Using $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1}$ we can write

$$e^{\boldsymbol{A}} = \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k = \sum_{k=0}^{\infty} \frac{1}{k!}(\boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1})^k = \boldsymbol{T}\left(\sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{\Lambda}^k\right)\boldsymbol{T}^{-1} = \boldsymbol{T}e^{\boldsymbol{\Lambda}}\boldsymbol{T}^{-1},$$

and hence

$$e^{\boldsymbol{A}} = \boldsymbol{T}\begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}\boldsymbol{T}^{-1}$$

2 Computing matrix exponential for general square matrices

2.1 Using Jordan normal form

Let be $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ then the matrix exponential can be computed starting from Jordan normal form (or Jordan canonical form):

Theorem 2 (Jordan normal form) *Any square matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is similar to a block diagonal matrix $\boldsymbol{J}$, i.e. $\boldsymbol{T}^{-1}\boldsymbol{A}\boldsymbol{T} = \boldsymbol{J}$ where*

$$\boldsymbol{J} = \begin{pmatrix} \boldsymbol{J}_1 & & & \\ & \boldsymbol{J}_2 & & \\ & & \ddots & \\ & & & \boldsymbol{J}_m \end{pmatrix} \qquad \textit{and} \qquad \boldsymbol{J}_k = \begin{pmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{pmatrix}$$

The column of $\boldsymbol{T} = [\boldsymbol{t}_{1,1}, \boldsymbol{t}_{1,2}, \ldots, \boldsymbol{t}_{m,n_m}, \boldsymbol{t}_{m,n_m-1}]$ are generalized eigenvectors, i.e.

$$\boldsymbol{A}\boldsymbol{t}_{k,j} = \begin{cases} \lambda_k \boldsymbol{t}_{k,j} & \textit{if } j = 1 \\ \lambda_k \boldsymbol{t}_{k,j} + \boldsymbol{t}_{k,j-1} & \textit{if } j > 1 \end{cases} \tag{3}$$

Using Jordan normal form $\boldsymbol{A} = \boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^{-1}$ we can write

$$
\begin{aligned}
e^{\boldsymbol{A}} &= \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{A}^k = \sum_{k=0}^{\infty} \frac{1}{k!}(\boldsymbol{T}\boldsymbol{\Lambda}\boldsymbol{T}^{-1})^k \\
&= \boldsymbol{T} \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{J}_1^k & & & \\ & \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{J}_2 & & \\ & & \ddots & \\ & & & \sum_{k=0}^{\infty} \frac{1}{k!}\boldsymbol{J}_m \end{pmatrix} \boldsymbol{T}^{-1} \\
&= \boldsymbol{T} \begin{pmatrix} e^{\boldsymbol{J}_1} & & & \\ & e^{\boldsymbol{J}_2} & & \\ & & \ddots & \\ & & & e^{\boldsymbol{J}_m} \end{pmatrix} \boldsymbol{T}^{-1}
\end{aligned}
$$

Thus, the problem is to find the matrix exponential of a Jordan block

$$
\begin{aligned}
\boldsymbol{J}_\lambda &= \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} \qquad (4) \\
&= \lambda \boldsymbol{I} + \boldsymbol{N}
\end{aligned}
$$

The matrix $\boldsymbol{N}$ has the property:

$$\boldsymbol{N}^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ & 0 & \ddots & 1 \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix}$$

and in general $\boldsymbol{N}^k$ as ones on the k-th upper diagonal and is the null matrix if $k \geq n$ the dimension of the matrix. Using (4) we have

$$\begin{aligned}
e^{\boldsymbol{J}_\lambda} &= \sum_{k=0}^{\infty} \frac{1}{k!} \boldsymbol{J}_\lambda^k = \sum_{k=0}^{\infty} \frac{1}{k!} (\lambda \boldsymbol{I} + \boldsymbol{N})^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} \lambda^{k-j} \boldsymbol{N}^j \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{1}{(k-j)!j!} \lambda^{k-j} \boldsymbol{N}^j \\
&= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(k-j)!j!} \lambda^{k-j} \boldsymbol{N}^j \mathbb{1}_{k-j} \qquad \left[\quad \mathbb{1}_i = \begin{cases} 1 & \text{if } i \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \right] \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \boldsymbol{N}^j \sum_{k=0}^{\infty} \frac{1}{(k-j)!} \lambda^{k-j} \mathbb{1}_{k-j} \\
&= \sum_{j=0}^{\infty} \frac{1}{j!} \boldsymbol{N}^j \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k = e^\lambda \sum_{j=0}^{n-1} \frac{1}{j!} \boldsymbol{N}^j
\end{aligned}$$

or explicit

$$\begin{aligned}
e^{\boldsymbol{J}_\lambda} &= e^\lambda \left(\boldsymbol{I} + \frac{1}{1!} \boldsymbol{N} + \frac{1}{2!} \boldsymbol{N}^2 + \cdots + \frac{1}{(n-1)!} \boldsymbol{N}^{n-1} \right), \\
&= e^\lambda \begin{pmatrix} 1 & 1/1! & & 1/(n-1)! \\ & 1 & \ddots & \\ & & \ddots & 1/1! \\ & & & 1 \end{pmatrix}
\end{aligned}$$