The *z* Transform

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Role of the *z* Transform

- Recall that the **Laplace Transform** was a crucial mathematical tool in the analysis and synthesis of **continuous-time** control systems which are described using **linear time-invariant (LTI) differential equations**
- The ***z* Transform** plays a similar role in the analysis and synthesis of **discrete-time** control systems which are described using **linear time-invariant (LTI) difference equations**
- With the *z* transform method, the solutions to the LTI difference equations become algebraic in nature (now in *z* rather than *s*)
- For the rest of the class, we will only consider discrete-time systems (sampling only, no amplitude quantization) that have periodic sampling
- Representation of Discrete-Time signals:
  - Let *T* be the sampling period, a sequence of discrete value sampled from *x(t):*

$$x(kT) = x(0)\ ,\ x(T),\ x(2T),\ x(3T),\ \cdots$$

Explicit expression with *T*

$$x(k) = x(0)\ ,\ x(1),\ x(2),\ x(3),\ \cdots$$

Implicit expression with *T*

Definition of the *z* Transform

- Consider a time function *x(t)* for *t>0*, the sampled values of *x(t)*, if the sampling period was *T* is expressed by:

$$x(kT) = x(0)\ ,\quad x(T),\ \ x(2T), \cdots \qquad\qquad x(k) = x(0)\ ,\quad x(1),\ \ x(2), \cdots$$

- The z transform of *x(t)* is denoted by *X*(*z*) and described by:

$$\boxed{\mathcal{Z}\left[x(t)\right] = X(z) = \sum_{k=0}^{\infty} x(kT) z^{-k} = \sum_{k=0}^{\infty} x(k) z^{-k}}$$

Blast from the past!

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Where: z is a complex variable

$\mathcal{Z}$ is the *z* Transform Operator

X(z) is the *z* Transform of *x(t)*

- Because the z transform can be expressed as a series, the z^{-k} indicates the position in time at which the amplitude *x*(*kT*) occurs
- z Transform is also linear! $\mathcal{Z}\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 X_1(z) + a_2 X_2(z)$

Geometric Series

- Before we continue, we need to refresh a fairly important theorem in mathematics!
- To motivate, let's tell a joke!

An infinite number of mathematicians walk into a bar...

The first one orders a beer.

The second one orders half a beer.

The third one orders a fourth of a beer.

The bartender stops them, pours two beers and says, "You're all a bunch of idiots."

- The punchline? $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$
- It is derived from the Geometric Power Series where:

$$\boxed{1+x+x^2+x^3+\cdots=\frac{1}{1-x}}$$

Series converges if |x|<1

$$\boxed{1+2x+3x^2+\cdots=\frac{1}{(1-x)^2}}$$

  - Remember this series, we will use it again shortly!

zT of Elementary Functions

- The **Unit-Step Function: (Unit-Step Sequence)**

$$x(t)=\begin{cases}1(t) & t\geq 0\\ 0 & t<0\end{cases}$$

$$x(kT)=x(k)=1 \qquad k=0,1,2,3,\cdots$$

$$X(z)=\sum_{k=0}^{\infty}1z^{-k}=1+z^{-1}+z^{-2}+z^{-3}+\cdots$$

$$=\frac{1}{1-z^{-1}}$$

$$=\frac{z}{z-1}$$

Laplace Transform

$$X(s)=\frac{1}{s}$$

- The **Unit-Ramp Function:**

$$x(t)=\begin{cases}t & t\geq 0\\ 0 & t<0\end{cases}$$

$$x(kT)=kT \qquad k=0,1,2,3,\cdots$$

$$x(k)=k$$

$$X(z)=\sum_{k=0}^{\infty}kTz^{-k}=T\left(z^{-1}+2z^{-2}+3z^{-3}+\cdots\right)$$

$$=Tz^{-1}\left(1+2z^{-1}+3z^{-2}+\cdots\right)$$

$$=\frac{Tz^{-1}}{\left(1-z^{-1}\right)^{2}}$$

$$=\frac{Tz}{\left(z-1\right)^{2}}$$

$$X(s)=\frac{1}{s^{2}}$$

zT of Elementary Functions

- The **Exponential Function:**

$$x(t)=\begin{cases} e^{-at} & t\geq 0 \\ 0 & t<0 \end{cases}$$

$$x(kT)=e^{-akT} \qquad k=0,1,2,3,\cdots$$

$$X(z)=\sum_{k=0}^{\infty} e^{-akT}z^{-k}=1+e^{-aT}z^{-1}+e^{-2aT}z^{-2}+e^{-3aT}z^{-3}+\cdots$$

$$=\frac{1}{1-e^{-aT}z^{-1}}$$

$$=\frac{z}{z-e^{-aT}}$$

$$X(s)=\frac{1}{s+a}$$

- The **Sinusoidal Function:**

$$x(t)=\begin{cases} \sin\omega t & t\geq 0 \\ 0 & t<0 \end{cases}$$

$$\sin\omega t=\frac{1}{2j}\left(e^{j\omega t}-e^{-j\omega t}\right)$$

$$X(z)=\mathcal{Z}\left[\frac{1}{2j}\left(e^{j\omega t}-e^{-j\omega t}\right)\right]$$

$$=\frac{1}{2j}\left(\frac{1}{1-e^{j\omega T}z^{-1}}-\frac{1}{1-e^{-j\omega T}z^{-1}}\right)$$

$$=\frac{z^{-1}\sin\omega T}{1-2z^{-1}\cos\omega T+z^{-2}}=\frac{z\sin\omega T}{z^{2}-2z\cos\omega T+1}$$

$$X(s)=\frac{\omega}{s^{2}+\omega^{2}}$$

zT of Elementary Functions

- The **Polynomial Function**:

$$x(k) = \begin{cases} a^k & k = 0,1,2,3,\cdots \\ 0 & k < 0 \end{cases}$$

$$X(z) = \sum_{k=0}^{\infty} a^k z^{-k} = 1 + az^{-1} + a^2 z^{-2} + a^3 z^{-3} + \cdots$$

$$= \frac{1}{1 - az^{-1}}$$

$$= \frac{z}{z - a}$$

No Laplace Equivalent!

- The **Unit-Impulse Sequence**:

$$x(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \qquad X(z) = 1$$

Similar to Unit-Impulse Function

$$X(s) = 1$$

Properties of zT

- Similar to the Laplace Transform, there are several properties for zT
  - Unlike LT, there is no more differentiation and integration properties as they no longer exist in discrete signals
- Linearity:

$$\mathcal{Z}\left[a_1 x_1(t) + a_2 x_2(t)\right] = a_1 X_1(z) + a_2 X_2(z)$$

- Multiplicative (Scaling):

$$\mathcal{Z}\left[a^k x(k)\right] = X(a^{-1}z)$$

- Time Shift:

$$\mathcal{Z}\left[x(t-nT)\right] = z^{-n} X(z)$$

$$\mathcal{Z}\left[x(t+nT)\right] = z^{n}\left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k}\right]$$

zT Pairs

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	–	–	Kronecker delta $\delta_0(k)$ 1 $k=0$ 0 $k\neq 0$	1
2.	–	–	$\delta_0(n-k)$ 1 $n=k$ 0 $n\neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

zT Pairs

12.	$\frac{2}{(s+a)^3}$	t^2e^{-at}	$(kT)^2e^{-akT}$	$\frac{T^2e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.	–	–	a^k	$\frac{1}{1-az^{-1}}$
19.	–	–	a^{k-1} $k=1, 2, 3, \dots$	$\frac{z^{-1}}{1-az^{-1}}$
20.	–	–	ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.	–	–	k^2a^{k-1}	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.	–	–	k^3a^{k-1}	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$

Exercise:

- Find the z-Transform of: $x(t) = 1(t-T)$

  $x(t) = 1(t-4T)$

$$\mathcal{Z}\left[1(t-T)\right] = z^{-1}\mathcal{Z}\left[1(t)\right] = z^{-1}\frac{1}{1-z^{-1}} = \frac{z^{-1}}{1-z^{-1}}$$

$$\mathcal{Z}\left[1(t-4T)\right] = z^{-4}\mathcal{Z}\left[1(t)\right] = z^{-4}\frac{1}{1-z^{-1}} = \frac{z^{-4}}{1-z^{-1}}$$

Exercise

- Consider a function *y(k)* which is defined as a sum of functions *x(h)* such that

$$y(k)=\sum_{h=0}^{k} x(h)$$

Find the *z* transform of *y(k)*

$$y(k)=x(0)+x(1)+\cdots+x(k-1)+x(k)$$
$$y(k-1)=x(0)+x(1)+\cdots+x(k-1)$$

Subtracting:

$$y(k)-y(k-1)=x(k)$$

Taking zT:

$$\mathcal{Z}\left[y(k)-y(k-1)\right]=\mathcal{Z}\left[x(k)\right]$$
$$Y(z)-z^{-1}Y(z)=X(z)$$
$$Y(z)=\frac{1}{1-z^{-1}}X(z)$$

Initial & Final Value Theorem (IVT, FVT)

- **IVT:** If *x(t)* has the *z* transform *X(z)* and if the limit exists, $\lim_{z\to\infty} X(z)$

Then the initial value *x(0)* of *x(t)* or *x(k)* is given by:

$$\boxed{x(0) = \lim_{z\to\infty} X(z)}$$

- Why? Easy!

$$X(z) = \sum_{k=0}^{\infty} x(k) z^{-k} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

- **FVT:** Suppose that *x(k)*, where *x(k)=0* for *k<0*, has the *z* transform *X(z)* and that all the poles of *X(z)* lie **inside the unit circle**, with exception of the pole at *z*=1. The final value of x(k) as k approaches infinity is:

$$\boxed{\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[\left(1 - z^{-1}\right) X(z)\right]}$$

- Why?

$$\mathcal{Z}\left[x(k)\right] = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$$

$$\mathcal{Z}\left[x(k-1)\right] = z^{-1} X(z) = \sum_{k=0}^{\infty} x(k-1) z^{-k}$$

$$\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k} = X(z) - z^{-1} X(z)$$

$$\lim_{z\to 1}\left[\sum_{k=0}^{\infty} x(k) z^{-k} - \sum_{k=0}^{\infty} x(k-1) z^{-k}\right] = \lim_{z\to 1}\left[\left(1 - z^{-1}\right) X(z)\right]$$

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[\left(1 - z^{-1}\right) X(z)\right]$$

Exercise

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[\left(1-z^{-1}\right)X(z)\right]$$

- Determine the final value of *x(t)* (or *x(k)*) of $X(z) = \dfrac{1}{1-z^{-1}} - \dfrac{1}{1-e^{-aT}z^{-1}}$ $\quad a > 0$

$$x(\infty) = \lim_{z\to 1}\left[\left(1-z^{-1}\right)\left[\frac{1}{1-z^{-1}} - \frac{1}{1-e^{-aT}z^{-1}}\right]\right]$$

$$= \lim_{z\to 1}\left[1 - \frac{1-z^{-1}}{1-e^{-aT}z^{-1}}\right] = 1$$

For the given *X(z)*, it was actually the z Transform of:

$$x(t) = 1 - e^{-at}$$

zT Properties

	$x(t)$ or $x(k)$	$\mathcal{Z}\{x(t)\}$ or $\mathcal{Z}\{x(k)\}$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t)+bx_2(t)$	$aX_1(z)+bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z)-zx(0)$
4.	$x(t+2T)$	$z^2X(z)-z^2x(0)-zx(T)$
5.	$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
6.	$x(t+kT)$	$z^kX(z)-z^kx(0)-z^{k-1}x(T)-\ldots-zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z)-z^kx(0)-z^{k-1}x(1)-\ldots-zx(k1-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$

zT Properties

13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^{k}x(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^{k}x(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1}\left[\left(1-z^{-1}\right)X(z)\right]$ if $\left(1-z^{-1}\right)X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k)=x(k)-x(k-1)$	$\left(1-z^{-1}\right)X(z)$
19.	$\Delta x(k)=x(k+1)-x(k)$	$(z-1)X(z)-zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^{m}x(k)$	$\left(-z\frac{d}{dz}\right)^{m}X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$