example on walk step 6

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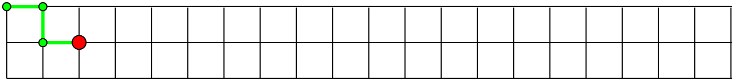

Example on walk step 7

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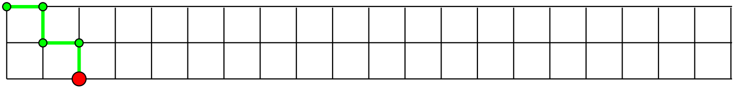

Example on walk step 8

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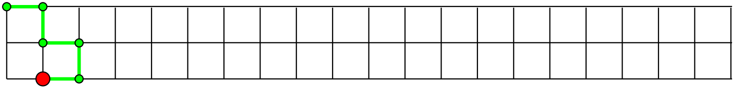

Example on walk step 9

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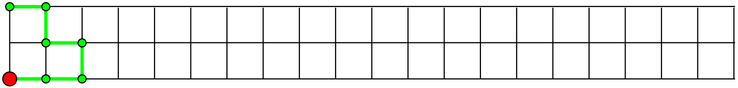

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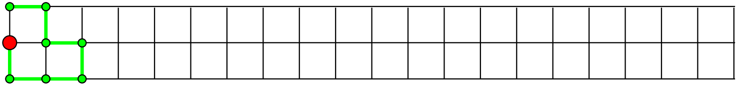

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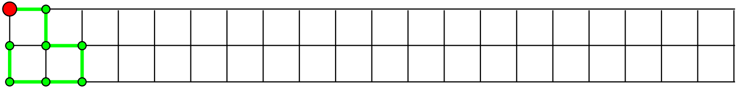

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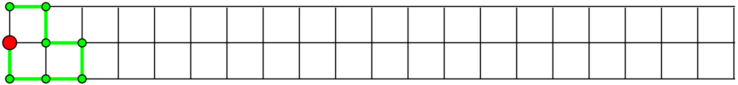
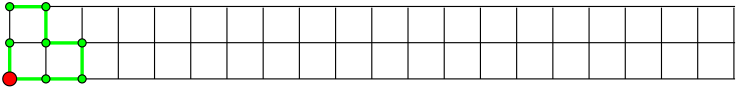
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



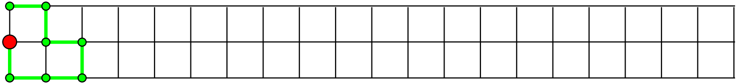
Example on walk step 13

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



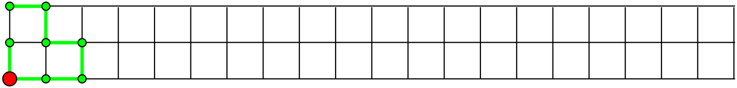
Example on walk step 14

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



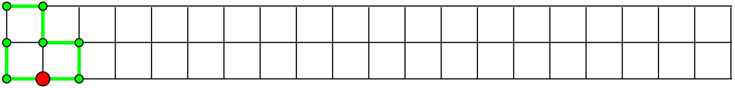
Example on walk step 15

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



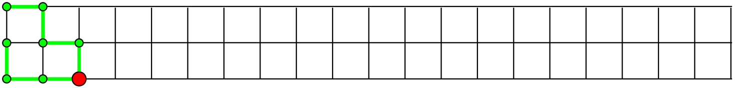
Example on walk step 16

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



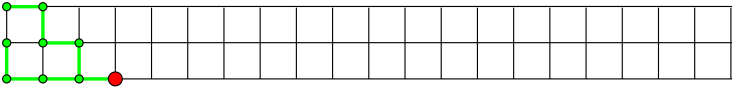
Example on walk step 17

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



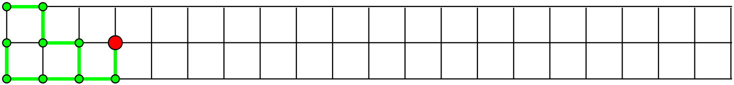
Example on walk step 18

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



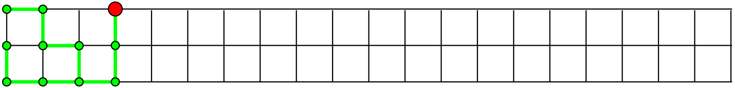
Example on walk step 19

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



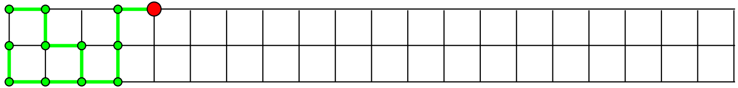
Example on walk step 20

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



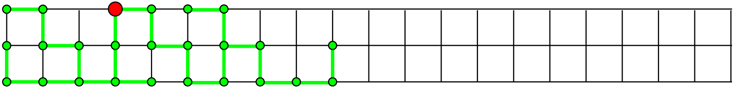
Example on walk step 21

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



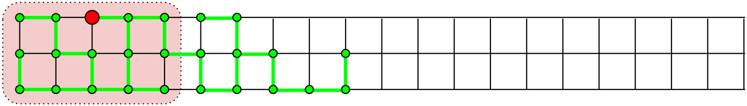
Example on walk step 64

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



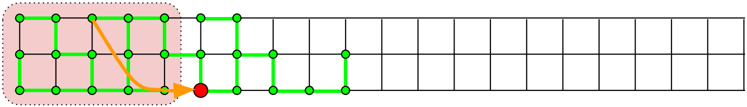
Example on walk step 65

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



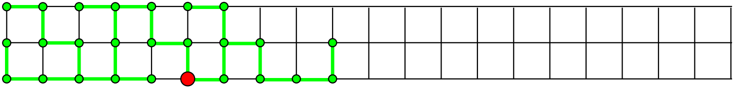
Example on walk step 65 $\rightarrow$ 120

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



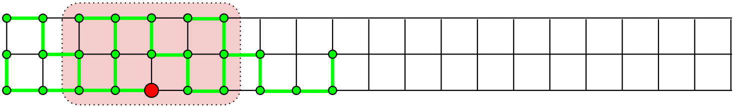
Example on walk step 120

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



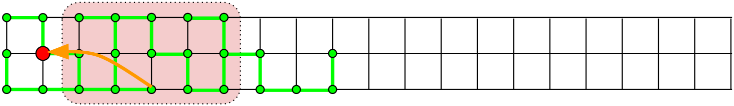
Example on walk step 121

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



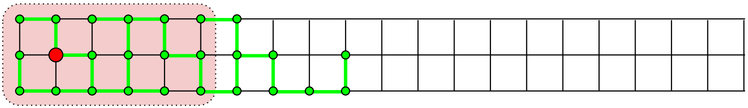
Example on walk step 121 $\to$ 127

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



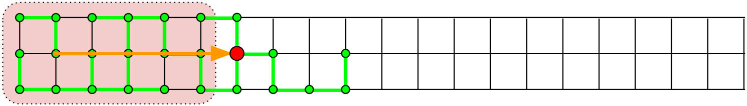
Example on walk step 127

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 2$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



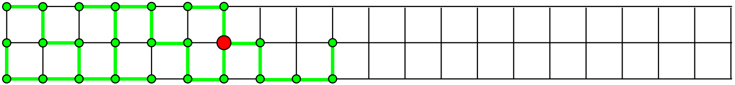
Example on walk step 127 $\rightarrow$ 204

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 2$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



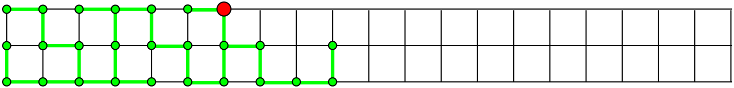
Example on walk step 204

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



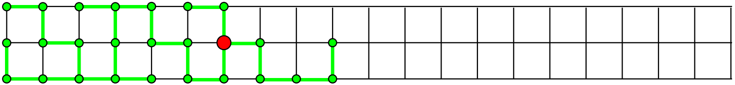
Example on walk step 205

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



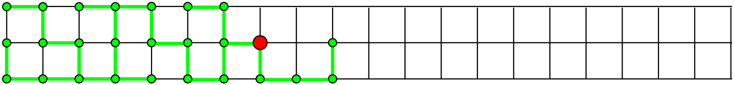
Example on walk step 206

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



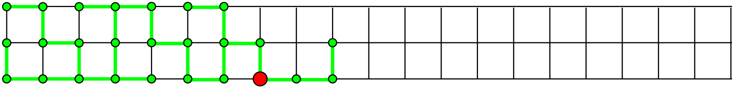
Example on walk step 207

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



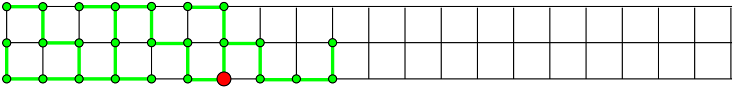
Example on walk step 208

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



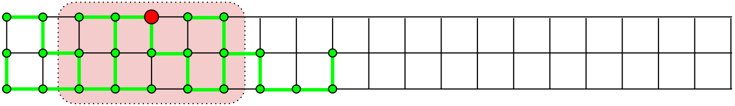
Example on walk step 209

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



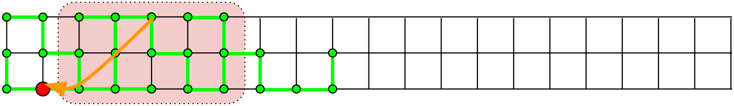
Example on walk step 213

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



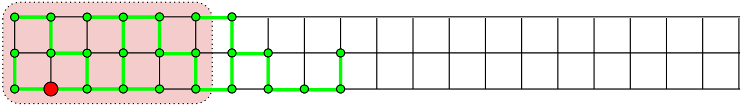
Example on walk step 213 $\to$ 246

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



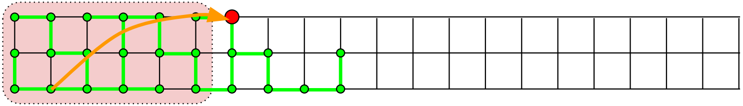
Example on walk step 246

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 2$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



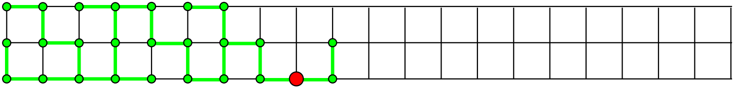
Example on walk step 246 $\rightarrow$ 307

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 2$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



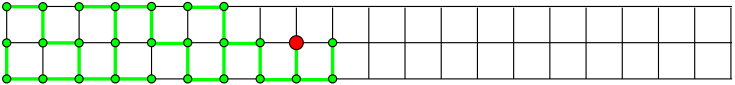
Example on walk step 313

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



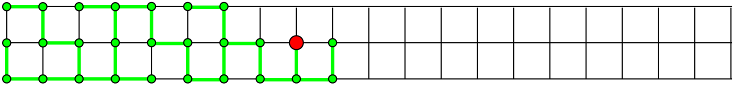
Example on walk step 314

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



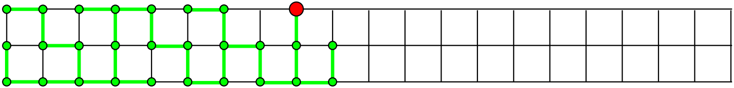
Example on walk step 316

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example on walk step 317

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^{3}$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 0$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example on walk step 318

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
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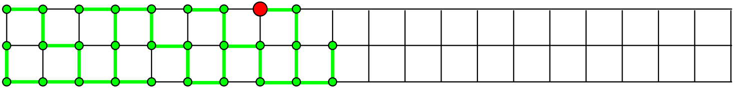
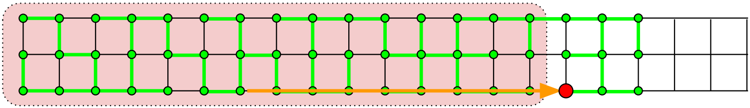


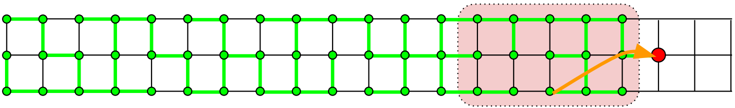
Example on walk step 485 $\rightarrow$ 821

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 3$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



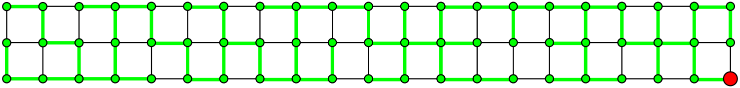
Example on walk step 821 $\rightarrow$ 826

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$ and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - $i^* \leftarrow 1$
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- Return the edges used to visit each vertex for the first time.



Example on walk step 875

- For each $v \in V(G)$, let $S_v^{(0)} = \{v\}$
  and pick shortcutters $\{S_v^{(i)}\}_{i=1}^3$ with $v \in S_v^{(i)} \subseteq V(G)$
- Pick an arbitrary vertex $u \in V(G)$.
- While there is an unvisited vertex
  - **DONE**
  - Sample the edge that the random walk starting at u uses to exit $S_u^{(i^*)}$.
  - Replace u with the endpoint of this edge.
- **Return** the edges used to visit each vertex for the first time.



Summary of existing shortcutting-based algorithms

σ_0: number of shortcutters

m: number of edges

n: number of vertices

Algorithm	σ_0	Shortcutting method	Runtime
[Bro89, Ald90]			$O(mn)$
[KM09]			$O(m\sqrt{n})$
[MST15]			$O(m^{4/3})$
This work			$O(m^{1+1/\sigma_0})$

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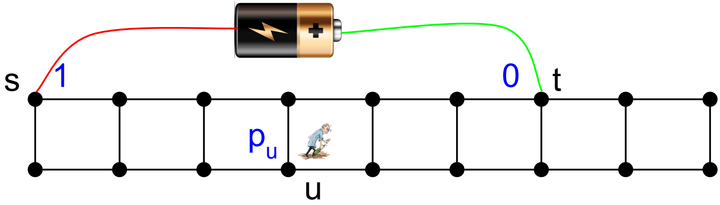
Using Laplacian solvers to calculate hitting probabilities

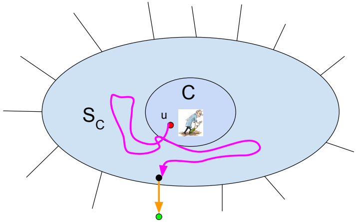
$$p_u = \Pr_u[\text{ random walk starting at } u \text{ hits } s \text{ before } t]$$



Can compute all p_us in $\tilde{O}(m)$ time! [ST14]

Shortcutting methods

- **Given:** a shortcutter S_C with $C \subseteq S_C \subseteq V(G)$
- **Goal:** sample the edge that the random walk starting at u uses to exit S_C for some $u \in C$



Shortcutting method	Preprocessing	Query

Offline shortcutting

- **Given:** a shortcutter S_C with $C \subseteq S_C \subseteq V(G)$
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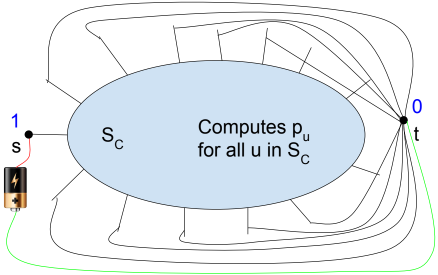

Shortcutting method	Preprocessing	Query
Offline	$\tilde{O}(\lvert E(S_C)\rvert \lvert \partial S_C \rvert)$	$O(\log n)$

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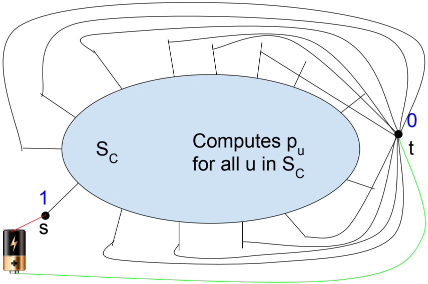

Shortcutting method	Preprocessing	Query
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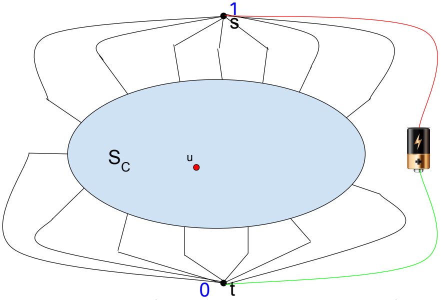

Shortcutting method	Preprocessing	Query
Offline	$\tilde{O}(\lvert E(S_C)\rvert \lvert \partial S_C \rvert)$	$O(\log n)$
Online	None	$\tilde{O}(\lvert E(S_C)\rvert)$

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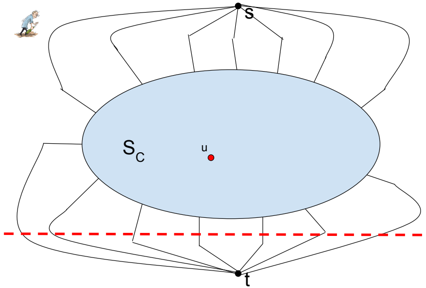

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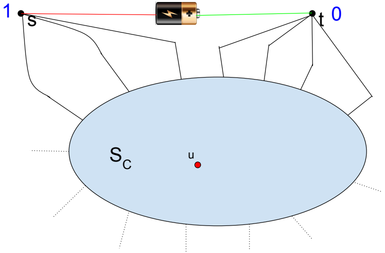

Shortcutting method	Preprocessing	Query
Offline	$\tilde{O}(\lvert E(S_C)\rvert \lvert\partial S_C\rvert)$	$O(\log n)$
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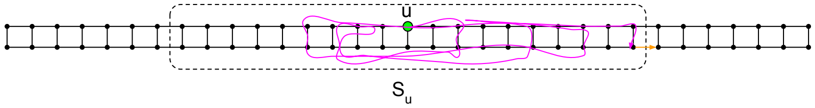
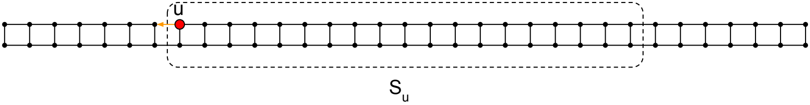
Using online shortcutting

Shortcutter work: $\tilde{O}(|E(S_u)|)$, random walk work $\Omega(|E(S_u)|^2)$ ☺



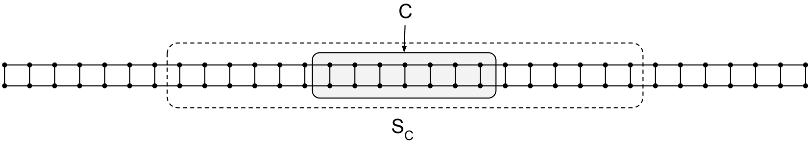
Using online shortcutting

Shortcutter work: $\tilde{\Omega}(|E(S_u)|)$, random walk work can be $O(1)$ ☹



Using online shortcutting

Core should be "well-separated" from the boundary of the shortcutter



Bounding work of online shortcutting

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Most random walk steps occur far away from an unvisited vertex

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Most random walk steps occur far away from an unvisited vertex

Lemma (Random walk bound)

Consider a random walk starting at an arbitrary vertex in a graph I and an edge $\{u, v\} = f \in E(I)$. The

- *expected number of times the random walk traverses f from $u \to v$*
- *before the distance R-neighborhood of u is covered*
- *is at most $\tilde{O}(c_f R)$, where c_f is the conductance of the edge f*

Schur complements

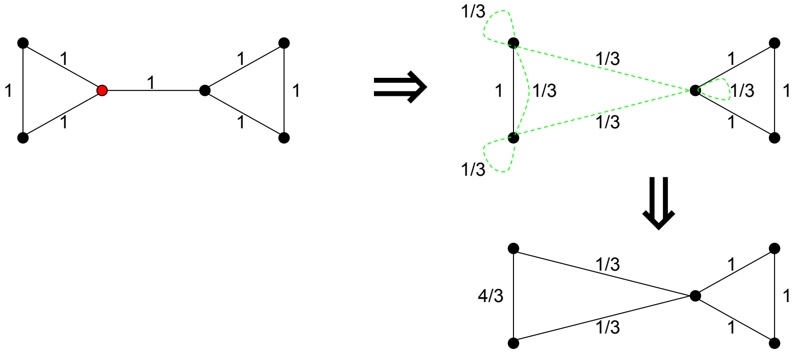
Let $\texttt{Schur}(G, S)$ be a graph with Laplacian matrix $A - BD^{-1}C$:

$$L = \begin{array}{cc} \left[\begin{array}{c|c} A := L[S,S] & B \\ \hline C := L[\bar{S},S] & D \end{array}\right] & \begin{array}{c} S \\ \bar{S} \end{array} \\ \begin{array}{cc} \quad S \quad & \quad \bar{S} \end{array} & \end{array}$$

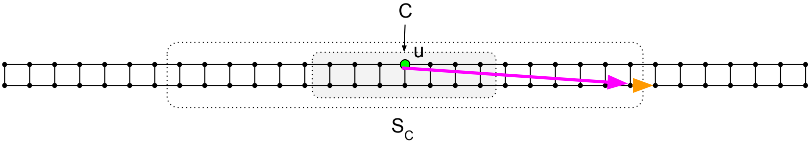
Graph-theoretic interpretation of Schur complements

$H := \texttt{Schur}(G, S)$: a graph with $V(H) = S$ and $E(H)$ and weights defined with the property that the following two distributions are equivalent:

- the list of vertices visited by a random walk in H
- the list of vertices visited by a random walk in G with all vertices outside of S omitted



Charging shortcutter uses to Schur complement crossings



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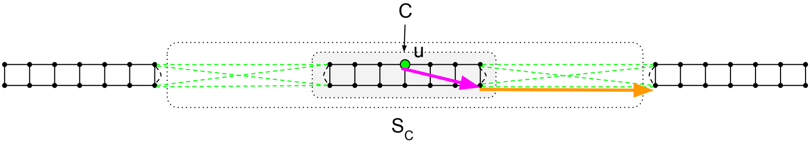

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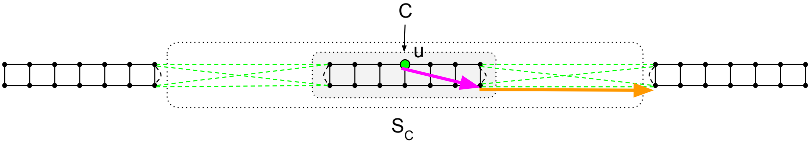
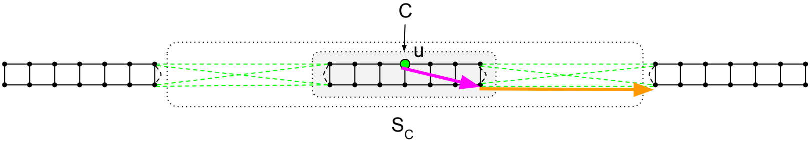


$R_i := m^{i/(\sigma_0+1)}$. To get almost-linear time,

Charging shortcutter uses to Schur complement crossings



$R_i := m^{i/(\sigma_0+1)}$. To get almost-linear time,

- total green conductance $\leq \frac{1}{R_i}$ for $S_C^{(i)}$

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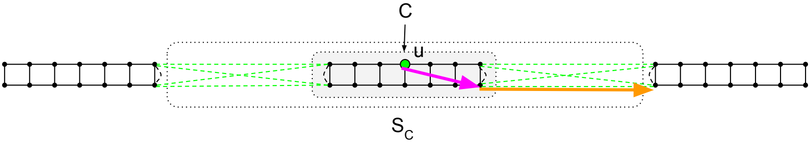

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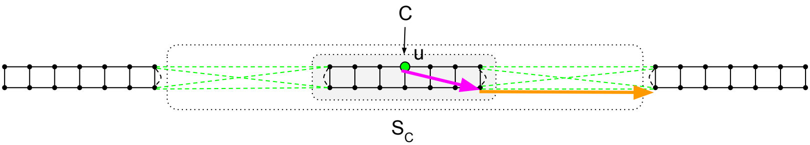

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- total green conductance $\leq \frac{1}{R_i}$ for $S_C^{(i)}$
- and distance to unvisited vertex $\leq R_{i+1}$
- implies number of uses is at most (distance to unvisited vertex)(weight of edges) $\leq \tilde{O}(R_{i+1}/R_i) = m^{o(1)}$ by edge crossing bound

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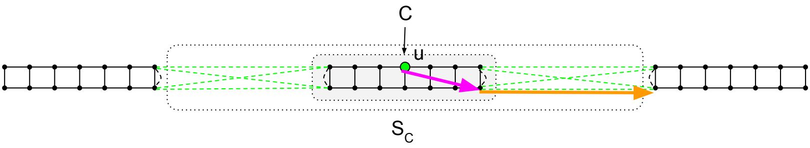

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- work $\leq$ (total shortcutter size)(max number of uses) $\leq (m^{1+o(1)})m^{o(1)} = m^{1+o(1)}$

Conclusion

- An $m^{1+o(1)}\alpha^{o(1)}$-time algorithm for generating weighted uniformly random spanning trees
- Overcame barriers in graph-partitioning based approaches from before by using Schur complements
- Paradigm relevant for other problems?

Questions?

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