Notes on applying super momentum balance law to constraint GW waveform (Update on Daily Progress)

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I. LOCAL SUPER MOMENTUM BALANCE LAW

The local super-momentum balance law for asymptotically flat space-time is given by

$$\mathcal{F}(\theta,\phi)\Big|_{u_1}^{u_2} := -\int_{u_1}^{u_2} du\left[|\dot{\sigma^o}|^2 - \Re\left(\eth^2\dot{\bar{\sigma}}^o\right)\right](u,\theta,\phi) = \Re\left[\Psi_2^o + \bar{\sigma}^o\dot{\sigma^o}\right](u_1,\theta,\phi) - \Re\left[\Psi_2^o + \bar{\sigma}^o\dot{\sigma^o}\right](u_2,\theta,\phi) \tag{1.1}$$

where the Weyl scalars is given by,

$$2\Re\,\Psi_2^o(u,\theta,\phi) = \lim_{r\to+\infty} r^3\, C_{abcd}n^a l^b n^c l^d \tag{1.2}$$

$$\Psi_4^o(u,\theta,\phi) = \lim_{r\to+\infty} r\, C_{abcd}n^a\bar{m}^b n^c\bar{m}^d = \ddot{\bar{\sigma}}^o = r\left(\ddot{h}_+ + i\,\ddot{h}_\times\right) \tag{1.3}$$

We simplify the above equation for the case for perturbed black hole using the results in ref [1]. Equation 2.59 in ref[1] is the mass aspect of asymptotically flat space-time.

$$\Psi = \psi_2^0 + \eth^2\bar{\sigma}^o + \sigma^o\dot{\bar{\sigma}}^o \tag{1.4}$$

and the spherical harmonic expansion of the above result is

$$\Psi = \Psi^0 + \Psi^i Y^0_{1\,i} + +\Psi^{i\,j}Y^0_{2\,i\,j} + ... \tag{1.5}$$

Bondi identified the interior mass and three-momentum with the $l = 0$ and $l = 1$ harmonic contributions is given by

$$M_B = \frac{-c^2}{2\sqrt{2}\,G}\Psi^0 \tag{1.6}$$

$$P^i = -\frac{c^3}{6\,G}\Psi^i \tag{1.7}$$

where M_B, P^i are the interior mass and angular momentum of the space time respectively. Also,

$$\begin{aligned} Y_0^0 &= 1 \\ Y^0_{1\,i} &= -c_i \\ Y^0_{2\,i\,j} &= 3c_ic_j - 2\delta_{i\,j} \\ c_i &= -\sqrt{2}\{\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)\} = -\sqrt{2}\,\hat{r} \end{aligned} \tag{1.8}$$

Using the above results, equation 1.4-1.7 at leading order we can relate mass and energy momentum of interior space time to ψ_2.

$$-2\sqrt{2}\,M_B - 6\vec{P}.\hat{r} + ... = \Re\left[\psi_2^0 + \eth^2\bar{\sigma}^o + \sigma^o\dot{\bar{\sigma}}^o\right] \tag{1.9}$$

Here we have just included only the mass and linear momentum terms, further terms includes angular momentum of interior space time (i.e. $J^k = -(\sqrt{2}c^3/12G)\Im(\psi_1^{0\,k})$). We can use the above result in the local balance law, to rewrite the local balance law. Substituting ψ_2 from equation 1.9 into equation 1.1 and as a first step just considering change in mass we have

$$-\int_{u_1}^{u_2} du\left[|\dot{\sigma^o}|^2 - \Re\left(\eth^2\dot{\bar{\sigma}}^o\right)\right](u,\theta,\phi) = -\left[\left(2\sqrt{2}M_B + \eth^2\bar{\sigma}^o\right)(u_1,\theta,\phi) - \left(2\sqrt{2}M_B + \eth^2\bar{\sigma}^o\right)(u_2,\theta,\phi)\right] \tag{1.10}$$

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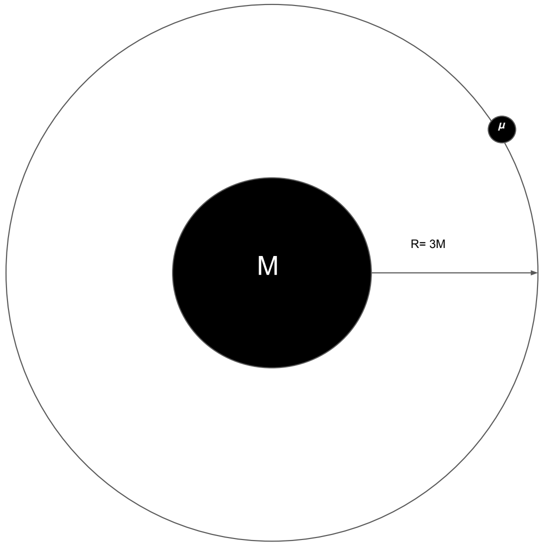

FIG. 1: The figure

$$= \int_{-\infty}^{t} du \left(2\sqrt{2}\,\dot{M}_B + \eth^2\bar{\sigma}^o\right) \tag{1.11}$$

$$\implies -|\dot{\sigma^o}|^2 + \Re\left(\eth^2\bar{\sigma}^o\right) = 2\sqrt{2}\,\dot{M}_B + \eth^2\bar{\sigma}^o \tag{1.12}$$

$$\implies -|\dot{\sigma^o}|^2 = 2\sqrt{2}\,\dot{M}_B \tag{1.13}$$

Here we have considered only the mass loss rate to the gravitational wave amplitude. Given the gravitational wave amplitude, above relation can be used to motivate the mass loss rate of interior space time and hence we get a insight for ψ_2, which appear in the local super-momentum balance law.

II. WEYL SCALAR Ψ_2^o FOR BOB AS PERTURBER CROSS THE LIGHT RING

In Backwards One-Body (BOB) [2] description of black hole binary merger, the system is described as a single black hole spacetime with a inspiraling in perturber. At the leading order we can just assume that the mass of the perturber and the source is evolving. We first look at how the mass loss rate in related to GW luminosity at leading order for a non-spinning black hole with inspiraling in perturber

A. Evolution of Ψ_2^o for perturbed Swarzchild black hole

We assume that the perturber is loosing mass at given rate $\dot{\epsilon}$. Given this and looking at Bianchi identity on asymptotically flat space time, we can also deduce that the source mass is evolving as $\dot{\epsilon}^2$. In this model where we assume both perturber and source mass evolving in time, we can relate this to gravitational wave strain. The gravitational wave luminosity is given by,

$$\mathcal{L} = -\frac{dE}{dt} = \lim_{r\to+\infty} \frac{r^2}{16\pi} \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \left| \int_{-\infty}^{\infty} dt' \Psi_4^{l\,m} \right|^2$$

$$\text{for l=2, m=2 mode} \tag{2.1}$$

$$\mathcal{L} = \lim_{r\to+\infty} \frac{r^2}{16\pi} \left| \int_{-\infty}^{\infty} dt' \Psi_4^{2\,2} \right|^2 \tag{2.2}$$

And the News function is related to Ψ_4 by

$$\sigma = \int_{-\infty}^{t} dt' \Psi_4^{2\,2} \tag{2.3}$$

Now let us look at leading order, how this luminosity evolves with time.

$$\frac{d\mathcal{L}_{GW}}{dt} = \frac{1}{16\pi}\frac{d}{dt}\Big|\int_{-\infty}^{\infty} dt'\Psi_4^{22}\Big|^2 = \frac{1}{16\pi}\frac{d}{dt}\Big[\int_{-\infty}^{t} dt'\bar{\Psi}_4^{22}(t')\int_{-\infty}^{t} dt''\Psi_4^{22}(t'')\Big]$$
$$= \frac{1}{16\pi}\Big[\bar{\Psi}_4^{22}(t)\int_{-\infty}^{t} dt'\Psi_4^{22}(t') + \Psi_4^{22}(t)\int_{-\infty}^{t} dt'\bar{\Psi}_4^{22}(t')\Big]$$
$$= \frac{1}{16\pi}\Big[\ddot{\bar{\sigma}}(t)\dot{\sigma}(t) + \dot{\bar{\sigma}}(t)\ddot{\sigma}(t)\Big] = \frac{1}{8\pi}\Re\Big[\ddot{\bar{\sigma}}(t)\dot{\sigma}(t)\Big] \tag{2.4}$$

1. Luminosity from a binary system

Now let us look at how at leading order the luminosity of binary source which is shedding mass evolve with time

$$\mathcal{L}_{GW} = \frac{32}{5}\frac{M^3\mu^2}{r^5}$$
$$\frac{d\mathcal{L}_{GW}}{dt} = \frac{32M^3\mu^2}{5r^5}\Big[\frac{3}{M}\frac{dM}{dt} + \frac{2}{\mu}\frac{d\mu}{dt} - \frac{5}{r}\frac{dr}{dt}\Big] \tag{2.5}$$

The radiation reaction has a negligible effect on dynamics of the perturber inside ISCO, so we model the rate of mass loss as source of rate of Luminosity change. The Mass of the central black holes is much larger then the mass of the perturber.

2. Mass, Momentum and Angular momentum loss of a perturbed spacetime

We can use the time evolution of mass aspect, equation 2.58 in ref[1] to relate the news amplitude to the mass and angular momentum loss.

$$\dot{\Psi} = \dot{\sigma}\dot{\bar{\sigma}} + k\phi_2^0\bar{\phi}_2^0 \tag{2.6}$$

In the absence of electromagnetic fields we set the second term in right hand side to zero. The above equation then is related to mass and angular momentum loss of the perturbed space time with the radiated GWs strain.

$$\dot{\Psi} = -\frac{2\sqrt{2}G}{c^2}\dot{M} - \frac{6G}{c^3}\dot{P}^i Y^0_{1\,i} + ... \tag{2.7}$$

We can use the above results to get an insight into how mass and angular momentum are related. For the case of gravitational wave emitted the merger waveform is well described by BOB model. We have

$$|\psi_4| = A_p \text{sech}[\gamma(t-t_p)] \tag{2.8}$$

Since $|\sigma| \approx |\psi|/\omega$ for quasi circular systems. Where $\omega_{l\,m} = m\,\Omega$ and

$$\Omega = \Big\{\Omega_0^4 + k\Big[\tanh\Big(\frac{t-t_p}{\tau}\Big) - \tanh\Big(\frac{t_0-t_p}{\tau}\Big)\Big]\Big\}^{1/4} \tag{2.9}$$

where the constant k is given by

$$k = \Bigg[\frac{\Omega^4_{QNM} - \Omega^4_0}{1-\tanh[(t_0-t_p)/\tau]}\Bigg] \tag{2.10}$$

Using equation 2.6 and 2.7 at leading order, we can relate the mass loss rate to the time derivative of GWs strain. Setting $c = G = 1$ we see that equation

$$|\dot{\sigma}|^2 = -2\sqrt{2}\dot{M} = \frac{|\psi_4|^2}{4\Omega^2} \tag{2.11}$$

To look at how the Mass evolves near the peak, we Taylor expand $|\dot{\sigma}|^2$ around the peak.

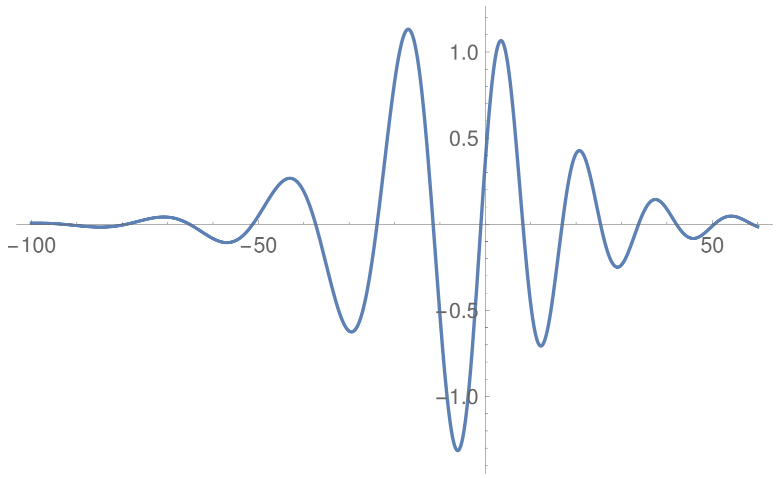

FIG. 2: The figure above shows stain amplitude from BOB

3. *Motivating ψ_2 from Bianchi identity on asymptotically flat space-time*

One of the Bianchi identity on asymptotically flat space-time is given by,

$$\dot{\psi}_2 = -\eth^2 \dot{\bar{\sigma}}^o - \sigma^o \ddot{\bar{\sigma}}^o \tag{2.12}$$

Now for σ in the above expression is related to ψ_4 in the following way.

$$\ddot{\bar{\sigma}}^o = r\left(\ddot{h}_+ + i\,\ddot{h}_\times\right) = \psi_4^0 \tag{2.13}$$

From the BOB model we have the $\psi_4^{2\,2}$

$$\psi_4^{22} = |\psi_4^{22}|\, e^{2i\Phi} \tag{2.14}$$

where Φ is given by Eqn. 10 in ref[2]. Now ψ_4^0 is given,

$$\psi_4 = \psi_4^{22}\,{}_{-2}Y^{22}(\theta,\phi) \tag{2.15}$$

where

$${}_{-2}Y^{2\pm 2}(\theta,\phi) = \frac{1}{8}\sqrt{\frac{5}{\pi}}\,(1 \pm \cos(\theta))^2\, e^{\pm 2\,i\,\Phi} \tag{2.16}$$

Now we can use equation 2.13, 2.15 and the expression for ψ_4 for BOB in local super momentum balance low. The operator $\eth$ acts on spin-weighted spherical harmonics to increase the spin weight by 1 (i.e $\eth^2_{-2}Y^{2\pm 2}(\theta,\phi) = {}_0Y^{2\pm 2}(\theta,\phi)$, which are the ordinary spherical harmonics). Let us look at each term in local super momentum balance law.

$$\sigma = h_+ + i\,h_\times = \frac{\psi_4}{\omega^2} \tag{2.17}$$

Since amplitude is slowly varying function of time, each time derivative in the above equation brings one ω in front. So we have the following relations.

$$\dot{\sigma}^0 = \frac{\psi_4^0}{\omega} \tag{2.18}$$

So the quantity inside the integral in local balance law can be written as

$$-\int_{u_1}^{u_2} du\left[|\dot{\sigma^o}|^2 - \Re\left(\eth^2\dot{\bar{\sigma}}^o\right)\right](u,\theta,\phi) = -\int_{u_1}^{u_2} du\left[\left|\frac{\psi_4^0}{\omega}\right|^2 - \Re\left(\eth^2\frac{\bar{\psi}_4^0}{\omega}\right)\right](u,\theta,\phi) \tag{2.19}$$

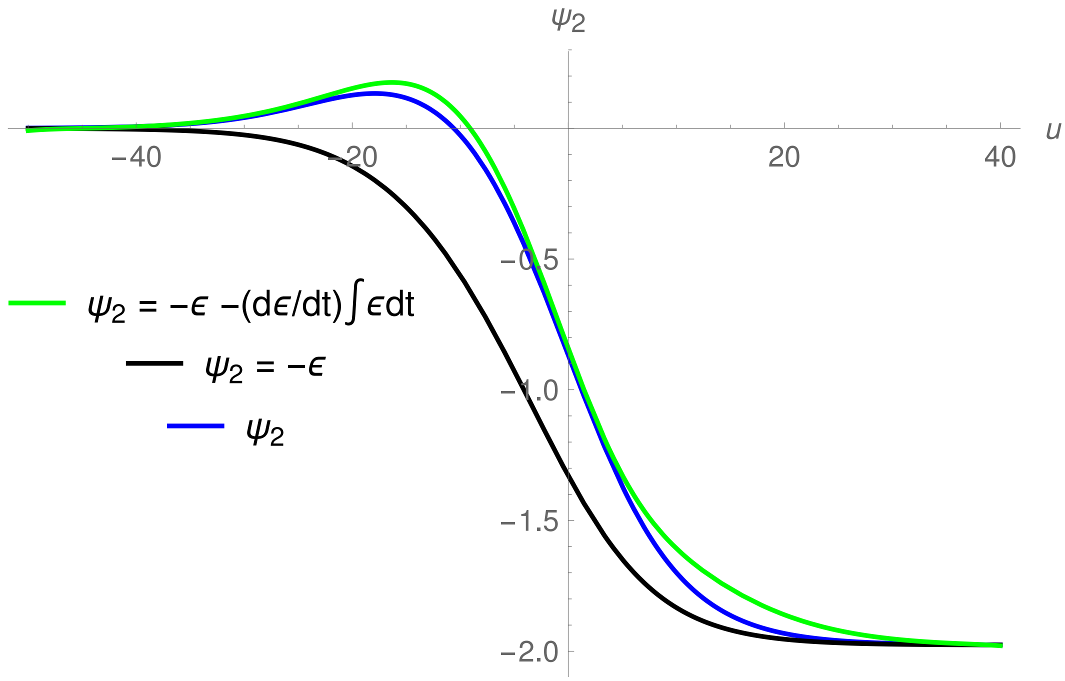

FIG. 3: The figure above show ψ_2. The blue curve show ψ_2 as it is calculated in Eqn[2.24]. Two other curves shows ψ_2 constructed to mass loss in perturber and the black hole. $\psi_2 = \int_{t_1}^{t_2} \left|\frac{dh}{dt}\right| dt$

We can now use ψ_4 from BOB model. Spin weighted spherical harmonics are independent of time so they can be taken out of integral in the above equation. The above equation can be rewritten as

$$-\int_{u_1}^{u_2} du \left[\left| \frac{\psi_4^0}{\omega} \right|^2 - \Re\left(\eth^2 \frac{\bar{\psi}_4^0}{\omega} \right) \right](u,\theta,\phi) = \Re\left(\eth^2_{-2} Y^{22}(\theta,\phi) \int_{u_1}^{u_2} du \frac{\bar{\psi}_4^{22}}{\omega}(u) \right) - \left| {}_{-2}Y^{22}(\theta,\phi) \right|^2 \int_{u_1}^{u_2} du \left| \frac{\psi_4^{22}}{\omega} \right|^2 (u) \tag{2.20}$$

The above expression, which is the left hand side of super momentum balance can be written as

$$P(u_2,u_1,\theta,\phi) = \Re\left({}_0Y^{22}(\theta,\phi) \int_{u_1}^{u_2} du \frac{\bar{\psi}_4^{22}}{\omega}(u) \right) - \left| {}_{-2}Y^{22}(\theta,\phi) \right|^2 \int_{u_1}^{u_2} du \left| \frac{\psi_4^{22}}{\omega} \right|^2 (u) \tag{2.21}$$

Let us now look at the right hand side of the super momentum balance law

$$\Re\left[\Psi_2^o + \bar{\sigma}^o \dot{\sigma^o} \right](u_1,\theta,\phi) - \Re\left[\Psi_2^o + \bar{\sigma}^o \dot{\sigma^o} \right](u_2,\theta,\phi) \tag{2.22}$$

We can rewrite the above expression as

$$Q(u_2,u_1,\theta,\phi) = \Re\left[\left| {}_{-2}Y^{22}(\theta,\phi) \right|^2 \frac{\left|\psi_4^{22}\right|^2}{\omega^3} \right](u_1) - \Re\left[\left| {}_{-2}Y^{22}(\theta,\phi) \right|^2 \frac{\left|\psi_4^{22}\right|^2}{\omega^3} \right](u_2) \tag{2.23}$$

We can now plot for $P(u_2,u_1,\theta,\phi)$ and $Q(u_2,u_1,\theta,\phi)$ for a given model for ψ_2 as a function on initial and final time. For a given waveform the above equation can also be used to show what ψ should look like for a given model.

$$\Psi_2^o(u_2,\theta,\phi) - \Psi_2^o(u_1,\theta,\phi) = P(u_2,u_1,\theta,\phi) - Q(u_2,u_1,\theta,\phi) \tag{2.24}$$

References

[1] T. M. Adamo, C. Kozameh, and E. T. Newman, Living Reviews in Relativity **12** (2009), 10.12942/lrr-2009-6.

[2] S. T. McWilliams, Physical Review Letters **122** (2019), 10.1103/physrevlett.122.191102.