Notes on BWM reseduals

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I. FOURIER TRANSFORM FOR POLYNOMIAL INSIDE A BOUNDED DOMAIN

Let $f(x)$ be defined as:

$$f(x) = \begin{cases} 1 & x \in [-1, 1] \\ 0 & x \in (-\infty, 1) \cup (1, \infty) \end{cases}$$

and $P_N(x)$ be the polynomial function defined as:

$$P_N(x) = \sum_{n=0}^{N} a_n x^n$$

The product of these two functions gives a polynomial function which is zero out side the interval $x \in [-1, 1]$

$$G(x) = f(x) P_N(x)$$

The Fourier transform $\mathscr{F}[G(x)]$ is given by convolution of Fourier transforms of two functions:

$$\mathscr{F}(G(x)) = \mathscr{F}(f(x)) * \mathscr{F}(P_N(x))$$

The Fourier transform of polynomial is given by

$$\mathscr{F}(P_N(x)) = \mathscr{F}\left(\sum_{n=0}^{N} a_n x^n\right) = \sum_{n=0}^{N} a_n \mathscr{F}(x^n) = \sqrt{2\pi} \sum_{n=0}^{N} a_n \left(i^n \delta^{(n)}(\omega)\right)$$

The Fourier transform of $f(x)$ is given by

$$\mathscr{F}(f(x)) = \frac{\sin(\omega)}{\omega}$$

Now the Fourier transform is given

$$\begin{aligned}
\mathscr{F}(G(x)) &= \mathscr{F}(f(x)) * \mathscr{F}(P_N(x)) \\
&= \left(\sqrt{2\pi} \sum_{n=0}^{N} a_n i^n \delta^{(n)}(\omega)\right) * \left(\frac{\sin(\omega)}{\omega}\right) \\
&= \int_{-\infty}^{\infty} \sqrt{2\pi} \sum_{n=0}^{N} a_n i^n \delta^{(n)}(\tau) \frac{\sin(\omega - \tau)}{\omega - \tau} d\tau \\
&= \sqrt{2\pi} \sum_{n=0}^{N} a_n i^n \int_{-\infty}^{\infty} \delta^{(n)}(\tau) \frac{\sin(\omega - \tau)}{\omega - \tau} d\tau \\
&= \sqrt{2\pi} \sum_{n=0}^{N} a_n i^n \int_{-\infty}^{\infty} \left[\frac{\sin(\omega - \tau)}{\omega - \tau}\right] \delta^{(n)}(\tau) d\tau \\
&= \sqrt{2\pi} \sum_{n=0}^{N} (-1)^n a_n i^n \int_{-\infty}^{\infty} \frac{\partial^n}{\partial \tau^n} \left[\frac{\sin(\omega - \tau)}{\omega - \tau}\right] \delta(\tau) d\tau \\
&= \sqrt{2\pi} \sum_{n=0}^{N} (-1)^{n+1} a_n i^n \int_{-\infty}^{\infty} \frac{\partial^n}{\partial \omega^n} \left[\frac{\sin(\omega - \tau)}{\omega - \tau}\right] \delta(\tau) d\tau \\
&= \sqrt{2\pi} \sum_{n=0}^{N} (-1)^{n+1} a_n i^n \frac{\partial^n}{\partial \omega^n} \int_{-\infty}^{\infty} \left[\frac{\sin(\omega - \tau)}{\omega - \tau}\right] \delta(\tau) d\tau \\
&= \sqrt{2\pi} \sum_{n=0}^{N} (-1)^{n+1} a_n i^n \frac{\partial^n}{\partial \omega^n} \left[\frac{\sin(\omega)}{\omega}\right]
\end{aligned}$$

$$\boxed{\int [x^n f(x)]\, \delta^{(n)}(x) dx = (-1)^n \int \frac{\partial^n [x^n f(x)]}{\partial x^n} \delta(x) dx}$$

So the Fourier transform of any polynomial function nonzero only inside the interval $x \in [1, 1]$ is given by

$$\boxed{\mathscr{F}(G(x)) = \sqrt{2\pi} \sum_{n=0}^{N} (-1)^{n+1} a_n i^n \frac{\partial^n}{\partial \omega^n} \left[\frac{\sin(\omega)}{\omega}\right]}$$