Linear Algebra
Assignment 2

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Question 1

(Page 92, Q26) Determine whether the stochastic matrix P is regular. Then find the steady state matrix $\overline{X}$ of the Markov chain with matrix of the transition probabilities P, where

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{5} & 1 \\ \frac{1}{3} & \frac{1}{5} & 0 \\ \frac{1}{6} & \frac{3}{5} & 0 \end{bmatrix}.$$

Solution:

1. To determine whether the stochastic matrix P is regular, we need to check whether the matrix P^n has all positive elements for some n. We have

$$P^2 = \begin{bmatrix} \frac{29}{60} & \frac{37}{50} & \frac{1}{2} \\ \frac{7}{30} & \frac{8}{75} & \frac{1}{3} \\ \frac{17}{60} & \frac{23}{150} & \frac{1}{6} \end{bmatrix}$$

   Since all elements of P^2 are positive, the stochastic matrix P is regular.

2. To find the steady state matrix $\overline{X} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ of the Markov chain with matrix of the transition probabilities P, we need to solve the equation

$$P\overline{X} = \overline{X}$$

   which is equivalent to

$$(P - I)\overline{X} = \mathbf{0}$$

   where I is the identity matrix. We have

$$\overline{X} \in \ker(P - I) = \operatorname{span}\left\{\begin{bmatrix} 12 \\ 5 \\ 5 \end{bmatrix}\right\},$$

   and by $x_1 + x_2 + x_3 = 1$, we have

$$\overline{X} = \begin{bmatrix} \frac{12}{22} \\ \frac{5}{22} \\ \frac{5}{22} \end{bmatrix}$$

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Question 2

(Page 92, Q28) Determine whether the stochastic matrix P is regular. Then find the steady state matrix $\overline{X}$ of the Markov chain with matrix of the transition probabilities P, where

$$P = \begin{bmatrix} 0.1 & 0 & 0.3 \\ 0.7 & 1 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}.$$

Solution: To determine whether the stochastic matrix P is regular, we need to check whether the matrix P^n has all positive elements for some n. We have

$$P_{12}^n = P_{32}^n = 0, \forall n > 1$$

Therefore, the stochastic matrix P is not regular, and the steady state matrix $\overline{X}$ does not exist.

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Question 3

(Page 94, Q42) Find the steady state matrix $\overline{X}$ of the absorbing Markov chain with matrix of transition probabilities P, where

$$P = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.2 & 1 & 0 \\ 0.7 & 0 & 1 \end{bmatrix}.$$

Solution: Use the matrix equation $P\overline{X} = \overline{X}$, along with the equation $x_1 + x_2 + x_3 + x_4 = 1$ to write the system of linear equations

$$\begin{cases} 0.1x_1 = x_1 \\ 0.2x_1 + x_2 = x_2 \\ 0.7x_1 + x_3 = x_3 \end{cases}.$$

The solution of this system is $x_1 = 0, x_2 = 1 - t, x_3 = t$, where $t \in \mathbb{R}$, and $0 \leqslant t \leqslant 1$.
So, the steady state matrix is

$$\overline{X} = \begin{bmatrix} 0 \\ 1 - t \\ t \end{bmatrix}$$

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Question 4

(Page 102, Q10) A code braeker intercepted the encoded message below.

$$45 \ -35 \ 38 \ -30 \ 18 \ -18 \ 35 \ -30 \ 81 \ -60 \ 42 \ -28 \ 75 \ -55 \ 2 \ -2 \ 22 \ -21 \ 15 \ -10$$

Let the inverse of the encoding matrix be

$$A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

(a) You know that the $\begin{bmatrix} 45 & -35 \end{bmatrix} A^{-1} = \begin{bmatrix} 10 & 15 \end{bmatrix}$ and $\begin{bmatrix} 38 & -30 \end{bmatrix} A^{-1} = \begin{bmatrix} 8 & 14 \end{bmatrix}$. Write and solve two systems of equations to find $w, x, y,$ and z.

(b) Decode the message.

Solution:

(a) By the given information, we have

$$\begin{cases} 45w - 35y = 10 \\ 45x - 35z = 15 \end{cases}$$

and

$$\begin{cases} 38w - 30y = 8 \\ 38x - 30z = 14 \end{cases}$$

Solving these two systems of equations, we get $w = 1, x = -2, y = 1, z = -3$.

(b) Decoding the message by putting the encoded message into the matrix form, we have

$$\begin{bmatrix} 45 & -35 \\ 38 & -30 \\ 18 & -18 \\ 35 & -30 \\ 81 & -60 \\ 42 & -28 \\ 75 & -55 \\ 2 & -2 \\ 22 & -21 \\ 15 & -10 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 10 & 15 \\ 8 & 14 \\ 0 & 18 \\ 5 & 20 \\ 21 & 18 \\ 14 & 0 \\ 20 & 15 \\ 0 & 2 \\ 1 & 19 \\ 5 & 0 \end{bmatrix}$$

The decoded message is 10 15 8 14 0 18 5 20 21 18 14 0 20 15 0 2 1 19 5 0.
which means "**JOHN RETURN TO BASE** " be the code table at page 94.

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Question 5

(Page 103, Q26) Find the least squares regression lline for the data

$$(0, 6), (4, 3), (5, 0), (8, -4), (10, -5)$$

Solution: The matrices X and Y are

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 5 \\ 1 & 8 \\ 1 & 10 \end{bmatrix}, \quad Y = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -4 \\ -5 \end{bmatrix}$$

Then, by the normal equation $X^T XA = X^T Y$, we have

$$\begin{bmatrix} 5 & 27 \\ 27 & 205 \end{bmatrix} A$$

$$= \begin{bmatrix} 0 \\ -70 \end{bmatrix}$$

Solving this system of equations, we get

$$A = \begin{bmatrix} \frac{945}{148} \\ -\frac{175}{148} \end{bmatrix}$$

Therefore, the least squares regression line is

$$y = \frac{945}{148} - \frac{175}{148}x.$$

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Question 6

(Page 116, Q24) Use expansion by cofactors to find the determinant of the matrix A, where

$$A = \begin{bmatrix} 0.1 & 0.2 & 0.3 \\ -0.3 & 0.2 & 0.2 \\ 0.5 & 0.4 & 0.4 \end{bmatrix}.$$

Solution: The cofactor expansion by the first row of A is

$$\det(A) = 0.1 \begin{vmatrix} 0.2 & 0.2 \\ 0.4 & 0.4 \end{vmatrix} - 0.2 \begin{vmatrix} -0.3 & 0.2 \\ 0.5 & 0.4 \end{vmatrix} + 0.3 \begin{vmatrix} -0.3 & 0.2 \\ 0.5 & 0.4 \end{vmatrix}$$

Solving these determinants, we get

$$\det(A) = 0.1 \cdot 0 - 0.2(-0.22) + 0.3(-0.22) = 0 + 0.044 - 0.066 = -0.022$$

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Question 7

(Page 116, Q28) Use expansion by cofactors to find the determinant of the matrix A, where

$$A = \begin{bmatrix} 3 & 0 & 7 & 0 \\ 2 & 6 & 11 & 12 \\ 4 & 1 & -1 & 2 \\ 1 & 5 & 2 & 10 \end{bmatrix}.$$

Solution: The cofactor expansion by the first row of A is

$$\det(A) = 3 \begin{vmatrix} 6 & 11 & 12 \\ 1 & -1 & 2 \\ 5 & 2 & 10 \end{vmatrix} - 0 + 7 \begin{vmatrix} 2 & 11 & 12 \\ 4 & -1 & 2 \\ 1 & 2 & 10 \end{vmatrix} - 0 \quad = 3 \cdot 0 - 7 \cdot (-338) = 2366.$$

Note that the first term is zero because the first column is linearly dependent on the third column.

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