L-14: Orthogonal Vectors and Subspaces

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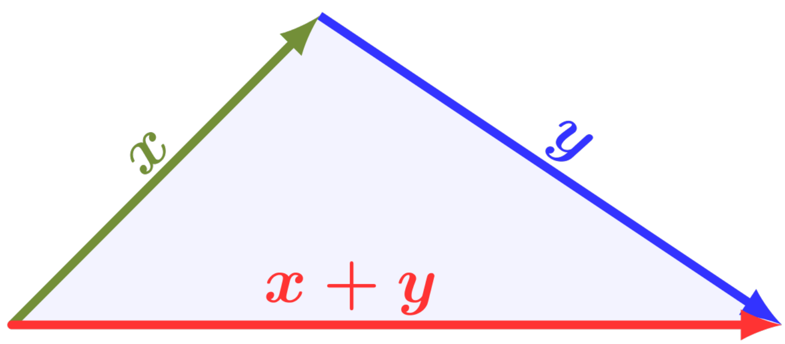
1 Orthogonal Vectors

1.1 What are orthogonal vectors?

Orthogonal vectors are vectors which lie at an angle of 90 degrees to each other. The dot product of such vectors is zero.

Consider two vectors at right angle to each other.



Now, according to pythagoras: $\|x\|^2 + \|y\|^2 = \|x + y\|^2$

This can be rewritten in vectoresque form as:

$x^T x + y^T y = (x + y)^T (x + y)$

$x^T x + y^T y = x^T x + x^T y + y^T y + y^T x$

$x^T y + y^T x = 0$

$2x^T y = 0 \implies x^T y = 0$

This proves that dot product is zero for orthogonal vectors.

1.2 Orthogonal Subspaces

Two subspaces S and T are orthogonal, if and only if all vectors in S are orthogonal to all vectors in T.

Note: If two subspaces overlap (excluding zero vector), they are not considered as orthogonal subspaces; a vector cannot to orthogonal to itself.

1.2.1 Row space is orthogonal to null space

Why?

$x \in n(A)$

$Ax = 0$

$$\begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{bmatrix} x = 0$$

Which means,
$row_1 * x, row_2 * x, row_3 * x ... row_m * x = 0$

So, we get

$(\alpha_1 * row_1 + \alpha_2 * row_2 + ... \alpha_m * row_m)x = 0$

Therefore,

$Col - Space * x \in n(A) = 0$

Q.E.D

1.2.2 Column space is orthogonal to null space of A^T

Similar proof, do yourself.

1 **Important Note**

Nullspace and rowspace are orthogonal complements in R^n.

2 Solving Ax=b when there's no solution

This problem might arise when there are more equations than the number of unknowns to be found. Bottom rows will be zero, as soon as we get n pivots. b might not have zeros in the last rows.

2.1 Useful matrix for this

Ax=b might not be solvable

$A^T A\hat{x} = A^T b$ will have a solution

We get this using projections. Next lesson.

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Important Notes

$N(A^T A) = N(A)$
$r(A^T A) = r(A)$
$A^T A$ is invertible only if A has independent columns.