Computational Nonlinear Mechanics
Assignment 3: Large elasto-plastic deformations

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Contents

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Computer assignment 3: Large elasto-plastic deformations

The goal of this computer assignment is that you should practise how to program a elasto-plastic constitutive model and implement it into your finite element program developed in the Computer assignment 2.

a Implemement a elasto-plastic material model according to the Lecture notes but with Neo-Hooke elasticity. To be specific:

- Elasticity defined by the free energy in (6.27) Bonet and Wood

$$\Psi_{\mathrm{e}} = \frac{\mu}{2}\left(I_{C_{\mathrm{e}}} - 3\right) - \mu \ln J_{\mathrm{e}} + \frac{\lambda}{2}\left(\ln J_{\mathrm{e}}\right)^2$$

  with $I_{C_{\mathrm{e}}} = \boldsymbol{I} : \boldsymbol{C}_{\mathrm{e}}$ and the elastic Lame constants $\mu = G = E/(2(1+\nu))$, $\lambda = E\,\nu/((1-2\nu)\,(1+\nu))$.

- Yield function of von Mises type

$$\phi = \sqrt{\frac{3}{2}}|\bar{\boldsymbol{M}}_{\mathrm{dev}}| - (\tau_{\mathrm{y}} + H\,\bar{\epsilon}_{\mathrm{p}})$$

  with $\bar{\epsilon}_{\mathrm{p}} = \int_0^t \dot{\gamma}\,\mathrm{d}t$

- Loading/unloading conditions

$$\phi \leq 0\,, \quad \dot{\gamma} \geq 0\,, \quad \dot{\gamma}\phi = 0 \tag{1}$$

- Associative evolutions eqn

$$\bar{\boldsymbol{L}}_{\mathrm{p}} = \dot{\gamma}\frac{\partial \phi}{\partial \bar{\boldsymbol{M}}} \tag{2}$$

- Apply the backward Euler integration technique but use the semi-explicit simplification described in the Lecture notes
- The constitutive driver should be a subroutine with a structure similar to the following

```
[S2,state_var]  = neo_hooke_plast(C,state_var_old,para);
```

  i.e. input: deformation gradient $\boldsymbol{C}$, state variables from previous time step, material parameters; output: 2nd Piola Kirchhoff stress, updated state variables

- Use numerical differentiation to obtain the stiffness $\mathrm{d}\boldsymbol{S}/\mathrm{d}\boldsymbol{E}$. Do not forget to symmetrize this tensor!
- Use the following values for the material parameters: $E = 200{\cdot}10^3$ [MPa], $\nu = 0.3$, $\tau_{\mathrm{y}} = 400$ [MPa], $H = E/20$.
- Test the subroutine by using the program test_const.m that can be downloaded from the Study portal. Plot the Cauchy stress component σ_{12} against $\epsilon_{12} = F_{12}$ for a uniaxial strain control $\boldsymbol{F} = F_{12}\boldsymbol{e}_1 \otimes \boldsymbol{E}_2$ for $F_{12} \in [0\ 0.2]$

b Implement the model into your finite element program. To check that the code is running work with the small problem with input data

```
Ex,Ey,Coord,Edof,Dof
```

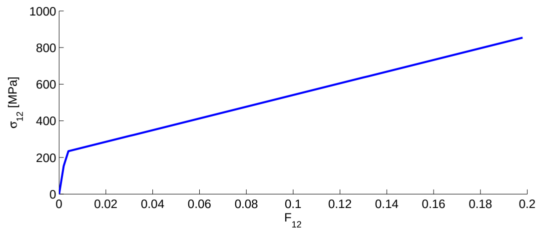


Figure 1: Test of constitutive driver.

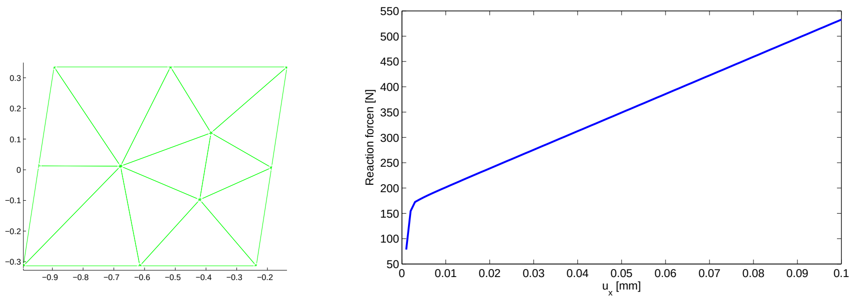


Figure 2: Result FE test with coarse mesh.

given in the file `small_mesh_data.mat`. The loading is the same as for computer assignment 2, i.e., the lower boundary is fixed while the nodes on the upper boundary are displaced in the horizontal (x) direction to $u_{\max} = 0.1$ [mm]. Assume the thickness $t = 1$ [mm]. Plot the deformed mesh as well as the total horizontal reaction force vs displacement on the upper nodes.

c The third subtask is to simulate a forming process and with remaining deformations due to plasticity after unloading (spring-back). Use the same geometry and and same type of loading as in Computer assignment 2, i.e. shearing of a plate with a hole. Move the upper boundary in the horizontal direction to $u_{\max} = 5$ [mm] and then unload until the reaction force in the horizontal direction is ≈ 0. Plot the total horizontal reaction force vs displacement on the upper nodes and also plot the final shape of the plate after unloading.

  Hint: Use a timestep length such that atleast 5 timesteps are taken before plasticity starts in the structure. Then when the plasticity has started to develop longer timesteps can be taken. But then smaller timesteps must also be taken at the start of the unloading. To improve the convergence use the old increment as the initial guess of the FE iterations, i.e., ${}^{n+1}\boldsymbol{x} = {}^{n}\boldsymbol{x} + ({}^{n}\boldsymbol{x} - {}^{n-1}\boldsymbol{x})$.

- The final subtask is to implement either the plasticity model in Chapter 5 (logarithmic formulation) or Chapter 6 (hypoelasticity-rotational neutralized formulation). Compare the results with the previously obtained results. Use numerical differentiation when you find it suitable to limit the implementation time.

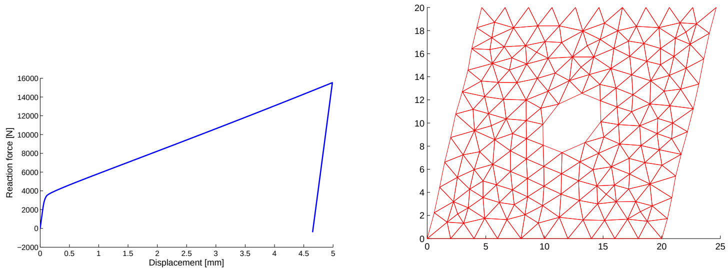


Figure 3: Result FE test with coarse mesh.

1 Task A

According to equation (6.27) in [1], the free energy for compressible Neo-Hookean material is defined as follows:

$$\Psi = \frac{\mu}{2}\left(I_{C_e} - 3\right) - \mu \ln J_e + \frac{\lambda}{2}\left(\ln J_e\right)^2 \tag{6.27}$$

with $I_{C_e} = \underset{\sim}{\boldsymbol{I}} : \underset{\sim}{\boldsymbol{C}}_e$ and the elastic Lame constants $\mu = G = 0.5E/(1+\nu)$, $\lambda = E\nu/(1-2\nu)/(1+\nu)$.

Yield function of von Mises type:

$$\phi = \sqrt{\frac{3}{2}}\left|\underset{\sim}{\overline{\boldsymbol{M}}}_{dev}\right| - \left(\tau_y - H\bar{\varepsilon}_p\right) \tag{1}$$

with $\bar{\varepsilon}_p = \int_0^t \dot{\gamma}\, dt$.

Mandel stress is defined as

$$\underset{\sim}{\overline{\boldsymbol{M}}} = \underset{\sim}{\boldsymbol{C}}_e \cdot \underset{\sim}{\widetilde{\boldsymbol{S}}} \tag{2}$$

with intermediate 2nd Piola-Kirchhoff stress defined as a pull-back of the Kirchhoff stress to the intermediate configuration as follows:

$$\underset{\sim}{\widetilde{\boldsymbol{S}}} = \underset{\sim}{\boldsymbol{F}}_e^{-1} \cdot \underset{\sim}{\boldsymbol{\tau}} \cdot \underset{\sim}{\boldsymbol{F}}_e^{-T} = 2\frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{C}}_e} \tag{3}$$

and intermediate elastic deformation tensor

$$\underset{\sim}{\boldsymbol{C}}_e = \underset{\sim}{\boldsymbol{F}}_p^{-T} \cdot \underset{\sim}{\boldsymbol{C}} \cdot \underset{\sim}{\boldsymbol{F}}_p^{-1} \tag{4}$$

and plastic part of the deformation gradient

$$\underset{\sim}{\boldsymbol{F}} = \underset{\sim}{\boldsymbol{F}}_e \cdot \underset{\sim}{\boldsymbol{F}}_p \tag{5}$$

Let us derive expression for $\underset{\sim}{\widetilde{\boldsymbol{S}}}$:

$$\underset{\sim}{\widetilde{\boldsymbol{S}}} = 2\frac{\partial \Psi}{\partial \underset{\sim}{\boldsymbol{C}}_e} = 2\frac{\partial \Psi}{\partial I_{C_e}} \cdot \frac{\partial I_{C_e}}{\partial \underset{\sim}{\boldsymbol{C}}_e} + 2\frac{\partial \Psi}{\partial J_e} \cdot \frac{\partial J_e}{\partial \underset{\sim}{\boldsymbol{C}}_e} \tag{6}$$

$$\frac{\partial \Psi}{\partial I_{C_e}} = \frac{\mu}{2} \tag{7}$$

$$\frac{\partial I_{C_e}}{\partial \underset{\sim}{\boldsymbol{C}}_e} = \underset{\sim}{\boldsymbol{I}} \tag{8}$$

$$\frac{\partial \Psi}{\partial J_e} = -\frac{\mu}{J_e} + \frac{\lambda}{J_e}\ln J_e \tag{9}$$

$$\frac{\partial J_e}{\partial \underset{\sim}{\boldsymbol{C}}_e} = \frac{J_e}{2}\underset{\sim}{\boldsymbol{C}}_e^{-1} \tag{10}$$

Therefore,

$$\underset{\sim}{\widetilde{\boldsymbol{S}}} = \mu \underset{\sim}{\boldsymbol{I}} + \left(\lambda \ln J_e - \mu\right)\underset{\sim}{\boldsymbol{C}}_e^{-1} \tag{11}$$

At this point we can assess the yield function. The Kuhn-Tucker conditions are

$$\phi \le 0, \quad \dot{\gamma} \ge 0, \quad \dot{\gamma}\phi = 0 \tag{12}$$

The evolution equation is of associative type:

$$\underset{\sim}{\overline{\boldsymbol{L}}}_p = \dot{\gamma}\frac{\partial \phi}{\partial \underset{\sim}{\overline{\boldsymbol{M}}}} = \dot{\gamma}\sqrt{\frac{3}{2}}\frac{\underset{\sim}{\overline{\boldsymbol{M}}}_{dev}}{\left|\underset{\sim}{\overline{\boldsymbol{M}}}_{dev}\right|} = \dot{\gamma}\underset{\sim}{\boldsymbol{\nu}} \tag{13}$$

Apply backward Euler integration:

$$\frac{{}^{n+1}\underset{\sim}{\boldsymbol{F}}_p - {}^{n}\underset{\sim}{\boldsymbol{F}}_p}{\Delta t}\,{}^{n+1}\underset{\sim}{\boldsymbol{F}}_p^{-1} = \frac{\Delta\gamma}{\Delta t}\,{}^{n+1}\underset{\sim}{\boldsymbol{\nu}} \tag{14}$$

$$\underset{\sim}{\boldsymbol{I}} - {}^{n}\underset{\sim}{\boldsymbol{F}}_p \cdot {}^{n+1}\underset{\sim}{\boldsymbol{F}}_p^{-1} \approx \Delta\gamma\,{}^{n+1}\underset{\sim}{\boldsymbol{\nu}} \tag{15}$$

Apply semi-explicit simplification ${}^{n+1}\underset{\sim}{\boldsymbol{\nu}} \approx {}^{n}\underset{\sim}{\boldsymbol{\nu}}$:

$$\underset{\sim}{\boldsymbol{I}} - {}^{n}\underset{\sim}{\boldsymbol{F}}_p \cdot {}^{n+1}\underset{\sim}{\boldsymbol{F}}_p^{-1} \approx \Delta\gamma\,{}^{n}\underset{\sim}{\boldsymbol{\nu}} \tag{16}$$

$${}^{n+1}\underset{\sim}{\boldsymbol{F}}_p^{-1} \approx {}^{n}\underset{\sim}{\boldsymbol{F}}_p^{-1}\left(\underset{\sim}{\boldsymbol{I}} - \Delta\gamma\,{}^{n}\underset{\sim}{\boldsymbol{\nu}}\right) \tag{17}$$

Solve the simplified local constitutive problem:

$$\phi\left(\Delta\gamma\right) = \sqrt{\frac{3}{2}}\left|\overline{\underset{\sim}{\boldsymbol{M}}}_{dev}\right| - \left[\tau_y - H\left({}^{n}\overline{\varepsilon}_p + \Delta\gamma\right)\right] \tag{18}$$

with

$$\overline{\underset{\sim}{\boldsymbol{M}}} = \underset{\sim}{\boldsymbol{C}}_e \cdot \widetilde{\underset{\sim}{\boldsymbol{S}}} = \underset{\sim}{\boldsymbol{C}}_e\left(\underset{\sim}{\boldsymbol{F}}_p^{-1}\left(\Delta\gamma\right), \underset{\sim}{\boldsymbol{C}}\right) \cdot \widetilde{\underset{\sim}{\boldsymbol{S}}}\left(\underset{\sim}{\boldsymbol{C}}_e\left(\underset{\sim}{\boldsymbol{F}}_p^{-1}\left(\Delta\gamma\right), \underset{\sim}{\boldsymbol{C}}\right)\right) \tag{19}$$

using Newton–Raphson method:

$$\phi\left(\Delta\gamma_{n+1}\right) \approx \phi\left(\Delta\gamma_n\right) + \frac{d\,\phi\left(\Delta\gamma_n\right)}{d\,\Delta\gamma}\blacktriangle\gamma = 0 \tag{20}$$

$$\blacktriangle\gamma = -\left[\frac{d\,\phi\left(\Delta\gamma_n\right)}{d\,\Delta\gamma}\right]^{-1} \cdot \phi\left(\Delta\gamma_n\right) \tag{21}$$

$$\Delta\gamma_{n+1} = \Delta\gamma_n + \blacktriangle\gamma \tag{22}$$

where the derivative of the yield function can be expanded to

$$\frac{d\phi}{d\gamma} = \frac{\partial\phi}{\partial\overline{\underset{\sim}{\boldsymbol{M}}}} : \frac{\partial\overline{\underset{\sim}{\boldsymbol{M}}}}{\partial\underset{\sim}{\boldsymbol{C}}_e} : \frac{\partial\underset{\sim}{\boldsymbol{C}}_e}{\partial\underset{\sim}{\boldsymbol{F}}_p^{-1}} : \frac{d\underset{\sim}{\boldsymbol{F}}_p^{-1}}{d\Delta\gamma} + \frac{\partial\phi}{\partial\Delta\gamma} \tag{23}$$

with $\dfrac{\partial\phi}{\partial\Delta\gamma} = -H$ and the rest of the derivatives given by equations (34)–(37) of [2], in which

$$\frac{\partial\widetilde{\underset{\sim}{\boldsymbol{S}}}}{\partial\underset{\sim}{\boldsymbol{C}}_e} = \frac{\lambda}{2}\underset{\sim}{\boldsymbol{C}}_e^{-1} \otimes \underset{\sim}{\boldsymbol{C}}_e^{-1} + \left(\mu - \lambda\ln J_e\right)\underset{\sim}{\boldsymbol{C}}_e^{-1}\overline{\otimes}\underset{\sim}{\boldsymbol{C}}_e^{-1} \tag{24}$$

Finally, compute the 2nd Piola-Kirchhoff stress

$$\underset{\sim}{\boldsymbol{S}} = \underset{\sim}{\boldsymbol{F}}_p^{-1} \cdot \widetilde{\underset{\sim}{\boldsymbol{S}}} \cdot \underset{\sim}{\boldsymbol{F}}_p^{-T} \tag{25}$$

and the material stiffness

$$\underset{\approx}{\boldsymbol{C}} = 2\frac{d\underset{\sim}{\boldsymbol{S}}}{d\underset{\sim}{\boldsymbol{C}}} \tag{26}$$

Note: if computed numerically, $\underset{\approx}{\boldsymbol{C}}$ must be symmetrised to get correct element stiffness.

The material model was implemented in Matlab and can be found in `neo_hooke_plast.m` (see section 5).

Figure 1 shows the test of the constitutive driver, where component of Cauchy stress plotted against engineering strain for the situation of uniaxial strain control $\underset{\sim}{\boldsymbol{F}} = F_{12}\underline{\boldsymbol{e}}_1 \otimes \underline{\boldsymbol{E}}_2$.

2 Task B

The given boundary value problem has been solved using Matlab script `test_plate_shear.m` (see section 5). Figure 2 shows total reaction force plotted against displacement of the top nodes and the shape of the plate before and after the deformation.

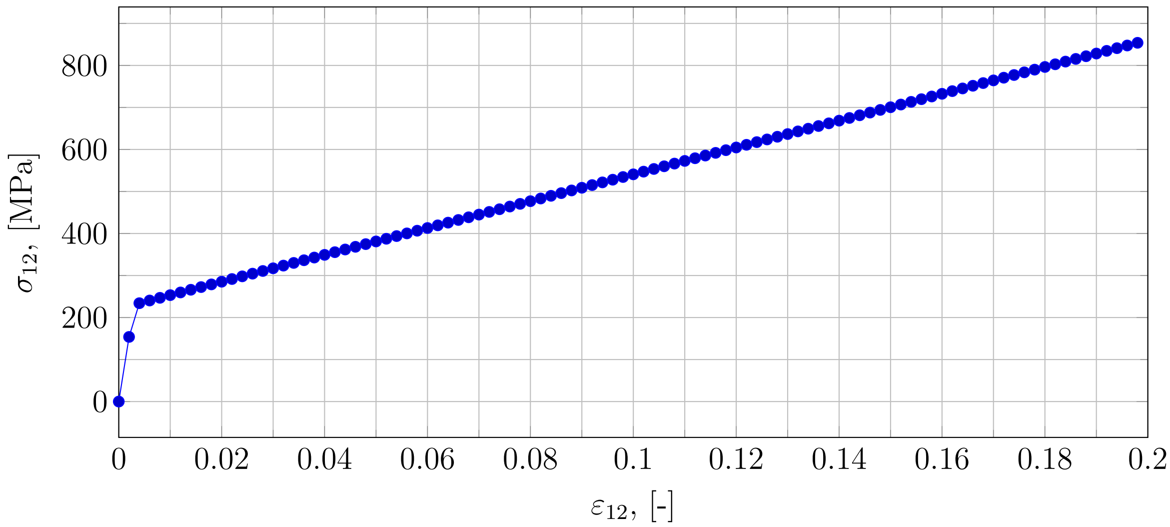


Figure 1: Cauchy stress component σ_{12} versus strain $\varepsilon_{12} = F_{12}$.

3 Task C

The second boundary value problem has been solved using Matlab script `forming_process.m` (see section 5). Figure 3 shows total reaction force plotted against displacement of the top nodes and the shape of the plate before and after the deformation.

4 Task D

The logarithmic strain model from chapter 5 [2] was implemented in Matlab function `neo_hooke_plast_log_strain.m` (see section 5) and applied to solve the same boundary value problem as in section 3. Figure 4 compares the total reaction force plotted against displacement of the top nodes for the two models. The two models yield virtually the same graphs.

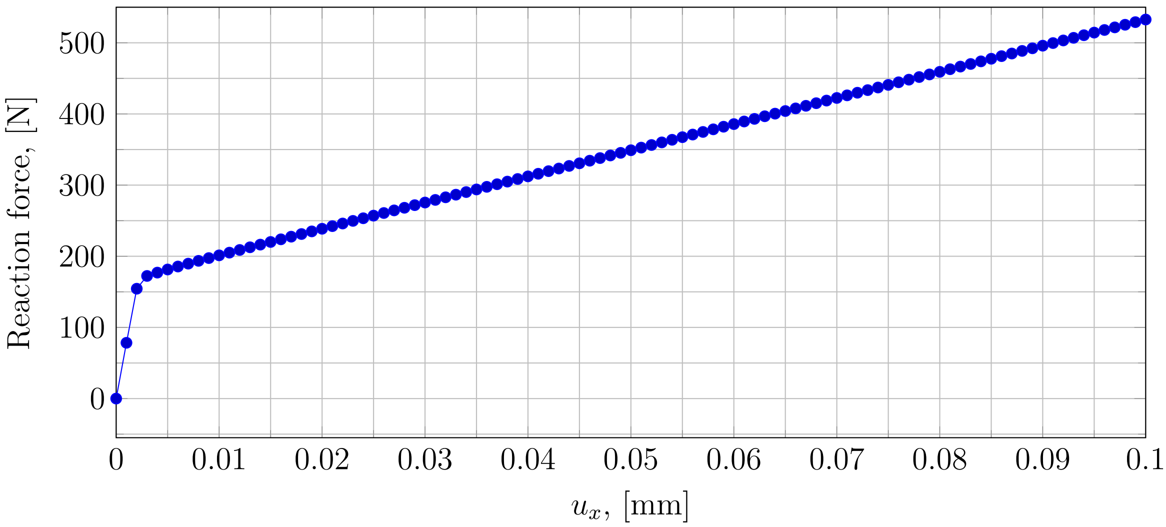
5 Matlab code

The Matlab code can be found in the Git repository on Github.

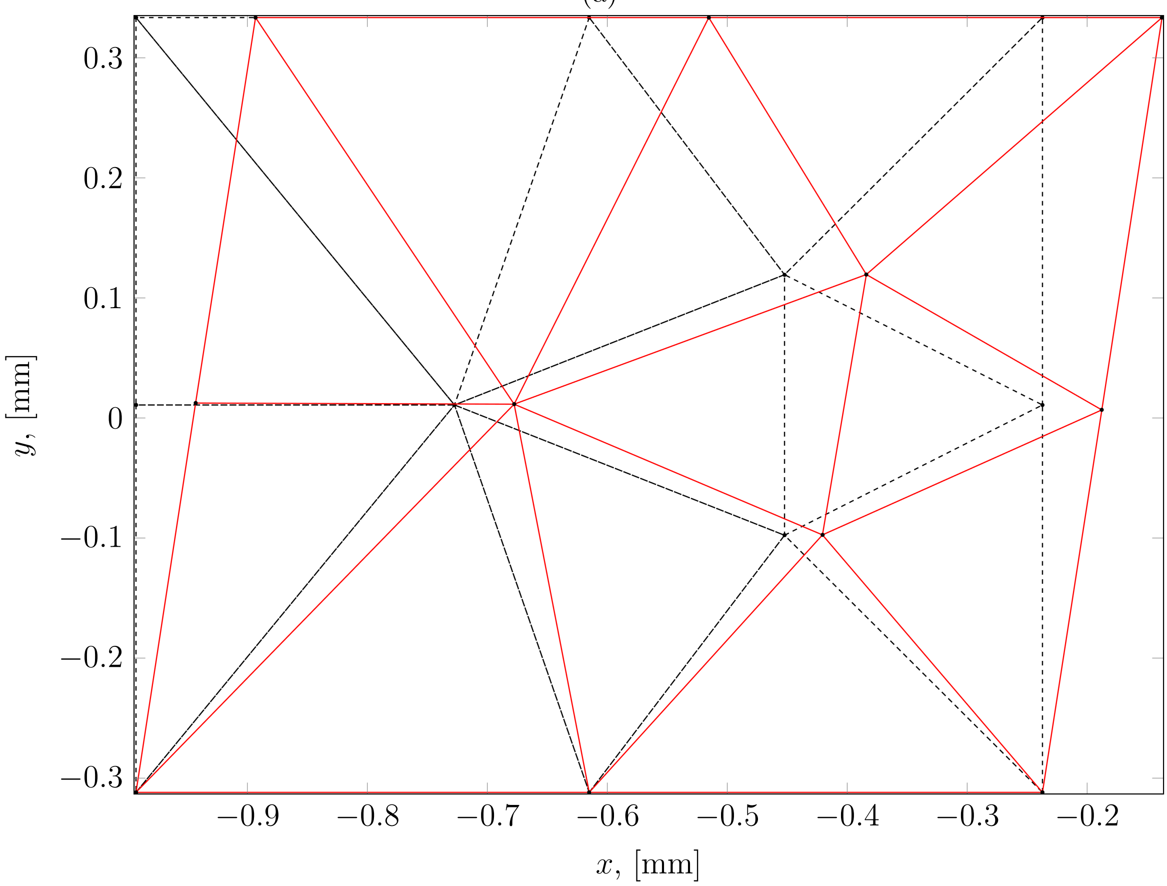
References

[1] J. Bonet and R. D. Wood. *Nonlinear continuum mechanics for finite element analysis.* Cambridge University Press, second edition, 2008.

[2] M. Ekh. Lecture notes on large elasto-plastic deformations, March 2016.

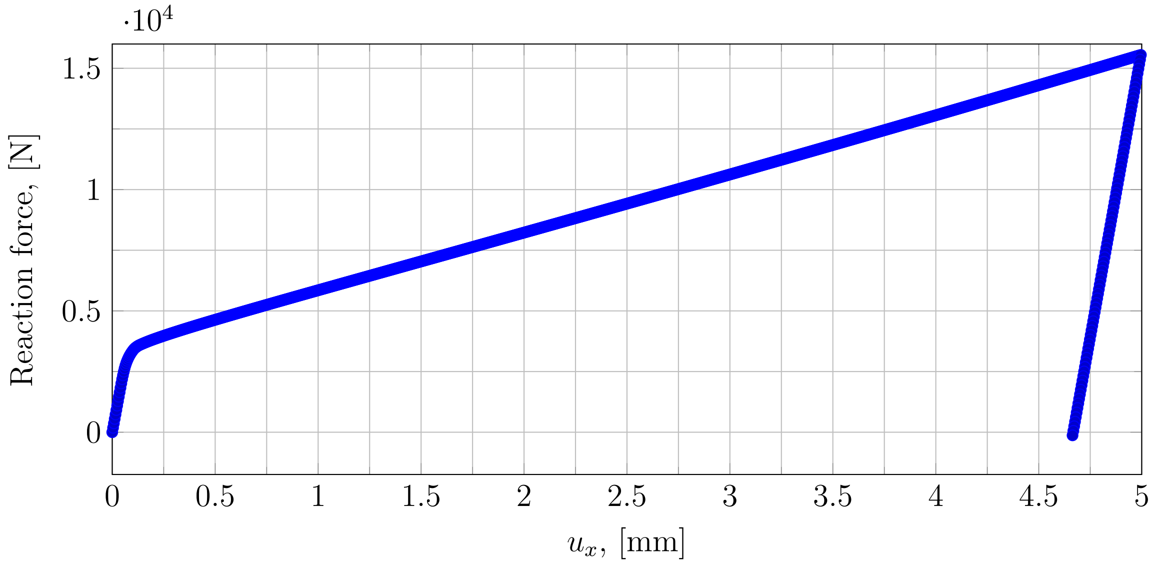


(a)

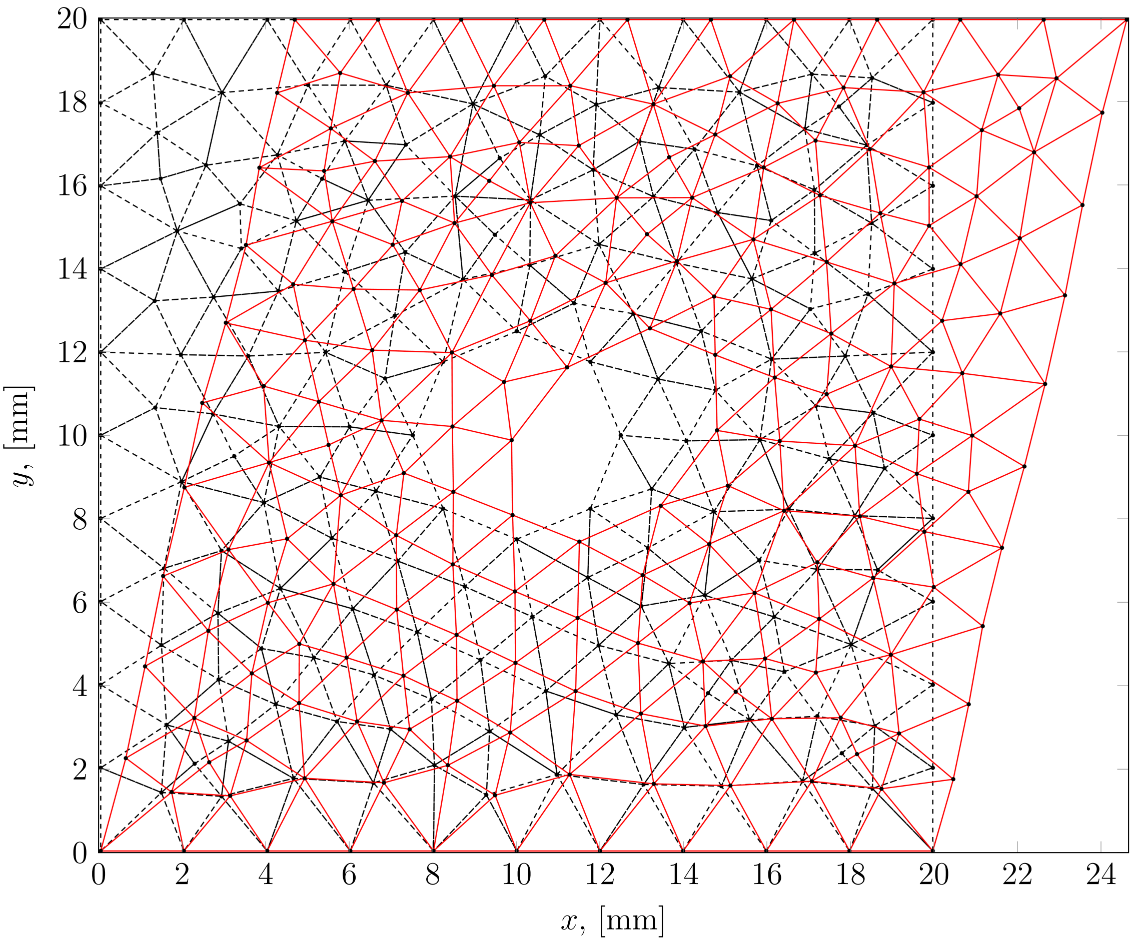


(b)

Figure 2: (a) Reaction force versus displacement and (b) deformed mesh.



(a)



(b)

Figure 3: (a) Reaction force versus displacement and (b) deformed mesh.

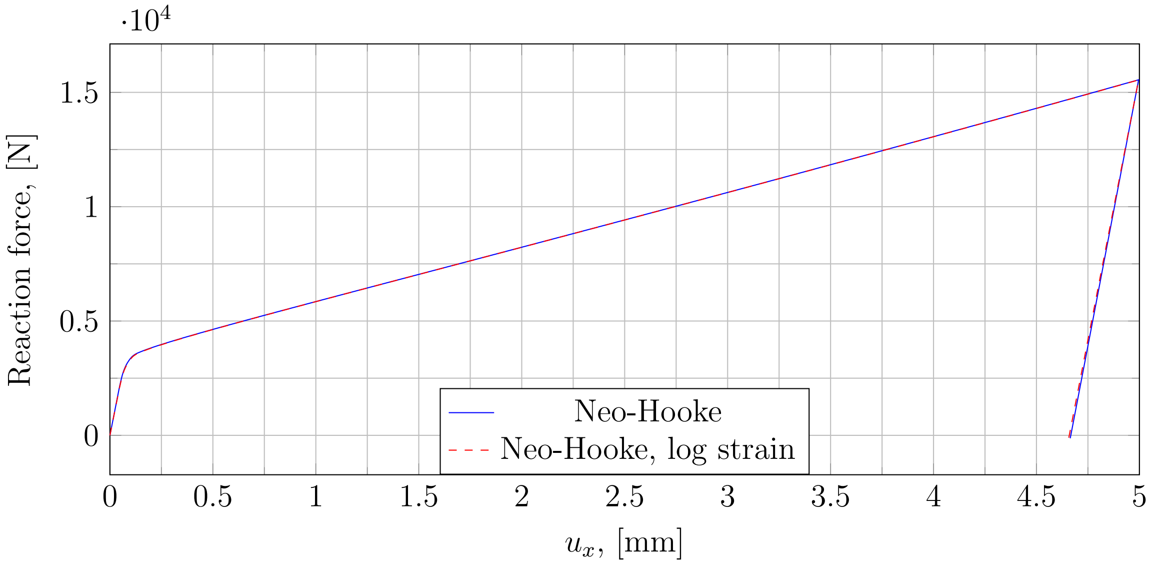


Figure 4: Reaction force versus displacement.