LETTERS TO NATURE

PHYSICAL SCIENCES

Floating Orbits, Superradiant Scattering and the Black-hole Bomb

Penrose[1] and Christodoulou[2] have shown how, in principle, rotational energy can be extracted from a black hole by orbiting and fissioning particles. Recently, Misner[3] has pointed out that waves can also extract rotational energy ("superradiant scattering" in which an impinging wave is amplified as it scatters off a rotating hole). As one application of superradiant scattering, Misner has suggested the possible existence of "floating orbits", that is, orbits in which a particle radiatively extracts energy from the hole at the same rate as it radiates energy to infinity; thereby it experiences zero net radiation reaction.

Here we point out a second application of superradiant scattering which we call the "black-hole bomb". We also present the chief results of quantitative analyses of superradiant scattering, floating orbits, and the black-hole bomb, for the case of scalar waves. Quantitative calculations are restricted to the scalar case because the scalar wave equation is separable[4] in the Kerr gravitational field of a black hole, whereas the gravitational wave equation appears not to be. We expect the gravitational case to resemble the scalar case qualitatively if not quantitatively.

The scalar wave equation

$$\Box\Phi = 4\pi T \qquad (1)$$

(T a scalar charge density) separates in Boyer–Lindquist coordinates (S. A. T., unpublished) by writing

$$\Phi = e^{-i\omega t}\, e^{im\varphi}\, S^m{}_l(\theta)\, \psi(r)/r \qquad (2)$$

with $S^m{}_l$ an oblate spheroidal harmonic (D. R. Brill and colleagues, unpublished). We define a new radial coordinate r^* by $dr^*/dr = r^2/(r^2-2Mr+a^2)$, so that the equation for the radial function takes the form

$$d^2\psi/dr^{*2} - W(r^*)\psi = \text{(source term)} \qquad (3)$$

The mass of the black hole is M and its angular momentum is aM. The effective potential $W(r^*)$ is negative at infinity and near the event horizon $r=r_+$, so travelling waves exist in those regions. In between, $W(r^*)$ is positive, that is, it becomes a potential barrier (J. M. Bardeen, W. H. P. and S. A. T., unpublished).

At infinity the asymptotic solutions for ψ are $\exp[-i\omega(t\pm r^*)]$ corresponding to ingoing ("+") and outgoing ("−") waves. By convention we set $G=c=1$; also we take the real part of Φ as the physical field, which permits the convention $\omega \geq 0$ without loss of generality. On the horizon the asymptotic solutions are $\exp[-i(\omega t \pm kr^*)]$ where $k=[-W(r^*=-\infty)]^{\frac{1}{2}}$. The correct boundary condition on the horizon is not that the waves appear ingoing in the coordinate frame, but rather that the wave be physically ingoing in the frames of all physical observers, who are dragged around the hole by its rotation. If $m>0$ and $0<\omega<m\omega_{\text{horizon}}$, where $\omega_{\text{horizon}} \equiv$(angular velocity of "dragging" at the horizon)$=(a/2Mr_{\text{horizon}})$, this physically ingoing condition corresponds to a "coordinate outgoing" wave, $\exp[-i(\omega t - kr^*)]$ (C. M. Misner, unpublished).

We now consider a wave which is incident on the black hole. Normally, a part of the wave's energy reflects off the potential barrier $W(r^*)$, while the rest leaks through and is lost down the hole, so that the outgoing wave is weaker than the ingoing wave. If, however, m and ω are in the anomalous range $0<\omega<m\omega_{\text{horizon}}$, the wave on the inside of the barrier is coordinate outgoing, it reinforces the reflected wave on the outside of the barrier, and there is thus more outgoing wave energy than ingoing. The extra energy comes from the rotational energy of the black hole. The amount of amplification is never very large, because there is always a potential barrier separating the travelling-wave regions.

Fig. 1 shows the results of our numerical integrations of equation (3) for the most favourable case, a maximally rotating black hole with $a=M$. The maximal amplification is a few tenths of a per cent in energy and occurs for low modes $[l=m\sim\varsigma(1)]$ and for wave frequencies $\omega\sim(0.8 \text{ to } 1.0)\, m\omega_{\text{horizon}}$. For a maximally rotating hole of mass M,

$$\omega_{\text{horizon}} \approx 10^5 \text{ rad/s } (M_\odot/M) \qquad (4)$$

By itself, a few tenths of a per cent is unimpressive; but any amplification mechanism admits improvement by positive feedback. To illustrate, in a rather speculative vein, we propose the "black-hole bomb" (closely related to a recent suggestion of Zel'dovich[6]): locate a rotating black hole and construct a spherical mirror around it. The mirror must reflect low-frequency radio waves (equation (4); and we now make the transition from scalar to electromagnetic fields) with reflectivity $\gtrsim 99.8\%$, so that in one reflexion and subsequent superradiant scattering there is a net amplification. The system is then

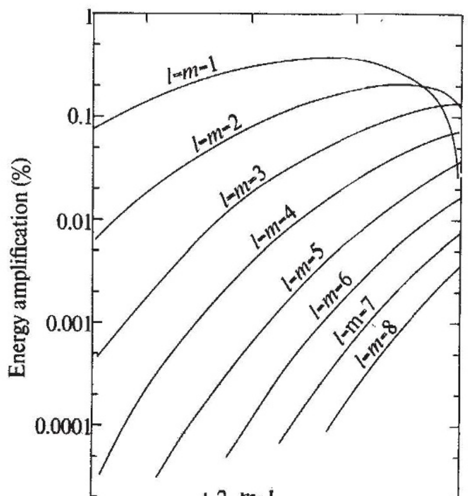

Fig. 1 Superradiant scattering of scalar radiation by maximally-rotating black hole. Radiation modes with axial eigenvalue $m>0$ and angular frequency $\omega<m\omega_{\text{horizon}}$ are amplified by the hole, not absorbed by it. The fractional wave energy added by the hole is here shown as a function of wave frequency for the most favourable modes.

unstable against a number of exponentially growing electromagnetic modes which will be initiated by random "seed fields" (thermal noise). Because a typical amplification in one reflexion is $\sim 10^{-3}$, the e-folding time is roughly $\tau \sim 10^{3} L/c$ where L is the radius of the mirror. For ease of construction, the mirror should not be too close to the hole; for a hole of $M_{\odot}$ an appropriate choice might be $L \sim 10^{3}$ km, so that $\tau \sim 3$ s. As the mode grows, electromagnetic pressure on the mirror increases until the mirror explodes, releasing the trapped electromagnetic energy in a time $\sim L/c$. This is the black-hole bomb. Alternatively a port hole in the mirror can be periodically opened, and the resultant radio flux rectified and used as a source of electric power.

Others may care to speculate on the possibility that nature provides her own mirror. The amplified wave frequencies are far below the plasma frequency of the interstellar medium, so that waves would reflect off the boundary of an evacuated cavity surrounding the hole; we are tempted to invoke radiation pressure to maintain the evacuation.

We turn now to "floating orbits". If a particle is in a stable circular orbit around the hole, it generates radiation both outward to infinity as a source term in equation (3), and also (physically) inward into the hole. If the particle is in a direct orbit, co-rotating with the hole, it generates only modes with $m>0$. For holes with $a>0.359\ M$, radiation from all direct, stable, circular orbits satisfied the anomalous boundary conditions $0<\omega<m\omega_{\mathrm{horizon}}$ for all l, m. The radiation energy balance of a particle in such an orbit has two parts: the power radiated to infinity, and the power extracted from (not deposited into) the rotating hole. The question of floating depends on the detailed numerical balance of these contributions. If at some radius more energy is extracted than is radiated, the particle will gradually spiral outward until the energy credits and debits are in balance. At this "floating" radius, the particle gradually radiates away the black hole's rotational energy; as a decreases, the radius of the lowest stable orbit moves outward with respect to the floating radius. When the two are equal, floating ceases, and the particle plunges into the hole.

The results of our calculations for scalar radiation are as follows: a system whose dominant radiation is in $m=1$ modes can float around any black hole with $a \gtrsim 0.985\ M$; for an $a=m$ hole, the floating radius is at $r \approx 1.4\ M$. A system whose dominant modes are $m=2$ can float for $a \gtrsim 0.9995$; for $a=M$ the floating radius is $r \approx 1.16\ M$. Systems radiating substantially in $m \geq 3$ modes cannot float at any radius for any $a \leq M$. In the particular case of a point particle in a circular equatorial orbit, it is not difficult to calculate the relative coupling of the source to various modes, and exhibit the energy "balance sheet". Table 1 shows this for the most favourable of all scalar-wave cases: an $a=M$ hole, with the particle very near the lowest stable orbit, $x \equiv r-M \ll M$. Although the first two modes give a net credit, the particle couples too strongly to non-floating modes and there is no net floating. If the particle were smeared out in azimuthal angle φ, these higher modes would be suppressed and floating would occur.

For the physical case of gravitational (not scalar) radiation, there is no $l=m=1$ radiation, and the numerical details for higher modes will be different. We do not know if there will be floating for the $l=m=2$ mode (or higher modes for that matter); our scalar results suggest only that gravitational floating is not implausible, and might conceivably enter into the dynamics of material processes near rotating black holes. (Some recent unpublished work by D. M. Chitre and R. H. Price, and by M. Davis and colleagues, suggests that source coupling to high, presumably non-floating, modes is weaker for gravitational than for scalar fields.)

Finally, we mention a curious aspect of our numerical results: in Fig. 1, the amplification factor with $l=m>1$ does not go to zero as $\omega \to m\omega_{\mathrm{horizon}}$. Because the amplification factor is negative for $\omega > m\omega_{\mathrm{horizon}}$ it must be a discontinuous function of frequency at $m\omega_{\mathrm{horizon}}$. This discontinuity is an artefact of taking $a=M$. For a slightly lower value, we expect the discontinuity to disappear, with the curves of Fig. 1 showing a sharp turnover very near $\omega/m\omega_{\mathrm{horizon}}=1$.

Table 1 Energy Balance for Scalar Radiation from a Particle in Close Circular Orbit

Mode (l, m)	Power from hole* (credit)	Power lost to infinity* (debit)	Net energy loss (or gain)
1, 1	(0.074)	≈ 0	(0.074)
2, 2	(0.081)	0.032	(0.049)
3, 3	(0.062)	0.070	0.008
4, 4	(0.042)	0.089	0.047
5, 5	(0.027)	0.091	0.064
6, 6	(0.016)	0.087	0.071
7, 7	(0.010)	0.081	0.071
8, 8	(0.006)	0.073	0.067
All modes with $l>m$	≈ 0	≈ 0	≈ 0
Total for modes $l \leq 8$	(0.318)	0.523	0.205

* In units of $(c^{5}/G)\ (\mu/M)^{2}\ x$ where $x=(c^{2}r/GM-1) \ll 1$ and μ is the particle's scalar charge. It is an artefact of the scalar case that power$\to 0$ as $x \to 0$.

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1 Penrose, R., *Revista Del Nuovo Cimento*, **1**, 252 (1969).
2 Christodoulou, D., *Phys. Rev. Lett.*, **25**, 1596 (1970).
3 Misner, C. W., *Phys. Rev. Lett.*, **28**, 994 (1972).
4 Carter, B., *Phys. Rev.*, **174**, 1559 (1968).
5 Boyer, R. H., and Lindquist, R. W., *J. Math. Phys.*, **8**, 265 (1967).
6 Zel'dovich, Ya. B., *JETP Lett.*, **14**, 270 (1971).

Whistling Beaches and Seabed Sand Transport

EXPERIMENTS on the whistling sand of Porth Oer, Caernarvonshire, show that the property of whistling or squeaking, when kicked, scuffed or walked on, arises from the very narrow particle size distribution, combined with fairly spherical particle shape[1]. These two factors enable the sand to yield in shear along thin slip planes, instead of throughout its bulk, and the drag of uniform particles over one another results in an audible note of regular pitch. There is a tendency for the dilation caused by one particle riding up and over its neighbour to be communicated along the shear plane, so that all the particles at the plane move in unison.

There are several other beaches on which whistling sand occurs. Since carrying out the work reported in ref. 1 we have collected data concerning 32 sites on the coast of the British Isles where such sand may be found. We have visited all of these sites and have taken shape and size distribution measurements on sand samples. These will be published in full later.

We wish now to draw attention to a close correlation between the occurrence of whistling sand and the position of the landward end of "bed-load partings" in the sand of the continental shelf[2]. Kenyon and Stride[2] report sand movements on the seabed which are caused by both tidal (oscillatory) transport and current (unidirectional) transport. They followed such movements in detail by analysing a large number of wide-scan echo-sounding records of the seabed obtained by the oceanographical survey ships RRS Discovery I and RRS Discovery II on coastal waters survey voyages between 1958 and 1969. The records show sand waves, sand ribbons and sand patches, from which the sand transport directions may be obtained. Certain lines are, in effect, "watersheds" for the flow of sand over the seabed, sand flowing away on each side roughly at