Linear Algebra
Assignment 3

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March 24, 2025

Question 1

(CH3.1 P116 Q18) Find the determinant of the maxtrix A using the method of expansion by cofactors.

(a) the third row and

(b) the first column.

, where

$$A = \begin{bmatrix} -3 & 4 & 2 \\ 6 & 3 & 1 \\ 4 & -7 & -8 \end{bmatrix}.$$

Solution:

(a)

$$\det(A) = 4\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 7\begin{vmatrix} -3 & 2 \\ 6 & 1 \end{vmatrix} - 8\begin{vmatrix} -3 & 4 \\ 6 & 3 \end{vmatrix} \quad = 4(4-6) + 7(-3-12) - 8(-9-24) = -8 - 105 + 264 \quad = 151.$$

(b)

$$\det(A) = -3\begin{vmatrix} 3 & 1 \\ -7 & -8 \end{vmatrix} - 6\begin{vmatrix} 4 & 2 \\ -7 & -8 \end{vmatrix} + 4\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} \quad = -3(-24+7) - 6(-32+14) + 4(4-6) = -91 + 108 - 8 \quad = 9.$$

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Question 2

(CH3.1 P117 Q52) Find the values of λ for which the determinant is zero.

$$\begin{vmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 3 \\ 2 & 2 & \lambda - 2 \end{vmatrix}$$

Solution:

$$\text{determinant} = \lambda\begin{vmatrix} \lambda & 3 \\ 2 & \lambda - 2 \end{vmatrix} - 0 + 1\begin{vmatrix} 0 & \lambda \\ 2 & 2 \end{vmatrix} \quad = \lambda(\lambda(\lambda-2) - 6) - 2\lambda = (\lambda+2)\lambda(\lambda-4)$$

Therefore, for $\lambda = -2, 0, 4$, the determinant is zero.

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Question 3

(CH3.2 P124 Q30) Use elementary row or column operations to find the determinant.

$$\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & -6 \end{vmatrix}$$

Solution:

$$\begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 6 & 1 & -6 \end{vmatrix} = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & -15 & 8 \end{vmatrix} = \begin{vmatrix} 3 & 8 & -7 \\ 0 & -5 & 4 \\ 0 & 0 & -8 \end{vmatrix} = 3 \cdot (-5) \cdot (-8) = 120.$$

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Question 4

(CH3.3 P131 Q10) Use the fact that $|cA| = c^n |A|$ to evaluate the determinant of the $n \times n$ matrix A.

$$A = \begin{vmatrix} 4 & 16 & 0 \\ 12 & -8 & 8 \\ 16 & 20 & -4 \end{vmatrix}$$

Solution:

$$|A| = 4^3 \left\| \begin{matrix} 1 & 4 & 0 \\ 3 & -2 & 2 \\ 4 & 5 & -1 \end{matrix} \right\| = 64 \cdot \left(1 \cdot \begin{vmatrix} -2 & 2 \\ 5 & -1 \end{vmatrix} - 4 \cdot \begin{vmatrix} 3 & 2 \\ 4 & -1 \end{vmatrix}\right) = 64 \cdot (1 \cdot (2 - 10) - 4 \cdot (-3 - 8)) = 64 \cdot 36 = 2304.$$

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Question 5

(CH3.3 P132 Q48) Find

(a) $|A^T|$

(b) $|A^2|$

(c) $|AA^T|$

(d) $|2A|$

(e) $|A^{-1}|$

for the matrix A.

$$A = \begin{vmatrix} 4 & 1 & 9 \\ -1 & 0 & -2 \\ -3 & 3 & 0 \end{vmatrix}$$

Solution:

(a)

$$|A^T| = |A| = 1 \cdot \begin{vmatrix} 1 & 9 \\ 3 & 0 \end{vmatrix} + 0 + 2 \cdot \begin{vmatrix} 4 & 1 \\ -3 & 3 \end{vmatrix} = -27 + 30 = 3.$$

(b)

$$|A^2| = |A|^2 = 3^2 = 9.$$

(c)

$$\left|AA^T\right| = |A| \cdot \left|A^T\right| = 3 \cdot 3 = 9.$$

(d)

$$|2A| = 2^3 \cdot |A| = 8 \cdot 3 = 24.$$

(e)

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{3}.$$

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Question 6

(CH3.4 P142 Q8) Find the adjoint of the matrix A. Then use the adjoint to find the inverse of A (if possible)).

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Solution:

1. $$\operatorname{adj}(A) = \begin{bmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{bmatrix}.$$

2. By $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$, we have

$$A^{-1} = \frac{1}{-3} \begin{bmatrix} -1 & -1 & -1 & 2 \\ -1 & -1 & 2 & -1 \\ -1 & 2 & -1 & -1 \\ 2 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

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Question 7

(CH3.4 P142 Q18) Use Cramer's Rule to solve (if possible) the system of equations.

$$4x - 2y + 3z = -2$$
$$2x + 2y + 5z = 16$$
$$8x - 5y - 2z = 4$$

Solution:

1. Solve for Δ

$$\Delta = \begin{vmatrix} 4 & -2 & 3 \\ 2 & 2 & 5 \\ 8 & -5 & -2 \end{vmatrix} = 4\begin{vmatrix} 2 & 5 \\ -5 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 8 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & 2 \\ 8 & -5 \end{vmatrix} = 4(-4+25) + 2(-4-40) + 3(-10-16) \quad = -82.$$

2. Solve for Δ_x to find x

$$\begin{aligned} \Delta_x &= \begin{vmatrix} -2 & -2 & 5 \\ 16 & 2 & 5 \\ 4 & -5 & -2 \end{vmatrix} \\ &= -2\begin{vmatrix} 2 & 5 \\ -5 & -2 \end{vmatrix} + 2\begin{vmatrix} 16 & 5 \\ 4 & -2 \end{vmatrix} + 5\begin{vmatrix} 16 & 2 \\ 4 & -5 \end{vmatrix} \\ &= -2(-4+25) + 2(-32-20) + 5(-80-8) \quad = -586 \end{aligned}$$

Therefore, $x = \frac{\Delta_x}{\Delta} = \frac{-586}{-82} = \frac{293}{41}$.

3. Solve for Δ_y to find y

$$\begin{aligned} \Delta_y &= \begin{vmatrix} 4 & -2 & 3 \\ 2 & 16 & 5 \\ 8 & 4 & -2 \end{vmatrix} \\ &= 4\begin{vmatrix} 16 & 5 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 8 & -2 \end{vmatrix} + 3\begin{vmatrix} 2 & 16 \\ 8 & 4 \end{vmatrix} \\ &= 4(-32-20) + 2(-4-40) + 3(8-128) \quad = -656 \end{aligned}$$

Therefore, $y = \frac{\Delta_y}{\Delta} = \frac{-656}{-82} = 8$.

4. Solve for Δ_z to find z

$$\begin{aligned} \Delta_z &= \begin{vmatrix} 4 & -2 & -2 \\ 2 & 2 & 16 \\ 8 & -5 & 4 \end{vmatrix} \\ &= 4\begin{vmatrix} 2 & 16 \\ -5 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & 16 \\ 8 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 8 & -5 \end{vmatrix} \\ &= 4(8+80) + 2(16-128) - 2(-10-16) \quad = 180 \end{aligned}$$

Therefore, $z = \frac{\Delta_z}{\Delta} = \frac{180}{-82} = -\frac{90}{41}$.

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Question 8

(CH3.4 P142 Q36) Determine whether the points are collinear.

$$(-1,-3), (-4,7), (2,-13)$$

Solution:

$$\begin{vmatrix} -1 & -3 & 1 \\ -4 & 7 & 1 \\ 2 & -13 & 1 \end{vmatrix} = \begin{vmatrix} -1 & -3 & 1 \\ 0 & 19 & -3 \\ 0 & -19 & 3 \end{vmatrix} = 0.$$

Since the determinant is zero, the points are collinear.

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Question 9

(CH3.4 P142 Q48) Determine whether the points are coplanar.

$$(1, 2, 3), (-1, 0, 1), (0, -2, -5), (2, 6, 11)$$

Solution:

$$\begin{vmatrix} 1 & 2 & 3 & 1 \\ -1 & 0 & 1 & 1 \\ 0 & -2 & -5 & 1 \\ 2 & 6 & 11 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & -5 & 1 \\ 0 & 2 & 5 & -1 \end{vmatrix} = 0.$$

Since the determinant is zero, the points are coplanar.

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Question 10

(CH4.2 P166 Q24) Determine whether the set

$$V = \left\{ \left(x, \frac{1}{2}x \right) \middle| \, x \text{ is a real number} \right\}$$

together with the standard operations, is a vector space. If it is not, identify at least one of the ten vector space axioms that fails.

Solution: Let's exaimine the ten vector space axioms.

1. $\forall \boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} \in V$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right)$ and $\boldsymbol{v} = \left(y, \frac{1}{2}y\right) \in V$.
   Then $\boldsymbol{u} + \boldsymbol{v} = \left(x + y, \frac{1}{2}x + \frac{1}{2}y\right) \in V$.

2. $\forall \boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right)$ and $\boldsymbol{v} = \left(y, \frac{1}{2}y\right) \in V$.
   Then $\boldsymbol{u} + \boldsymbol{v} = \left(x + y, \frac{1}{2}x + \frac{1}{2}y\right) = \left(y + x, \frac{1}{2}y + \frac{1}{2}x\right) = \boldsymbol{v} + \boldsymbol{u}$.

3. $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V, \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right), \boldsymbol{v} = \left(y, \frac{1}{2}y\right), \boldsymbol{w} = \left(z, \frac{1}{2}z\right) \in V$.
   Then $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = \left(x, \frac{1}{2}x\right) + \left(y + z, \frac{1}{2}y + \frac{1}{2}z\right) = \left(x + y + z, \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z\right)$
   and $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \left(x + y, \frac{1}{2}x + \frac{1}{2}y\right) + \left(z, \frac{1}{2}z\right) = \left(x + y + z, \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}z\right)$.

4. $\exists\, \mathbf{0} \in V$, s.t. $\forall \boldsymbol{u} \in V$, $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right)$ and $\mathbf{0} = (0,0) \in V$
   Then $\boldsymbol{u} + \mathbf{0} = \left(x, \frac{1}{2}x\right) + (0,0) = \left(x, \frac{1}{2}x\right) = \boldsymbol{u}$.

5. $\forall \boldsymbol{u} \in V$, $\exists - \boldsymbol{u} \in V$, s.t. $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right) \in V$
   Then $-\boldsymbol{u} = \left(-x, -\frac{1}{2}x\right) \in V$
   And $\boldsymbol{u} + (-\boldsymbol{u}) = \left(x, \frac{1}{2}x\right) + \left(-x, -\frac{1}{2}x\right) = (0,0) = \mathbf{0}$.

6. $\forall \boldsymbol{u} \in V, \forall c \in \mathbb{R}$, $c \cdot \boldsymbol{u} \in V$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right) \in V$ and $c \in \mathbb{R}$
   Then $c \cdot \boldsymbol{u} = c \cdot \left(x, \frac{1}{2}x\right) = \left(c \cdot x, c \cdot \frac{1}{2}x\right) \in V$.

7. $\forall \boldsymbol{u}, \boldsymbol{v} \in V$, $c \in \mathbb{R}$, $c \cdot (\boldsymbol{u} + \boldsymbol{v}) = c \cdot \boldsymbol{u} + c \cdot \boldsymbol{v}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right), \boldsymbol{v} = \left(y, \frac{1}{2}y\right) \in V$ and $c \in \mathbb{R}$
   Then $c \cdot (\boldsymbol{u} + \boldsymbol{v}) = c \cdot \left(x + y, \frac{1}{2}x + \frac{1}{2}y\right) = \left(c \cdot (x+y), c \cdot \frac{1}{2}x + c \cdot \frac{1}{2}y\right)$
   and $c \cdot \boldsymbol{u} + c \cdot \boldsymbol{v} = \left(c \cdot x, c \cdot \frac{1}{2}x\right) + \left(c \cdot y, c \cdot \frac{1}{2}y\right) = \left(c \cdot (x+y), c \cdot \frac{1}{2}x + c \cdot \frac{1}{2}y\right)$.

8. $\forall \boldsymbol{u} \in V, \forall c, d \in \mathbb{R}$, $(c + d) \cdot \boldsymbol{u} = c \cdot \boldsymbol{u} + d \cdot \boldsymbol{u}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right) \in V$ and $c, d \in \mathbb{R}$
   Then $(c + d) \cdot \boldsymbol{u} = (c + d) \cdot \left(x, \frac{1}{2}x\right) = \left((c + d) \cdot x, (c + d) \cdot \frac{1}{2}x\right)$
   and $c \cdot \boldsymbol{u} + d \cdot \boldsymbol{u} = \left(c \cdot x, c \cdot \frac{1}{2}x\right) + \left(d \cdot x, d \cdot \frac{1}{2}x\right) = \left((c + d) \cdot x, (c + d) \cdot \frac{1}{2}x\right)$.

9. $\forall \boldsymbol{u} \in V, \forall c, d \in \mathbb{R}$, $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
   Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right) \in V$ and $c, d \in \mathbb{R}$
   Then $c(d\boldsymbol{u}) = c(d \cdot \left(x, \frac{1}{2}x\right)) = c \cdot \left(d \cdot x, d \cdot \frac{1}{2}x\right) = \left(c \cdot d \cdot x, c \cdot d \cdot \frac{1}{2}x\right)$
   and $(cd)\boldsymbol{u} = (cd) \cdot \left(x, \frac{1}{2}x\right) = \left(cd \cdot x, cd \cdot \frac{1}{2}x\right)$.

10. $\forall \boldsymbol{u} \in V$, $1 \cdot \boldsymbol{u} = \boldsymbol{u}$
    Let $\boldsymbol{u} = \left(x, \frac{1}{2}x\right) \in V$
    Then $1 \cdot \boldsymbol{u} = 1 \cdot \left(x, \frac{1}{2}x\right) = \left(1 \cdot x, 1 \cdot \frac{1}{2}x\right) = \left(x, \frac{1}{2}x\right) = \boldsymbol{u}$.

Since all the ten vector space axioms hold, the set V together with the standard operations is a vector space.

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Question 11

(CH4.2 P166 Q26) Determine whether the set

$$V = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \middle| a, b, c \in F \right\}$$

together with the standard operations, is a vector space. If it is not, identify at least one of the ten vector space axioms that fails.

Solution: Let's exaimine the ten vector space axioms. For the ease of discussion, assume $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & 0 \end{bmatrix}$ and $C = \begin{bmatrix} m & n \\ p & 0 \end{bmatrix}$ are arbitrary elements in V. And c, d are arbitrary elements in $\mathbb{R}$.

1. $A + B \in V$
   $A + B = \begin{bmatrix} a+x & b+y \\ c+z & 0 \end{bmatrix} \in V.$

2. $A + B = B + A$

   $$A + B = \begin{bmatrix} a+x & b+y \\ c+z & 0 \end{bmatrix} = \begin{bmatrix} x+a & y+b \\ z+c & 0 \end{bmatrix} = B + A.$$

3. $A + (B + C) = (A + B) + C$
   $A + (B + C) = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} x+m & y+n \\ z+p & 0 \end{bmatrix} = \begin{bmatrix} a+x+m & b+y+n \\ c+z+p & 0 \end{bmatrix}$
   and $(A + B) + C = \begin{bmatrix} a+x & b+y \\ c+z & 0 \end{bmatrix} + \begin{bmatrix} m & n \\ p & 0 \end{bmatrix} = \begin{bmatrix} a+x+m & b+y+n \\ c+z+p & 0 \end{bmatrix}.$

4. $\exists \mathbf{0} \in V$, s.t. $\forall A \in V$, $A + \mathbf{0} = A$
   Let $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in V$
   Then $A + \mathbf{0} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = A.$

5. $\forall A \in V, \exists -A \in V$, s.t. $A + (-A) = \mathbf{0}$
   Let $-A = \begin{bmatrix} -a & -b \\ -c & 0 \end{bmatrix} \in V$
   Then $A + (-A) = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} + \begin{bmatrix} -a & -b \\ -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}.$

6. $c \cdot A \in V$
   $c \cdot A = \begin{bmatrix} c \cdot a & c \cdot b \\ c \cdot c & 0 \end{bmatrix} \in V.$

7. $c \cdot (A + B) = c \cdot A + c \cdot B$
   $c \cdot (A + B) = c \cdot \begin{bmatrix} a+x & b+y \\ c+z & 0 \end{bmatrix} = \begin{bmatrix} c \cdot (a+x) & c \cdot (b+y) \\ c \cdot (c+z) & 0 \end{bmatrix}$
   and $c \cdot A + c \cdot B = \begin{bmatrix} c \cdot a & c \cdot b \\ c \cdot c & 0 \end{bmatrix} + \begin{bmatrix} c \cdot x & c \cdot y \\ c \cdot z & 0 \end{bmatrix} = \begin{bmatrix} c \cdot (a+x) & c \cdot (b+y) \\ c \cdot (c+z) & 0 \end{bmatrix}.$

8. $(c + d) \cdot A = c \cdot A + d \cdot A$
   $(c + d) \cdot A = (c + d) \cdot \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} (c+d) \cdot a & (c+d) \cdot b \\ (c+d) \cdot c & 0 \end{bmatrix}$
   and $c \cdot A + d \cdot A = \begin{bmatrix} c \cdot a & c \cdot b \\ c \cdot c & 0 \end{bmatrix} + \begin{bmatrix} d \cdot a & d \cdot b \\ d \cdot c & 0 \end{bmatrix} = \begin{bmatrix} (c+d) \cdot a & (c+d) \cdot b \\ (c+d) \cdot c & 0 \end{bmatrix}.$

9. $c \cdot (d \cdot A) = (cd) \cdot A$

   $c \cdot (d \cdot A) = c \cdot \begin{bmatrix} d \cdot a & d \cdot b \\ d \cdot c & 0 \end{bmatrix} = \begin{bmatrix} c \cdot (d \cdot a) & c \cdot (d \cdot b) \\ c \cdot (d \cdot c) & 0 \end{bmatrix}$

   and $(cd) \cdot A = (cd) \cdot \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} (cd) \cdot a & (cd) \cdot b \\ (cd) \cdot c & 0 \end{bmatrix}$.

10. $1 \cdot A = A$

    $1 \cdot A = 1 \cdot \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot a & 1 \cdot b \\ 1 \cdot c & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = A$.

Since all the ten vector space axioms hold, the set V together with the standard operations is a vector space.

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Question 12

(CH4.2 P166 Q30) Determine whether the set

$$V = \left\{ \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} \middle| \, a, b, c \in F \right\}$$

together with the standard operations, is a vector space. If it is not, identify at least one of the ten vector space axioms that fails.

Solution: Let's exaimine the ten vector space axioms. For the ease of discussion, assume $A = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & x & y & z \\ x & 0 & y & z \\ x & y & 0 & z \\ x & y & z & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & m & n & p \\ m & 0 & n & p \\ m & n & 0 & p \\ m & n & p & 0 \end{bmatrix}$ are arbitrary elements in V. And c, d are arbitrary elements in $\mathbb{R}$.

1. $A + B \in V$

   $$A + B = \begin{bmatrix} 0 & a+x & b+y & c+z \\ a+x & 0 & b+y & c+z \\ a+x & b+y & 0 & c+z \\ a+x & b+y & c+z & 0 \end{bmatrix} \in V.$$

2. $A + B = B + A$

   $$A + B = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} + \begin{bmatrix} 0 & x & y & z \\ x & 0 & y & z \\ x & y & 0 & z \\ x & y & z & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a+x & b+y & c+z \\ a+x & 0 & b+y & c+z \\ a+x & b+y & 0 & c+z \\ a+x & b+y & c+z & 0 \end{bmatrix} = \begin{bmatrix} 0 & x+a & y+b & z+c \\ x+a & 0 & y+b & z+c \\ x+a & y+b & 0 & z+c \\ x+a & y+b & z+c & 0 \end{bmatrix} = B + A.$$

3. $A + (B + C) = (A + B) + C$

$$A + (B + C) = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} + \begin{bmatrix} 0 & x+m & y+n & z+p \\ x+m & 0 & y+n & z+p \\ x+m & y+n & 0 & z+p \\ x+m & y+n & z+p & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a+x+m & b+y+n & c+z+p \\ a+x+m & 0 & b+y+n & c+z+p \\ a+x+m & b+y+n & 0 & c+z+p \\ a+x+m & b+y+n & c+z+p & 0 \end{bmatrix}$$

and $(A + B) + C = \begin{bmatrix} 0 & a+x & b+y & c+z \\ a+x & 0 & b+y & c+z \\ a+x & b+y & 0 & c+z \\ a+x & b+y & c+z & 0 \end{bmatrix} + \begin{bmatrix} 0 & m & n & p \\ m & 0 & n & p \\ m & n & 0 & p \\ m & n & p & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & a+x+m & b+y+n & c+z+p \\ a+x+m & 0 & b+y+n & c+z+p \\ a+x+m & b+y+n & 0 & c+z+p \\ a+x+m & b+y+n & c+z+p & 0 \end{bmatrix}.$$

4. $\exists\, \mathbf{0} \in V$, s.t. $\forall A \in V$, $A + \mathbf{0} = A$

Let $\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in V$

Then $A + \mathbf{0} = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} = A.$

5. $\forall A \in V$, $\exists -A \in V$, s.t. $A + (-A) = \mathbf{0}$

Let $-A = \begin{bmatrix} 0 & -a & -b & -c \\ -a & 0 & -b & -c \\ -a & -b & 0 & -c \\ -a & -b & -c & 0 \end{bmatrix} \in V$

Then $A + (-A) = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b & -c \\ -a & 0 & -b & -c \\ -a & -b & 0 & -c \\ -a & -b & -c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$

6. $c \cdot A \in V$

$$c \cdot A = \begin{bmatrix} 0 & c \cdot a & c \cdot b & c \cdot c \\ c \cdot a & 0 & c \cdot b & c \cdot c \\ c \cdot a & c \cdot b & 0 & c \cdot c \\ c \cdot a & c \cdot b & c \cdot c & 0 \end{bmatrix} \in V.$$

7. $c \cdot (A + B) = c \cdot A + c \cdot B$

$$c \cdot (A + B) = c \cdot \begin{bmatrix} 0 & a+x & b+y & c+z \\ a+x & 0 & b+y & c+z \\ a+x & b+y & 0 & c+z \\ a+x & b+y & c+z & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \cdot (a+x) & c \cdot (b+y) & c \cdot (c+z) \\ c \cdot (a+x) & 0 & c \cdot (b+y) & c \cdot (c+z) \\ c \cdot (a+x) & c \cdot (b+y) & 0 & c \cdot (c+z) \\ c \cdot (a+x) & c \cdot (b+y) & c \cdot (c+z) & 0 \end{bmatrix}$$

and $$c \cdot A + c \cdot B = \begin{bmatrix} 0 & c \cdot a & c \cdot b & c \cdot c \\ c \cdot a & 0 & c \cdot b & c \cdot c \\ c \cdot a & c \cdot b & 0 & c \cdot c \\ c \cdot a & c \cdot b & c \cdot c & 0 \end{bmatrix} + \begin{bmatrix} 0 & c \cdot x & c \cdot y & c \cdot z \\ c \cdot x & 0 & c \cdot y & c \cdot z \\ c \cdot x & c \cdot y & 0 & c \cdot z \\ c \cdot x & c \cdot y & c \cdot z & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & c \cdot (a+x) & c \cdot (b+y) & c \cdot (c+z) \\ c \cdot (a+x) & 0 & c \cdot (b+y) & c \cdot (c+z) \\ c \cdot (a+x) & c \cdot (b+y) & 0 & c \cdot (c+z) \\ c \cdot (a+x) & c \cdot (b+y) & c \cdot (c+z) & 0 \end{bmatrix}.$$

8. $(c + d) \cdot A = c \cdot A + d \cdot A$

$$(c+d) \cdot A = (c+d) \cdot \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & (c+d) \cdot a & (c+d) \cdot b & (c+d) \cdot c \\ (c+d) \cdot a & 0 & (c+d) \cdot b & (c+d) \cdot c \\ (c+d) \cdot a & (c+d) \cdot b & 0 & (c+d) \cdot c \\ (c+d) \cdot a & (c+d) \cdot b & (c+d) \cdot c & 0 \end{bmatrix}$$

and $$c \cdot A + d \cdot A = \begin{bmatrix} 0 & c \cdot a & c \cdot b & c \cdot c \\ c \cdot a & 0 & c \cdot b & c \cdot c \\ c \cdot a & c \cdot b & 0 & c \cdot c \\ c \cdot a & c \cdot b & c \cdot c & 0 \end{bmatrix} + \begin{bmatrix} 0 & d \cdot a & d \cdot b & d \cdot c \\ d \cdot a & 0 & d \cdot b & d \cdot c \\ d \cdot a & d \cdot b & 0 & d \cdot c \\ d \cdot a & d \cdot b & d \cdot c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (c+d) \cdot a & (c+d) \cdot b & (c+d) \cdot c \\ (c+d) \cdot a & 0 & (c+d) \cdot b & (c+d) \cdot c \\ (c+d) \cdot a & (c+d) \cdot b & 0 & (c+d) \cdot c \\ (c+d) \cdot a & (c+d) \cdot b & (c+d) \cdot c & 0 \end{bmatrix}.$$

9. $c \cdot (d \cdot A) = (cd) \cdot A$

$$c \cdot (d \cdot A) = c \cdot \begin{bmatrix} 0 & d \cdot a & d \cdot b & d \cdot c \\ d \cdot a & 0 & d \cdot b & d \cdot c \\ d \cdot a & d \cdot b & 0 & d \cdot c \\ d \cdot a & d \cdot b & d \cdot c & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \cdot (d \cdot a) & c \cdot (d \cdot b) & c \cdot (d \cdot c) \\ c \cdot (d \cdot a) & 0 & c \cdot (d \cdot b) & c \cdot (d \cdot c) \\ c \cdot (d \cdot a) & c \cdot (d \cdot b) & 0 & c \cdot (d \cdot c) \\ c \cdot (d \cdot a) & c \cdot (d \cdot b) & c \cdot (d \cdot c) & 0 \end{bmatrix}$$

and $$(cd) \cdot A = (cd) \cdot \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & (cd) \cdot a & (cd) \cdot b & (cd) \cdot c \\ (cd) \cdot a & 0 & (cd) \cdot b & (cd) \cdot c \\ (cd) \cdot a & (cd) \cdot b & 0 & (cd) \cdot c \\ (cd) \cdot a & (cd) \cdot b & (cd) \cdot c & 0 \end{bmatrix}.$$

10. $1 \cdot A = A$

$$1 \cdot A = 1 \cdot \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1\cdot a & 1\cdot b & 1\cdot c \\ 1\cdot a & 0 & 1\cdot b & 1\cdot c \\ 1\cdot a & 1\cdot b & 0 & 1\cdot c \\ 1\cdot a & 1\cdot b & 1\cdot c & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b & c \\ a & 0 & b & c \\ a & b & 0 & c \\ a & b & c & 0 \end{bmatrix} = A.$$

Since all the ten vector space axioms hold, the set V together with the standard operations is a vector space.

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Question 13

(CH4.3 P173 Q2) Verify that W is a subspace of V, assume that V has the standard operations.

$$W = \left\{(x, y, 4x - 5y) \mid x, y \in \mathbb{R}\right\} \subseteq \mathbb{R}^3 = V.$$

Solution: Let $\boldsymbol{u} = (x_1, y_1, 4x_1 - 5y_1)$ and $\boldsymbol{v} = (x_2, y_2, 4x_2 - 5y_2)$ be arbitrary elements in W and $c, d \in \mathbb{R}$.

$$\begin{aligned} c \cdot \boldsymbol{u} + d \cdot \boldsymbol{v} &= c \cdot (x_1, y_1, 4x_1 - 5y_1) + d \cdot (x_2, y_2, 4x_2 - 5y_2) \\ &= (c \cdot x_1 + d \cdot x_2, c \cdot y_1 + d \cdot y_2, c \cdot (4x_1 - 5y_1) + d \cdot (4x_2 - 5y_2)) \in V \end{aligned}$$

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Question 14

(CH4.3 P173 Q4) Verify that W is a subspace of V, assume that V has the standard operations.

$$W = \left\{ \left(\begin{bmatrix} a & b \\ a - 2b & 0 \\ 0 & c \end{bmatrix} \right) \middle| \, a, b, c \in \mathbb{R} \right\} \subseteq M_{3 \cdot 2}(\mathbb{R}) = V.$$

Solution: Let $A = \begin{bmatrix} a_1 & b_1 \\ a_1 - 2b_1 & 0 \\ 0 & c_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ a_2 - 2b_2 & 0 \\ 0 & c_2 \end{bmatrix}$ be arbitrary elements in W and $c, d \in \mathbb{R}$.

$$\begin{aligned} c \cdot A + d \cdot B &= c \cdot \begin{bmatrix} a_1 & b_1 \\ a_1 - 2b_1 & 0 \\ 0 & c_1 \end{bmatrix} + d \cdot \begin{bmatrix} a_2 & b_2 \\ a_2 - 2b_2 & 0 \\ 0 & c_2 \end{bmatrix} \\ &= \begin{bmatrix} c \cdot a_1 + d \cdot a_2 & c \cdot b_1 + d \cdot b_2 \\ c \cdot (a_1 - 2b_1) + d \cdot (a_2 - 2b_2) & 0 \\ 0 & c \cdot c_1 + d \cdot c_2 \end{bmatrix} \in V \end{aligned}$$

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Question 15

(CH4.3 P173 Q32) Determine whether the subset W of $M_{n\times n}(F)$ is a subspace of $M_{n\times n}(F)$ with the standard operations. Justify your answer.

$$W \text{ is the set of all } A \in M_{n\times n}(F) \text{ s.t. } AB = BA, \text{ for a given } B \in M_{n\times n}(F).$$

Solution: $\forall A_1, A_2 \in W, \forall c, d \in F$

1. $$\begin{aligned} & A_1 \in W \text{ and } A_2 \in W \\ \Longrightarrow\ & A_1B = BA_1 \text{ and } A_2B = BA_2 \\ \Longrightarrow\ & (A_1 + A_2)B = A_1B + A_2B = BA_1 + BA_2 = B(A_1 + A_2) \\ \Longrightarrow\ & A_1 + A_2 \in W \end{aligned}$$

2. $$(c \cdot A_1)B = c \cdot (A_1B) = c \cdot (BA_1) = B(c \cdot A_1) \implies c \cdot A_1 \in W$$

Therefore, W is a subspace of $M_{n\times n}(F)$ with the standard operations.

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Question 16

(CH4.3 P173 Q38) Determine whether the set W is a subspace of $\mathbb{R}^3$ with the standard operations. Justify your answer.

$$W = \{(x_1, x_2, 4) \mid x_1, x_2 \in \mathbb{R}\}$$

Solution: W is not a subspace of $\mathbb{R}^3$ with the standard operations.
Let $\boldsymbol{u} = (1, 2, 4) \in W$ and $c = -1 \in \mathbb{R}$.
Then $c \cdot \boldsymbol{u} = -1 \cdot (1, 2, 4) = (-1, -2, -4) \notin W$.
Therefore, W is not a subspace of $\mathbb{R}^3$ with the standard operations.

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