Why does this formula work?

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1}$$

$C_{11} = ei - hf$ [(n-1) entries]

det A → product of n entries.
C^T → product of n-1 entries

Check :-

$$A A^{-1} = I$$

$$A C^T = \det(A) I$$

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} C_{11} & \cdots & C_{n1} \\ C_{12} & & C_{n2} \\ \vdots & & \vdots \\ C_{1n} & & C_{nn} \end{bmatrix} = \begin{bmatrix} \det A & & 0 \\ & \det A & \\ 0 & & \det A \end{bmatrix}$$

$$(\det A)_{11} = a_{11} C_{11} + a_{12} C_{12} \cdots a_{1n} C_{1n}.$$

For ~~2x~~ (2,2) position = $(\det A)_{22}$ [Similar for (n, n)]

Why are off-diagonal elements zero?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$A_s = \begin{bmatrix} a & b \\ a & b \end{bmatrix}$ → we are doing something similar to $|A_s|$

[Look at book]

$|A_s| = ab - ba$
$|A_s| = 0$

This is what's happening in row × column.

∴ $a_{11}C_{n1} + a_{12}C_{n2} \dots a_{1n}C_{nn}$ is similar to taking determinant of a screwed up matrix with ~~values~~ identical first and last rows.

∴ We proved the inverse formula.

Second application is to

$$Ax = b$$

$x = A^{-1}b$

$$\boxed{x = \frac{1}{\det A} C^T b}$$

CRAMER's RULE :-

$x_1 = \dfrac{\det B_1}{\det A}$ [we are multiplying cofactors wi (product cofactor rule). We set || of some matrix B_1]

$x_2 = \dfrac{\det B_2}{\det A} \quad \dots \quad x_j = \dfrac{\det B_j}{\det A}$

$B_1 = \begin{bmatrix} | & n-1 \\ b & \text{columns} \\ & \text{of} \\ | & A \end{bmatrix}$ {A with column 1 r with R.H.S}

$\boxed{B_j = A \text{ with column } j \text{ replaced by } b.}$

Try to use elimination instead of both these formulas.

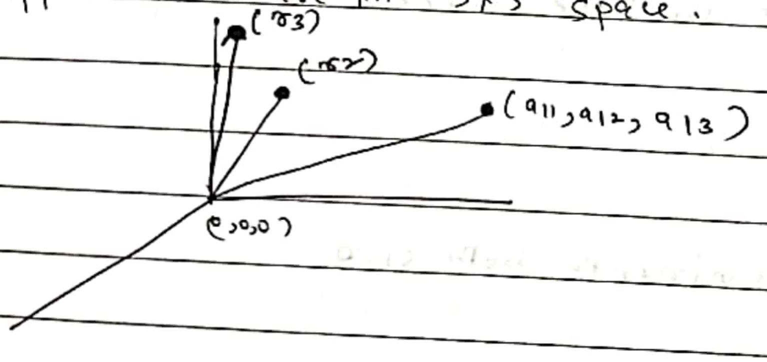
(more computation time)

Third Application =

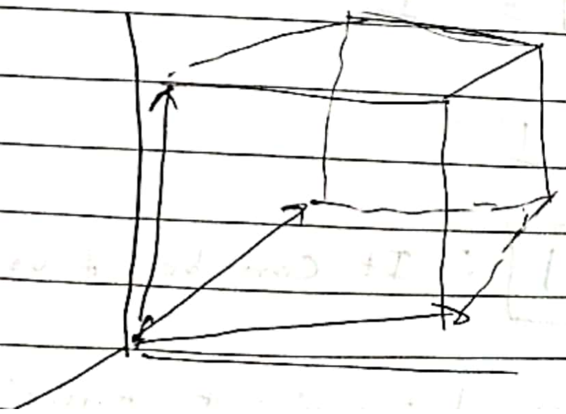
det A = volume of box

which box?

Suppose we are in 3×3 space.



The rows are co-ordinates of edge of box. (parallelopipe)



This volume is given by the determinant.

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Sign of determinant will tell right-handed box or left-handed box (cyclic order).

Special case

$A = I$

In this ~~as~~ case points are $(1,0,0)$ $(0,1,0)$, $(0,0,1)$

In this case we have unit cube.

Now suppose :-

$A = Q$ (orthogonal matrix : orthonormal columns)

$\det A = \det A^T$ (we can take colums or rows)

$\therefore$ ~~Ea~~

We will again have a cube unit.
But it will be turned in space.

$Q^T Q = I$

Take determinants both side.

$\det(Q^TQ) = 1$

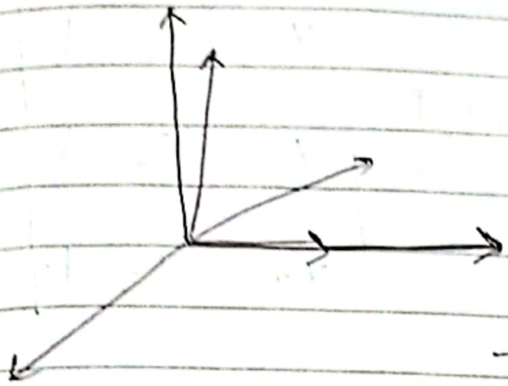
$|Q^T||Q| = 1$

$|Q|^2 = 1$

$\boxed{|Q| = \pm 1}$ $\therefore$ It can be 1 or -1.

[$\therefore$ For cubes determinant equals volume].

For non-cubes :



Suppose we double the edge of the cube. Volume also doubles.

$\Downarrow$

If we double first row of matrix, determinant also doubles.
[Rule 3 (a)]

Volume satisfies rule 3(a)

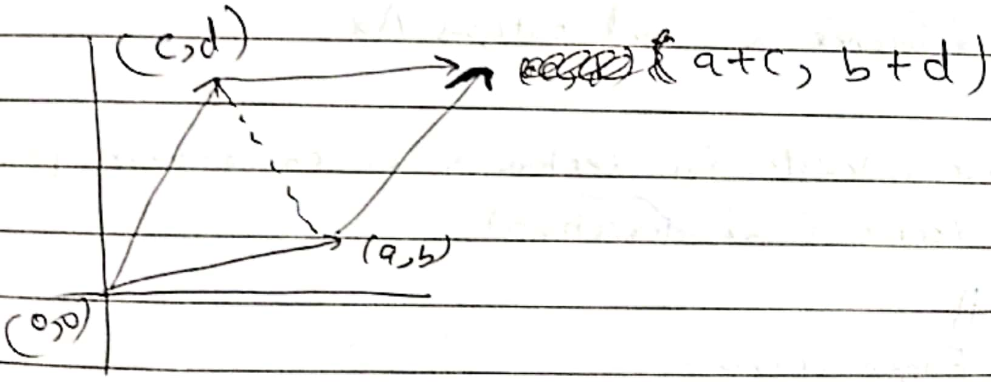
$|\det A| =$ volume of box

Properties {1✓, 2✓ (abs sure), $3a$✓, $\underline{\underline{3b}}$}

Proof 3(b) :-

Consider 2×2

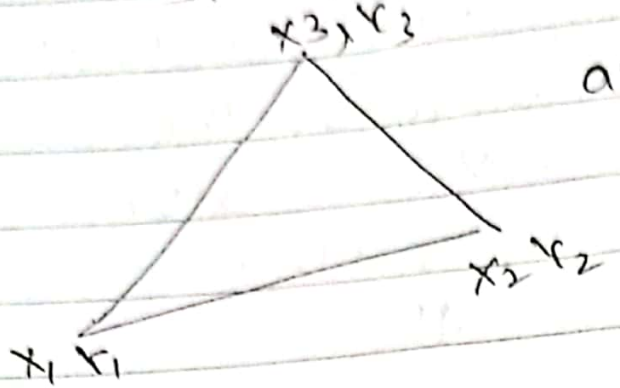
$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$



$$\text{area} = \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$\boxed{\text{For } \triangle\text{le} = \frac{1}{2}(ad-bc)}$

Suppose trigle does not start from orig.



$$\text{area} = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

\# **Note :-**

We might perform $r_2 \to r_2 - r_1$ and $r_3 \to r_3 - r_2$ to kill one's. This is similar to shifting the triangle to origin.