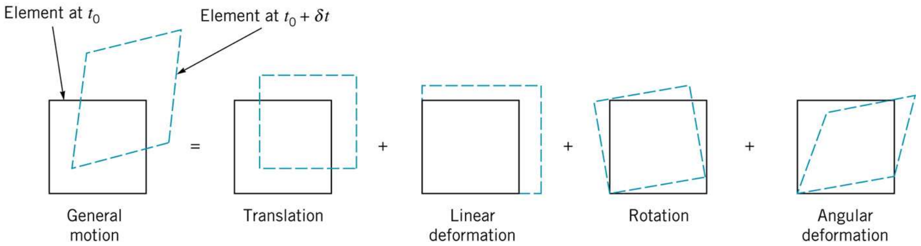
FUNDAMENTALS OF FLUID MECHANICS

Chapter 6 Differential Analysis of Fluid Flow

MAIN TOPICS

6.1 Fluid Element Kinematics

- Fluid Translation: The element moves from one point to another.
- Fluid Rotation: The element rotates about any or all of the x,y,z axes.
- Fluid Deformation:
  - ⇨Angular Deformation: The element's angles between the sides change.
  - ⇨Linear Deformation:The element's sides stretch or contract.



Fluid Translation velocity and acceleration

- The velocity of a fluid particle can be expressed

$$V = V(x, y, z, t) = u\hat{i} + v\hat{j} + w\hat{k} \qquad \text{Velocity field}$$

- The total acceleration of the particle is given by

$$a = \frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z} \qquad \text{Acceleration field}$$

$a = \dfrac{D\vec{V}}{Dt}$ is called the material , or substantial derivative.

Physical Significance

$$\vec{a} = \frac{D\vec{V}}{Dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{Local Acceleration}} + \underbrace{u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}}_{\text{Convective Acceleration}}$$

Total Acceleration of a particle

$$\vec{a} = \frac{D\vec{V}}{Dt} = \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}$$

Scalar Component

Rectangular coordinates system

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

Cylindrical coordinates system

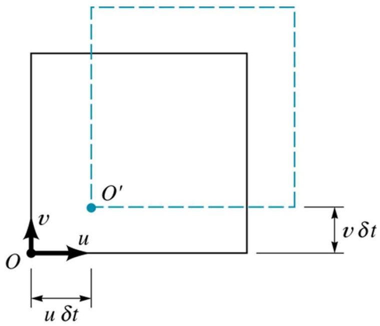
$$a_r = \frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta} - \frac{V^2{}_\theta}{r} + V_z\frac{\partial V_r}{\partial z}$$

$$a_\theta = \frac{\partial V_\theta}{\partial t} + V_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r V_\theta}{r} + V_z\frac{\partial V_\theta}{\partial z}$$

$$a_z = \frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}$$

Linear Translation

❖ All points in the element have the same velocity (which is only true if there are no velocity gradients), then the element will simply translate from one position to another.



7

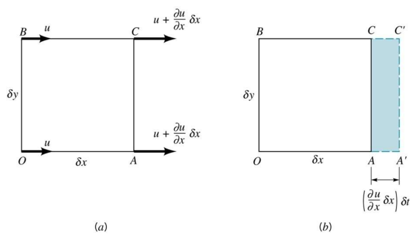
Linear Deformation 1/2

❖ The shape of the fluid element, described by the angles at its vertices, remains unchanged, since all right angles continue to be right angles.

❖ A change in the x dimension requires a nonzero value of $\partial u / \partial x$

❖ A ……………… y $\partial v / \partial y$

❖ A ……………… z $\partial w / \partial z$



8

Linear Deformation 2/2

- The change in length of the sides may produce change in volume of the element.

The change in $\delta V = \left(\frac{\partial u}{\partial x}\delta x\right)(\delta y \delta z)(\delta t)$

The rate at which the δV is changing per unit volume due to gradient $\partial u / \partial x$

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \lim_{\delta t \to 0}\left[\frac{(\partial u/\partial x)\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

If $\partial v/\partial y$ and $\partial w/\partial z$ are involved

Volumetric dilatation rate $\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot V$

9

Angular Rotation 1/4

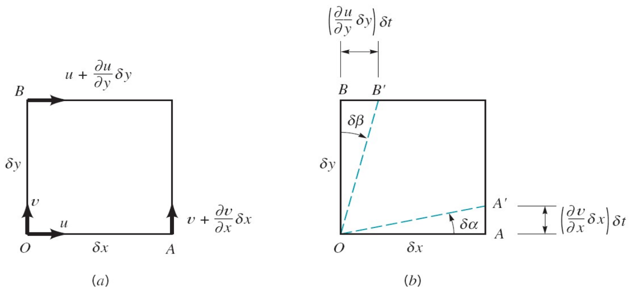
The angular velocity of line OA $\omega_{OA} = \lim_{\delta t \to 0}\frac{\delta \alpha}{\delta t}$

For small angles $\tan \delta\alpha \approx \delta\alpha = \frac{\left(\frac{\partial v}{\partial x}\right)\delta x \delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$

$\omega_{OA} = \frac{\partial v}{\partial x}$ **CCW**

$\omega_{OB} = \frac{\partial u}{\partial y}$ **CW**

"-" for CW



10

Angular Rotation 2/4

The rotation of the element about the z-axis is defined as the average of the angular velocities ω_{OA} and ω_{OB} of the two mutually perpendicular lines OA and OB.

$$\omega_z = \frac{1}{2}(\omega_{OA} + \omega_{OB}) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

In vector form $\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

Angular Rotation 3/4

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\omega = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right]$$

$$\omega = \frac{1}{2} curl V = \frac{1}{2}\nabla \times V = \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]\hat{i} + \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]\hat{j} + \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\hat{k}$$

Defining vorticity $\zeta = 2\omega = \nabla \times V$

Defining irrotation $\nabla \times V = 0$

Angular Rotation 4/4

$$\omega = \frac{1}{2} curlV = \frac{1}{2}\nabla \times V = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right]\hat{i} + \frac{1}{2}\left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right]\hat{j} + \frac{1}{2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\hat{k}$$

Vorticity

- Defining Vorticity ζ which **is a measurement of the rotation of a fluid element** as it moves in the flow field:

$$\zeta = 2\omega = curlV = \nabla \times V$$

$$\vec{\omega} = \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}\right] = \frac{1}{2}\nabla \times \vec{V}$$

- In cylindrical coordinates system:

$$\nabla \times \vec{V} = \vec{e}_r\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z}\right) + \vec{e}_\theta\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right) + \vec{e}_z\left(\frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

Angular Deformation 1/2

❖ Angular deformation of a particle is given by the sum of the two angular deformation

$$\delta\gamma = \delta\alpha + \delta\beta$$

$$\delta\xi = \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta t - u\delta t = \frac{\partial u}{\partial y}\delta y\delta t \qquad \delta\eta = \left(v + \frac{\partial v}{\partial x}\delta x\right)\delta t - v\delta t = \frac{\partial v}{\partial x}\delta x\delta t$$

$$\delta\alpha = \delta\eta / \delta x \qquad \delta\beta = \delta\xi / \delta y$$

ξ（Xi） η（Eta）

Rate of shearing strain or the rate of angular deformation

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0} \frac{\left(\frac{\partial v}{\partial x}\delta t + \frac{\partial u}{\partial y}\delta t\right)}{\delta t} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

Angular Deformation 2/2

❖ The rate of angular deformation in xy plane

$$\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

❖ The rate of angular deformation in yz plane

$$\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

❖ The rate of angular deformation in zx plane

$$\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Example 6.1 Vorticity

- For a certain two-dimensional flow field the velocity is given by the equation
- is flow irrotational?

$$V = (x^2 - y^2)\hat{i} - 2xy\hat{j}$$

Example 6.1 Solution

$$u = x^2 - y^2 \qquad v = -2xy \qquad w = 0$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

This flow is irrotational

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[(-2y) - (-2y)\right] = 0$$

6.2 Conservation of Mass 1/5

- With field representation, the property fields are defined by continuous functions of the space coordinates and time.
- To derive the differential equation for conservation of mass in rectangular and in cylindrical coordinate system.
- The derivation is carried out by applying conservation of mass to a differential control volume.

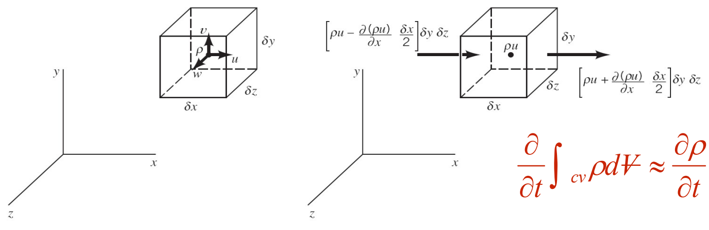
With the control volume representation of the conservation of mass

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho V \cdot \hat{n} dA = 0$$

The differential form of continuity equation???

Conservation of Mass 2/5

- The CV chosen is an infinitesimal cube with sides of length δx, δy, and δz.



(a) (b)

$$\frac{\partial}{\partial t}\int_{cv} \rho d\forall \approx \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$$

$$\rho u\Big|_{x+\left(\frac{dx}{2}\right)} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \qquad \rho u\Big|_{x-\left(\frac{\delta x}{2}\right)} = \rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}$$

Conservation of Mass 3/5

Net rate of mass
Outflow in x-direction

$$=\left[\rho u+\frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\delta z-\left[\rho u-\frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\delta z=\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z$$

Net rate of mass
Outflow in y-direction

$$=\cdots\cdots=\frac{\partial(\rho v)}{\partial y}\delta x\delta y\delta z$$

Net rate of mass
Outflow in z-direction

$$=\cdots\cdots=\frac{\partial(\rho w)}{\partial z}\delta x\delta y\delta z$$

21

Conservation of Mass 4/5

Net rate of mass Outflow

$$\left[\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z=0$$

The differential equation for conservation of mass

Continuity equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0$$

22

Conservation of Mass 5/5

❖ Incompressible fluid

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot V=0$$

❖ Steady flow

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\nabla\cdot\rho V=0$$

Example 6.2 Continuity Equation

- The velocity components for a certain incompressible, steady flow field are

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z$$
$$w = ?$$

Determine the form of the z component, w, required to satisfy the continuity equation.

Example 6.2 Solution

The continuity equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$

$$\frac{\partial u}{\partial x}=2x$$

$$\frac{\partial v}{\partial y}=x+z$$

$$\frac{\partial w}{\partial z}=-2x-(x+z)=-3x-z$$

$$\Rightarrow w=-3xz-\frac{z^2}{2}+f(x,y)$$

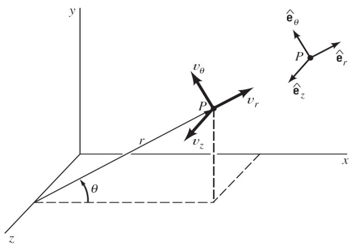
Conservation of Mass
Cylindrical Coordinate System 1/3

- The CV chosen is an infinitesimal cube with sides of length *dr, rdθ*, and *dz*.
- The net rate of mass flux out through the control surface

$$\left[\rho V_r + r\frac{\partial \rho V_r}{\partial r}+\frac{\partial \rho V_\theta}{\partial \theta}+r\frac{\partial \rho V_z}{\partial z}\right]\delta r\delta\theta\delta z$$

- The rate of change of mass inside the control volume

$$\frac{\partial \rho}{\partial t} r d\theta dr dz$$



Conservation of Mass
Cylindrical Coordinate System 2/3

The continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta}+\frac{\partial(\rho V_z)}{\partial z}=0$$

By "Del" operator

$$\nabla=\vec{e}_r\frac{\partial}{\partial r}+\vec{e}_\theta\frac{1}{r}\frac{\partial}{\partial \theta}+\vec{e}_z\frac{\partial}{\partial z}$$

The continuity equation becomes

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\rho\vec{V}=0$$

27

Conservation of Mass
Cylindrical Coordinate System 3/3

- Incompressible fluid

$$\frac{1}{r}\frac{\partial(rV_r)}{\partial r}+\frac{1}{r}\frac{\partial(V_\theta)}{\partial \theta}+\frac{\partial(V_z)}{\partial z}=\nabla\cdot\vec{V}=0$$

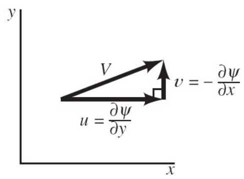
- Steady flow

$$\frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r}+\frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta}+\frac{\partial(\rho V_z)}{\partial z}=\nabla\cdot\rho\vec{V}=0$$

28

Stream Function 1/6

- **Streamlines?** Lines tangent to the instantaneous velocity vectors at every point.
- Stream function ψ (x,y) [Psi]? Used to represent the velocity component u(x,y,t) and v(x,y,t) of a two-dimensional incompressible flow.
- Define a function ψ(x,y), called the stream function, which relates the velocities shown by the figure in the margin as



$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

Stream Function 2/6

- The stream function Ψ(x, y) satisfies the two-dimensional form of the incompressible continuity equation

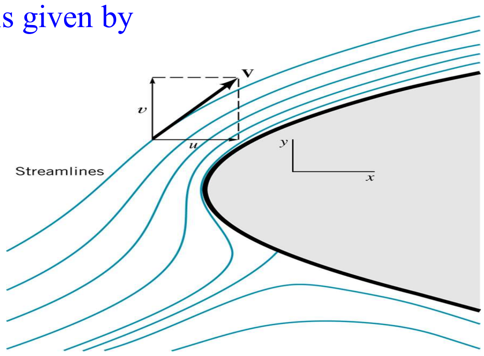
$$\frac{\partial u}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial v}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

- **ψ(x,y)?** Still unknown for a particular problem, but at least we have simplify the analysis by having to determine only one unknown, **ψ(x,y)**, rather than the two function u(x,y) and v(x,y).

Stream Function 3/6

- Another advantage of using stream function is related to the fact that line along which ψ **(x,y) =constant** are streamlines.
- How to prove ? From the definition of the streamline that the slope at any point along a streamline is given by

$$\frac{dy}{dx} = \frac{v}{u}$$



Velocity and velocity component along a streamline

Stream Function 4/6

- The change of $\psi(x,y)$ as we move from one point (x,y) to a nearly point (x+dx, y+dy) is given by $\psi(x,y)=C$

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -vdx + udy$$

$$>> d\psi = 0 \qquad >> -vdx + udy = 0$$

Along a line of constant Ψ

$$\frac{dy}{dx} = \frac{v}{u}$$

This is the definition for a streamline. Thus, if we know the function $\Psi(x,y)$ we can plot lines of constant Ψ to provide the family of streamlines that are helpful in visualizing the pattern of flow. There are an infinite number of streamlines that make up a particular flow field, since for each constant value assigned to Ψ a streamline can be drawn.

Stream Function 5/6

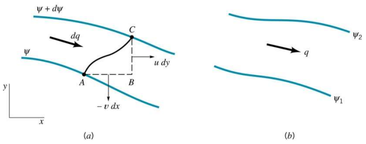
- The actual numerical value associated with a particular streamline is not of particular significance, but the change in the value of ψ is related to the volume rate of flow.
- For a unit depth, the flow rate across BC is

$$q_1 = \int_{y_1}^{y_2} u dy = \int_{y_1}^{y_2} \frac{\partial \psi}{\partial y} dy = \int_{\psi_r}^{\psi_2} d\psi = \psi_2 - \psi_r$$

- For a unit depth, the flow rate across AB is

$$q_2 = -\int_{x_1}^{x_2} v dx = \int_{x_1}^{x_2} \frac{\partial \psi}{\partial x} dx = \int_{\psi_1}^{\psi_r} d\psi = \psi_r - \psi_1$$

$$q = q_1 + q_2 = \psi_2 - \psi_1$$



Stream Function 6/6

- Thus the volume flow rate between any two streamlines can be written as the difference between the constant values of ψ defining two streamlines.
- The velocity will be relatively high wherever the streamlines are close together, and relatively low wherever the streamlines are far apart.

Stream Function
Cylindrical Coordinate System

❖ For a two-dimensional, incompressible flow in the r**θ** plane, conservation of mass can be written as:

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}=0$$

❖ The velocity components can be related to the stream function, ψ(r,**θ**) through the equation

$$v_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta=-\frac{\partial\psi}{\partial r}$$

35

Example 6.3 Stream Function

- The velocity component in a steady, incompressible, two dimensional flow field are

$$u = 2y \qquad v = 4x$$

Determine the corresponding stream function and show on a sketch several streamlines. Indicate the direction of glow along the streamlines.

36

Example 6.3 Solution

From the definition of the stream function

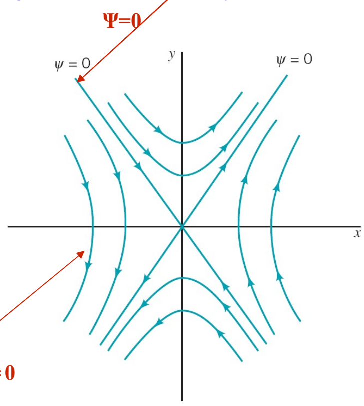
$$u = \frac{\partial \psi}{\partial y} = 2y \qquad v = -\frac{\partial \psi}{\partial x} = 4x$$

$$\psi = y^2 + f_1(x) \qquad \psi = -2x^2 + f_2(y)$$

$$\psi = -2x^2 + y^2 + C$$

For simplicity, we set C=0

$$\psi = -2x^2 + y^2$$



$$\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1$$

37

Orthogonal Coordinate Systems

h_i: metric stretching factors

Arc length change

$$\mathbf{ds} = \mathbf{a}_{x1}h_1dx_1 + \mathbf{a}_{x2}h_2dx_2 + \mathbf{a}_{x3}h_3dx_3$$

Arc length ds

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = (h_1\,dx_1)^2 + (h_2\,dx_2)^2 + (h_3\,dx_3)^2$$

$$dv = h_1h_2h_3dx_1dx_2dx_3$$

$$ds_1 = h_2h_3dx_2dx_3$$

$$ds_2 = h_1h_3dx_1dx_3$$

$$ds_3 = h_1h_2dx_1dx_2$$

An orthogonal coordinate systems, the components of the gradient of a scalar ϕ

$$\nabla\phi \qquad \frac{1}{h_1}\frac{\partial\phi}{\partial x_1} \qquad \frac{1}{h_2}\frac{\partial\phi}{\partial x_2} \qquad \frac{1}{h_3}\frac{\partial\phi}{\partial x_3}$$

The divergence of vector A

$$\operatorname{div}\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x_1}(h_2h_3A_1) + \frac{\partial}{\partial x_2}(h_3h_1A_2) + \frac{\partial}{\partial x_3}(h_1h_2A_3)\right]$$

$$Let \quad A = \nabla\phi$$

$$\nabla^2\phi = \nabla\bullet\nabla\phi$$

$$\nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \mathbf{a}_{u_1}h_1 & \mathbf{a}_{u_2}h_2 & \mathbf{a}_{u_3}h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}.$$

B=curl **A**

$$B_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial x_2}(h_3A_3) - \frac{\partial}{\partial x_3}(h_2A_2)\right]$$

$$B_2 = \frac{1}{h_1h_3}\left[\frac{\partial}{\partial x_3}(h_1A_1) - \frac{\partial}{\partial x_1}(h_3A_3)\right]$$

$$B_3 = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}(h_2A_2) - \frac{\partial}{\partial x_2}(h_1A_1)\right]$$

Cartesian Coordinates

$(x_1, x_2, x_3) = (x, y, z)$

$(h_1, h_2, h_3) = (1, 1, 1)$

$V=(V_x, V_y, V_z)=(u, v, w)$

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

$$\text{div}\,\mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\text{curl}\,\mathbf{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\boxed{\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}}$$

$$\mathbf{V}\cdot\nabla = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

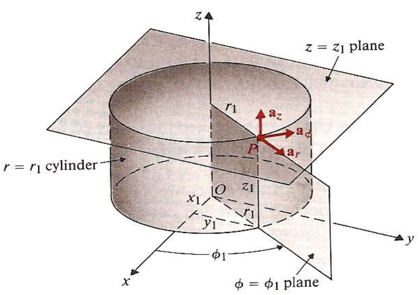
Cylindrical Polar Coordinates

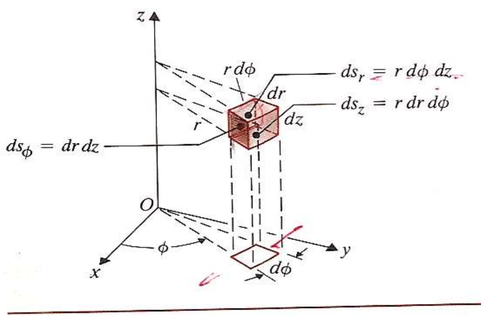
$(x_1, x_2, x_3) = (r, \theta, z)$

$(h_1, h_2, h_3) = (1, r, 1)$

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$V=(v_r, v_\theta, v_z)$ $\qquad$ $dv = r\,dr\,d\theta\,dz$





$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{\partial\phi}{\partial z}\right)$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z}$$

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}$$

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

Spherical Polar Coordinates

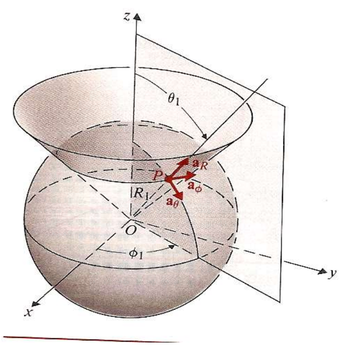
$(x_1, x_2, x_3) = (r, \theta, \phi)$

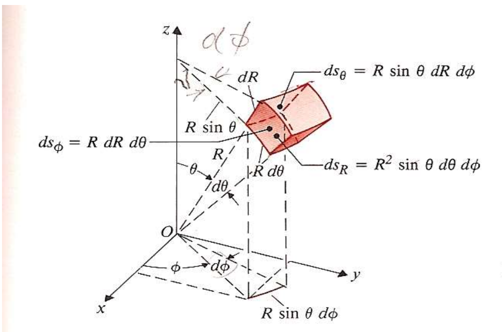
$(h_1, h_2, h_3) = (1, r, r\sin\theta)$

$x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\cos\phi,\ \ z = r\cos\theta$

$V = (v_r, v_\theta, v_\phi)$

$dv = r^2\sin\theta\, dr\, d\theta\, d\phi$





$$\nabla\phi = \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\lambda}\right)$$

$$\text{div}\,\mathbf{V} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(v_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\lambda}{\partial\lambda}$$

$$\mathbf{V}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + \frac{v_\lambda}{r\sin\theta}\frac{\partial}{\partial\lambda}$$

$$\nabla^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\lambda^2}$$

Orthogonal Coordinate Systems

Three Basic Orthogonal Coordinate Systems

Coordinate System Relations		Cartesian Coordinates (x, y, z)	Cylindrical Coordinates (r, ϕ, z)	Spherical Coordinates (R, θ, ϕ)
Base vectors	$\mathbf{a}_{u_1}$	$\mathbf{a}_x$	$\mathbf{a}_r$	$\mathbf{a}_R$
	$\mathbf{a}_{u_2}$	$\mathbf{a}_y$	$\mathbf{a}_\phi$	$\mathbf{a}_\theta$
	$\mathbf{a}_{u_3}$	$\mathbf{a}_z$	$\mathbf{a}_z$	$\mathbf{a}_\phi$
Metric coefficients	h_1	1	1	1
	h_2	1	r	R
	h_3	1	1	$R\sin\theta$
Differential volume	dv	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

6.3 Conservation of Linear Momentum

❖ Applying Newton's second law to control volume

$$F = \left.\frac{DP}{Dt}\right|_{SYS} \qquad P = \int_{system} \mathrm{V}\, dm$$

$$\sum F_{\substack{contents\ of\ the\\ control\ volume}} = \frac{\partial}{\partial t}\int_{CV} \rho \mathrm{V}\, d\forall + \int_{CS} \rho \mathrm{V}(V \cdot \hat{n} dA)$$

$$\delta F = \frac{\mathrm{D}(V\delta m)}{\mathrm{D}t} = \delta m\left(\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z}\right)$$

$$= \delta m \frac{DV}{Dt} = \delta m a$$

For a ***infinitesimal system of mass dm****, what's the* ***The differential form of linear momentum equation?***

Forces Acting on Element 1/2

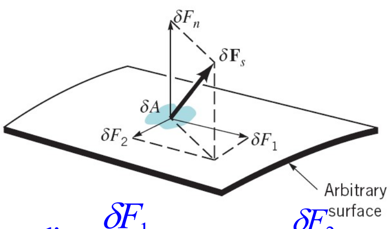
❖ The forces acting on a fluid element may be classified as body forces and surface forces; surface forces include **normal forces and tangential** (shear) forces.

$$\delta F = \delta \vec{F}_S + \delta \vec{F}_B$$

$$= \delta F_{sx}\vec{i} + \delta F_{sy}\vec{j} + \delta F_{sz}\vec{k}$$

$$+ \delta F_{bx}\vec{i} + \delta F_{by}\vec{j} + \delta F_{bz}\vec{k}$$

Surface forces acting on a fluid element can be described in terms of normal and shearing stresses.



$$\sigma_n = \lim_{\delta A \to 0}\frac{\delta F_n}{\delta A} \qquad \tau_1 = \lim_{\delta A \to 0}\frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta A \to 0}\frac{\delta F_2}{\delta A}$$

Forces Acting on Element 2/2

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x \delta y \delta z$$

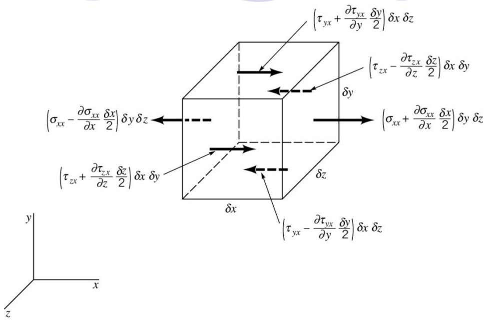
$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x \delta y \delta z$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x \delta y \delta z$$

$$\delta F_{bx} = \rho g_x \delta x \delta y \delta z$$

$$\delta F_{by} = \rho g_y \delta x \delta y \delta z$$

$$\delta F_{bz} = \rho g_z \delta x \delta y \delta z$$



Equation of Motion

Equation of Motion

$\delta F_x = \delta m a_x \qquad \delta F_y = \delta m a_y \qquad \delta F_z = \delta m a_z$

$$\begin{cases} \rho g_x + \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} = \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right) \\ \rho g_y + \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} = \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right) \\ \rho g_z + \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} = \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right) \end{cases}$$

These are the differential equations of motion for any fluid satisfying the continuum assumption.

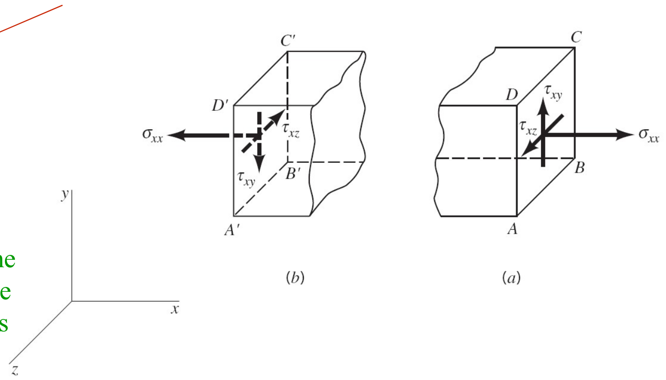
How to solve u,v,w ?

Double Subscript Notation for Stresses

τ_{xy}

The direction of the stress

The direction of the normal to the plane on which the stress acts



6.4 Inviscid Flow

- ❖ Shearing stresses develop in a moving fluid because of the viscosity of the fluid.
- ❖ For some common fluid, such as air and water, the viscosity is small, and therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity.
- ❖ Flow fields in which the shearing stresses are assumed to be negligible are said to be inviscid, nonviscous, or frictionless.

Define the pressure, p, as the negative of the normal stress

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

Euler's Equation of Motion

❖ Under **frictionless condition**, the equations of motion are reduced to **Euler's Equation:**

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\Rightarrow \quad \rho\vec{g} - \nabla p = \rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right)$$

Bernoulli Equation 1/3

❖ Euler's equation for steady flow along a streamline is

$$\rho g - \nabla p = \rho(V\cdot\nabla)V$$

Selecting the coordinate system with the z-axis vertical so that the acceleration of gravity vector can be expressed as

$$\vec{g} = -g\nabla z$$

$$(V\cdot\nabla)V = \frac{1}{2}\nabla(V\cdot V) - V\times(\nabla\times V)$$ **Vector identity ….**

$$-\rho g\nabla z - \nabla p = \frac{\rho}{2}\nabla(V\cdot V) - \rho(V\times\nabla\times V)$$

Bernoulli Equation 2/3

$$\frac{\nabla p}{\rho}+\frac{1}{2}\nabla\left(V^2\right)+g\nabla z=V\times\left(\nabla\times V\right)$$

$V\times\left(\nabla\times V\right)$ perpendicular to V

$$\xrightarrow{\cdot ds}\ \frac{\nabla p}{\rho}\cdot ds+\frac{1}{2}\nabla\left(V^2\right)\cdot ds+g\nabla z\cdot ds=\left[V\times\left(\nabla\times V\right)\right]\cdot ds$$

With $ds=dx\hat{i}+dy\hat{j}+dz\hat{k}$

$$\nabla p\cdot ds=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz=dp$$

Bernoulli Equation 3/3

$$\frac{\nabla p}{\rho}\cdot ds+\frac{1}{2}\nabla\left(V^2\right)\cdot ds+g\nabla z\cdot ds=0$$

$$\Rightarrow \frac{dp}{\rho}+\frac{1}{2}d\left(V^2\right)+gdz=0$$

Integrating … $\int\frac{dp}{\rho}+\frac{V^2}{2}+gz=\text{cons tan t}$

For steady inviscid, incompressible fluid (commonly called ideal fluids) **along a streamline**

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=\text{cons tan t}$$ **Bernoulli equation**

Irrotational Flow 1/4

❖ Irrotation ? The irrotational condition is

$$\nabla \times V = 0$$

➪ In rectangular coordinates system

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

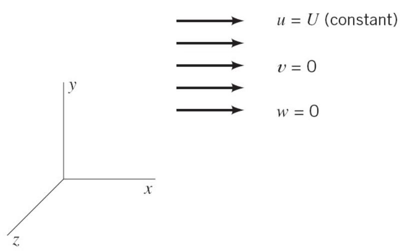
➪ In cylindrical coordinates system

$$\frac{1}{r}\frac{\partial V_z}{\partial \theta} - \frac{\partial V_\theta}{\partial z} = \frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = \frac{1}{r}\frac{\partial r V_\theta}{\partial r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} = 0$$

Irrotational Flow 2/4

❖ A general flow field would not be irrotational flow.

❖ A special uniform flow field is an example of an irrotation flow



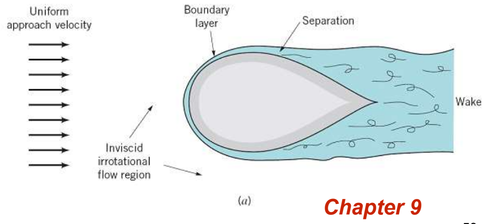
Irrotational Flow 3/4

A general flow field

⇨ A solid body is placed in a uniform stream of fluid. Far away from the body remain uniform, and in this far region the flow is irrotational.

⇨ The flow around the body remains irrotational except very near the boundary.

⇨ Near the boundary the velocity changes rapidly from zero at the boundary (no-slip condition) to some relatively large value in a short distance from the boundary.



Chapter 9

Irrotational Flow 4/4

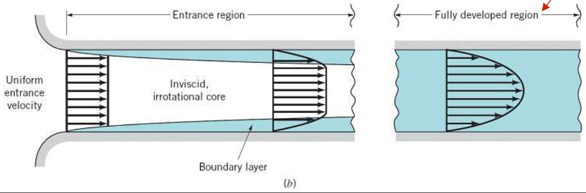
❖ A general flow field

⇨ Flow from a large reservoir enters a pipe through a streamlined entrance where the velocity distribution is essentially uniform. Thus, at entrance the flow is irrotational. (b)

⇨ In the central core of the pipe the flow remains irrotational for some distance.

⇨ The boundary layer will develop along the wall and grow in thickness until it fills the pipe.

Viscous forces are dominant



Chapter 8

Bernoulli Equation for Irrotational Flow 1/3

- The Bernoulli equation for **steady, incompressible, and inviscid flow** is

$$\frac{p}{\rho}+\frac{V^2}{2}+gz=constant$$

- The equation can be applied between **any two points on the same streamline**. In general, **the value of the constant will vary from streamline to streamline**.
- Under additional **irrotational condition**, *the Bernoulli equation ?* Starting with Euler's equation in vector form

$$(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-g\vec{k}=\frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)-\vec{V}\times\cancel{\left(\nabla\times\vec{V}\right)}$$

ZERO Regardless of the direction of ds

Bernoulli Equation for Irrotational Flow 2/3

- With **irrotaional** condition $\nabla\times\vec{V}=0$

$$(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-g\vec{k}=\frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)-\vec{V}\times\cancel{\left(\nabla\times\vec{V}\right)}$$

$$\frac{1}{2}\nabla\left(\vec{V}\cdot\vec{V}\right)=\frac{1}{2}\nabla\left(V^2\right)=-\frac{1}{\rho}\nabla p-g\vec{k}\quad\cdot\, d\vec{r}$$

$$\frac{1}{2}\nabla\left(V^2\right)\cdot d\vec{r}=-\frac{1}{\rho}\nabla p\cdot d\vec{r}-g\vec{k}\cdot d\vec{r}$$

$$>>\frac{1}{2}d\left(V^2\right)=-\frac{dp}{\rho}-gdz>>\frac{dp}{\rho}+\frac{1}{2}d\left(V^2\right)+gdz=0$$

Bernoulli Equation for Irrotational Flow 3/3

Integrating for incompressible flow

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \quad \Rightarrow \quad \frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant}$$

This equation is valid between any two points in a steady, incompressible, inviscid, and irrotational flow.

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

Velocity Potential $\phi(x,y,z,t)$ 1/4

- The stream function for two-dimensional incompressible flow is $\psi(x,y)$ which satisfies the continuous condtions
- For an irrotational flow, the velocity components can be expressed in terms of a scalar function $\phi(x,y,z,t)$ as

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

where $\phi(x,y,z,t)$ is called the velocity potential.

Velocity Potential $\Phi(x,y,z,t)$ 2/4

❖ In vector form

$$\vec{V} = \nabla\phi$$

❖ For an incompressible flow

$$\nabla \cdot \vec{V} = 0$$

called a potential flow

For incompressible, irrotational flow

$$\vec{V} = \nabla\phi \Rightarrow \nabla \cdot \vec{V} = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Laplace's equation

Laplacian operator

Velocity Potential $\phi(x,y,z,t)$ 3/4

❖ Inviscid, incompressible, irrotational fields are governed by Laplace's equation.

❖ This type flow is commonly called a potential flow.

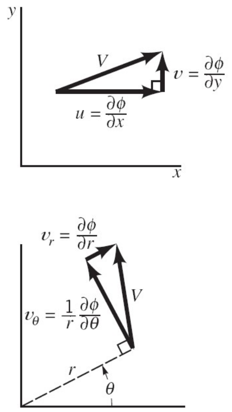
❖ To complete the mathematical formulation of a given problem, boundary conditions have to be specified. These are usually velocities specified on the boundaries of the flow field of interest.

Velocity Potential Φ(x,y,z,t) 4/4

- In cylindrical coordinate, r, θ, and z

$$v_r = \frac{\partial \phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \qquad v_z = \frac{\partial \phi}{\partial z}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$



67

Example 6.4 Velocity Potential and Inviscid Flow Pressure 1/2

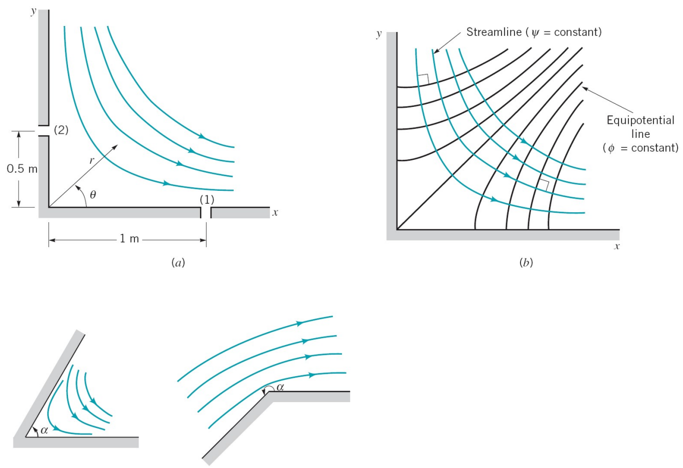
- The two-dimensional flow of a nonviscous, incompressible fluid in the vicinity of the 90° corner of Figure E6.4(a) is described by the stream function

$$\psi = 2r^2 \sin 2\theta$$

Where ψ has units of m^2/s when r is in meters. (a) Determine, if possible, the corresponding velocity potential. (b) If the pressure at point (1) on the wall is 30 kPa, what is the pressure at point (2)? Assume the fluid density is 10^3 kg/m^3 and the x-y plane is horizontal – that is, there is no difference in elevation between points (1) and (2).

68

Example 6.4 Velocity Potential and Inviscid Flow Pressure 2/2



■ FIGURE E6.4

Example 6.4 Solution 1/2

$$\psi = 2r^2 \sin 2\theta$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r\cos 2\theta$$

$$v_\theta = -\frac{\partial \psi}{\partial r} = -4r\sin 2\theta$$

$$v_r = \frac{\partial \phi}{\partial r} = 4r\cos 2\theta \Rightarrow \phi = 2r^2\cos 2\theta + f_1(\theta)$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -4r\sin 2\theta \Rightarrow \phi = 2r^2\cos 2\theta + f_2(r)$$

$$\phi = 2r^2\cos 2\theta + C$$

Let C=0

$$\phi = 2r^2\cos 2\theta$$

Example 6.4 Solution 2/2

Bernoulli equation between points (1) and (2) with no elevation change

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}\Rightarrow p_2=p_1+\frac{\rho}{2}(V_1^2-V_2^2)$$

$$\left.\begin{aligned} &V^2=v_r^2+v_\theta^2\\ &V_1^2=\ldots=16\mathrm{m}^2/\mathrm{s}^2\\ &V_2^2=\ldots=4\mathrm{m}^2/\mathrm{s}^2\end{aligned}\right\}\quad p_2=\ldots=36\mathrm{kPa}$$

$$\psi=2r^2\sin 2\theta=4r^2\cos\theta\sin\theta=4xy$$

$$\phi=2r^2\cos 2\theta=2r^2(\cos^2\theta-\sin^2\theta)=2(x^2-y^2)$$

6.5 Some Basic, Plane Potential Flow

- Since Laplace's equation is linear, various solutions can be added to obtain other solution – that is , if $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are two solutions to Laplace's equation, then $\phi = c_1\phi_1 + c_2\phi_2$ is also solution.
- The practical implication of this result is that if we have certain basic solution we can combine them to obtain more complicated and interesting solutions.
- Several basic velocity potentials, which describe some relatively simple flows, will be determined.
- These basic velocity potential will be combined to represent complicate flows.

Laplace's Equation 1/2

- For two-dimensional, incompressible flow

$$xy:\quad u=\frac{\partial\psi}{\partial y}\qquad v=-\frac{\partial\psi}{\partial x}\qquad (1)\qquad r\theta:\quad v_r=\frac{\partial\psi}{r\partial\theta}\qquad v_\theta=-\frac{\partial\psi}{\partial r}\qquad (1a)$$

- For two-dimensional, irrotational flow

$$xy:\quad u=\frac{\partial\phi}{\partial x}\qquad v=\frac{\partial\phi}{\partial y}\qquad (2)\quad r\theta:\quad v_r=\frac{\partial\phi}{\partial r}\qquad v_\theta=\frac{\partial\phi}{r\partial\theta}\qquad (2a)$$

(1) + irrotational condition … $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0\Rightarrow\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$

(2) + continuity equation … $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\Rightarrow\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$

Laplace's Equation 2/2

- For a two-dimensional incompressible flow, we can define a stream function ψ; if the flow is also irrotational, ψ will satisfy Laplace's equation.
- For an irrotational flow, we can define a velocity potential ϕ; if the flow is also incompressible, ϕ will satisfy Laplace's equation.
- Any function ϕ or ψ that satisfies Laplace's equation represents a possible two-dimensional, incompressible, irrotational flow field.

Φ and Ψ 1/2

- For ψ=constant, $d\psi$ =0 and $d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0$
- The slope of a streamline – a line of constant ψ

$$\left.\frac{dy}{dx}\right)_{\psi} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = \frac{v}{u}$$

- Along a line of constant ϕ, $d\phi$=0 and $d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0$
- The slope of a potential line – a line of constant ϕ

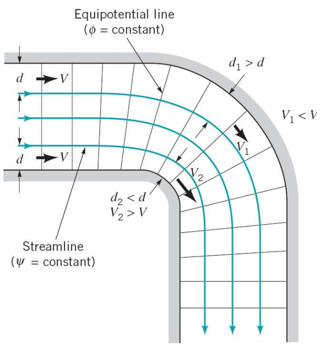
$$\left.\frac{dy}{dx}\right)_{\phi} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v}$$

Line of constant Ψ and constant Φ are orthogonal.

75

Φ and Ψ 2/2

- The lines of constant ϕ (called equipotential lines) are orthogonal to lines of constant ψ (streamlines) at all points where they intersect.
- For any potential flow a "flow net" can be drawn that consists of a family of streamlines and equipotential lines.
- Velocities can be estimated from the flow net, since the velocity is inversely proportional to the streamline spacing



Flow net for 90° bend

76

Uniform Flow 1/2

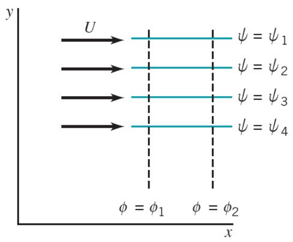
❖ A uniform flow is a simplest plane flow for which the streamlines are all straight and parallel, and the magnitude of the velocity is constant.

u=U and v=0

$$\frac{\partial \phi}{\partial x} = U, \qquad \frac{\partial \phi}{\partial y} = 0 \qquad \Rightarrow \phi = Ux + C$$

set C=0, $\phi = Ux$

$$\frac{\partial \psi}{\partial y} = U, \qquad \frac{\partial \psi}{\partial x} = 0 \qquad \Rightarrow \psi = Uy$$

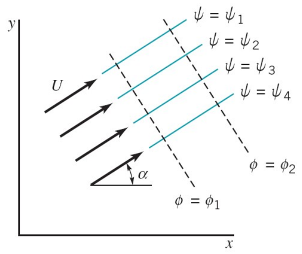


(a)

Uniform Flow 2/2

❖ For a uniform flow of constant velocity V, inclined to an angle α to the x-axis (b).

$$\psi = (U\cos\alpha)y - (U\sin\alpha)x$$
$$\phi = (U\sin\alpha)y + (U\cos\alpha)x$$



(b)

$$\vec{V} = u\vec{i} + v\vec{j} = U\cos\alpha\,\vec{i} + U\sin\alpha\,\vec{j}$$

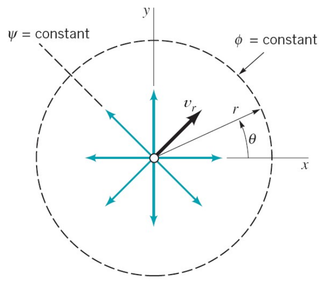
Source and Sink 1/2

❖ For a source flow (from origin radially) with volume flow rate per unit depth m **($m = 2\pi r v_r$)**

$$v_r = \frac{m}{2\pi r} \qquad v_\theta = 0 \Rightarrow \psi = \frac{m}{2\pi}\theta \qquad \Phi = \frac{m}{2\pi}\ln r$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

$$v_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$



Source and Sink 2/2

❖ For a sink flow (toward origin radially) with volume flow rate per unit depth *m*

$$v_r = -\frac{m}{2\pi r} \qquad v_\theta = 0 \Rightarrow \psi = -\frac{m}{2\pi}\theta \qquad \Phi = -\frac{m}{2\pi}\ln r$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

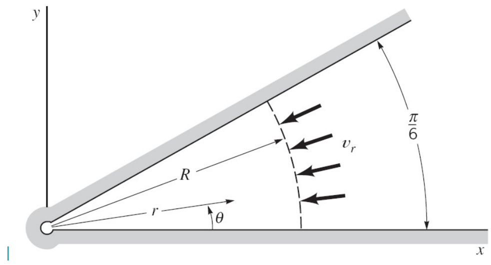
$$v_r = \frac{\partial \phi}{\partial r} \quad \text{and} \quad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Example 6.5 Potential Flow - Sink

- A nonviscous, incompressible fluid flows between wedge-shaped walls into a small opening as shown in Figure E6.5. The velocity potential (in m^2/s), which approximately describes this flow is

$$\phi = -2 \ln r$$

Determine the volume rate of flow (per unit length) into the opening.



81

Example 6.5 Solution

The components of velocity

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

The flowrate per unit width

$$q = \int_0^{\pi/6} v_r R d\theta = -\int_0^{\pi/6} \left(\frac{2}{R}\right) R d\theta$$

$$= -\frac{\pi}{3} = -1.05 m^2/s$$

82

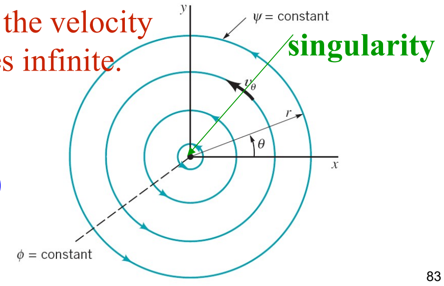
Vortex

- A vortex represents a flow in which the streamlines are concentric circles.
- Vortex motion can be either rotational or irrotational.
- For an irrotational vortex (ccw, center at origin) with vortex strength $K=\Gamma/2\pi$

At r=0, the velocity becomes infinite.

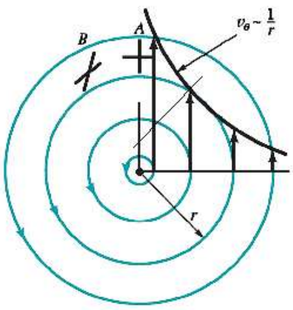
$$v_r = 0 \qquad v_\theta = \frac{K}{r}$$

$$\Rightarrow \psi = -K \ln r \qquad \Phi = K\theta$$



Free Vortex 1/2

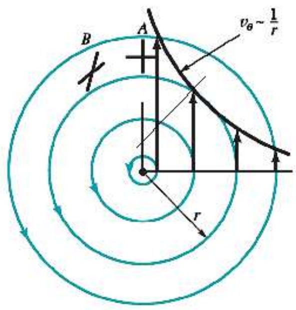
- Free (Irrotational) vortex (a) is that rotation refers to the orientation of a fluid element and not the path followed by the element.
- A pair of small sticks were placed in the flow field at location A, the sticks would rotate as they as they move to location B.



(a)

Free Vortex 2/2

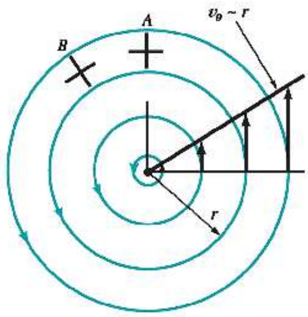
- One of the sticks, the one that is aligned the streamline, would follow a circular path and rotate in a counterclockwise direction
- The other rotates in a clockwise direction due to the nature of the flow field – that is, the part of the stick nearest the origin moves faster than the opposite end.
- The average velocity of the two sticks is zero.



(a)

Forced Vortex

- If the flow were rotating as a rigid body, such that $v_\theta=K_1 r$ where K_1 is a constant.
- Force vortex is rotational and cannot be described with a velocity potential.
- Force vortex is commonly called a rotational vortex.



(b)

Combined Vortex

❖ A combine vortex is one with a forced vortex as a central core and a velocity distribution corresponding to that of a free vortex outside the core.

$$v_\theta = \frac{K}{r} \qquad r > r_0$$

$$v_\theta = r\omega \qquad r \le r_0$$

Circulation 1/3

❖ Circulation Γ is defined as the **line integral of the tangential velocity component about any closed curve** fixed in the flow:

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} = \int_A (\nabla \times \vec{V})_Z \, dA = \int_A 2\omega_Z \, dA$$

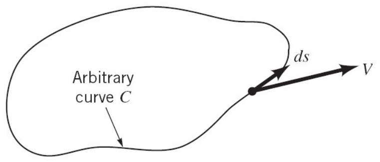
where the $d\vec{s}$ is an element vector tangent to the curve and having length ***ds*** of the element of arc. It's positive corresponds to a c.c.w. direction of integration around the curve.

Circulation 2/3

❖ For irrotational flow , Γ =0

$$\Gamma = \oint_c \vec{V} \cdot d\vec{s} = \int_A (\nabla \times \vec{V}) dA = \oint_c \nabla\phi \cdot d\vec{s} = \oint_c d\phi = 0$$

For irrotational flow , Γ =0



The circulation around any path that does not include the singular point at the origin will be zero.

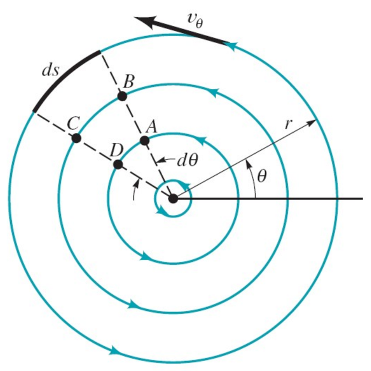
Circulation 3/3

❖ For free vortex $v_\theta = \dfrac{K}{r}$

$$\Gamma = \int_0^{2\pi} \frac{K}{r}(r d\theta) = 2\pi K$$

$$K = \Gamma / 2\pi$$

$$\psi = -\frac{\Gamma}{2\pi} \ln r \qquad \Phi = \frac{\Gamma}{2\pi}\theta$$



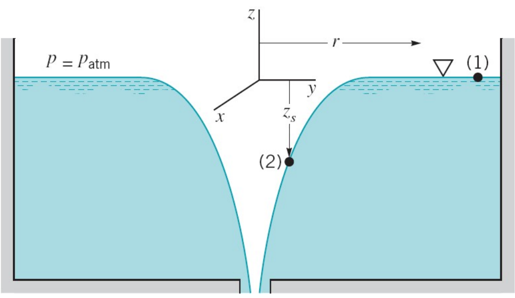
The circulation around any path that encloses singularities will be nozero.

Example 6.6 Potential Flow – Free Vortex

- A liquid drains from a large tank through a small opening as illustrated in Figure E6.6. A vortex forms whose velocity distribution away from the tank opening can be approximated as that of a free vortex having a velocity potential

$$\Phi = \frac{\Gamma}{2\pi}\theta$$

Determine an expression relating the surface shape to the strength of the vortex as specified by the circulation Γ.



Example 6.6 Solution

Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

At free surface $p_1 = p_2 = 0$

$$\frac{V_1^2}{2g} = \frac{V_2^2}{2g} + z_s$$

Far from the origin at point (1), $V_1 = v_\theta = 0$

$$v_{2\theta} = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} = \frac{\Gamma}{2\pi r}$$

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$

Doublet 1/2

❖ For a doublet (produced mathematically by allowing a source and a sink of numerically equal strength to merge) with a strength m

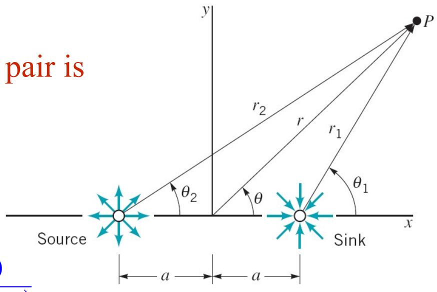
The combined stream function for the pair is

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$\tan\theta_1 = \frac{r\sin\theta}{(r\cos\theta - a)} \quad \text{and} \quad \tan\theta_2 = \frac{r\sin\theta}{(r\cos\theta + a)}$$

$$\Rightarrow \tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{(r^2 - a^2)} \Rightarrow \psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$



Doublet 2/2

$$a \to 0 \qquad \psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)}$$

The so-called doublet is formed by letting $a \to 0$, $m \to \infty$

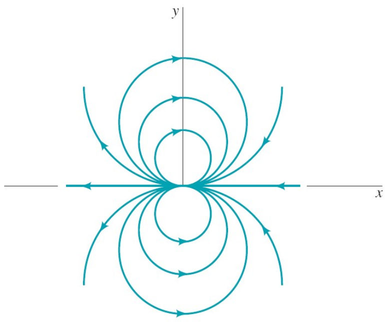
$$\frac{r}{r^2 - a^2} \to \frac{1}{r}$$

$$\psi = -\frac{K\sin\theta}{r} \qquad \phi = \frac{K\cos\theta}{r}$$

$$K = \frac{ma}{\pi}$$

Streamlines for a Doublet

❖ Plots of lines of constant ψ reveal that the streamlines for a doublet are circles through the origin tangent to the x-axis.



95

■ **TABLE 6.1**

Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see **Fig. 6.16*b***)	$\phi = U(x \cos \alpha + y \sin \alpha)$	$\psi = U(y \cos \alpha - x \sin \alpha)$	$u = U \cos \alpha$ $v = U \sin \alpha$
Source or sink (see **Fig. 6.17**) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi} \ln r$	$\psi = \dfrac{m}{2\pi} \theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see **Fig. 6.18**) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi} \theta$	$\psi = -\dfrac{\Gamma}{2\pi} \ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see **Fig. 6.23**)	$\phi = \dfrac{K \cos \theta}{r}$	$\psi = -\dfrac{K \sin \theta}{r}$	$v_r = -\dfrac{K \cos \theta}{r^2}$ $v_\theta = -\dfrac{K \sin \theta}{r^2}$

[a]Velocity components are related to the velocity potential and stream function through the relationships:

$u = \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y} \quad v = \dfrac{\partial \phi}{\partial y} = -\dfrac{\partial \psi}{\partial x} \quad v_r = \dfrac{\partial \phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} \quad v_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = -\dfrac{\partial \psi}{\partial r}.$

96

6.6 Superposition of Elementary Plane Flows 1/2

- ❖ Potential flows are governed by Laplace's equation, which is a linear partial differential equation.
- ❖ Various basic velocity potentials and stream function, ⎞ and ⎠, can be combined to form new potentials and stream functions.

$$\psi_3=\psi_1+\psi_2 \qquad \phi_3=\phi_1+\phi_2$$

97

Superposition of Elementary Plane Flows 2/2

- ❖ Any streamline in an inviscid flow field can be considered as a solid boundary, since the conditions along a solid boundary and as streamline are the same – that is, there is no flow through the boundary or the streamline.
- ❖ We can combine some of the basic velocity potentials or stream functions to yield a streamline that corresponds to a particular body shape of interest, that combination can be used to describe in detail the flow around that body.

<u>**Methods of superposition**</u>

98

Half-Body : Uniform Stream + Source 1/4

$$\psi = \psi_{uniform-flow} + \psi_{source} = Ur\sin\theta + \frac{m}{2\pi}\theta$$

$$\phi = \phi_{uniform-flow} + \phi_{source} = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

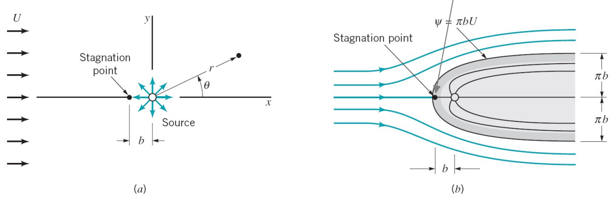
$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta + \frac{m}{2\pi r}$$

$$v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$$

The stagnation point occurs at r=b, and $\theta=\pi$, where $v_r=v_\theta=0$.

$$v_r = -U + \frac{m}{2\pi b} = 0 \Rightarrow b = \frac{m}{2\pi U}$$

The combination of a uniform flow and a source can be used to describe the flow around a streamlined body placed in a uniform stream.



Half-Body : Uniform Stream + Source 2/4

The value of the stream function at the stagnation point can be obtained by evaluating ψ at r=b θ=π

$$\psi_{stagnation} = \frac{m}{2} \qquad \frac{m}{2} = \pi bU$$

The equation of the streamline passing through the stagnation point is

$$\pi bU = Ur\sin\theta + bU\theta$$

$$r = \frac{b(\pi-\theta)}{\sin\theta}$$

The streamline can be replaced by a solid boundary. The body is open at the downstream end, and thus is called a HALF-BODY. The combination of a uniform flow and a source can be used to describe the flow around a streamlined body placed in a uniform stream.

Half-Body : Uniform Stream + Source 3/4

$$r = \frac{b(\pi - \theta)}{\sin\theta} \Rightarrow y = r\sin\theta \;=\; b(\pi - \theta)$$

As $\theta \to 0$ or $\theta = \to 2\pi$ the half-width approaches $\pm b\pi$. The width of the half-body asymptotically approach $2\pi b$.
The velocity components at any point

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\cos\theta + \frac{m}{2\pi r} \qquad v_\theta = -\frac{\partial\psi}{\partial r} = -U\sin\theta$$

$$b = \frac{m}{2\pi U}$$

$$V^2 = v_r^2 + v_\theta^2 = U^2 + \frac{Um\cos\theta}{\pi r} + \left(\frac{m}{2\pi r}\right)^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

Half-Body : Uniform Stream + Source 4/4

❖ With the velocity known, the pressure at any point can be determined from the Bernoulli equation

$$\underbrace{p_0 + \frac{1}{2}\rho U^2}_{\text{Far from the body}} = p + \frac{1}{2}\rho V^2$$

Where elevation change have been neglected.

Example 6.7 Potential Flow – Half-Body

- The shape of a hill arising from a plain can be approximated with the top section of a half-body as is illustrated in Figure E6.7(a). The height of the hill approaches 60m as shown. (a) When a 64km/hr wind blows toward the hill, what is the magnitude of the air velocity at a point on the hill directly above the origin [point (2)]? (b) What is the elevation of point (2) above the plain and what is the difference in pressure between point (1) on the plain far from the hill and point (2)? Assume an air density of $1.23 kg/m^3$?

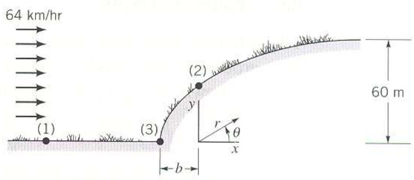


FIGURE E6.7a

103

Example 6.7 Solution[1/2]

The velocity is

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

At point (2), $\theta=\pi/2$

$$r = \frac{b(\pi - \theta)}{\sin\theta} = \frac{\pi b}{2}$$

$$V_2^2 = U^2\left(1 + \frac{b^2}{(\pi b/2)^2}\right) = U^2\left(1 + \frac{4}{\pi^2}\right), \quad V_2 = 76 km/hr$$

The elevation at (2) above the plain is $y_2 = \frac{\pi b}{2} = \frac{60m}{2} = 30m$

$\pi b = 60m$

104

Example 6.7 Solution[2/2]

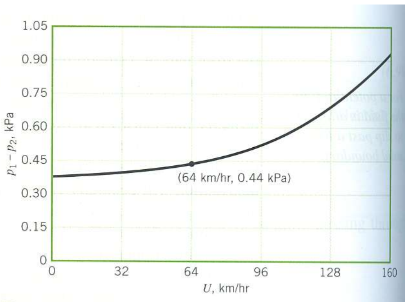
From the Bernoulli equation

$$\frac{p_1}{\gamma}+\frac{V_1^2}{2g}+z_1=\frac{p_2}{\gamma}+\frac{V_2^2}{2g}+z_2$$

$$p_1-p_2=\frac{\rho}{2}(V_2^2-V_1^2)+\gamma(y_2-y_1)=\ldots=440.9N/m^2=0.44kPa$$

$$V_1=(64km/hr)\left(\frac{1000m/km}{3600s/hr}\right)=17.8m/s$$

$$V_2=(76km/hr)\left(\frac{1000m/km}{3600s/hr}\right)=21.1m/s$$



■ FIGURE E6.7b

Rankine Oval: Uniform Stream + Doublet [1/3]

$$\psi=Ur\sin\theta-\frac{m}{2\pi}(\theta_1-\theta_2)$$

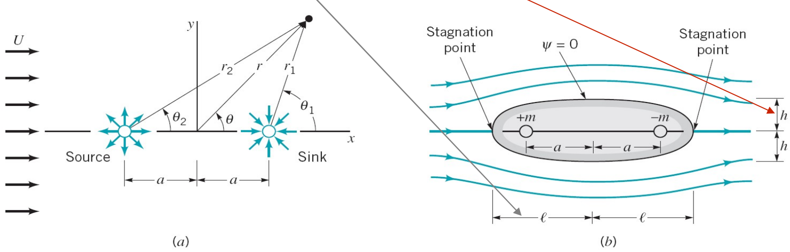
$$\Phi=Ur\cos\theta-\frac{m}{2\pi}(\ln r_1-\ln r_2)$$

$$\psi=Ur\sin\theta-\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2-a^2}\right)$$

$$\psi=Uy-\frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2+y^2-a^2}\right)$$

The corresponding streamlines for this flow field are obtained by setting ψ=constant. It is discovered that the streamline forms a closed body of length 2ℓ and width 2h.

Rankine ovals



Rankine Oval: Uniform Stream + Doublet 2/3

- The stagnation points occur at the upstream and downstream ends of the body. These points can be located by determining where along the x axis the velocity is zero.
- The stagnation points correspond to the points where the uniform velocity, the source velocity, and the sink velocity all combine to give a zero velocity.
- The locations of the stagnation points depend on the value of a, m, and U. at y=0, **V**=0 $l=|x|$

The body half-length ℓ

Dimensionless

$$\ell = \left(\frac{ma}{\pi U} + a^2 \right)^{1/2} \Rightarrow \frac{\ell}{a} = \left(\frac{m}{\pi U a} + 1 \right)^{1/2}$$

Rankine Oval: Uniform Stream + Doublet 3/3

- The body half-width, h, can be obtained by determining the value of y where the y axis intersects the Ψ=0 streamline. Thus, with Ψ=0, x=0, and y=h.

The body half-width 2h

$$h = \frac{h^2 - a^2}{2a} \tan \frac{2\pi U h}{m} \Rightarrow \frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right] \tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right]$$

Dimensionless

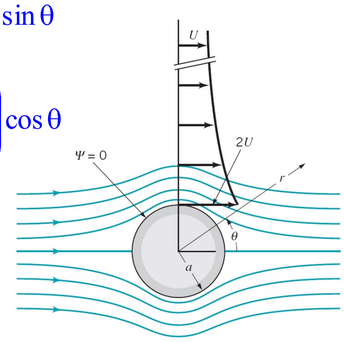
Flow around a Circular Cylinder 1/4

❖ When the distance between the source-sink pair approaches zero, the shape of the rankine oval becomes more blunt and in fact approaches a circular shape.

$$\psi = Ur\sin\theta - \frac{K\sin\theta}{r} \quad \Rightarrow \quad \psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta$$

$$\Phi = Ur\cos\theta + \frac{K\cos\theta}{r} \quad \quad \Phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta$$

In order for the stream function to represent flow around a circular cylinder
$\begin{cases} \Psi = \text{constant for } r=a \\ \Psi = 0 \text{ for } r=a \quad K=Ua^2 \end{cases}$



Flow around a Circular Cylinder 2/4

The velocity components

$$v_r = \frac{\partial\Phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$v_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

On the surface of the cylinder (r=a)

$$v_r = 0$$

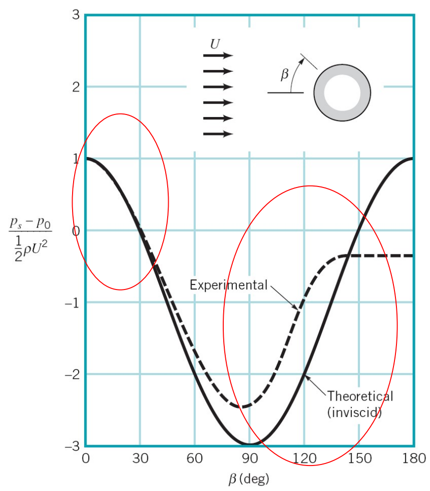
$$v_\theta = -2U\sin\theta$$

The pressure distribution on the cylinder surface can be obtained from the Bernoulli equation

$$\underbrace{p_0 + \frac{1}{2}\rho U^2}_{\text{Far from the body}} = p_s + \frac{1}{2}\rho v_{\theta s}^2 \quad \Rightarrow \quad p_s = p_0 + \frac{1}{2}\rho U^2(1 - 4\sin^2\theta)$$

$$v_\theta = -2U\sin\theta$$

Flow around a Circular Cylinder 3/4

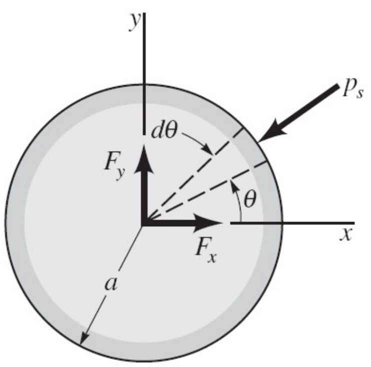


On the upstream part of the cylinder, there is approximate agreement between the potential flow and the experimental results. Because of the viscous boundary layer that develops on the cylinder, the main flow separates from the surface of the cylinder, leading to the large difference between the theoretical, frictionless solution and the experimental results on the downstream side of the cylinder.

Flow around a Circular Cylinder 4/4

$$F_x = -\int_0^{2\pi} p_s \cos\theta a d\theta$$

$$F_y = -\int_0^{2\pi} p_s \sin\theta a d\theta$$



Flow around a Circular Cylinder + Free Vortex 1/4

❖ Adding a free vortex to the stream function or velocity potential for the flow around a cylinder.

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r \qquad \Phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

The circle r=a will still be a streamline, since the streamlines for the added free vortex are all circular.

The tangential velocity on the surface of the cylinder

$$v_\theta = -\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = -2U\sin\theta + \frac{\Gamma}{2\pi a}$$

113

Flow around a Circular Cylinder + Free Vortex 2/4

❖ This type of flow could be approximately created by placing a rotating cylinder in a uniform stream.

❖ Because of the presence of viscosity in any real fluid, the fluid in contacting with the rotating cylinder would rotate with the same velocity as the cylinder, and the resulting flow field would resemble that developed by the combination of a uniform flow past a cylinder and a free vortex.

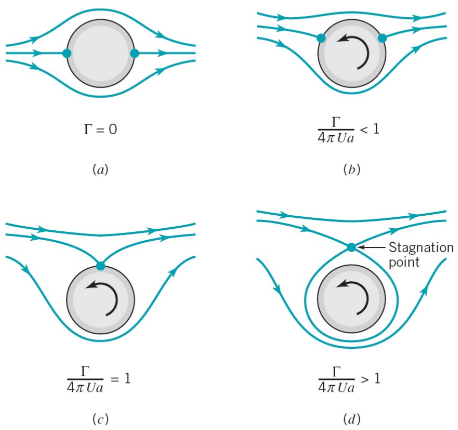
114

Flow around a Circular Cylinder + Free Vortex 3/4

- A variety of streamline patterns can be developed, depending on the vortex strength Γ.
- The location of stagnation points on a circular cylinder (a) without circulation; (b, c, d) with circulation.

$$\sin\theta_{stag} = \frac{\Gamma}{4\pi U a}$$

$v_\theta = 0 \qquad \theta = \theta_{stag}$



Flow around a Circular Cylinder + Free Vortex 4/4

- For the cylinder with circulation, the surface pressure, p_s, is obtained from the Bernoulli equation

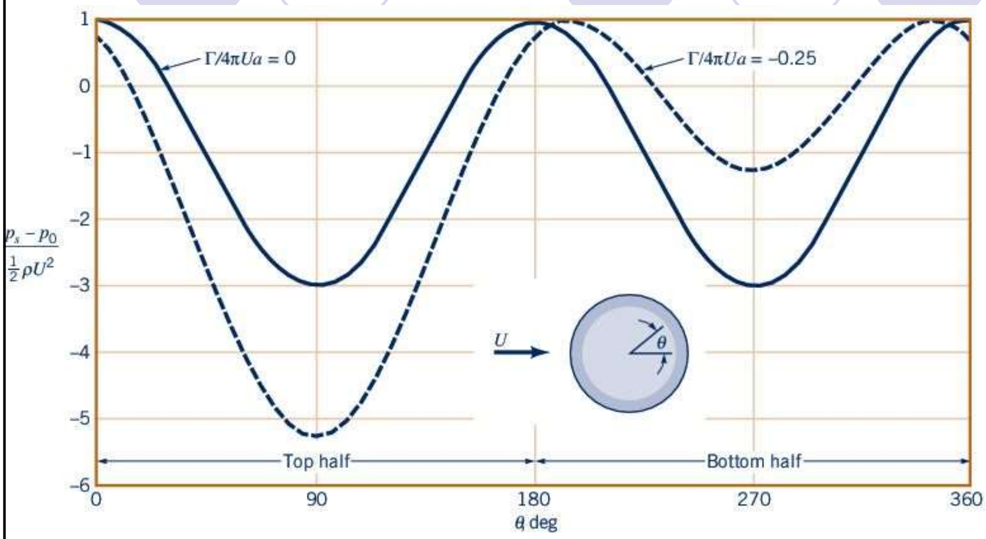
$$p_0 + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$

$$p_s = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi a U} - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\right)$$

Drag $F_x = -\int_0^{2\pi} p_s \cos\theta a d\theta = 0$

Lift $F_y = -\int_0^{2\pi} p_s \sin\theta a d\theta = -\rho U\Gamma$

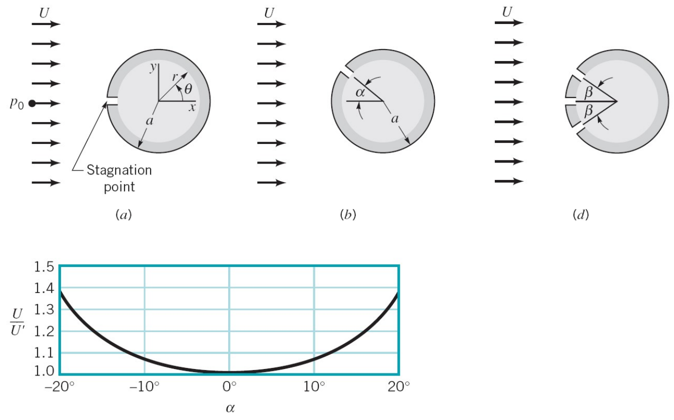
Pressure distribution on a cylinder with and without circulation



Example 6.8 Potential Flow – Cylinder 1/2

- When a circular cylinder is placed in a uniform stream, a stagnation point is created on the cylinder as is shown in Figure E6.8(a). If a small hole is located at this point, the stagnation pressure, p_{stag}, can be measured and used to determine the approach velocity, U. (a) Show how p_{stag} and U are related. (b) If the cylinder is misaligned by an angle α (Figure E6.8(b)), but the measured pressure still interpreted as the stagnation pressure, determine an expression for the ratio of the true velocity, U, to the predicted velocity, U'. Plot this ratio as a function of α for the range $-20^\circ \leqq \alpha \leqq 20^\circ$.

Example 6.8 Potential Flow – Cylinder 2/2



■ FIGURE E6.8

Example 6.8 Solution 1/2

The Bernoulli equation between a point on the stagnation streamline upstream from the cylinder and the stagnation point

$$\frac{p_0}{\gamma}+\frac{U^2}{2g}=\frac{p_{stag}}{\gamma}$$

The difference between the pressure at the stagnation point and the upstream pressure

$$\Rightarrow U=\left[\frac{2}{\rho}(p_{stag}-p_0)\right]^{1/2}$$

If the cylinder is misaligned by an angle, α, the pressure actually measured, p_a, will be different from the stagnation pressure.

$$U'=\left[\frac{2}{\rho}(p_a-p_0)\right]^{1/2}\Rightarrow\frac{U(\text{true})}{U'(\text{predicted})}=\left(\frac{p_{stag}-p_0}{p_a-p_0}\right)^{1/2}$$

Example 6.8 Solution[2/2]

On the surface of the cylinder (r=a) $v_\theta = -2U\sin\theta$

The Bernoulli equation between a point upstream if the cylinder and the point on the cylinder where r=a, θ=α.

$$p_0 + \frac{1}{2}\rho U^2 = p_a + \frac{1}{2}\rho(-2U\sin\alpha)^2$$

$$p_a - p_0 = \frac{1}{2}\rho U^2(1-4\sin^2\alpha)$$

$$p_{stag} - p_0 = \frac{1}{2}\rho U^2$$

$$\frac{U}{U'} = (1-4\sin^2\alpha)^{-1/2}$$

121

6.8 Viscous Flow

❖ To incorporate viscous effects into the differential analysis of fluid motion

General equation of motion Eq(6-50)

$$\rho g_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

⇧Stress-Deformation Relationship ⇨⇨⇨

122

Stress-Deformation Relationship 1/2

- For (incompressible) Newtonian fluid flow, the stresses must be expressed in terms of the velocity and pressure field.

Cartesian coordinates

$$\sigma_{xx} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p - \frac{2}{3}\mu\nabla\cdot\vec{V} + 2\mu\frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

Stress-Deformation Relationship 2/2

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial v_z}{\partial z}$$

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right)$$

$$\tau_{rz} = \tau_{zr} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)$$

Cylindrical polar coordinates (Newtonian incompressible fluid)

Introduced into the differential equation of motion….

The Navier-Stokes Equations 1/5

❖ These obtained equations of motion are called the Navier-Stokes Equations.

Cartesian coordinates

$$\rho\frac{Du}{Dt}=\rho g_x-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]$$

$$\rho\frac{Dv}{Dt}=\rho g_y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]$$

$$\rho\frac{Dw}{Dt}=\rho g_z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial zx}\right)\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z}-\frac{2}{3}\nabla\cdot\vec{V}\right)\right]$$

125

The Navier-Stokes Equations 2/5

Cylindrical polar coordinates

(Newtonian incompressible fluid with constant μ)

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\rho g_r+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right)-\frac{v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)$$

$$=-\frac{1}{r}\frac{\partial p}{\partial\theta}+\rho g_\theta+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right)-\frac{v_\theta}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}++\frac{v_\theta}{r}\frac{\partial v_z}{\partial\theta}+v_z\frac{\partial v_z}{\partial z}\right)$$

$$=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

126

The Navier-Stokes Equations 3/5

❖ Under **incompressible flow with constant viscosity conditions**, the Navier-Stokes equations are reduced to:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

The Navier-Stokes Equations 4/5

❖ Under **frictionless condition**, the equations of motion are reduced to **Euler's Equation:**

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z$$

$$\Rightarrow \quad \rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla p$$

The Navier-Stokes Equations 5/5

❖ The Navier-Stokes equations apply to both laminar and turbulent flow, but for turbulent flow each velocity component fluctuates randomly with respect to time and this added complication makes an analytical solution intractable.

❖ The exact solutions referred to are for laminar flows in which the velocity is either independent of time (steady flow) or dependent on time (unsteady flow) in a well-defined manner.

6.9 Some Simple Solutions for Viscous, Incompressible Fluids

❖ A principal difficulty in solving the Navier-Stokes equations is because of their nonlinearity arising from the **convective acceleration terms**.

❖ There are no general analytical schemes for solving nonlinear partial differential equations.

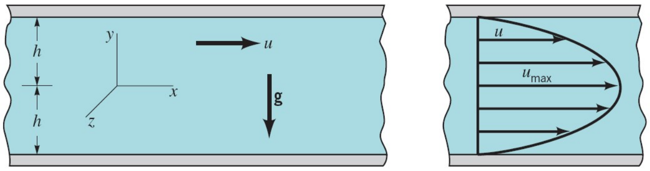
❖ There are a few special cases for which the convective acceleration vanishes. In these cases exact solution are often possible.

Steady, Laminar Flow between Fixed Parallel Plates 1/5

❖ The Navier-Stokes equations reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \qquad 0 = -\frac{\partial p}{\partial z}$$



(a) (b)

Steady, Laminar Flow between Fixed Parallel Plates 2/5

$$\left.\begin{aligned} 0 &= -\frac{\partial p}{\partial y} - \rho g \\ 0 &= -\frac{\partial p}{\partial z} \end{aligned}\right\} \xrightarrow{\text{Integrating}} p = -\rho g y + f_1(x)$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \xrightarrow{\text{Integrating}}$$

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2 \qquad c_1\,? \qquad c_2\,?$$

Steady, Laminar Flow between Fixed Parallel Plates 3/5

❖ With the boundary conditions u=0 at y=-h u=0 at y=h

$$c_1 = 0,\ c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2$$

Velocity distribution

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)\left(y^2 - h^2\right)$$

133

❖ Shear stress distribution

$$\tau_{yx} = \mu\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial p}{\partial x}\right)y$$

❖ Volume flow rate

$$q = \int_{-h}^{h} u\,dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

$$\frac{\partial p}{\partial x} = \text{constant} = \frac{p_2 - p_1}{\ell} = -\frac{\Delta p}{\ell}$$

$$>> q = \frac{2h^3 \Delta p}{3\mu\ell}$$

134

Steady, Laminar Flow between Fixed Parallel Plates 5/5

❖ Average velocity

$$V_{average} = \frac{q}{2h} = \frac{h^2 \Delta p}{3\mu\ell}$$

❖ Point of maximum velocity

$$\frac{du}{dy} = 0 \quad \textbf{at y=0}$$

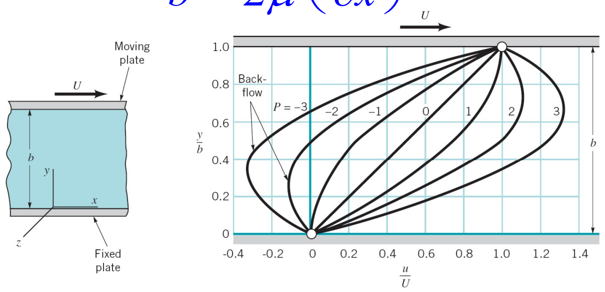
$$u = u_{max} = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{3}{2} V_{average}$$

135

Couette Flow 1/3

❖ Since only the **boundary conditions have changed**, there is **no need to repeat the entire analysis** of the "both plates stationary" case.

$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by)$$



(a) (b)

136

Couette Flow 2/3

❖ The boundary conditions for the moving plate case are

u=0 at y=0

u=U at y=b

$$\Rightarrow c_1 = \frac{U}{b} - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)b \qquad c_2 = 0$$

Velocity distribution

$$u = \frac{Uy}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 - \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)by$$

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left[1-\left(\frac{y}{b}\right)\right]$$

$$\boxed{P = \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)}$$

137

Couette Flow 3/3

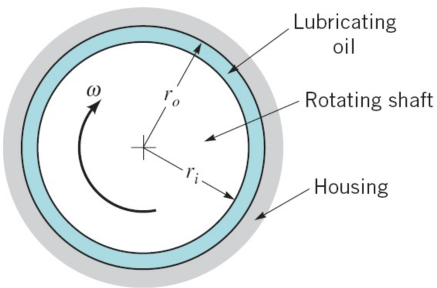
❖ **Simplest type of Couette flow**

$$\frac{\partial p}{\partial x} = 0 \Rightarrow u = U\frac{y}{b}$$

This flow can be approximated by the flow between closely spaced concentric cylinder is fixed and the other cylinder rotates with a constant angular velocity.

$$U = r_i\omega \qquad b = r_o - r_i$$

$$\tau = \mu r_i\omega/(r_o - r)$$

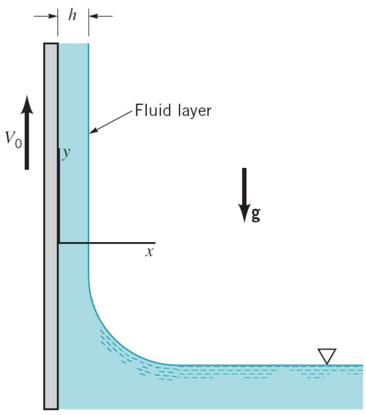


Flow in the narrow gap of a journal bearing.

138

Example 6.9 Plane Couette Flow

- A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0, as illustrated in Figure E6.9(a). Because of viscous forces the belt picks up a film of fluid of thickness h. Gravity tends to make the fluid drain down the belt. Use the Navier Stokes equations to determine an expression for the average velocity of the fluid film as it is dragged up the belt. Assume that the flow is laminar, steady, and fully developed.



Example 6.9 Solution 1/2

Since the flow is assumed to be fully developed, the only velocity component is in the y direction so that u=w=0.

From the continuity equation

$\frac{\partial v}{\partial y} = 0,$ and for steady flow, so that $v=v(x)$

$\rightarrow \frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial z} = 0$

$$0 = -\rho g + \mu \frac{d^2 v}{dx^2}$$

Integrating

$$\frac{dv}{dx} = \frac{\gamma}{\mu} x + c_1$$

$$\tau_{xy} = \mu\left(\frac{dv}{dx}\right)$$

Example 6.9 Solution 2/2

$$\tau_{xy} = 0 \text{ at } x = h \xrightarrow{} c_1 = -\frac{\gamma h}{\mu}$$

Integrating

$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + c_2$$

$$v = V_0 \text{ at } x = 0 \xrightarrow{} c_2 = V_0$$

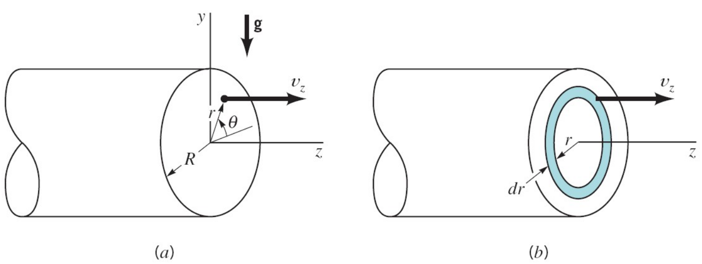
$$v = \frac{\gamma}{2\mu}x^2 - \frac{\gamma h}{\mu}x + V_0$$

141

Steady, Laminar Flow in Circular Tubes 1/5 (Poiseuille Flow)

❖ Consider the flow through a horizontal circular tube of radius R.

$$v_r = 0,\ v_\theta = 0 \Rightarrow continuity\ \ Eq.\ \frac{\partial v_z}{\partial z} = 0 \quad \mathrm{v_z = v_z(r)}$$



142

Steady, Laminar Flow in Circular Tubes 2/5

Navier – Stokes equation reduced to

$$\left.\begin{aligned}0 &= -\rho g \sin\theta - \frac{\partial p}{\partial r}\\ 0 &= -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}\end{aligned}\right\} \xrightarrow{\text{Integrating}} \begin{aligned}p &= -\rho g (r\sin\theta) + f_1(z)\\ p &= -\rho g y + f_1(z)\end{aligned}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right) \xrightarrow{\text{Integrating}}$$

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2 \qquad c_1\,? \qquad c_2\,?$$

Steady, Laminar Flow in Circular Tubes 3/5

At r=0, the velocity v_z is finite. At r=R, the velocity v_z is zero.

$$c_1 = 0, c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)R^2$$

Velocity distribution $\Rightarrow$ $$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left(r^2 - R^2\right)$$

Steady, Laminar Flow in Circular Tubes 4/5

❖ The shear stress distribution

$$\tau_{rz} = \mu \frac{dv_z}{dr} = \frac{r}{2}\left(\frac{\partial p}{\partial z}\right)$$

❖ Volume flow rate

$$Q = \int_0^R v_z 2\pi r dr = ... = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right)$$

$$\frac{\partial p}{\partial z} = \text{constant} = \frac{p_2 - p_1}{\ell} = -\Delta p / \ell$$

$$>> Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{\pi R^4}{8\mu}\left(\frac{\Delta p}{\ell}\right) = \frac{\pi \Delta p D^4}{128\mu\ell}$$

145

Steady, Laminar Flow in Circular Tubes 5/5

❖ Average velocity

$$V_{average} = \frac{Q}{A} = \frac{Q}{\pi R^2} = \frac{R^2 \Delta p}{8\mu\ell}$$

❖ Point of maximum velocity

$$\frac{dv_z}{dr} = 0 \quad \textbf{at r=0}$$

$$v_{max} = -\frac{R^2 \Delta p}{4\mu\ell} = 2V_{average} \Rightarrow \frac{v_z}{v_{max}} = 1 - \left(\frac{r}{R}\right)^2$$

146

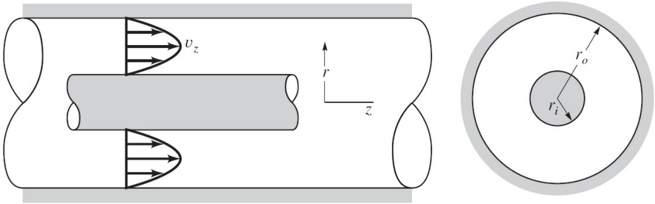
Steady, Axial, Laminar Flow in an Annulus 1/2

For steady, laminar flow in circular tubes

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right) r^2 + c_1 \ln r + c_2 \qquad c_1\,? \qquad c_2\,?$$

Boundary conditions

$$v_z = 0 \;,\; \text{at } r = r_o$$
$$v_z = 0 \;,\; \text{at } r = r_i$$



147

Steady, Axial, Laminar Flow in an Annulus 2/2

The velocity distribution

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)} \ln\frac{r}{r_o}\right]$$

The volume rate of flow

$$Q = \int_{r_i}^{r_o} v_z (2\pi r)dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^{\,4} - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

The maximum velocity occurs at $r = r_m$

$$\frac{\partial v_z}{\partial r} = 0 \rightarrow \quad r_m = \left[\frac{r_o^2 - r_i^2}{2\ln(r_o/r_i)}\right]^{1/2}$$

148