Local Class Field Theory

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January 2020

1 Preliminary

1.1 Setting

Here, we refer to a local field as a non-Archimedean one. Suppose K be a local field with valuation v_K normalized such that $v_K(K^*) = \mathbb{Z}$. We denote the only maximal ideal $\mathfrak{P}_K$ in K with uniformizer π_K. Finally we denote $\mathcal{O}_K$ as the valuation ring and $U_K^{(n)} := 1 + \mathfrak{P}_K^n, U_K := U_K^{(0)}$. We denote $\widetilde{K}$ the maximal unramified extension of K, k the residue field of $\widetilde{K}$ and $\bar{k}$ be the algebraic closure of k. Note that $\bar{k}$ is also the residue field of $\widetilde{K}$. Thus we have the following inclusion diagram.

$$\begin{tikzcd}
\widetilde{K} \arrow[r, two heads, "\text{mod } \mathfrak{P}_{\widetilde{K}}"] & \bar{k} \\
K \arrow[u, hook] \arrow[r, two heads, "\text{mod } \mathfrak{P}_K"] & k \arrow[u, hook]
\end{tikzcd}$$

1.2 Procyclic Galois Group

Since any finite sub-extension of $\widetilde{K}$ share the same uniformizer with K and $\pi_K \in K$ is fixed by $\mathsf{Gal}(\widetilde{K}/K)$, we only need to specify how automorphism acts on residue field and thus.

$$\mathsf{Gal}(\widetilde{K}/K) \cong \mathsf{Gal}(\bar{k}/k)$$

The residue field k is finite, thus easier to study.

Definition 1.1 (Procyclic Group)**.** *Let's first see a useful structure, the procyclic group,*

$$\widehat{\mathbb{Z}} := \varprojlim_n \mathbb{Z}/n\mathbb{Z} := \left\{(a_n \mod n)_{1\leq n} : \forall m \middle| n, a_n = a_m \mod n\right\}.$$

An element in such projective limit could be thought as the following diagram where each arrow respect modular equivalence.

$$\begin{tikzcd}
\cdots & a_4 \arrow[r] \arrow[rrd] & a_2 \arrow[rd] & \\
\cdots & a_6 \arrow[r] \arrow[ru] \arrow[rr, bend right] & a_3 \arrow[r] & a_1 \\
& \cdots & a_5 \arrow[ru] &
\end{tikzcd}$$

Theorem 1.2. *Suppose $k \cong \mathbb{F}_q$. We have a natural isomorphism,*

$$\mathsf{Gal}(\bar{k}/k) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} =: \widehat{\mathbb{Z}}.$$

For $g \mapsto (a_n)_{n\geq 1}$, we have

$$g\Big|_{\mathbb{F}_{q^n}} : x \mapsto x^{q^{a_n}}, \text{ for all } n.$$

Proof. First of all, the map is clearly multiplicative. Then, by assumption, $\bar{k} \cong \bar{\mathbb{F}}_q = \bigcup_{n\geq 1} \mathbb{F}_{q^n}$. For $g \in \mathsf{Gal}(\bar{k}/k)$ restricting on finite subextension, we have,

$$g\Big|_{\mathbb{F}_{q^n}} \in \langle \phi_q \rangle,$$

where $\phi_q : x \mapsto x^q$. Therefore, the map is surjective. On the other hand, if $g\Big|_{\mathbb{F}_{q^n}} = \mathsf{id}$, then $x^{q^{a_n}} = x, \forall x \in \mathbb{F}_{q^n} \subseteq \bar{k}$. Therefore $a_n = 1 \mod n$ and we have trivial kernel. $\square$

We could also regard $\widehat{\mathbb{Z}}$ as product of p-adic integers. Namely,

$$\varprojlim_n \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}} \xrightarrow{\sim} \prod_{p \text{ primes}} \mathbb{Z}_p,$$
$$(a_n)_{n\geq 1} \longmapsto (a_p)_{p \text{ primes}}.$$

This is trivially injective, and indeed surjective due to Chinese remainder theorem. For any $n \in \mathbb{Z}_{>0}$, write $n = \prod_{p \text{ primes}} p^{e_p}$ the prime factorization with $\forall' p : e_p = 0$, then we have,

$$\widehat{Z}/n\widehat{\mathbb{Z}} \cong \prod_{p \text{ primes}} \mathbb{Z}_p/n\mathbb{Z}_p \cong \prod_{p \text{ primes}} \mathbb{Z}_p/p^{e_p}\mathbb{Z}_p \cong \mathbb{Z}/n\mathbb{Z}.$$

Equip $\widehat{\mathbb{Z}}$ with the product topology from all $\mathbb{Z}_p$ topology, those open groups within are exactly $n\widehat{\mathbb{Z}}$ where $n \in \mathbb{Z}$. On the other hand, given any (possibly infinite) Galois extension, Ω/K, we could also equip $\mathsf{Gal}(\Omega/K)$ with Krull topology, which is generated by the open sets $\sigma\mathsf{Gal}(\Omega/L)$, where $\sigma \in \mathsf{Gal}(\Omega/K)$ and L/K is any finite subextension.

Theorem 1.3 (Infinite Galois Theory)**.** *There is an bijective correspondence,*

$$\{M/K : \textit{subextension of } \Omega/K\} \longleftrightarrow \{H \leq \mathsf{Gal}(\Omega/K) : H \textit{ is closed}\},$$

given by $M/K \mapsto \mathsf{Gal}(\Omega/M)$ *and* $\{x \in \Omega : x \textit{ fixed by } H\} \leftarrow\!\shortmid H$.

Denote $\mathcal{F}(H)$ the fixed field of H, then $\mathsf{Gal}(\Omega/\mathcal{F}(H)) = \bar{H}$ is the topological closure of H, represented in the following diagram.

$$\begin{tikzcd}
\Omega \arrow[rr, leftrightarrow] & & \{e\} \arrow[d, hook] \\
& & H \arrow[d, hook] \arrow[dll] \\
\mathcal{F}(H) \arrow[uu, hook] \arrow[rr, leftrightarrow] & & \bar{H} = \mathsf{Gal}(\Omega/\mathcal{F}(H)) \arrow[d, hook] \\
K \arrow[u, hook] \arrow[rr, leftrightarrow] & & \mathsf{Gal}(\Omega/K)
\end{tikzcd}$$

Proposition 1.4. *Consider,* $\sigma \in \mathsf{Gal}(\Omega/K)$*, and* Σ *the fixed field of* $\langle\sigma\rangle$*, then* $\mathsf{Gal}(\Omega/\Sigma)$ *is a quotient of the procyclic group,* $\widehat{\mathbb{Z}}$.

Proof. Denote $G = \mathsf{Gal}(\Omega/\Sigma)$, for every $n \in \mathbb{Z}_{>0}$ we have a surjective homomorphism,

$$\begin{aligned} \mathbb{Z}/n\mathbb{Z} &\longrightarrow G/G^n, \\ 1 \mod n &\longmapsto \sigma \mod G^n. \end{aligned}$$

Running through n we get a continuous surjective homomorphism,

$$\widehat{\mathbb{Z}} \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} \longrightarrow \varprojlim_n G/G^n \cong G.$$

□

1.3 Herbrand Quotient

Definition 1.5. *Suppose* A *is a multiplicative* G *module, and we denote* $a^g := *(g,a), \forall g \in G, a \in A$*. Then, define the following objects,*

$$\begin{aligned} &N_G : A \to N_G A, \quad a \mapsto \prod_{g \in G} a^g, \\ &A_G := \{a \in A : a^g = a, \forall g \in G\}, \quad I_G A := \left\{a^{g-1} : a \in A, g \in G\right\} \\ &H^0(G,A) := A_G/N_G A, \quad H^{-1}(G,A) := \ker N_G / I_G A. \end{aligned}$$

If $G = \langle\sigma\rangle$ is cyclic, then

$$I_G A = \left\{a^{\sigma^m - 1} : m \in \mathbb{Z}\right\} = \left\{a^{(\sigma-1)(\sum_{0 \leq i < m} \sigma^i)} : m \in \mathbb{Z}\right\} = A^{\sigma-1}.$$

Suppose there is a exact sequence of G module,

$$1 \to A \to B \to C \to 1.$$

Then there is a exact hexagon.

$$\begin{tikzcd}
& H^0(G,A) \arrow[r] & H^0(G,B) \arrow[dr] & \\
H^{-1}(G,C) \arrow[ur] & & & H^0(G,C) \arrow[dl] \\
& H^{-1}(G,B) \arrow[ul] & H^{-1}(G,A) \arrow[l] &
\end{tikzcd}$$

We define the Herbrand quotient as fraction of diagonal sizes, $h(G,A) := \frac{\#H^0(G,A)}{\#H^{-1}(G,A)}$ when both are finite.

With cyclic condition, if A is finite, the exact sequences

$$1 \to A_G \to A \overset{\sigma-1}{\to} I_G A \to 1, \quad 1 \to \ker N_G \to A \overset{N_G}{\to} N_G A \to 1,$$

would yield $\#A = \#A_G \cdot \#I_G A = \# \ker N_G \cdot \#N_G A$ thus $h(G,A) = 1$.

Proposition 1.6. *Suppose A is possibly infinite but G is finite and $1 \to A \to B \to C \to 1$ is exact, then we have,*

$$h(G,B) = h(G,A) \cdot h(G,C).$$

Finally, we see some useful facts without proof.

Theorem 1.7. *Suppose L/K cyclic extension of local fields. Then,*

$$\#H^0(\mathsf{Gal}(L/K), L^\times) = [L : K], \quad \#H^{-1}(\mathsf{Gal}(L/K), L^\times) = 1.$$

Theorem 1.8. *Suppose L/K is a finite unramified extension, then*

$$\#H^0(\mathsf{Gal}(L/K), U_L^{(n)}) = \#H^{-1}(\mathsf{Gal}(L/K), U_L^{(n)}) = 1,$$

for all $n \in \mathbb{Z}_{\geq 0}$.

2 The reciprocity homomorphism

Our goal in this part is to construct a canonical homomorphism

$$r_{L/K} : G(L/K) \to K^\times / N_{L/K} L^\times.$$

To do so, the concept about "Frobenius automorphism" plays an important role. Let L/K be finite Galois, $\phi_K \in G(\widetilde{K}/K)$ be a Frobenius automorphism and let d_K be an isomorphism

$$d_K : G(\widetilde{K}/K) \xrightarrow{\sim} \widehat{\mathbb{Z}}$$

such that $d_K(\phi_K) = 1 \in \widehat{\mathbb{Z}}$. Then we could define another map

$$\begin{aligned} d_{L/K} : G(\widetilde{L}/K) &\longrightarrow \widehat{\mathbb{Z}} \\ \sigma &\longmapsto d_K(\sigma \mid_{\widetilde{K}}). \end{aligned}$$

In particular, if $\phi_L \in G(\widetilde{L}/L)$ is a Frobenius automorphism, then $\phi_L \mid_{\widetilde{K}} = \phi_K^{f(L/K)}$. Thus

$$d_{L/K}(\phi_L) = d_K(\phi_K^{f(L/K)}) = f(L/K).$$

We define a subset of $G(\widetilde{L}/K)$:

$$\mathrm{Frob}(\widetilde{L}/K) := \{\sigma \in G(\widetilde{L}/K) \mid d_{L/K}(\sigma) \in \mathbb{N}\} = d_{L/K}^{-1}(\mathbb{N}).$$

In other words, $\sigma \in \mathrm{Frob}(\widetilde{L}/K)$ iff $\sigma \mid_K = \phi_K^n$ for some $n \in \mathbb{N}$.

Proposition 2.1. *For a finite Galois extension L/K, the mapping*

$$\begin{aligned} \mathit{Frob}(\widetilde{L}/K) &\longrightarrow G(L/K) \\ \sigma &\longmapsto \sigma \mid_L \end{aligned}$$

is surjective.

Thus every element $\sigma \in G(L/K)$ can be extended to an element in $\mathrm{Frob}(\widetilde{L}/K)$.

Proposition 2.2. *Let $\sigma \in \mathit{Frob}(\widetilde{L}/K)$ and let $\Sigma = \{a \in \widetilde{L} \mid a^\sigma = a\}$ be the fixed field of σ. Then*

1. $\widetilde{\Sigma} = \widetilde{L}$ *(i.e. Their maximal unramified extension coincide.)*
2. $f(\Sigma/K) = d_{L/K}(\sigma)$
3. $e(\Sigma/K) = e(L/K)$.

In particular, Σ/K is a finite extension since

$$[\Sigma : K] = f(\Sigma/K)e(\Sigma/K) = d_{L/K}(\sigma)e(L/K) < \infty.$$

Definition 2.3. *The reciprocity map is defined by*

$$\begin{aligned} r_{\widetilde{L}/K} : \mathit{Frob}(\widetilde{L}/K) &\longrightarrow K^\times/N_{L/K}(L^\times) \\ \sigma &\longmapsto N_{\Sigma/K}(\pi_\Sigma) \mod N_{L/K}(L^\times), \end{aligned}$$

where Σ is the fixed field of σ.

One can show that $r_{\widetilde{L}/K}$ is well-defined, that is, if π_Σ and π'_Σ are both uniformizers of Σ, then

$$N_{\Sigma/K}(\pi_\Sigma) \equiv N_{\Sigma/K}(\pi'_\Sigma) \mod N_{L/K}(L^\times).$$

Proposition 2.4. *If L/K is finite Galois, then the reciprocity map $r_{\widetilde{L}/K}$ is multiplicative.*

This proposition is not trivial: we need to prove that for $i = 1, 2, 3$, given $\sigma_i \in \mathrm{Frob}(\widetilde{L}/K)$ and Σ_i is the fixed field of σ_i, then

$$N_{\Sigma_1/K}(\pi_{\Sigma_1})N_{\Sigma_2/K}(\pi_{\Sigma_2}) \equiv N_{\Sigma_3/K}(\pi_{\Sigma_3}) \mod N_{L/K}(L^\times).$$

Proposition 2.5. *For every finite Galois extension L/K, there is a canonical homomorphism*

$$\begin{aligned} r_{L/K} : G(L/K) &\longrightarrow K^\times/N_{L/K}(L^\times) \\ \sigma &\longmapsto N_{\Sigma/K}(\pi_\Sigma) \mod N_{L/K}(L^\times), \end{aligned}$$

where Σ is the fixed field of an extension $\widetilde{\sigma} \in Frob(\widetilde{L}/K)$ of $\sigma \in G(L/K)$.

We claim that this map is well-defined. (It is a homomorphism since $r_{\widetilde{L}/K}$ is.)

Proof. Let $\widetilde{\sigma}, \widetilde{\sigma}' \in \mathrm{Frob}(\widetilde{L}/K)$ be two extensions of $\sigma \in G(L/K)$ and Σ, Σ' be their fixed fields, resp. If $d_{L/K}(\widetilde{\sigma}) = d_{L/K}(\widetilde{\sigma}')$, then $\widetilde{\sigma} = \widetilde{\sigma}'$ are identical in both L and $\widetilde{K}$. Thus they are also identical in their compositum $\widetilde{L}$, that is, $\widetilde{\sigma} = \widetilde{\sigma}'$. Therefore WLOG, we may assume

$$d_{L/K}(\widetilde{\sigma}) < d_{L/K}(\widetilde{\sigma}').$$

Let $\tau = \widetilde{\sigma}'\widetilde{\sigma}^{-1}$, then $d_{L/K}(\tau) = d_{L/K}(\widetilde{\sigma}') - d_{L/K}(\widetilde{\sigma}) \in \mathbb{N}$. So $\tau \in \mathrm{Frob}(\widetilde{L}/K)$. Let T be the fixed field of τ. Since $\tau(a) = a$ for all $a \in L$, we have $T \supseteq L$. In particular,

$$N_{T/K}(\pi_T) = N_{L/K}(N_{T/L}(\pi_T)) \in N_{L/K}(L^\times).$$

So

$$N_{\Sigma'/K}(\pi_{\Sigma'}) \equiv N_{T/K}(\pi_T)N_{\Sigma/K}(\pi_\Sigma) \equiv N_{\Sigma/K}(\pi_\Sigma) \mod N_{L/K}(L^\times).$$

□

3 Local Reciprocity Law

Let $G(L/K)'$ be the commutator subgroup of $G(L/K)$,

and let $G(L/K)^{ab} = G(L/K)/G(L/K)'$.

Since $r_{L/K}(G(L/K)') = \{1\} \subset K^\times/N_{L/K}L^\times$,

$r_{L/K}$ naturally induces the homomorphism

$$\hat{r}_{L/K} : G(L/K)^{ab} \to K^\times/N_{L/K}L^\times.$$

Theorem 3.1 (Local Reciprocity Law)**.** *For every finite Galois extension L/K of non-archimedean local field, $\hat{r}_{L/K}$ is an isomorphism.*

Proposition 3.2. *If L/K is a finite unramified extension, then $r_{L/K}$ is an isomorphism.*

Proof. Since $\widetilde{L} = \widetilde{K}$ and $\widetilde{K}^{\phi_K} = K$

$\Rightarrow r_{L/K}(\phi_K \mid_L) \equiv \pi_K \mod N_{L/K}L^\times$.

Note that v_K induces an surjective homomorphism

$$K^\times / N_{L/K}L^\times \to \mathbb{Z}/n\mathbb{Z}, n = [L : K].$$

Since L/K is a cyclic extension,

$\Rightarrow \# K^\times / N_{L/K}L^\times = \# H^0(G(L/K), L^\times) = n$,

$\Rightarrow K^\times / N_{L/K}L^\times \simeq \mathbb{Z}/n\mathbb{Z}$, and so π_K generates $K^\times / N_{L/K}L^\times$.

Since $\phi_K \mid_L$ generates $G(L/K)$ and $\# G(L/K) = \# K^\times / N_{L/K}L^\times$,

$\Rightarrow r_{L/K}$ is an isomorphism. $\square$

Proposition 3.3. *If L/K is a finite totally ramified cyclic extension, then $r_{L/K}$ is an isomorphism.*

Proof. Since L/K is totally ramified, therefore $K = L \cap \widetilde{K}$.

$$\begin{tikzcd}[arrows=dash]
 & \widetilde{L} = L\widetilde{K} \arrow[dl] \arrow[dr] & \\
L \arrow[dr] & & \widetilde{K} \arrow[dl] \\
 & K = L \cap \widetilde{K} &
\end{tikzcd}$$

We have an isomorphism

$$G(\widetilde{L}/\widetilde{K}) \xrightarrow{\sim} G(L/K), \text{ by restriction } \mid_L .$$

Let $\widetilde{\sigma}$ be a generator of $G(\widetilde{L}/\widetilde{K})$, $\sigma = \widetilde{\sigma} \mid_L$,

and let $\sigma' = \widetilde{\sigma}\phi_L \in G(\widetilde{L}/K)$, $\Sigma = \widetilde{L}^{\sigma'}$, so $\sigma' \mid_L = \widetilde{\sigma} \mid_L = \sigma$.

We know $f_{\Sigma/K} = d_{L/K}(\sigma') = d_K(\sigma' \mid_K) = d_K(\phi_L \mid_{\widetilde{K}}) = f_{L/K} = 1$.

Let M/K be a finite galois subextension of $\widetilde{L}/K$ containing Σ and L, and let $M_0 = M \cap \widetilde{K}$,

$\Rightarrow \widetilde{M} = \widetilde{L} = \widetilde{\Sigma}$, therefore we have

$$\begin{tikzcd}
G(\widetilde{L}/\widetilde{K}) \arrow[r, dash, "="] \arrow[d, dash, "\simeq"', "|_L"] & G(\widetilde{M}/\widetilde{K}) \arrow[r, dash, "="] \arrow[d, dash, "\simeq"', "|_M"] & G(\widetilde{\Sigma}/\widetilde{K}) \arrow[d, dash, "\simeq"', "|_\Sigma"] \\
G(L/K) & G(M/M_0) \arrow[l, "|_L"'] \arrow[r, "|_\Sigma"] & G(\Sigma/K)
\end{tikzcd}$$

Hence $N_{M/M_0} |_\Sigma = N_{\Sigma/K}$ and $N_{M/M_0} |_L = N_{L/K}$.

Claim : If $r_{L/K}(\sigma^m) \equiv 1 \mod N_{L/K}L^\times$ whit $0 \le m < n = [L : K]$ then $m = 0$.

Assume this. Then $r_{L/K}$ is injective.

Since $\#K^\times/N_{L/K}L^\times = n = \#G(L/K)$, so $r_{L/K}$ is an isomorphism.

If now $r_{L/K}(\sigma^m) \equiv 1 \mod N_{L/K}L^\times$ with $0 \le m < n$.

Let $\pi_L \in L$, $\pi_\Sigma \in \Sigma$ be prime elements,

$\Rightarrow \pi_L$, π_Σ are both prime elements of M,

$\Rightarrow \pi_\Sigma^m = u\pi_L^m$ for some $u \in U_M$,

$\Rightarrow 1 \equiv r_{L/K}(\sigma^m) \equiv N_{\Sigma/K}(\pi_\Sigma^m)$

$\equiv N_{M/M_0}(u)N_{L/K}(\pi_L^m) \equiv N_{M/M_0}(u) \mod N_{L/K}L^\times$.

$\Rightarrow \exists \epsilon \in U_L$ such that $N_{L/K}(\epsilon) = N_{M/M_0}(u)$.

Since $G(M/M_0)$ is cyclic, so $\#H^{-1}(G(L/K), L^\times) = 1$,

$\Rightarrow \exists a \in M^\times$ such that $u^{-1}\epsilon = a^{\widetilde{\sigma}-1}$,

$\Rightarrow (\pi_L^m\epsilon)^{\widetilde{\sigma}-1} = (\pi_L^m\epsilon)^{\sigma'-1} = (\pi_\Sigma^m a^{\widetilde{\sigma}-1})^{\sigma'-1} = (a^{\sigma'-1})^{\widetilde{\sigma}-1}$,

$\Rightarrow b := \pi_L^m\epsilon a^{\sigma'-1} \in M_0^\times$ with $v_M(b) = m$.

Since $v_M(b) = e_{M/M_0}v_{M_0}(b) = nv_{M_0}(b)$,

$\Rightarrow n \mid m$, therefore $m = 0$. $\square$

Proposition 3.4. *Let L/K and L'/K' be finite galois extensions such that K'/K and L'/L are finite separable extensions. Then we have the commutative* *diagram*

$$\begin{array}{ccc} G(L'/K') & \xrightarrow{\ |_L\ } & G(L/K) \\ \Big\downarrow{\scriptstyle r_{L'/K'}} & & \Big\downarrow{\scriptstyle r_{L/K}} \\ K'^{\times}/N_{L'/K'}L'^{\times} & \xrightarrow{N_{K'/K}} & K^{\times}/N_{L/K}L^{\times} \end{array}$$

Proof. Given $\sigma' \in G(L'/K')$, let $\sigma = \sigma' \mid_L \in G(L/K)$.

If $\widetilde{\sigma'} \in Frob(\widetilde{L'}/K')$ such that $\widetilde{\sigma'} \mid_{L'} = \sigma'$,

$\Rightarrow \widetilde{\sigma} := \widetilde{\sigma'} \mid_{\widetilde{L}} \in Frob(\widetilde{L}/K)$ (since $d_{L/K}(\widetilde{\sigma}) = f_{K'/K} d_{L'/K'}(\widetilde{\sigma'}) \in \mathbb{N}$)

and $\widetilde{\sigma} \mid_L = \sigma$.

Let $\Sigma' = \widetilde{L'}^{\widetilde{\sigma'}}$, and let $\Sigma = \widetilde{L}^{\widetilde{\sigma}} = \Sigma' \cap \widetilde{L} = \Sigma' \cap \widetilde{\Sigma}$,

$\Rightarrow f_{\Sigma'/\Sigma} = 1$.

If now $\pi_{\Sigma'}$ is a prime element of Σ'

then $\pi_\Sigma := N_{\Sigma'/\Sigma}(\pi_{\Sigma'})$ is a prime element of Σ,

$\Rightarrow N_{\Sigma/K}(\pi_\Sigma) = N_{\Sigma/K}(N_{\Sigma'/\Sigma}(\pi_{\Sigma'})) = N_{K'/K}(N_{\Sigma'/K'}(\pi_{\Sigma'}))$. □

Corollary 3.5. *Let M/K be a galois subextension of a finite galois extension L/K. Then we have the commutative exact diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(L/M) & \longrightarrow & G(L/K) & \longrightarrow & G(M/K) & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle r_{L/M}} & & \Big\downarrow{\scriptstyle r_{L/K}} & & \Big\downarrow{\scriptstyle r_{M/K}} & & \\ & & M^{\times}/N_{L/M}L^{\times} & \xrightarrow{N_{M/K}} & K^{\times}/N_{L/K}L^{\times} & \longrightarrow & K^{\times}/N_{M/K}M^{\times} & \longrightarrow & 1 \end{array}$$

Proof. Since $K^{\times}/N_{M/K}M^{\times} \simeq \frac{K^{\times}/N_{L/K}L^{\times}}{N_{M/K}M^{\times}/N_{L/K}L^{\times}}$. □

Proof of Local Reciprocity Law.

Case 1 : Suppose L/K is a cyclic extension.

Let $M = L \cap \widetilde{K}$,

$\Rightarrow L/M$ is a totally ramified cyclic extension,

and M/K is a unramified extension.

Since $L/K, L/M, M/K$ are cyclic extension,

$\Rightarrow \#K^{\times}/N_{L/K}L^{\times} = (\#M^{\times}/N_{L/M}L^{\times})(\#K^{\times}/N_{M/K}M^{\times})$.

Thus, we have

$$\begin{tikzcd}
1 \arrow[r] & G(L/M) \arrow[r] \arrow[d, "r_{L/M}", "\simeq"'] & G(L/K) \arrow[r] \arrow[d, "r_{L/K}"] & G(M/K) \arrow[r] \arrow[d, "r_{M/K}", "\simeq"'] & 1 \\
1 \arrow[r] & M^\times/N_{L/M}L^\times \arrow[r, "N_{M/K}"] & K^\times/N_{L/K}L^\times \arrow[r] & K^\times/N_{M/K}M^\times \arrow[r] & 1
\end{tikzcd}$$

Hence $\hat{r}_{L/K} = r_{L/K}$ is an isomorphism.

Case 2 : Suppose L/K is an abelian extension but not cyclic.

We prove this by induction on $[L : K]$.

Note that $G(L/K) = \bigoplus_{i=1}^{\ell} H_i$ for some cyclic subgroups H_i, $\ell > 1$.

Let $M_i = L^{H_i}$. Then by induction assumption, we have

$$\begin{tikzcd}
1 \arrow[r] & G(L/M_i) \arrow[r] \arrow[d, "r_{L/M_i}", "\simeq"'] & G(L/K) \arrow[r] \arrow[d, "r_{L/K}"] & G(M_i/K) \arrow[r] \arrow[d, "r_{M_i/K}", "\simeq"'] & 1 \\
 & M_i^\times/N_{L/M_i}L^\times \arrow[r, "N_{M_i/K}"] & K^\times/N_{L/K}L^\times \arrow[r] & K^\times/N_{M_i/K}M^\times \arrow[r] & 1
\end{tikzcd}$$

$\Rightarrow r_{L/K}$ is surjective, and $\ker(r_{L/K}) \subset \bigcap_{i=1}^{\ell} G(L/M_i) = \bigcap_{i=1}^{\ell} H_i = \{1\}$

$\Rightarrow \hat{r}_{L/K} = r_{L/K}$ is an isomorphism.

Case 3 : Suppose L/K is not abelian extension.

We prove this by induction on $[L : K]$.

Let H be the commutator subgroup of $G(L/K)$, and let $M = L^H$.

In fact, a finite galois extension of non-archimedean local fields

is always a solvable extension. Hence $\{1\} \neq H \neq G(L/K)$.

By induction assumption, we have

$$\begin{tikzcd}
1 \arrow[r] & G(L/M) \arrow[r] \arrow[d, "r_{L/M}", "surjective"'] & G(L/K) \arrow[r] \arrow[d, "r_{L/K}"] & G(M/K) \arrow[r] \arrow[d, "r_{M/K}", "\simeq"'] & 1 \\
 & M^\times/N_{L/M}L^\times \arrow[r, "N_{M/K}"] & K^\times/N_{L/K}L^\times \arrow[r] & K^\times/N_{M/K}M^\times \arrow[r] & 1
\end{tikzcd}$$

$\Rightarrow r_{L/K}$ is surjective, and $\ker(r_{L/K}) = G(L/M) = H$,

$\Rightarrow \hat{r}_{L/K}$ is an isomorphism. $\square$

Corollary 3.6. *Let L/K be a finite galois extension of non-archimedean local field, and let L^{ab}/K be the maximal abelian subextension in L/K. Then*

$N_{L/K}L^\times = N_{L^{ab}/K}{L^{ab}}^\times$.

Proof. Since $G(L/K)^{ab} = G(L^{ab}/K)$. $\square$

References

[1] J. Neukirch, *Algebraic Number Theory,* Springer, 1999