Grover's Search

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Overview

Intended Outcome. Leverage matrix analysis and used it on Grover's search and BBBV bound. Understand how would quantum evolution corresponds to our matrix method.

- Preliminary
- Quantum "rules"
- Preimage finding problem
- Housholder transform
- Grover's search
- BBBV's bound
- Consequences

Preliminary

Definition (Lebesgue Space)

Given a measure space X, μ its p-Labesgue space collects all finite energy mapping to $\mathbb{C}$,

$$\mathcal{L}^p(X) := \left\{ f : X \to \mathbb{C}, \int_X |f|^p \mathrm{d}\mu < \infty \right\}.$$

Specifically, we denote $\mathbb{C}^X := \mathcal{L}^2(X)$ with inner product,

$$\langle x, y \rangle := \int_X \bar{f} \cdot g \mathrm{d}\mu,$$

and norm $\|x\| := \langle x, x \rangle$.

Preliminary

Theorem (Riesz, Fisher)

The $\mathbb{C}^X$ *is topologically complete, i.e. any Cauchy sequence,* $\{x_i\}_{i\in\mathbb{N}} \subseteq \mathbb{C}^X$ *with* $\forall \epsilon > 0 \exists N, \|x_n - x_m\| < \epsilon, \forall n, m > N$*, has limit,*

$$\lim_{i\to N} x_i \in \mathbb{C}^X.$$

If X is finite, then we could choose μ as discrete measure, then

$$\mathbb{C}^X = \left\{ f : X \to \mathbb{C}, \sum_{x\in X} |f(x)|^2 < \infty \right\}.$$

For any $x \in X$, denote ket $|x\rangle : \begin{cases} x \mapsto 1 \\ x' \mapsto 0, \text{ for } x \neq x' \end{cases}$ and bra $\langle x| := |x\rangle^* := \langle x, \cdot\rangle \in \mathrm{Hom}(\mathbb{C}^X, \mathbb{C})$.

Preliminary

Definition

Given a linear map between Hilbert spaces $L : V \to W$, it's adjoint $L^* : W \to V$ is the only operator [Riesz, Frechet] satisfying,

$$\langle Lv, w \rangle_W = \langle v, L^* w \rangle_V.$$

When $W = V$ and $L \circ L^* = L^* \circ L = \mathrm{Id}$, L is said to be unitary.

If V, W are finite dimensional $\mathcal{L}^2$ spaces, then operator could be represented as a matrix and its adjoint is just the conjugate transpose matrix. The unitary operators are just those unitary matrices.

Preliminary

Definition (local operators)

Consider $\mathbb{C}^{\{0,1\}^J}$, $J \subseteq \mathbb{N}$ composed of countably many 2 dimensional Lebesgue spaces, i.e. it could be written as the tensor space,

$$\mathbb{C}^{\{0,1\}^K} = \bigotimes_{j \in J} \mathbb{C}^{\{0,1\}}.$$

A operator L is said to be r-local iff, $\exists R \subseteq J$ of size r and $L_R \in \mathrm{GL}(\mathbb{C}^{\{0,1\}^R})$ that

$$L = \left(\bigotimes_{j \in J - R} \mathrm{Id} \right) \otimes L_R.$$

Quantum "Rules"

Definition (quantum circuit)

A quantum circuit with n-qubits and t gates is composed of a sequence of $O(1)$-local unitary operators $\{U_i\}_{1\leq i\leq t} \subseteq \mathbb{C}^{\{0,1\}^n}$. With initial input state $|\psi_0\rangle$, the circuit induces a unitary evolution $\{|\psi_i\rangle\}_{0\leq i\leq n}$ as following,

$$|\psi_i\rangle = U_i|\psi_{i-1}\rangle.$$

Definition (measurement)

Given a quantum state $|\psi\rangle \in \mathbb{C}^X$ measurement is specified by a set of orthogonal projection operators $\{P_j\}_{j\in J}$, i.e.

- $\bigoplus_{j\in J} \mathrm{Im}(P_j) = \mathbb{C}^X$,
- $\mathrm{Im}(P_i) \perp \mathrm{Im}(P_j)$, for $i \neq j$,
- $P_j^2 = P_j, \forall j \in J$.

The measurement result would yield j with probability $\|P_j|\psi\rangle\|^2$.

Quantum "Rules"

For example, take Hadamard tansform $H^{\otimes n}$ as example,

$$H^{\otimes n} = \frac{1}{\sqrt{2^n}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{\otimes n}.$$

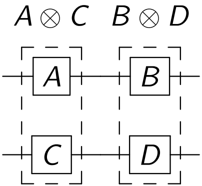
This maps as follows,

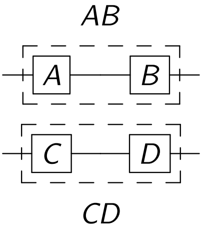
$$|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{x' \in X} (-1)^{x \cdot x'} |x'\rangle.$$

Specifically define $|+\rangle = H|0\rangle$ and $|-\rangle = H|1\rangle$, Also we have,

$$H|0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x' \in X} |x'\rangle.$$

Quantum "Rules"

$(A \otimes C)(B \otimes D) \sim$ 

$=$ 

$\sim AB \otimes CD$

First Look

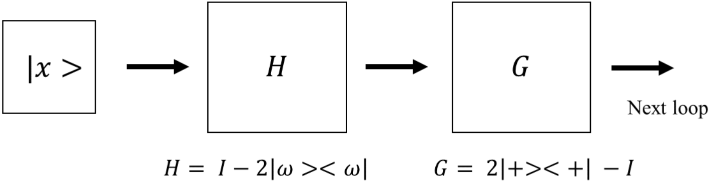
Definition (preimage finding problem)

In a preimage finding finding problem, one is given access to the quantum oracle of $f : \{0,1\}^n \to \{0,1\}$ in the following form,

$$O_{\pm} : |x\rangle \to (-1)^{f(x)}|x\rangle,$$

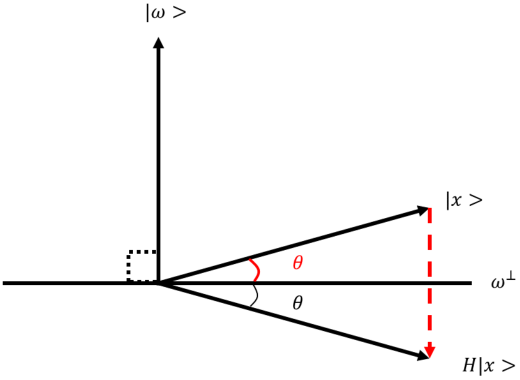
and asked to find any x_0 such that $f(x_0) = 1$.

First Look



(Here H represent Householder transform instead of Hadamard.)

First Look



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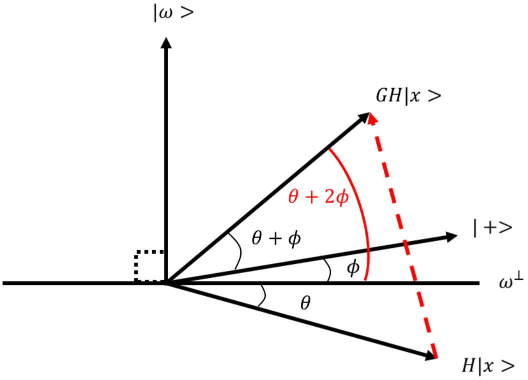
First Look



(Here *H* represent Householder transform instead of Hadamard.)

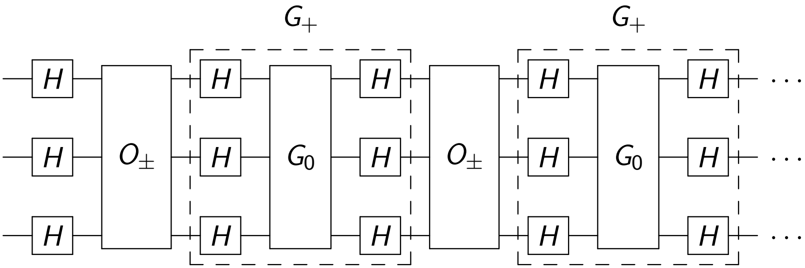
Unstructural Search

Theorem (Grover's search)

The preimage finding problem could be solved in $O(\sqrt{\frac{N}{k}})$ quantum queries and $O(n\sqrt{\frac{N}{k}})$ gates where $N := 2^n$ and $k = \#\{x : f(x) = 1\}$ with the following algorithm,

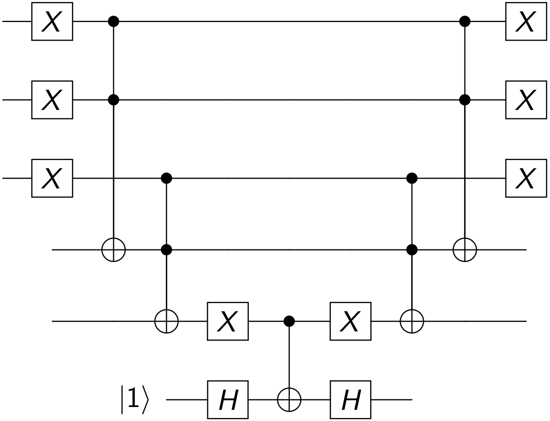
- *Initialize the state $|\psi_0\rangle = |0\rangle$.*
- *Apply Hadamard transform $|\psi_1\rangle = H|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_{i\in\{0,1\}^n} |i\rangle$.*
- *Apply $G_+ \circ O_\pm$ for $O(\sqrt{\frac{N}{k}})$ iterations, $|\psi_{i+1}\rangle = G_+ \circ O_\pm|\psi_i\rangle$, where*
$$G_+ = 2|+\rangle\langle+| - I.$$
- *Measure the final state in computational basis with the projection matrices*
$$\{P_x := |x\rangle\langle x|\}_{x\in\{0,1\}^n} \subseteq \mathbb{C}^{\{0,1\}^n}.$$

Unstructural Search



Unstructural Search

$G_0 =$



Unstructural Search

Proof.

Well-definedness. First of all, since $\sigma(G_n) = \{\pm 1\}$ and Hermitian, G_+ is unitary. Also we have,

$$G_+ = 2|+\rangle\langle +| - I = H \circ G_0 \circ H,$$

where

$$G_0 := 2|0\rangle\langle 0| - I : \begin{cases} |x\rangle \mapsto -|x\rangle, & \text{for } x \neq 0 \\ |0\rangle \mapsto |0\rangle. \end{cases}$$

Check that G_0 could be constructed in $O(n)$ gates!

Convergence. Suppose

$$|\psi_i\rangle = \sum_{x:f(x)=1} a_i|x\rangle + \sum_{x:f(x)=0} b_i|x\rangle.$$

Define mean value $\mu_i := \frac{-k}{N} a_i + \frac{(N-k)}{N} b_i$.

Unstructural Search

Proof.

We have,

$$a_{i+1} = 2\mu_i + a_i = \frac{N-2k}{N} a_i + \frac{2N-2k}{N} b_i, \text{ and } b_i = \sqrt{\frac{1-ka_i^2}{N-k}}.$$

Suppose $ka_i^2 = \sin^2(\theta_i)$, then $a_i = \frac{\sin(\theta_i)}{\sqrt{k}}$ and $b_i = \frac{\cos(\theta_i)}{\sqrt{N-k}}$.

$$\begin{aligned}\sin(\theta_{i+1}) &= \frac{N-2k}{N\sqrt{k}} a_i + \frac{2N-2k}{N\sqrt{k}} b_i \\ &= \frac{N-2k}{N} \sin(\theta_i) + \frac{2\sqrt{(N-k)k}}{N} \cos(\theta_i) \\ &= \sin(\theta_i + \phi),\end{aligned}$$

where $\sin(\phi) = \frac{2\sqrt{(N-k)k}}{N}$.

Unstructural Search

Proof.

Note that $\phi \in \Theta\left(\sqrt{\frac{k}{N}}\right)$ is independent of i and thus, in order that

$$\begin{aligned}\Omega(1) &\leq \Pr[\text{measure result } x : f(x) = 1] \\ &= ka_i^2 = k\sin^2(\theta_i) = k\sin^2(\theta_1 + (i-1)\phi),\end{aligned}$$

the iteration count $i \in \Theta(\sqrt{\frac{N}{k}})$ would be enough. Also, since each iteration would cost $\Theta(n)$ gates, in total $\Theta(n\sqrt{\frac{N}{k}})$ gates are enough. $\square$

Query Lower Bound

We now introduce a restricted version of BBBV bound for pre-image finding problem.

Theorem (Bennett, Bernstein, Brassard, Vazirani)

In order to solve for pre-image finding problem with $k = 1$, $\Omega(\sqrt{N})$ quantum queries are required.

Lemma

Suppose $|\phi\rangle, |\psi\rangle \in \mathbb{C}^X$ with $\||\phi\rangle\| = \||\psi\rangle\| = 1$ and $\||\phi\rangle - |\psi\rangle\| \leq \epsilon$. Given measurement matrices $\{P_j\}_{j\in J}$ the probability difference is bounded as following,

$$\sum_{j\in J} \left| \|P_j|\phi\rangle\|^2 - \|P_j|\psi\rangle\|^2 \right| = \sum_{j\in J} |\langle\phi|P_j|\phi\rangle - \langle\psi|P_j|\psi\rangle| \leq 4\epsilon.$$

Query Lower Bound

Proof.

Suppose $|\delta\rangle = |\phi\rangle - |\psi\rangle$ and,

$$P_j|\phi\rangle = \alpha_j, \quad P_j|\psi\rangle = \beta_j, \quad \gamma_j = P_j|\delta\rangle = \alpha_j - \beta_j.$$

Then

$$\begin{aligned} \alpha_j\alpha_j^* - \beta_j\beta_j^* &= (\beta_j + \gamma_j)(\beta_j^* + \gamma_j^*) - \beta_j\beta_j^* \\ &= \beta_j\beta_j^* + \gamma_j\gamma_j^* + \beta_j\gamma_j^* + \gamma_j\beta_j^* - \beta_j\beta_j^* \\ &= \gamma_j\gamma_j^* + \beta_j\gamma_j^* + \gamma_j\beta_j^*. \end{aligned}$$

Therefore,

$$\sum_{j\in J} |\langle\phi|P_j|\phi\rangle - \langle\psi|P_j|\psi\rangle| \leq \sum_{j\in J} \left|\gamma_j\gamma_j^* + \beta_j\gamma_j^* + \gamma_j\beta_j^*\right|$$
$$\leq \langle\delta|\delta\rangle + \langle\psi|\delta\rangle + \langle\delta|\psi\rangle \leq \epsilon^2 + 2\epsilon \leq 4\epsilon.$$

Query Lower Bound

Proof of main theorem.

Any q-query algorithm is of form

$$U_q O_\pm U_{q-1} \dots U_1 O_\pm U_0.$$

For some $\tilde{x} \in X$ pick $f(x) = \begin{cases} 1, & \text{for } x = \tilde{x}, \\ 0, & \text{otherwise.} \end{cases}$ Let

$$|\psi_{\tilde{x}}^i\rangle = O_\pm U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle,$$
$$|\psi_0^i\rangle = U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle.$$

We are bounding $\| U_q |\psi_{\tilde{x}}^q\rangle - U_q |\psi_0^q\rangle \| = \| |\psi_{\tilde{x}}^q\rangle - |\psi_0^q\rangle \|$.

Query Lower Bound

Proof of main theorem.

Note that,

$$
\begin{aligned}
& \| |\psi_{\tilde{x}}^{i}\rangle - |\psi_{0}^{i}\rangle \| \\
& \leq \| O_{\pm} U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle - U_{i-1} |\psi_{0}^{i-1}\rangle \| \\
& \leq \| O_{\pm} U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle - U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle \| + \| U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle - U_{i-1} |\psi_{0}^{i-1}\rangle \| \\
& \leq \| O_{\pm} U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle - U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle \| + \| |\psi_{\tilde{x}}^{i-1}\rangle - |\psi_{0}^{i-1}\rangle \|.
\end{aligned}
$$

Let $U_{i-1}|\psi_{\tilde{x}}^{i-1}\rangle = \sum_x \alpha_x |x\rangle$, then

$$\| O_{\pm} U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle - U_{i-1} |\psi_{\tilde{x}}^{i-1}\rangle \| \leq 2|\alpha_{\tilde{x}}|.$$

By Cauchy-Schwartz Inequality when running all possilbe $\tilde{x}$,

$$\sum_{\tilde{x} \in X} |\alpha_{\tilde{x}}| \leq \sqrt{\sum_{\tilde{x} \in X} |\alpha_{\tilde{x}}^2| \cdot N} \leq \sqrt{N}.$$

Query Lower Bound

Proof of main theorem.

Therefore,

$$\begin{aligned}
&\sum_{\tilde{x}\in X} \| |\psi^i_{\tilde{x}}\rangle - |\psi^i_0\rangle \| \\
&\leq \sum_{\tilde{x}\in X} \left(\| O_{\pm} U_{i-1} |\psi^{i-1}_{\tilde{x}}\rangle - U_{i-1} |\psi^{i-1}_{\tilde{x}}\rangle \| + \| |\psi^{i-1}_{\tilde{x}}\rangle - |\psi^{i-1}_0\rangle \| \right) \\
&\leq \sum_{\tilde{x}\in X} \left(\sqrt{N} + \| |\psi^{i-1}_{\tilde{x}}\rangle - |\psi^{i-1}_0\rangle \| \right) \leq q\sqrt{N},
\end{aligned}$$

We have that $\mathbb{E}_{\tilde{x} \xleftarrow{\$} X} [\| |\psi^i_{\tilde{x}}\rangle - |\psi^i_0\rangle \|] \leq \frac{q}{\sqrt{N}}$. Together with previous lemma, in order to find preimage with $\Omega(1)$ probability, we require $q \geq \Omega(\sqrt{N})$. $\square$

Consequences

Corollary

The post quantum security for any symmetrical cryptosystem is cut in half.

Corollary (Brassard, Hoyer, Tapp)

Hash collision could be found in $O(N^{\frac{1}{3}})$ *queries.*

Corollary (complexity theoretic result)

There is an quantum oracle A *that*

$$\mathrm{NP}^A \not\subseteq \mathrm{BQP}^A.$$

Summary and Take-aways

- Quantum computation is just unitary evolution driven by local unitaries.
- The Grover diffusion operator G_+ is in fact negative Householder transform.
- For upper bound proof, need to prove for well-definedness and convergence speed.
- For lower bound proof, need to consider general case. (any potential algorithm)
- Grover search produces quadratic and only quadratic speedup, quantum computation is not the cure-all!