Neo-Hookean VUMAT and UMAT details

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Using the notation $J = \det \mathbf{F}$ as is done in the Abaqus documentation, with

$$\mathbf{B}_{\text{dis}} = J^{-2/3}\mathbf{B}, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^{\top}, \quad (\mathbf{B}_{\text{dis}})_0 = \mathbf{B}_{\text{dis}} - \frac{1}{3}\text{tr}\,\mathbf{B}_{\text{dis}}$$

is the distortional part of the left Cauchy-Green tensor, meaning without and volumetric contribution. The Cauchy stress is given by

$$\mathbf{T} = J^{-1}\big[G(\mathbf{B}_{\text{dis}})_0 + K(\ln J)\mathbf{1}\big]$$

for this nearly incompressible neo-Hookean material model. Generally the bulk modulus is much larger than the shear modulus to enforce the near incompressibility, $K \gg G$. For the UMAT implementation the material Jacobian is required, and as per the Abaqus documentation, it is defined for rate-form constitutive laws exactly by

$$\mathbb{C} = \frac{1}{J}\frac{\partial \Delta(J\mathbf{T})}{\partial \Delta \mathbf{E}},$$

and for total-form constitutive laws by

$$\delta(J\mathbf{T}) = J\left(\mathbb{C} \colon \delta\mathbf{D} + \delta\mathbf{W}\mathbf{T} - \mathbf{T}\delta\mathbf{W}\right).$$

In this case we use the approximation

$$\delta(JT_{ij}) = J\mathbb{C}_{ijkl}\text{sym}\left(\delta F_{km}F^{-1}_{ml}\right)$$

where we have used $\delta\mathbf{D} = \text{sym}\left(\delta\mathbf{F}\mathbf{F}^{-1}\right)$ and neglected the spin terms. Ignoring the symmetry of $\delta\mathbf{D}$ *momentarily* we have

$$\delta(JT_{ij})F_{lm} = J\mathbb{C}_{ijkl}\delta F_{km},$$

and enforcing symmetry again, implies

$$\mathbb{C}_{ijkl} = \frac{1}{2}J^{-1}\left(F_{lm}\frac{\partial(JT_{ij})}{\partial F_{km}} + F_{km}\frac{\partial(JT_{ij})}{\partial F_{lm}}\right).$$

For this particular material model we have

$$\frac{\partial(JT_{ij})}{\partial F_{km}} = GJ^{-2/3}\left(\delta_{ik}F_{jm} + F_{im}\delta_{jk} - \frac{2}{3}\delta_{ij}F_{km} - \frac{2}{3}B_{ij}F^{-1}_{mk} + \frac{2}{9}(\text{tr}\,\mathbf{B})\delta_{ij}F^{-1}_{mk}\right) + K\delta_{ij}F^{-1}_{mk}$$

which leads to

$$\mathbb{C}_{ijkl} = \frac{G}{J}\Bigg[\frac{1}{2}\left(\delta_{ik}B_{\text{dis},jl} + \delta_{jk}B_{\text{dis},il} + \delta_{il}B_{\text{dis},jk} + \delta_{jl}B_{\text{dis},ik}\right) - \frac{2}{3}\delta_{ij}B_{\text{dis},kl} - \frac{2}{3}B_{\text{dis},ij}\delta_{kl} + \frac{2}{9}(\text{tr}\,\mathbf{B}_{\text{dis}})\delta_{ij}\delta_{kl}\Bigg] + \frac{K}{J}\delta_{ij}\delta_{kl}$$