State Observers

EPD 30.114 ADVANCED FEEDBACK & CONTROL

State Observation & Estimation

- When designing feedback controllers (Pole-Placement), all state variables were assumed to be present
- In practice, not all state variables are available for feedback or they might be noisy or unreliable
  - Need to estimate unavailable or unknown state variables
- **OBSERVATION:**
  - Process of estimating state variables in the *deterministic* case (e.g. Luenberger Observers)
- **ESTIMATION:**
  - Process of estimate state variables in the *stochastic* case (e.g. Kalman Filters)
- If the state observer observes ALL state variables, it is a **Full-Order State Observer**
  - Many times this is unnecessary as observation is needed only for the unmeasurable states
- An observer that estimates fewer that the dimension of the state vector is a **Reduced-Order State Observer**
  - If the order of the reduced-order state observer is the minimum possible, it is called the **Minimum-Order State Observer**

State Observer (Luenberger Observer)

- The state observer estimates the state variables based on the measurements of the output and control variables
  - It is also called Luenberger Observer in honour of David Luenberger who formulated the concept in 1964
  - Recall the concept of Observability. A state observer can only be designed if and only if the observability condition is satisfied (will be shown)

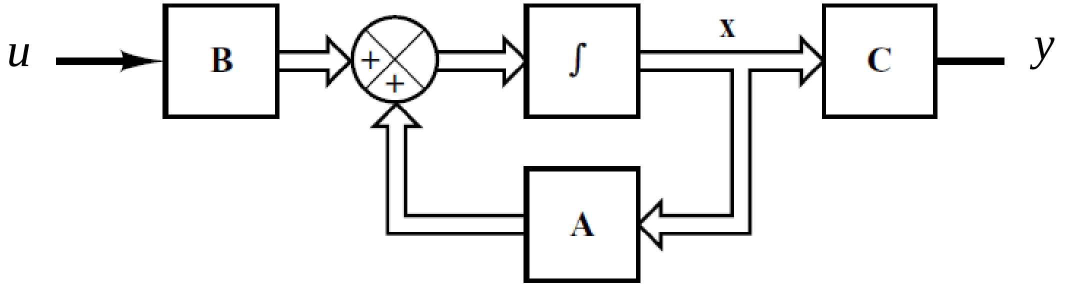
$$\mathbf{O}_B = \left[\mathbf{C}^* \quad \mathbf{A}^*\mathbf{C}^* \quad \cdots \quad (\mathbf{A}^*)^{n-1}\mathbf{C}^*\right]$$

Must be at least rank *n*

- Consider the standard plant: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

$$y = \mathbf{C}\mathbf{x}$$

  - All feedback controllers we are explored so far require the state vector **x** to be known
  - Observers attempt to reconstruct the state vector *x* through measurement of output *y* and control input *u* with help of a ‘duplicated’ mathematical model of the system
  - They are designed to be asymptotically stable so that the observation error is zero at steady state



Full-Order State Observer

- In addition to duplicating the dynamics of the system, we would like the observer to compensate for inaccuracies in **A** and **B** as well as lack of knowledge of the initial error e(0)
  - Let's define $\tilde{\mathbf{x}}$ as the state observation/estimation of the observer
  - An observer gain $\mathbf{K}_e$ is used as a weighting matrix to the observation error: $y-\tilde{y}=y-\mathbf{C}\tilde{\mathbf{x}}$
    - It continuously corrects the model output and improves performance of the observer

$$\dot{\tilde{\mathbf{x}}}=\mathbf{A}\tilde{\mathbf{x}}+\mathbf{B}u+\mathbf{K}_e\left(y-\mathbf{C}\tilde{\mathbf{x}}\right)$$
$$=\left(\mathbf{A}-\mathbf{K}_e\mathbf{C}\right)\tilde{\mathbf{x}}+\mathbf{B}u+\mathbf{K}_e y$$

Observer dynamics (inputs: *u* & *y*)

$$\dot{\mathbf{x}}=\mathbf{A}\mathbf{x}+\mathbf{B}u$$
$$y=\mathbf{C}\mathbf{x}$$

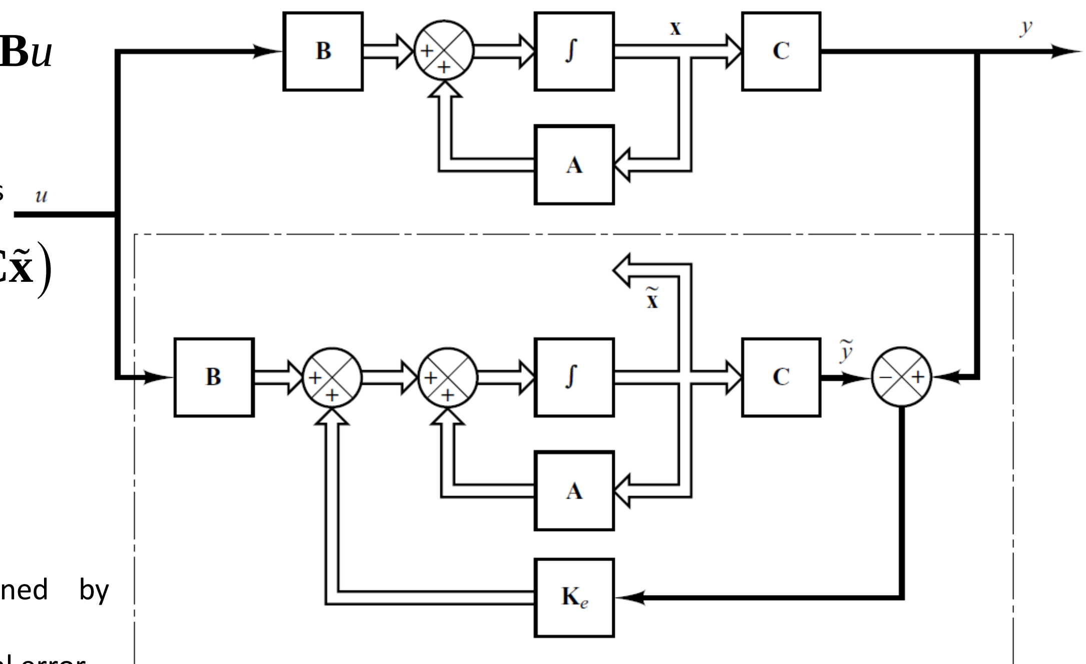
System dynamics

$$\dot{\mathbf{x}}-\dot{\tilde{\mathbf{x}}}=\mathbf{A}\mathbf{x}+\mathbf{B}u-\mathbf{A}\tilde{\mathbf{x}}-\mathbf{B}u-\mathbf{K}_e\left(\mathbf{C}\mathbf{x}-\mathbf{C}\tilde{\mathbf{x}}\right)$$
$$=\left(\mathbf{A}-\mathbf{K}_e\mathbf{C}\right)\left(\mathbf{x}-\tilde{\mathbf{x}}\right)$$

Defining: $\mathbf{e}=\mathbf{x}-\tilde{\mathbf{x}}$

$$\boxed{\dot{\mathbf{e}}=\left(\mathbf{A}-\mathbf{K}_e\mathbf{C}\right)\mathbf{e}}$$

- Dynamic performance of the error dynamics governed by eigenvalues of **A-K$_e$C**
- If **A-K$_e$C** is stable, the error will converge to zero for any initial error



Full-order state observer

Determining Observer Gain

- Now that we have the error dynamics for the full-state observer system, how to solve for $\mathbf{K_e}$? $\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)\mathbf{e}$
- From properties of matrix conjugate transpose:

$$(\mathbf{A}+\mathbf{B})^* = \mathbf{A}^* + \mathbf{B}^*$$
$$(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$$
$$\left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)^* = \left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right)$$

- In addition, eigenvalues are <u>not</u> affected by conjugate transpose
  - Eigenvalues of $\left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)$ are the same with $\left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)^* = \left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right)$
  - And more importantly the characteristic equations are the same:

$$\left|s\mathbf{I} - \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)\right| = \left|s\mathbf{I} - \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)^*\right|$$
$$= \left|s\mathbf{I} - \left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right)\right|$$

- Recall that we have been expertly solving using the Pole-Placement to solve equations of the form: $\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}$
- This means you can use the normal **Pole-Placement techniques** to determine the required $\mathbf{K_e}$ for a specified eigenvalues for the observer
  - You will have to make the following substitutions: $\dot{\mathbf{e}}^* = \mathbf{e}^*\left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right)$

$$\mathbf{A} = \mathbf{A}^* \quad , \quad \mathbf{B} = \mathbf{C}^* \quad , \quad \mathbf{K} = \mathbf{K_e}^*$$

Going One Step Even Further

- This is a startling result! But we can go deeper!
- Recall when solving $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$, there was mention on when we could find a **K** to place the eigenvalues anywhere we wanted?
  - It was called **State Controllability** and the controllability matrix $\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}$
- If this analysis was applied to the error dynamics of the observer:

$$\dot{\mathbf{z}} = (\mathbf{A}^* - \mathbf{C}^* \mathbf{K}_e^*)\mathbf{z}$$

  - A generic state space system under state feedback that produces this equation is:

$$\dot{\mathbf{z}} = \mathbf{A}^* \mathbf{z} + \mathbf{C}^* \mathrm{v}$$

$$w = \mathbf{B}^* \mathbf{z}$$

$$v = -\mathbf{K}_{\mathbf{e}}^* \mathbf{z}$$

  - In order to place eigenvalues at specific locations in **A*-C*K$_\mathbf{e}$***, this system must be completely state controllable
    - The condition is that the matrix should be rank *n* $\mathbf{C}_O = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix}$
    - This is incidentally the condition for **complete observability** of the <u>ORIGINAL System</u>
    - Hence, in order to design a full state observer for a system, it must be completely observable
    - Recall the concept of Duality. The above generic system is the dual system of the original system

Controllable & Observable Canonical Forms

- Hence, it is also the reason why CCF and OCF are so closely related

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 \,\vdots\, b_{n-1} - a_{n-1} b_0 \,\vdots\, \cdots \,\vdots\, b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

CCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

OCF

- Recall the expression of CCF allows state-feedback gains to be computed directly during Pole-Placement

- Same approach in OCF allows observer gains to be computed directly during Pole-Placement

Design By Pole-Placement for Observers

- To determine the Observer Gain Matrix, $\mathbf{K}_e$, the error dynamics is re-expressed into a form that is used extensively in Pole-Placement design for state feedback controllers

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \Rightarrow \dot{\mathbf{z}} = (\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*)\mathbf{z}$$

Eigenvalues of $(\mathbf{A} - \mathbf{K}_e\mathbf{C})$ are the same of $(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*)$

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$$

Pole-placement design 'standard' form

- When designing the observer, chose desired eigenvalues such that it responds at least 2 to 5 times faster than the closed-loop system

- There are 3 ways to approach Pole-Placement design for Observer Design:
  - M1: Using the Observable Canonical Form as a start
  - M2: Using direct substitution & comparing with desired c.e. (30.101 method)
    - Because of the direct approach, operating in the original expression is possible: $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e}$
    - Not recommended for high order systems since it is not easily implemented/scalable on a computer program
  - M3: Using Ackermann's Formula

Method 1: Direct Computation from OCF

- Consider the system is in OCF:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \cdot\\ \cdot\\ \cdot\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}0 & 0 & \cdots & 0 & -a_n\\ 1 & 0 & \cdots & 0 & -a_{n-1}\\ \cdot & \cdot & & \cdot & \cdot\\ \cdot & \cdot & & \cdot & \cdot\\ \cdot & \cdot & & \cdot & \cdot\\ 0 & 0 & \cdots & 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \cdot\\ \cdot\\ \cdot\\ x_n\end{bmatrix} + \begin{bmatrix}b_n - a_n b_0\\ b_{n-1} - a_{n-1} b_0\\ \cdot\\ \cdot\\ \cdot\\ b_1 - a_1 b_0\end{bmatrix} u$$

$$y = \begin{bmatrix}0 & 0 & \cdots & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \cdot\\ \cdot\\ \cdot\\ x_{n-1}\\ x_n\end{bmatrix} + b_0 u$$

- We can choose a set of desired eigenvalues (observer dynamics) at $-\mu_1, -\mu_2, \ldots, -\mu_n$.

- Desired c.e:

$$(s+\mu_1)(s+\mu_2)\cdots(s+\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

- Define $\mathbf{K_e}$: $\mathbf{K}_e = \begin{bmatrix}\beta_n\\ \beta_{n-1}\\ \vdots\\ \beta_1\end{bmatrix}$ $\quad \mathbf{K}_e{}^* = \begin{bmatrix}\beta_n & \beta_{n-1} & \cdots & \beta_1\end{bmatrix}$

- c.e. of system: $\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \Rightarrow \dot{\mathbf{e}}^* = \mathbf{e}^*(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e{}^*)$

$$\left|s\mathbf{I} - \mathbf{A}^* + \mathbf{C}^*\mathbf{K}_e{}^*\right| = \left|s\mathbf{I} - \begin{bmatrix}0 & 1 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -a_n & -a_{n-1} & \cdots & -a_1\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \vdots\\ 1\end{bmatrix}\begin{bmatrix}\beta_n & \beta_{n-1} & \cdots & \beta_1\end{bmatrix}\right| = \begin{vmatrix}s & -1 & \cdots & 0\\ 0 & s & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_n+\beta_n & a_{n-1}+\beta_{n-1} & & s+a_1+\beta_1\end{vmatrix}$$

$$\left|s\mathbf{I} - \mathbf{A}^* + \mathbf{C}^*\mathbf{K}_e{}^*\right| = s^n + (a_1+\beta_1)s^{n-1} + \cdots + (a_{n-1}+\beta_{n-1})s + (a_n+\beta_n) = 0$$

Equating coefficients: $(a_1+\beta_1) = \alpha_1\ ,\ \cdots,\ (a_n+\beta_n) = \alpha_n$ $\quad \boxed{\mathbf{K}_e{}^* = \begin{bmatrix}\alpha_n - a_n & \alpha_{n-1} - a_{n-1} & \cdots & \alpha_1 - a_1\end{bmatrix}}$

Method 2: Direct Substitution Method (30.101)

- If the system is low order ($n \leq 3$), direct substitution of the gain matrix **K** into the desired characteristic equation may be simpler
- For a n=3 system, the gain may be expressed as:

$$\mathbf{K}_e = \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \end{bmatrix}$$

- Substituting directly into desired c.e: (Desired eigenvalues are $-\mu_1$, $-\mu_2$, $-\mu_3$)

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} \right| = \left(s + \mu_1\right)\left(s + \mu_2\right)\left(s + \mu_3\right) = 0$$

- Both sides of the c.e. are polynomials in s and the unknown k1, k2 and k3 can be computed by comparing coefficients
- This method is great for n=2,3 but is very tedious for higher n
- If system is not completely controllable, the gain matrix cannot be determined (no solution exists).

Method 3: Using Ackermann's Formula

- Adapting Ackermann's approach for the observer dynamics:

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)\mathbf{e} \Rightarrow \dot{\mathbf{e}}^* = \mathbf{e}^*\left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right) \Rightarrow \dot{\mathbf{z}} = \left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right)\mathbf{z}$$

$$\mathbf{K}_e^* = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}\begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & \left(\mathbf{A}^*\right)^{n-1}\mathbf{C}^* \end{bmatrix}^{-1}\phi\left(\mathbf{A}^*\right)$$

$$\begin{aligned} \mathbf{K}_e &= \phi\left(\mathbf{A}^*\right)^* \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= \phi\left(\mathbf{A}\right) \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}$$

Exercise

- For the following system, design a full-order state observer where the desired eigenvalues of the observer matrix are at -10 and -10.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad \mathbf{A} = \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$y = \mathbf{C}\mathbf{x}$$

Let's check observability of the system before continuing!

$$\mathbf{O_B} = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 20 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Observability matrix is rank 2. Hence system is completely observable

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

System already in OCF!

$$y = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

$$a_2 = -20$$

$$a_1 = 0$$

Desired C.E:

$$(s+\mu_1)(s+\mu_2) = (s+10)(s+10) = s^2 + 20s + 100$$

$$= s^2 + \alpha_1 s + \alpha_2 = 0$$

$$\alpha_1 = 20$$

$$\alpha_2 = 100$$

$$\mathbf{K}_e = \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} \qquad \mathbf{K}_e{}^* = \begin{bmatrix} \beta_2 & \beta_1 \end{bmatrix}$$

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\mathbf{e} \Rightarrow \dot{\mathbf{e}}^* = \mathbf{e}^*(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e{}^*)$$

$$\left|s\mathbf{I} - \mathbf{A}^* + \mathbf{C}^*\mathbf{K}_e{}^*\right| = \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} \beta_2 & \beta_1 \end{bmatrix} \right| = \begin{vmatrix} s & -1 \\ a_2 + \beta_2 & s + a_1 + \beta_1 \end{vmatrix}$$

$$= s^2 + (a_1 + \beta_1)s + (a_2 + \beta_2)$$

$$\boxed{\begin{aligned} \mathbf{K}_e{}^* &= \begin{bmatrix} \alpha_2 - a_2 & \alpha_1 - a_1 \end{bmatrix} \\ &= \begin{bmatrix} 100 + 20 & 20 - 0 \end{bmatrix} = \begin{bmatrix} 120 & 20 \end{bmatrix} \end{aligned}}$$

Exercise (Use Direct Substitution)

$$\mathbf{K}_e = \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix}$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = (s+10)(s+10) = s^2 + 20s + 100 = 0$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = \left| s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \beta_2 \\ 0 & \beta_1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s & -20+\beta_2 \\ -1 & s+\beta_1 \end{bmatrix} \right|$$

$$= s^2 + \beta_1 s + (-20 + \beta_2)$$

$$\beta_1 = 20$$
$$\beta_2 = 120$$

$$\boxed{\mathbf{K}_e = \begin{bmatrix} 120 \\ 20 \end{bmatrix}}$$

Exercise (Use Ackermann's Formula)

$$\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)\mathbf{e} \Rightarrow \dot{\mathbf{e}}^* = \mathbf{e}^*\left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right) \Rightarrow \dot{\mathbf{z}} = \left(\mathbf{A}^* - \mathbf{C}^*\mathbf{K}_e^*\right)\mathbf{z}$$

$$\mathbf{K}_e = \phi(\mathbf{A})\begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}$$

Desired C.E: $(s+\mu_1)(s+\mu_2) = (s+10)(s+10) = s^2 + 20s + 100$ $\quad\quad \phi(s) = s^2 + 20s + 100$

$$= s^2 + \alpha_1 s + \alpha_2 = 0$$

$$\phi(\mathbf{A}) = \mathbf{A}^2 + 20\mathbf{A} + 100\mathbf{I} = \begin{bmatrix}0 & 20\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 20\\ 1 & 0\end{bmatrix} + 20\begin{bmatrix}0 & 20\\ 1 & 0\end{bmatrix} + 100\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = \begin{bmatrix}20 & 0\\ 0 & 20\end{bmatrix} + \begin{bmatrix}0 & 400\\ 20 & 0\end{bmatrix} + \begin{bmatrix}100 & 0\\ 0 & 100\end{bmatrix}$$

$$= \begin{bmatrix}120 & 400\\ 20 & 120\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\end{bmatrix} = \begin{bmatrix}0 \quad 1\\ \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0 & 20\\ 1 & 0\end{bmatrix}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \quad\quad \begin{bmatrix}\mathbf{C}\\ \mathbf{CA}\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}^{-1} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

$$\mathbf{K}_e = \begin{bmatrix}120 & 400\\ 20 & 120\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$= \begin{bmatrix}120 & 400\\ 20 & 120\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}120\\ 20\end{bmatrix}$$

Visualizing the State Observer

- Let's start with our system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, $y = \mathbf{C}\mathbf{x}$ $\quad \mathbf{A} = \begin{bmatrix} 0 & -5 \\ 1 & -4 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$
  - We'll like to see how the observer poles affect the state observer performance
    - Try to design two observers: One with poles at -10, -10 and other at -1, -1.
  - This system has stable eigenvalues (-2±j)

Let's check observability of the system before continuing!

$$\mathbf{O_B} = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -4 \end{bmatrix}$$

Observability matrix is rank 2. Hence system is completely observable

System already in OCF! $\quad a_2 = 5$, $a_1 = 4$

Desired C.E (poles @ -10,-10:

$$(s+\mu_1)(s+\mu_2) = (s+10)(s+10) = s^2 + 20s + 100$$

$$= s^2 + \alpha_1 s + \alpha_2 = 0$$

$\alpha_1 = 20$

$\alpha_2 = 100$

$$\mathbf{K}_e^* = \begin{bmatrix} \alpha_2 - a_2 & \alpha_1 - a_1 \end{bmatrix}$$

$$= \begin{bmatrix} 100-5 & 20-4 \end{bmatrix} = \begin{bmatrix} 95 & 16 \end{bmatrix}$$

Desired C.E (poles @ -1,-1:

$$(s+\mu_1)(s+\mu_2) = (s+1)(s+1) = s^2 + 2s + 1$$

$$= s^2 + \alpha_1 s + \alpha_2 = 0$$

$\alpha_1 = 2$

$\alpha_2 = 1$

$$\mathbf{K}_e^* = \begin{bmatrix} \alpha_2 - a_2 & \alpha_1 - a_1 \end{bmatrix}$$

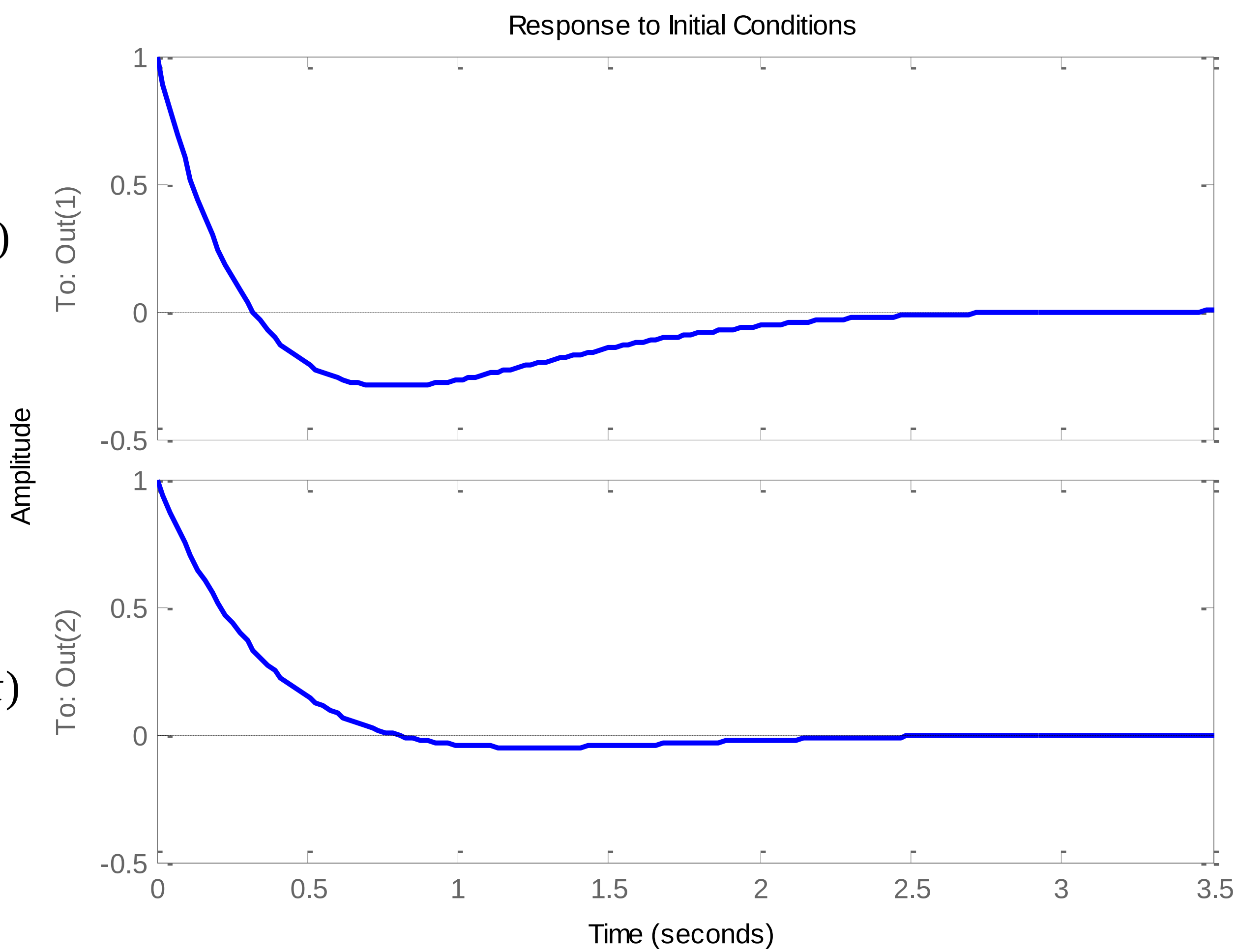
$$= \begin{bmatrix} 1-5 & 2-4 \end{bmatrix} = \begin{bmatrix} -4 & -2 \end{bmatrix}$$

Visualizing the State Observer

- First let's see how the system is responding due a specified initial condition: **x**(0)=[1 1]^T

$x_1(t)$

$x_2(t)$



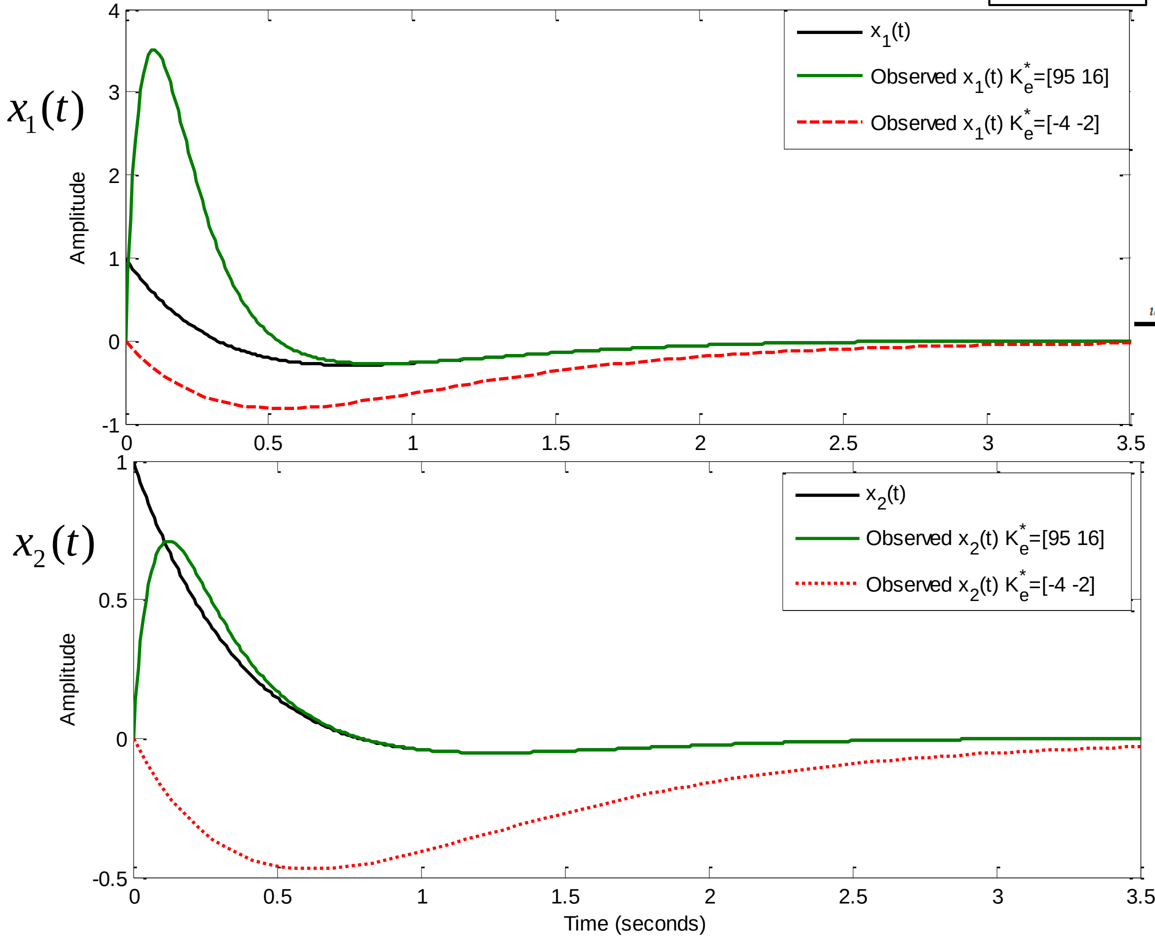
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

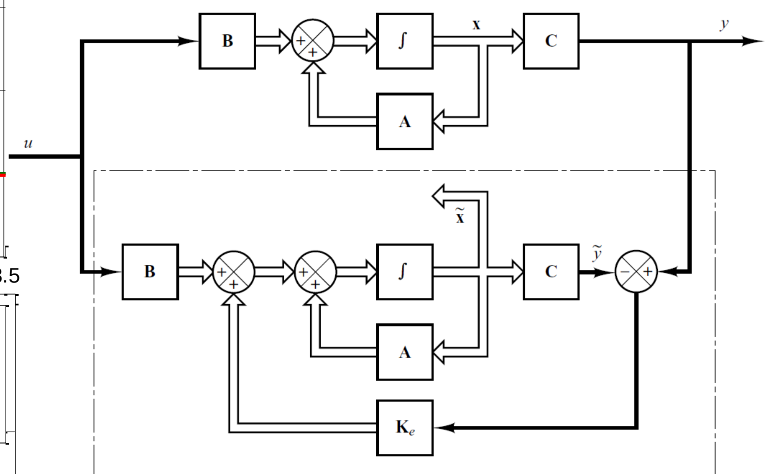
$$y = \mathbf{C}\mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} 0 & -5 \\ 1 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Visualizing the State Observer

- Now, let's see how the state observer is performing
  - Note that the state observer has no idea on the value of **x**(0). So as an initial guess, it will assume its initial observation value is zero. $\boxed{\tilde{\mathbf{x}}(0) = 0}$





Full-order state observer