L-21: Eigenvalues and Eigenvectors

Aahan Singh Charak
Computer Science Grad

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1 What are we looking at here?

Consider a matrix A. Treat it like a function which takes in an input vector x. The function performs matrix multiplication and regurgitates something out. Suppose we get a multiple of the input vector as the output of the function. It would look something like this:

$A.x = \lambda x$

Here x is the eigenvector of matrix A and λ is the eigenvalue for that eigenvector.

We will now discuss some examples to understand these concepts:

1.1 Singular matrix example

If A is a singular matrix, there exists an element in the nullspace of A which is non-zero. x is one of those elements. The eigenvalue in this case will be λ=0.

1.2 Projection matrix example

Suppose A has a column space which is an n-dimensional plane. Projection matrix P projects on this plane. Any vector x in the plane will give the vector when fed to the projection matrix. $\therefore P.x = 1.x$.

Here, eigenvector=x and eigenvalue=1.

Vector perpendicular to the plane will be another eigen vector for P, as $P.x = 0.x$

Here, eigenvalue=0 and eigenvector=x.

1 **Example 1**

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

What are the eigenvalues and eigenvectors

Sol: **(a)** x=[1,1] is an evector and $\lambda = 1$ is an evalue.
(b) x=[-1,1] is an evector and $\lambda = -1$ is an evalue.

1.3 Relation between eigen values and trace of a matrix.

$\sum \lambda = \sum A_{ii}$

Where, i is the row number in an n*n dimensional squared matrix.

2 Finding eigenvalues and eigenvectors

2.1 Eigenvectors

We know that,

$Ax = \lambda x$

$(A - \lambda I)x = 0$

If an eigenvector x exists, then A-λI must be singular. We can substitute λ to get a solution.

2.2 Eigenvalues

Since $A - \lambda I$ is singular, we can simply equate the determinant to 0 in order to find eigenvalues.

$\det Singular = 0$

$\det A - \lambda I = 0$

So, we first find eigenvalues and then find eigenvectors by substitution.

3 A good property

If we add nI to A, λ increases by n. Eigenvectors on the other hand, remain the same.

Proof:

If, $Ax = \lambda x$
Then, (A+nI)x=$\lambda x + nx$=$(\lambda + n)$x.

4 A property which doesn't exist tho.

Suppose, Ax=λx, and
Bx=αx

But, (A+B)x=$(\lambda + \alpha)x$ does not always exist.

x has to be the eigenvector for both matrices simultaneously for this property to exist. So, this property is not universal for all matrices A and B with eigenvectors and eigenvalues existing. Same can also be said for A.B

More examples can be seen in supplementary notes 1.