5

CHARACTERIZATION AND MODELLING OF CLOCKS

Designing complex systems of clocks as synchronization networks is not an obvious task, especially if aiming at guaranteeing the quality of service demanded by modern applications. Several years ago, specifying the quality requirements of clocks for telecommunications was often limited to not more than specifying their fractional deviation from the nominal frequency. More recently, since the introduction of SDH and of advanced digital services, clocks have been requested to comply with much more stringent and complex requirements for accuracy and stability, as specified by international standard bodies. A knowledge effusion occurred from the world of oscillator specialists to the world of telecommunication engineers, that imported methods and models conceived for time and frequency metrology and for the characterization of state-of-the-art oscillators.

Therefore, in order to describe the behaviour of clocks in synchronization networks and to accurately specify their characteristics, first it is necessary to identify a proper mathematical model of the clock and of the timing signals generated and distributed. This chapter supplies this basic knowledge: models of autonomous and slave clocks are defined and the mathematical tools for time and frequency stability characterization are provided. Finally, chains of slave clocks are considered, in order to assess the performance of timing transfer along synchronization trails.

5.1 CLOCKS AND TIMING SIGNALS

A clock is a device able to supply a *timing signal*, or *chronosignal*, i.e. a pseudo-periodic (ideally periodic) signal usable to control the timing of actions. Under different forms, clocks are widely spread in the everyday life of everybody and penetrated a surprisingly wide range of applications. Among the many possible examples, clocks supply the timing to digital hardware systems (gates, chips or boards), where the operation of different modules must be synchronized to ensure the proper transfer of binary symbols, and to telecommunications systems, where digital signals are multiplexed, transmitted and switched. Chapter 1 outlined several different contexts in which clocks play a central role in telecommunications, driving synchronization processes at different levels.

In principle, clocks consist simply of a generator of oscillations based on any periodic physical phenomenon. Among the most common examples we may mention the swinging of a pendulum or a wheel in mechanical clocks, the vibration of atoms in a crystal around their minimum-energy position in quartz clocks or the radiation associated with specific quantic atomic transitions in atomic clocks. *Autonomous clocks* supply a timing signal generated from an internal oscillator, running independently of external influence. In *slave clocks*, conversely, the oscillator is controlled by an external reference.

Timing signals are usually described as sine waves for ease of mathematical analysis, without loss of generality. Thus, a general expression modelling the pseudo-periodic timing signal $s(t)$ at the output of a clock is given by [5.1][5.2]

$$s(t) = A(t)\sin\Phi(t) \tag{5.1}$$

where $A(t)$ is the instantaneous amplitude and $\Phi(t)$ is the *total phase*.

The instantaneous amplitude in the most common cases can be considered constant, or

$$A(t) = A_n + \varepsilon(t) \qquad \left|\frac{\varepsilon(t)}{A_n}\right| \ll 1 \tag{5.2}$$

where A_n is the nominal amplitude and $\varepsilon(t)$ is the deviation from the nominal value. However, $\varepsilon(t)$ can be safely neglected in our following analysis. In fact, output stages of clocks are able to control accurately the amplitude of output signals. On the other hand, the position of significant instants does not depend on signal amplitude modulation, at least if trigger events are properly chosen.

More interesting is to notice that the timing information of the signal $s(t)$ is carried by its total phase, or equivalently by its instantaneous frequency $\nu(t)$[1], which is given by

$$\nu(t) = \frac{1}{2\pi}\frac{d\Phi(t)}{dt} \tag{5.3}$$

5.2 TIMING SIGNAL MODEL AND BASIC QUANTITIES

Several models have been proposed in literature [5.1]–[5.4] to describe the behaviour of the instantaneous frequency $\nu(t)$ in actual clocks. A common and comprehensive model is the following

$$\begin{aligned}\nu(t) &= \nu_0 + \nu_d(t) + \nu_a(t)\\ &= \nu_n + \Delta\nu + \sum_{k=1}^{K-1}\frac{q_k t^k}{k!} + \frac{1}{2\pi}\frac{d\varphi(t)}{dt}\end{aligned} \tag{5.4}$$

The *nominal frequency* ν_n (design goal) and the *starting frequency offset* $\Delta\nu$ (also called *syntonization error*) make up the starting frequency ν_0 of the oscillator:

$$\nu_0 = \nu_n + \Delta\nu \tag{5.5}$$

The starting frequency offset $\Delta\nu$ depends on the precision of the initial calibration.

[1] In most parts of the literature on time and frequency stability characterization, the symbol $\nu(t)$ is used to denote the instantaneous total frequency of a timing signal, while f is used to denote the Fourier frequency (see Section 5.6). In this chapter and in the following ones, we adhere to this notation.

The term $\nu_d(t)$ is the deterministic (for a given oscillator) time-dependent component of the clock total frequency, modelling as a power series the *frequency drift* mainly due to oscillator ageing, that is

$$\nu_d(t) = \sum_{k=1}^{K-1}\frac{q_k t^k}{k!} \tag{5.6}$$

The coefficients q_k $(k = 1, 2, \ldots, K-1)$ are time-independent. They are random variables considering a set of clocks, but fixed parameters for a given oscillator. The frequency drift in real clocks is due to complex phenomena and strongly dependent on the particular clock under measurement [5.5][5.6], but for practical purposes and for the sake of simplicity the summation above is often truncated to the first term, so that

$$\nu_d(t) \cong q_1 t = D\nu_n t \tag{5.7}$$

where D is the *linear fractional frequency drift rate*.

Finally, the term $\nu_a(t)$ is the random (aleatory) time-dependent component of the total instantaneous frequency, that is

$$\nu_a(t) = \frac{1}{2\pi}\frac{d\varphi(t)}{dt} \tag{5.8}$$

where $\dot\varphi(\tau)/(2\pi)$ and $\varphi(t)$ are stochastic processes, respectively the random frequency deviation and the random phase deviation, modelling oscillator intrinsic phase noise sources.

From Equations (5.3), (5.4) and (5.7), approximating the frequency drift to its linear term, the model for the total phase results

$$\Phi(t) = 2\pi(\nu_n + \Delta\nu)t + \pi D\nu_n t^2 + \varphi(t) + \Phi_0 \tag{5.9}$$

where $\Phi_0 = \Phi(0) - \varphi(0)$. In the ideal case, the total phase is linearly increasing with time, that is

$$\Phi(t) = 2\pi\nu_n t + \Phi_0 \tag{5.10}$$

Two functions, strictly related to $\dot\varphi(t)$ and $\varphi(t)$, are used in treating random frequency and time fluctuations of clocks: the *random fractional frequency deviation* $y(t)$ and the *random time deviation* $x(t)$, defined as

$$\begin{aligned}y(t) &= \frac{1}{2\pi\nu_n}\frac{d\varphi(t)}{dt}\\ x(t) &= \frac{\varphi(t)}{2\pi\nu_n}\end{aligned} \tag{5.11}$$

Please note that the above two functions express the sole random phase and frequency fluctuations of the clock, apart from the frequency offset and drift. Together with the model and definitions provided previously, they have been conceived and widely adopted by specialists since the 1960s [5.1][5.7]. More recently, needs arisen in particular in the telecommunications field, for the design of synchronization equipment and networks, led to the introduction of the following other basic functions, more oriented to the timing aspects of clocks.

The generated *Time* function $\mathrm{T}(t)$ of a clock is defined, in terms of its total instantaneous phase, as

$$\mathrm{T}(t) = \frac{\Phi(t)}{2\pi \nu_n} \tag{5.12}$$

For an ideal clock, $\mathrm{T}_{id}(t) = t$ holds, as expected.

For a given clock, the *Time Error* function $\mathrm{TE}(t)$ between its time $\mathrm{T}(t)$ and a reference time $\mathrm{T}_{ref}(t)$ is defined as

$$\mathrm{TE}(t) = \mathrm{T}(t) - \mathrm{T}_{ref}(t) \tag{5.13}$$

In practice, the reference clock is supposed to be much better performing than the clock under test and supplies the time reference for the measurement process itself. Thus, the Equation (5.13) should be actually rewritten as

$$\mathrm{TE}[\mathrm{T}_{ref}(t)] = \mathrm{T}[\mathrm{T}_{ref}(t)] - \mathrm{T}_{ref}(t) \tag{5.14}$$

In the following, however, we will omit to point out that all quantities are actually measured at times dictated by the reference clock and we will write expressions as in Equation (5.13), to keep formulas simple. If we assume to have an ideal clock as reference, Equation (5.13) becomes $\mathrm{TE}(t) = \mathrm{T}(t) - t$.

It is worthwhile noticing, at this point, the relationship between the quantities $x(t)$ and $\mathrm{TE}(t)$. Both represent a time deviation of the clock under test from a reference time, but the former expresses the sole random phase fluctuations, while the latter takes into account also the effects of any frequency offset and drift. Revealing the deterministic components in the TE measured data may be not straightforward and the result can highly depend on the parameter estimation technique adopted, as it will be shown in Chapter 7.

The *Time Interval* function $\mathrm{TI}_t(\tau)$ is defined as

$$\mathrm{TI}_t(\tau) = \mathrm{T}(t+\tau) - \mathrm{T}(t) \tag{5.15}$$

and represents the measure of a time interval τ, starting at time t, accomplished by the clock under test (again supposing to have an ideal reference clock). In other words, it expresses how long the clock under test perceives a time interval of ideal length τ starting at ideal time t.

The time error variation over an interval of duration τ starting at time t is called *Time Interval Error* function $\mathrm{TIE}_t(\tau)$ and is defined as

$$\begin{aligned}\mathrm{TIE}_t(\tau) &= [\mathrm{T}(t+\tau) - \mathrm{T}(t)] - [\mathrm{T}_{ref}(t+\tau) - \mathrm{T}_{ref}(t)] \\ &= \mathrm{TE}(t+\tau) - \mathrm{TE}(t)\end{aligned} \tag{5.16}$$

It represents the error committed by the clock under test in measuring an interval τ, starting at time t, with respect to the reference clock. If the reference clock is ideal, the last equation can be also written as $\mathrm{TIE}_t(\tau) = \mathrm{TI}_t(\tau) - \tau$.

The example plots provided in Figures 5.1 and 5.2 clarify the meaning of functions $\mathrm{T}(t)$, $\mathrm{TE}(t)$, $\mathrm{TI}_{t_0}(\tau)$ and $\mathrm{TIE}_{t_0}(\tau)$ and the relationships among them.

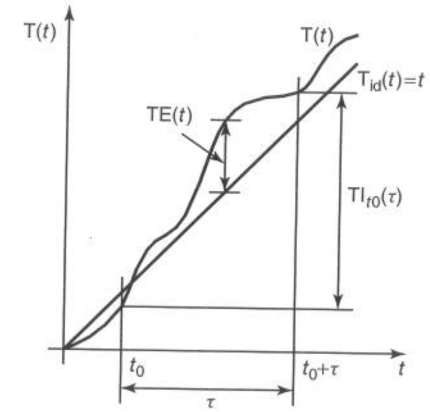

Figure 5.1. Example plot of the functions $\mathrm{T}(t)$, $\mathrm{T}_{id}(t) = t$, $\mathrm{TE}(t)$ and $\mathrm{TI}_{t_0}(\tau)$

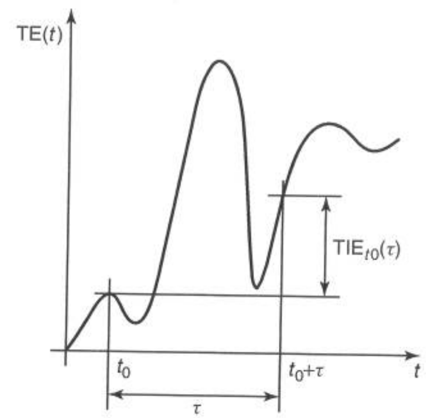

Figure 5.2. Example plot of the functions $\mathrm{TE}(t)$ and $\mathrm{TIE}_{t_0}(\tau)$

5.3 BASIC CONCEPTS OF QUALITY OF CLOCKS: STABILITY AND ACCURACY

Characterizing the *quality* of a clock (or equivalently of its timing signal) is one of the most debated issues in practical applications of clocks. In the most common sense, one simple term is used to refer to clock quality: the *precision*, somehow denoting how much the clock under test is close to a reference clock (in time or in frequency). Nevertheless, the term 'precision' is even not included in the ISO International Vocabulary of Metrology [5.8] and its use is therefore not recommended.

The quality of a clock is usually defined by means of two other basic terms: the *stability* and the *accuracy*. Although it must be recognized that researchers and engineers working in different fields may understand these two terms with subtly different meanings, the definitions provided in the following paragraphs can be considered quite general and widely accepted.

5.3.1 Stability

The *stability* of a measuring instrument is its ability to maintain constant its metrological characteristics with time (ISO International Vocabulary of Metrology [5.8]). The *stability of a clock*, therefore, is its capacity of generating a time interval (or a frequency) with constant value (such a value may be different from the nominal value, but it is intended constant in time).

In other words, the stability of a clock deals with the measurement of random and deterministic variations of its instantaneous frequency (or of the time generated) compared to the nominal value (i.e., in practice, to that of a reference clock), *over a given observation interval*. For example, a stability measure of the instantaneous frequency based on the concept of variance σ^2, over an observation interval τ, informs about the instantaneous-frequency variance that is expected by collecting measurements along a time interval τ. With reference to the mathematical model of Equation (5.4), the clock time stability depends on the random phase noise $\varphi(t)$, the frequency offset $\Delta\nu$ and the frequency drift coefficients q_k. The relative weights of such parameters in affecting stability depend on the observation interval: if this is short, the frequency drift and even offset are negligible.

When the observation interval τ is small, the expression *short-term* stability is commonly used, otherwise the expression *long-term* stability applies. What should be the meaning of the word 'small' (i.e., where is the border between short and long term) depends on the specific application. For example, in the time metrology field, it is common to consider observation intervals longer than one day as long-term, while in telecommunications applications observation intervals above 100 s are definitely considered to fall in the long term as well. To further clarify the concept of short-term and long-term stability, Figure 5.3 shows two examples of instantaneous frequency plot. The timing signal in the upper graph is rather stable in the short term, but quite drifty in the long term; on the contrary, the one in the lower graph has a very poor short-term stability, but is very stable in the long term.

Since the 1960s, several quantities have been defined aiming at characterizing clock stability. With different properties, they highlight distinct phenomena in the phase noise or they are more oriented to specific applications. Details will be given in the next sections.

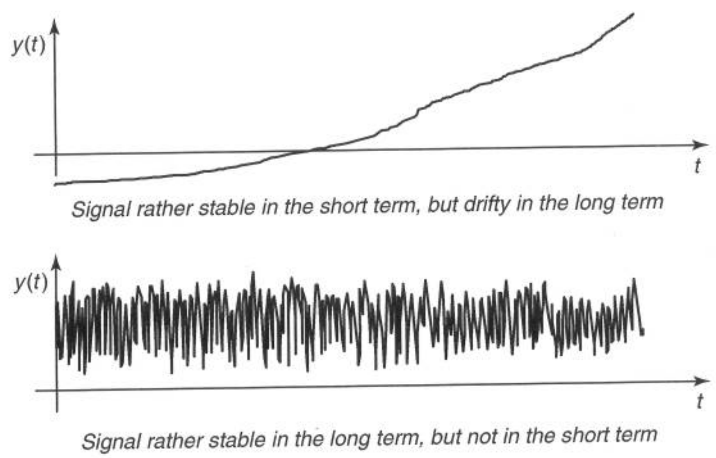

Figure 5.3. Examples of long-term and short-term stability

5.3.2 Accuracy

The *accuracy of measurement*, on the other hand, denotes the closeness of the agreement between the result of a measurement and a true value of the measurand (ISO International Vocabulary of Metrology [5.8]). In the context of clock quality definition, thus, accuracy of a time scale means its agreement with UTC, while accuracy of the frequency delivered by a frequency standard means its agreement with the nominal value. Qualitatively speaking, accuracy tells how much a clock is 'right'.

According to common use in telecommunications, we define the *accuracy of a clock* as the maximum time or frequency error, which may be measured in general *over the whole clock life* (e.g., 20 years), unless differently specified. Therefore, the *time accuracy* means how well a clock agrees with UTC over the specified period, while the *frequency accuracy* is the maximum frequency error $\Delta\nu_{\max}$ compared to the nominal value ν_n. It must be pointed out that also the accuracy depends in principle on the above mentioned parameters $\varphi(t)$, $\Delta\nu$ and the coefficients q_k, but in this case the observation interval is so long that in practice the only relevant quantities are the frequency offset and drift. The frequency accuracy is usually expressed by the a-dimensional ratio $\Delta\nu_{\max}/\nu_n$ and is often measured in 10^{-6} units [μHz/Hz], in the engineering practice called also [parts per million] and abbreviated as [ppm][2].

Finally, to further clarify the difference between the concepts of stability and accuracy, let us point out that a clock could have a significant frequency error (and thus poor frequency accuracy), while still keeping constant this frequency error during its lifetime (and thus featuring perfect frequency stability).

For example, a hydrogen-MASER clock typically has better frequency stability than a caesium-beam clock from second to second or from hour to hour, but often not from month to month and longer. On the other hand, the typical caesium-beam clock is more accurate than the hydrogen-MASER clock. Quartz-oscillators can be very stable in the short term, but they drift in frequency and do not feature the frequency accuracy of atomic clocks [5.9].

5.4 AUTONOMOUS CLOCKS

An *autonomous clock* is a stand-alone device able to generate a timing signal, suitable for the measurement of time intervals, starting from some periodic physical phenomenon that runs independently of external influence.

Examples of autonomous clocks are the atomic frequency standards (such as the rubidium, or the caesium-beam or the hydrogen-MASER oscillators) and the crystal quartz oscillators. Some of them (the rubidium and the quartz oscillators) may also run locked to a reference timing signal and thus work as slave clocks.

A simplified model of autonomous clock is shown in Figure 5.4. Here, according to the model defined in Equation (5.9), the total output phase $\Phi(t)$ is made of three deterministic terms modelling the phase generated by an ideal oscillator, the frequency offset $\Delta\nu$ and the linear frequency drift D, together with the random phase noise $\varphi(t)$.

[2] The unit [ppm] is not an International System (SI) unit, and therefore its use is deprecated by time and frequency metrologists. In spite of this, we still use it in this book, following the common practice among telecommunications engineers.

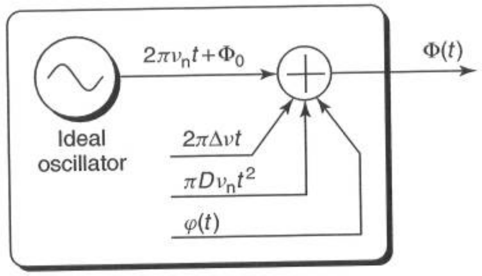

Figure 5.4. Simplified model of autonomous clock

The starting frequency offset $\Delta\nu$ is as smaller as more accurate the clock calibration procedures have been during the production process. Such calibration procedures aim at making the clock generate an output frequency as closest as possible to the nominal frequency.

As far as the frequency drift coefficient D is concerned, it is worthwhile noticing that it is practically null in the case of caesium-beam oscillators, while it cannot be neglected in rubidium and quartz oscillators, at least in the most demanding applications. Moreover, accurate characterization of quartz oscillators needs to consider at least the first two drift coefficients (i.e., linear and quadratic).

Finally, details on the statistical characteristics of the random phase-noise process $\varphi(t)$, as it can be measured on actual clocks, will be provided in Section 5.9.

5.5 SLAVE CLOCKS

A *slave clock* is a device able to generate a timing signal, suitable for the measurement of time intervals, having phase (or much less frequently frequency) controlled by a reference timing signal at its input. The Phase-Locked Loop (PLL) and the Frequency-Locked Loop (FLL) are examples of slave clocks. In the former case, the slave clock outputs a timing signal that is synchronous with the input signal, owing to the feedback control on the phase error between them. In the latter case, the slave clock outputs a timing signal mesochronous with the input signal, because the feedback control acts on the frequency error between them.

Slave clocks are very widely employed in synchronization networks or in digital telecommunications equipment. Their major role is to recover timing from the input reference signal and to maintain output timing as close to the source-clock's timing as possible. This implies two basic functions. First, the slave clock must follow faithfully the source-clock's timing, recovering it from the input reference signal, even though this may be affected by transmission impairments of any kind. Second, the slave clock must be able to maintain adequate timekeeping even after a major timing-reference failure.

The widespread application of slave clocks in synchronization networks and in digital telecommunications equipment calls for a somehow detailed description of their models and properties. Therefore, the next subsections deal with the main characteristics of PLLs and the operation modes of slave clocks in synchronization networks.

5.5.1 Phase-Locked Loop Fundamentals

There is a huge literature dealing with PLL theory and application. Among the many books that have been written, the following ones can be definitely recommended. The pioneering work of Viterbi on the non-linear theory of PLLs is reported in [5.10]. The fundamental research of Lindsey, really a synchronization guru, is reported in [5.11], which, although not recent, contains a wealth of information on advanced topics of PLL theory that cannot be found anywhere else. The book by Blanchard [5.12] focuses on the application of PLLs to coherent receiver design. The books by Gardner [5.13] and Best [5.14] are aimed at engineers and stress practical aspects. The book by Horowitz and Hill [5.15][3], as the title itself says, deals with circuit implementation practice. The book by Meyr and Ascheid [5.16], finally, deals with phase and frequency control systems widely used in digital communications and provides a deep overview on the whole subject. In this section, the introductory sections of the book by Meyr and Ascheid have been followed, to provide an overview just on the very basics of PLL. We omitted most analytical details, in order to stress some practical results useful for PLL application in slave clocks for synchronization network. For further details the reader is referred to that book and other literature.

5.5.1.1 Automatic Phase Control and PLL Principle

A device implementing the phase locking to the reference timing signal is commonly based on a loop architecture, based on the negative feedback principle, where the output signal keeps tracking the phase fluctuations of the input reference. Such a device is called *Phase-Locked Loop* (PLL).

A phase-locked loop is a control system used to automatically adjust the phase of a locally generated signal $\hat{s}(t)$ to the phase of an incoming signal $s(t)$. As previously done, let us assume that the two signals are sine waves, respectively given by

$$s(t) = \sin[\omega_0 t + \theta(t)]$$
$$\hat{s}(t) = \sin[\omega_0 t + \hat{\theta}(t)] \tag{5.17}$$

where the phases $\theta(t)$ and $\hat{\theta}(t)$ are slowly varying with respect to the nominal angular frequency ω_0, i.e.

$$\left|\frac{d\theta(t)}{dt}\right| \ll \omega_0, \quad \left|\frac{d\hat{\theta}(t)}{dt}\right| \ll \omega_0 \tag{5.18}$$

Any difference in instantaneous frequency among the two signals is included in the time-varying function $\hat{\theta}(t)$, as

$$\hat{\omega}_0 = \omega_0 + \frac{d\hat{\theta}(t)}{dt} \tag{5.19}$$

The goal is to adjust the total phase of the output signal $\hat{s}(t)$

$$\hat{\Phi}(t) = \omega_0 t + \hat{\theta}(t) \tag{5.20}$$

[3] We cannot refrain from recommending the textbook by Horowitz and Hill as an outstanding source of information on an astonishing wide spectrum of electronic circuit design techniques. In that book, the so-called 'random-open and look' test always yields surprising results, enabling us to find unexpected and instructive topics.

to that of the input reference signal $s(t)$

$$\Phi(t) = \omega_0 t + \theta(t) \tag{5.21}$$

Such two signals can be phase-aligned by an automatic control system if we are able to generate a control signal as a function of the phase error between them. An easy way to achieve this is to simply multiply the two signals, as

$$\begin{aligned}&\sin[\omega_0 t + \theta(t)] \cdot \sin[\omega_0 t + \hat{\theta}(t)] \\ &= \frac{1}{2}\cos[\theta(t) - \hat{\theta}(t)] - \frac{1}{2}\cos[2\omega_0 t + \theta(t) + \hat{\theta}(t)]\end{aligned} \tag{5.22}$$

Since the phase $\theta(t)$ is slowly varying compared to $2\omega_0$, the second cosine term can be easily removed by low-pass filtering. The first cosine term provides a measure of the phase difference $\theta(t) - \hat{\theta}(t)$, but, being an even function, does not provide the key information whether $\theta(t)$ is larger than $\hat{\theta}(t)$ or *vice versa*. A way to get an odd function of the phase difference $\theta(t) - \hat{\theta}(t)$ is to advance the locally generated signal $\hat{s}(t)$ by $\pi/2$ before multiplication by the input reference signal, to yield

$$\begin{aligned}&\sin[\omega_0 t + \theta(t)] \cdot \sin\left[\omega_0 t + \frac{\pi}{2} + \hat{\theta}(t)\right] \\ &= \frac{1}{2}\sin[\theta(t) - \hat{\theta}(t)] + \frac{1}{2}\sin[2\omega_0 t + \theta(t) + \hat{\theta}(t)]\end{aligned} \tag{5.23}$$

Then, if the phase error $\theta(t) - \hat{\theta}(t)$ is non zero, an error signal with the same sign as the phase error is produced. This error signal is filtered and then applied to an oscillator whose output frequency can be varied by the voltage applied to it: a Voltage-Controlled Oscillator (VCO). When the control voltage equals zero, the VCO runs at its quiescent (free-run) frequency ω_0. A positive or negative control voltage causes the VCO to increase or decrease, respectively, its instantaneous angular frequency $d\hat{\theta}(t)/dt$, thereby forcing the phase error to decrease or increase accordingly.

Hence, note that, in such a control system, zero-phase error occurs when the locally generated VCO signal and the input signal $s(t)$ have a phase difference of $\pi/2$. In order to output a signal $\hat{s}(t)$ with the same phase as the input signal $s(t)$, a phase shift of $\pi/2$ must be applied to the signal directly output by the VCO.

5.5.1.2 Scheme of PLL and Baseband Model

The automatic phase control system presented above is generally referred to as Phase-Locked Loop (PLL). Its block diagram, in the most basic form, is shown in Figure 5.5. The building blocks are a multiplier (*phase detector*), a *low-pass filter* having transfer function $F(s)$ (in the Laplace domain) and a *voltage-controlled oscillator* (VCO).

Please note that a signal multiplier is not the only way to generate an output functioning as measure of phase error between two inputs. Two broad categories of phase detectors are *multiplier-type* circuits and *sequential-logic* circuits. The ideal multiplier is a useful analytical model for phase detector, but in its basic form is rarely used in actual equipment. Multiplier-type phase detectors fully utilize the signal waveform and are capable of operating on signals deeply buried in noise. An example of sequential-logic phase detector, on the other hand, is the reset–set flip-flop. Sequential-logic phase detectors operate on signal level crossings only and provide a linear phase detection characteristic. For the

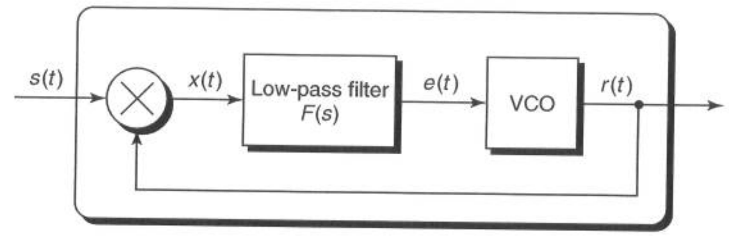

Figure 5.5. Block diagram of a basic phase-locked loop

sake of simplicity, in the following analysis PLL systems based on an ideal multiplier phase detector will be considered.

If the input reference signal and the signal output by the VCO are given respectively by[4]

$$\begin{aligned}s(t) &= \sqrt{2}A\sin\Phi(t) \\ r(t) &= \sqrt{2}K_1\cos\hat{\Phi}(t)\end{aligned} \tag{5.24}$$

where A [V] and K_1 [V] are their root-mean-square values, then the output of the multiplier, discarding the sum-frequency term, is given by

$$x(t) = AK_1K_m\sin[\theta(t) - \hat{\theta}(t)] \tag{5.25}$$

where K_m [V^{-1}] is the multiplier gain. The frequency of the VCO is function of the low-pass filtered error voltage $e(t)$. Therefore

$$\frac{d\hat{\Phi}(t)}{dt} = \omega_0 + K_0 e(t) \tag{5.26}$$

where K_0 [s^{-1} V^{-1}] is the VCO gain factor. From the definition of $\hat{\Phi}(t)$, we get also

$$\frac{d\hat{\theta}(t)}{dt} = K_0 e(t) \tag{5.27}$$

The analysis of the model yields thus to the following dynamic equation for the phase error $\phi(t) = \theta(t) - \hat{\theta}(t)$

$$\frac{d\phi(t)}{dt} = \frac{d\theta(t)}{dt} - KA\int_0^t f(t-u)\sin\phi(u)\,du \tag{5.28}$$

where $K = K_0K_mK_1$ [s^{-1} V^{-1}]. The product KA is usually called *loop gain*.

Therefore, the equivalent mathematical model of the phase-locked loop represented in Figure 5.5 is the control system depicted in Figure 5.6, obeying the dynamic Equation (5.28). This model is a *baseband* model of the PLL, because centred on the

[4] The only difference between the signal $r(t)$, output directly by the VCO, and the output signal $\hat{s}(t)$ defined before is the $\pi/2$-phase shift and the peak value $\sqrt{2}K_1$.

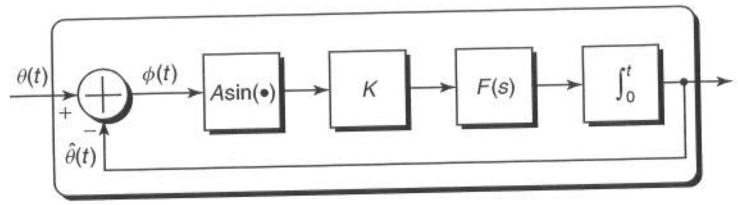

Figure 5.6. Baseband model of the phase-locked loop

phase error $\phi(t)$ independently of the quiescent frequency ω_0. Moreover, it is a *nonlinear* system model, which makes difficult its mathematical analysis.

5.5.1.3 *Linear Model of PLL and Transfer Functions*

If the phase error $\phi(t)$ is small (PLL locked to the reference) i.e. $\phi(t) \ll 1$ rad then the approximation

$$\sin\phi \approx \phi \tag{5.29}$$

can be used and the non-linearity can be disregarded, as shown in the plot of Figure 5.7 where the two functions are compared for $-\pi \le \phi \le +\pi$.

Under this approximation, the system in Figure 5.5 becomes linear and thus can be analysed with standard techniques. Hence, the linearized baseband model of the phase-locked loop, in the Laplace domain, is depicted in Figure 5.8. The functions in the time domain $\phi(t)$, $\theta(t)$ and $\hat{\theta}(t)$ have been replaced with their Laplace transforms denoted as $\phi(s)$, $\theta(s)$ and $\hat{\theta}(s)$, respectively.

From Equation (5.28), linearized by the approximation of Equation (5.29), we get the *closed-loop transfer function* of the PLL

$$H(s) = \frac{\hat{\theta}(s)}{\theta(s)} = \frac{KAF(s)}{s + KAF(s)} \tag{5.30}$$

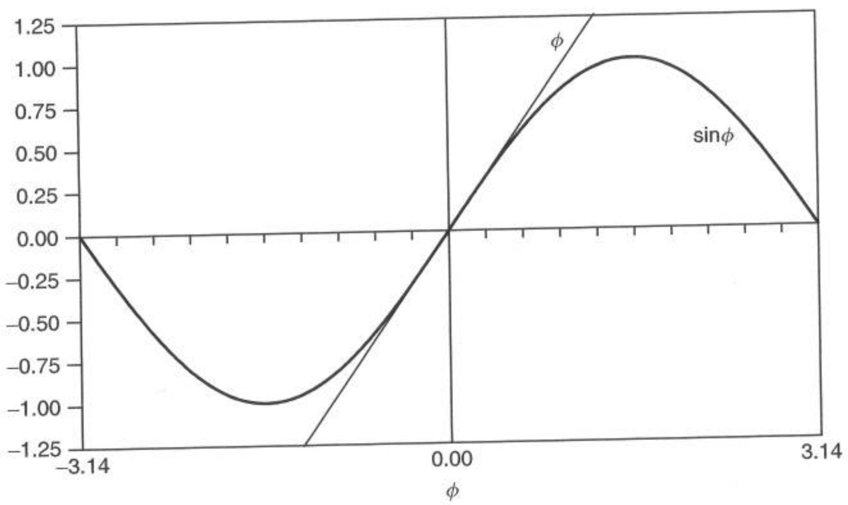

Figure 5.7. Compared plot of the functions $\sin(\phi)$ and ϕ for $-3.14 \le \phi \le 3.14$

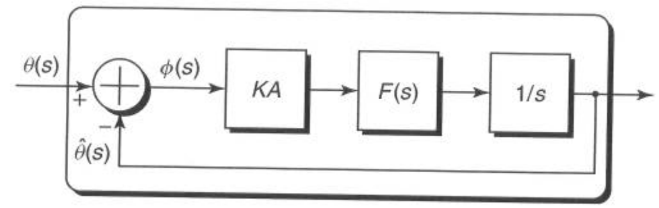

Figure 5.8. Linearized baseband model in the Laplace domain of the phase-locked loop

By contrast, the *open-loop transfer function*

$$G_0(s) = \frac{KAF(s)}{s} \tag{5.31}$$

yields $\hat{\theta}(s)/\theta(s)$ when the feedback from the integrator to the sum node is open. Hence, the two important basic transfer functions of the phase-locked loop are the following

$$\frac{\hat{\theta}(s)}{\theta(s)} = H(s) = \frac{G_0(s)}{1 + G_0(s)} \tag{5.32}$$

$$\frac{\phi(s)}{\theta(s)} = \frac{\theta(s) - \hat{\theta}(s)}{\theta(s)} = 1 - H(s) = \frac{1}{1 + \dfrac{KAF(s)}{s}} = \frac{1}{1 + G_0(s)} \tag{5.33}$$

5.5.1.4 *PLL Type and Order*

The characteristics of the phase-locked loop are determined by the kind of filter $F(s)$.

A PLL without filter at all, i.e. with $F(s) = 1$, is called a first-order loop. One with a single-pole low-pass filter is a second-order loop. In general, a loop with a filter having $k - 1$ poles is called a kth-order loop. The order of the PLL is thus the degree of the polynomial at the denominator of the closed-loop transfer function $H(s)$ in Equation (5.30).

In control theory, feedback systems are also distinguished by their *type*. If the open-loop transfer function $G_0(s)$ has k poles at the origin ($s = 0$), i.e. k integrators, the phase-locked loop is a type-k system. Loop order can be higher then type, since all poles contribute to the order, but not to the type if they are not at the origin.

In the PLL model of Figure 5.8, the VCO is represented by an integrator. Therefore, the loop filter must have $k - 1$ poles at the origin to yield a type-k loop.

5.5.1.5 *Steady-State Phase Error*

The steady-state phase error can be evaluated from Equation (5.33) by application of the final-value theorem, which states that

$$\lim_{t\to\infty} \phi(t) = \lim_{s\to 0} s\phi(s) \tag{5.34}$$

Three notable cases to consider are the phase step, the frequency step and the frequency ramp as input signal $\theta(t)$.

In the case of *phase step* on the input, we have

$$\theta(t) = \Delta\theta \leftrightarrow \theta(s) = \frac{\Delta\theta}{s} \tag{5.35}$$

and

$$\lim_{t\to\infty}\phi(t)=\lim_{s\to 0}s\frac{\Delta\theta}{s}\frac{1}{1+\dfrac{KAF(s)}{s}}=0 \quad \text{for } F(0)\neq 0 \tag{5.36}$$

Therefore, the PLL damps any input phase step to zero, even if the loop filter $F(s)$ is a pure constant.

In the case of *frequency step* on the input, the input phase is a ramp as

$$\theta(t)=\Delta\omega t \leftrightarrow \theta(s)=\frac{\Delta\omega}{s^2} \tag{5.37}$$

and we have

$$\lim_{t\to\infty}\phi(t)=\lim_{s\to 0}s\frac{\Delta\omega}{s^2}\frac{1}{1+\dfrac{KAF(s)}{s}}=\lim_{s\to 0}\frac{\Delta\omega}{s+KAF(s)} \tag{5.38}$$

Therefore, to keep the steady-state phase error small, the value $F(0)$ should be as large as possible. In particular, it is clear that the PLL is able to cancel any ramp phase error, resulting from a frequency step applied at $t=0$, if $F(s)$ has at least one pole at $s=0$, i.e. the loop is of type 2.

In the case of *frequency ramp* on the input, as it happens for example receiving a constant-frequency signal from a source moving with constant radial acceleration relative to the receiver, the input signal is

$$\theta(t)=\frac{\Delta\dot{\omega}t^2}{2} \leftrightarrow \theta(s)=\frac{\Delta\dot{\omega}}{s^3} \tag{5.39}$$

and we have

$$\lim_{t\to\infty}\phi(t)=\lim_{s\to 0}s\frac{\Delta\dot{\omega}}{s^3}\frac{1}{1+\dfrac{KAF(s)}{s}}=\lim_{s\to 0}\frac{\Delta\dot{\omega}}{s^2+sKAF(s)} \tag{5.40}$$

Therefore, the PLL can track an input frequency ramp with zero steady-state phase error only if $F(s)$ has at least two poles at $s=0$, i.e. the loop is of type 3. If a single-pole filter $F_1(s)$ is used, a residual phase error equal to

$$\lim_{t\to\infty}\phi(t)=\frac{\Delta\dot{\omega}}{KAF_1(0)} \tag{5.41}$$

remains.

5.5.1.6 *Stability of the Feedback System*

A basic requirement for any feedback system is *stability*, which for our purposes can be defined in the sense that a system should respond to any bounded input signal with a bounded output signal. From basic control theory, the simplified Nyquist criterion states that a feedback system is stable if the *phase margin*[5]

[5] In this and in the following formulas, the *frequency response* $G(\omega)$ of the system is used, which is obtained by replacing the complex variable s in the transfer function $G(s)$ by the pure imaginary part $j\omega$. Therefore, $G(\omega)$ is a complex number for any given real angular frequency $\omega=2\pi f$. The name *frequency response* is

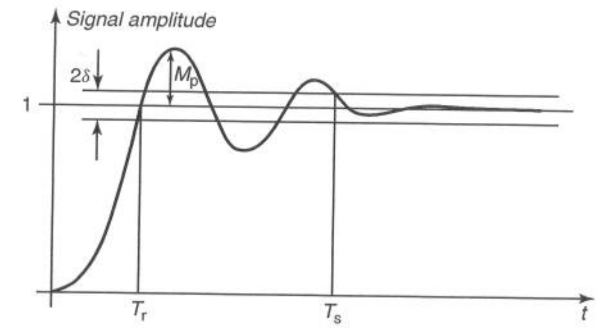

Figure 5.9. Example of system response to a unitary step input signal in the time domain and descriptive measures of the transient behaviour

$$\phi_R \hat{=} \arg[G_0(\omega_c)]-(-\pi) \tag{5.42}$$

is positive, where ω_c is the *crossover frequency*, at which the log magnitude of the open-loop frequency response function $G_0(\omega)$ crosses the 0-dB gain axis, i.e.

$$|G_0(\omega_c)|=1 \tag{5.43}$$

5.5.1.7 *Transient Response of the System*

The exact transient response of the system for any input signal can be evaluated by solving the dynamic Equation (5.28). Under the linear approximation, the response can be also evaluated by convolution of the input signal with the inverse transform of $H(s)$ (5.30), as

$$\hat{\theta}(t)=\theta(t)*h(t) \tag{5.44}$$

Typically, a step input signal is considered to evaluate the system response in the time domain. Descriptive measures of the transient response to a step input signal are the rise time T_r, the settling time T_s and the peak overshoot M_p, as illustrated in Figure 5.9. The rise time T_r is the time needed for the transient response to reach 100% of the step amplitude. The settling time T_s is the time needed for the transient response to settle within an error not higher than $\pm\delta\%$ around the step amplitude (a small value such as $\delta=5\%$ is customary). The peak overshoot M_p is the maximum deviation of the transient response from the step amplitude.

due to the fact that if a sinusoidal signal

$$s_{in}(t)=\mathrm{Re}\{A_{in}e^{j\omega t}\}=A_{in}\cos\omega t$$

is input into a linear system characterized by the transfer function $G(s)$, then the steady-state response is the signal

$$s_{out}(t)=\mathrm{Re}\left\{|G(\omega)|A_{in}e^{j\omega t+j\arg[G(\omega)]}\right\}=A_{out}\cos(\omega t+\phi_{out}).$$

In other words, amplitude and phase of the output sine wave are related to the input amplitude and phase by the frequency response $G(\omega)$ as

$$A_{out}=|G(\omega)|A_{in}$$

$$\phi_{out}=\arg[G(\omega)]$$

What is interesting is that the transient response measures above can be approximately estimated, knowing the crossover frequency ω_c and the phase margin ϕ_R of the open-loop transfer functions. To this aim, we note that Equation (5.32) can be rather crudely approximated as a single-pole low-pass function around the crossover frequency ω_c (a slope of $G_0(\omega)$ greater than 40 dB/decade about $\omega \cong \omega_c$ would not yield sufficient phase margin to guarantee system stability) and thus

$$H(s) \approx \frac{1}{1 + s/\omega_c} \leftrightarrow h(t) \approx \omega_c e^{-\omega_c t} \quad \text{for } t \geq 0 \tag{5.45}$$

Hence, for $\delta = 5\%$, we obtain the approximated value of the settling time

$$T_s \cong \frac{3}{\omega_c} \tag{5.46}$$

Moreover, qualitatively, we notice that zero phase margin (i.e., $G_0(\omega_c) = -1$) corresponds to a pair of imaginary poles at $s \pm j\omega_c$ and therefore to an everlasting undamped sinusoidal oscillation as response to a step input signal (100% peak overshoot). A small phase margin ϕ_R corresponds to a large peak overshoot. A large phase margin ϕ_R means safe system stability and thus small peak overshoot.

5.5.2 Second-Order Phase-Locked Loop

Second-order PLLs are the most common in practical systems. As shown in the previous section, a single integrator in the loop filter allows to track a constant frequency offset between the input signal and the local oscillator without residual steady-state phase error. Also in this section, the book by Meyr and Ascheid [5.16] has been mostly followed to provide a simple overview on the topic.

Two types of single-pole filter are particularly worthwhile to consider in this analysis:

$$F_1(s) = \frac{1 + sT_2}{sT_1} \tag{5.47}$$

$$F_2(s) = \frac{1 + sT_2}{1 + sT_1} \tag{5.48}$$

The former filter $F_1(s)$ contains a perfect integrator and may be realized with active circuitry, i.e. a dc high-gain operational amplifier (*active filter*). On the other hand, a *passive filter* realization can approximate perfect integration by a low-pass filter with a pole at $s = -1/T_1$, as in the latter filter $F_2(s)$. A PLL incorporating passive filter is called an *imperfect second-order loop* and, in steady state, yields a fixed phase error in response to an input frequency offset.

With active filter implementation $F_1(s)$, from Equation (5.30) the overall closed-loop transfer function results

$$H_1(s) = \frac{KA(1 + sT_2)}{s^2 T_1 + sKAT_2 + KA} \tag{5.49}$$

while, in case of passive filter $F_2(s)$, the transfer function

$$H_2(s) = \frac{KA(1 + sT_2)}{s^2 T_1 + s(1 + KAT_2) + KA} \tag{5.50}$$

is obtained.

More generally, in second-order PLLs the single-pole loop filter is of the form

$$F(s) = \frac{A + sB}{C + sD} \tag{5.51}$$

The most common practical realizations of the loop-filter general form above are the following four:

- passive filter with a pole at $s = -1/T_1$ ($A = 1$, $B = 0$, $C = 1$, $D = T_1$);
- passive filter with a pole at $s = -1/T_1$ and a zero at $s = -1/T_2$ (5.48) ($A = 1$, $B = T_2$, $C = 1$, $D = T_1$);
- active filter with a pole at $s = 0$ ($A = 1$, $B = 0$, $C = 0$, $D = T_1$);
- active filter with a pole at $s = 0$ and a zero at $s = -1/T_2$ (5.47) ($A = 1$, $B = T_2$, $C = 0$, $D = T_1$)

A more practical way to write general second-order closed-loop transfer functions is the form

$$H(s) = \frac{Es + F}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.52}$$

where ω_n is the *natural angular frequency*, ζ is the *damping ratio*, E and F are constants dependent on ω_n and ζ. The natural angular frequency ω_n is the frequency at which, in the PLL open-loop transfer function, the extension of the line with -40 dB/decade slope crosses the 0-dB line. As it will be shown in the next paragraphs, ω_n and ζ are fair descriptors of the PLL transient response, because they are related to the crossover frequency ω_c and the phase margin ϕ_R.

The formulas giving the values of the parameters E, F, ω_n, ζ, T_1 and T_2 are provided in Table 5.1, for the four cases of loop-filter transfer function above.

The magnitude of the closed-loop transfer function (5.52) is plotted in the graph of Figure 5.10, in the case of a perfect second-order PLL (active filter), as function of the normalized angular frequency ω/ω_n and for six values of the damping ratio ζ.

It is evident that $H(s)$ is low-pass type, with asymptotic slope -20 dB/decade for $\omega \gg \omega_n$: the PLL is able to cut off high-frequency phase fluctuations on the input signal. A practical measure of the PLL *bandwidth* is the cut-off frequency B [Hz] for which input phase fluctuations are transferred to the output attenuated by a factor $\sqrt{2}$, i.e.

$$20\,\mathrm{Log}_{10}|H(2\pi B)| = -3 \text{ dB} \tag{5.53}$$

This measure is very common in practical measurements on slave clocks. In terms of natural angular frequency ω_n and damping ratio ζ, it results:

$$\begin{aligned} B &= \frac{\omega_n}{2\pi}\sqrt{2\zeta^2 + 1 + \sqrt{(2\zeta^2 + 1)^2 + 1}} \\ &\cong \frac{\omega_n}{\pi}\zeta \qquad \text{for } \zeta > 2 \end{aligned} \tag{5.54}$$

Also of interest is the *overshoot* around $\omega \cong \omega_n$ (maximum gain): here, phase fluctuations on the input signal are amplified, instead of being reduced. It is evident that this

Table 5.1 Parameters of second-order PLL transfer function $H(s)$ (5.52) in the cases of perfect (active filter) and imperfect (passive filter) integrator

	Passive Filter		Active Filter	
Loop filter $F(s)$	$\frac{1}{1+sT_1}$	$\frac{1+sT_2}{1+sT_1}$	$\frac{1}{sT_1}$	$\frac{1+sT_2}{sT_1}$
E	0	$2\zeta\omega_n\left(1-\frac{\omega_n}{KA}\frac{1}{2\zeta}\right)$	$2\zeta\omega_n$	$2\zeta\omega_n$
F	ω_n^2	ω_n^2	ω_n^2	ω_n^2
ω_n	$\sqrt{\frac{KA}{T_1}}$	$\sqrt{\frac{KA}{T_1}}$	$\sqrt{\frac{KA}{T_1}}$	$\sqrt{\frac{KA}{T_1}}$
ζ	$\frac{1}{2}\sqrt{\frac{1}{KAT_1}}=\frac{\omega_n}{2KA}$	$\frac{T_2}{2}\sqrt{\frac{KA}{T_1}}\left(1+\frac{1}{KAT_2}\right)$ $=\frac{T_2\omega_n}{2}+\frac{\omega_n}{2KA}$	0	$\frac{T_2}{2}\sqrt{\frac{KA}{T_1}}=\frac{T_2\omega_n}{2}$
T_1	$\frac{KA}{\omega_n^2}$	$\frac{KA}{\omega_n^2}$	$\frac{KA}{\omega_n^2}$	$\frac{KA}{\omega_n^2}$
T_2	0	$\frac{2\zeta}{\omega_n}\left(1-\frac{\omega_n}{KA}\frac{1}{2\zeta}\right)$	0	$\frac{2\zeta}{\omega_n}$

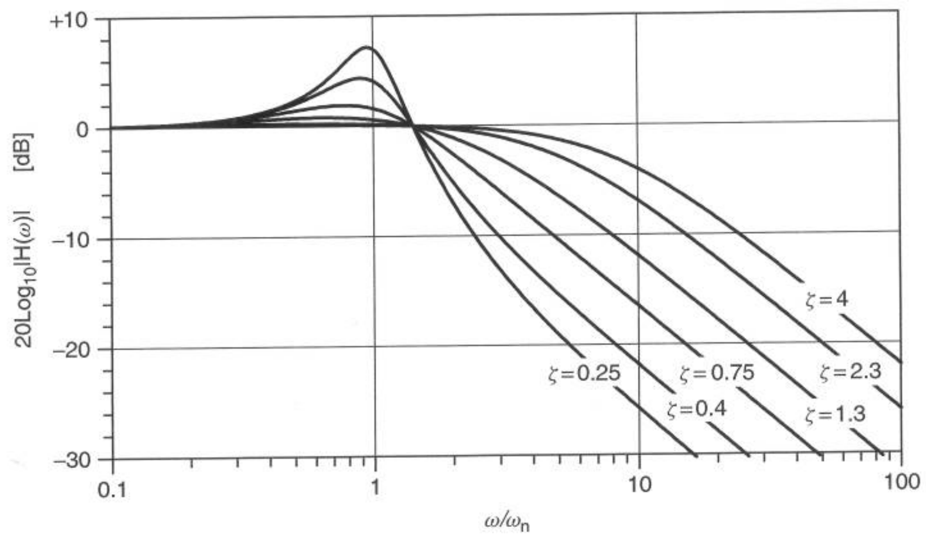

Figure 5.10. Magnitude of the closed-loop transfer function (5.52) of a perfect second-order PLL (active filter), as function of the normalized angular frequency ω/ω_n and for $\zeta = 0.25$, 0.4, 0.75, 1.3, 2.3, 4

is an unwanted property in slave clocks. Along a chain of slave clocks with the same ω_n, cumulated phase noise around this frequency gets considerably amplified and may become annoying even if the overshoot of the single clock is small. Hence, the overshoot size is usually less than 0.1 dB in slave clocks for synchronization networks.

As far as loop stability is concerned, we note that a second-order phase-locked loop is stable for all values of KA, because all poles of the closed-loop transfer function $H(s)$ lie in the left half-plane for all values of KA (necessary and sufficient condition for system stability).

As far as the transient response is concerned, we remind that the measures T_r, T_s and M_p introduced previously can be estimated as functions of the crossover frequency ω_c and the phase margin ϕ_R. These two quantities are related to the natural frequency ω_n and to the damping ratio ζ. Leaving the proof to [5.16], in second-order PLLs the following approximated relationships hold

$$\omega_c \cong 2\zeta\omega_n \qquad \text{(active filter)}$$

$$\omega_c \cong 2\zeta\omega_n\left(1-\frac{\omega_n}{KA}\frac{1}{2\zeta}\right) \qquad \text{(passive filter)} \tag{5.55}$$

$$\zeta \cong \frac{\phi_R}{100} \quad \text{for } \zeta < 0.7 \tag{5.56}$$

with ϕ_R expressed in degrees (e.g., if $\phi_R = 60°$, then approximately $\zeta \cong 0.6$). For $\zeta > 0.8$, the damping ratio ζ significantly departs from $\phi_R/100$ and becomes much higher than it. As stated before for a generic PLL, we point out again that large phase margin (i.e., large damping ratio) means safe system stability and thus small peak overshoot.

5.5.3 Third-Order, Type-3 Phase-Locked Loop

Loops with type higher than 2 are sometimes used, because they feature improved tracking capability of high-dynamics phase fluctuations. We remind that a type-3 PLL, characterized by a loop filter $F(s)$ with two poles at $s = 0$, can track a frequency ramp on the input with zero steady-state phase error. Also in this section, the book by Meyr and Ascheid [5.16] has been followed to provide a simple overview on the topic.

Loop stability considerations yield that two zeros are needed in the open-loop transfer function if $F(s)$ has two poles at $s = 0$. Therefore, the open-loop transfer function is of the form

$$G_0(s) = \frac{KAF(s)}{s} = \frac{KA(1+sT_2)(1+sT_3)}{s(sT_1)^2} \tag{5.57}$$

and the closed-loop transfer functions result

$$H(s) = \frac{\hat{\theta}(s)}{\theta(s)} = \frac{\omega_c(s+\alpha\omega_c)(s+\alpha_1\omega_c)}{s^3+\omega_c(s+\alpha\omega_c)(s+\alpha_1\omega_c)}$$

$$1-H(s) = \frac{\phi(s)}{\theta(s)} = \frac{s^3}{s^3+\omega_c(s+\alpha\omega_c)(s+\alpha_1\omega_c)} \tag{5.58}$$

where

$$\omega_c = KA\left(\frac{T_2}{T_1}\right)\left(\frac{T_3}{T_1}\right), \quad \alpha = \frac{1}{\omega_c T_2}, \quad \alpha_1 = \frac{1}{\omega_c T_3} \tag{5.59}$$

Also in this case, ω_c is the crossover frequency.

As far as stability is concerned, system analysis reported in [5.16] shows that, while first- and second-order PLLs are unconditionally stable, a third-order, type-3 PLL is stable if the gain KA is larger than a minimum value, evaluated as

$$KA > \frac{1}{2T_2}\left(\frac{T_1}{T_2}\right)^2 \tag{5.60}$$

in the common case in which the two zeros are coincident ($T_2 = T_3$). Therefore, to ensure stability in practical systems, the input signal amplitude A must be kept above a minimum value by means of some automatic gain control system.

As far as the transient response is concerned, analysis carried out in [5.16] yields an interesting result: the transient response of a third-order type-3 loop is well approximated by that of a second-order type-2 loop having equal crossover frequency ω_c and phase margin ϕ_R. Actually, this is true also for systems of any order, provided that the response is primarily due to a *dominant* pair of complex poles in the closed-loop transfer function[6].

This result is useful in first-draft designing third-order loops by second-order approximation. Given for example settling time T_s and peak overshoot M_p, the designer can derive the corresponding second-order loop damping ratio ζ from numerical tables (an example graphical plot is reported in [5.16]). Using this value of ζ, as shown in [5.16], the open-loop transfer functions of a third-order type-3 PLL and of a second-order PLL, both characterized by same crossover frequency ω_c and phase margin ϕ_R, are given respectively by (frequencies normalized to ω_c)

$$G_{0/3}(s) = \frac{\left(\dfrac{s}{\omega_c} + \alpha\right)^2}{\left(\dfrac{s}{\omega_c}\right)^3}, \quad G_{0/2}(s) = \frac{\dfrac{s}{\omega_c} + \left(\dfrac{1}{2\zeta}\right)^2}{\left(\dfrac{s}{\omega_c}\right)^2} \tag{5.61}$$

with

$$\omega_c = 2\zeta\omega_n, \quad \zeta = \frac{1}{2}\sqrt{\frac{1-\alpha^2}{2\alpha}} \tag{5.62}$$

Therefore, using the value of ζ obtained to solve the equation above for α, the third-order loop transfer function can be readily designed from the second-order approximation.

5.5.4 PLL Performance with Input Additive Noise

Characterizing the PLL behaviour in the case the input signal is affected by some additive noise $n(t)$ is not a simple task, at least in the general case. Meyr and Ascheid in their

[6] As a rule of thumb, a *complex conjugate pair of poles* in the closed-loop transfer function $H(s)$ can be considered *dominant* if at least one of the following conditions is fulfilled:

- the absolute values of the real parts of the additional poles are at least 10 times larger than the absolute value of the real part of the dominant pole pair;
- in the case a pole is located near the dominant pair, there is a zero close to it, forming then a zero-pole doublet.

book [5.16] analyse this problem with the fairly general assumption of narrow-band, band-pass noise.

The problem can be greatly simplified by the further assumption that the input noise has small amplitude, so that the non-linearity in the phase detector model can be approximated by a linear characteristic, as in Equation (5.29). The linearized baseband model of the PLL with input additive noise is then shown in Figure 5.11. The only difference with the scheme of Figure 5.8 is the input noise that the PLL has to track, modelled by the additive noise term $n'(t)$[7] inserted *after* the phase-detector node, and some constant factors grouped and moved around the loop. Here, the VCO gain factor K_0 [$\mathrm{s^{-1}\,V^{-1}}$] and the phase detector gain $K_d = AK_mK_1$ [V] are used.

In this analysis, the noise $n(t)$ added on the input signal is assumed to be narrow-band, band-pass, Gaussian and zero-mean. A narrow-band band-pass noise process is defined in the frequency domain by the property that its spectrum is negligibly small everywhere except in the frequency range $|\omega - \omega_0| < \pi B_{IF} \ll \omega_0$, where ω_0 is the central frequency and B_{IF} is the equivalent noise bandwidth, which will be defined formally in the next paragraph. Then, the input additive noise $n(t)$ can be expressed as

$$\begin{aligned} n(t) &= \sqrt{2}n_c(t)\cos\omega_0 t - \sqrt{2}n_s(t)\sin\omega_0 t \\ &= \sqrt{2}\mathrm{Re}\{\vec{n}_L(t)e^{j\omega_0 t}\} \end{aligned} \tag{5.63}$$

where $n_c(t)$ and $n_s(t)$ are two zero-mean, stationary, Gaussian, base-band (i.e., slowly variant) random processes. The complex vector

$$\vec{n}_L(t) = n_c(t) + jn_s(t) \tag{5.64}$$

is called *complex envelope* of the noise process.

The equivalent noise bandwidth B_{IF} [Hz] is so defined that, if we approximate the noise power spectral density $S_n(\omega)$ by two rectangles for the positive and negative frequency axes of width $2\pi B_{IF}$ and amplitude $S_{n_c}(0)$, the area will be the same as the area under $S_n(\omega)$, i.e.

$$\int_{-\infty}^{\infty} S_n(\omega)d\omega = S_{n_c}(0)2\pi B_{IF} + S_{n_c}(0)2\pi B_{IF} \tag{5.65}$$

Hence,

$$B_{IF} = \frac{1}{2\pi}\int_0^{\infty}\frac{S_n(\omega)}{S_{n_c}(0)}d\omega \quad [\mathrm{Hz}] \tag{5.66}$$

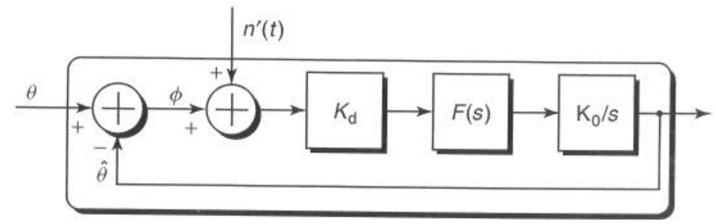

Figure 5.11. Linearized model of phase-locked loop with input additive noise

[7] The context should make clear that, in this case, the prime sign does not denote derivative.