Linear Algebra
Assignment 1

110307039 財管四 黃柏維

March 1, 2025

Question 1

Find the inverse of the following matrices:

$$A = \begin{bmatrix} -\frac{5}{6} & \frac{1}{3} & \frac{11}{6} \\ 0 & \frac{2}{3} & 2 \\ 1 & -\frac{1}{2} & -\frac{5}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ -4 & 4 & 0 \end{bmatrix}$$

Solution:

1. For matrix A, let
$$A = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{A}_2 \\ \boldsymbol{A}_3 \end{bmatrix}$$
where $\boldsymbol{A}_1,\ \boldsymbol{A}_2,\ \boldsymbol{A}_3$ are the rows of matrix A. Then we have
$$-12\boldsymbol{A}_1 - \frac{3}{2}\boldsymbol{A}_2 - 10\boldsymbol{A}_3 = \boldsymbol{0},$$
which implies that the rows of A are linearly dependent, therefore the inverse of matrix A does not exist.

2. For matrix B, we have
$$\det(B) = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ -4 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 5 \\ 2 & 2 & 4 \\ 0 & 8 & 8 \end{vmatrix} = 8 \cdot \begin{vmatrix} 3 & 0 & 3 \\ 2 & 0 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$
Since the determinant of B is zero, the inverse of matrix B does not exists.

😄

Question 2

Find $(AB)^{-1}, (A^T)^{-1}$ and $(2A)^{-1}$ where

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 1 & 3 \\ 4 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Solution: The inverse of matrix A and B are

$$A^{-1} = \frac{1}{61}\begin{bmatrix} 5 & -8 & -14 \\ -12 & 7 & 3 \\ 4 & 18 & -1 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

Therefore, we have

1. $(AB)^{-1} = B^{-1}A^{-1} = \frac{1}{61}\begin{bmatrix} 5 & -8 & -14 \\ -17 & 15 & -11 \\ 21 & 3 & 10 \end{bmatrix}.$

2. $(A^T)^{-1} = (A^{-1})^T = \frac{1}{61}\begin{bmatrix} 5 & -12 & 4 \\ -8 & 7 & 18 \\ -14 & 3 & -1 \end{bmatrix}$.

3. $(2A)^{-1} = 2^{-1}A^{-1} = \frac{1}{122}\begin{bmatrix} 5 & -8 & -14 \\ -12 & 7 & 3 \\ 4 & 18 & -1 \end{bmatrix}$.

😄

Question 3

Find a sequence of elementary matrices that can be used to write the matrix A in row-echelon form, where

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 3 & -2 & -4 \end{bmatrix}$$

Solution: We firstly do the following row operations:

$$1. \quad \begin{matrix} R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix} \implies A \overset{r}{\sim} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & -11 & -4 \end{bmatrix}$$

and

$$2. \quad R_3 \leftarrow R_3 - 11R_2 \implies A \overset{r}{\sim} \begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 7 \end{bmatrix} \triangleq U.$$

Therefore, the matrix A can be written in row-echelon form by

$$E_2E_1A = U,$$

where

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 11 & 1 \end{bmatrix}.$$

😄

Question 4

Find and LU-factorization of the matrix A, where

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix}$$

Solution: We can factorize the matrix A by

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 10 & 12 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 7 \end{bmatrix} = LU.$$

Question 5

Use and LU-factorization to solve the system of linear equations

$$\begin{cases} 2x_1 = 4 \\ -2x_1 + x_2 - x_3 = -4 \\ 6x_1 + 2x_2 + x_3 = 15 \\ -x_4 = -1 \end{cases}$$

Solution: The system of linear equations can be written in matrix form as

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 15 \\ -1 \end{bmatrix}.$$

Let the equation be $A\boldsymbol{x} = \boldsymbol{b}$. We can factorize the coefficient matrix by

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ -2 & 1 & -1 & 0 \\ 6 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = LU.$$

Further let $U\boldsymbol{x} = \boldsymbol{y}$, where $\boldsymbol{y} = [y_1,\ y_2,\ y_3,\ y_4]^T$. We can solve the system of linear equations by firstly solving $L\boldsymbol{y} = \boldsymbol{b}$ and then solving $U\boldsymbol{x} = \boldsymbol{y}$.

$$\begin{cases} y_1 = 4 \\ -y_1 + y_2 = -4 \\ 3y_1 + y_3 = 15 \\ y_4 = -1 \end{cases} \implies \begin{cases} y_1 = 4 \\ y_2 = 0 \\ y_3 = 3 \\ y_4 = -1 \end{cases}$$

$$\begin{cases} 2x_1 = 4 \\ x_2 - x_3 = 0 \\ 3x_1 = 3 \\ -x_4 = -1 \end{cases} \implies \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 1 \end{cases}.$$