L-17: Orthogonal Matrices and Gram Schmidt

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1 Orthogonal Matrices

1.1 Orthonormal Vectors

Unit vectors which are prependicular to each other are called orthonormal vectors.

$$q_i{}^T.q_j = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

The self dot product is 1 as it is a unit vector. So length squared is zero.

1.2 What are orthogonal matrices?

Consider an n-dimensional vector space with n orthogonal basis $q_1, q_2, q_3 ... q_n$

The orthogonal matrices has these basis vectors in its columns.

$Q = [q_1\ q_2\ ....\ q_n]$

$$Q_T.Q = \begin{bmatrix} q_1{}^T \\ . \\ . \\ q_n{}^T \end{bmatrix} [q_1\ q_2\ ....\ q_n] = I$$

Solve the identity proof yourself.

1.3 Examples of orthogonal matrices

1)

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2)

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

3)

$$Q = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

4) Rectangular example.

$$Q = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 2 & -1 \\ 2 & 2 \end{bmatrix}$$

1.4 Some properties

Suppose Q is an orthogonal matrix with orthonormal vectors as its columns. Projection matrix for projecting a vector b onto its column space is given by:

$P = Q(Q^TQ)^{-1}Q^T$

$P = QQ^T$

Now, suppose Q is an orthogonal matrix. All columns are orthonormal and independent. This means in an n dimensional space, we get n pivots after elimination. So the column space will be the entire n dimensional space.

In this case P=I, as we won't have the need to project a vector as it would be already in the n-dimensional space. We can verify properties like $P^T = P \ and \ P^2 = P$.

1.5 Normal Equation

$A^TA\hat{x} = A^Tb$
$Q^TQ\hat{x} = Q^Tb$

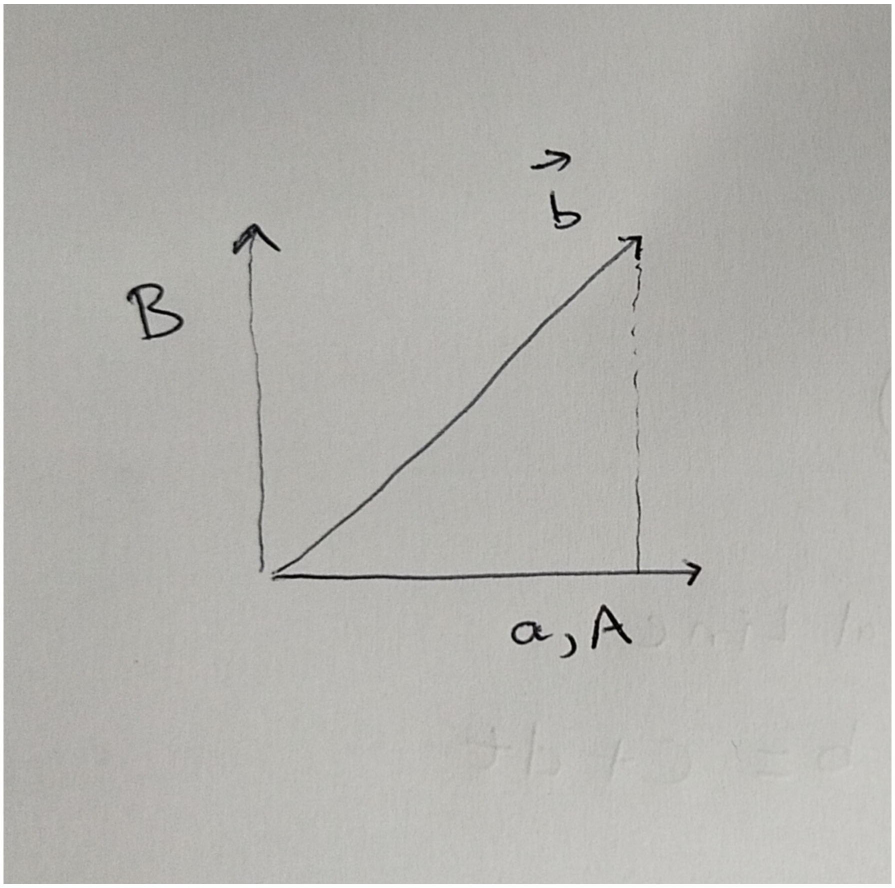
$\hat{x} = Q^Tb$

$\hat{x_i} = {q_i}^Tb$ [i=ith component]

2 Gram Schmidt Process

In mathematics, particularly linear algebra and numerical analysis, the Gram–Schmidt process is a method for orthonormalizing a set of vectors in an inner product space, most commonly the Euclidean space R^n equipped with the standard inner product.

2.1 Two dimensional intuition



We have to make both vectors orthogonal. Lets take a vector as reference. Let's say say vector A is equal to a.

Suppose B is perpendicular to A vector.

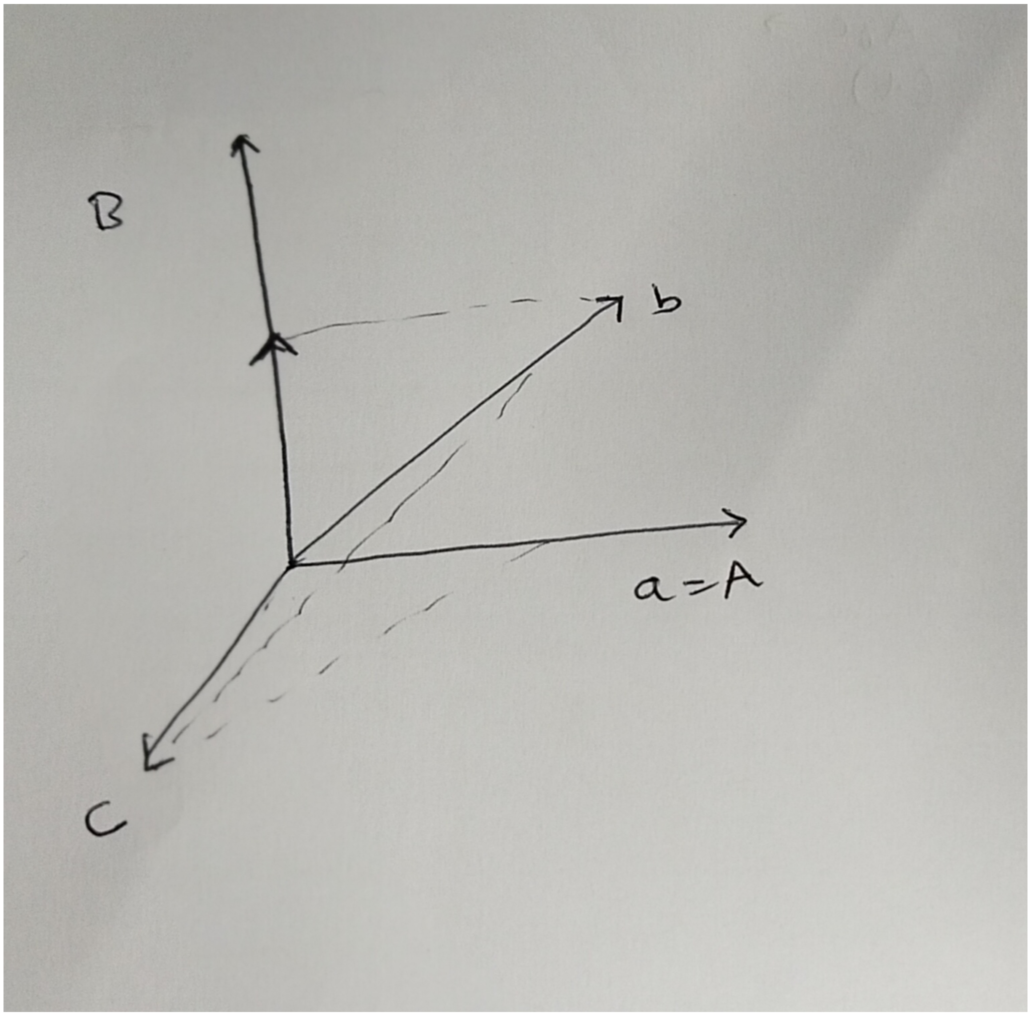
B is the error projection of b.
Therefore, B=e

B=b-e [parallelogram law]

B=$b - \hat{x}a$[Proved in projection onto subspace lecture]
Or
$B = b - \hat{x}A$

$\therefore B = b - \frac{A^T b}{A^T A} A$ [Gram's formula]

2.2 Three dimensional intuition



Here, $A \perp B \perp C$

Like 2d intuition, A=a and $B = b - \frac{A^T b}{A^T A} A$

For C, we have to think differently.

Projection of C on A is given by, $p_1 = \frac{A^T c}{A^T A} A$

Projection of C on B is given by, $p_2 = \frac{B^T c}{B^T B} B$
Now, these two vectors are perpendicular as A and B are perpendicular. They will become basis for a plane. A vector line on the plane would be linear combination of the above mentioned vectors, which is:

$p_p lane = p1 + p2$

Now, this is equivalent to projection of c on that plane. The perpendicular vector is given by:

$C = c - (\frac{A^T c}{A^T A} A) - (\frac{B^T c}{B^T B} B)$

3 Extra

3.1 Numerical Example

$$a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$Q = [q1, q2]$$

$$Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

3.2 Relation between C(Q) and C(S)

The column space of both matrices is the same. A and B are perpendicular. They form a 2d column space. It's just that Q matrix is more refined than the above mentioned vectors,as it contains unit vectors in the directions of A and B. Thus, it has basis for C(S).