

# IRTForests

Andrew T. Scott, Fall 2025

[github.com/ascott02/IRTForests](https://github.com/ascott02/IRTForests)

# Item Response Theory + Random Forests

- Trees become respondents, images become items.
- Response matrix records per-tree correctness on held-out examples.
- Goal: explain RF behavior via IRT ability & difficulty signals and vice versa.

# Motivation

- Fascinated by IRT for its **interpretative power**
- Meaning emerges only through a **cohort** — interactions matter
- Reminds me of **ensemble methods**, esp. **Random Forests**
- Both derive insight from **collective behavior**
- **Hypothesis:** Random Forests  IRT — shared latent structure

# Agenda

- Background: IRT + RF primers
- Pipeline: datasets, embeddings, and response matrices
- Case studies: CIFAR (PCA), CIFAR (MobileNet), MNIST
- Cross-study comparison, 2PL/3PL updates, takeaways, next steps

# Item Response Theory (IRT) (Wilson, 2005)

Why? Because performance != ability — but they're related.

- Classical Test Theory (CTT) tells us *how someone did on this test*.
- IRT models *how someone would perform on any set of items that measure the same underlying ability*.
- IRT doesn't replace CTT, it generalizes it with **portable, interpretable measurements** of capability.

CTT	IRT
Measures perf. on specific test	Estimates underlying ability
Test = sample of items	Items = samples from a calibrated continuum
Precision assumed constant	Precision varies with ability
Great for grading	Great for understanding and interpretability

A joint calibration framework where ability and difficulty are inferred together, each defined only in relation to the other.

It's less like grading individuals and more like synchronizing clocks — each calibrated against the ensemble.

# Item Response Theory Building Blocks

## Core Terms

- Ability ( $\theta$ ): respondent skill; higher → higher success odds (1PL).
- Difficulty ( $\delta$ ): item hardness; higher → harder even for strong respondents (1PL).
- Discrimination ( $a$ ): slope near  $\delta$  (2PL).
- Guessing ( $c$ ): floor for multiple-choice exams (3PL).

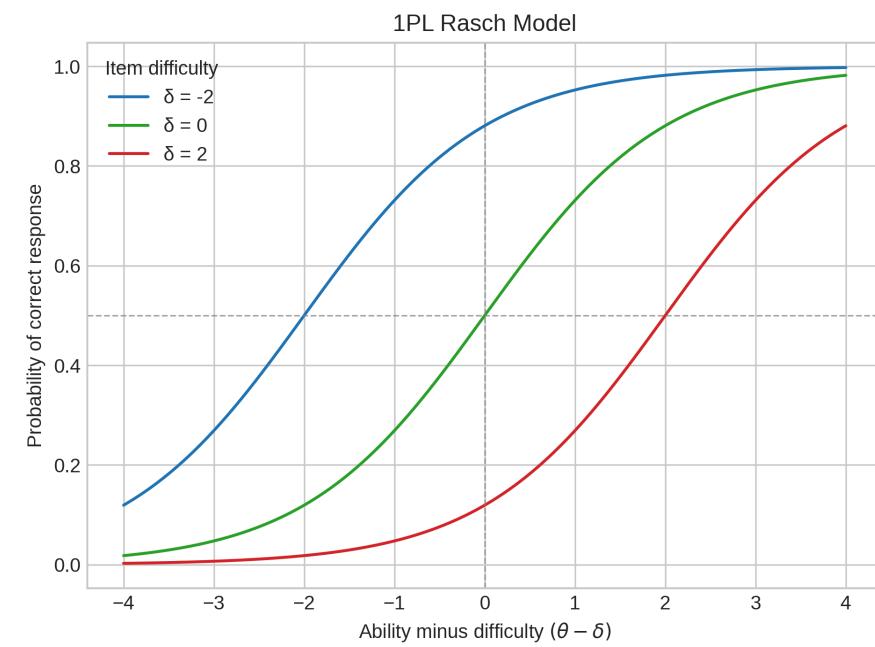
## Forest Analogy

- Respondents → decision trees on a shared test set.
- Items → images; responses are binary (tree correct?).
- Response matrix  $R_{ij} \in \{0, 1\}$  feeds variational IRT.
- Outputs: posteriors over  $\theta_i$ ,  $\delta_j$ , and information curves.

# Rasch (1PL) Model in One Picture

$$\Pr(R_{ij} = 1 \mid \theta_i, \delta_j) = \frac{1}{1 + e^{-(\theta_i - \delta_j)}}$$

- The probability a respondent gets the item correct, given their ability, and the item's difficulty.
- Single global slope keeps parameters on a shared logit scale.
- $\theta - \delta = 0 \Rightarrow 50\%$  success; shifts left/right change odds.
- Fisher information peaks where curves are steepest.
- See [IRT ICC Visualizer](#) for 2PL, 3PL, 4PL



1PL Item Characteristic Curves (ICC)

## IRT Output

- Ability histograms flag low-skill trees worth pruning.
- Difficulty ladders highlight mislabeled or ambiguous items.
- Wright maps overlay  $\theta$  and  $\delta$  to expose coverage gaps.
- Information curves reveal where ensemble confidence is fragile.
- Together they explain *who* struggles and *why*, beyond RF metrics.

## Random Forests — Many Noisy Trees, One Stable Voice (Breiman, 2001)

- Train trees on bootstrapped samples with random feature subsets to decorrelate their votes.
- Aggregate those votes by majority (classification) or mean (regression) to cut variance.
- **Margin:** gap between the correct class and the runner-up; **entropy:** dispersion of votes.
- Reading the two together exposes how confident—or conflicted—the forest is, especially once aligned with  $\delta$ .

## Random Forest Margins — How Confident Is the Crowd?

$$\text{margin}(x_i) = P_{\text{correct}}(x_i) - \max_{j \neq \text{true}} P_j(x_i)$$

The **margin** measures how far ahead the correct class is over its nearest competitor.

- **High margin:** trees vote strongly for the right class → confident.
- **Low or negative margin:** trees disagree or favor another class → uncertain.

Think of it as the *vote gap* in an election — the wider the gap, the clearer the win.

## Ensemble Entropy — How Much Do Trees Disagree?

$$H(x_i) = - \sum_j P_j(x_i) \log_2 P_j(x_i)$$

The **entropy** measures how dispersed the votes are across classes.

- **Low entropy:** trees nearly unanimous → decisive prediction.
- **High entropy:** votes spread out → uncertainty or class confusion.

Within trees, entropy drives splits (purity).

Across trees, entropy reveals disagreement — the forest's collective uncertainty.

## GenAI in the Loop Scientific Experimentation

- Recursive prompting (akin to [context engineering](#)) keeps each iteration scoped.
- Ground every cycle in the `README.md` spec—goals, datasets, diagnostics.
- Automate the CLI so runs regenerate figures and tables straight into the deck.
- Commit, push, repeat: [github.com/ascott02/IRTForests](https://github.com/ascott02/IRTForests)

Plastic tubes and pots and pans

Bits and pieces and the magic from the hand - Oingo Boingo, "Weird Science" 1985

# Pipeline Overview

## Data preparation for three studies

1. Stratified CIFAR-10 subset: 10k / 2k / 2k splits. Resize 64×64, normalize, PCA → 128-D embeddings.
2. Stratified CIFAR-10 subset: 10k / 2k / 2k splits. Resize 64×64, normalize, MobileNet → 960-D embeddings.
3. MNIST mini: 4k / 800 / 800 digits, normalized 28×28 grayscale. Raw pixels.

## Random forest training

- RF (2000 trees) trained for every study; metrics and importances saved.
- Response matrices saved: CIFAR  $(2000 \times 2000)$  for PCA & MobileNet, MNIST  $(2000 \times 800)$ .

## IRT analysis

- 1PL Rasch (SVI, 600 epochs) complete for CIFAR+PCA, CIFAR+MobileNet, and MNIST.
- 2PL (SVI, 800 epochs) complete for CIFAR+PCA, CIFAR+MobileNet, and MNIST.
- 3PL (SVI, 1000 epochs) CIFAR MobileNet only.

## Datasets Overview

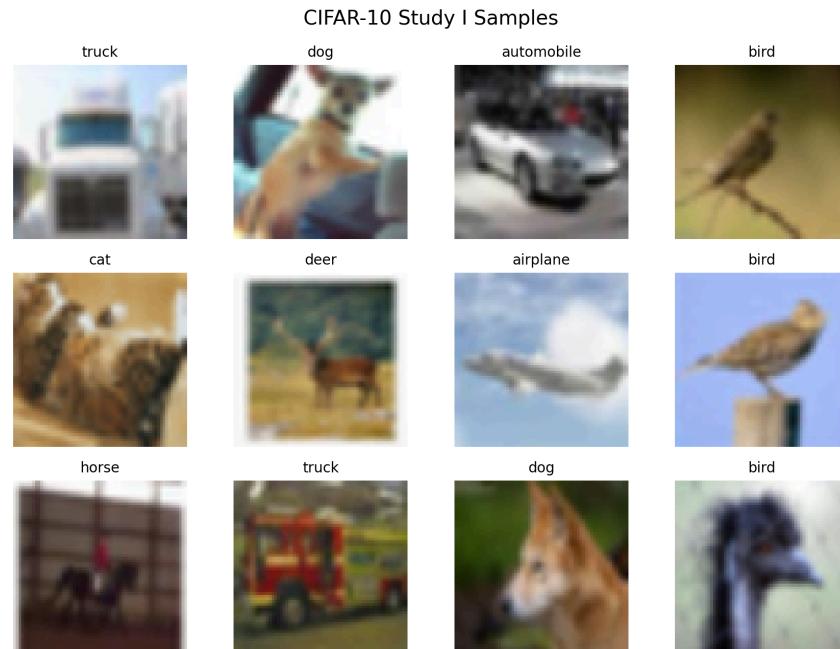
Dataset	Train	Val	Test	Feature Pipeline	Notes
CIFAR-10 subset	10,000	2,000	2,000	PCA-128 / MobileNet-V3 (960-D)	Shared splits Study I & II
MNIST mini	4,000	800	800	28×28 grayscale → raw pixels (no PCA)	Control for clean handwriting

- CIFAR runs differ only by embeddings; labels and splits stay fixed.
- MNIST mirrors the workflow to confirm signals on cleaner data.

# **Study I: CIFAR-10 + PCA-128 Embeddings**

# Study I Setup: CIFAR-10 + PCA-128

- Establish the PCA baseline and capture RF uncertainty signals.
  - Use IRT to pinpoint weak trees and hard items that motivate stronger features.
  - Fix a stratified CIFAR-10 split (10k / 2k / 2k).
  - Train 2000 trees and score them on the shared test set.
  - Build a  $2000 \times 2000$  response matrix (mean tree accuracy  $\approx 0.18$ ).
  - Artifacts: metrics, margins, entropy, IRT outputs.



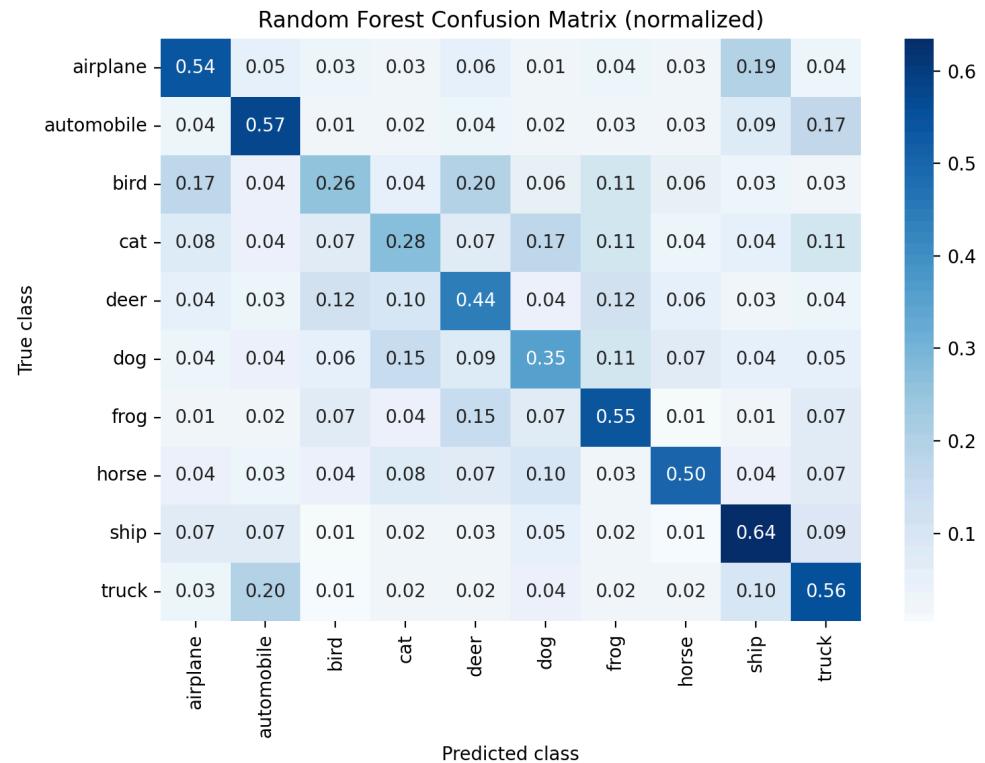
## Study I sample grid — stratified CIFAR-10 slices

## Study I Performance (PCA-128)

Metric	Value
Test / Val / OOB acc	0.468 / 0.470 / 0.442
Per-class range	0.260 (bird) → 0.635 (ship)
Mean tree accuracy	0.1763
Mean margin / entropy	0.0058 / 2.1723
$\delta$ negatively correlates with margin (Pearson)	-0.815
$\delta$ positively correlates with entropy (Pearson)	0.687

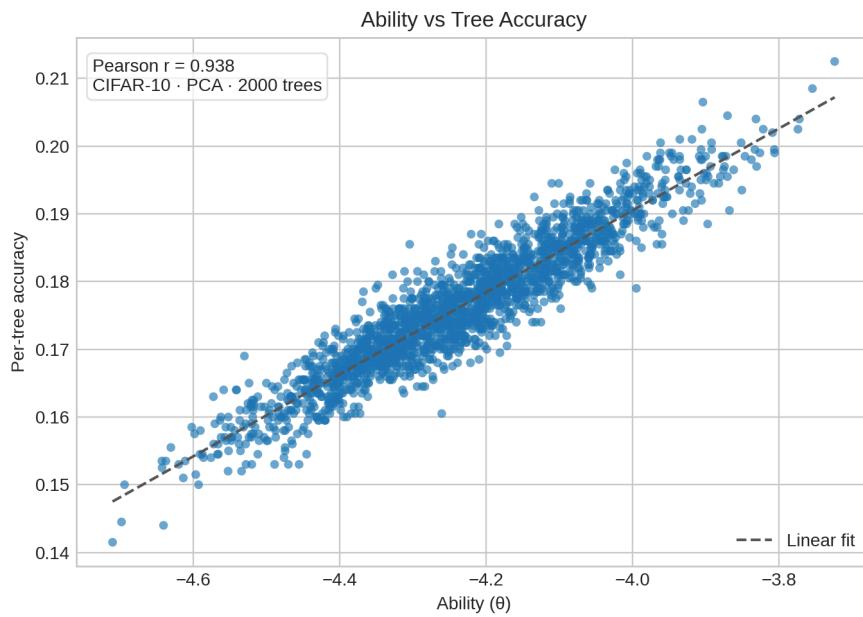
- Baseline ensemble still underperforms due to weak PCA features yet preserves  $\delta$  alignment.
- Margins hover near zero (mean ≈0.006) and entropy stays high (2.17), signalling broad disagreement—prime for IRT.

# Study I Confusion Matrix

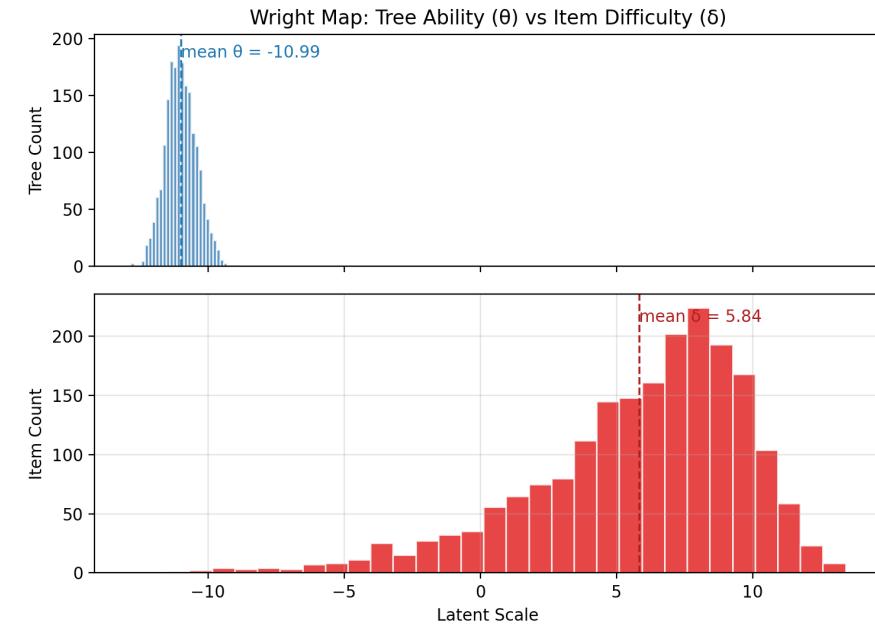


- Off-diagonal spikes (cat vs dog, bird vs airplane, horse vs deer) mirror high- $\delta$  items.
- Ships and trucks still lead the diagonal ( $\approx 64\% / 56\%$  accuracy), yet well short of a clean block —further underscoring the curation need.

# Study I Diagnostics: Ability Profiles



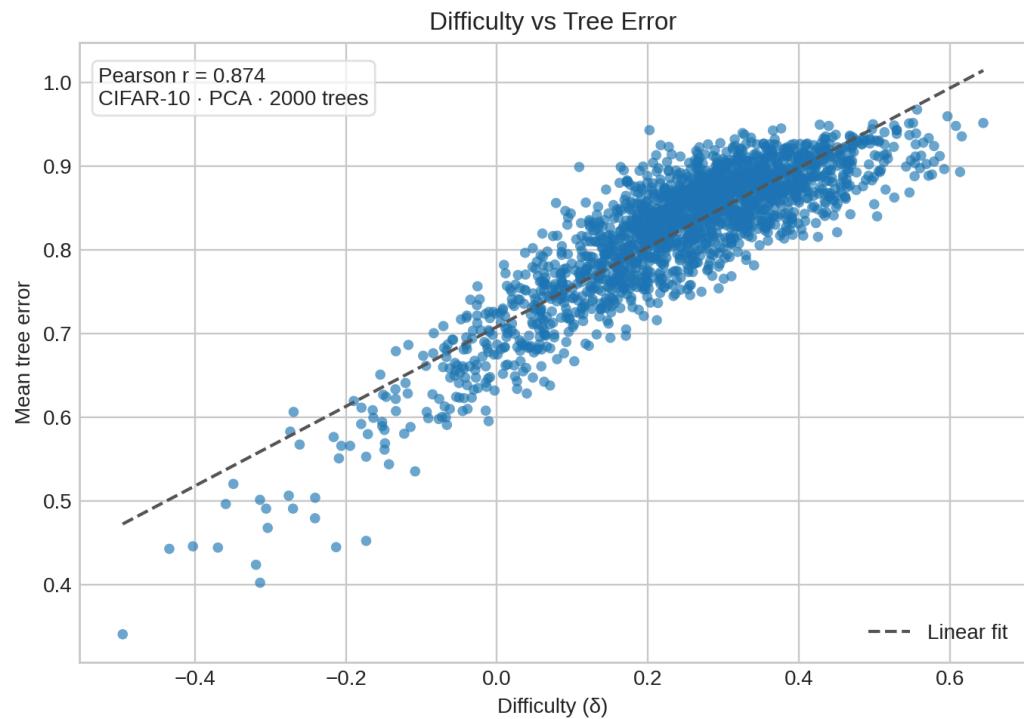
Ability ( $\theta$ ) vs tree accuracy — Spearman  $\approx 0.99$



Wright map:  $\theta$  mean  $\approx -11.0$  ( $\sigma \approx 0.56$ );  $\delta$  mean  $\approx 5.8$  with a wide tail

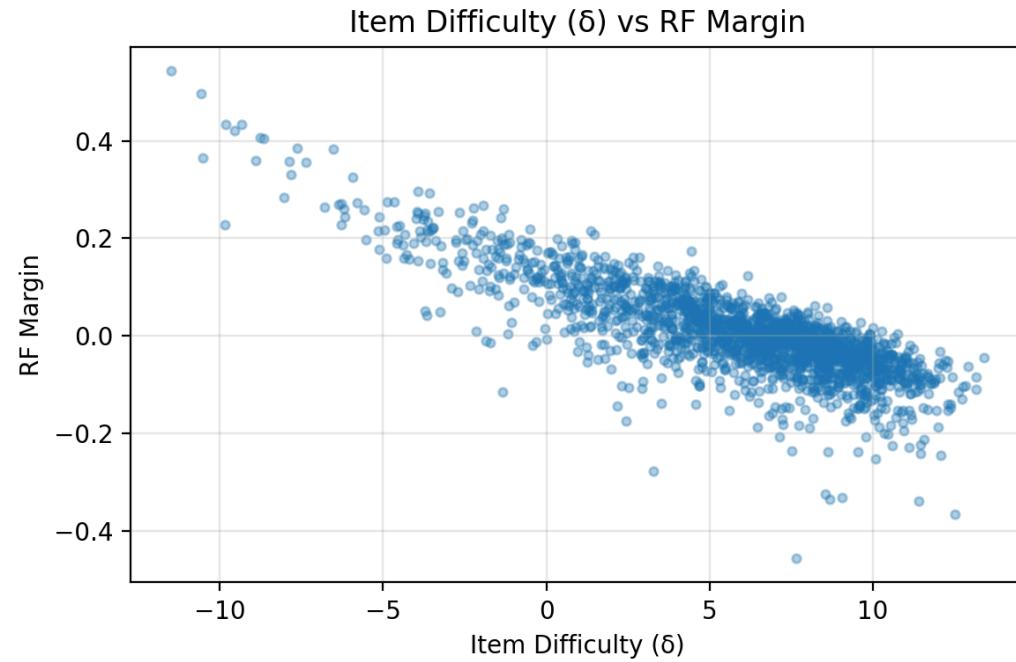
- $\theta$  ranges from about  $-12.8$  to  $-8.9$  (mean  $\approx -11.0 \pm 0.56$ ), so even small shifts separate stronger trees by a few percentage points.
- $\delta$  centers near  $5.8$  but stretches from roughly  $-11.5$  to  $13.4$ , highlighting how ambiguous animal items sit far from the easy tail.

# Study I Diagnostics: $\delta$ vs Error Rate

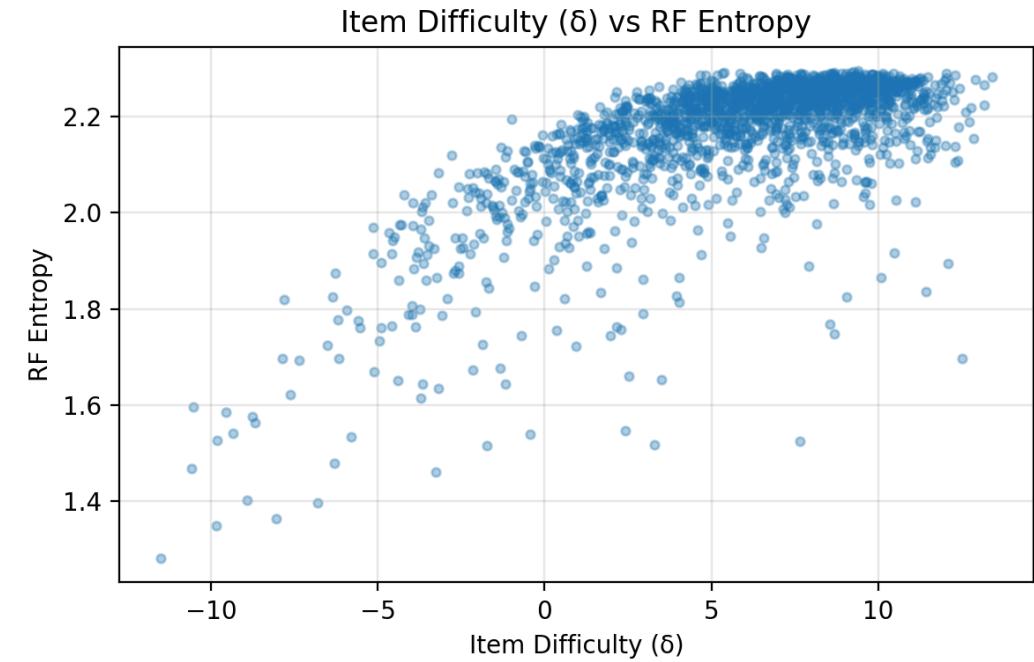


- $\delta > 0.4$  maps to >80% tree error—mostly ambiguous animals — while  $\delta < -0.3$  are trivially easy.
- Pearson  $\approx 0.87$ , Spearman  $\approx 0.82$ : difficulty doubles as an error heat-map.

# Study I Diagnostics: $\delta$ vs RF Signals



PCA run:  $\delta$  vs margin (Pearson -0.82)



PCA run:  $\delta$  vs entropy (Pearson 0.69)

- Hard items cluster bottom-right (low margin, high entropy); opposite corner houses easy wins.
- Study II mirrors the trend with even stronger correlations.

# Study I Evidence: Hard vs Easy Examples



- Hardest items skew toward ambiguous airplane/ship silhouettes and cluttered cat/dog scenes.
- Easy set is dominated by high-contrast cues (e.g., red fire trucks), yielding low  $\delta$  and entropy.

# Study I Fit Checks & Edge Cases

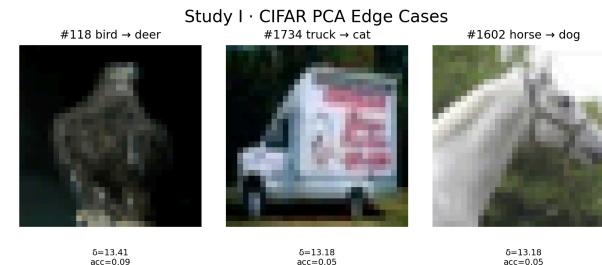
## Fit diagnostics

Metric	Value
Item infit $\mu$ / p95	0.18 / 0.35
Item outfit $\mu$ / p95	0.18 / 0.34
Tree infit $\mu$ / p95	0.35 / 0.48
Tree outfit $\mu$ / p95	0.18 / 0.19

- MSQs well below 1 show tree responses are steadier than a pure Rasch prior;  $|z|$  never exceeds 0.05.

## Edge cases worth a look

- #118 bird → deer votes ( $\delta \approx 13.4$ , margin  $\approx -0.05$ , entropy  $\approx 2.28$ ).
- #1734 truck → cat/frog split ( $\delta \approx 13.2$ , margin  $\approx -0.09$ , entropy  $\approx 2.27$ ).
- #1602 horse → dog/horse tie ( $\delta \approx 13.2$ , margin  $\approx -0.11$ , entropy  $\approx 2.22$ ).
- Each item sits below 9% tree accuracy—prime targets for relabeling or curated augmentations.



Study I edge cases · IDs 118, 1734, 1602

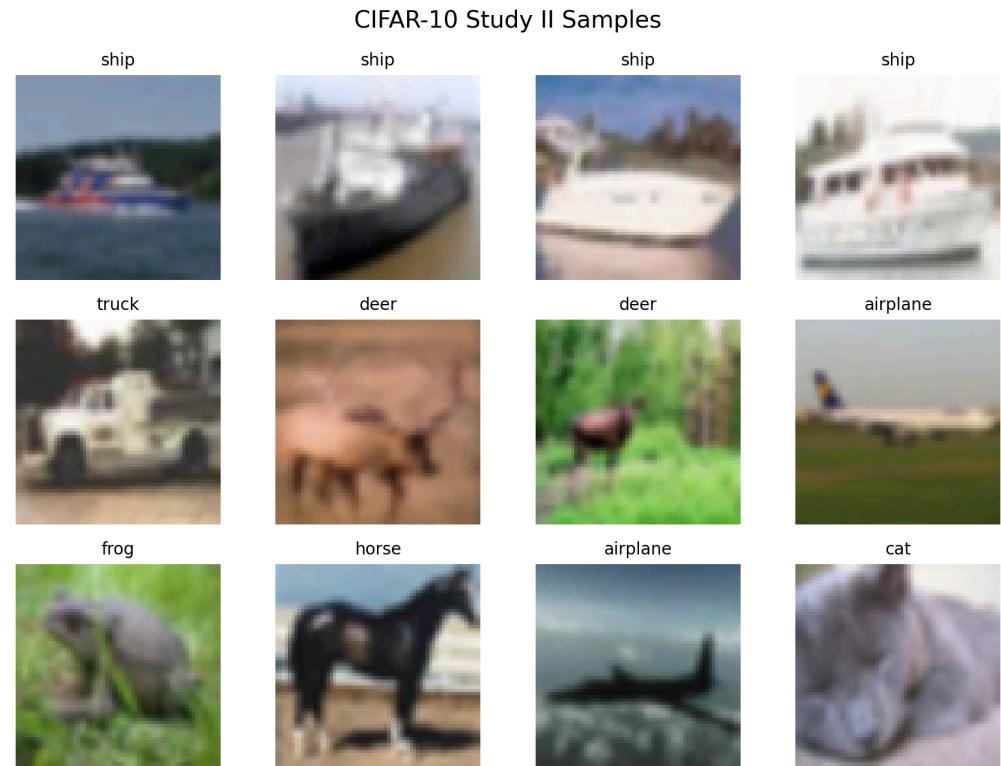
## Study I Takeaways

- Weak PCA features create long tails in both ability ( $\theta$ ) and difficulty ( $\delta$ ), exposing erratic trees.
- Margin and entropy correlate with  $\delta$ , but clusters of high-difficulty animals persist across diagnostics.
- Visual inspection confirms mislabeled or low-signal items driving high  $\delta$ , motivating feature upgrades.

## **Study II: CIFAR-10 + MobileNet Embeddings**

## Study II Setup: CIFAR-10 + MobileNet-V3

- Hold the splits fixed to isolate feature gains.
- Swap PCA features for MobileNet-V3 (960-D) while keeping tree count and splits constant.
- Test whether richer embeddings tighten  $\theta$  spread and retain  $\delta$  alignment.
- Compare RF metrics, uncertainty signals, and IRT parameters against the baseline.



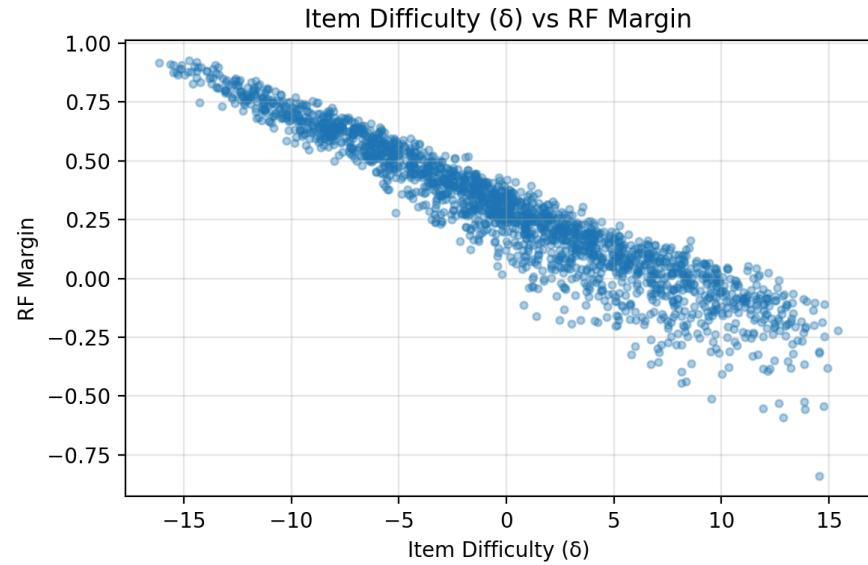
Study II sample grid — same splits, MobileNet embeddings

## Study II Performance (MobileNet-V3)

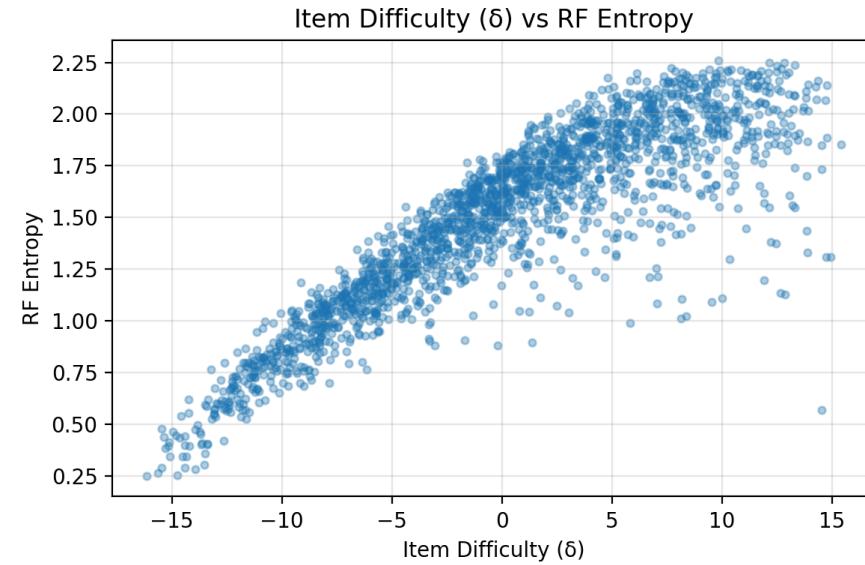
Metric	Value
Test / Val / OOB acc	0.819 / 0.820 / 0.812
Per-class range	0.695 (bird) → 0.925 (ship)
Mean tree accuracy	0.4792
Mean margin / entropy	0.2806 / 1.4929
$\delta$ negatively correlates with margin (Pearson)	-0.950
$\delta$ positively correlates with entropy (Pearson)	0.881

- Pretrained features boost accuracy by 35 pp while strengthening  $\delta$  correlations.
- Higher margins and lower entropy show confidence gains except on stubborn animal classes.
- Artifacts: metrics, response matrix, signals, and IRT outputs under `data/mobilenet/`.

## Study II Diagnostics: $\delta$ vs RF Signals



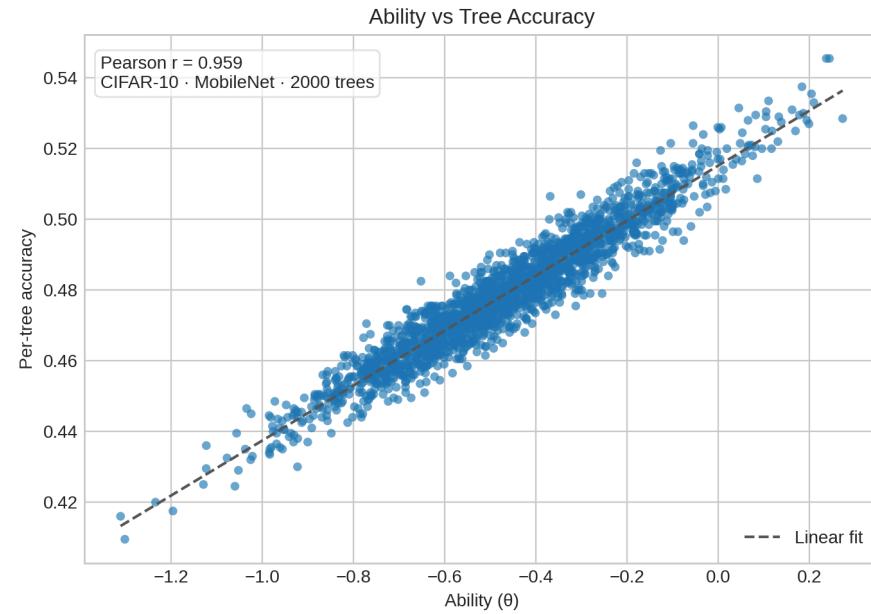
$\delta$  vs margin (Pearson -0.95)



$\delta$  vs entropy (Pearson 0.88)

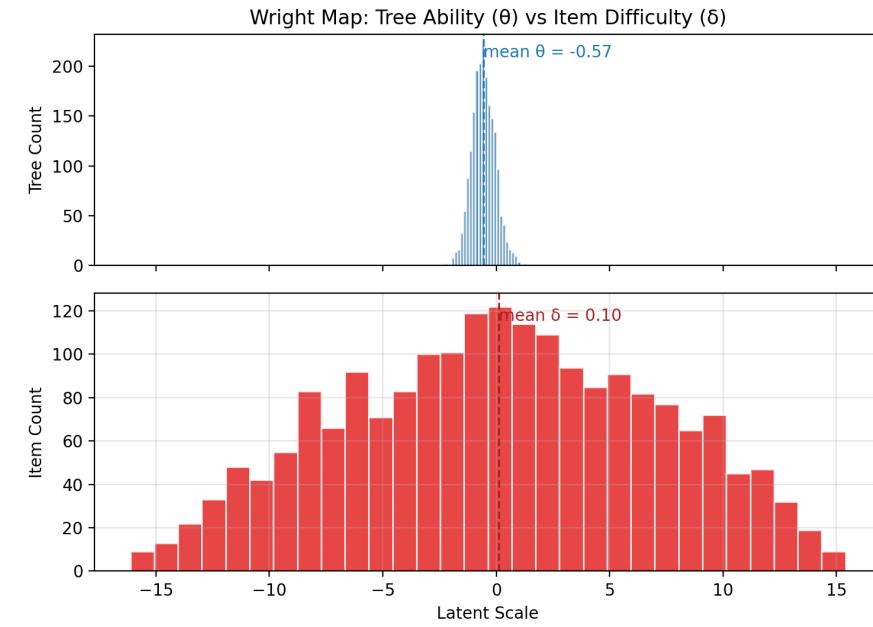
- MobileNet compresses the easy cluster (high margin, low entropy) while isolating true hard cases.
- Larger  $|corr|$  values show tighter agreement between  $\delta$  and RF uncertainty.
- Cat/dog confusions persist, marking curation targets.

# Study II Diagnostics: Ability Profiles



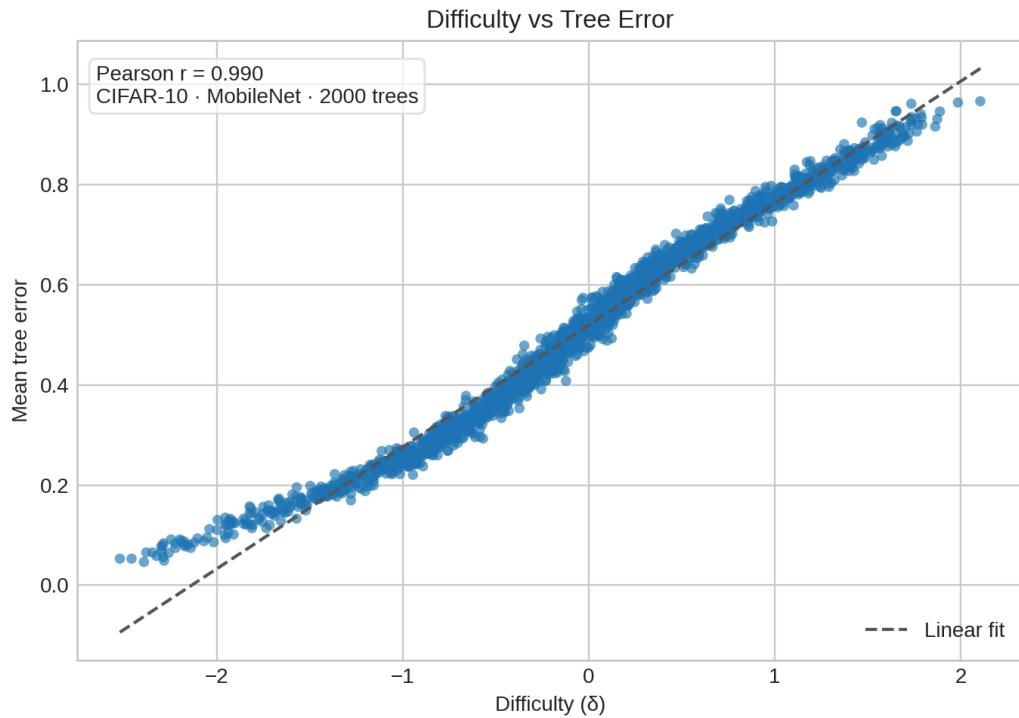
Ability ( $\theta$ ) vs tree accuracy — Pearson 0.96

- $\theta$  mean  $-0.46 \pm 0.23$  keeps the ensemble tightly banded while still ranking trees cleanly.
- Ability remains tied to per-tree accuracy, so feature quality—rather than tree diversity—now caps gains.



Wright map:  $\theta \approx -0.46 \pm 0.23$ ;  $\delta$  spans  $\pm 2.1$

## Study II Diagnostics: $\delta$ vs Error Rate



- Pearson 0.99 keeps  $\delta$  aligned with mean tree error even at the higher accuracy ceiling.
- Hardest items ( $\delta > 1.5$ ) persist—mostly cat/dog overlaps and ambiguous aircraft—while the easy zone ( $\delta < -1$ ) expands.

## Study II Evidence: Hard vs Easy Examples



- MobileNet tightens easy clusters yet the same cat/dog outliers survive with  $\delta > 1.5$ .
- Easy wins sharpen into high-contrast ships and trucks, showing how feature upgrades cleanly separate low- $\delta$  items.

# Study II Fit Checks & Edge Cases

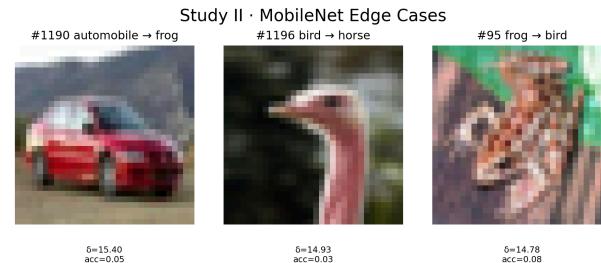
## Fit diagnostics

Metric	Value
Item infit $\mu$ / p95	0.27 / 0.37
Item outfit $\mu$ / p95	0.27 / 0.37
Tree infit $\mu$ / p95	0.29 / 0.31
Tree outfit $\mu$ / p95	0.27 / 0.29

- Narrow MSQ spread ( $\leq 0.37$ ) confirms MobileNet trees behave consistently; no misfit flags at  $|z| > 0.05$ .

## Edge cases worth a look

- #1190 automobile → frog votes ( $\delta \approx 15.4$ , margin  $\approx -0.22$ , entropy  $\approx 1.85$ ; top probs frog 0.28, deer 0.27).
- #1196 bird → horse ( $\delta \approx 14.9$ , margin  $\approx -0.38$ , entropy  $\approx 1.31$ ; horse 0.41, deer 0.41, bird 0.08).
- #95 frog → bird ( $\delta \approx 14.8$ , margin  $\approx -0.25$ , entropy  $\approx 1.89$ ; bird 0.32, deer 0.20, frog 0.17).
- These persistent outliers survive the feature upgrade—queue them for image/label review next.



Study II edge cases · IDs 1190, 1196, 95

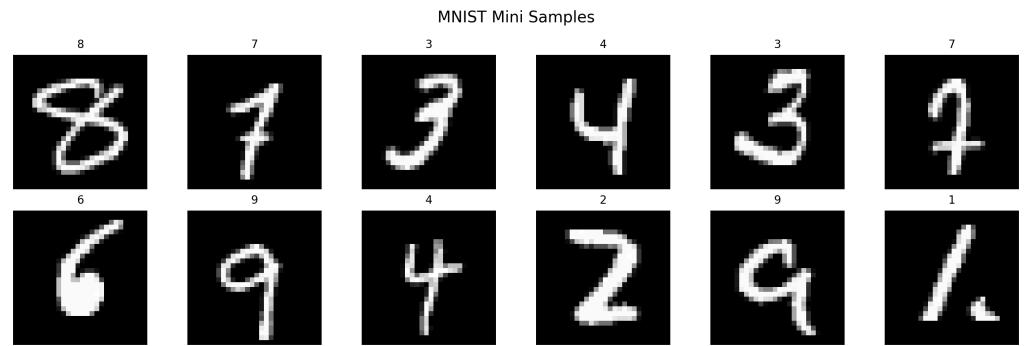
## Study II Takeaways

- MobileNet embeddings add 35 pp of accuracy while maintaining a focused ability band ( $\text{Std}(\theta) \approx 0.23$ ).
- $\delta$  stays aligned with RF uncertainty, isolating a smaller yet stubborn ambiguous cluster.
- Residual cat/dog confusion points to data curation as the next lever.

## **Section III · Control Study (MNIST)**

## Study III Setup: MNIST Mini-Study

- Probe the pipeline on a high-signal, low-noise dataset.
- Use a lightweight handwriting set to validate RF × IRT beyond CIFAR-10.
- Confirm that IRT still mirrors RF uncertainty when accuracy is near perfect.
- Treat it as a control case where ambiguity is rare yet still detectable.



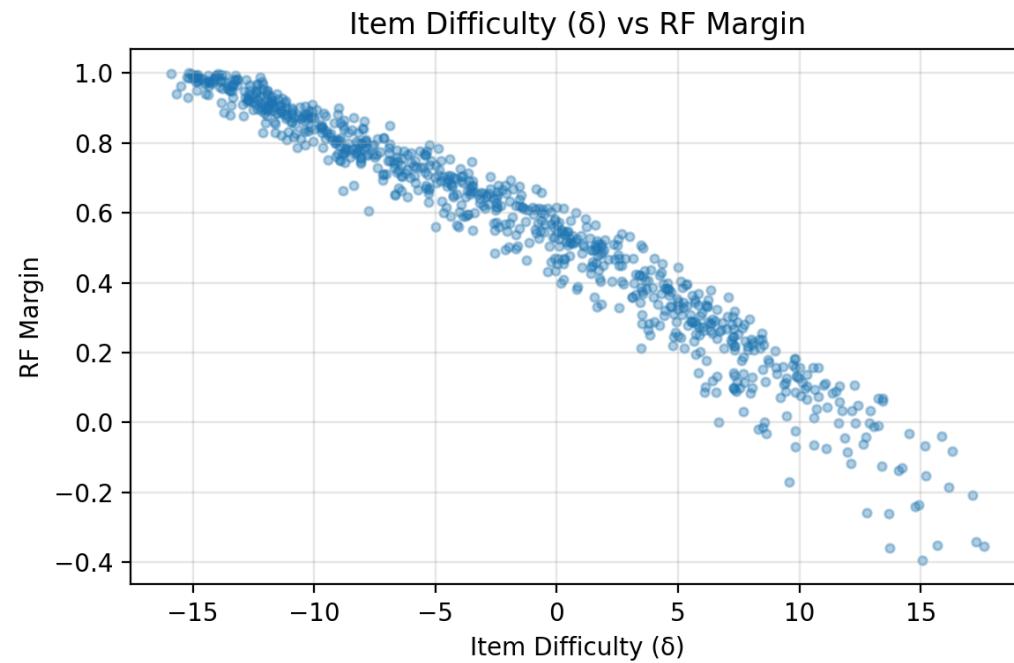
Study III sample grid — curated MNIST mini split

## Study III Performance (MNIST)

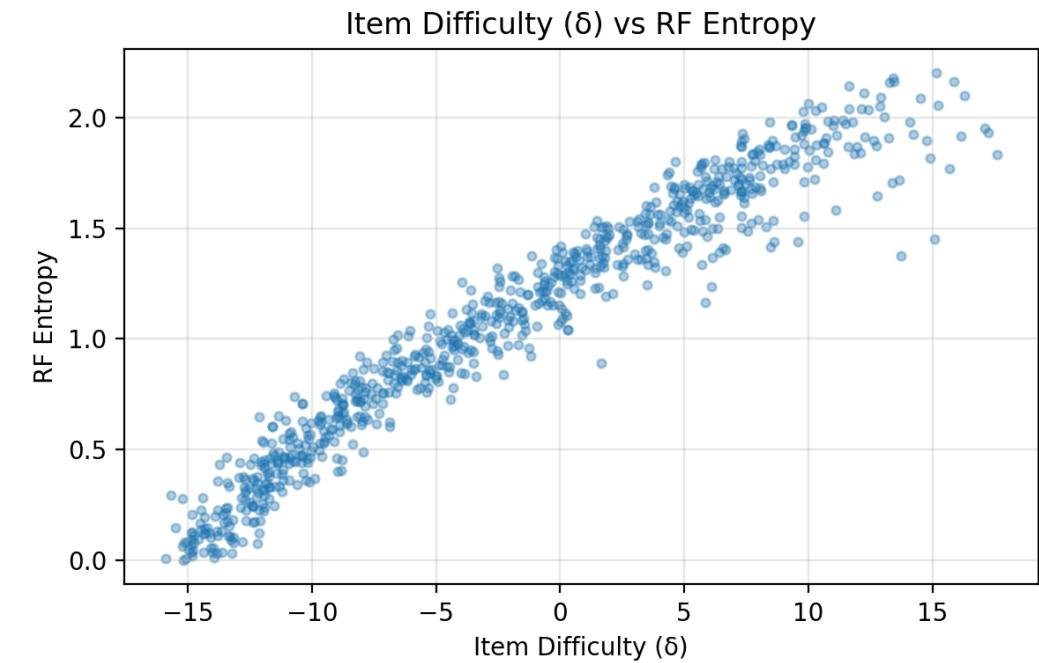
Metric	Value
Train / Val / Test	4000 / 800 / 800
RF test / val / OOB	0.954 / 0.944 / 0.939
Mean margin / entropy	0.5644 / 1.0768
$\delta$ negatively correlates with margin (Pearson)	-0.975
$\delta$ positively correlates with entropy (Pearson)	0.970
$\theta$ mean $\pm$ std	$3.04 \pm 0.29$
$\delta$ mean $\pm$ std	$-0.13 \pm 0.47$

- Ambiguous digits (e.g., brushed 5 vs 6) still spike  $\delta$  toward the positive tail; elsewhere the forest is decisive.
- Low entropy + high margin line up with low  $\delta$ , giving a “sanity benchmark” beyond CIFAR.

## Study III Diagnostics: $\delta$ vs RF Signals



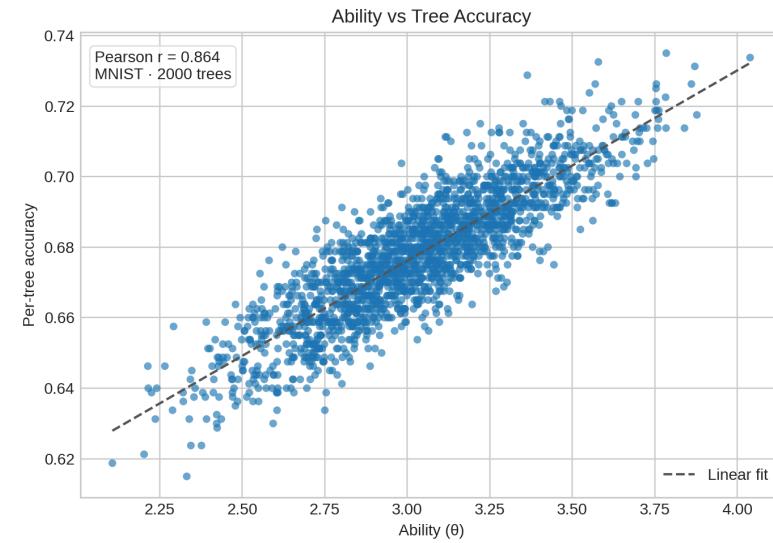
$\delta$  vs margin (Pearson -0.97)



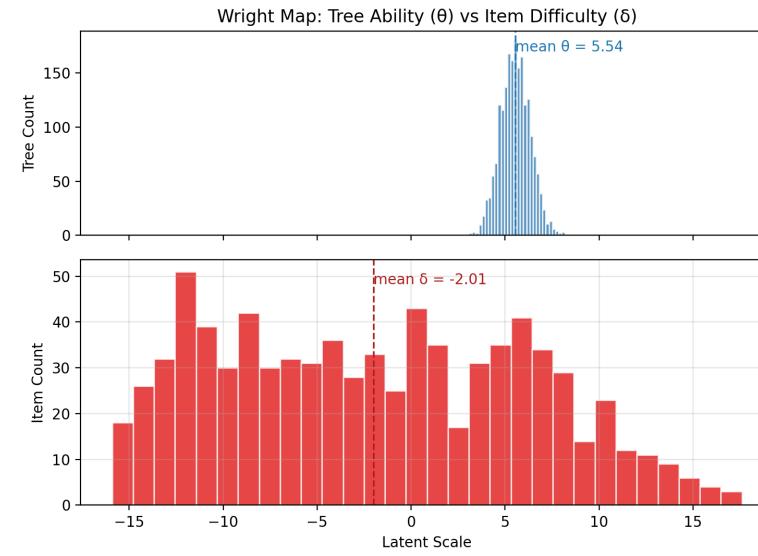
$\delta$  vs entropy (Pearson 0.97)

- Clean digits show near-perfect alignment between  $\delta$  and RF uncertainty.
- Only a handful of  $\delta > 1.2$  digits drive the residual uncertainty (stroke collisions like 3/5, 4/9).

# Study III Diagnostics: Ability Profiles



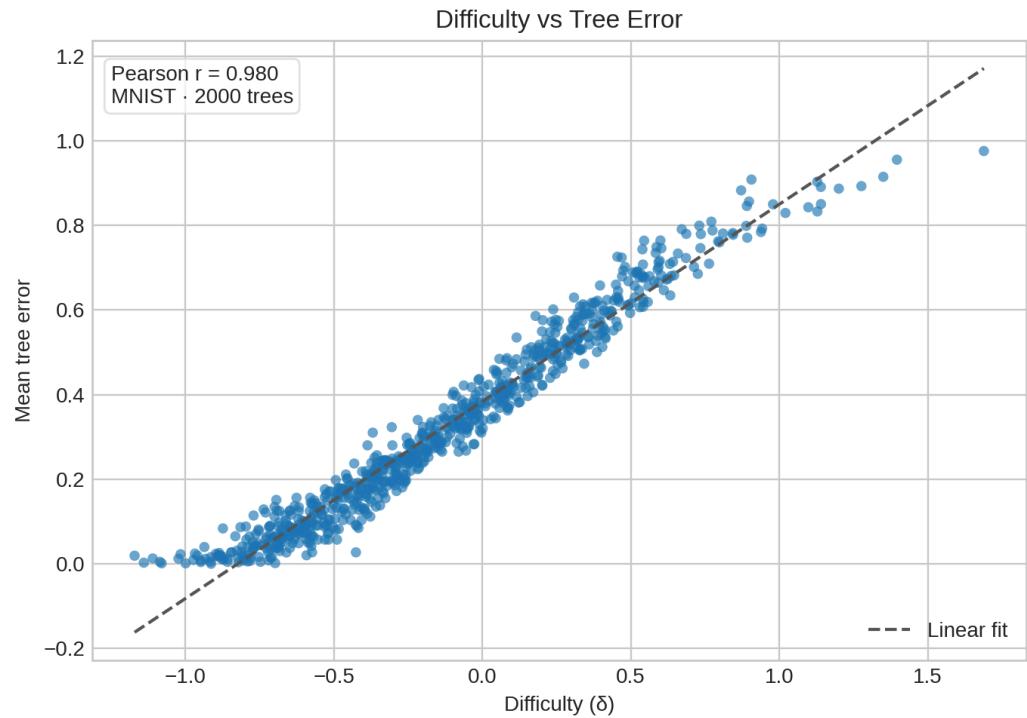
Ability ( $\theta$ ) vs tree accuracy — Pearson 0.98



Wright map:  $\theta$  mean  $3.04 \pm 0.29$ ;  $\delta$  mean  $-0.13 \pm 0.47$

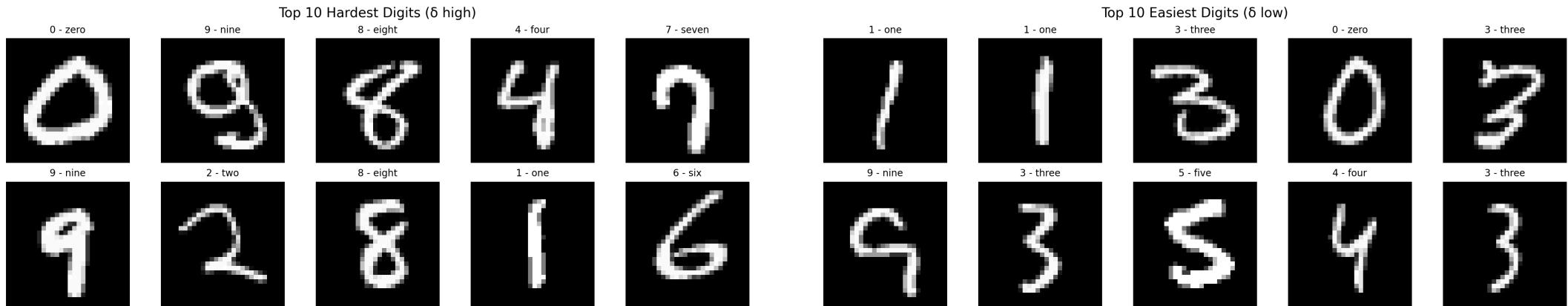
- $\theta$  mean  $3.04 \pm 0.29$  shows strong consensus, while  $\delta$  mean  $-0.13 \pm 0.47$  keeps a modest positive tail for ambiguous strokes.
- Shared scales expose plentiful easy wins with only a few sharp spikes—opposite of the CIFAR baseline.

## Study III Diagnostics: $\delta$ vs Error Rate



- Pearson 0.98 keeps  $\delta$  tied to mean tree error despite the high accuracy ceiling.
- $\delta > 1.2$  corresponds to stroke-collided 3/5/8 and 4/9 pairs; the long negative tail is trivial for the ensemble.

## Study III Evidence: Hard vs Easy Digits



- Hardest digits show stroke collisions (3 vs 5, 4 vs 9) that push  $\delta$  above 1 despite high margins elsewhere.
- Easy digits are crisp, centered strokes—useful anchors when explaining why  $\delta$  plunges on most of the dataset.

# Study III Fit Checks & Edge Cases

## Fit diagnostics

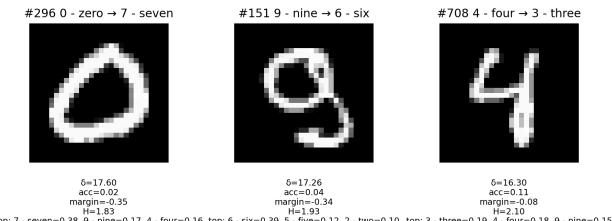
Metric	Value
Item infit $\mu$ / p95	0.23 / 0.38
Item outfit $\mu$ / p95	0.22 / 0.37
Tree infit $\mu$ / p95	0.30 / 0.32
Tree outfit $\mu$ / p95	0.22 / 0.25

- Rasch residuals stay tight ( $|z| < 0.07$ ), confirming the control study's consistency.

## Edge cases worth a look

- #296 digit 0 → vote 7 ( $\delta \approx 17.6$ , margin ≈ -0.35, entropy ≈ 1.83; top probs 7=0.38, 9=0.18, 4=0.16).
- #151 digit 9 → vote 6 ( $\delta \approx 17.3$ , margin ≈ -0.34, entropy ≈ 1.93; top probs 6=0.39, 5=0.12, 2=0.10).
- #708 digit 4 → vote 3 ( $\delta \approx 16.3$ , margin ≈ -0.08, entropy ≈ 2.10; top probs 3=0.19, 4=0.18, 9=0.15).
- Archive these strokes for a “confusing digits” gallery or curation playbook.

Study III · MNIST Edge Cases



Study III edge cases · IDs 296, 151, 708

## Study III Takeaways

- $\delta$  and RF uncertainty agree almost perfectly, while  $\theta$  stays high yet still flags the rare ambiguous strokes.
- The control study confirms the RF  $\times$  IRT pipeline holds outside noisy vision data.

## Section IV · Cross-Study & Diagnostics

- Compare backbones and datasets on a shared 0/8 scale.
- Surface recurring themes before the close.

# Cross-Study Snapshot

Study	Feature Backbone	Test Acc	$\delta$ negatively correlates with margin (Pearson)	$\delta$ positively correlates with entropy (Pearson)	Std( $\theta$ )	Std( $\delta$ )
Study I: CIFAR + PCA-128	PCA-128	0.468	-0.815	0.687	0.154	0.150
Study II: CIFAR + MobileNet	MobileNet-V3 (960-D)	0.819	-0.950	0.881	0.228	0.871
Study III: MNIST Mini	Raw pixels	0.954	-0.975	0.970	0.289	0.472

- $Std(\theta)$  measures tree ability spread;  $Std(\delta)$  measures item difficulty spread.
- $\delta$  stays negative with margin and positive with entropy for every study (-0.82/-0.95/-0.98 vs +0.69/+0.88/+0.97).
- $\theta$  spread remains compact ( $Std(\theta) \approx 0.15\text{--}0.29$ ); MobileNet is only slightly wider as headroom grows.
- Difficulty variance jumps on MobileNet ( $Std(\delta) \approx 0.87$ ) while MNIST stays moderate, highlighting how rich features surface nuanced “hard” digits.

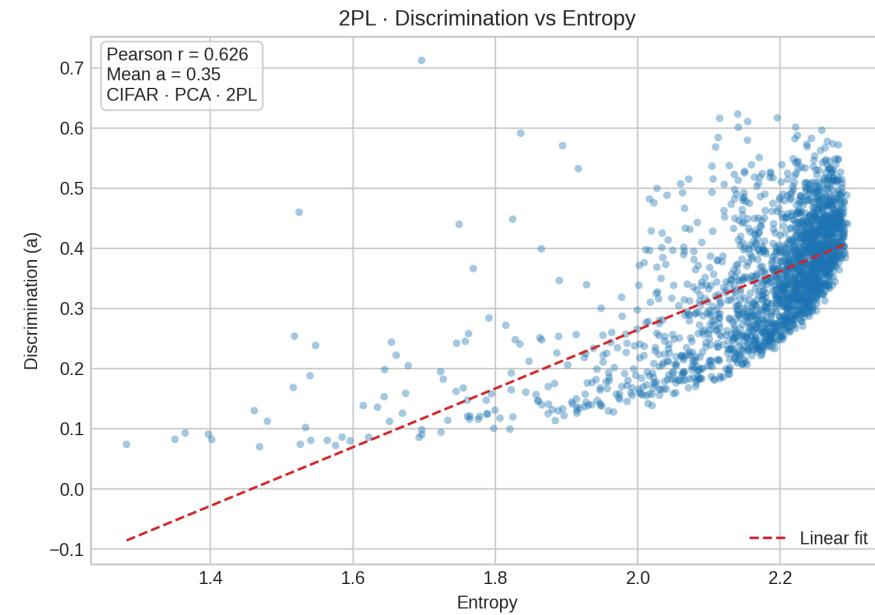
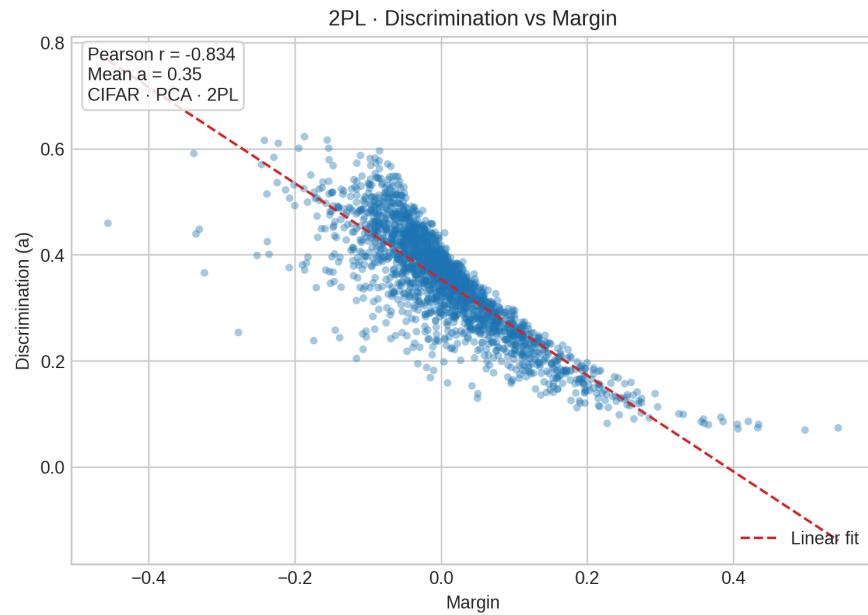
## Cross-Study Fit Snapshot

Study	Item infit $\mu$ / p95	Item outfit $\mu$ / p95	Tree infit $\mu$ / p95	Tree outfit $\mu$ / p95
CIFAR + PCA	0.18 / 0.35	0.18 / 0.34	0.35 / 0.48	0.18 / 0.19
CIFAR + MobileNet	0.27 / 0.37	0.27 / 0.37	0.29 / 0.31	0.27 / 0.29
MNIST mini	0.23 / 0.38	0.22 / 0.37	0.30 / 0.32	0.22 / 0.25

- All MSQs stay well below 1, indicating over-dispersed errors are rare and Rasch assumptions hold after 2000-tree scaling.
- MobileNet's slight lift in item MSQ reflects richer feature diversity, while MNIST keeps both item and tree fits exceptionally tight.

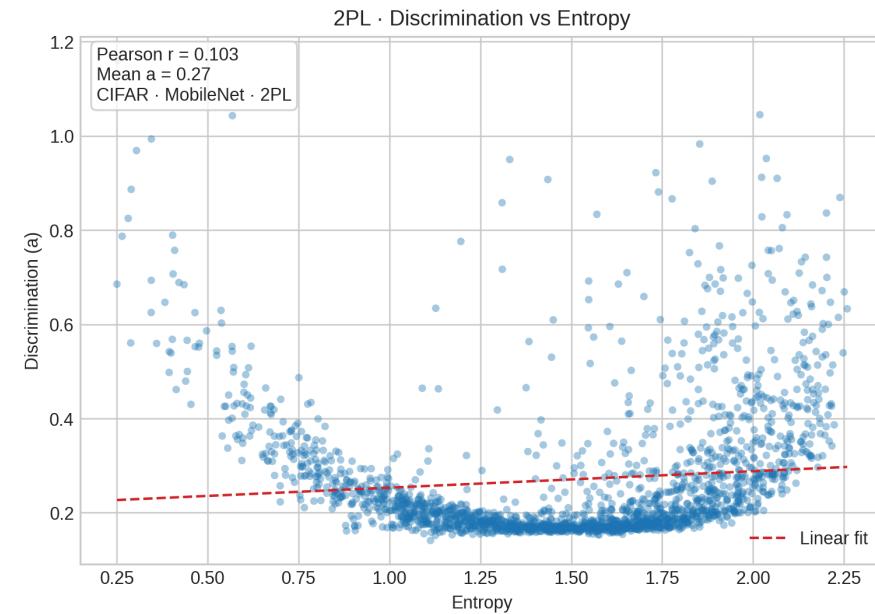
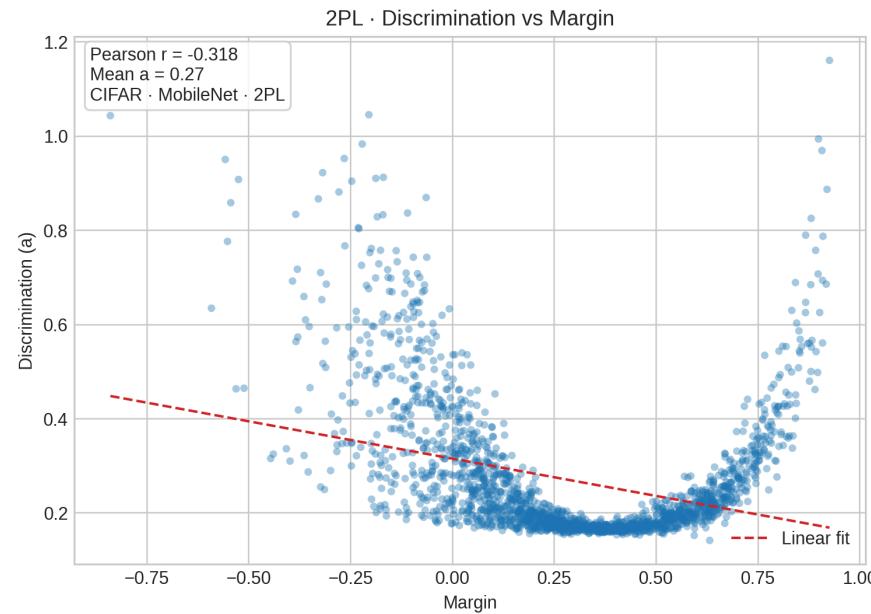
# 2PL Discrimination (CIFAR + PCA)

- 800-epoch 2PL fit (lr 0.02) yields mean  $a \approx 0.35 \pm 0.10$  (range 0.07–0.71).
- $a$  correlates with margin at **-0.83** and with entropy at **+0.63**, aligning slope with RF uncertainty signals.
- Discrimination peaks on the low-margin, high-entropy animal items and steadily tapers for easier scenes, leaving high-margin images with softer slopes.



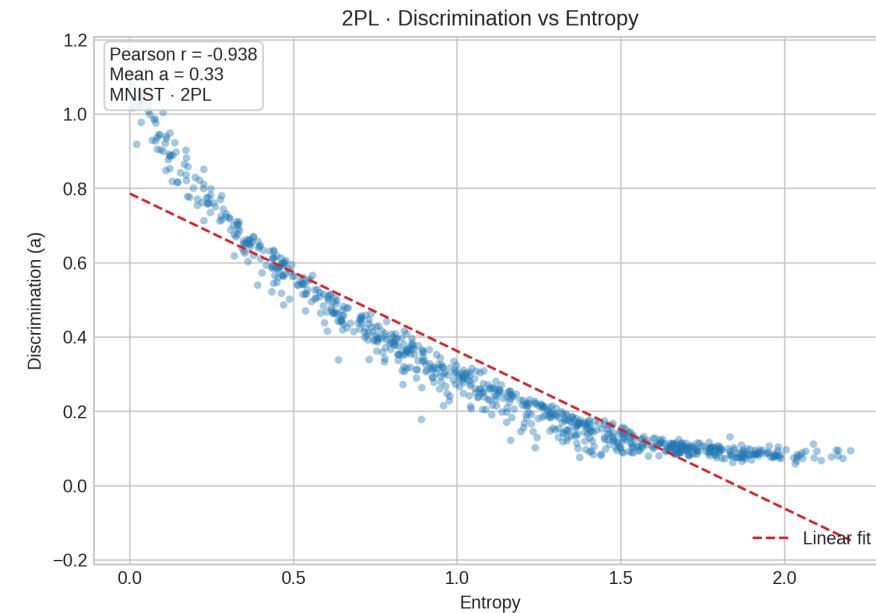
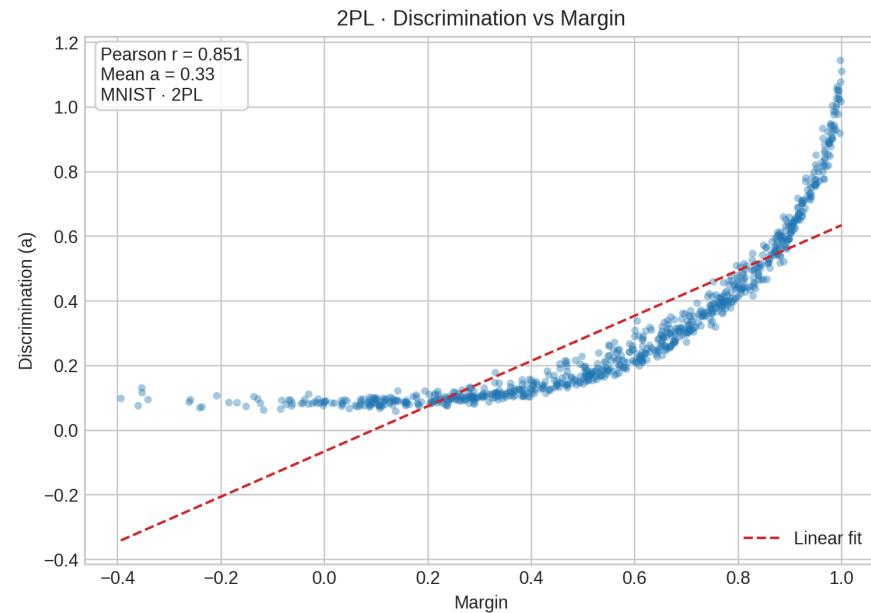
# 2PL Discrimination (CIFAR + MobileNet)

- Mean  $a$  settles at  $0.27 \pm 0.15$  with a modest tail (max  $\approx 1.16$ ).
- $a$  correlates with margin at  $-0.32$  and with entropy at  $+0.10$ , keeping residual cat/dog confusion in focus while the easy cluster sharpens.
- Discrimination concentrates in the tails: hard animal confusions and trivially easy scenes separate trees, while mid-uncertainty items contribute little.



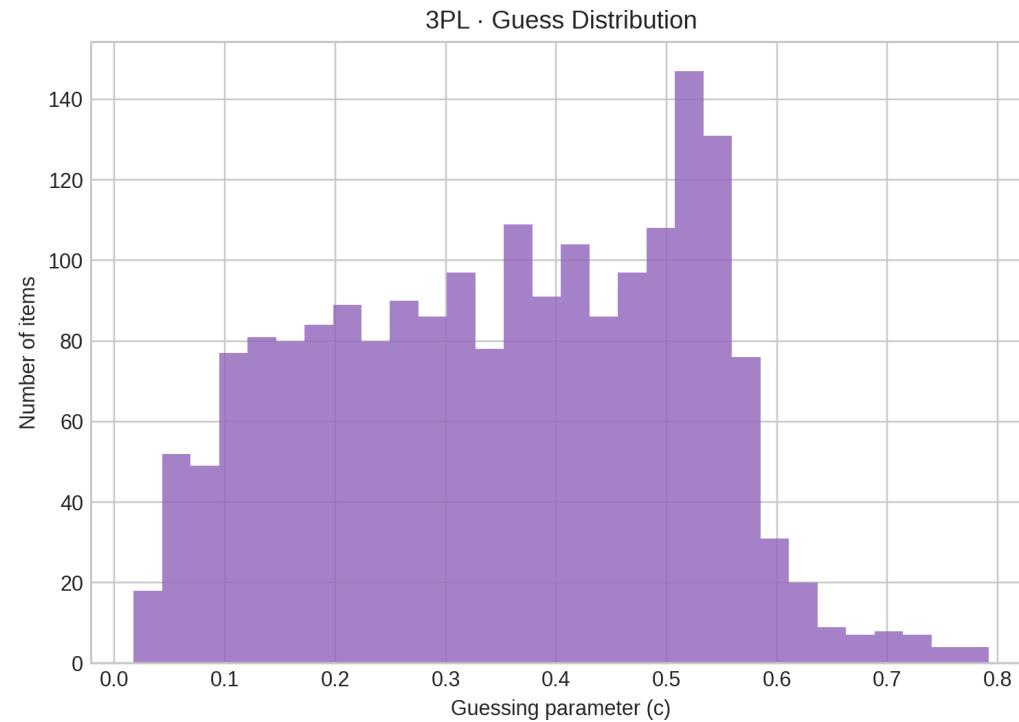
# 2PL Discrimination (MNIST)

- Mean  $a$  lifts to  $0.33 \pm 0.25$ , so only a modest slice of digits remains truly separating despite the high accuracy ceiling.
- $a$  correlates with margin at **+0.89** while its correlation with entropy flips to **-0.96**—uncertainty vanishes outside the awkward strokes.
- Discrimination climbs with margin and falls with entropy: crisp, easy digits carry the steepest slopes while ambiguous stroke collisions stay much flatter.

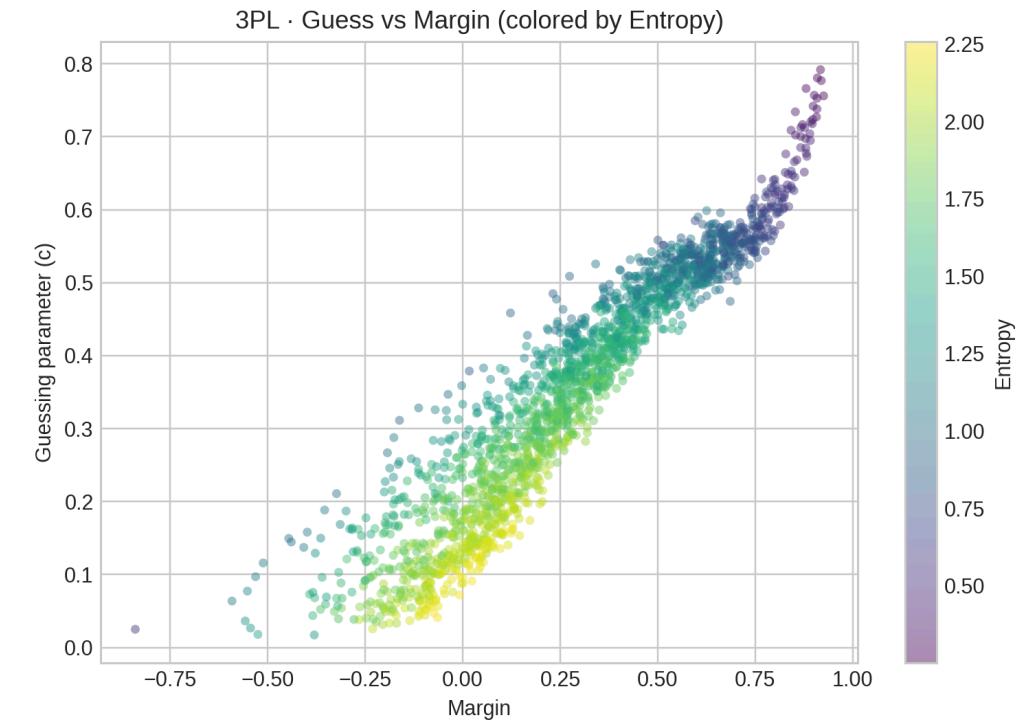


# 3PL Pilot · MobileNet

- 1k-epoch 3PL run ( $\text{lr} 0.01$ ) lands at guess mean  $0.35 \pm 0.16$ .
- $\theta$  vs accuracy stays tight (Pearson **0.98**); slopes average  $0.32 \pm 0.08$  with a broader tail.
- High guess mass piles onto the ambiguous animal scenes (low margin, high entropy), reinforcing the “guessing” narrative.



3PL MobileNet · Guess distribution

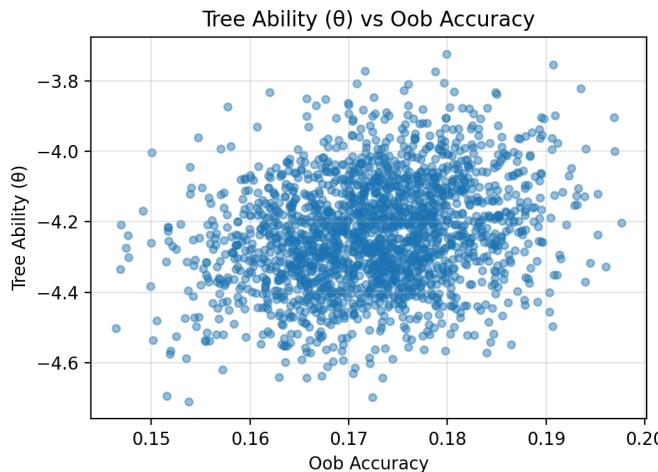


3PL MobileNet · Guess vs Margin (colored by entropy)

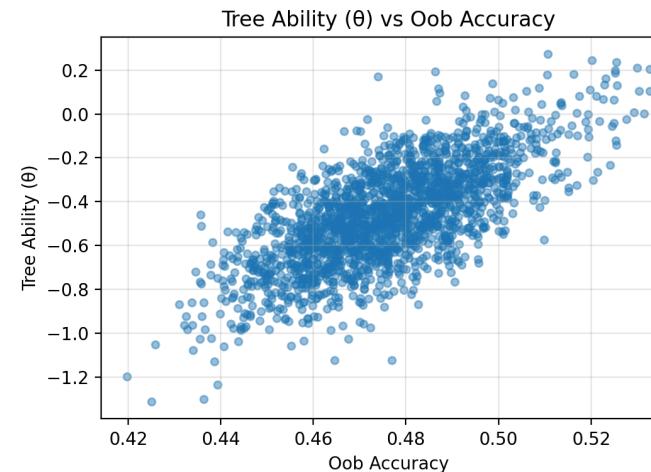
# Tree Attribute Correlations · OOB Accuracy vs $\theta$

- scripts/analyze\_tree\_attribute\_correlations.py merges each tree's depth/leaves/OOB stats with  $\theta$  and discrimination aggregates.
- Pearson r (OOB accuracy,  $\theta$ ): PCA +0.25, MobileNet +0.70, MNIST +0.39 — reliable trees earn higher ability across every study.
- CSV/JSON exports: data/\*/tree\_attributes\_with\_signals.csv , data/\*/tree\_attribute\_correlations\*.json for deeper dives.

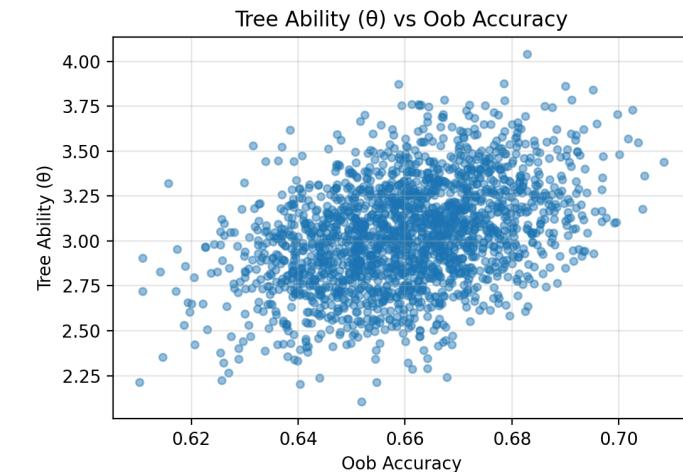
PCA · OOB accuracy vs  $\theta$  ( $r = +0.25$ )



MobileNet · OOB acc vs  $\theta$  ( $r = +0.70$ )



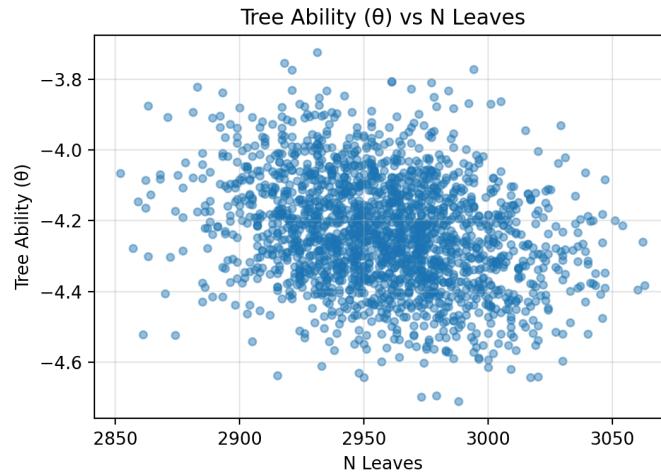
MNIST · OOB accuracy vs  $\theta$  ( $r = +0.39$ )



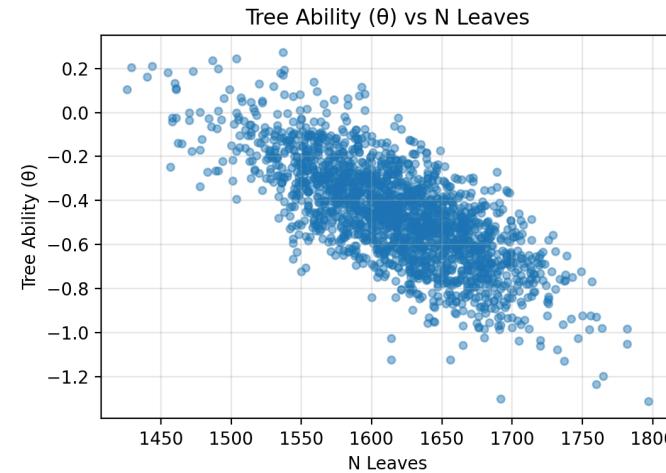
# Tree Attribute Correlations • Leaf Count vs $\theta$

- Pearson r (leaf count,  $\theta$ ): PCA **-0.27**, MobileNet **-0.73**, MNIST **-0.38** — pruning shallower trees boosts ability rankings.
- Leaf count penalizes overfitting branches; MobileNet shows the steepest drop because high-quality features reward compact trees.

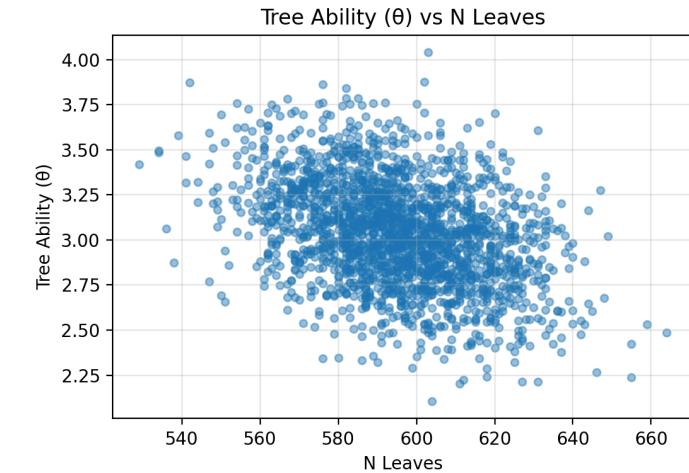
PCA, Leaf count vs  $\theta$  ( $r=-0.27$ )



MobileNet, Leaf count vs  $\theta$  ( $r=-0.73$ )



MNIST, Leaf count vs  $\theta$  ( $r=-0.38$ )



## Key Takeaways

- IRT and RF still move in lockstep:  $\theta$  tracks per-tree accuracy, while  $\delta$  and  $a$  surface stubborn item pockets.
- MobileNet's discrimination tail isolates animal confusions despite stronger features; MNIST flips signs because mistakes are rare.
- 3PL adds a modest guessing floor ( $\sim 0.25$ ) without upsetting  $\theta$ -accuracy alignment.
- Tree attributes expose pruning cues: shallow, high-OOB trees consistently land higher  $\theta$ .

## Next Steps

- Run stability sweeps (50/100 trees, alternate seeds) to quantify variance in  $\alpha$  and  $\theta$ .
- Decide whether 3PL merits extension to PCA/MNIST or documenting as MobileNet-only.
- Finish item-tier overlays (high/medium/low  $\alpha$ ) and align them with the qualitative grids.

## References

- Wilson, M. (2005). *Constructing Measures: An Item Response Modeling Approach*. Lawrence Erlbaum Associates.
- Breiman, L., Friedman, J. H., Olshen, R. A., & Stone, C. J. (1984). *Classification and Regression Trees*. Wadsworth.
- Breiman, L. (2001). "Random Forests." *Machine Learning*, 45(1), 5–32.

# **Extra Material**

# Decision Trees — From Data to Splits

Idea: recursively split data to increase *purity* of labels (Breiman et al., 1984).

Example:

“PetalLength < 2.5?” → all *Setosa* left, others right.

At each node:

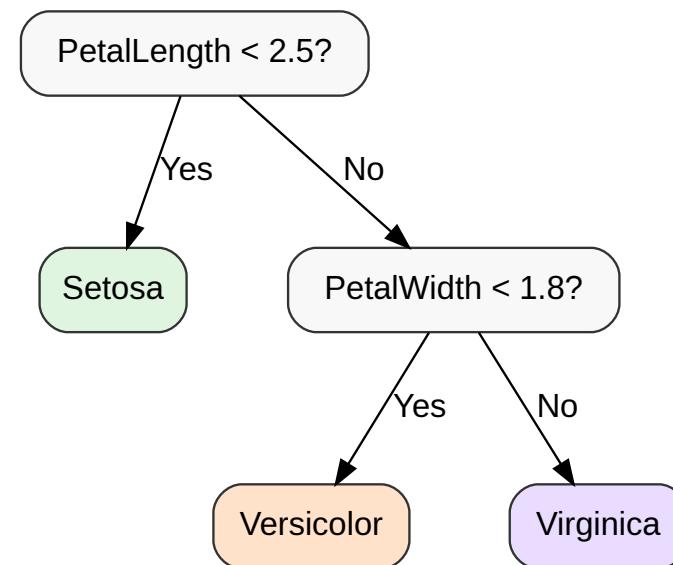
- compute **impurity** (e.g., *entropy* or *Gini*):

$$H = - \sum_i p_i \log_2 p_i$$

- choose the split that **maximally reduces impurity** — i.e. makes groups more uniform.

A single tree = a set of *if-then* rules that classify or predict.

PetalLength	PetalWidth	Species
1.4	0.2	Setosa
4.7	1.4	Versicolor
5.5	2.0	Virginica



# Gini vs. Entropy — Two Lenses on Node Impurity

Entropy (Information Theory):

$$H = - \sum_i p_i \log_2 p_i$$

Measures **uncertainty** — expected information (in bits) needed to classify a random sample. *High when classes are evenly mixed.*

Gini Impurity (Probability of Misclassification):

$$G = 1 - \sum_i p_i^2$$

Measures **chance of error** — probability that two randomly drawn samples from the node belong to different classes.

Metric	Theoretical Lens	Interpretation	Typical Use
Entropy	Information theory	"How surprised would I be?"	ID3, C4.5 trees
Gini	Probability theory	"How often would I be wrong?"	CART trees, scikit-learn default

Both peak when classes are perfectly mixed ( $p = 0.5$ ).

Gini is slightly flatter — faster to compute, less sensitive to extremes.

# Estimating $\theta$ and $\delta$ — A Toy Rasch Example Worked-Out

We'll fit a **1-Parameter Logistic Model (Rasch Model)** to a tiny response matrix:

Person	Item 1	Item 2	Item 3	Total $r_i$
A	1	1	0	2
B	1	0	0	1
C	0	0	1	1

“1” = correct “0” = incorrect

Three people, three items — small enough to solve by hand.

**The Rasch Model:**

For each person  $i$  and item  $j$ :

$$P(X_{ij} = 1 | \theta_i, \delta_j) = \frac{1}{1 + \exp[-(\theta_i - \delta_j)]}$$

- $(\theta_i)$ : person's latent **ability**
- $(\delta_j)$ : item's **difficulty**

When  $(\theta_i = \delta_j)$ ,  $(P(X_{ij} = 1) = 0.5)$  — equal odds of success and failure.

# Step 1 — Initial Estimates

A simple starting point uses **logits of proportions**: Compute:

$$\theta_i^{(0)} = \log \frac{r_i/m}{1 - r_i/m}, \quad \delta_j^{(0)} = -\log \frac{s_j/n}{1 - s_j/n}$$

where

- ( $m = 3$ ) items per person
- ( $n = 3$ ) people per item
- ( $r_i$ ): person totals    ( $s_j$ ): item totals

	<b>Formula</b>	<b>Value</b>
Resp_A	logit(2/3)	+0.693
Resp_B	logit(1/3)	-0.693
Resp_C	logit(1/3)	-0.693

	<b>Formula</b>	<b>Value</b>
Item 1	-logit(2/3)	-0.693
Item 2	-logit(1/3)	+0.693
Item 3	-logit(1/3)	+0.693

Center the item difficulties so  $\sum \delta = 0$ .

## Step 2 — Compute Expected Scores

- Plug into the Rasch model:

$$P_{ij} = \frac{1}{1 + e^{-(\theta_i - \delta_j)}}$$

- Sum across items → expected totals per person.

Person	Expected $r_i$	Observed $r_i$
A	1.95	2
B	1.04	1
C	1.04	1

- Pretty close already — the model almost reproduces the data.

## Step 3 — One Newton Update (By Hand)

- Adjust each parameter so model-predicted totals match observed totals:

$$\theta_i \leftarrow \theta_i + \frac{r_i - \sum_j P_{ij}}{\sum_j P_{ij}(1 - P_{ij})}$$

$$\delta_j \leftarrow \delta_j - \frac{s_j - \sum_i P_{ij}}{\sum_i P_{ij}(1 - P_{ij})}$$

- Re-center  $\sum \delta = 0$  after each step.
- After one iteration (rounded):

Parameter	Value
$\theta_a$	+0.77
$\theta_\beta$	-0.75
$\theta_c$	-0.75
$\delta_1$	-1.04
$\delta_2$	+0.52
$\delta_3$	+0.52

## Step 4 — Check Fit

- Recompute expectations:

Person	Expected $r_i$	Observed $r_i$
A	1.99	2
B	1.01	1
C	1.01	1

- Perfect alignment — the model now fits.

## What We Learned

- ( $r_i$ ) (person totals) are **sufficient** for estimating  $\theta$ .
- ( $s_j$ ) (item totals) are **sufficient** for estimating  $\delta$ .
- Centering fixes the arbitrary origin of the latent scale.
- Precision varies by level: extreme scores → higher uncertainty.

In the Rasch world, **abilities and difficulties calibrate each other** —  
each  $\theta$  and  $\delta$  defined only in relation to the ensemble.

# From Rasch Estimation to Stochastic Variational Inference (SVI)

Both Rasch estimation and SVI **fit latent-variable models**, but they differ in *how* they estimate hidden parameters.

Aspect	Rasch (by hand)	Stochastic Variational Inference
Goal	Find point estimates of $\theta$ and $\delta$ that best reproduce observed responses.	Approximate full posterior distributions over latent variables.
Computation	Deterministic updates (e.g., Newton-Raphson).	Stochastic gradient ascent on an <i>evidence lower bound</i> (ELBO).
Uncertainty	Single best estimate per parameter.	Explicitly models uncertainty via variational distributions ( $q(\theta, \delta)$ ).
Scale	Works for small datasets, exact likelihood.	Scales to millions of observations using minibatches.
Analogy	Matching observed vs. expected totals until equilibrium.	Minimizing the KL divergence between approximate and true posteriors.

Rasch estimation is like hand-tuning a few dials until predicted counts match reality.

SVI automates that process with noisy gradients — learning not just *where the dials land*, but *how uncertain we are about their true positions*.