Linear Algebra
Assignment 4

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Question 1

(CH4.4 P184 Q6) For the matrices

$$A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix}$$

in $M_{2\times 2}$, determine whether matrix

$$C = \begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix}$$

is a linear combination of A and B.

Solution: Let $C = xA + yB$ for some $x, y \in \mathbb{R}$. Then we have the system of equations:

$$x \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} + y \begin{bmatrix} 0 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 9 & 11 \end{bmatrix}$$

$$\implies \begin{cases} 2x + 0y = 6 \\ -3x + 5y = 2 \\ 4x + 1y = 9 \\ 1x - 2y = 11 \end{cases}$$

Because the above system of equations are not consistent, C is not a linear combination of A and B.

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Question 2

(CH4.4 P184 Q18) Determine whether the set

$$S = \{(-1, 2), (2, -1), (1, 1)\}$$

spans $\mathbb{R}^2$. If the set does not span $\mathbb{R}^2$, then give a geometric description of the subspace that it does span.

Solution: We first form the matrix A whose rows are the vectors in S:

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \\ 1 & 1 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \sim A.$$

Since the rank of A is 2, which is equal to the dimension of $\mathbb{R}^2$, we conclude that S spans $\mathbb{R}^2$.

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Question 3

(CH4.4 P184 Q26) Determine whether the set

$$S = \{-2x + x^2, 8x + x^3, -x^2 + x^3, -4 + x^2\}$$

spans P_3.

Solution: We first form the matrix A whose rows are the coefficients of the vectors in S:

$$A = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 8 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -4 & 0 & 1 & 0 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim A.$$

$\because \operatorname{rank}(A) = 3 < \operatorname{rank}(P_3) = 4$

$\therefore S$ does not span P_3.

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Question 4

(CH4.5 P194 Q56) Determine whether

$$S = \{\left(\frac{2}{3}, \frac{5}{2}, 1\right), \left(1, \frac{3}{2}, 0\right), (2, 12, 6)\}$$

is a basis for $\mathbb{R}^3$. If it is, write $\boldsymbol{u} = (8, 3, 8)$ as a linear combination of the vectors in S.

Solution: We first form the matrix A whose rows are the vectors in S:

$$A = \begin{bmatrix} \frac{2}{3} & \frac{5}{2} & 1 \\ 1 & \frac{3}{2} & 0 \\ 2 & 12 & 6 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \sim A.$$

$\because \operatorname{rank}(A) = 2 < \operatorname{rank}(\mathbb{R}^3) = 3$

$\therefore S$ is not a basis for $\mathbb{R}^3$.

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Question 5

(CH4.5 P194 Q68) Find all subsets of the set

$$S = \{(1,3,-2),(-4,1,1),(-2,7,-3),(2,1,1)\}$$

that form a bsis for $\mathbb{R}^3$.

Solution: Let the vectors in S be $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ in order.

Then we can test whether the matrices A_1, A_2, A_3, A_4 are rank 3, where A_i is the matrix whose rows are the vectors in S excluding $\boldsymbol{v}_i$:

$$A_1 = \begin{bmatrix} -4 & 1 & 1 \\ -2 & 7 & -3 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 7 & -3 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 3 & -2 \\ -4 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 1 & 3 & -2 \\ -4 & 1 & 1 \\ -2 & 7 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{5}{13} \\ 0 & 1 & -\frac{7}{13} \\ 0 & 0 & 0 \end{bmatrix}.$$

$\because \operatorname{rank}(A_1) = \operatorname{rank}(A_2) = \operatorname{rank}(A_3) = 3 = \operatorname{rank}(\mathbb{R}^3)$ and $\operatorname{rank}(A_4) = 2 < \operatorname{rank}(\mathbb{R}^3) = 3$

$\therefore$ The subsets $\{\boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}, \{\boldsymbol{v}_1, \boldsymbol{v}_3, \boldsymbol{v}_4\}, \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$ form a basis for $\mathbb{R}^3$.

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Question 6

(CH4.5 P194 Q70) Find a basis for $\mathbb{R}^3$ that includes the vectors $(1,0,2)$ and $(0,1,1)$.

Solution: We can directly try a standard vector $\boldsymbol{e}_3 = (0,0,1)$.

$$A \triangleq \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$\therefore S = \{(1,0,2),(0,1,1),(0,0,1)\}$ is a basis for $\mathbb{R}^3$.

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Question 7

(CH4.6 P205 Q14) Find a basis for the subspace of $\mathbb{R}^3$ spanned by

$$S = \{(2,3,-1),(1,3,-9),(0,1,5)\}.$$

Solution: We first form the matrix A whose rows are the vectors in S:

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 3 & -9 \\ 0 & 1 & 5 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim A.$$

$\therefore$ S is a basis for the subspace of $\mathbb{R}^3$ spanned by S.

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Question 8

(CH4.6 P205 Q20) Find a basis for the subspace of $\mathbb{R}^4$ spanned by

$$S = \{(2,4,-3,-2),(-2,-3,2,-5),(1,3,-2,2),(-1,-5,3,5)\}.$$

Solution: We first form the matrix A whose rows are the vectors in S:

$$A = \begin{bmatrix} 2 & 4 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix},$$

which can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -13 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim A.$$

$\therefore$ $\{(2,4,-3,-2),(-2,-3,2,-5),(1,3,-2,2)\}$ is a basis for the subspace of $\mathbb{R}^4$ spanned by S.

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Question 9

(CH4.6 P205 Q38) Find the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ -2 & -8 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nullspace of A is therefore spanned by S, where

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

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Question 10

(CH4.6 P205 Q40) Find the nullspace of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix}.$$

Solution:

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The nullspace of A is therefore spanned by S, where

$$S = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

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Question 11

(CH4.6 P206 Q42) Use the fact that matrices A and B are row-equivalent to

(a) Find the rank and nullity of A.

(b) Find a basis for the nullspace of A.

(c) Find a basis for the row space of A.

(d) Find a basis for the column space of A.

(e) Determine whether the rows of A are linearly independent.

(f) Let the columns of A be denoted by $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4,$ and $\boldsymbol{a}_5$. Determine whether each set is linearly independent:

(i) $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4\}$

(ii) $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$

(iii) $\{\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_5\}$

,where

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

(a) $\text{rank}(A) = \text{rank}(B) = 3,$
$\text{nullity}(A) = \text{nullity}(B) = 2.$

(b) $\text{N}(A) = \text{N}(B) = \text{span}(S_N)$, where

$$S_N = \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}.$$

(c) $\text{R}(A) = \text{R}(B) = \text{span}(S_R)$, where

$$S_R = \left\{ \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -5 \end{bmatrix} \right\}.$$

(d) C(A) is spanned by the independent columns of A, which can be found by the correspondant pivot columns of B,

$\therefore$ C(A) = span(S_C), where

$$S_C = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}.$$

(e) The rows of A are not linearly independent, since the rank of A is less than the number of rows.

(f) We can easily check the answers by looking at the correspondant columns of B

(i) $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4\}$ is linearly independent.

(ii) $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3\}$ is not linearly independent.

(iii) $\{\boldsymbol{a}_1, \boldsymbol{a}_3, \boldsymbol{a}_5\}$ is not linearly independent.

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Question 12

(CH4.6 P206 Q62) Determine whether

$$\boldsymbol{b} = \begin{bmatrix} -9 \\ 11 \\ -25 \end{bmatrix}$$

is in the column space of

$$A = \begin{bmatrix} 5 & 4 & 4 \\ -3 & 1 & -2 \\ 1 & 0 & 8 \end{bmatrix}.$$

If it is, write $\boldsymbol{b}$ as a linear combination of the column vectors of A.

Solution:

$$A \triangleq \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \boldsymbol{a}_3 \end{bmatrix}$$

Do row opeartions on $[\,A\,|\,\boldsymbol{b}\,]$:

$$[\,A\,|\,\boldsymbol{b}\,] \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}.$$

$\therefore \boldsymbol{b} = -\boldsymbol{a}_1 + 2\boldsymbol{a}_2 - 3\boldsymbol{a}_3$.

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