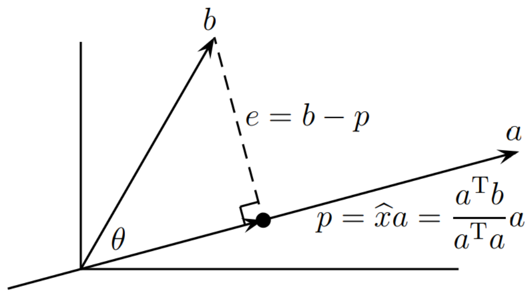
L-15: Projections Onto Subspaces

Aahan Singh Charak
Computer Science Grad

October 27, 2022

1 2d Projections



p = projection of b onto a

p=$\hat{x}a$

Also, a $\perp b \implies a \perp (b - \hat{x}a)$

$a_T(b - \hat{x}a) = 0$

$\therefore \hat{x}a^T a = a^T b$

$$\hat{x} = \frac{a^t b}{a^t a} \tag{1}$$

$\therefore p = a(\frac{a^T b}{a^T a})$

1 **Important Notes**

If b is doubled, projection is doubled too.

If a is doubled, nothing changes.(line onto which we are projecting remains the same.)

1.1 Projection Matrix

It is a matrix which acts on input b and projects it on a.

proj p=Pb

$$\therefore P = \frac{aa^T}{a^T a} \quad (2)$$

2 **Properties of projection matrix**

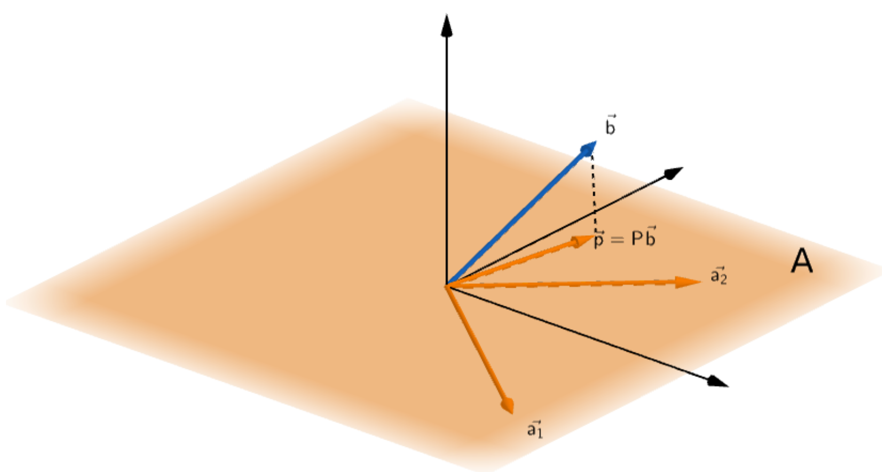
- C(P) is the line through a. According to projection formula, we multiply P with b. So, we are always getting a projection on a.
- r(P)=1 (col*row=rank 1 matrix, a and it's transpose)
- $P = P^T$
- $P^2 = P$ (self projection, obvious)

2 Why Project?

Suppose, Ax=b has no solution. This means that b doesn't lie in A's column space. So, now we project b onto C(A) using a projection matrix.

Therefore, $A\hat{x} = p$, has a solution because we projected b onto C(A). p is the best solution, as it it closest to b, due to being a perpendicular projection.

3 3d Projections



a1 and a2 are the two basis vectors for the plane. They are independent and their linear combinations span the entire 2d plane.

Plane is the column space of A.

A=[a1 a2]

Now, $e \perp C(A)$

p can be written as a linear combination of a1 and a2.

$\therefore p = a1\hat{x1} + a2\hat{x2}$

$\therefore p = A\hat{x}$

Problem is to find the right combinations of the columns of A so that error vector is perpendicular to the plane.We have to find $\hat{x}$

Now, $e \perp plane \implies e \perp a1 \; and \; e \perp a2$

$a1^T(b - A\hat{x}) = 0 \; and \; a2^T(b - A\hat{x}) = 0$

Writing this in matrix form, we get :

$$\begin{bmatrix} a1^T \\ a2^T \end{bmatrix} (b - A\hat{x}) = 0$$

Or

$A^T(b - A\hat{x}) = 0$

Now,

$A^T A\hat{x} = A^T b$

$\therefore \hat{x} = (A^T A)^{-1} A^T b$

$$p = A(A^T A)^{-1} A^T b \tag{3}$$

3.1 3d Projection Matrix

$$P = A(A^T A)^{-1} A^T \tag{4}$$

3 **Properties of 3d projection matrix**

- $P^T = P$
- $P^2 = P$

4 Proofs

4.1 e is perpendicular to C(A)

$A^T e = 0$

$\therefore e \in N(A^T)$

$Now, N(A^T) \perp C(A)$

Therefore, e is perpendicular to C(A).

Q.E.D

4.2 $P^2 = P$

$P^2 = A(A^T A)^{-1} A^T A (A^T A)^{-1} A^T$

$= A(A^T A)^{-1} I A^T$

$= A(A^T A)^{-1} A^T = P$

Q.E.D