Notes on applying super momentum balance law to constraint GW waveform
(Update on Daily Progress)

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I.   LOCAL SUPER MOMENTUM BALANCE LAW

The local super-momentum balance law for asymptotically flat space-time is given by

$$\mathcal{F}(\theta,\phi)\Big|_{u_1}^{u_2} := -\int_{u_1}^{u_2} du \left[|\dot{\sigma}^o|^2 - \mathfrak{R}\left(\eth^2 \dot{\bar{\sigma}}^o\right)\right](u,\theta,\phi) = \mathfrak{R}\left[\Psi_2^o + \bar{\sigma}^o \dot{\sigma}^o\right](u_1,\theta,\phi) - \mathfrak{R}\left[\Psi_2^o + \bar{\sigma}^o \dot{\sigma}^o\right](u_2,\theta,\phi) \tag{1.1}$$

where

$$2\mathfrak{R}\,\Psi_2^o(u,\theta,\phi) = \lim_{r\to+\infty} r^3\, C_{abcd} n^a l^b n^c l^d \tag{1.2}$$

$$\Psi_4^o(u,\theta,\phi) = \lim_{r\to+\infty} r\, C_{abcd} n^a \bar{m}^b n^c \bar{m}^d = \ddot{\bar{\sigma}}^o = r\left(\ddot{h}_+ + i\,\ddot{h}_\times\right) \tag{1.3}$$

We simplify the above equation for the case for perturbed black hole using the results in [add ref]. Equation 2.59 in ref[add ref] the mass aspect

$$\Psi = \psi_2^0 + \eth^2 \bar{\sigma}^o + \sigma^o \dot{\bar{\sigma}}^o \tag{1.4}$$

and the spherical harmonic expansion of the above result is

$$\Psi = \Psi^0 + \Psi^i Y_{1i}^0 + +\Psi^{ij} Y_{2ij}^0 \tag{1.5}$$

The interior mass and three-momentum with the $l=0$ and $l=1$ harmonic contributions is given by

$$M_B = \frac{-c^2}{2\sqrt{2}\,G}\Psi^0 \tag{1.6}$$

$$P^i = -\frac{c^3}{6\,G}\Psi^i \tag{1.7}$$

Using the above results, at leading order we can relate mass and energy momentum of interior space time to ψ_2 as

$$-2\sqrt{2}\,M_B = \mathfrak{R}\left[\psi_2^0 + \eth^2 \bar{\sigma}^o + \sigma^o \dot{\bar{\sigma}}^o\right] \tag{1.8}$$

We can use the above result in the local balance law, to rewrite the local balance law. As a first step just considering change in mass we have

$$-\int_{u_1}^{u_2} du \left[|\dot{\sigma}^o|^2 - \mathfrak{R}\left(\eth^2 \dot{\bar{\sigma}}^o\right)\right](u,\theta,\phi) = -\left[\left(2\sqrt{2}M_B + \eth^2\bar{\sigma}^o\right)(u_1,\theta,\phi) - \left(2\sqrt{2}M_B + \eth^2\bar{\sigma}^o\right)(u_2,\theta,\phi)\right] \tag{1.9}$$

$$= \int_{-\infty}^{t} du \left(2\sqrt{2}\,\dot{M}_B + \eth^2\bar{\sigma}^o\right) \tag{1.10}$$

$$\implies -|\dot{\sigma}^o|^2 + \mathfrak{R}\left(\eth^2\bar{\sigma}^o\right) = 2\sqrt{2}\,\dot{M}_B + \eth^2\bar{\sigma}^o \tag{1.11}$$

$$\implies -|\dot{\sigma}^o|^2 = 2\sqrt{2}\,\dot{M}_B \tag{1.12}$$

Higher order corrections can be obtained in a similar way

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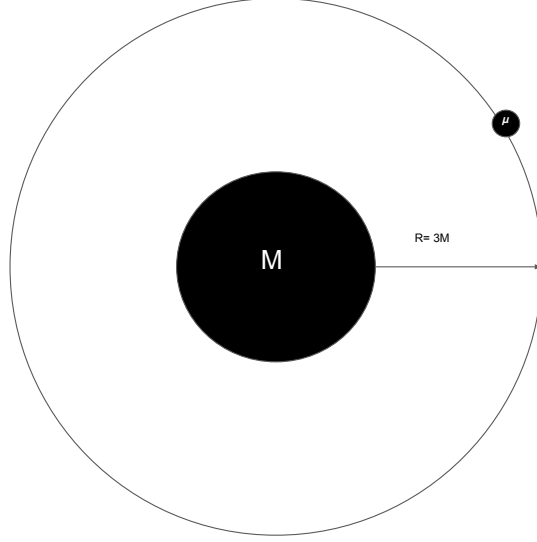†Electronic address: sean.mcwilliams@mail.wvu.edu

FIG. 1: The figure

II.   WEYL SCALAR Ψ_2^o FOR BOB AS PERTURBER CROSS THE LIGHT RING

In Backwards One-Body (BOB)[add ref] description of black hole binary merger, the system is described as a single black hole spacetime with a inspiraling in perturber. At the leading order we can just assume that the mass of the perturber and the source is evolving.

A.   Evolution of Ψ_2^o for perturbed Swarzchild black hole

We assume that the perturber is loosing mass at given rate $\dot{\epsilon}$. Given this and looking at Bianchi identity on asymptotically flat space time, we can also deduce that the source mass is evolving as $\dot{\epsilon}^2$. In this model where we assume both perturber and source mass evolving in time, we can relate this to gravitational wave strain. The gravitational wave luminosity is given by,

$$\mathcal{L} = -\frac{dE}{dt} = \lim_{r \to +\infty} \frac{r^2}{16\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \Big| \int_{-\infty}^{\infty} dt' \Psi_4^{lm} \Big|^2$$

for l=2, m=2 mode $\qquad\qquad$ (2.1)

$$\mathcal{L} = \lim_{r \to +\infty} \frac{r^2}{16\pi} \Big| \int_{-\infty}^{\infty} dt' \Psi_4^{2m} \Big|^2 \qquad (2.2)$$

And the News function in related to Ψ_4 by

$$\sigma = \int_{-\infty}^{t} dt' \Psi_4^{22} \qquad (2.3)$$

Now let us look at how this luminosity evolves with time.

$$\frac{d\mathcal{L}_{GW}}{dt} = \frac{1}{16\pi} \frac{d}{dt} \Big| \int_{-\infty}^{\infty} dt' \Psi_4^{22} \Big|^2 = \frac{1}{16\pi} \frac{d}{dt} \Big[\int_{-\infty}^{t} dt' \bar{\Psi}_4^{22}(t') \int_{-\infty}^{t} dt'' \Psi_4^{22}(t'') \Big] \qquad (2.4)$$

$$= \frac{1}{16\pi} \Big[\bar{\Psi}_4^{22}(t) \int_{-\infty}^{t} dt' \Psi_4^{22}(t') + \Psi_4^{22}(t) \int_{-\infty}^{t} dt' \bar{\Psi}_4^{22}(t') \Big]$$

$$= \frac{1}{16\pi} \Big[\ddot{\bar{\sigma}}(t)\dot{\sigma}(t) + \dot{\sigma}(t)\ddot{\sigma}(t) \Big] = \frac{1}{8\pi} \Re \Big[\ddot{\bar{\sigma}}(t)\dot{\sigma}(t) \Big]$$

1. Luminosity from a binary system

Now let us look at how at leading the luminosity of binary source which is shedding mass evolve with time

$$\mathcal{L}_{GW} = \frac{32}{5} \frac{M^3 \mu^2}{r^5}$$

$$\frac{d\mathcal{L}_{GW}}{dt} = \frac{32 M^3 \mu^2}{5r^5} \left[\frac{3}{M} \frac{dM}{dt} + \frac{2}{\mu} \frac{d\mu}{dt} - \frac{5}{r} \frac{dr}{dt} \right] \tag{2.5}$$

The radiation reaction has a negligible effect on dynamics of the perturber inside ISCO, so we model the rate of mass loss as source of rate of Luminosity change. The Mass of the central black holes is much larger then the mass of the perturber so in the above equation

2. Mass, Momentum and Angular momentum loss of a perturbed spacetime

We can use the time evolution of mass aspect, equation 2.58 in ref[add] to relate the news amplitude to the mass and angular momentum loss.

$$\dot{\Psi} = \dot{\sigma}\dot{\bar{\sigma}} + k\phi_2^0 \bar{\phi}_2^0 \tag{2.6}$$

In the absence of electromagnetic fields we set the second term in right hand side to zero. The above equation then is related to mass and angular momentum loss of the perturbed space time with the radiated GWs strain.

$$\dot{\Psi} = -\frac{2\sqrt{2}G}{c^2} \dot{M} - \frac{6G}{c^3} \dot{P}^i Y_{1\,i}^0 + \dots \tag{2.7}$$

We can use the above results to get an insight into how mass and angular momentum are related. For the case of gravitational wave emitted the merger the waveform is well described by BOB model. We have

$$|\psi_4| = A_p \mathrm{sech}[\gamma(t - t_p)] \tag{2.8}$$

Since $|\sigma| \approx |\psi|/\omega$ for quasi circular systems. Where ω is given by

$$\Omega = \left\{ \Omega_0^4 + k \left[\tanh\left(\frac{t - t_p}{\tau}\right) - \tanh\left(\frac{t_0 - t_p}{\tau}\right) \right] \right\}^{1/4} \tag{2.9}$$

where the constant k is given by

$$k = \left[\frac{\Omega_{QNM}^4 - \Omega_0^4}{1 - \tanh[(t_0 - t_p)/\tau]} \right] \tag{2.10}$$