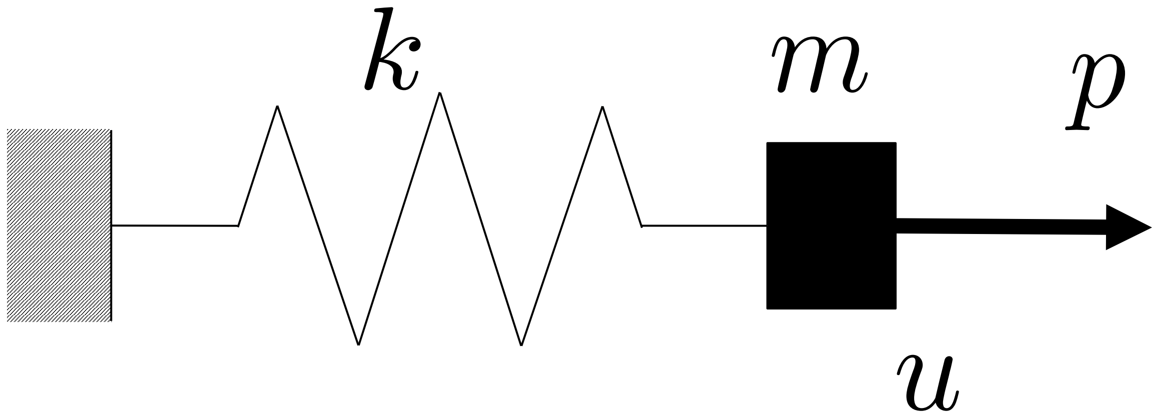
# Resonance Frequency

Philippe Büchler
Can Gökgöl

Institute for Surgical Technology and Biomechanics
University of Bern, Switzerland

# Tacoma Narrows Bridge

# Natural frequencies




- Equation of motion:

$$m\ddot{u} + ku - p = 0$$

- When load is release, natural frequency:

$$\omega = \sqrt{\frac{k}{m}}$$

# Natural frequencies

- Large structure have many natural frequencies. The dynamic response based on natural frequencies
- Equation of motion

$$M\ddot{u} + I - P = 0$$

- For an undamped system: $I = Ku$
- Solutions to this equation have the form: $u = \phi e^{i\omega t}$
- Substituting this into the equation of motion yields the eigenvalue problem:

$$K\phi = \omega^2 M\phi$$

$$M = LL^T$$

$$L^{-1}KL^{-T}\phi = \omega^2\phi$$

- This system has n eigenvalues, where n is the number of degrees of freedom in the finite element model.

# Two possible techniques



$$L^{-1}\mathbf{K}\mathbf{L}^{-T}\phi = \omega^2\phi$$

↓

Calculate the eigenvalues and vectors of the matrix

**A**

$$M\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} - P = 0$$

↓

Solve the equation of motion (including inertia)

**B**

# Eigenvalue extraction (A)

UNIVERSITÄT BERN

- The problem can be written as:

$$L^{-1}KL^{-T}\phi = \omega^2\phi$$

$$A\phi = \lambda\phi$$

$$(A - \lambda I)\phi = 0$$

- The linear system of equation only has a solution if the matrix coefficient is invertible;

$$\det(A - \lambda I) \neq 0$$

- Solving this equation gives the eigenvalues will give the eigenvalues of *A*. It is a polynomial function of $\lambda$ of degree n.
- For each eigenvalue, the corresponding eigenvector can be calculated as the solution of the linear system:

$$(A - \lambda_i I)X = 0$$

# Explicit time integration (B)

UNIVERSITÄT BERN

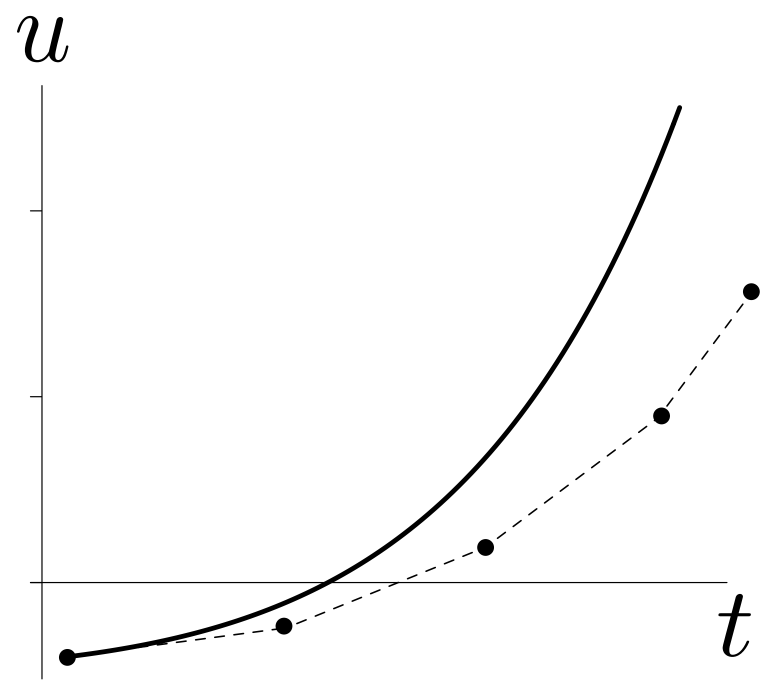
- Explicitly for dynamic situation
  - High speed problems
  - Complex contact problems
  - Material failure
  - …

$$M\ddot{u} + Ku - P = 0$$

$$\ddot{u} = M^{-1}(P - Ku)$$

$$\dot{u}_{t+\Delta t} = \dot{u}_t + \ddot{u}\,\Delta t$$

$$u_{t+\Delta t} = u_t + \dot{u}\,\Delta t$$




- Conditionally stable:

$$\Delta t_{\text{stable}} = \frac{L^e}{c_d} \qquad c_d = \sqrt{\frac{E}{\rho}}$$

# Implicit time integration



- Implicit solver for smooth problem
  - For quasi-static problems

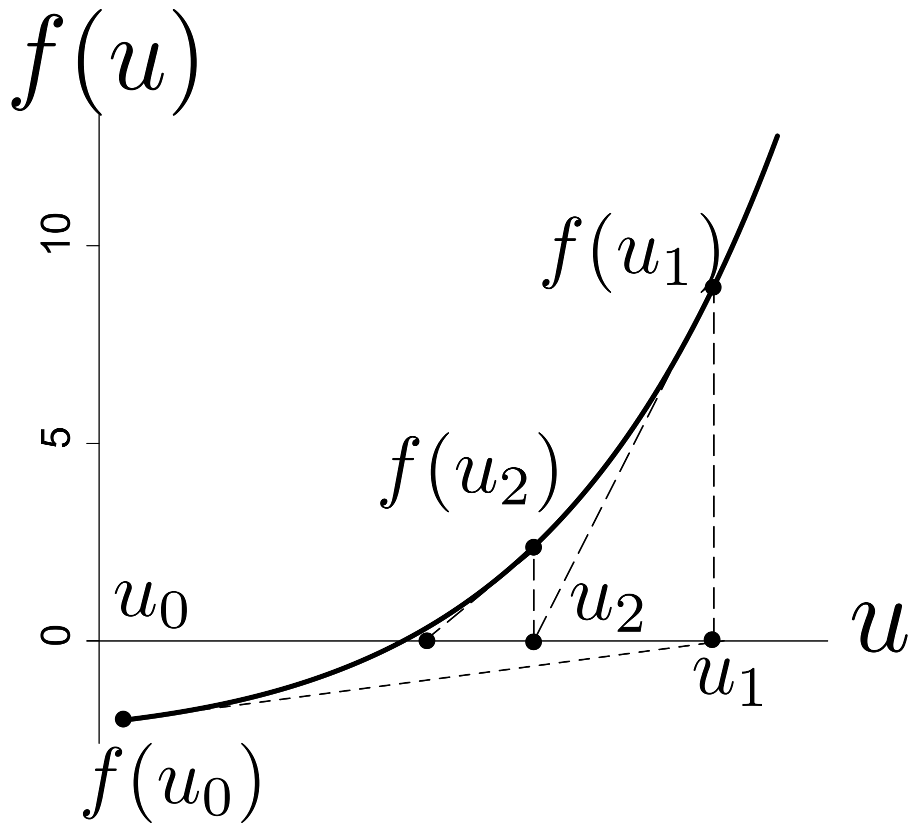
$$Ku - P = 0$$

  - Newton solver

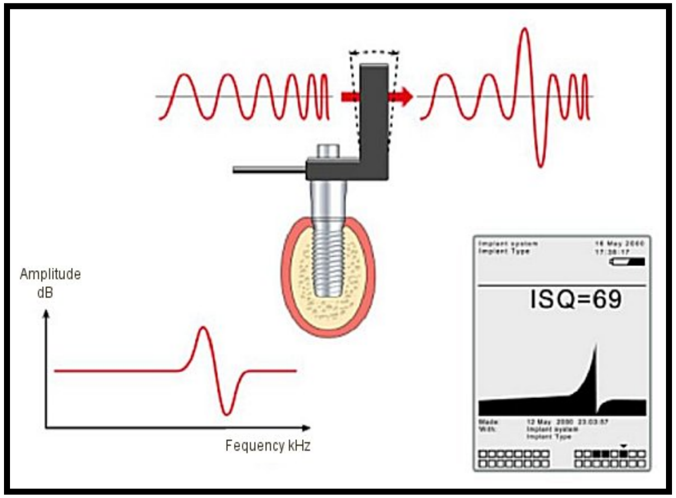
$$f(u) = Ku - P = 0$$

$$f(u) = -e^u u + 2 = 0$$

  - Unconditionally stable



# Application to dental implants



# Assignment – Natural frequency of a beam



- Find the first natural frequencies of a simple beam with two different methods
  - Analytical solution
  - Extraction of the eigenvalues
  - Explicit calculation of the motion and signal processing