Linear Algebra
Assignment 8

110307039 財管四 黃柏維

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Question 1

(CH6.4 Q12) Find the matrix A' for T relative to the basis B', where

$$\begin{aligned} &T : \mathbb{R}^3 \to \mathbb{R}^3, \\ &T(x,y,z) = (x, x+2y, x+y+3z), \\ &B' = \{(1,-1,0),(0,0,1),(0,1,-1)\} \end{aligned}$$

Solution:

$$\begin{aligned} T(1,-1,0) &= (1,-1,0) \\ T(0,0,1) &= (0,-2,3) \\ T(0,1,-1) &= (0,2,-2) \end{aligned}$$

Thus the matrix A' is

$$A' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 3 \\ 0 & 2 & -2 \end{bmatrix}$$

Question 2

(CH6.4 Q14) Let $B = \{(1,1),(-2,3)\}, B' = \{(1,-1),(0,1)\}, and\, [\boldsymbol{v}]_{B'} = 1 \quad -3^T$, and let

$$A = \begin{bmatrix} 3 & 2 \\ 0 & 4 \end{bmatrix}$$

be the matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ relative to B.

(a) Find the transition matrix P from B' to B.

(b) Use the matrices P and A to find $[\boldsymbol{v}]_B$ and $[T(\boldsymbol{v})]_B$, where $[\boldsymbol{v}]_{B'} = -1 \quad 2^T$.

(c) Find P^{-1} and A' (the matrix for T relative to B').

(d) Find $\left[T(\boldsymbol{v})\right]_{B'}$ two ways.

Solution:

(a)

$$P = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}, \qquad [\mathbf{x}]_B = P\,[\mathbf{x}]_{B'}.$$

(b) $[\mathbf{v}]_{B'} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$:

$$[\mathbf{v}]_B = P\,[\mathbf{v}]_{B'} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \qquad [T(\mathbf{v})]_B = A\,[\mathbf{v}]_B = \begin{bmatrix} \frac{17}{5} \\ \frac{16}{5} \end{bmatrix}.$$

(c)

$$P^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \qquad A' = P^{-1}AP = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}.$$

(d)

$$[\,T(\mathbf{v})\,]_{B'} = A'\,[\mathbf{v}]_{B'} = \begin{bmatrix} 3 & 0 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \end{bmatrix},$$

$$[\,T(\mathbf{v})\,]_{B'} = P^{-1}[\,T(\mathbf{v})\,]_B = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{17}{5} \\ \frac{16}{5} \end{bmatrix} = \begin{bmatrix} -3 \\ 10 \end{bmatrix}.$$

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Question 3

(CH6.4 Q22) Use the matrix P to show that the matrices A and A' are similar.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A' = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution: To prove that A and A' are similar we must show that $A' = P^{-1}AP$.

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. Compute P^{-1}.

Because P is unit upper-triangular, its inverse is also unit upper-triangular:

$$P^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

2. Evaluate $P^{-1}AP$.

$$P^{-1}A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$P^{-1}AP = \begin{bmatrix} 5 & -3 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} = A'.$$

3. Conclusion.

Since $A' = P^{-1}AP$ (equivalently $A = PAP^{-1}$), the matrices A and A' are similar. Therefore they represent the same linear transformation with respect to two different bases.

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Question 4

(CH7.1 Q6) Verify that λ_i is an eigenvalue of A and that $\boldsymbol{x}_i$ is a corresponding eigenvector.

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad \begin{aligned} \lambda_1 &= 4, & \boldsymbol{x}_1 &= (1,0,0) \\ \lambda_2 &= 2, & \boldsymbol{x}_2 &= (1,2,0) \\ \lambda_3 &= 3, & \boldsymbol{x}_3 &= (-2,1,1) \end{aligned}$$

Solution: **Verification that each λ_i is an eigenvalue of A with eigenvector $\mathbf{x}_i$.**

$$A = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \qquad \mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{x}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

(a) *Eigenpair* $\lambda_1 = 4,\ \mathbf{x}_1$:

$$A\mathbf{x}_1 = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \mathbf{x}_1.$$

(b) *Eigenpair* $\lambda_2 = 2,\ \mathbf{x}_2$:

$$A\mathbf{x}_2 = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4-2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \lambda_2 \mathbf{x}_2.$$

(c) *Eigenpair* $\lambda_3 = 3,\ \mathbf{x}_3$:

$$A\mathbf{x}_3 = \begin{bmatrix} 4 & -1 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8-1+3 \\ 2+1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \lambda_3 \mathbf{x}_3.$$

$$A\mathbf{x}_i = \lambda_i \mathbf{x}_i \quad \text{for } i = 1,2,3.$$

Hence each λ_i is an eigenvalue of A and each $\mathbf{x}_i$ is a corresponding eigenvector.

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Question 5

(CH7.1 Q12) Determine whether $\boldsymbol{x}$ is an eigenvector of A.

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$$

(a) $\boldsymbol{x} = (1, 1, 0)$

(b) $\boldsymbol{x} = (-5, 2, 1)$

(c) $\boldsymbol{x} = (0, 0, 0)$

(d) $\boldsymbol{x} = (2\sqrt{6} - 3, -2\sqrt{6} + 6, 3)$

Solution: **Test for each candidate vector x whether $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ.**

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$$

(a) $\mathbf{x} = (1, 1, 0)^\mathsf{T}$

$$A\mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \qquad \frac{1}{1} \neq \frac{-2}{1},\ \mathbf{x}_3 = 0.$$

The three coordinates are not in a common ratio, so no scalar λ satisfies $A\mathbf{x} = \lambda\mathbf{x}$. **Not an eigenvector.**

(b) $\mathbf{x} = (-5, 2, 1)^\mathsf{T}$

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 8 \\ -20 \end{bmatrix}, \qquad \frac{0}{-5} \neq \frac{8}{2}.$$

No common ratio exists. **Not an eigenvector.**

(c) $\mathbf{x} = \mathbf{0}$

By definition the zero vector can *never* be an eigenvector (eigenvectors must be non-zero). **Not an eigenvector.**

(d) $\mathbf{x} = (2\sqrt{6} - 3,\ -2\sqrt{6} + 6,\ 3)^\mathsf{T}$

$$A\mathbf{x} = \begin{bmatrix} 2\sqrt{6} + 12 \\ -4\sqrt{6} \\ 12\sqrt{6} - 24 \end{bmatrix}, \qquad \frac{2\sqrt{6} + 12}{2\sqrt{6} - 3} \neq \frac{-4\sqrt{6}}{-2\sqrt{6} + 6}.$$

Again the coordinates fail to share a common ratio. **Not an eigenvector.**

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Question 6

(CH7.1 Q26) Find (a) the characteristic equation and (b) the eigenvalues of (and corresponding eigenvectors) of the matrix

$$A = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -2 & \frac{13}{2} & -10 \\ \frac{3}{2} & -\frac{9}{2} & 8 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -2 & \frac{13}{2} & -10 \\ \frac{3}{2} & -\frac{9}{2} & 8 \end{bmatrix}$$

(a) **Characteristic equation**

$$\det\bigl(A - \lambda I\bigr) = \lambda^3 - \frac{31}{2}\lambda^2 + \frac{59}{4}\lambda - \frac{29}{8} = \frac{1}{8}(2\lambda - 1)^2(2\lambda - 29) = 0.$$

(b) **Eigenvalues and eigenvectors**

$$\lambda_1 = \frac{1}{2} \qquad \text{(algebraic multiplicity 2)}$$

$$\text{eigenvectors: span}\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}\right\};$$

$$\lambda_2 = \frac{29}{2} \qquad \text{(algebraic multiplicity 1)}$$

$$\text{eigenvectors: span}\left\{\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}\right\}.$$

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Question 7

(CH7.2 Q14) Find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal. Verify that $P^{-1}AP$ is a diagonal matrix with the eigenvalues on the main diagonal.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 4 & 0 \\ 0 & 4 & 4 \end{bmatrix}$$

(a) **Characteristic polynomial**

$$\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda & 0 & 0\\ 4 & 4-\lambda & 0\\ 0 & 4 & 4-\lambda\end{bmatrix}=(2-\lambda)\big((4-\lambda)^2\big)=(2-\lambda)(4-\lambda)^2.$$

Hence

$$\lambda_1=2,\qquad \lambda_2=\lambda_3=4 \text{ (algebraic multiplicity 2).}$$

(b) **Eigenvectors**

$$\lambda=2:(A-2I)\mathbf{x}=0 \implies \mathbf{x}=t\begin{bmatrix}1\\-2\\4\end{bmatrix},\ \text{so } E_2=\text{span}\left\{\begin{bmatrix}1\\-2\\4\end{bmatrix}\right\},\ \dim E_2=1.$$

$$\lambda=4:(A-4I)\mathbf{x}=0 \implies \mathbf{x}=s\begin{bmatrix}0\\0\\1\end{bmatrix},\ \text{so } E_4=\text{span}\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\},\ \dim E_4=1.$$

(c) **Diagonalizability test**

To diagonalize A we need

$$\sum_{\lambda}\dim E_\lambda=n=3.$$

Here $\dim E_2+\dim E_4=1+1=2<3$, so A does *not* possess three linearly independent eigenvectors.

(d) **Conclusion**

Because the geometric multiplicity of $\lambda=4$ is $1<2$ (its algebraic multiplicity), matrix A is *not* diagonalizable. Consequently, there is no nonsingular matrix P for which $P^{-1}AP$ is diagonal.

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Question 8

(CH7.2 Q20) Show that the matrix A is not diagonalizable, where

$$A=\begin{bmatrix}3 & 2 & -2\\ 0 & -2 & 3\\ 0 & 0 & -2\end{bmatrix}$$

Solution:

1. Characteristic polynomial

Because A is upper-triangular, its eigenvalues are the diagonal entries. Formally,

$$\chi_A(\lambda)=\det(A-\lambda I)=(3-\lambda)(-2-\lambda)^2.$$

$$\lambda_1 = 3 \text{ (alg. mult. 1)}, \qquad \lambda_2 = -2 \text{ (alg. mult. 2)}$$

2. Eigenspaces

(a) $\lambda = 3$

$$A - 3I = \begin{bmatrix} 0 & 2 & -2 \\ 0 & -5 & 3 \\ 0 & 0 & -5 \end{bmatrix}, \qquad (A - 3I)\mathbf{x} = 0 \Longrightarrow \mathbf{x} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$E_3 = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \qquad \dim E_3 = 1.$$

(b) $\lambda = -2$

$$A + 2I = \begin{bmatrix} 5 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (A + 2I)\mathbf{x} = 0 \Longrightarrow \mathbf{x} = s \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix}.$$

$$E_{-2} = \operatorname{span}\left\{ \begin{bmatrix} 2 \\ -5 \\ 0 \end{bmatrix} \right\}, \qquad \dim E_{-2} = 1.$$

3. Diagonalizability test

A matrix of size $n = 3$ is diagonalizable iff the sum of the geometric multiplicities equals n.

$$\dim E_3 + \dim E_{-2} = 1 + 1 = 2 < 3.$$

Hence A has only two linearly independent eigenvectors.

$$A \text{ is } \textit{not} \text{ diagonalizable}$$

The failure arises because the geometric multiplicity of $\lambda = -2$ (1) is strictly less than its algebraic multiplicity (2).

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Question 9

(CH7.2 Q26) Find the eigenvalues of the matrix A and determine whether there is a sufficient number of eigenvalues to guarantee the matrix is diagonalizable by Theorem 7.6, where

$$A = \begin{bmatrix} 4 & 3 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix},$$

and Theorem 7.6 states that

> If an $n \times n$ matrix A has n *distinct* eigenvalues, then the corresponding eigenvectors are linearly independent and A is diagonalizable.

Solution:

(a) **Eigenvalues**

Because A is upper-triangular, its eigenvalues are the diagonal entries:

$$\lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = -2.$$

(b) **Test for diagonalizability via Theorem 7.6**

The matrix is 3×3 ($n = 3$) and has n *distinct* eigenvalues $\{4, 1, -2\}$.

> Therefore, by Theorem 7.6, the three corresponding eigenvectors are linearly independent, and A is diagonalizable.

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Question 10

(CH7.2 Q30) Find a basis B for the domain of T such that the matrix for T relative to B is diagonal.

$$T : P_2 \longrightarrow P_2 : T(c + bx + ax^2) = (3c + a) + (2b + 3a)x + ax^2$$

Solution:

$$T\big(c + bx + ax^2\big) = (3c + a) \ + \ (2b + 3a)x \ + \ a\,x^2, \qquad P_2 \equiv \{\text{polynomials of degree } \le 2\}.$$

1. Matrix of T in the standard basis $S = \{1, x, x^2\}$

Writing a polynomial $p = c + bx + ax^2$ as the column vector $[c\ b\ a]^\mathsf{T}$, the action of T is

$$\begin{bmatrix} c \\ b \\ a \end{bmatrix} \xmapsto{T} \begin{bmatrix} 3c + a \\ 2b + 3a \\ a \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}}_{A=[T]_S} \begin{bmatrix} c \\ b \\ a \end{bmatrix}.$$

2. Eigenvalues of A

Because A is upper-triangular, its eigenvalues are the diagonal entries:

$$\lambda_1 = 3, \qquad \lambda_2 = 2, \qquad \lambda_3 = 1 \quad \text{(all distinct)}.$$

3. Eigenvectors

$$\lambda_1 = 3 : (A - 3I) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \ (p_1 = 1).$$

$$\lambda_2 = 2 : (A - 2I) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \ (p_2 = x).$$

$$\lambda_3 = 1 : (A - I) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \implies \mathbf{v}_3 = \begin{bmatrix} -1 \\ -6 \\ 2 \end{bmatrix} \ (p_3 = 2x^2 - 6x - 1).$$

4. A diagonalizing basis and the diagonal matrix

Set

$$B = \{\, p_1 = 1,\ p_2 = x,\ p_3 = 2x^2 - 6x - 1 \,\}.$$

Since the three eigenvalues are distinct, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, so B is a basis of P_2.

With $P = [\, \mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3 \,]$ the change-of-basis matrix from B to S,

$$P^{-1}AP = \operatorname{diag}(3, 2, 1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$[T]_B = \operatorname{diag}(3, 2, 1)$$

Hence T is diagonal in the eigenbasis B.

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