$$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

$= \cancel{c \times a} + a \times d + \cancel{0 \times d} + \cancel{ax}$.

Singular matrix $(\det = 0)$

$$= ad + \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix}$$

or exchange rows

$$= ad + (-1)\begin{vmatrix} c & 0 \\ 0 & b \end{vmatrix}$$

$$= ad - bc$$

Q.E.D

[# Method :-
(a) Take each row at a time and ~~form~~ split row into n (no of col's) pieces].

$\therefore$ For $\mathbb{R}^{3\times3}$

1 row $\to$ 3 pieces $\to$ 3 pieces (9 pieces) (2nd row) $\to$ 3 pieces (27 pieces) (3rd row).

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \text{How many non-zero col matrix.}$$

$\Downarrow$

e.g

$$\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}$$

Idea :-

[one entry from each row and column]

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$$= a_{11} a_{22} a_{33} + (-1) \begin{vmatrix} a_{11} & 0 & 0 \\ 0 & -a_{32} & 0 \\ 0 & 0 & a_{23} \end{vmatrix}$$

$$= a_{11} a_{22} a_{33} - a_{11} a_{32} a_{23}$$

$$= a_{11} (a_{22} a_{33} - a_{32} a_{23})$$

+

other columns.

$$\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$= -a_{21} \times a_{12} \times a_{33} + a_{31} \times a_{12} \times a_{23} \text{ (two flips)}$$

$$= a_{12} (a_{23} a_{31} - a_{21} a_{33})$$

+

$$\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}$$

$$= + a_{32} \times a_{21} \times a_{13} - a_{31} \times a_{22} \times a_{13}$$

$$= a_{13} (a_{32} a_{21} - a_{31} a_{22}).$$

$$\therefore |A| = a_{11} (a_{22} a_{33} - a_{32} a_{23}) - a_{12} (a_{21} a_{33} - a_{23} a_{31}) + a_{13} (a_{32} a_{21} - a_{31} a_{22})$$

General Formula :

(for $n \times n$ dimensional matrix)

$$\det A = \sum_{n!\ \text{terms}} \pm$$

~~2x2 → 2 terms (ad − bc)~~
~~3x3 →~~

terms

$2\times2 \Rightarrow 2$ term (ad − bc)

$3\times3 \Rightarrow 6$ (without simplification)

$4\times4 = 4! = 24$ terms.

$\therefore n\times n = n!$ terms.

(half plus, half minus).

Proof :

1st element → n ways (fix column)

2nd element → $(n-1)$ ways.

$\vdots$

n^{th} element → 1 way.

$\therefore$ Total ways $= n \times (n-1) \times (n-2) \cdots 1$

$= n!$ ways.

$$\boxed{\therefore \det A = \sum_{n!\ \text{permutation}} \pm a_{1\alpha} a_{2\beta} a_{3\gamma} \ldots a_{n\omega}}$$

[$\alpha, \beta, \gamma, \ldots \omega$ = Some permutation of 1 to n column.]

#

$a_{11}, a_{12}, a_{13} = \alpha = 1, \beta = 2, \gamma = 3$

Perm = [1, 2, 3]

Example :-

$\mathbb{R}^{2\times 2}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \times d + b(-c)$$

$$= ad - bc$$

example :- Tridiagonal matrix.

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$|A_1| = 1, \quad |A_2| = 0, \quad |A_3| = 0 - 1 + 0$
$= -1$

$|A_4| = 1 \cdot |A_3| + 1(-|A_3|) + 0 + 0$

$= 1 \times (-1) + 1(-[0 + 0 + 0])$
$= -1$

$|A_4| = 1 \cdot |A_3| - 1(|A_2|)$ → only with non-zero elements.

$\cancel{|A_n| = |A_{n-1}|}$

$\boxed{|A_n| = |A_{n-1}| - |A_{n-2}|}$ → True for all.

$\boxed{|A_4| = -1}$, $\boxed{|A_5| = 0}$, $\boxed{|A_6| = 1}$,

$|A^7| = 1$

Series.

$1, 0, -1, -1, 0, 1,$ | 1 → repeating afterwards

Period = 6 (after which start repeating)