

Linear Algebra

Assignment 4

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Question 1

(CH4.4 P184 Q6) For the matrices

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$$

in $M_{2 \times 2}$, determine whether matrix

$$C = \begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix}$$

is a linear combination of A and B .

Solution: Let $C = xA + yB$ for some $x, y \in \mathbb{R}$. Then we have the system of equations:

$$\begin{aligned} x \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix} &= \begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix} \\ \Rightarrow \begin{cases} 2x + 0y = 6 \\ -3x + 5y = 2 \\ 4x + 1y = 9 \\ 1x - 2y = 11 \end{cases} \end{aligned}$$

Because the above system of equations are not consistent, C is not a linear combination of A and B .



Question 2

(CH4.4 P184 Q18) Determine whether the set

$$S = \{(-1, 2), (2, -1), (1, 1)\}$$

spans \mathbb{R}^2 . If the set does not span \mathbb{R}^2 , then give a geometric description of the subspace that it does span.

Solution: We first form the matrix A whose rows are the vectors in S :

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim A.$$

Since the rank of A is 2, which is equal to the dimension of \mathbb{R}^2 , we conclude that S spans \mathbb{R}^2 .



Question 3

(CH4.4 P184 Q26) Determine whether the set

$$S = \{-2x + x^2, 8x + x^3, -x^2 + x^3, -4 + x^2\}$$

spans P_3 .

Solution: We first form the matrix A whose rows are the coefficients of the vectors in S :

$$A = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 8 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -4 & 0 & 1 & 0 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim A.$$

$$\therefore \text{rank}(A) = 3 < \text{rank}(P_3) = 4$$

$\therefore S$ does not span P_3 .



Question 4

(CH4.5 P194 Q56) Determine whether

$$S = \left\{ \left(\frac{2}{3}, \frac{5}{2}, 1 \right), \left(1, \frac{3}{2}, 0 \right), (2, 12, 6) \right\}$$

is a basis for \mathbb{R}^3 . If it is, write $\mathbf{u} = (8, 3, 8)$ as a linear combination of the vectors in S .

Solution: We first form the matrix A whose rows are the vectors in S :

$$A = \begin{bmatrix} \frac{2}{3} & \frac{5}{2} & 1 \\ 1 & \frac{3}{2} & 0 \\ 2 & 12 & 6 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \sim A.$$

$$\therefore \text{rank}(A) = 2 < \text{rank}(\mathbb{R}^3) = 3$$

$\therefore S$ is not a basis for \mathbb{R}^3 .



Question 5

(CH4.5 P194 Q68) Find all subsets of the set

$$S = \{(1, 3, -2), (-4, 1, 1), (-2, 7, -3), (2, 1, 1)\}$$

that form a basis for \mathbb{R}^3 .

Solution: Let the vectors in S be $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in order.

Then we can test whether the matrices A_1, A_2, A_3, A_4 are rank 3, where A_i is the matrix whose rows are the vectors in S excluding \mathbf{v}_i :

$$A_1 = \begin{bmatrix} -4 & 1 & 1 \\ -2 & 7 & -3 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 7 & -3 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 3 & -2 \\ -4 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 3 & -2 \\ -4 & 1 & 1 \\ -2 & 7 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{13} \\ 0 & 1 & -\frac{7}{13} \\ 0 & 0 & 0 \end{bmatrix}.$$

$\therefore \text{rank}(A_1) = \text{rank}(A_2) = \text{rank}(A_3) = 3 = \text{rank}(\mathbb{R}^3)$ and $\text{rank}(A_4) = 2 < \text{rank}(\mathbb{R}^3) = 3$

\therefore The subsets $\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}, \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}, \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ form a basis for \mathbb{R}^3 .



Question 6

(CH4.5 P194 Q70) Find a basis for \mathbb{R}^3 that includes the vectors $(1, 0, 2)$ and $(0, 1, 1)$.

Solution: We can directly try a standard vector $\mathbf{e}_3 = (0, 0, 1)$.

$$A \triangleq \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\therefore S = \{(1, 0, 2), (0, 1, 1), (0, 0, 1)\}$ is a basis for \mathbb{R}^3 .



Question 7

(CH4.6 P205 Q14) Find a basis for the subspace of \mathbb{R}^3 spanned by

$$S = \{(2, 3, -1), (1, 3, -9), (0, 1, 5)\}.$$

Solution: We first form the matrix A whose rows are the vectors in S :

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 3 & -9 \\ 0 & 1 & 5 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim A.$$

$\therefore S$ is a basis for the subspace of \mathbb{R}^3 spanned by S .



Question 8

(CH4.6 P205 Q20) Find a basis for the subspace of \mathbb{R}^4 spanned by

$$S = \{(2, 4, -3, -2), (-2, -3, 2, -5), (1, 3, -2, 2), (-1, -5, 3, 5)\}.$$

Solution: We first form the matrix A whose rows are the vectors in S :

$$A = \begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim A.$$

$\therefore \{(2, 4, -3, -2), (-2, -3, 2, -5), (1, 3, -2, 2)\}$ is a basis for the subspace of \mathbb{R}^4 spanned by S .



Question 9

(CH4.6 P205 Q38) Find the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullspace of A is therefore spanned by S , where

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$



Question 10

(CH4.6 P205 Q40) Find the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The nullspace of A is therefore spanned by S , where

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$



Question 11

(CH4.6 P206 Q42) Use the fact that matrices A and B are row-equivalent to

- (a) Find the rank and nullity of A .
- (b) Find a basis for the nullspace of A .
- (c) Find a basis for the row space of A .
- (d) Find a basis for the column space of A .
- (e) Determine whether the rows of A are linearly independent.
- (f) Let the columns of A be denoted by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$, and \mathbf{a}_5 .
Determine whether each set is linearly independent:

(i) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$

(ii) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$

(iii) $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$

,where

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

- (a) $\text{rank}(A) = \text{rank}(B) = 3$,
 $\text{nullity}(A) = \text{nullity}(B) = 2$.

- (b) $N(A) = N(B) = \text{span}(S_N)$, where

$$S_N = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

- (c) $R(A) = R(B) = \text{span}(S_R)$, where

$$S_R = \left\{ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -5 \end{bmatrix} \right\}.$$

- (d) $C(A)$ is spanned by the independent columns of A , which can be found by the correspondant pivot columns of B ,
 $\therefore C(A) = \text{span}(S_C)$, where

$$S_C = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

- (e) The rows of A are not linearly independent, since the rank of A is less than the number of rows.

- (f) We can easily check the answers by looking at the correspondant columns of B

- (i) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4\}$ is linearly independent.

- (ii) $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ is not linearly independent.

- (iii) $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is not linearly independent.



Question 12

(CH4.6 P206 Q62) Determine whether

$$\mathbf{b} = \begin{bmatrix} -9 \\ 11 \\ -25 \end{bmatrix}$$

is in the column space of

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -3 & 1 & -2 \\ 1 & 0 & 8 \end{bmatrix}.$$

If it is, write \mathbf{b} as a linear combination of the column vectors of A .

Solution:

$$A \triangleq \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$$

Do row operations on $[A | \mathbf{b}]$:

$$[A | \mathbf{b}] \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}.$$

$$\therefore \mathbf{b} = -\mathbf{a}_1 + 2\mathbf{a}_2 - 3\mathbf{a}_3.$$

