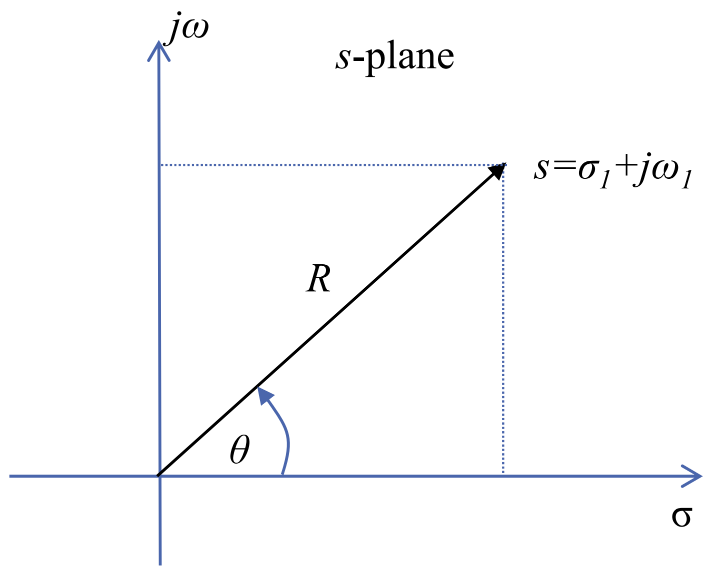
Recap of Controls Fundamentals

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Complex Variables

- Complex Variable: $\boxed{s = \sigma + j\omega}$
- Complex Function: $G(s) = G_x + jG_y$

$$\boxed{G(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{N(s)}{D(s)}}$$



- *ZEROS*: $\boxed{s=-z_1, s=-z_2, \cdots, s=-z_m}$ *m* zeros
- *POLES*: $\boxed{s=-p_1, s=-p_2, \cdots, s=-p_n}$ *n* poles

Differential Equations

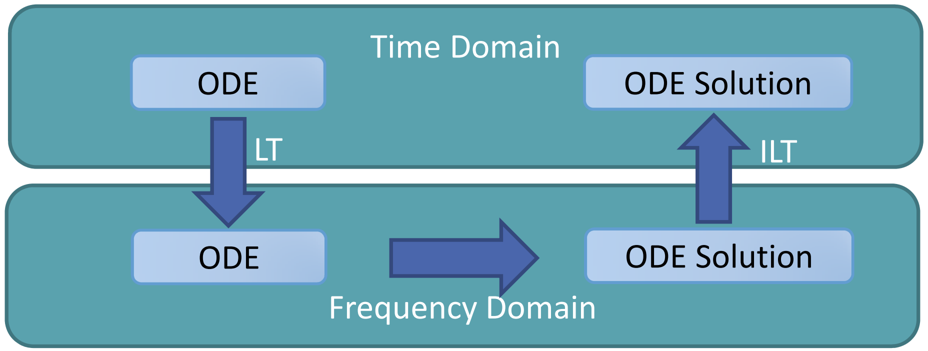
- Linear vs Non-Linear
  - Dependent variable and its derivatives appear as linear combination.

$$\frac{d^2x}{dt^2}+\cos 2t\frac{dx}{dt}+10x=0 \qquad \left(x^2-1\right)\frac{d^2x}{dt^2}+\frac{dx}{dt}+3x=0$$

- Time Varying vs Time Invariant
  - Coefficients are constants (independent of *t*)

$$\frac{d^2x}{dt^2}+3\frac{dx}{dt}+10x=0 \qquad \frac{d^2x}{dt^2}+\cos 2t\frac{dx}{dt}+10x=0$$

- **Linear, Time-Invariant** (LTI) Systems
  - Superposition holds
  - Output of the system is independent of the current time
- Solving LTI Systems:



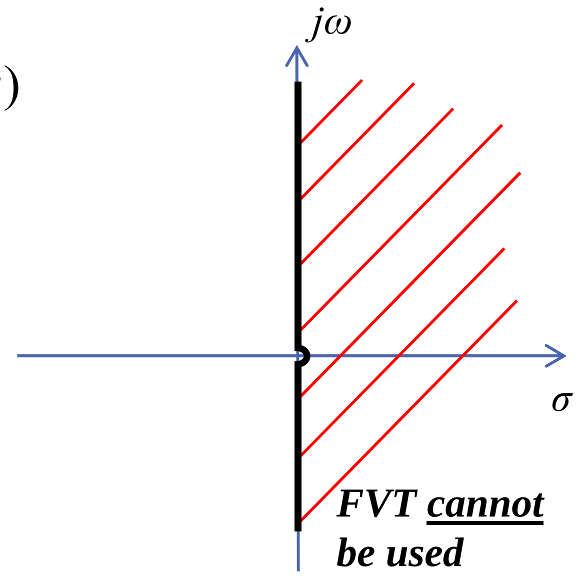
Laplace Transform

- For a time function such that $f(t) = 0$ for $t<0$

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

- Properties: Linearity, Frequency shift, Time shift, Scaling, Differentiation, Integration
- LT Table Pairs (No need to memorize)
- Initial Value Theorem (IVT) & Final Value Theorem (FVT)
  - Conditions where they can be applied

$$f\left(0^+\right) = \lim_{s\to\infty} sF(s) \qquad\qquad f\left(\infty\right) = \lim_{t\to\infty} f\left(t\right) = \lim_{s\to 0} sF(s)$$



Inverse Laplace Transform

- Process of finding time function $f(t)$ from $F(s)$ $$f(t) = \mathcal{L}^{-1}\{F(s)\}$$
- For **rational functions** of $F(s)$, the Inverse Laplace Transform (ILT) can be computed using **partial fractions decomposition**
  - Step 1: Express $F(s)$ as a **proper** rational function: $F(s)=N(s)/D(s)$
    - $N(s)$ and $D(s)$ are polynomials in s
  - Step 2: Check the roots of $D(s)$
    - Case A: Roots are Real & Distinct, Case B: Roots are Real & Repetitive, Case C: Roots are Complex Conjugates, Case D: Combination of A,B and C
  - Step 3: Use Laplace Transform Table Pairs to infer $f(t)$ from $F(s)$

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$

8	$\frac{1}{(n-1)!} t^{n-1}e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at}-e^{-bt})$	$\frac{1}{(s+a)(s+b)}$

System Modelling

- Describing system characteristics using a set of equations
- <u>Mathematical Modeling Procedure:</u>
  - Draw schematic diagram (e.g. FBD, circuit diagram,…) of the system and its components and define variables
  - Using physical laws (Newton's laws, Kirchoff's, etc) write equations for each component and combine them
  - Verify model with experiments
- Mechanical Systems (Newton's Law)
  - Inertia/Mass, Spring, Dampers
  - Translational & Rotational
  - Constitutive Equations
  - Energy Method

$$f_k = k\left(x_1 - x_2\right)$$

$$f_b = b\left(\dot{x}_1 - \dot{x}_2\right)$$

$$\sum f = m\ddot{x}$$

$$\tau_k = k\left(\theta_1 - \theta_2\right)$$

$$\tau_b = b\left(\dot{\theta}_1 - \dot{\theta}_2\right)$$

$$\sum \tau = J\ddot{\theta}$$

- Electrical Systems (Kirchhoff's/Ohm's Law)
  - Inductor, Capacitor, Resistor
  - Mechanical-Electrical Analogy

Transfer Function

- **Transfer Function** of a <u>LTI system</u> is defined as the ratio of the Laplace Transform of the output (response) to the Laplace Transform of the input (driving) under assumptions that <u>all initial conditions are zero</u>

$$a_0 \frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_{n-1} \frac{dy}{dt} + a_n y = b_0 \frac{d^m x}{dt^m} + b_1 \frac{d^{m-1} x}{dt^{m-1}} + \ldots + b_{m-1} \frac{dx}{dt} + b_m x$$

$$\text{Transfer function} = G(s) = \left.\frac{\mathcal{L}[\text{Output}]}{\mathcal{L}[\text{Input}]}\right|_{\text{Zero Initial Conditions}} = \left.\frac{\mathcal{L}[y(t)]}{\mathcal{L}[x(t)]}\right|_{\text{Zero Initial Conditions}}$$

$$= \frac{Y(\mathrm{s})}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \ldots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n}$$

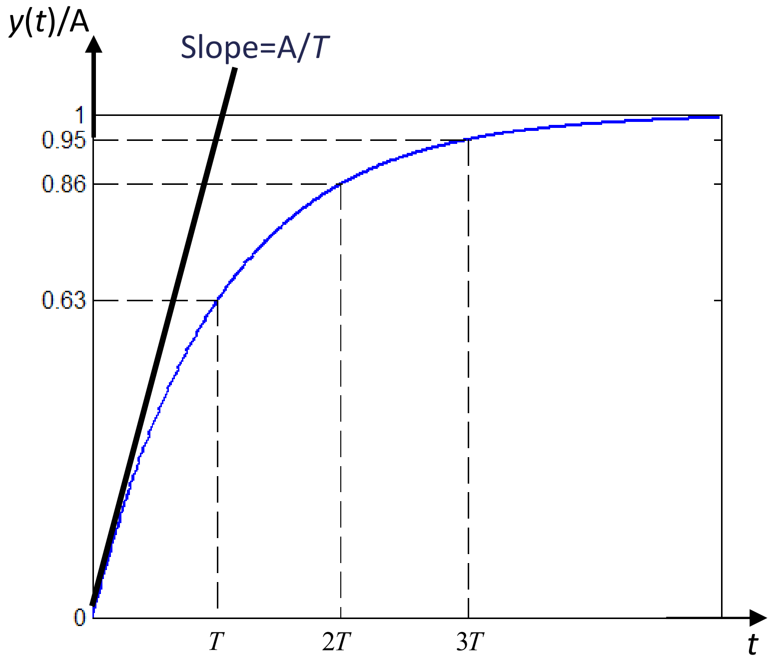
- System is represented by an algebraic expression in *s*
- Highest power of s in the denominator (C.E.) is the order of the system
- Transfer Function = Laplace Transform of the impulse-response function of a system

$$\mathcal{L}^{-1}[G(\mathrm{s})] = g(t)$$

LTI System Response (1st Order Systems)

- Transient & Steady-State Response
  - Refers to the process generated in going from an initial state to the final state
  - The way in which the system output behaves as *t* approaches infinity
- 1st Order Systems
  - Time constant is *T=RC*.

$$G(s) = \frac{1}{RCs+1} = \frac{1}{Ts+1}$$



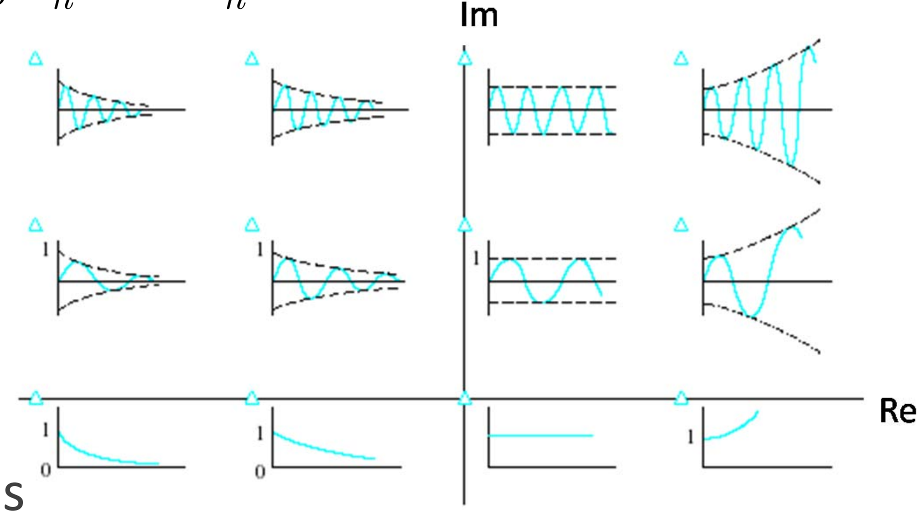
at t=T, system attain 63.2% of its final value

$$t_s = 4T$$

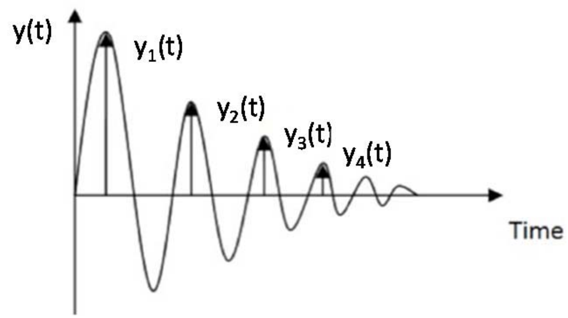
response within 2% of its final value

LTI System Response (2nd Order Systems)

- Standard Form: $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
- Characteristic Equation: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
  - **Damping Ratio**
    - No-damp if $\zeta = 0$
    - Under-damp if $0 < \zeta < 1$
    - Critically-damp if $\zeta = 1$
    - Over-damp if $\zeta > 1$
  - **Natural Frequency**



- Transient Response Specifications
  - Peak Time, Steady-State Error, Maximum Overshoot, Settling Time
- Logarithmic Decrement Method
- Dominant poles



Linearization

- Expanding the non-linear function into a Taylor series about an operating point (**equilibrium or steady-state**) and retaining only the linear term
  - Variables must deviate only slightly from the operating condition
- Consider $z = f(x)$
- If the operating point is $(\bar{x}, \bar{z})$,then *z* can be expressed using the Taylor series as:

$$z = f(x) = f(\bar{x}) + \left.\frac{df(x)}{dx}\right|_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2!} \left.\frac{d^2 f(x)}{dx^2}\right|_{x=\bar{x}} (x - \bar{x})^2 + \cdots$$

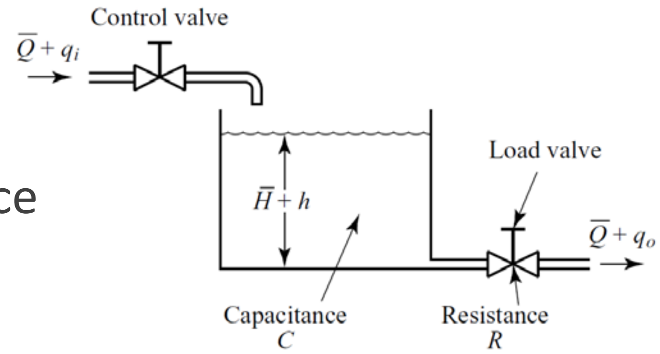
- If $x - \bar{x}$ is small, the higher order terms can be neglected and note that $\bar{z} = f(\bar{x})$

$$z - \bar{z} = \left.\frac{df(x)}{dx}\right|_{x=\bar{x}} (x - \bar{x})$$

$$\hat{z} = \left.\frac{df(x)}{dx}\right|_{x=\bar{x}} \hat{x}$$

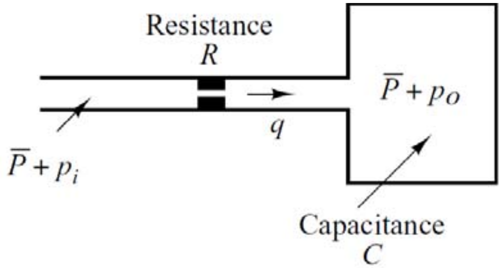
Modelling of Fluid & Thermal Systems

- **Fluids** (Gases, Pneumatic + Liquids, Hydraulic)
  - Assuming incompressible flow
  - Fluid Resistance, Fluid Capacitance and Fluid Inertance

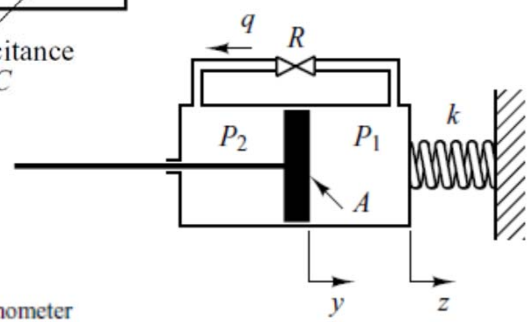


$$P_a - P_b = QR \qquad Q = C\frac{d}{dt}(P_a - P_b) \qquad P_a - P_b = I\frac{dQ}{dt}$$

- Liquid-Level Systems
  - Continuity of Flow, Newton's 2nd Law

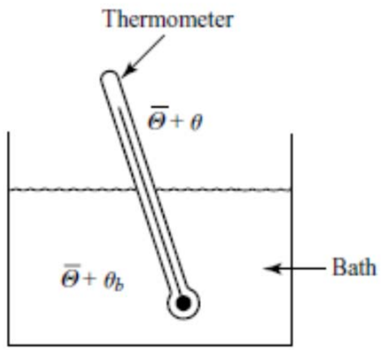


- Pneumatic Pressure-vessel Systems
  - Continuity of Mass Flow, Newton's 2nd Law



- Hydraulic Systems
  - Continuity of Mass Flow, Newton's 2nd Law

- **Thermal** (Conduction, Convection & Radiation)
  - Thermal Resistance, Thermal Capacitance

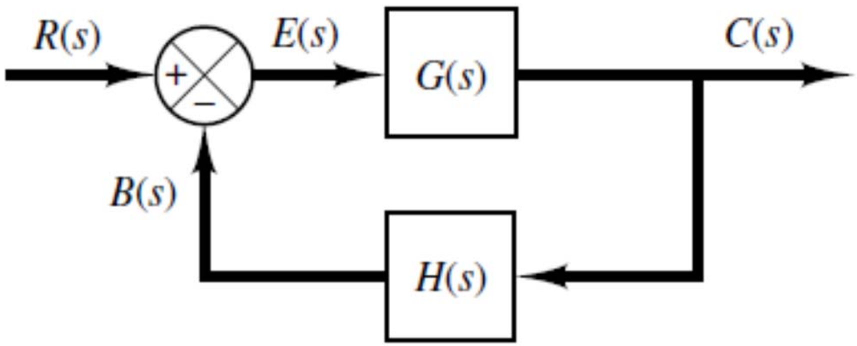


$$\theta_a - \theta_b = qR \qquad q = C\frac{d}{dt}(\theta_a - \theta_b)$$

Block Diagrams

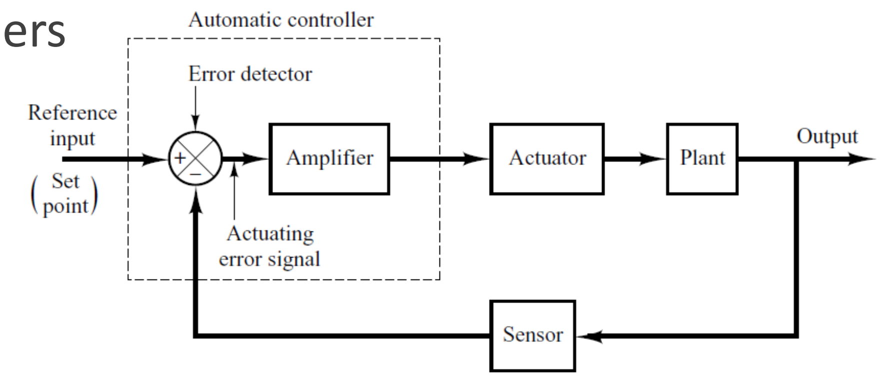
- A pictorial representation of the functions performed by each component of the system
  - Functional blocks is a symbol for mathematical operations on the input signal to the block that produces the output.
  - The **transfer functions** of the components are usually entered in the blocks
  - Signals can only pass in the direction as specified by the arrow
- Feedback Block Diagrams
  - Open-Loop Transfer Function $$\text{OLTF} = \frac{B(s)}{E(\mathrm{s})} = G(s)H(s)$$
  - Feedforward Transfer Function $$\text{FTF} = \frac{C(s)}{E(\mathrm{s})} = G(s)$$
  - Closed-Loop Transfer Function

$$\boxed{\text{CLTF} = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}}$$



Automatic Control & PID Controllers

- An automatic controller compares the actual value of the plant/system output with the desired value, determines the deviation and produces a control signal that reduce to deviation to zero or a small value.
  - Improve transient response, Enhance steady-state performance, Augment or introduce stability into the system
  - **P**roportional-**I**ntegral-**D**erivative **PID** controllers



- PID 3-term control
  - **Proportional**: Depends on **PRESENT** error
  - **Integral**: Accumulation of **PAST** error
  - **Derivative**: Prediction of **FUTURE** error

Time Domain

$$u(t) = K_p e(t) + K_i \int_{-\infty}^{t} e(t)dt + K_d \frac{de(t)}{dt}$$

$$= K_p \left[e(t) + \frac{1}{T_i} \int_{-\infty}^{t} e(t)dt + T_d \frac{de(t)}{dt} \right]$$

Laplace Domain

$$U(s) = K_p E(s) + \frac{K_i}{s} E(s) + K_d s E(s)$$

$$G_c(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s$$

K_p = proportional gain

$K_i = K_p / T_i$ = integral gain

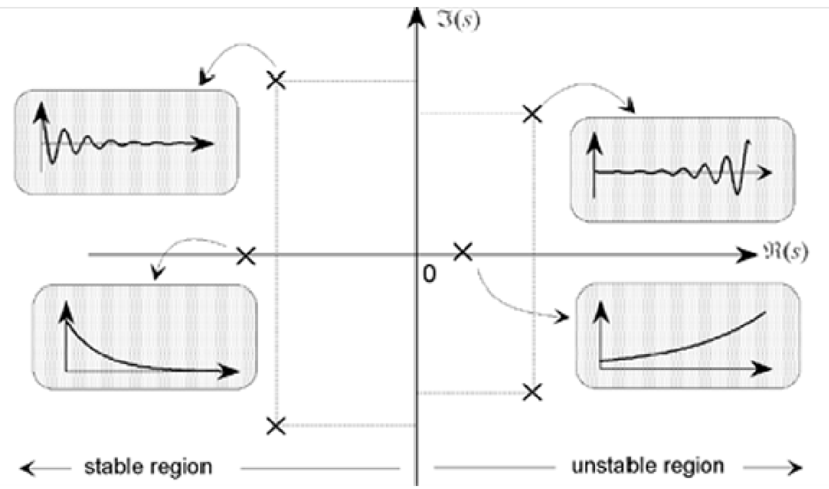
$K_d = K_p T_d$ = derivative gain

System Stability

- LTI system is **STABLE** if **ALL** roots of the transfer function denominator polynomial (aka **Characteristic Equation**) have **NEGATIVE real** parts (ALL POLES IN THE LEFT HAND S-PLANE). It is unstable otherwise.

- Roots of the **Characteristic Equation** D(s)=0 determine stability of the (closed-loop) system

- **Routh-Hurwitz Stability Criterion**
  - **A System is STABLE IF AND ONLY IF ALL ELEMENTS in the first column of the Routh Array are positive (Necessary and Sufficient Condition for Stability)**
  - All coefficients of the first column of the Routh's matrix must be positive (no sign changes)
  - Special cases:
    - Coefficient is zero (replace is small positive number)
    - All coefficients in a row is zero (use aux polynomial)

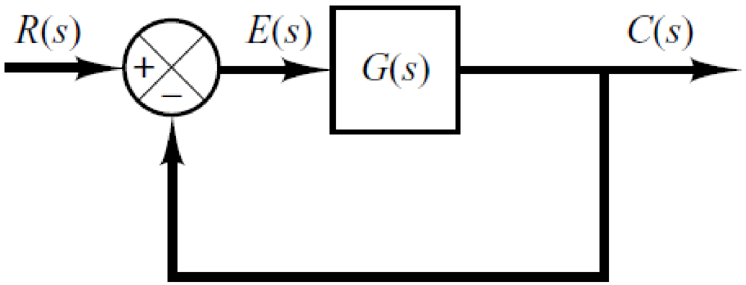


System Types

- Control systems may be classified according to their ability to follow:
  - Step inputs, Ramp inputs, Parabolic inputs

$$G(s) = K\frac{(\tau_1 s+1)(\tau_2 s+1)\cdots(\tau_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_n s+1)}$$

- Transfer function involves the term s^N in the denominator, representing pole multiplicity N at the origin



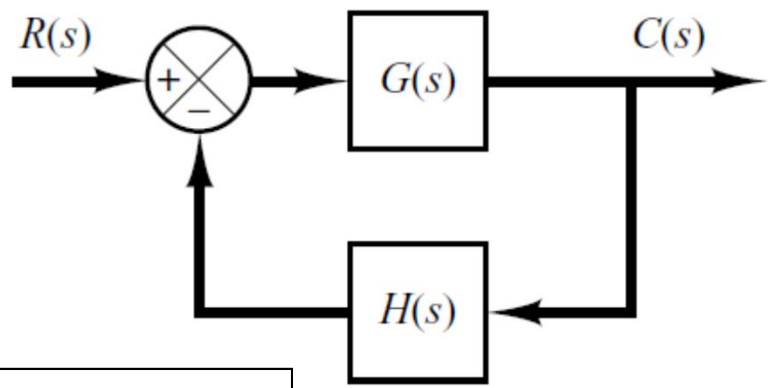
- A system is classified from this pole multiplicity
  - Type 0 System: N=0, Type 1 System: N=1, Type 2 System: N=2
- Remember: **SYSTEM ORDER ≠ SYSTEM TYPE**
- Static Position/Velocity/Acceleration Error Constant (K_P,K_V,K_A)

	Steady-State Error		
	Step Input r(t)=1	Ramp Input r(t)=t	Parabolic Input r(t)=0.5t^2
Type 0 System	$\frac{1}{1+K_P}$	∞	∞
Type 1 System	0	$\frac{1}{K_V}$	∞
Type 2 System	0	0	$\frac{1}{K_A}$

Root Locus Analysis

- Root Locus is a graphical method to examine how the roots of a system (closed-loop poles) change with variation of system parameters, such as the gain within a feedback system
  - Designer can predict and evaluate the effects on the location of the closed-loop poles due to:
    - Variation of the gain value, Adding open-loop poles (integrators) and zeros (derivatives)

- The goal is to visualize the **locus** (A set of points satisfying a condition) of roots of the **characteristic equation** of the **closed-loop system** as a **gain** is varied from **zero** to **infinity**.
  - CLTF: $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)H(s)}$
  - Characteristic Equation: $1+G(s)H(s)=0$ $\quad \boxed{G(s)H(s)=-1}$



- ***G(s)H(s)*** Is a complex function:
  - <u>Angle Condition</u>: $\boxed{\angle G(s)H(s) = \pm 180^\circ (2k+1) \qquad k = 0,1,2,\cdots}$
  - <u>Magnitude Condition</u>: $\boxed{|G(s)H(s)| = 1}$

Root Locus Analysis

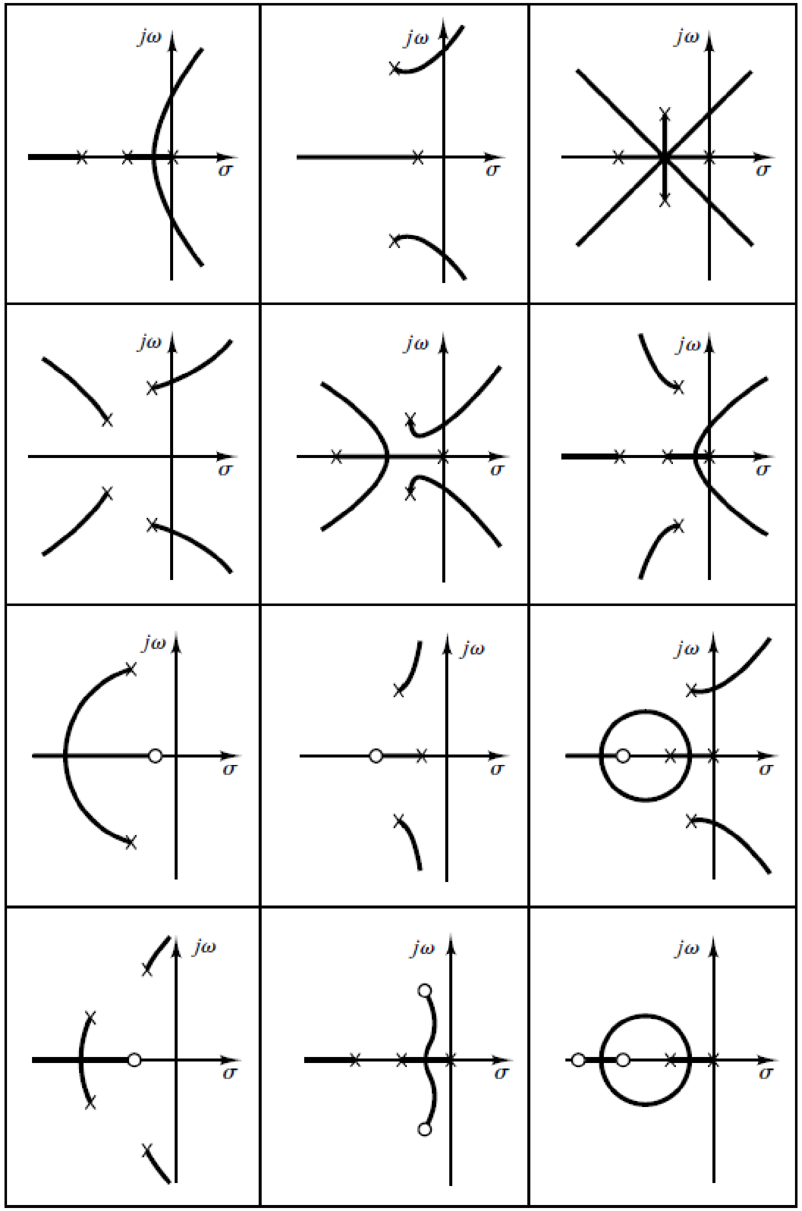
$$G(s)H(s) = -1$$

$$K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = -1 = K\frac{N(s)}{D(s)}$$

- The root locus for the system are the loci of the closed-loop poles as the gain *K* is varied from 0 to Infinity.
- **General Rules for construction Root Loci:**
  1. Rearrange the C.E. such that the gain *K* appears as a multiplicative factor
  2. Locate the open-loop poles (D(s)=0) and zeros (N(s)=0) of onto the s-plane
  3. Determine # of loci and root loci on the real axis
  4. Determine the asymptotes of the root locus
  5. Find break-away and break-in points
  6. Determine angle of departure (or arrival) of the root locus from a complex pole (or complex zero)
  7. Find the points where the root loci crosses the imaginary axis (if any)
  8. The magnitude condition will be used to determine the location of the closed-loop poles for a specific value of *K*.

Root Locus Analysis

- Pattern of root loci depends on the relative separation of **open-loop poles and zeros**
- If the number of open-loop poles exceeds the number of open loop zeros by 3 or more, there is a value of *K* beyond which the root loci enters the RHP (Unstable)
- A stable system will have **ALL** its **closed-loop poles** on the **LHP**.



Bode Diagrams

- Transfer function (TF) of a LTI system under sinusoidal excitation (input) can be represented by a **Bode Diagram** (2 graphs)
  - **Magnitude** (*Logarithmic*) of TF versus applied **Frequency** (*Logarithmic*)
  - **Phase angle** (*linear*) of TF versus applied **Frequency** (*Logarithmic*)

$$K\frac{N(j\omega)}{D(j\omega)} = G(j\omega) = |G(j\omega)|\angle G(j\omega)$$

$$dB = 20\log_{10}|G(j\omega)| = 20\log_{10}|N(j\omega)| - 20\log_{10}|D(j\omega)|$$

$$\phi = \angle G(j\omega) = \angle N(j\omega) - \angle D(j\omega)$$

- Bode Diagrams of:
  - Constant, Integral Factor, Derivative Factor, First Order, Second Order

- Graphing Procedure:
  - Given $G(j\omega)$, decompose the function as a product of elementary factors (K, first-order, second-order). All expressions in 'Bode Form'.
  - Identify corner-frequencies for each elemental factors and construct asymptotes in the log-magnitude curves. The composite magnitude-frequency plot is a composite of individual curves.
  - The phase-frequency is also constructed by adding all the individual phase-frequency plots

Interpreting Bode Diagrams

- Transfer functions having neither poles or zeros in RHP are minimum-phase TFs. Systems with minimum-phase TFs are minimum-phase systems.
  - For a **minimum-phase TF**, the transfer function can be **uniquely** determined from the **Bode Magnitude-frequency plot**.

- Determine System Types and Static Error Constants from Bode Diagrams
  - For a given system, **only one** of the static error constants is **finite** and significant. This also directly determines the **system type**.
  - Both information can be determined by observing the **low-frequency** region of the **Bode diagram (magnitude-frequency plot)**

- Stability Margins
  - Bode Diagrams can be used to infer stability of a system in unity feedback due to variation in the system gain *K*
    - *G(s)* must be stable (no RHP poles)
    - *G(s)* is a minimum phase Transfer Function
  - **Gain Margin, K_g**
    - The amount of Gain that can be raised before instability results
  - **Phase Margin, γ**
    - The amount of additional phase lag required to bring the system into instability
  - For stability, **Gain Margin** and **Phase Margin** of the ***Bode Diagram*** of the **OPEN-LOOP SYSTEM**, *G(s)*, must be **POSITIVE**