# Inverse *z* Transform

EPD 30.114 ADVANCED FEEDBACK & CONTROL

# The Inverse *z* Transform

- Just as the Laplace Transform and the inverse Laplace Transform play crucial roles in understanding and designing continuous-time control systems, to analyze and synthesize discrete-time control systems, the inverse *z* Transform is required
- The inverse *z* transform of $X(z)$ will yield the corresponding time sequence $x(k)$ ONLY
  - It does not infer information of x(t) because that information is not encoded since z transform takes on sampled information.
- What this means is that the inverse z transform of $X(z)$ yields a unique $x(k)$ but **not** a unique $x(t)$
  - The inverse z transform yields a time sequence that specifies values of $x(t)$ ONLY at discrete instants of time $t$=0, $T$, 2$T$, etc
  - It says nothing about the values of $x(t)$ at all other times
  - Hence many different time functions $x(t)$ can have the same $x(kT)$ and subsequently $X(z)$

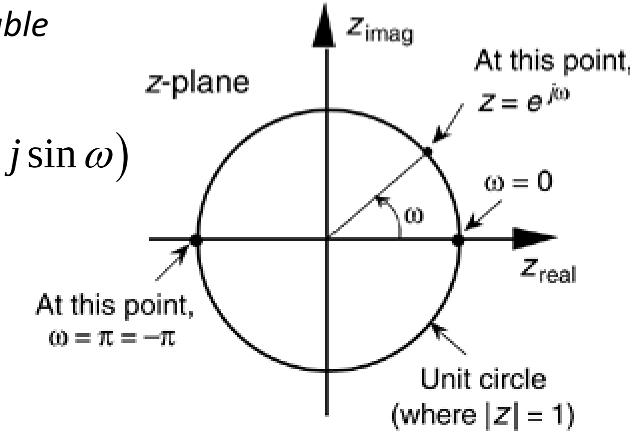
# Poles & Zeros of TF and in the z Plane

- In engineering applications of the z Transform, the discrete Transfer Function may be expressed as a ratio of polynomials in z,

$$G(z)=\frac{Y(z)}{X(z)}=\frac{b_0 z^m+b_1 z^{m-1}+\cdots+b_m}{z^n+a_1 z^{n-1}+\cdots+a_n} \qquad (m\le n)$$

*z: complex variable*

$$\begin{aligned} z &= Ae^{j\omega} \\ &= A(\cos\omega+j\sin\omega) \\ &= \sigma+j\varpi \end{aligned}$$




  - Or alternatively:

$$G(z)=\frac{Y(z)}{X(z)}=\frac{b_0(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Where $p_i$'s ($i=1,2,\ldots n$) are the **poles** of *G(z)* and the $z_i$'s ($j=1,2,\ldots,m$) are the **zeros** of *G(z)*

- However, in control engineering and signal processing, it is frequently also expressed as ratio of polynomials in $z^{-1}$

$$\boxed{G(z)=\frac{Y(z)}{X(z)}=\frac{b_0 z^{-(n-m)}+b_1 z^{-(n-m+1)}+\cdots+b_m z^{-n}}{1+a_1 z^{-1}+a_2 z^{-2}+\cdots+a_n z^{-n}}}$$

$z^{-1}$ is interpreted as the unit delay operator

# Discrete Transfer Function Expressions

- While the 2 methods of expressing a TF are equivalent, you will need to be careful with using it to determine poles and zeros
- Consider: $$G(z)=\frac{z^2+0.5z}{z^2+3z+2}=\frac{z(z+0.5)}{(z+1)(z+2)}$$
  - Here G(z) has poles at *z=-1* and *z=-2*.
  - G(z) has zeros at *z=0* and *z=-0.5.*
- Using the equivalent expression in $z^{-1}$: $$G(z)=\frac{1+0.5z^{-1}}{1+3z^{-1}+2z^{-2}}=\frac{1+0.5z^{-1}}{(1+z^{-1})(1+2z^{-1})}$$
  - The zero at z=0 is **not** explicitly shown!
- When determining the poles and zeros of a TF, express *G(z)* as ratio of polynomials in *z*
- When trying to determine the inverse z transform, as it will be shown, it will be desirable to express *G(z)* as a ratio of polynomials in $z^{-1}$

# Methods of Inverse z Transform

- There are 4 main methods for obtaining inverse z transforms:
  - Direct Division Method
    - No such approach exists for inverse Laplace Transform
  - Computational Method
    - A numerical based approach to obtain, sequentially the inverse z transform. No closed form solution is presented
  - Partial Fraction Expansion + Table Method
    - Similar to the approach used for Laplace Transform. Closed form solution is possible
  - Inversion Integral Method
    - Derived from the theory of complex variables
    - In many occurrences, it is similar to using Partial Fraction Expansion approach

# Direct Division Method

- This method works by expanding X(z) into an infinite power series in $z^{-1}$
- As this method does not produce a closed-form expression for the inverse, it is normally used only
  - to find the first few terms of *x(k)*
  - *x(k)* only has values for the first few k values
  - *x(k)* has a pattern that can be understood with the first few k values
- Recall that X(z) can be expanded into a power series in z-1

$$X(z)=\sum_{k=0}^{\infty} x(kT)z^{-k}=x(0)+x(T)z^{-1}+x(2T)z^{-2}+\cdots+x(kT)z^{-k}+\cdots$$

- Hence if X(z) is given in the form of a rational function, the expansion into an infinite power series in $z^{-1}$ can be accomplished by simply dividing the numerator by the denominator of the function
  - Both the numerator and denominator of *X(z)* needs to be written in increasing powers of $z^{-1}$

# Exercise

- Find x(k) for *k=0,1,2,3* and 4 when *X(z)* is given by $$X(z)=\frac{10z+5}{(z-1)(z-0.2)}$$

Rewrite X(z) as ratio of polynomials in $z^{-1}$

$$X(z)=\frac{10z^{-1}+5z^{-2}}{1-1.2z^{-1}+0.2z^{-2}}$$

Divide numerator by denominator

$$1-1.2z^{-1}+0.2z^{-2}\overline{\big)10z^{-1}+5z^{-2}}$$

$$\begin{array}{r l}
 & \quad 10z^{-1} \\
1-1.2z^{-1}+0.2z^{-2} & \overline{\big)10z^{-1}+5z^{-2}\qquad\qquad} \\
 & \quad \underline{10z^{-1}-12z^{-2}+2z^{-3}} \\
 & \qquad\qquad 17z^{-2}-2z^{-3}
\end{array}$$

$$\begin{array}{r l}
 & \quad 10z^{-1}+17z^{-2} \\
1-1.2z^{-1}+0.2z^{-2} & \overline{\big)10z^{-1}+5z^{-2}\qquad\qquad\qquad\qquad} \\
 & \quad \underline{10z^{-1}-12z^{-2}+2z^{-3}} \\
 & \qquad\qquad 17z^{-2}-2z^{-3} \\
 & \qquad\qquad \underline{17z^{-2}-20.4z^{-3}+3.4z^{-4}} \\
 & \qquad\qquad\qquad\qquad 18.4z^{-3}-3.4z^{-4}
\end{array}$$

$$X(z)=10z^{-1}+17z^{-2}+18.4z^{-3}+18.68z^{-4}+\cdots$$

Compare to: $$X(z)=\sum_{k=0}^{\infty}x(kT)z^{-k}=x(0)+x(T)z^{-1}+x(2T)z^{-2}+\cdots+x(kT)z^{-k}+\cdots$$

$$\Rightarrow x(0)=0,\quad x(T)=10,\quad x(2T)=17,\quad x(3T)=18.4,\quad\cdots$$

# Computational Method

- An alternative to the Direct Division Method (which is not numerically friendly to computers) is to make use of the unique property of the discrete impulse function (Kronecker delta input) to transform the transfer function to a difference equation for it to be solved sequentially
- A discrete transfer function can be expressed into a difference equation $G(z)=\dfrac{Y(z)}{X(z)}$
- If the input was assumed to be a Kronecker delta input, *X(z)=1*
- And, we get: $Y(z)=G(z)$
- Because *G(z)*, which contains powers of $z/z^{-1}$, can be expressed as a difference equation containing *y(k)* and *x(k)*, *y(k)* can be incrementally obtained by solving for the difference equation.
- This approach is best illustrated with an example:
  - Find the inverse z transform of:

$$G(z)=\frac{0.4673z^{-1}-0.3393z^{-2}}{1-1.5327z^{-1}+0.6607z^{-2}}$$

# Exercise

- Find the inverse *z* transform of the following system iteratively.

$$G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}}$$

Rewriting: $$G(z) = \frac{0.4673z - 0.3393}{z^2 - 1.5327z + 0.6607} = \frac{Y(z)}{X(z)}$$

$$\left(z^2 - 1.5327z + 0.6607\right)Y(z) = \left(0.4673z - 0.3393\right)X(z)$$

Taking IzT: $$y(k+2) - 1.5327y(k+1) + 0.6607y(k) = 0.4673x(k+1) - 0.3393x(k)$$

Difference equation!

Let's assume *X(z)* is the Kronecker delta input: $X(z) = 1,\ Y(z) = G(z)$

$$x(0) = 1,\ x(k) = 0 \quad for \quad k \neq 0 \qquad\qquad y(k) = g(k)$$

# Exercise

- Find the inverse *z* transform of the following system iteratively.

$$G(z) = \frac{0.4673z^{-1} - 0.3393z^{-2}}{1 - 1.5327z^{-1} + 0.6607z^{-2}}$$

$$\boxed{y(k+2) - 1.5327y(k+1) + 0.6607y(k) = 0.4673x(k+1) - 0.3393x(k)}$$

Determine initial data *y(0)* and *y(1):* $x(0) = 1,\ \ x(k) = 0 \ \ \mathit{for} \ \ k \neq 0$

$k = -2$ $\quad y(0) - 1.5327y(-1) + 0.6607y(-2) = 0.4673x(-1) - 0.3393x(-2)$

$$y(0) = 0$$

$k = -1$ $\quad y(1) - 1.5327y(0) + 0.6607y(-1) = 0.4673x(0) - 0.3393x(-1)$

$$y(1) = 0.4673x(0) = 0.4673$$

You now have the first 2 values of the inverse z transform of Y(z) (which is also now G(z)) You also have the difference equation as above. So to get other values, you can solve the difference equation iteratively.

# Partial-Fraction-Expansion Method

- Similar approach to the Partial-Fraction-Expansion Method for inverse Laplace Transform
  - Requires all terms in the partial fraction expansion be easily recognizable in the table of z transform pairs

- To find the inverse z Transform,
  - if X(z) has 1 or more zeros at the origin (z=0), then X(z)/z or X(z) is expanded into a sum of first or second order terms by partial fractions
    - Case A: Roots are Real & Distinct
    - Case B: Roots are Real & Repetitive
    - Case C: Roots are Complex Conjugates
    - Case D: Combination of A,B and C
  - A z transform table is used to find the corresponding time function of each expanded term

- Remember, to expand *X(z)* into partial fractions, you will need to factor the denominator polynomials and find poles of X(z)

$$X(z) = \frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

# Partial-Fraction-Expansion Method

- A commonly used procedure for the case where all the poles are simple order (real & distinct) and at least 1 zero at the origin ($b_m$=0) is to expand *X(z)/z* into partial fractions

$$\frac{X(z)}{z} = \frac{a_1}{(z-p_1)} + \frac{a_2}{(z-p_2)} + \cdots + \frac{a_n}{(z-p_n)}$$

  - Coefficients $a_i$ can be determined by computing the residues:

$$a_i = \left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}$$

- If *X(z)/z* involves multiple poles, for example a double (repeated) pole at $z=p_1$ and no other poles:

$$\frac{X(z)}{z} = \frac{c_1}{(z-p_1)^2} + \frac{c_2}{(z-p_1)}$$

  - where:

$$c_1 = \left[(z-p_1)^2\frac{X(z)}{z}\right]_{z=p_1} \qquad c_2 = \left\{\frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]\right\}_{z=p_1}$$

# Exercise

- Find the inverse $z$ transform of $X(z)$ using the partial fraction expansion. $a$ is a constant and $T$ is the sampling period.

$$X(z)=\frac{\left(1-e^{-aT}\right)z}{(z-1)\left(z-e^{-aT}\right)}$$

Zeros? $z=0$

Poles? $z=1,\quad z=e^{-aT}$

$$\frac{X(z)}{z}=\frac{\left(1-e^{-aT}\right)}{(z-1)\left(z-e^{-aT}\right)}=\frac{1}{(z-1)}-\frac{1}{\left(z-e^{-aT}\right)}$$

$1(k)$ $\qquad \dfrac{1}{1-z^{-1}}$

$$X(z)=\frac{z}{(z-1)}-\frac{z}{\left(z-e^{-aT}\right)}=\frac{1}{\left(1-z^{-1}\right)}-\frac{1}{\left(1-e^{-aT}z^{-1}\right)}$$

Using zT tables, $\quad x(kT)=1-e^{-akT} \qquad k=0,1,2,3,\cdots$

# Advanced Exercise

- Find the inverse $z$ transform of $X(z)$ using the partial fraction expansion.

$$X(z)=\frac{z^2+z+2}{(z-1)\left(z^2-z+1\right)}$$

Zeros? $z=\dfrac{-1\pm j\sqrt{7}}{2}$

Poles? $z=1,\quad z=\dfrac{1\pm j\sqrt{3}}{2}$

$$X(z)=\frac{z^2+z+2}{(z-1)\left(z^2-z+1\right)}=\frac{4}{z-1}+\frac{-3z+2}{z^2-z+1}=\frac{4z^{-1}}{1-z^{-1}}+\frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}$$

Quadratic poles are complex conjugates:
Looking at the zT Tables….

| | |
|---|---|
| $e^{-akT}\sin\omega kT$ | $\dfrac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$ |
| $e^{-akT}\cos\omega kT$ | $\dfrac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$ |

$e^{-2aT}=1,\quad \cos\omega T=0.5$

$\Rightarrow e^{-aT}=1$

$\Rightarrow \omega T=\pi/3$

$\Rightarrow \sin\omega T=\sqrt{3}/2$

Considering the sine term and matching the denominator: $\dfrac{\left(\sqrt{3}/2\right)z^{-1}}{1-z^{-1}+z^{-2}}$

Considering the cosine term and matching the denominator: $\dfrac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}$

# Advanced Exercise

| | |
|---|---|
| $e^{-akT}\sin\omega kT$ | $\dfrac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$ |
| $e^{-akT}\cos\omega kT$ | $\dfrac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$ |

- Find the inverse *z* transform of *X(z)* using the partial fraction expansion.

$$X(z)=\frac{z^2+z+2}{(z-1)\left(z^2-z+1\right)}$$

Zeros? $z=\dfrac{-1\pm j\sqrt{7}}{2}$

Poles? $z=1,\quad z=\dfrac{1\pm j\sqrt{3}}{2}$

$$X(z)=\frac{z^2+z+2}{(z-1)\left(z^2-z+1\right)}=\frac{4}{z-1}+\frac{-3z+2}{z^2-z+1}=\frac{4z^{-1}}{1-z^{-1}}+\frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}$$

From the cosine term, trying to match the required expression for the 2nd order term:

$$-3z^{-1}\left(\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right)=\frac{-3z^{-1}+1.5z^{-2}}{1-z^{-1}+z^{-2}}$$

Because the expression is $-3z^{-1}+2z^{-2}$:

$$\frac{-3z^{-1}+2z^{-2}}{1-z^{-1}+z^{-2}}=\frac{-3z^{-1}+1.5z^{-2}}{1-z^{-1}+z^{-2}}+\frac{0.5z^{-2}}{1-z^{-1}+z^{-2}}$$

$$=-3z^{-1}\left(\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right)+\frac{0.5z^{-1}}{(\sqrt{3}/2)}\left(\frac{(\sqrt{3}/2)z^{-1}}{1-z^{-1}+z^{-2}}\right)$$

$$=-3z^{-1}\left(\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right)+\frac{1}{\sqrt{3}}z^{-1}\left(\frac{(\sqrt{3}/2)z^{-1}}{1-z^{-1}+z^{-2}}\right)$$

# Advanced Exercise

| | |
|---|---|
| $e^{-akT}\sin\omega kT$ | $\dfrac{e^{-aT}z^{-1}\sin\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$ |
| $e^{-akT}\cos\omega kT$ | $\dfrac{1-e^{-aT}z^{-1}\cos\omega T}{1-2e^{-aT}z^{-1}\cos\omega T+e^{-2aT}z^{-2}}$ |

- Find the inverse *z* transform of *X(z)* using the partial fraction expansion.

$$X(z)=4z^{-1}\left(\frac{1}{1-z^{-1}}\right)-3z^{-1}\left(\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right)+\frac{1}{\sqrt{3}}z^{-1}\left(\frac{(\sqrt{3}/2)z^{-1}}{1-z^{-1}+z^{-2}}\right)\qquad X(z)=\frac{z^2+z+2}{(z-1)\left(z^2-z+1\right)}$$

$$\mathcal{Z}^{-1}\left[4z^{-1}\left(\frac{1}{1-z^{-1}}\right)\right]=4\left(1^{k-1}\right)\quad k=1,2,\ldots$$

$$\mathcal{Z}^{-1}\left[\left(\frac{1-0.5z^{-1}}{1-z^{-1}+z^{-2}}\right)\right]=1^k\cos\frac{k\pi}{3}\quad k=0,1,2,3,\cdots$$

$$\mathcal{Z}^{-1}\left[\left(\frac{(\sqrt{3}/2)z^{-1}}{1-z^{-1}+z^{-2}}\right)\right]=1^k\sin\frac{k\pi}{3}\quad k=0,1,2,3,\cdots$$

$$x(k)=4\left(1^{k-1}\right)-3\left(1^{k-1}\right)\cos\frac{(k-1)\pi}{3}+\frac{1}{\sqrt{3}}\left(1^{k-1}\right)\sin\frac{(k-1)\pi}{3}$$

$$x(k)=\begin{cases}4-3\cos\dfrac{(k-1)\pi}{3}+\dfrac{1}{\sqrt{3}}\sin\dfrac{(k-1)\pi}{3} & k=1,2,3,\cdots\\ 0 & k\le 0\end{cases}$$

# Inversion Integral Method

- This method uses some relationships from the theory of complex variables and in some cases allow inverse z to be obtained quickly
- The inversion integral for the z transform X(z) is given by:

$$\mathcal{Z}^{-1}\left[X(z)\right] = x(kT) = x(k) = \frac{1}{2\pi j}\oint X(z)z^{k-1}dz$$

- What the equation above means is that, for a system with *m* poles:

$$x(kT) = x(k) = K_1 + K_2 + \cdots + K_m$$
$$= \sum_{i=1}^{m}\left[\text{residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1}\right]$$

  - $K_1, K_2, \dots K_m$ are the residues of $X(z)z^{k-1}$ at poles $z_1, z_2, \dots, z_m$

- For simple poles: $$K = \lim_{z\to z_i}\left[(z - z_i)X(z)z^{k-1}\right]$$
- For multiple poles: $$K = \frac{1}{(q-1)!}\lim_{z\to z_j}\frac{d^{q-1}}{dz^{q-1}}\left[(z - z_j)^q X(z)z^{k-1}\right]$$

Multiple pole $z_j$ of order $q$

*k* are nonnegative integer values

# Exercise

- Use the inversion integral method to obtain *x(kT).*

$$X(z)=\frac{z\left(1-e^{-aT}\right)}{(z-1)\left(z-e^{-aT}\right)}$$

$$X(z)z^{k-1}=\frac{\left(1-e^{-aT}\right)z^{k}}{(z-1)\left(z-e^{-aT}\right)}$$

For $k$=0,1,2,… $X(z)z^{k-1}$ has 2 simple poles:

$$z=z_1=1$$

$$z=z_2=e^{-aT}$$

$$x(kT)=x(k)=\sum_{i=1}^{2}\left[\text{residue of }\frac{\left(1-e^{-aT}\right)z^{k}}{(z-1)\left(z-e^{-aT}\right)}\text{ at pole }z=z_i\right]=K_1+K_2$$

$$K_1=\lim_{z\to z_1}\left[(z-z_1)\frac{\left(1-e^{-aT}\right)z^{k}}{(z-1)\left(z-e^{-aT}\right)}\right]=\lim_{z\to 1}\left[(z-1)\frac{\left(1-e^{-aT}\right)z^{k}}{(z-1)\left(z-e^{-aT}\right)}\right]=1$$

$$K_2=\lim_{z\to z_2}\left[(z-z_2)\frac{\left(1-e^{-aT}\right)z^{k}}{(z-1)\left(z-e^{-aT}\right)}\right]=\lim_{z\to e^{-aT}}\left[\left(z-e^{-aT}\right)\frac{\left(1-e^{-aT}\right)z^{k}}{(z-1)\left(z-e^{-aT}\right)}\right]=-e^{-akT}$$

$$x(kT)=x(k)=K_1+K_2=1-e^{-akT}\qquad k=0,1,2,3,\cdots$$

# Advanced Exercise

- Use the inversion integral method to obtain *x(kT).*

$$X(z) = \frac{z^2}{(z-1)^2(z-e^{-aT})}$$

$$X(z)z^{k-1} = \frac{z^2 z^{k-1}}{(z-1)^2(z-e^{-aT})} = \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})}$$

For $k$=0,1,2,… $X(z)z^{k-1}$ has 1 simple pole and a double pole:

$$z = z_1 = e^{-aT}$$

$$z = z_2 = 1 \ \text{(double)}$$

$$x(kT) = x(k) = \sum_{i=1}^{2}\left[\text{residue of } \frac{z^{k+1}}{(z-1)^2(z-e^{-aT})} \text{ at pole } z = z_i\right] = K_1 + K_2$$

$$K_1 = \lim_{z\to z_1}\left[(z-z_1)\frac{z^{k+1}}{(z-1)^2(z-e^{-aT})}\right] = \lim_{z\to e^{-aT}}\left[(z-e^{-aT})\frac{z^{k+1}}{(z-1)^2(z-e^{-aT})}\right]$$

$$= \frac{e^{-aT(k+1)}}{(e^{-aT}-1)^2} = \frac{e^{-aT(k+1)}}{(1-e^{-aT})^2}$$

# Advanced Exercise

- Use the inversion integral method to obtain *x(kT)*.

$$X(z)=\frac{z^2}{(z-1)^2\left(z-e^{-aT}\right)}$$

$$K_2=\frac{1}{(2-1)!}\lim_{z\to z_2}\frac{d}{dz}\left[(z-z_2)^2\frac{z^{k+1}}{(z-1)^2\left(z-e^{-aT}\right)}\right]=\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^{k+1}}{(z-1)^2\left(z-e^{-aT}\right)}\right]$$

$$=\lim_{z\to 1}\frac{d}{dz}\left[\frac{z^{k+1}}{\left(z-e^{-aT}\right)}\right]=\lim_{z\to 1}\frac{(k+1)z^k\left(z-e^{-aT}\right)-z^{k+1}}{\left(z-e^{-aT}\right)^2}=\frac{(k+1)\left(1-e^{-aT}\right)-1}{\left(1-e^{-aT}\right)^2}$$

$$=\frac{k-ke^{-aT}+1-e^{-aT}-1}{\left(1-e^{-aT}\right)^2}=\frac{k\left(1-e^{-aT}\right)-e^{-aT}}{\left(1-e^{-aT}\right)^2}=\frac{k}{1-e^{-aT}}-\frac{e^{-aT}}{\left(1-e^{-aT}\right)^2}$$

$$x(kT)=x(k)=K_1+K_2=\frac{e^{-aT(k+1)}}{\left(1-e^{-aT}\right)^2}+\frac{k}{1-e^{-aT}}-\frac{e^{-aT}}{\left(1-e^{-aT}\right)^2}\qquad k=0,1,2,3,\cdots$$

$$=\frac{k}{\left(1-e^{-aT}\right)}-\frac{e^{-aT}\left(1-e^{-akT}\right)}{\left(1-e^{-aT}\right)^2}$$

# Be Wary of Poles at the Origin

- When deriving the expressions for $X(z)z^{k-1}$, it is important to check if the poles of that expression changes due to values of $k$.
- This is particularly important when $X(z)$ has poles or zeros at the origin and this will affect the expression of $X(z)z^{k-1}$ and increase the number of residues needed to construct the expression for $x(kT)$.
- For example: $X(z)=\dfrac{10}{(z-1)(z-2)}$
  - Then: $X(z)z^{k-1}=\dfrac{10z^{k-1}}{(z-1)(z-2)}$
  - Notice that for $k=0$ and for $k$=1,2,3,… the number of poles of the expression changes. Hence you will need to perform the inversion integral **TWICE**…

***k*=0**

$$X(z)z^{-1}=\frac{10z^{-1}}{(z-1)(z-2)}=\frac{10}{(z-1)(z-2)z}$$

3 poles: $z$=1,$z$=2 and $z$=0

***k*=1,2,3,…**

$$X(z)z^{k-1}=\frac{10z^{k-1}}{(z-1)(z-2)}$$

2 poles: $z$=1 and $z$=2

# Illustrative Example

- Use the inversion integral method to obtain the inverse z transform of:

$$X(z) = \frac{10}{(z-1)(z-2)}$$

**$k$=0**

$$X(z)z^{-1} = \frac{10z^{-1}}{(z-1)(z-2)} = \frac{10}{(z-1)(z-2)z}$$

**3 poles: $z_1 = 1$, $z_2 = 2$ and $z_3 = 0$**

$$x(0) = K_1 + K_2 + K_3$$

$$= \sum_{i=1}^{m} \left[ \text{residue of } X(z)z^{-1} \text{ at pole } z = z_i \text{ of } X(z)z^{-1} \right]$$

$$K_1 = \lim_{z \to z_1} \left[ (z - z_1) X(z) z^{-1} \right] = \lim_{z \to z_1} \left[ \frac{10}{(z-2)z} \right] = -10$$

$$K_2 = \lim_{z \to z_2} \left[ (z - z_2) X(z) z^{-1} \right] = \lim_{z \to z_2} \left[ \frac{10}{(z-1)z} \right] = 5$$

$$K_3 = \lim_{z \to z_3} \left[ (z - z_3) X(z) z^{-1} \right] = \lim_{z \to z_3} \left[ \frac{10}{(z-1)(z-2)} \right] = 5$$

$$x(0) = K_1 + K_2 + K_3 = -10 + 5 + 5 = 0$$

**$k$=1,2,3,…**

$$X(z)z^{k-1} = \frac{10z^{k-1}}{(z-1)(z-2)}$$

**2 poles: $z_1 = 1$ and $z_2 = 2$**

$$x(k) = K_{11} + K_{22}$$

$$= \sum_{i=1}^{m} \left[ \text{residue of } X(z)z^{k-1} \text{ at pole } z = z_i \text{ of } X(z)z^{k-1} \right]$$

$$K_{11} = \lim_{z \to z_1} \left[ (z - z_1) X(z) z^{k-1} \right] = \lim_{z \to z_1} \left[ \frac{10z^{k-1}}{(z-2)} \right] = -10$$

$$K_{22} = \lim_{z \to z_2} \left[ (z - z_2) X(z) z^{k-1} \right] = \lim_{z \to z_2} \left[ \frac{10z^{k-1}}{(z-1)} \right] = 10\left(2^{k-1}\right)$$

$$x(k) = K_{11} + K_{22} = -10 + 10\left(2^{k-1}\right)$$

$$\boxed{\Rightarrow x(k) = \begin{cases} 0 & k = 0 \\ -10 + 10\left(2^{k-1}\right) & k = 1, 2, 3, \ldots \end{cases}}$$

# Comprehensive Exercise

- Obtain the inverse z Transform of: $X(z)=\dfrac{z(z+2)}{(z-1)^2}$

Using all 4 methods!

**Direct Division Method**

$$X(z)=\frac{z(z+2)}{(z-1)^2}=\frac{1+2z^{-1}}{1-2z^{-1}+z^{-2}}$$

Rewrite X(z) as ratio of 2 polynomials in $z^{-1}$

$$1-2z^{-1}+z^{-2}\,\overline{)\,1+2z^{-1}}$$

$$\begin{array}{r}
1 \\
1-2z^{-1}+z^{-2}\,\overline{)\,1+2z^{-1}\phantom{+z^{-2}}} \\
\underline{1-2z^{-1}+z^{-2}} \\
4z^{-1}-z^{-2}
\end{array}$$

$$\begin{array}{r}
1+4z^{-1} \\
1-2z^{-1}+z^{-2}\,\overline{)\,1+2z^{-1}\phantom{-8z^{-2}+4z^{-3}}} \\
\underline{1-2z^{-1}+z^{-2}} \\
4z^{-1}-z^{-2} \\
\underline{4z^{-1}-8z^{-2}+4z^{-3}} \\
7z^{-2}-4z^{-3}
\end{array}$$

$$X(z)=1+4z^{-1}+7z^{-2}+10z^{-3}+\cdots$$

Compare to: $$X(z)=\sum_{k=0}^{\infty}x(k)z^{-k}=x(0)+x(1)z^{-1}+x(2)z^{-2}+\cdots+x(k)z^{-k}+\cdots$$

$$\Rightarrow x(0)=1,\quad x(1)=4,\quad x(2)=7,\quad x(3)=10,\quad \cdots$$

# Comprehensive Exercise

- Obtain the inverse z Transform of: $X(z) = \dfrac{z(z+2)}{(z-1)^2}$

Using all 4 methods!

**Computational Method**

$$G(z) = \frac{z(z+2)}{(z-1)^2} = \frac{X(z)}{Y(z)}$$

$$\left(z^2 - 2z + 1\right) X(z) = \left[z^2 + 2z\right] Y(z)$$

$$x(k+2) - 2x(k+1) + x(k) = y(k+2) + 2y(k+1)$$

Difference Equation!

Let's assume *Y(z)* is the Kronecker delta input: $Y(z) = 1, \;\; X(z) = G(z)$

$$y(0) = 1, \;\; y(k) = 0 \;\;\; for \;\; k \neq 0$$

$$x(k) = g(k)$$

# Comprehensive Exercise

- Obtain the inverse z Transform of: $X(z) = \dfrac{z(z+2)}{(z-1)^2}$

Using all 4 methods!

**Computational Method**

$$x(k+2) - 2x(k+1) + x(k) = y(k+2) + 2y(k+1)$$

Determine initial data *x(0)* and *x(1)*:

$$y(0) = 1, \;\; y(k) = 0 \quad for \quad k \neq 0$$

$k = -2$

$$x(0) - 2x(-1) + x(-2) = y(0) + 2y(-1)$$

$$x(0) = 1$$

$k = -1$

$$x(1) - 2x(0) + x(-1) = y(1) + 2y(0)$$

$$x(1) = 2 + 2 = 4$$

$k = 0$

$$x(2) - 2x(1) + x(0) = y(2) + 2y(1)$$

$$x(2) - 8 + 1 = 0$$

$$x(2) = 7$$

# Comprehensive Exercise

- Obtain the inverse z Transform of: $X(z)=\dfrac{z(z+2)}{(z-1)^2}$

Using all 4 methods!

**Partial-Fraction Expansion Method**

$$X(z)=\frac{z(z+2)}{(z-1)^2}=1+4\left(\frac{1}{z-1}\right)+3\left(\frac{1}{(z-1)^2}\right)=1+4\left(\frac{z^{-1}}{1-z^{-1}}\right)+3\left(\frac{z^{-2}}{(1-z^{-1})^2}\right)$$

Taking inverse zT:

$$\mathcal{Z}^{-1}\left[4\left(\frac{z^{-1}}{1-z^{-1}}\right)\right]=\begin{cases}0 & k=0\\ 4 & k=1,2,3,\ldots\end{cases}$$

$$\mathcal{Z}^{-1}[1]=\begin{cases}1 & k=0\\ 0 & k=1,2,3,\ldots\end{cases}$$

$$\mathcal{Z}^{-1}\left[3\left(\frac{z^{-2}}{(1-z^{-1})^2}\right)\right]=\begin{cases}0 & k=0\\ 3(k-1) & k=1,2,3,\ldots\end{cases}$$

$$x(0)=1$$

$$x(k)=4+3(k-1)=3k+1 \qquad k=1,2,3,\ldots$$

$$\boxed{\Rightarrow x(k)=3k+1 \qquad k=0,1,2,3,\ldots}$$

# Comprehensive Exercise

- Obtain the inverse z Transform of: $X(z) = \dfrac{z(z+2)}{(z-1)^2}$

Using all 4 methods!

**Inversion Integral Method**

$$X(z)z^{k-1} = \frac{z^k(z+2)}{(z-1)^2}$$

For $k$=0,1,2,… $X(z)z^{k-1}$ has a double poles at $z_1=1$:

$$x(k) = \left[\text{residue of } \frac{z^k(z+2)}{(z-1)^2} \text{ at pole } z = z_1\right] = K$$

$$K = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2\frac{z^k(z+2)}{(z-1)^2}\right] = \lim_{z\to 1}\frac{d}{dz}\left[z^k(z+2)\right]$$

$$= \lim_{z\to 1}(k+1)z^k + 2kz^{k-1}$$

$$= (k+1) + 2k = 3k+1$$

$$x(k) = 3k+1 \qquad k = 0,1,2,3,\ldots$$