Gravitational waves in general relativity
VII. Waves from axi-symmetric isolated systems

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(*Received* 8 *January* 1962—*Revised* 2 *April* 1962)

This paper is divided into four parts. In part A, some general considerations about gravitational radiation are followed by a treatment of the scalar wave equation in the manner later to be applied to Einstein's field equations.

In part B, a co-ordinate system is specified which is suitable for investigation of outgoing gravitational waves from an isolated axi-symmetric reflexion-symmetric system. The metric is expanded in negative powers of a suitably defined radial co-ordinate r, and the vacuum field equations are investigated in detail. It is shown that the flow of information to infinity is controlled by a single function of two variables called the *news function*. Together with initial conditions specified on a light cone, this function fully defines the behaviour of the system. No constraints of any kind are encountered.

In part C, the transformations leaving the metric in the chosen form are determined. An investigation of the corresponding transformations in Minkowski space suggests that no generality is lost by assuming that the transformations, like the metric, may be expanded in negative powers of r.

In part D, the mass of the system is defined in a way which in static metrics agrees with the usual definition. The principal result of the paper is then deduced, namely, that *the mass of a system is constant if and only if there is no news; if there is news, the mass decreases monotonically so long as it continues.* The linear approximation is next discussed, chiefly for its heuristic value, and employed in the analysis of a receiver for gravitational waves. Sandwich waves are constructed, and certain non-radiative but non-static solutions are discussed. This part concludes with a tentative classification of time-dependent solutions of the types considered.

PART A. GENERAL CONSIDERATIONS

BY H. BONDI, F.R.S.

1. INTRODUCTION

A great deal of work has been done on gravitational waves. In the first instance the linearized theory has been developed extensively, but it seems doubtful whether its results can be fully trusted. The non-linearity of the gravitational field is one of its most characteristic properties, and it is likely that at least some of the crucial properties of the field show themselves only through the non-linear terms. Moreover, it is never entirely clear whether solutions derived by the usual method of linear approximation necessarily correspond in every case to exact solutions, or whether there might be spurious linear solutions which are not in any sense approximations to exact ones. Next, although a good deal is known about exact plane and cylindrical wave solutions, it is doubtful whether these necessarily display the most important characteristics of physically significant waves, that is, of waves from bounded sources. General relativity is a peculiarly complete theory and may not give sensible solutions for situations too far removed from what is physically reasonable. The simplest field due to a finite source is spherically symmetrical, but Birkhoff's theorem shows that a spherically symmetrical empty-space field is necessarily static. Therefore there cannot be truly spherically symmetrical waves, and thus any description of radiation from a finite system must necessarily involve three co-ordinates significantly. This enormously complicates the mathematical difficulties and thus we have to make use of methods of approximation.

2. Causality

The equations of general relativity, like those of most other wave theories, are symmetrical in time. The choice of the retarded solution is as arbitrary in the gravitational case as any other, but whereas in electromagnetic theory it is the direct appeal to experience that forces us to the retarded solution, no such appeal is possible in the gravitational case. All we can say is that, if our usual notions of causality and the flow of time are not to be upset by gravitational waves, if we are to suppose, in other words, that even the most carefully constructed gravitational receiver would not enable us to look into the future, then we are forced to prescribe purely retarded solutions in this theory as well.

The boundary conditions adopted, therefore, are that we have an isolated material system in an empty space that tends to flatness at infinity, where only outgoing waves are present, and we examine the changes of space which are determined by changes in the material system enclosed.

It might be argued that a closed material system cannot undergo any change if it has been isolated for sufficiently long, but this is incorrect. For the system may have an equation of state containing the time explicitly. Moreover, this time dependence may contain a random element and so produce motions in the system that could not have been forecast from outside. This lack of the possibility of forecasting is an important and characteristic point. It is well known that the solutions of hyperbolic equations, such as we are dealing with in general relativity, need not be analytic. On the contrary, it is typical of hyperbolic systems that non-analytic behaviour can be propagated, though only along characteristics, which are the wave fronts defined by the system of equations. It will, of course, be realized that an analytic function of time is one whose entire future can be forecast from an arbitrarily small section of time, whereas a non-analytic function is one whose future is undetermined. We shall, therefore, expect to find in our work that the behaviour of the system can be described by functions that need not be analytic and can thus contain the effects of the possible 'time-bomb' character of the system enclosed.

3. The loss of mass

The symmetry properties of gravitational waves are restricted by conservation laws. The conservation of mass effectively prohibits purely spherically symmetrical waves and, similarly, the conservation of momentum prohibits waves of dipole symmetry. The lowest kind of symmetry which we can associate with gravitational waves is that of a quadrupole. However, the significance of this result is substantially reduced by the non-linearity of the equations. In a linear theory the absence of a purely spherically symmetrical mode implies that there can be no spherically symmetrical component of any wave motion at all, but this is not so in a non-linear theory; on the contrary, perhaps the most important character of gravitational waves concerns just this. For a wave to be a wave in any real physical sense it must convey energy: accordingly, an outgoing wave must diminish the energy of the source and, therefore, its mass.

Contemplate now a transmitter quiescent for a semi-infinite period (so that during this time we have a static situation), then emitting by moving in a suitable way for a finite period, with the field eventually returning again to a static situation. If the waves are real physical waves, i.e. if they carry energy, then in the final situation the transmitter must have less mass than in the initial situation. But the mass is the spherically symmetrical part of the gravitational field and therefore a diminution in mass means a change in the spherically symmetrical component. No change can be expected if the whole situation is purely spherically symmetrical throughout by virtue of Birkhoff's theorem. However, if the field was initially spherically symmetrical and is eventually spherically symmetrical, yet there is an intermediate non-spherically symmetrical wave-emitting period, a change of mass may occur, that is, a change in the coefficient of the r^{-1} term in the static solution. Thus we would expect the higher terms to react back through the non-linearities and to produce an effect on the spherically symmetrical term which represents the mass and thus the energy of the source. The situation is in marked contrast to the electromagnetic case, for there the one thing that cannot be radiated away at all is source strength, that is, charge. In the gravitational case the source strength must diminish if the wave carries energy. The case here mentioned of a transmitter initially and eventually quiescent is the case that will receive most attention in the rest of this paper, because in this case, and in this case alone, we are concerned with initial and final situations that are static and therefore well understood.

The loss of energy, that is the loss of mass, is immediately connected with the problem of the availability to receivers of the radiated energy. This raises the question of what constitutes a receiver for gravitational waves and how much energy it can absorb from such a wave. The matter is discussed more fully in section D of this paper, but it may perhaps be worth pointing out now that in electromagnetic theory we are familiar with the distinction between near-field transfer as, say, in the case of electromagnetic induction of energy, and radiative wave transfer of energy. The distinction between these two types of transfer is normally clear-cut in the electromagnetic case, though even there difficulties can occur (Bondi 1961). We shall see that these difficulties are very much greater in the gravitational case.

4. Huygens's principle and the change of wave form

Different kinds of waves show a number of different properties and one of the most important of these is whether or not they adhere to Huygens's principle. This means, briefly, whether after the end of excitation the wave rings on ('has a tail'), or whether with the end of excitation there is an end to the wave motion, propagated throughout space with the fundamental velocity. It is well known that the ordinary d'Alembert wave equation with an odd number of spatial dimensions satisfies Huygens's principle but that in an even number of spatial dimensions or with suitable additive terms the wave equation does not satisfy this principle, and therefore its solutions then possess a tail. Whether gravitational waves have tails is one of the questions that will be investigated in the course of this paper.

5. Method of treatment

The extreme complexity of the field equation for empty space makes it clear that a method of expansion should be used to examine the problem. The method that will be used here is that of expansion in negative powers of a suitably defined radius. This seems to be a very suitable method for a wave problem, and the difficulties that have previously stood in the way of such an analysis are avoided by the choice of a suitable system of co-ordinates. The problem of convergence is, naturally, always a very real problem in such a method. However, arguments will be given suggesting that the difficulties arising from this can be contained. At the same time it must be emphasized that many of the strange features that appear in the course of the work are due to this method rather than inherent in the equations. It may therefore be useful to consider here the ordinary scalar wave equation using the same methods as will later be applied to the gravitational wave problem.* These methods are not the easiest methods for dealing with the scalar wave equation, but they seem to be the most promising for the gravitational wave case. Consider then

$$\partial^2 Q/\partial t^2 = \nabla^2 Q. \tag{1}$$

Separate now the part Q_n of Q proportional to the surface harmonic S_n and introduce a null variable u by the relation $u = t-r$. The wave equation (1) now takes the form

$$2\left(\frac{\partial^2 Q_n}{\partial r\,\partial u}+\frac{1}{r}\frac{\partial Q_n}{\partial u}\right) = \frac{\partial^2 Q_n}{\partial r^2}+\frac{2}{r}\frac{\partial Q_n}{\partial r}-\frac{n(n+1)}{r^2}Q_n. \tag{2}$$

We attempt to find solutions of this equation by an expansion in negative powers of r

$$Q_n = \sum_k \overset{k}{L}_n(u)\,r^{-k-1}. \tag{3}$$

Substitution readily yields the recurrence relation

$$2(k+1)\frac{\mathrm{d}\overset{k+1}{L}_n}{\mathrm{d}u} = (n-k)\,(n+k+1)\,\overset{k}{L}_n. \tag{4}$$

The class of solutions of (1) that we expect to be able to represent in form (3) is essentially the class of outward travelling waves, for with general inward travelling waves we would expect to find arbitrary functions of $t+r$, that is of $u+2r$, which only in the rarest cases would admit expansion in powers of r^{-1}. Indeed the Sommerfeld condition may be taken to be virtually equivalent to the validity of the expansion (3).

The recurrence relation (4) shows

(i) $\overset{0}{L}_n$ is the lowest $\overset{k}{L}_n$ occurring, so that we have a satisfactory expansion diminishing to zero at infinity.

* This approach is considered in detail by Friedlander (1962) in the following paper.

(ii) Since $\mathrm{d}\overset{n+1}{L_n}/\mathrm{d}u = 0$, the set of $\overset{k}{L_n}$ is divided into two disconnected parts, one from $\overset{0}{L_n}$ to $\overset{n}{L_n}$, the other from $\overset{n+1}{L_n}$ onwards.

(iii) $\overset{n}{L_n}$ is the only non zero term if we impose the condition that Q_n is static.

Consider again the combination of all the surface harmonics S_n. Calling $\overset{n}{L_n}$ the nth moment M_n of the generating distribution, a static system is therefore fully described by all its moments. Next, suppose the system to vary in time, with each of the moments a given function of the time or, rather, of the variable u. Again confining attention to the part proportional to $S_n (n \geqslant 1)$, we have

$$\overset{n}{L_n} = M_n(u) \text{ (given)}.$$

Then, by (4)

$$\overset{n-1}{L_n} = \frac{\mathrm{d}M_n}{\mathrm{d}u}, \quad \overset{n-2}{L_n} = \frac{n-1}{2n-1}\frac{\mathrm{d}^2M_n}{\mathrm{d}u^2}, \ldots, \quad \overset{0}{L_n} = \frac{2^n n!}{(2n)!}\frac{\mathrm{d}^nM_n}{\mathrm{d}u^n}. \tag{5}$$

The coefficient of r^{-1} in the expansion of Q, namely $\overset{0}{L} = \sum_1^\infty \overset{0}{L_n} S_n$, will be called the coefficient of the *radiative part* of the field. A field for which all the $\overset{0}{L_n}$ vanish is called non-radiative. In addition to static fields, all those fields for which M_n varies like a polynomial of degree not exceeding $(n-1)$ will also be non-radiative.

Suppose now that a field is originally static, is then radiative for a finite period, and finally is non-radiative again (a so-called sandwich wave). Therefore $\overset{0}{L}$ differs from zero only for this finite period of time. If a distant observer registers $\overset{0}{L}$ and knows that before the wave period the system was static then he can decide by applying (5) whether the final state is static or one of the more general non-radiative solutions. For the wave period will lead from a static situation to a static one only if not merely $\overset{0}{L_n}$ vanishes initially and finally, but also $\overset{1}{L_n}, \overset{2}{L_n}, \ldots, \overset{n-1}{L_n}$ vanish initially and finally.

Therefore the $(n-1)$ additional conditions (wave period $a \leqslant u \leqslant b$)

$$\int_a^b \overset{0}{L_n}(u)\,\mathrm{d}u = 0, \quad \int_a^b \mathrm{d}u \int_a^u \overset{0}{L_n}(u')\,\mathrm{d}u' = \int_a^b (b-u)\overset{0}{L_n}(u)\,\mathrm{d}u = 0, \ldots,$$

$$\int_a^b \mathrm{d}u \int_a^u \mathrm{d}u' \ldots \int_a^{u''} \overset{0}{L_n}(u''')\,\mathrm{d}u''' = \int_a^b \frac{(b-u)^{k-1}}{(k-1)!}\overset{0}{L_n}(u)\,\mathrm{d}u = 0 \quad (k = 1, 2, \ldots, n-1) \tag{6}$$

must be satisfied. This is an important result that will be required later, and so it may be worth stressing how this should be applied to the complete solution involving all S_n. Then a distant observer aware only of the complete coefficient $\overset{0}{L}$ of the entire $1/r$ term must apply the following necessary and sufficient conditions for a sandwich wave to lead from a configuration known to be static to a static final configuration:

(i) The S_0 part (i.e. spherically symmetrical part) of $\overset{0}{L}$ is independent of u for $u \geqslant b$.

(ii) The S_1 part (dipole part) of $\overset{0}{L}$ vanishes for $u \geqslant b$.

(iii) The S_2 part (quadrupole part) of $\overset{0}{L}$ vanishes for $u \geqslant b$ and its integral with u from a to b vanishes.

(iv) The S_n part of $\overset{0}{L}$ vanishes for $u \geqslant b$ and the $(n-1)$ conditions (6) apply to its integral over the radiative period (a, b).

Unless all these conditions are satisfied (note that n apply to the S_n part for $n \geqslant 1$) the final state will not be static, even if still non-radiative.

If we consider a more complicated, and especially a non-linear, hyperbolic equation, as we shall do later in this paper, then even if an analysis of this type, with expansion in negative powers of the radius (though not in surface harmonics) is possible, the right-hand side of the equation corresponding to (4) will be vastly more complicated. However, it turns out, in the case of the gravitational field in empty space, that $\mathrm{d}\overset{k+1}{L}/\mathrm{d}u$ is determined entirely by $\overset{0}{L}, \overset{1}{L}, \ldots, \overset{k}{L}$, and their angle derivatives. It need not cause surprise then to find an infinite set of conditions for the field to go from a static state to a static state via a radiative interlude, namely, the set corresponding to (i) to (iv), but with the various S_n unseparated. This situation suggests therefore that Huygens's principle applies to gravitational waves so that after the excitation a completely static situation sets in. However, this argument is no proof, and it is quite possible that there are 'tails' in the gravitational case. Perhaps one can pick up a hint of how these tails arise by returning to the linear case (equation (4)) and considering the series beyond $\overset{n}{L}_n$. It is clear that as we go along the series each term will introduce a new arbitrary constant, since each coefficient enters for the first time through its derivative. A simple calculation and summation of the series occurring shows that each of these arbitrary constants is multiplied by an expression of the form

$$\frac{S_n}{r^{n+p}} \frac{1}{u+2r} \left[\text{polynomial in } \frac{u}{u+2r} \text{ of degree } (n+p-1)\right], \tag{7}$$

where p is a positive integer. The process of summation and the final result show that the series occurring converge only over a limited range; in fact, r must be less than $\frac{1}{2}u$. However, it is ascertained by direct substitution that the sum (7) is everywhere a solution of equation (2). We may, if we wish, look at expression (7) as a particular type of combination of incoming and outgoing waves. It is clear that every one of expressions (7) for fixed values of the radius tends to zero as u becomes very large, that is to say, these terms all represent declining modes.

We shall not further investigate the meaning of terms of type (7), but conclude from this heuristic argument that equations of type (4) may generate expressions that tend to zero as $u \to \infty$ which might in the non-linear case represent tails. Moreover, they make it plausible that in general we may obtain useful series for u/r less than some positive constant.

PART B. A SUITABLE CO-ORDINATE SYSTEM

By H. Bondi and M. G. J. van der Burg

1. The character of the metric

In most investigations of specific systems in general relativity the work can be simplified considerably by a suitable choice of co-ordinate system. In the case of radiation from an isolated system, we are interested in the behaviour of the gravitational field at large distances. As it seems unlikely that a solution in closed form can be found, the main aim in the choice of the system of co-ordinates should be to make an expansion appropriate to large distances as simple as possible. Investigators have often been hindered by the appearance of terms in $\log r$ which prevent expansion in negative powers of r. It is highly desirable so to define the co-ordinates that such terms do not appear. There seem to be two distinct ways in which logarithmic terms arise. In the Schwarzschild and similar solutions, in the usual form, the equation for the radial null geodesics contains a logarithm of r. By using a co-ordinate constant along such radial null geodesics, the appearance of such a term can be avoided. Secondly, it is clear that gravitational waves spread with distance from the source. If the area of intersection of any bundle of these null geodesics with the surfaces of equal phase varies like r^2, then the decay of fields with distance may be expected to occur with suitable powers of $1/r$. If these areas varied like a more complicated function of r, as usually happens, one would expect this function, which often includes logarithmic terms, to appear also in the variation of the strength of the wave with distance.

The co-ordinate system that we shall use is designed to avoid both these possible sources of logarithmic terms. It will first be defined in an intuitive manner; later the definition will be made mathematically more precise. Throughout this paper we shall suppose the 4-space to be axially symmetrical and also reflexion symmetrical. We do not think it likely that this restriction, which materially simplifies the analysis, is too severe for the study of gravitational waves. From the axial symmetry the azimuth angle ϕ is readily defined in an invariant manner. Suppose we now put a source of light at a point O on the axis of symmetry and surround it by a small sphere on which we can produce the azimuth co-ordinate ϕ together with a co-latitude θ and a time co-ordinate u. We then define the u, θ, ϕ co-ordinates of an arbitrary event E to be the u, θ, ϕ co-ordinates of the event at which the light ray OE intersects the small sphere. In other words, along an outward radial light ray the three co-ordinates u, θ, ϕ are constant. If we wish to write down the metric for such a system of co-ordinates (in which the part referring to the azimuth angle ϕ appears separately) then we know that since only the co-ordinate r varies along a light ray, the term g_{11} of the metric tensor must vanish, the four co-ordinates u, r, θ, ϕ being denoted by 0, 1, 2, 3 in that order. Moreover, we must have

$$\Gamma^0_{11} = \Gamma^2_{11} = 0. \tag{8}$$

Owing to the restriction $g_{11} = 0$ this implies

$$g^{00}g_{01,1} + g^{02}g_{12,1} = g^{02}g_{01,1} + g^{22}g_{12,1} = 0, \tag{9}$$

Since neither g nor g_{33} can vanish, the equations are equivalent to

$$g_{12}(g_{12}g_{01,1}-g_{01}g_{12,1}) = g_{01}(g_{12}g_{01,1}-g_{01}g_{12,1}) = 0. \tag{10}$$

This implies either that $g_{01} = 0$, or that $g_{12} = 0$, or that g_{12}/g_{01} is independent of r. If $g_{01} = 0$, it may be shown that $g_{00} < 0$ (for otherwise the signature is incorrect), and thus that u is not a time-like co-ordinate, contrary to its definition. Therefore we reject the possibility $g_{01} = 0$. If g_{12}/g_{01} is independent of r then there exists a function $\tilde{u}(u,\theta)$ such that, with a suitable $\lambda(u,\theta)$,

$$\mathrm{d}\tilde{u} = \lambda[\mathrm{d}u + g_{12}\,\mathrm{d}\theta/g_{01}]. \tag{11}$$

Replacing u by $\tilde{u}$ reduces g_{12} to zero. Accordingly we have arrived at a metric distinguished by the condition

$$g_{11} = g_{12} = 0. \tag{12}$$

Although we have defined the u, θ, ϕ co-ordinatives fairly closely now, the radial co-ordinate r remains entirely indeterminate and can be replaced by any function of r, u and θ without changing the character of the metric or, indeed, the coefficients g_{22} and g_{33}. We accordingly define the co-ordinate r by the condition

$$r^4 \sin^2\theta = g_{22}g_{33} \tag{13}$$

which ensures that the area of the surface element u = constant r = constant is in fact $r^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\rho$. We can now write the metric in the form*

$$\begin{aligned}\mathrm{d}s^2 = {} & (Vr^{-1}\,\mathrm{e}^{2\beta} - U^2r^2\,\mathrm{e}^{2\gamma})\,\mathrm{d}u^2 + 2\,\mathrm{e}^{2\beta}\,\mathrm{d}u\,\mathrm{d}r \\ & + 2Ur^2\,\mathrm{e}^{2\gamma}\,\mathrm{d}u\,\mathrm{d}\theta - r^2(\mathrm{e}^{2\gamma}\,\mathrm{d}\theta^2 + \mathrm{e}^{-2\gamma}\sin^2\theta\,\mathrm{d}\phi^2).\end{aligned} \tag{14}$$

The peculiar form of the first coefficient is chosen for later convenience. The four functions U, V, β, γ, are functions of u, r and θ. All our investigations will be based on this form of the metric.

There is still considerable freedom in setting up such a metric, but this is somewhat reduced by the following consideration. In our problem, space tends to flatness at infinity. We infer from this that tetrad vectors may be chosen such that the physical components of the curvature tensor tend to zero at infinity. Accordingly, the mutual accelerations in a swarm of freely moving point particles at large distances can be made arbitrarily small by having the swarm sufficiently far away. We now choose such a swarm of particles so far away that the effect of the accelerations can be neglected for a period sufficient for our investigations, and attach the u and θ co-ordinates to this swarm of particles instead of to the small sphere. This ensures that at sufficiently large distances u is a time-like co-ordinate—that is, the coefficient of $\mathrm{d}u^2$ remains positive however large r is taken to be.†

* First given in Bondi (1960).

† The significance of this specification may be understood by considering what happens in flat space if the co-ordinate θ is fixed on a small sphere surrounding the origin, but is chosen to be a function of time. Then the 'searchlight beam' corresponding to a particular value of θ moves about in space. At large distances points with the same r, θ, ϕ co-ordinates, but different u co-ordinates will, owing to the motion of the 'searchlight beam', have a space-like rather than a time-like separation.

In any metric in polar co-ordinate form conditions must be imposed in the neighbourhood of the polar axis ($\sin\theta = 0$) in order to insure regularity there. It must be possible to choose a Minkowskian tangent metric which in turn implies that, for a small circle around the axis, the ratio of circumference to radius equals 2π to the second order in the radius, and also that the metric has no kink as the axis is crossed. In our case these conditions imply that, as $\sin\theta \to 0$,

$$V,\ \beta,\ U/\sin\theta,\ \gamma/\sin^2\theta \tag{15}$$

each equals a function of $\cos\theta$ regular at $\cos\theta = \pm 1$. The metric chosen is sufficiently general and chosen by such a physical reasoning that we may suppose it to be valid for sufficiently large distances from an isolated material system. In other words, while co-ordinate patches are the rule rather than the exception, we may reasonably expect that all the space sufficiently far from an isolated system may be covered by one patch, and that this may be expressed in the form of the co-ordinate system given. We next consider the field equations, then impose an outgoing wave condition, and finally, using both these, we further restrict the co-ordinated system.

2. The structure of the field equations

It is advantageous to consider the relations between the field equations arising from the Bianchi identities before considering the field equations themselves. First, note that the contravariant fundamental tensor for our metric (14) is given by

$$g^{\mu\nu} = \begin{pmatrix} 0 & e^{-2\beta} & 0 & 0 \\ e^{-2\beta} & -V e^{-2\beta} r^{-1} & U e^{-2\beta} & 0 \\ 0 & U e^{-2\beta} & -e^{-2\gamma} r^{-2} & 0 \\ 0 & 0 & 0 & -e^{2\gamma} r^{-2} \sin^{-2}\theta \end{pmatrix}. \tag{16}$$

A list of the three-index symbols is given in the appendix to this paper. For the moment we require only the result

$$g^{\alpha\epsilon}\Gamma^0_{\alpha\epsilon} = -2\,e^{-2\beta} r^{-1}. \tag{17}$$

The contracted Bianchi identities are

$$g^{\alpha\epsilon}(R_{\mu\alpha} - \tfrac{1}{2}g_{\mu\alpha}R)_{;\epsilon} = g^{\alpha\epsilon}(R_{\mu\alpha.\,\epsilon} - \tfrac{1}{2}R_{\alpha\epsilon,\,\mu} - \Gamma^\delta_{\alpha\epsilon} R_{\mu\delta}) = 0. \tag{18}$$

Suppose now that
$$R_{11} = R_{12} = R_{22} = R_{33} = 0. \tag{19}$$

Then the Bianchi identities reduce to

$$\left.\begin{aligned} \mu = 0:&\quad g^{01}R_{00,1} + g^{11}R_{01,1} + g^{12}(R_{01,2} + R_{02,1}) \\ &\qquad + g^{22}R_{02,2} - g^{\alpha\epsilon}\Gamma^0_{\alpha\epsilon}R_{00} - g^{\alpha\epsilon}\Gamma^1_{\alpha\epsilon}R_{01} - g^{\alpha\epsilon}\Gamma^2_{\alpha\epsilon}R_{02} = 0, \\ \mu = 1:&\quad -g^{\alpha\epsilon}\Gamma^0_{\alpha\epsilon}R_{01} = 0, \\ \mu = 2:&\quad g^{01}(R_{02,1} - R_{01,2}) - g^{\alpha\epsilon}\Gamma^0_{\alpha\epsilon}R_{02} = 0. \end{aligned}\right\} \tag{20}$$

We see that the four equations (19) are independent. They will be referred to as 'the main equations'. Next, we see from the second of equations (20) that R_{01} vanishes as a consequence of the main equations. Using this result and also equation (17), the last of equations (20) may therefore be written

$$e^{-2\beta}\left(\frac{\partial}{\partial r} + \frac{2}{r}\right) R_{02} = r^{-2}\,e^{-2\beta}\frac{\partial}{\partial r}(r^2 R_{02}) = 0. \tag{21}$$

Accordingly, as a consequence of the main equations (19) only, R_{02} is of the form $f(u,\theta)\,r^{-2}$. Therefore, if R_{02} vanishes for some value of r and all values of u and θ, it will vanish for all values of r. If this condition is satisfied, the first of equations (20) also reduces to the form (21). What we have said about R_{02} then applies equally to R_{00}. Thus, a complete set of field equations resolves into the four main equations (19), the trivial equation $R_{01}=0$, and the two supplementary conditions, as we shall call them, which only imply that R_{00} and R_{02} vanish for some finite value of r. If R_{00} and R_{02} can be expanded in powers of r then the supplementary conditions merely state that the coefficients of the r^{-2} terms vanish.

3. The main equations

We now write the main equations (19) in the following form:

$$0 = R_{11} = -4[\beta_1 - \tfrac{1}{2}r\gamma_1^2]\,r^{-1}. \tag{22}$$

$$0 = -2r^2R_{12} = [r^4\,\mathrm{e}^{2(\gamma-\beta)}U_1]_1 - 2r^2[\beta_{12}-\gamma_{12}+2\gamma_1\gamma_2-2\beta_2 r^{-1}-2\gamma_1\cot\theta]. \tag{23}$$

$$\left.\begin{aligned}0 = R_{22}\,\mathrm{e}^{2(\beta-\gamma)} - r^2R_3^3\,\mathrm{e}^{2\beta} &= 2V_1 + \tfrac{1}{2}r^4\,\mathrm{e}^{2(\gamma-\beta)}\,U_1^2\\ &\quad - r^2U_{12} - 4rU_2 - r^2\,U_1\cot\theta - 4r\,U\cot\theta\\ &\quad + 2\mathrm{e}^{2(\beta-\gamma)}[-1-(3\gamma_2-\beta_2)\cot\theta-\gamma_{22}+\beta_{22}+\beta_2^2+2\gamma_2(\gamma_2-\beta_2)].\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}0 = -R_3^3\,\mathrm{e}^{2\beta}r^2 &= 2r(r\gamma)_{01} + (1-r\gamma_1)\,V_1 - (r\gamma_{11}+\gamma_1)\,V\\ &\quad - r(1-r\gamma_1)\,U_2 - r^2(\cot\theta-\gamma_2)\,U_1\\ &\quad + r(2r\gamma_{12}+2\gamma_2+r\gamma_1\cot\theta-3\cot\theta)\,U\\ &\quad + \mathrm{e}^{2(\beta-\gamma)}[-1-(3\gamma_2-2\beta_2)\cot\theta-\gamma_{22}+2\gamma_2(\gamma_2-\beta_2)].\end{aligned}\right\} \tag{25}$$

Note that equations (22) to (24) involve only differentiation in the hypersurface $u=$ const. (hypersurface equations), while (25), the 'standard equation', contains a derivative with respect to u. Consider now the structure of these equations without, at first, worrying about the functions of integration. If, for some value of u, γ is given, equation (22) will determine β. Equation (23) will then determine U, and equation (24) will give V. From equation (25), the u-derivative of γ may then be deduced. Thus the function γ may be found at the next instant of u, and then we can again go through the whole cycle. In other words, given γ for some value of u, the future follows, apart from the question of functions of integration, each of which is an arbitrary function of u and θ, but independent of r. We can count them easily enough. Equation (22) determines β apart from an additive function $H(u,\theta)$. In equation (23), two such functions of integration occur, one of them being an addition $-6N(u,\theta)$ to $r^4\,\mathrm{e}^{2(\gamma-\beta)}\,U_1$, the other an addition $L(u,\theta)$ to U itself. In equation (24), a function $-2M(u,\theta)$ may be added to V. Finally, equation (25) determines γ_0 apart from a term $c_0(u,\theta)\,r^{-1}$ which goes out when the first term is formed. Thus γ contains a term $c(u,\theta)\,r^{-1}$ where $c_0(u,\theta)$ is not determined by (24).

There are, accordingly, a total of five such functions, and we can now re-state what we found before by saying that, given γ for one value of u, and given the five functions $H(u,\theta)$, $N(u,\theta)$, $L(u,\theta)$, $M(u,\theta)$, $c(u,\theta)$, the entire development is determined by the four main equations. We shall see below that, in certain circumstances, co-ordinate considerations serve to eliminate two of these five functions, and that the supplementary conditions yield two relations between the three surviving functions.

The significance of this structure of the equations is readily understood. If we know the situation for a particular value of u = constant (which represents a light cone opening out into the future) then we know everything about incoming waves owing to our knowledge of the various functions for all radii. If the system is to do anything new, then the information about this must be contained in the functions of u and θ that appear as functions of integration in our analysis. Thus we see that all the news there is appears in these functions which, as has been indicated, can be reduced to a single 'news function'.

4. The outgoing wave condition

If we wish to adopt the principle of causality—in other words, if we wish to eliminate inward travelling waves—we have to apply a suitable condition to γ, etc. This can be done in various ways. We may suppose, for instance, that at one value of u and for a little while beforehand, the entire system was static, that is, without any radiation whatsoever. In that case, the system can be described by the well-known Weyl metric, and it may be shown (see appendix) that, in this case, γ and the other variables have the form of power series in negative powers of r. Alternatively, we may say that the absence of inward flowing radiation is equivalent to the condition that γ (and the other variables) should be of the form, roughly speaking,

$$\gamma = \frac{f(t-r)}{r} + \frac{g(t-r)}{r^2} + \dots \tag{26}$$

which is equivalent to what we said before. Condition (26) is essentially equivalent to Sommerfeld's radiation condition which, in our case, means that

$$[\partial(r\gamma)/\partial r]_{u=\text{const.}} \to 0$$

as $r \to \infty$. Next consider the main equations in the light of (26). If γ has a term proportional to r^{-1}, together with other terms tending to zero more rapidly at infinity, then it follows from equation (22) that β remains bounded as $r \to \infty$. Similarly equation (23) then shows that U remains bounded as $r \to \infty$, and in fact $\lim U = L(u, \theta)$. Substituting this into equation (24) and integrating one finds that the leading term in V is proportional to r^2. Accordingly, the coefficient g_{00} of the metric is dominated by the second term, which is bound to make it negative for sufficiently large values of r. This is contrary to the way in which the co-ordinate system was defined according to which g_{00} had to remain positive for arbitrarily large values of r. Therefore $L(u, \theta) = 0$.* This disposes of one of our functions of integration. On this basis, the leading terms of the various functions are given by

$$\left.\begin{aligned} &\gamma = c(u,\theta)\, r^{-1} + \dots, \quad \beta = H(u,\theta) - \tfrac{1}{4}c^2 r^{-2} + \dots, \\ &U = 2H_2\, e^{2H} r^{-1} + \dots, \quad V = r\, e^{2H}[1 + 2H_2 \cot\theta + 4H_2^2 + 2H_{22}] + \dots \end{aligned}\right\} \tag{27}$$

* It might be suggested that a more general form of γ should be used with a leading term $Q(u, \theta)$ independent of r. However, Q does not affect the argument just given that $L = 0$ and, if $L = 0$, (14) proves that $Q_0 = 0$. The elementary transformation $\tan \frac{1}{2}\bar{\theta} = \exp \int e^{4Q} \sin^{-1}\theta \, d\theta$, with a consequent adjustment of r, then reduces Q to zero.

As a final restriction on the co-ordinate system, we reduce H to zero by a co-ordinate transformation

$$\left.\begin{aligned} u &= \overset{0}{a}(\bar{u},\bar{\theta}) + \overset{1}{a}(\bar{u},\bar{\theta})\,\bar{r}^{-1} + \dots, \\ r &= \bar{r} + \overset{0}{\rho}(\bar{u},\bar{\theta}) + \dots, \\ \theta &= \bar{\theta} + \overset{1}{g}(\bar{u},\bar{\theta})\,\bar{r}^{-1} + \dots. \end{aligned}\right\} \tag{28}$$

In order to preserve the character of the metric,

$$\left.\begin{aligned} \bar{g}_{11} &= 0\colon 2\overset{1}{a}\,\mathrm{e}^{2H} = -\left(\overset{1}{g}\right)^2, \\ \bar{g}_{12} &= 0\colon \overset{0}{a}_{\bar{\theta}}\,\mathrm{e}^{2H} + 2\overset{1}{g} = 0, \\ \bar{g}_{22}\,\bar{g}_{33} &= \bar{r}^4 \sin^2\bar{\theta}\colon 2\overset{0}{\rho} + \overset{1}{g}_{\bar{\theta}} + \overset{1}{g}\cot\bar{\theta} = \overset{0}{a}_{\bar{\theta}}\,H_2\,\mathrm{e}^{2H}, \\ \bar{g}_{01} &= \overset{0}{a}_{\bar{u}}\,\mathrm{e}^{2H} + O(\bar{r}^{-1}). \end{aligned}\right\} \tag{29}$$

For an arbitrary $\overset{0}{a}$ the second of these equations defines $\overset{1}{g}$, the first $\overset{1}{a}$, the third one $\overset{0}{\rho}$, and similarly the higher-order equations determine the higher-order coefficients. Then $\bar{H} = 0$ if $\overset{0}{a}$ is chosen so that

$$\overset{0}{a}_{\bar{u}} = \exp(-2H). \tag{30}$$

Thus a suitable transformation of type (28) reduces H to zero.

It may be advantageous to recapitulate briefly how successive restrictions have been imposed on the co-ordinate system:

(i) $\mathrm{d}u = \mathrm{d}\theta = \mathrm{d}\phi = 0$ is an outgoing light-ray. This implies $g_{11} = 0$.

(ii) u is time-like, so that $g_{00} > 0$, leading to $g_{12} = 0$.

(iii) Area of 2-surface $u = \text{const.}$ $r = \text{const.}$ restricted to equal $4\pi r^2$, used to define r and to fix $g_{22}\,g_{33} = r^4 \sin^2\theta$.

(iv) Field equations for empty space imposed, functions H, N, L, M, c isolated.

(v) Outgoing wave condition imposed. With (ii) this implies $L = 0$.

(vi) H reduced to zero by suitable transformation.

The range of possible transformations of co-ordinate systems satisfying (i) to (vi) is investigated in part C of this paper. It should be noted that in general the three surviving functions of integration c, M, N, cannot be reduced to zero.

If the form (26) for γ is substituted into the main equations it turns out that the other variables do not satisfy the radiation condition unless the r^{-2} term in γ vanishes. (This occurs also in the static case when the Weyl metric is translated into our form.) Carrying out this expansion* and substituting into the main equations we obtain the following relations:

If
$$\gamma = c(u,\theta)\,r^{-1} + [C(u,\theta) - \tfrac{1}{6}c^3]\,r^{-3} + \dots, \tag{31}$$

then
$$\left.\begin{aligned} U = &-(c_2 + 2c\cot\theta)\,r^{-2} + [2N(u,\theta) + 3cc_2 + 4c^2\cot\theta]\,r^{-3} \\ &+ \tfrac{1}{2}(3C_2 + 6C\cot\theta - 6cN - 8c^2c_2 - 8c^3\cot\theta)\,r^{-4} + \dots, \end{aligned}\right\} \tag{32}$$

* No assumption of analyticity is implied. It is probably sufficient if the remainder after the first few terms vanishes suitably at infinity.

$$\left.\begin{aligned}V = r-2M(u,\theta)-[N_2+N\cot\theta-c_2^2-4cc_2\cot\theta-\tfrac{1}{2}c^2(1+8\cot^2\theta)]\,r^{-1}\\ -\tfrac{1}{2}[C_{22}+3C_2\cot\theta-2C+6N(c_2+2c\cot\theta)\\ +8c(c_2^2+3cc_2+2c^2\cot^2\theta)]\,r^{-2}+\dots,\end{aligned}\right\} \quad (33)$$

$$4C_0 = 2c^2c_0+2cM+N\cot\theta-N_2. \quad (34)$$

Thus the form of γ is preserved and the development of the system is fully determined from initial conditions provided the functions c, N, M are known.

5. The supplementary conditions

The full form of the equations $R_{02}=0$ and $R_{00}=0$, which is exceedingly complicated, is given in the appendix. It is in no way obvious how, in general, the substitution of the main equations reduces the supplementary conditions to the inverse square-law term. However, on the basis of the expansions given above, the supplementary conditions reduce enormously. The sole surviving terms are of the r^{-2} form as, of course, they should be, and involve only relations between the three functions c, M and N, namely

$$M_0 = -c_0^2+\tfrac{1}{2}(c_{22}+3c_2\cot\theta-2c)_0, \quad (35)$$

$$-3N_0 = M_2+3cc_{02}+4cc_0\cot\theta+c_0c_2. \quad (36)$$

Thus we see that if M and N are given for one value of u, and c is given as a function of u and θ, the entire situation is fully determined. In other words, the flow of information in the system is entirely controlled by the single function c. In fact, as will be seen later, the whole character of the developments can be read off from equations (35) and (36). This is particularly satisfactory because these are in no way approximate equations, but are exact relations valid supposing only the series expansions to be valid. In order to identify M and N to some extent at least, we consider now static and therefore well understood metrics.

6. The static case

It is well known that the empty space axially symmetric static metric can always be reduced to Weyl's form

$$\mathrm{d}s^2 = \mathrm{e}^{2\psi}\,\mathrm{d}t^2-\mathrm{e}^{-2\psi}[\mathrm{e}^{2\sigma}(\mathrm{d}\rho^2+\mathrm{d}z^2)+\rho^2\,\mathrm{d}\phi^2], \quad (37)$$

where

$$\frac{\partial^2\psi}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial\psi}{\partial\rho}+\frac{\partial^2\psi}{\partial z^2} = 0, \quad (38)$$

and ψ determines σ by a relation not required here. It follows from (38) that if the metric is Minkowskian at infinity then

$$\psi = -\sum_{n=0}^{\infty} A^{(n)}R^{-n-1}P_n(\cos\Theta), \quad \rho = R\sin\Theta, \quad z = R\cos\Theta. \quad (39)$$

Here $A^{(0)}$ is the mass m of the system, $A^{(1)}$ is the dipole moment D, and

$$A^{(2)} = Q+\tfrac{1}{3}m^3, \quad (40)$$

where Q is the quadrupole moment, defined so as to vanish for the Weyl form of the Schwarzschild line element. A rather lengthy and tedious transformation is required to connect the Weyl metric with our metric*. Therefore only the result will be quoted here, in a form chosen to fit in with the transformation equations given in part C. These equations show that c is an arbitrary function of θ in the static case, given in the general case, as compared with the case $c = 0$, in terms of the transformation function α by

$$c = -\tfrac{1}{2}\alpha_{22} + \tfrac{1}{2}\alpha_2 \cot\theta. \tag{41}$$

Then we have

$$\left.\begin{aligned} M &= m, \\ N &= D\sin\theta - m\alpha_2, \\ C &= \tfrac{1}{2}Q\sin^2\theta - \alpha_2 D\sin\theta + \tfrac{1}{2}m\alpha_2^2. \end{aligned}\right\} \tag{42}$$

Equations (42) allow us to interpret the principal functions occurring in our metric in the static case. The function $M(u,\theta)$ will now be named the *mass aspect*, and the first of equations (42) shows that in the static case the mass aspect equals the mass of the system and is accordingly independent of both its arguments. The functions N and C are seen to be closely related to the dipole and quadrupole moments respectively.

7. The curvature tensor

In order to investigate the character of the solutions, it is necessary to know the behaviour of the curvature tensor, and, to be able to interpret it in physical terms, a tetrad formulation is required. Accordingly, we need a system of orthogonal unit vectors, one time-like, and three space-like. Moreover, to prevent spurious effects, we choose the unique set of vectors that, by parallel transport along the null geodesics $\mathrm{d}u = \mathrm{d}\theta = \mathrm{d}\phi = 0$, turns at infinity into

$$T^\mu = (1,0,0,0), \quad R^\mu = (-1,1,0,0), \quad S^\mu = \left(0,0,\frac{1}{r},0\right), \quad P^\mu = \left(0,0,0,\frac{1}{r\sin\theta}\right).$$

To find this set at a general point define

$$q = \int_r^\infty [\tfrac{1}{2}r\,\mathrm{e}^{\gamma}U_1 + \beta_2 r^{-1}\mathrm{e}^{2\beta-\gamma}]\,\mathrm{d}r = (c_2 + 2c\cot\theta)\,r^{-1} - [\tfrac{3}{2}N + 2c(c_2 + c\cot\theta)]\,r^{-2} + \dots. \tag{43}$$

The set can be shown to be

$$\begin{aligned}
T^\mu &= [1, \tfrac{1}{2}(1+q^2)\,\mathrm{e}^{-2\beta} - \tfrac{1}{2}Vr^{-1}, U + q\,\mathrm{e}^{-\gamma}r^{-1}, 0] \\
&= [1, Mr^{-1} + \tfrac{1}{2}(N_2 + N\cot\theta)\,r^{-2} + \dots, \tfrac{1}{2}Nr^{-3} + \dots, 0], \\
R^\mu &= [-1, \tfrac{1}{2}(1-q^2)\,\mathrm{e}^{-2\beta} + \tfrac{1}{2}Vr^{-1}, \; -U - q\,\mathrm{e}^{-\gamma}r^{-1}, 0] \\
&= [-1, 1 - Mr^{-1} - \tfrac{1}{4}(2N_2 + 2N\cot\theta - c^2)\,r^{-2} + \dots, -\tfrac{1}{2}Nr^{-3} + \dots, 0], \\
S^\mu &= [0, q\,\mathrm{e}^{-2\beta}, \mathrm{e}^{-\gamma}r^{-1}, 0] \\
&= [0, (c_2 + 2c\cot\theta)\,r^{-1} - \tfrac{1}{2}(3N + 4c(c_2 + c\cot\theta))\,r^{-2} + \dots, r^{-1} - cr^{-2} + \tfrac{1}{2}c^2r^{-3} + \dots, 0], \\
P^\mu &= [0,0,0,\mathrm{e}^{\gamma}r^{-1}\sin^{-1}\theta] \\
&= [0,0,0,r^{-1}\sin^{-1}\theta(1 + cr^{-1} + \tfrac{1}{2}c^2r^{-2} + \dots)].
\end{aligned} \tag{44}$$

* This is given in appendix 4.

Owing to the axial symmetry, the number of free components of the curvature tensor is reduced to 12. The surviving components have been worked out, with the use of the power series expansions for the field variables. Again it turns out that, to the approximation worked, several components coincide. We are then left with

$$\left.\begin{aligned} R_{(TTSS)} &= -R_{(TTPP)} = -R_{(TRSS)} = R_{(TRPP)} = R_{(RRSS)} \\ &= -R_{(RRPP)} = -c_{00}\, r^{-1} + \tfrac{1}{2}[c_{022} + c_{02}\cot\theta - 2c_0(1+2\cot^2\theta)]\, r^{-2} \ldots + , \\ R_{(TTRS)} &= -R_{(TRSR)} = -R_{(TSPP)} = R_{(RSPP)} = -(c_{02} + 2c_0 \cot\theta)\, r^{-2} + \ldots \\ R_{(TTRR)} &= -R_{(SSPP)} = -2(M + cc_0)\, r^{-3} + \ldots. \end{aligned}\right\} \tag{45}$$

PART C. PERMISSIBLE CO-ORDINATE TRANSFORMATIONS

By A. W. K. Metzner

1. Evaluation of the transformations by power series

The purpose of this part is to discuss the permissible transformations of co-ordinates that will leave unchanged the character of the metric discussed in part B. By this we mean not only that the form of the metric should remain the same but also that U, β and γ should tend to zero at infinity. It will be assumed throughout that not only the metric but also the co-ordinate transformations can be expanded in powers of r. Thus, expressing the ordinary unbarred co-ordinates in terms of barred ones, we have

$$\left.\begin{aligned} u &= a(\bar{u},\bar{\theta})\,\bar{r} + \overset{0}{a}(\bar{u},\bar{\theta}) + \overset{1}{a}(\bar{u},\bar{\theta})\,\bar{r}^{-1} + \ldots, \\ r &= K(\bar{u},\bar{\theta})\,\bar{r} + \overset{0}{\rho}(\bar{u},\bar{\theta}) + \overset{1}{\rho}(\bar{u},\bar{\theta})\,\bar{r}^{-1} + \ldots, \\ \theta &= g(\bar{u},\bar{\theta})\,\bar{r} + \overset{0}{g}(\bar{u},\bar{\theta}) + \overset{1}{g}(\bar{u},\bar{\theta})\,\bar{r}^{-1} + \ldots. \end{aligned}\right\} \tag{46}$$

Evaluating the components of the new barred metric tensor in terms of the old one by means of tensor transformations, we find

$$\bar{g}_{11} = K^2 g^2 \bar{r}^2 + \ldots = 0.$$

Since clearly $K \neq 0$, we must have $g = 0$. With this

$$\bar{g}_{11} = a^2 + 2aK + O(\bar{r}^{-1}) = 0.$$

Thus $a = 0$ or $a = -2K$. The second possibility merely corresponds to a reversal of time, that is, to a consideration of advanced rather than retarded solutions. We reject this, and take the first possibility. Continuing the analysis we have

$$\bar{g}_{00} = K^2\left(\partial \overset{0}{g}/\partial \bar{u}\right)^2 \bar{r}^2 + O(\bar{r}) = 1 + \ldots.$$

Hence
$$\overset{0}{g} = \overset{0}{g}(\bar{\theta}).$$

Next
$$\bar{g}_{22} = K^2\left(\mathrm{d}\overset{0}{g}/\mathrm{d}\bar{\theta}\right)^2 \bar{r}^2 + O(\bar{r}) = \bar{r}^2 + O(\bar{r}).$$

Thus
$$K\left(\mathrm{d}\overset{0}{g}/\mathrm{d}\bar{\theta}\right) = \pm 1.$$

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Furthermore

$$\bar{g}_{22}\bar{g}_{33} = g_{22}g_{33}\left(\mathrm{d}\overset{0}{g}/\mathrm{d}\bar{\theta}\right)^2 + O(\bar{r}^3) = K^4\bar{r}^4\sin^2\overset{0}{g} + O(\bar{r}^3) = \bar{r}^4\sin^2\bar{\theta}.$$

From the previous result

$$\frac{\mathrm{d}\overset{0}{g}}{\mathrm{d}\bar{\theta}} = \pm\frac{\sin\overset{0}{g}}{\sin\bar{\theta}}. \tag{47}$$

The minus signs are trivial alternatives and thus

$$\tan\left(\tfrac{1}{2}\overset{0}{g}\right) = \mathrm{e}^{-\nu}\tan\left(\tfrac{1}{2}\bar{\theta}\right) \quad (\nu = \text{const.}), \tag{48}$$

$$K = \cosh\nu + \cos\bar{\theta}\sinh\nu. \tag{49}$$

Finally, from these results,

$$\bar{g}_{01} = K\frac{\partial\overset{0}{a}}{\partial\bar{u}} + \dots = 1 + \dots.$$

Thus

$$\overset{0}{a} = \bar{u}/K(\bar{\theta}) + \alpha(\bar{\theta}), \tag{50}$$

where α is an arbitrary function of its argument. Continuing in this manner to compare coefficients, it turns out that no further freedom exists. Thus the entire range of transformations possible is described by the single constant ν introduced in equation (48), together with the single function α of a single variable introduced in equation (50). What do this constant and this function represent? The constant ν enters the function K whose form immediately suggests that we are dealing with aberration. At large distances, where space is effectively flat, the ν transformation is in fact fully equivalent to a Lorentz transformation corresponding to motion along the axis of symmetry with velocity $-\tanh\nu$. Therefore we can regard the K transformation as a generalized Lorentz transformation, that is to say, as a uniform motion of the material system relative to the fixture of our system of co-ordinates at large distances. The significance of the transformation introduced in equation (50) is a little less obvious. It will be recalled that our system of co-ordinates was fixed in the end by tying down light rays at infinity. In doing this we have dropped the restriction that all the light rays for every angle should originate at the same position. We can imagine, as it were, a different lamp for each angular co-ordinate. In order to keep to the various orthogonality conditions implied by our system of co-ordinates, we cannot introduce a very great deal of freedom there but a kind of static deviation. We can fix, as it were, a light for each angle at one time, and then the motions of these different lights are determined. In this way we account for the function of one variable introduced in equation (50).

The equations (48) to (50) may be applied to find the other terms in expansion (46) and may also be used in order to evaluate in the new co-ordinates the functions defining the metric that were introduced in part B. They tend to be somewhat complicated formulae and so are listed in the appendix.

2. Rigorous transformations of Minkowski space

The assumption that the power series (46) represent all possible transformations is open to criticism. No direct proof of this assumption is possible but it is made plausible by the fact, demonstrated below, that *all* transformations for flat space are of this form, together with the assumption made in part B that the coefficients of the general metric can be expanded in power series.

We may write the metric of flat space in the form

$$\mathrm{d}s^2 = \mathrm{d}\bar{u}^2 + 2\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{r} - \bar{r}^2(\mathrm{d}\bar{\theta}^2 + \sin^2\bar{\theta}\,\mathrm{d}\phi^2),$$

and ask for the most general transformation of this into co-ordinates (u, r, θ, ϕ) giving a metric as specified in part B. We first note that in 'mixed' co-ordinates $(u, \bar{r}, \theta, \phi)$ the metric still has $g_{11} = g_{12} = 0$. Thus taking $\bar{u} = \bar{u}(u, \bar{r}, \theta)$, etc., we have

$$(\bar{u}_{\bar{r}} + 1)^2 = 1 + \bar{r}^2\bar{\theta}^2_{\bar{r}},$$

$$u_\theta(\bar{u}_{\bar{r}} + 1) = \bar{r}^2\bar{\theta}_{\bar{r}}\,\bar{\theta}_\theta.$$

Solving for $\bar{u}_{\bar{r}}$ and $\bar{u}_\theta$, we find that the compatibility condition for these implies

$$\bar{\theta} = p(u,\theta) - \sin^{-1}[q(u,\theta)\,\bar{r}^{-1}], \tag{51}$$

$$\bar{u} = (\bar{r}^2 - q^2)^{\frac{1}{2}} - \bar{r} + s(u,\theta), \tag{52}$$

with $$s_\theta = qp_\theta.$$

The functions p, q, s are otherwise arbitrary, except for the usual regularity conditions and the need to make the old and new axes of symmetry coincide, i.e.

$$p \to 0, \quad q \to 0 \quad \text{as} \quad \theta \to 0; \quad p \to \pi, \quad q \to 0 \quad \text{as} \quad \theta \to \pi. \tag{53}$$

After evaluating the coefficient of $\mathrm{d}\theta^2$ we can find r and γ:

$$r^2\sin\theta = p_\theta\sin p(\bar{r}^2 - q^2) - (q\sin p)_\theta\,(\bar{r}^2 - q^2)^{\frac{1}{2}} + qq_\theta\cos p, \tag{54}$$

$$\mathrm{e}^{2\gamma} = \frac{[(\bar{r}^2 - q^2)^{\frac{1}{2}}\,p_\theta - q_\theta]\sin\theta}{(\bar{r}^2 - q^2)^{\frac{1}{2}}\sin p - q\cos p} \to \frac{p_\theta\sin\theta}{\sin p} \quad \text{as} \quad \bar{r} \to \infty. \tag{55}$$

Equation (54) shows that $\bar{r} \to \infty$ implies $r \to \infty$ unless $p_\theta \sin p$ is not positive, a case that must be excluded since otherwise the mapping is incomplete. The requirement that $\gamma \to 0$ as $r \to \infty$ then shows that $p_\theta \sin\theta = \sin p$, i.e. $\tan\frac{1}{2}p = \exp\nu(u)\tan\frac{1}{2}\theta$. The condition that g_{00} remain positive as $r \to \infty$ implies $p_u = 0$ and therefore $\nu = \text{const.}$

The r equation (54) can be solved for $\bar{r}$ and it is immediately clear that $\bar{r}$ can be expanded in powers of r as required provided r is sufficiently large if q_θ, q/θ and $q/(\pi - \theta)$ are bounded which is a consequence of the conditions imposed above. Moreover, $\bar{r} \to \infty$ uniformly as $r \to \infty$, and so the expansion of all functions is guaranteed, as required.

The work can readily be continued without resorting to the expansions. Next, $\beta \to 0$ as $r \to \infty$ implies $s_u = 1$, and so the transformation has been reduced to the generality found above by series expansion.

To interpret the α transformation ($p = \theta, s = -\alpha(\theta), q = -\alpha'(\theta)$) in flat space we use the exact equations which are valid everywhere. (In the general case only the series expansion is available and this precludes an approach to regions of small r.) The co-ordinates $\bar{r}, \bar{\theta}$ are ordinary spherical polars and thus the light ray $\theta = \text{const.}$ has equation

$$\bar{r}\sin(\bar{\theta} - \theta) = \alpha'(\theta). \tag{56}$$

Thus the ray is parallel to $\bar{\theta} = \theta$. If $\alpha'(\theta)/\cos\theta > 0$ it intersects the equatorial plane $\bar{\theta} = \frac{1}{2}\pi$ at distance $\alpha'(\theta)/\cos\theta$ from the origin, and equally if $\alpha'(\theta)/\cos\theta < 0$ it intersects the axis of symmetry at distance $-\alpha'(\theta)/\sin\theta$ from the origin.

The interpretation of the K transformation is identical in the general and flat space cases as a Lorentz transformation.

PART D. THE NATURE OF THE SOLUTIONS

By H. Bondi, F.R.S.

1. News and mass loss

Consider an axi-symmetric system that is static for $u \leqslant 0$. As has been shown, the Weyl metric for this system may be transformed to our form, with all the coefficients $c, M, N, C, \ldots$ independent of u. By (35) the vanishing of c_0 guarantees the vanishing of M_0, but by (36) M_2 has to vanish also to secure the constancy of N. Similarly, by (34), N has to be of the form (42) to assure the vanishing of C_0, and so on. If $c_0 = 0$ for $u > 0$, the system must remain static, but if c_0 begins to deviate from zero the other coefficients in turn begin to vary. Thus if anything happens at all at the source leading to changes in the field, it can only do so by affecting c_0, and vice versa. Thus all the news in the field is contained in c_0, which therefore merits the name *news function*. In general the structure of our equations indicates that if γ, M and N are known for $u = a$, and the news function c_0 is known for all u in $a \leqslant u \leqslant b$, then the system is fully determined in the interval $a \leqslant u \leqslant b$.

Next we define the mass $m(u)$ of the system as the mean value of $M(u,\theta)$ over the sphere

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta\,\mathrm{d}\theta. \tag{57}$$

We note that in the static case $m(u) = m$. Now we integrate (35) and obtain

$$m_0 = -\frac{1}{2}\int_0^\pi c_0^2\sin\theta\,\mathrm{d}\theta, \tag{58}$$

since the last term in (35) goes out on integration because of conditions (15). Thus we have the central result of this paper:

The mass of a system is constant if and only if there is no news. If there is news, the mass decreases monotonically as long as the news continues.

This result may appear to depend on the definition of mass given above, but this can be avoided if we confine our attention to systems initially and finally static, in which the physical significance of m as mass is clear and unambiguous. Thus a dynamic period interposed between two static periods is bound to imply a loss of mass. We can ascribe this in the only physically reasonable way to the emission of waves by the system. Note from (45) that the physical components of the Riemann tensor have an r^{-1} term if and only if $c_{00} \neq 0$.*

Clear-cut and precise though our result is for initially and eventually static systems, it depends for its validity on the possibility of return to a static state. At

* The reason for the appearance of c_{00} in the Riemann tensor against c_0 in (58) will be discussed in §5; very much the same situation occurs for electromagnetic waves.

first sight the conditions for such a return look rather forbidding. If the series expansion of equation (31) were taken beyond the terms given there and in equation (34), then it is easily seen that all the equations for the higher terms have the same general structure as equation (34). The derivative with respect to u of the coefficient of every term is determined by an expression involving the previous coefficients but not involving any differentiation with respect to u. Equations (35) and (36) must also be considered in this connexion. The problem is essentially this—suppose the system is static before $u = 0$, and then c is allowed to vary for a finite period of time at the end of which it becomes constant. Can we so choose the variation of c that at the end of this period of excitation we return to a situation that (but possibly for a tail) will eventually become static? Thus, when c stops varying, by equation (35) M stops varying. Equation (36) then implies that unless M_2 vanishes at this stage, N will go on varying linearly with time. The first condition on c is therefore that the final function M should be independent of θ. Next, consider the variation of N through the period of wave motion. Once again, the change in N must be such that when N ceases to vary, equation (34) implies that C ceases to vary. Continuing like this one obtains an infinity of conditions on c. To understand this we return to the last section of part A. There it was shown that if we are analyzing the wave equation by means of the first term, that is by the equivalent of c_0, then we first have to sort out the various angular dependences, and the part corresponding to P_n is then such that n conditions have to be applied to this part in order to ensure that the system goes from a static to a static situation. Owing to the non-linearity of our equations, the sorting out into the different types of angular dependence cannot be done in any exact form. What is, however, reasonably plausible is that when we deal with these different forms of angular dependence, the part involving P_n need only be pursued as far as the nth coefficient. For it does not matter if anything of this is left over for the higher coefficients. In accordance with the last equation of part A all this might do is to generate a tail to the wave.

To put it differently, if the ideas of the linear equation are applied to our case, then the structure of the equations of condition becomes reasonably clear. In part, these equations of condition make sure that we return from a static to a static case and do not embark on a non-radiative motion. For the rest, our equations may lead to the production of tails. These tails, it is true, will upset the convergence of our series but, once again following the first part, it appears that this difficulty need only arise when u is allowed to exceed $2r$. Therefore the method can be applied with reasonable confidence as long as we are dealing with a sandwich wave, that is, with c constant outside a period of finite length. If we want to investigate this situation, we have merely got to go sufficiently far out for r to exceed substantially one-half of the period of fluctuation. Thus the structure of our equations does not in this respect imply any essential difference from the scalar wave equation considered in part A.

2. Linearized form of the equations

For some purposes it is useful to consider the linear form of our equations. Since our co-ordinates have been chosen in order to simplify the non-linear problem, the relation of our metric to the usual linear ones requires elucidation. Assuming all our variables to be small and to satisfy the boundary conditions at infinity previously imposed, equation (22) shows that β is negligible, and equation (23) becomes

$$(r^4U_1)_1 = -2r^2(\partial/\partial\theta + 2\cot\theta)\,\gamma_1, \tag{59}$$

showing immediately that power series expansion of U implies the vanishing of the r^{-2} term in γ. The integration of (24) turns out not to be required beyond the zero order approximation $V = r$, and then (25) becomes

$$2\left(\gamma_{01}+\frac{\gamma_0}{r}\right)-\gamma_{11}-\frac{2\gamma_1}{r}+\left(\frac{\partial}{\partial\theta}-\cot\theta\right)\left(\frac{1}{2}\frac{\partial}{\partial r}+\frac{1}{r}\right)U = 0. \tag{60}$$

Since all the terms in this equation tend to zero at infinity at least like r^{-2}, no information is lost by multiplying it by r and then differentiating with respect to r. Similarly multiplication by

$$\partial/\partial\theta + 2\cot\theta$$

does not involve any loss of information owing to the regularity conditions on the axis. It is now possible to substitute for γ from (59), and after a little simplification we obtain

$$2\left(\frac{\partial}{\partial r}+\frac{5}{r}\right)\left(\frac{\partial}{\partial r}+\frac{3}{r}\right)U_{01} = \left(\frac{\partial}{\partial r}+\frac{5}{r}\right)\left(\frac{\partial}{\partial r}+\frac{3}{r}\right)^2U_1+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\right)\left(\frac{\partial}{\partial r}+\frac{3}{r}\right)U_1. \tag{61}$$

Inspection of (61) shows that it does not serve to determine the r^{-2} part of U. Multiplying the equation by r^5, integrating with respect to r and dividing by r^5 we obtain

$$2\left(\frac{\partial}{\partial r}+\frac{3}{r}\right)U_{01} = \left(\frac{\partial}{\partial r}+\frac{3}{r}\right)^2U_1+\frac{1}{r^2}\left(\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\right)U_1+\frac{l(u,\theta)}{r^5}, \tag{62}$$

where $l(u,\theta)$ is a function of integration which serves to make the r^{-2} part of U as indeterminate in (62) as it is in (61). If $\overline{U}(u,r,\theta)$ differs from U only by a term in r^{-2} and satisfies (62) with $l = 0$, put $r^2\overline{U}_1 = Q_2(u,r,\theta)$, substitute in (62), and integrate with respect to θ. Assuming the part of Q independent of θ (which is irrelevant for our purposes) to satisfy the same equation as the rest of Q, this is immediately seen to be equivalent to equation (2). Accordingly Q satisfies the scalar wave equation

$$\Box Q = 0. \tag{63}$$

The procedure for constructing a linear approximation to solutions of our main equations is therefore to take an axially symmetric solution of (63) representing outgoing waves tending to zero at infinity. By forming

$$\overline{U} = -\int_r^\infty \frac{Q_2}{r^2}\,\mathrm{d}r, \tag{64}$$

U is found except for the leading term and, by (59), γ is determined, again except for the leading term. Thus all the higher coefficients have been found, and N is known as the coefficient of r^{-3} in U, but c and M remain to be determined from the supplementary conditions which, in linearized form, are

$$-3N_0 = M_2, \tag{65}$$

$$M_0 = \tfrac{1}{2}(c_{22}+3c_2\cot\theta-2c)_0. \tag{66}$$

Since Q can be expanded as a series of $P_n(\cos\theta)$, (64) shows that $\overline{U}$ (and with it F) is a series in $P_n^{(1)}$ and, by (59), $\gamma - r^{-1}c$ correspondingly a series in $P_n^{(2)}$, so that the boundary conditions (15) on the axis are automatically satisfied except of course for the as yet undetermined M and c. Next, (65) determines M but for a function $h(u)$ which may be defined as the coefficient of P_0 in the expansion of M, the factors of P_n $(n \geqslant 1)$ being found from (65). The left-hand side of (66) thus being known, the news function c_0 is readily obtained by a double integration of (66), which yields no admissible complementary functions. Moreover, it can be seen (e.g. by expansion of c_0 in $P_n^{(2)}$) that the c so obtained will not satisfy the boundary conditions on the axis unless $\mathrm{d}h/\mathrm{d}u = 0$ and $\mathrm{d}^2 l(u)/\mathrm{d}u^2 = 0$, where $l(u)$ is the coefficient of $P_n^{(1)}$ in N. Thus the mass must be constant in time and the dipole moment must vary linearly in time. Both these are direct consequences of the conservation laws in the linear approximation.

The time-independent part of c cannot be determined from Q and is clearly irrelevant in the linear approximation. Putting, as before

$$Q = \sum_{k=0}^{\infty} \frac{\overset{k}{L}(u,\theta)}{r^{k+1}}, \tag{67}$$

we find in detail

$$6N = -\overset{1}{L}_2, \tag{68}$$

$$-4M = \overset{0}{L}_{22} + \overset{0}{L}_2 \cot\theta + \text{const.}, \tag{69}$$

$$2(c_0 \sin^2\theta)_2 = -\sin^2\theta \overset{0}{L}_{02}, \tag{70}$$

$$6C_2 + 12C\cot\theta = -\overset{2}{L}_2. \tag{71}$$

3. Non-radiative motions

Consider a system in which c_0 vanishes but M is not independent of θ, though, owing to the constancy of c, it is independent of u. A system in this state is clearly non-radiative both because of the vanishing of the leading terms of the curvature tensor and also because there is no loss of mass. On the other hand, it follows from equation (36) and from the basic equations that the system is not at rest. This is, therefore, a case of non-radiative motion. How can this be interpreted?

One very simple case is clearly illustrated by the work in part C. If a K transformation is applied to the Schwarzschild metric ($\gamma = \beta = U = 0$, $V = r - 2\overline{m}$) we find

$$M = \frac{\overline{m}}{(\cosh\nu + \cos\theta\sinh\nu)^3}. \tag{72}$$

Thus a mass $\overline{m}$ moving along the axis of symmetry with constant velocity $\tanh\nu$ produces a field with M given by (72) and therefore depending on θ. Also note that $m = \overline{m}\cosh\nu$ contains the correct contribution for the kinetic energy.

This interesting phenomenon represents a kind of Doppler shift of the mass aspect. It is plausible to suggest that this is not confined to the simple case displayed in equation (72). If we imagine that our material system consists of several masses moving in various directions, then there would be some form of a superposition of these

Doppler shifts, leading to M being a function with complicated angular dependence but independent of u. We therefore get the notion that there is a class of non-radiative motions in which different parts of the material system move with constant velocity in various directions. This case is easy to visualize when the different particles are sufficiently far away from each other for their own gravitational effects on each other to be negligible. Otherwise we know that their motions will not be uniform and that, in particular, oscillations are likely to occur. How would these show themselves in our equations? It follows from (72) applied to (36) that N is a linear function of u, that C is a quadratic function of u and so on. Of course this fits in perfectly with the picture of the moving mass. The dipole moment represented by N will increase linearly with time, the quadrupole moment represented by C will increase quadratically with time and so on. In the case of the moving single mass all these statements are strictly correct. In the more complicated cases where M, though independent of u, depends on angle in a more complex fashion, we will still have it that N increases linearly with u, C quadratically and so on. However, the more complicated angular dependence now implies that N no longer just represents the dipole moment and C no longer represents the quadrupole moment. If we could sort out the terms and put together all the terms representing the dipole moment and all the terms representing the quadrupole moment and so on (which of course because of the non-linearity are not independent of each other) then each of them would be represented by a whole power series in u. The accelerations which are known to occur may well be expressed by such power series converging for all values of u, or convergence may cease for some finite value of u. In either case, we see that the future behaviour of a system in this class is entirely determined by the present, i.e. there is no news. This might suggest identification of such behaviour with purely gravitational motions. This identification is strongly supported by the work of Infeld (1960), who found that a system of particles in motion need not radiate. On this basis, we may tentatively identify non-radiative motions on the one hand with the class of solutions in which M depends on angle but not on time, and on the other with motion under purely gravitational forces. The relation between the non-radiative motions discussed here and those discussed in part A is clear.

4. Construction of solutions

In this section we give a method for the construction of sandwich wave solutions. We shall suppose that the system is static except during an interval extending from $u = -1$ to $u = +1$. Thus outside this interval c_0 vanishes and within the interval it may be given in terms of the expansion

$$c_0 = \sum_{n=0}^{\infty} f_n(\mu) P_n(u), \quad -1 \leqslant u \leqslant 1, \quad \mu = \cos\theta. \tag{73}$$

In order to ensure continuity at the beginning and end of the interval, we must have

$$\sum_{n=0}^{\infty} f_{2n} = \sum_{n=0}^{\infty} f_{2n+1} = 0. \tag{74}$$

In order to fit the boundary conditions on the axis we have

$$f_n(1-\mu^2)^{-1} \quad \text{bounded as} \quad \mu \to \pm 1. \tag{75}$$

Now substitute in equation (35) and integrate throughout the interval of emission. We obtain for the change in M

$$[M] = -2\sum_{n=0}^{\infty}\frac{f_n^2}{2n+1}+\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}[(1-\mu^2)f_0]. \tag{76}$$

If we wish to go from a static to a static solution then the change in M must be independent of angle. It can readily be established that in order to allow for the divisibility of the first term on the right-hand side of (76) by $(1-\mu^2)^2$, as implied by (75), we must have

$$f_0 = -l(1-\mu^2)(3-\mu^2)+g(\mu), \quad l = \text{const.}, \quad g(\mu) \sim (1-\mu^2)^3. \tag{77}$$

Then, by substitution
$$[M] = -16l, \tag{78}$$

$$4[g(1-\mu^2)^{-1}-l(3-\mu^2)]^2+4\sum_{n=1}^{\infty}\frac{1}{2n+1}\left(\frac{f_n}{1-\mu^2}\right)^2-60l-\frac{2}{(1-\mu^2)^2}\frac{\mathrm{d}^2}{\mathrm{d}\mu^2}[(1-\mu^2)g(\mu)] = 0. \tag{79}$$

Equation (79) is a condition on the higher terms of the series, which together with (74) and (75) can be satisfied provided l is positive, which by (78) implies a loss of mass.

A simple example may illustrate this part of the method. Suppose

$$g = 0, \quad f_n = 0 \quad \text{unless} \quad n = 0, 2, 4, \quad f_0+f_2+f_4 = 0, \tag{80}$$

$$4(3-\mu^2)^2 l^2-60l+\frac{4}{5}\left(\frac{f_2}{1-\mu^2}\right)^2+\frac{4}{9}\left[\frac{f_2}{1-\mu^2}+l(3-\mu^2)\right]^2 = 0, \tag{81}$$

$$f_2 = \tfrac{5}{14}l\{3-\mu^2 \pm 3^{\frac{3}{2}}[14l^{-1}-(3-\mu^2)^2]^{\frac{1}{2}}\}(1-\mu^2), \tag{82}$$

which is real provided $0 \leqslant l \leqslant \frac{14}{9}$.

The method may be continued to evaluate the changes in N, C and so on. With sufficient terms in the series we can make N_0, C_0, etc., all vanish initially and finally. Though this method of ensuring change from a static system to a static system is laborious, it is very much like the method applicable to the scalar wave equation when this is treated as in part A. Also the degree of freedom left appears to be the same. Accordingly, it seems likely that the change from radiative to static system is not only possible, but immediate, i.e. that Huygens's principle applies to gravitational waves. It may be worth mentioning that the u derivative of each coefficient in the expansion of γ is given by an expression analogous to (34) in which the only coefficient to enter linearly is the immediate predecessor which enters through the associated Legendre operator for $P_n^{(2)}$. There is little doubt that further development of this method would lead to an improved understanding of the equations.

5. The reception of gravitational waves

In order to clarify the energy concept for gravitational waves the problem of their reception is now considered. The simplest receiver to discuss is freely falling, for otherwise one would not know whether some of the energy obtained from the field was not derived from the framework holding the receiver. In order that the gravitational terms entering should be easy to express, the receiver should be small and it should be as simple as possible. One wants, therefore, a device that can make use of the curvature tensor, that is, of the relative acceleration of neighbouring particles. Our ideal receiver of the simplest type is then a quadrupole receiver. It consists of two massive particles which are arranged with a motor between them, such that their distance from each other can be varied at will. Then the machinery in the receiver will absorb energy whenever the motion of the particles is such that the relative gravitational force, as given by the curvature tensor, does work in the motion in question. On the other hand, the receiver loses energy because, being a quadrupole of variable moment, it will itself radiate gravitational waves and so energy to space. The crucial question then is of how the particles should be moved in order to maximize the gain of energy, that is, the difference between energy received and energy re-radiated. In this respect the gravitational receiver is completely analogous to the electromagnetic receiver studied by Bondi (1961). Let the mass of each of the particles to M and let the distance of each from the common centre of mass be x. Then the quadrupole moment Q is given by

$$Q = 2Mx^2. \tag{83}$$

We now suppose the receiver to fall freely, moving so that its time axis coincides with the T axis of the tetrad used at the end of part B. With the quadrupole lined up along the P axis of the tetrad, the gravitational acceleration of each particle with respect to the mid-point is the product $xR_{(TTPP)}$, and so we can regard the particles as being acted upon by a Newtonian force

$$\pm Mc_{00}\, x/r. \tag{84}$$

Multiplying this by the velocity of the particles with respect to the mid-point and adding for the two particles, we obtain for the rate of doing work of the field on the receiver

$$\frac{2Mc_{00}\, x\dot{x}}{r} = \frac{1}{2}\frac{c_{00}}{r}Q_0. \tag{85}$$

Next we have to consider the re-radiation from the receiver. For this purpose the receiver has to be considered as a transmitter and all our previous work can be applied. To avoid confusion, the corresponding symbols will now be barred. Here, however, we have to proceed with caution, making a certain number of approximations in order to obtain manageable expressions and in order to be able to apply our previous work. In as far as this work was exact it never identified the precise nature of the source. The quadrupole moment was not identified in the moving but only in the static case. We shall now suppose that we can regard our receiver as quasi-static, so that the identification of the quadrupole moment contained in equation (42) can be applied and, moreover, we shall suppose the entire radiation from the receiver to be so small that, until we come to the final stages, we may work in the linear approximation. Then, from equation (42), we have

$$\bar{C} = \tfrac{1}{2}Q(\bar{u})\sin^2\bar{\theta}. \tag{86}$$

From (34) $\quad 4\bar{C}_0 = \bar{N}\cot\bar{\theta} - \bar{N}_2 \quad$ so that $\quad \bar{N} = 8Q_0 \sin\bar{\theta}\cos\bar{\theta}. \qquad (87)$

Then (36) gives

$$\bar{M}_2 = -3\bar{N}_0. \quad \text{Thus} \quad \bar{M} = -3Q_{00}\sin^2\bar{\theta} + p(\bar{u}). \tag{88}$$

Substitute into (35) and obtain

$$\bar{c} = \tfrac{1}{2}Q_{00}\sin^2\bar{\theta}. \tag{89}$$

Still using the quasi-static approach and combining (35) and (57) we have

$$-4\bar{m}_0 = -2\int_0^\pi \bar{M}_0 \sin\bar{\theta}\,\mathrm{d}\bar{\theta} = 2\int_0^\pi \bar{c}_0^2 \sin\bar{\theta}\,\mathrm{d}\bar{\theta} = \tfrac{1}{2}Q_{000}^2\int_0^\pi \sin^5\bar{\theta}\,\mathrm{d}\bar{\theta} = \tfrac{8}{15}Q_{000}^2. \tag{90}$$

Thus the rate of radiation of energy is

$$-\bar{m}_0 = \tfrac{2}{15}Q_{000}^2. \tag{91}$$

This is a well-known result.

The limitations of our approach are evident. We have supposed a linear superposition of incident radiation and re-radiation and the whole treatment of the re-radiation has been distinctly crude. Nevertheless, it seems unlikely that when the incident wave is weak there should be any major mistake in this calculation. We arrive, therefore, at the answer that the total amount of energy received in the interval of reception is given by

$$\int \mathrm{d}u\left[\frac{1}{2}\frac{c_{00}}{r}Q_0 - \tfrac{2}{15}Q_{000}^2\right]. \tag{92}$$

This expression is completely equivalent to the corresponding expression for the reception of electromagnetic waves (equation (3) of Bondi 1961) and the entire analysis given in that paper can be applied here. The method in brief is to use the usual approach of the calculus of variations to find that variation of quadrupole moment Q with time that will maximize expression (92). The result is easily obtained and shows that if initially c was constant then the maximum possible rate of absorption of energy is given by

$$\frac{15}{16}\frac{(c-c_{\text{initial}})^2}{r^2}. \tag{93}$$

Comparing this with the electromagnetic case, where we suppose both receiving and transmitting aerials to be magnetic dipoles, one sees that this expression is identical (apart from a numerical factor) with the electromagnetic one, provided c is replaced by the time derivative of the current in the transmitter coil. We are now faced with precisely the same problem as arises in electromagnetism in the case, there very unusual, in which after the period of transmission the time derivative of the transmitting current does not equal its value before the period of transmission. It will be recalled that in that case no unambiguous treatment of energy reception can be carried out unless the near field (induction) is considered as well as wave field. However, in the gravitational case this is not the unusual but the only possible situation. For we cannot have c returning to its initial value after the period of transmission, except, possibly, in a few isolated directions. This can immediately be verified by referring to equation (35) and considering its significance along the axis. Since c vanishes on the axis, the first term on the right-hand side vanishes there. Accordingly a change in M on the axis is entirely due to a change in c there. Since M, in going from a static to a static situation, must change independently of angle, as was previously shown, it follows that c, at least near the axis, must differ from its previous value. This result can also be established in a more comprehensive way by looking at equation (73). The change in c is given by $2f_0$. Equation (77) shows that in no circumstances can f_0 vanish, which establishes the same result. Thus, as far as energy is concerned we must be very careful in the consideration of reception of energy from a wave. Energy can only be taken from the whole field, including the non-wave parts. Serious though this consideration is, one must remember that the change in c following upon the change in M is of the second order. Therefore the rate of energy reception can be approximated to by expression (93), particularly for an oscillatory type of wave. One must, however, be careful not to take this too far for otherwise the receiver will appear to be able to continue to absorb energy *ad infinitum* after the wave has passed, that is, after c has taken on a fixed value different from its initial value. Looked at in a different way, expression (93) underlines what has previously been said about the desirability of considering only sandwich waves. The curvature tensor is proportional to c_{00}, the rate of loss of mass to c_0^2, and the rate of energy absorption to the change in c itself. If one did not stick to sandwich waves, then one might be faced with the absurd situation of c varying linearly with time. This would have the effect that there was no wave term in the curvature tensor but that, nevertheless, there was a constant loss of mass and a constant ability to absorb energy. But all these difficulties arise only from the consideration of rather unnatural situations. As long as one is determined to work only with c constant initially and finally, that is with a sandwich wave, the internal connexions between the change in c, its first and its second derivative are enough to ensure that one obtains only sensible results.

A problem that is as little solved in the general case in gravitational theory as in electromagnetic theory is the maximum permissible proximity of receivers. In order to reconcile equation (93) with an energy loss proportional to c_0^2 requires one to make a statement about receivers, in order that they may not interfere with each other, having to be a distance of the order of the wavelength apart. Artificial as the restriction to harmonic waves and perfectly definite wavelength is in the electromagnetic case, there the linearity enables one to get over the worst consequences. In the gravitational case no such escape is in sight and the question of the maximum permissible number of receivers remains in a rather unsatisfactory state. One can, however, proceed rather differently to show that the energy transmitted can indeed eventually be absorbed. Suppose the transmitting region is enclosed by a large empty region, which, in turn, is bounded by a material shell of matter beyond which again space is empty. Within the outer shell of matter the situation is exactly as described by our equations, provided we suppose the shell not to send out any radiation inwards. Suppose now, moreover, that outside the outer shell we have a spherically symmetrical solution, that is, a Schwarzschild solution with a necessarily fixed value of the mass. Then it will be possible to conceive of a transition from the metric inside the shell to the metric outside the shell involving certain pressures and densities and stresses in the shell. If the mass of the exterior solution is made large enough one can always ensure that the density within the shell is everywhere positive and large compared with the stresses, that is, that one has a physical situation. We can hence speak of the outermost solution as defining the mass of the entire system of central transmitter together with shell, and this is constant in time. If then we have an initially static situation in the interior and equally an eventually static situation which means a final mass of the transmitter necessarily less than its initial mass, then the mass of the shell must increase by exactly the amount by which the mass of the transmitter has diminished. Hence such a shell constitutes a perfectly matched absorbing receiver.

6. The classification of time variation

The work of this paper allows one to discriminate between various types of time variation of empty space fields surrounding isolated material system. Although only axially symmetric cases have been considered here, the generalization of the work of this paper to arbitrary systems by Sachs (1962) enables one to apply this classification without restriction.

(i) *Radiative class*

This is characterized by the existence of news, and the non-vanishing of the news function c_0 defines the class. A mass loss necessarily occurs, and in general the physical components of the Riemann tensor $\sim r^{-1}$. An exception occurs only if $c_{00} = 0$ although $c_0 \neq 0$. Though this seems to be a case of little physical significance one should perhaps put into *subclass* (i*): *mass loss without radiative Riemann tensor*.

(ii) *Time-dependent systems without news*

This occurs whenever some terms in the field are time dependent, but there is no news ($c_0 = 0$) and accordingly no mass loss ($M_0 = 0$). The physical interpretation of this class is perhaps the biggest outstanding problem in the subject. What distinguishes locally those source motions that do not give rise to a radiation field from those that do? The methods of this paper do not lend themselves to answering this question but perhaps the alternatives might be stated.

It may be that there is no locally significant way of distinguishing between these types of motion, unsatisfactory as this would be. If there is such a locally significant distinction it seems likely that it will be related to whether one is dealing with free gravitational motion. The lack of radiation for freely falling particles emerges from Infeld's work, but one would like to generalize this to non-singular equations of state. The most clear-cut case then would seem to be pressure-free dust ($T^{\mu\nu} = \rho v^{\mu} v^{\nu}$), but beyond this it is tempting to suggest that perfectly elastic equations of state do not lead to radiation. Pursuing this line of thought one is driven to the following conclusion:

If the distinction between radiative and non-radiative motions is locally significant then the clearest self-consistent distinction appears to be between cases where the equations of state do not involve the time explicitly and are time reversible (no dissipation), and others. A system of the second kind clearly contains news, for either the time enters explicitly into the equation of state (time bomb), or, through the action of dissipation, the system continually reaches *new* states in which its behaviour is not a consequence of its previous behaviour ('fatigue'). A system of the first kind does not contain news in this sense. Its future is a clear consequence of its past, and it would seem difficult to draw a distinguishing line between different systems of this kind though conceivably the pressure-free gas might be only non-radiative material, all others radiating if in motion.

The distant field of time dependent systems without news could be divided into two subclasses:

(*a*) $M_2 \neq 0$ (natural non-radiative moving system),

(*b*) $M_2 = 0$ (non-natural non-radiative moving system).

The presence of moving masses in (ii *a*) is a clear consequence of the case discussed in § 3, but the time dependence may only enter the coefficient through N or C or later in the series, corresponding to (ii *b*). The necessarily very peculiar cancellation of mass motion terms leads to the name suggested.

(iii) *Stationary systems*

Time-independent metrics not reflexion symmetric and

(iv) *Static systems*

are well known.

The author is deeply indebted to many colleagues for help in understanding and presenting the subject of this paper, particularly to Dr F. A. E. Pirani and to Dr R. K. Sachs, who has recently been able to extend this theory to the general case without any symmetries in a paper now in the press.

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Appendix

1. *List of 3-index symbols*

$$\Gamma^0_{00} = 2\beta_0 + \frac{V}{2r^2} - \frac{V_1}{2r} - \frac{\beta_1 V}{r} + r^2 e^{2(\beta-\gamma)} U\left(U_1 + \frac{U}{r} + \gamma_1 U\right) = -\frac{M + cc_0}{r^2} + \dots,$$

$$\Gamma^0_{01} = \Gamma^0_{11} = \Gamma^0_{12} = 0,$$

$$\Gamma^0_{02} = \beta_2 - r^2 e^{2(\gamma-\beta)}\left(\tfrac{1}{2}U_1 + \frac{U}{r} + \gamma_1 U\right) = \frac{N}{r^2} + \dots,$$

$$\Gamma^0_{22} = r\,e^{2(\gamma-\beta)}(1 + r\gamma_1) = r + c + \frac{1}{2}\frac{c^2}{r} + \dots,$$

$$\Gamma^0_{33} = r\,e^{-2(\gamma+\beta)}\sin^2\theta(1 - r\gamma_1) = \sin^2\theta\left(r - c + \frac{1}{2}\frac{c^2}{r} + \dots\right),$$

$$\Gamma^1_{00} = \frac{V_0}{2r} - \frac{\beta_0 V}{r} - \frac{V^2}{2r^3} + \frac{VV_1}{2r^2} + \frac{\beta_1 V^2}{r^2} - \frac{UV_2}{2r} - \frac{\beta_2 UV}{r}$$

$$+ r^2 \mathrm{e}^{2(\gamma-\beta)} \left[U^2 \left(U_2 + \gamma_2 U - \frac{V}{r^2} - \frac{\gamma_1 V}{r} + \gamma_0 \right) - \frac{UU_1 V}{r} \right] = -\frac{M_0}{r} + \dots,$$

$$\Gamma^1_{01} = \frac{V_1}{2r} - \frac{V}{2r^2} + \frac{\beta_1 V}{r} - \beta_2 U - \tfrac{1}{2} r^2 \mathrm{e}^{2(\gamma-\beta)} UU_1 = \frac{M}{r^2} \dots +,$$

$$\Gamma^1_{11} = 2\beta_1 = \frac{c^2}{r^3} + \dots,$$

$$\Gamma^1_{02} = \frac{V_2}{2r} + r^2 \mathrm{e}^{2(\gamma-\beta)} \left[U \left(\frac{V}{r^2} + \frac{\gamma_1 V}{r} - \gamma_0 - U_2 - \gamma_2 U \right) + \frac{U_1 V}{2r} \right] = \frac{-M_2 + c_0(c_2 + 2c\cot\theta)}{r} + \dots,$$

$$\Gamma^1_{12} = \beta_2 + \tfrac{1}{2} r^2 \mathrm{e}^{2(\gamma-\beta)} U_1 = \frac{c_2 + 2c\cot\theta}{r} - \frac{6N + 5cc_2 + 8c^2\cot\theta}{2r^2} + \dots,$$

$$\Gamma^1_{22} = r^2 \mathrm{e}^{2(\gamma-\beta)} \left(\gamma_0 + U_2 + \gamma_2 U - \frac{V}{r^2} - \frac{\gamma_1 V}{r} \right) = -r(1 - c_0) + \dots,$$

$$\Gamma^1_{33} = r^2 \sin^2\theta\, \mathrm{e}^{-2(\gamma+\beta)} \left(-\gamma_0 + U\cot\theta - \gamma_2 U - \frac{V}{r^2} + \frac{\gamma_1 V}{r} \right) = -r\sin^2\theta(1 + c_0) + \dots,$$

$$\Gamma^2_{00} = -U_0 + U \left(2\beta_0 - 2\gamma_0 - U_2 - \gamma_2 U + \frac{V}{2r^2} - \frac{V_1}{2r} - \frac{\beta_1 V}{r} \right) + \mathrm{e}^{2(\beta-\gamma)} \frac{V_2 + 2\beta_2 V}{2r^3} + r^2 \mathrm{e}^{2(\gamma-\beta)}$$

$$\times U^2 \left(U_1 + \frac{U}{r} + \gamma_1 U \right) = \frac{c_{02} + 2c_0\cot\theta}{r^2} - \frac{2N_0 + c_0 c_2 + 3cc_{02} + 4cc_0\cot\theta + M_2}{r^3},$$

$$\Gamma^2_{01} = -\tfrac{1}{2} U_1 - \frac{U}{r} - \gamma_1 U + \frac{\beta_2}{r^2} \mathrm{e}^{2(\beta-\gamma)} = \frac{N}{r^4} + \dots,$$

$$\Gamma^2_{11} = 0,$$

$$\Gamma^2_{02} = \gamma_0 + \beta_2 U - r^2 \mathrm{e}^{2(\gamma-\beta)} U \left(\tfrac{1}{2} U_1 + \frac{U}{r} + \gamma_1 U \right) = \frac{c_0}{r} + O(r^{-3}),$$

$$\Gamma^2_{12} = \frac{1}{r} + \gamma_1 = \frac{1}{r} - \frac{c}{r^2} - \frac{3C}{r^4} + \dots,$$

$$\Gamma^2_{22} = \gamma_2 + r^2 \mathrm{e}^{2(\gamma-\beta)} U \left(\frac{1}{r} + \gamma_1 \right) = -\frac{2c\cot\theta}{r} + \dots,$$

$$\Gamma^2_{33} = r^2 \sin^2\theta\, \mathrm{e}^{-2(\gamma-\beta)} U \left(\frac{1}{r} - \gamma_1 \right) - \mathrm{e}^{-4\gamma} \sin^2\theta (\cot\theta - \gamma_2) = -\sin\theta\cos\theta \left(1 - \frac{2c}{r} \right) + \dots,$$

$$\Gamma^3_{03} = -\gamma_0 = -\frac{c_0}{r} - \frac{C_0}{r^3} + \dots,$$

$$\Gamma^3_{13} = \frac{1}{r} - \gamma_1 = \frac{1}{r} + \frac{c}{r^2} + \frac{3C}{r^4} + \dots,$$

$$\Gamma^3_{23} = \cot\theta - \gamma_2 = \cot\theta - \frac{c_2}{r} - \frac{C_2}{r^3} - \dots.$$

2. *The supplementary conditions*

$$
\begin{aligned}
R_{02} = {} & \beta_{02} - \gamma_{02} + 2\gamma_0\gamma_2 - 2\gamma_0\cot\theta - U(\beta_{22} + 2\beta_2^2 - 2\beta_2\gamma_2 + \beta_2\cot\theta) \\
& - \frac{V_{12}}{2r} + \frac{V_2}{2r^2} + (\gamma_1 - \beta_1)\frac{V_2}{r} + r^2 e^{2(\gamma-\beta)}\left[\tfrac{3}{2}UU_{12} + \frac{3UU_2}{r} + 2U\left(\gamma_{01} + \frac{\gamma_0}{r}\right) + \tfrac{1}{2}U_{01}\right. \\
& + 2\gamma_{12}U^2 + (\gamma_0 - \beta_0)U_1 + \gamma_1 UU_2 + (2\gamma_2 - \beta_2)UU_1 + U_1U_2 - \frac{U_{11}V}{2r} - \frac{UV_1 + 2U_1V}{r^2} \\
& \left. - \frac{\gamma_{11}UV + (\gamma_1 - \beta_1)U_1V + \gamma_1 UV_1}{r} - \frac{\gamma_1 UV}{r^2} + \frac{2\gamma_2 U^2}{r} + U\left(\tfrac{1}{2}U_1 + \frac{U}{r} + \gamma_1 U\right)\cot\theta\right] \\
& - \tfrac{1}{2}r^4 e^{4(\gamma-\beta)}UU_1^2,
\end{aligned}
$$

$$
\begin{aligned}
R_{00} = {} & \frac{2\beta_{01}V}{r} - \frac{VV_{11}}{2r^2} - \frac{\beta_{11}V^2}{r^2} - \frac{\beta_1 V^2}{r^3} - \frac{\beta_1 VV_1}{r^2} - \frac{V_0 - 2\beta_0 V}{r^2} \\
& + \frac{2\beta_{12}UV + \beta_2 UV_1 + \beta_1 U_2 V + 2\beta_1 UV_2}{r} + \frac{2\beta_2 UV}{r^2} - \frac{U_2V}{2r^2} + \frac{U_2V_1}{2r} - \frac{2UV_2}{r^2} - \frac{U_1V_2}{2r} \\
& - \frac{2\gamma_1 UV_2}{r} - 2\beta_{02}U - 2\beta_0 U_2 + 2\gamma_{02}U + 2\gamma_0 U_2 + U_{02} + UU_{22} + U_2^2 \\
& + 2(\gamma_2 - \beta_2)UU_2 + \frac{UV_{12}}{r} + (2\beta_2^2 - 2\beta_2\gamma_2 + \gamma_{22})U^2 + 2\gamma_0^2 \\
& - \cot\theta\left(2\beta_0 U - 2\gamma_0 U - U_0 - UU_2 - \gamma_2 U^2 + \frac{UV}{2r^2} - \frac{UV_1}{2r} - \frac{\beta_1 UV}{r}\right) \\
& + r^2 e^{2(\gamma-\beta)}\left[-UU_{01} - 2\left(\gamma_{01} + \frac{\gamma_0}{r}\right)U^2 - 2(\gamma_0 - \beta_0)UU_1 - 2U^2U_{12} - 2UU_1U_2\right. \\
& - 2\gamma_{12}U^3 - \frac{2}{r}\gamma_2 U^3 - 3\gamma_2 U^2U_1 + 2\beta_2 U^2U_1 + \frac{UU_{11}V}{r} + \frac{4UU_1V}{r^2} \\
& + 2(\gamma_1 - \beta_1)\frac{UU_1V}{r} + \frac{\gamma_{11}U^2V}{r} + \frac{\gamma_1 U^2V_1}{r} + \frac{\gamma_1 U^2V}{r^2} - \frac{3U^2U_2}{r} - \gamma_1 U^2U_2 \\
& \left. + \frac{U^2V_1}{r^2} + \frac{U_1^2V}{2r} - U^2\left(U_1 + \frac{U}{r} + \gamma_1 U\right)\cot\theta\right] + \tfrac{1}{2}r^4 e^{4(\gamma-\beta)}U^2U_1^2 \\
& - \frac{1}{2r^3}e^{2(\beta-\gamma)}[V_{22} + 2\beta_{22}V + (2\beta_2 - 2\gamma_2 + \cot\theta)(V_2 + 2\beta_2 V)].
\end{aligned}
$$

3. *Transformation formulae* (*part C*)

$$K(\bar\theta) = \cosh\nu + \sinh\nu\cos\bar\theta, \quad \alpha = \alpha(\bar\theta),$$

$$u = \left(\frac{\bar u}{K} + \alpha\right) - \frac{K}{2\bar r}\left(\frac{K'}{K^2}\bar u - \alpha'\right)^2 + O(\bar r^{-2}).$$

$$r = K\bar r + \frac{1}{K}[\bar u(1 - K\cosh\nu) + \tfrac{1}{2}K^2(K''\alpha' + K\alpha'')] + O(\bar r^{-1}),$$

$$\theta = 2\tan^{-1}(e^{-\nu}\tan\tfrac{1}{2}\bar\theta) + \frac{1}{\bar r}\left[\frac{K'}{K^2}\bar u - \alpha'\right] + O(\bar r^{-1}),$$

$$\bar c = \frac{c}{K} + \tfrac{1}{2}K\left[\alpha'\cot\bar\theta - \alpha'' - 2\alpha'\frac{K'}{K}\right],$$

$$\bar{c}_0 = \frac{c_0}{K^2}, \quad \bar{c}_{00} = \frac{c_{00}}{K^3},$$

$$\bar{M} = \frac{M}{K^3} - \frac{c_0}{K^2}\left[2\bar{u}\,\frac{1-K\cosh\nu}{K^2} - \tfrac{1}{2}K\alpha'' + \alpha' K' - \tfrac{3}{2}K\alpha'\cot\bar{\theta}\right]$$
$$- \frac{1}{K^2}c_{02}\left(\frac{K'}{K}\bar{u} - \alpha'\right) + \frac{1}{2K}c_{00}\left(\frac{K'}{K^2}\bar{u} - \alpha'\right)^2.$$

$$\bar{N} = \frac{N}{K^3} + \frac{1}{K^2}(M + cc_0)\left(\frac{K'}{K^2}\bar{u} - \alpha'\right) - \frac{1}{2K}[2c_0(K\cot\bar{\theta} - K') + c_{02}]$$
$$\times\left(\frac{K'}{K}\bar{u} - \alpha'\right)^2 + \tfrac{1}{6}c_{00}\left(\frac{K'}{K^2}\bar{u} - \alpha'\right)^3.$$

$$\bar{C} = \frac{C}{K^3} + \frac{N}{K^2}\left(\frac{K'}{K^2}\bar{u} - \alpha'\right) + \frac{M + cc_0}{2K}\left(\frac{K'}{K^2}\bar{u} - \alpha'\right)^2 - \tfrac{1}{6}[2c_0(K\cot\bar{\theta} - K') + c_{02}]$$
$$\times\left(\frac{K'}{K^2}\bar{u} - \alpha'\right)^3 + \tfrac{1}{24}c_{00}K\left(\frac{K'}{K^2}\bar{u} - \alpha'\right)^4.$$

4. *Transformation of the Weyl metric*

Write the metric (37) in the form

$$ds^2 = e^{2\psi}\,dt^2 - e^{2\sigma-2\psi}(dR^2 + R^2\,d\Theta^2) - e^{-2\psi}R^2\sin^2\Theta\,d\phi^2$$

and put
$$t = u + f(R,\theta), \quad \Theta = \Theta(R,\theta),$$

in order to make g_{11} and g_{12} vanish. Hence

$$e^{2\psi}f_R^2 = e^{2\sigma-2\psi}[1 + R^2\Theta_R^2], \quad e^{2\psi}f_R\,f_\theta = e^{2\sigma-2\psi}R^2\Theta_R\,\Theta_\theta.$$

Eliminate f:
$$R^2\Theta_R(e^{2\sigma-4\psi})_\theta = \Theta_\theta\left[e^{2\sigma-4\psi}\frac{R^4\Theta_R^2}{1 + R^2\Theta_R^2}\right]_R.$$

Let
$$e^{2\sigma-4\psi} = 1 + \frac{4m}{R} + \frac{p(\Theta)}{R^2} + \frac{q(\Theta)}{R^3} + \dots.$$

The case $c = 0$ corresponds to $\lim\limits_{R\to\infty}\Theta_\theta = 1.$

Then
$$\Theta = \theta + \frac{p'}{4R^2} + \frac{q' - 6mp'}{12R^3} + \dots,$$

$$f = R + 2m\log R - \frac{\tfrac{1}{2}p - 2m^2}{R} - \frac{\tfrac{1}{2}q - mp + 4m^3}{2R^2} + \dots.$$

Introduce r in the usual way (13)

$$r^4\sin^2\theta = e^{-2\psi}R^2\sin^2\Theta[e^{2\sigma-2\psi}R^2\Theta_\theta^2 - e^{2\psi}f_\theta^2].$$

Solve for R

$$R = r - m - \tfrac{1}{2}[\tfrac{1}{4}p'\cot\theta + \tfrac{1}{2}p - 3m^2 + \tfrac{1}{4}p'']\,r^{-1}$$
$$- \tfrac{1}{2}[\tfrac{1}{12}q'' + \tfrac{1}{12}q'\cot\theta + \tfrac{1}{2}q - mp + 4m^3]\,r^{-2} + \dots.$$

We now express p and q by (39) and (40) in terms of the dipole moment D and the quadrupole moment Q.

$$p = 4D\cos\Theta + m^2(7+\cos^2\Theta),$$

$$q = 2Q(3\cos^2\Theta - 1) + 4mD(3+\cos^2\Theta) + 6m^3(1+\cos^2\Theta).$$

Hence

$$R = r - m - \tfrac{1}{2}m^2(1-\cos^2\theta)\,r^{-1} - \tfrac{1}{2}m\sin^2\theta[2D\cos\theta + m^2]\,r^{-2} + \dots.$$

Finally
$$\frac{V}{r}e^{2\psi}f_R R_r = e^{2\psi} + e^{2\psi}\frac{(f_R R_\theta + f_\theta)^2}{R^2 e^{2\sigma-4\psi}\Theta_\theta^2 - f_\theta^2}.$$

Substitute and obtain

$$\frac{V}{r} = 1 - \frac{2m}{r} - \frac{2D\cos\theta}{r^2} - \frac{Q(3\cos^2\theta - 1)}{r^3} + \dots.$$

Thus from (33) $\quad M = m, \quad N = D\sin\theta, \quad C = \tfrac{1}{2}Q\sin^2\theta.$

With the transformation equations of appendix 3 this gives equations (42).