# Observed State Feedback Control

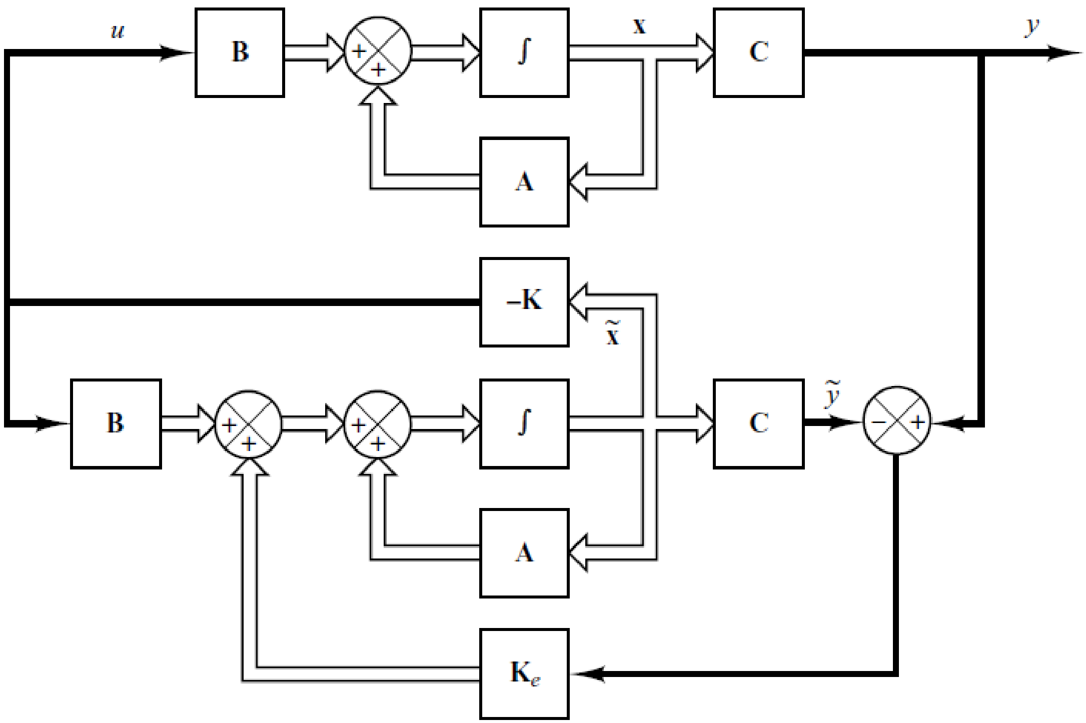
EPD 30.114 ADVANCED FEEDBACK & CONTROL

# State Feedback Control with Observer

- In the Pole-Placement design process, the actual state *x(t)* was assumed to be available
- Since in actual practice, *x(t)* may not be measurable so an observer was implemented and use the observed state for feedback
- Design is now a 2 stage process
  - Determine Feedback Gain **K** to yield desired closed-loop performance
  - Determine Observer Gain $\mathbf{K_e}$ to yield desired observer performance
- For this to be possible, the system needs to be **completely state controllable** and **completely observable**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$



# State Feedback Control with Observer

- The following system is both completely state controllable and completely observable:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

- The state feedback control is based on observed state of *x(t)*

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

- With this control, the state equation is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\tilde{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x} + \mathbf{B}\mathbf{K}\left(\mathbf{x} - \tilde{\mathbf{x}}\right)$$

- Defining the error vector: $\mathbf{e}(t) = \mathbf{x}(t) - \tilde{\mathbf{x}}(t)$
- The state equation is: $\boxed{\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e}}$
- Recall that the observer error dynamics is: $\boxed{\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)\mathbf{e}}$
- Augmenting:

$$\boxed{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{K}_\mathbf{e}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}}$$

# Separation Principle

- The augmented state equation describes both the dynamics of the observed state feedback control system AND the observer system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK} \\ \mathbf{0} & \mathbf{A} - \mathbf{K_e C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

- The c.e. is:

$$\left| \begin{bmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{K_e C} \end{bmatrix} \right| = 0$$

- Because the matrix is upper triangular,

$$\left| \begin{bmatrix} s\mathbf{I} - \mathbf{A} + \mathbf{BK} & -\mathbf{BK} \\ \mathbf{0} & s\mathbf{I} - \mathbf{A} + \mathbf{K_e C} \end{bmatrix} \right| = \left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| \left| s\mathbf{I} - \mathbf{A} + \mathbf{K_e C} \right| = 0$$

$$\left| s\mathbf{I} - \mathbf{A}^* + \mathbf{C}^* \mathbf{K_e}^* \right|$$

  - The closed-loop poles of the system consist of the poles of the state feedback system as well as the observer
  - State feedback design and observer design are **independent** of each other
  - Can be designed **separately** and combined
  - Total system order is doubled (from *n* to 2*n*)

# Transfer Function of Observer-Based Controller

- For a state-space system that is completely state controllable and observable, the equations of the observer is: $\dot{\tilde{\mathbf{x}}} = (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$
  - Under the control law: $u = -\mathbf{K}\tilde{\mathbf{x}}$
- Taking LT and zero IC:

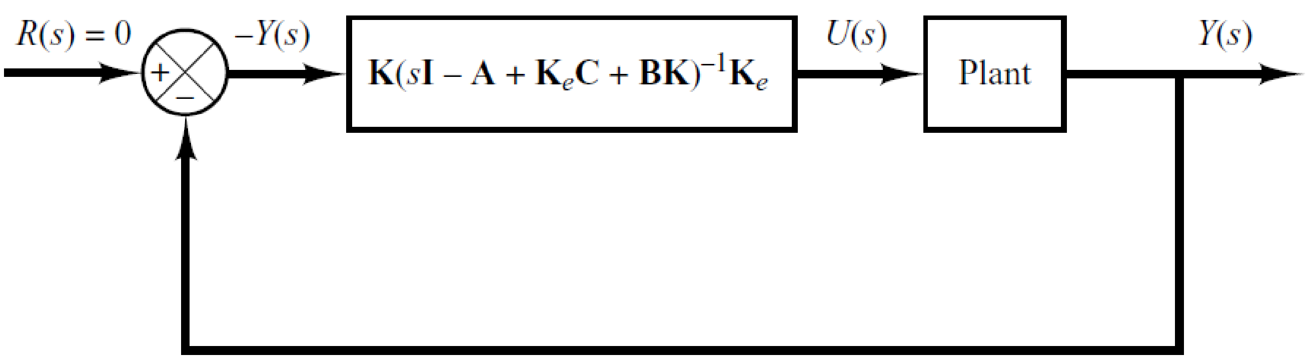
$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= (\mathbf{A} - \mathbf{K}_e\mathbf{C})\tilde{\mathbf{x}} - \mathbf{B}\mathbf{K}\tilde{\mathbf{x}} + \mathbf{K}_e y \\ &= (\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{B}\mathbf{K})\tilde{\mathbf{x}} + \mathbf{K}_e y\end{aligned}$$

- The expression for the control law is:

$$s\tilde{\mathbf{X}}(s) = (\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{B}\mathbf{K})\tilde{\mathbf{X}}(s) + \mathbf{K}_e Y(s)$$

$$\tilde{\mathbf{X}}(s) = (s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e Y(s)$$

- TF *U(s)/Y(s)*: $\dfrac{U(s)}{Y(s)} = -\mathbf{K}(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})^{-1}\mathbf{K}_e$ Acts like the feedback controller (Observer-Controller TF)




$(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{B}\mathbf{K})$

Needs to be chosen such that it is stable

# Comprehensive Exercise

- For the following 2nd order system, design a state feedback controller that uses a full state observer for state estimation. The desired poles of the observer should have a natural frequency of 20 rad/sec and damping ratio of 0.5. The closed loop poles should be at -1, -4.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -50 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Let's check controllability of the system before continuing!

$$\mathbf{C_o} = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} 0 \\ -50 \end{bmatrix} \\ -50 & \end{bmatrix} = \begin{bmatrix} 0 & -50 \\ -50 & 250 \end{bmatrix}$$

Controllability matrix is rank 2. Hence system is completely state controllable

Desired C.E (State Feedback): $(s+\mu_1)(s+\mu_2) = (s+1)(s+4)$

$$= s^2 + 5s + 4 = 0$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} 0 \\ -50 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s & -1 \\ -50k_1 & s+5-50k_2 \end{bmatrix} \right| = s^2 + (5 - 50k_2)s - 50k_1$$

$$k_1 = -\frac{4}{50}, \quad k_2 = 0$$

Let's check observability of the system before continuing!

$$\mathbf{O_B} = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \begin{bmatrix} 0 & 0 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 & \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Observability matrix is rank 2. Hence system is completely observable

Desired C.E (Observer): $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2(0.5)(20)s + 400$

$$s^2 + 20s + 400 = 0$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C}| = \left| s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix} + \begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} s+\beta_2 & -1 \\ \beta_1 & s+5 \end{bmatrix} \right| = s^2 + (5+\beta_2)s + 5\beta_2 + \beta_1$$

$$\beta_2 = 20 - 5 = 15$$

$$\beta_1 = 400 - 5(15) = 325$$

# Exercise (Try with another Pole-Placement)

- Observed state dynamics: $\dot{\tilde{\mathbf{x}}} = \left(\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{BK}\right)\tilde{\mathbf{x}} + \mathbf{K}_e y$

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{bmatrix} = \left(\begin{bmatrix}0 & 1\\ 0 & -5\end{bmatrix} - \begin{bmatrix}15\\ 325\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} - \begin{bmatrix}0\\ -50\end{bmatrix}\begin{bmatrix}-\frac{4}{50} & 0\end{bmatrix}\right)\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} + \begin{bmatrix}15\\ 325\end{bmatrix} y$$

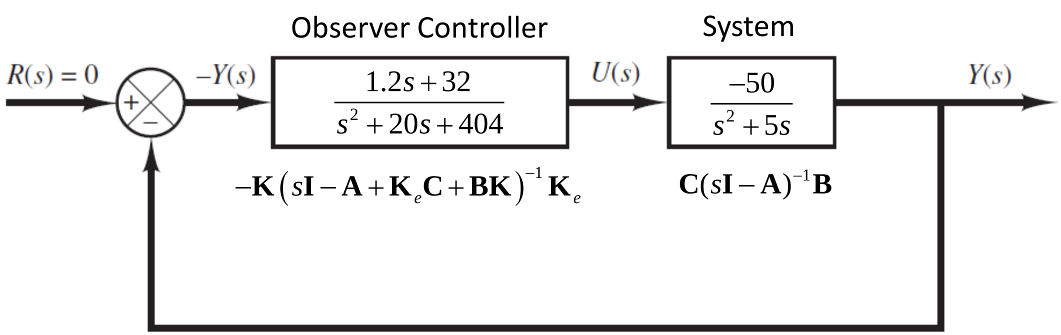
$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix}-15 & 1\\ -329 & -5\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix} + \begin{bmatrix}15\\ 325\end{bmatrix} y$$

- Observer poles: $-10 \pm 17.3j$ Controller poles: $-1, -4$
- Combined observer-based state feedback system stable? Check eigenvalues of: $\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{BK}$
  - Poles @ $-10 \pm 17.4j$ (STABLE)

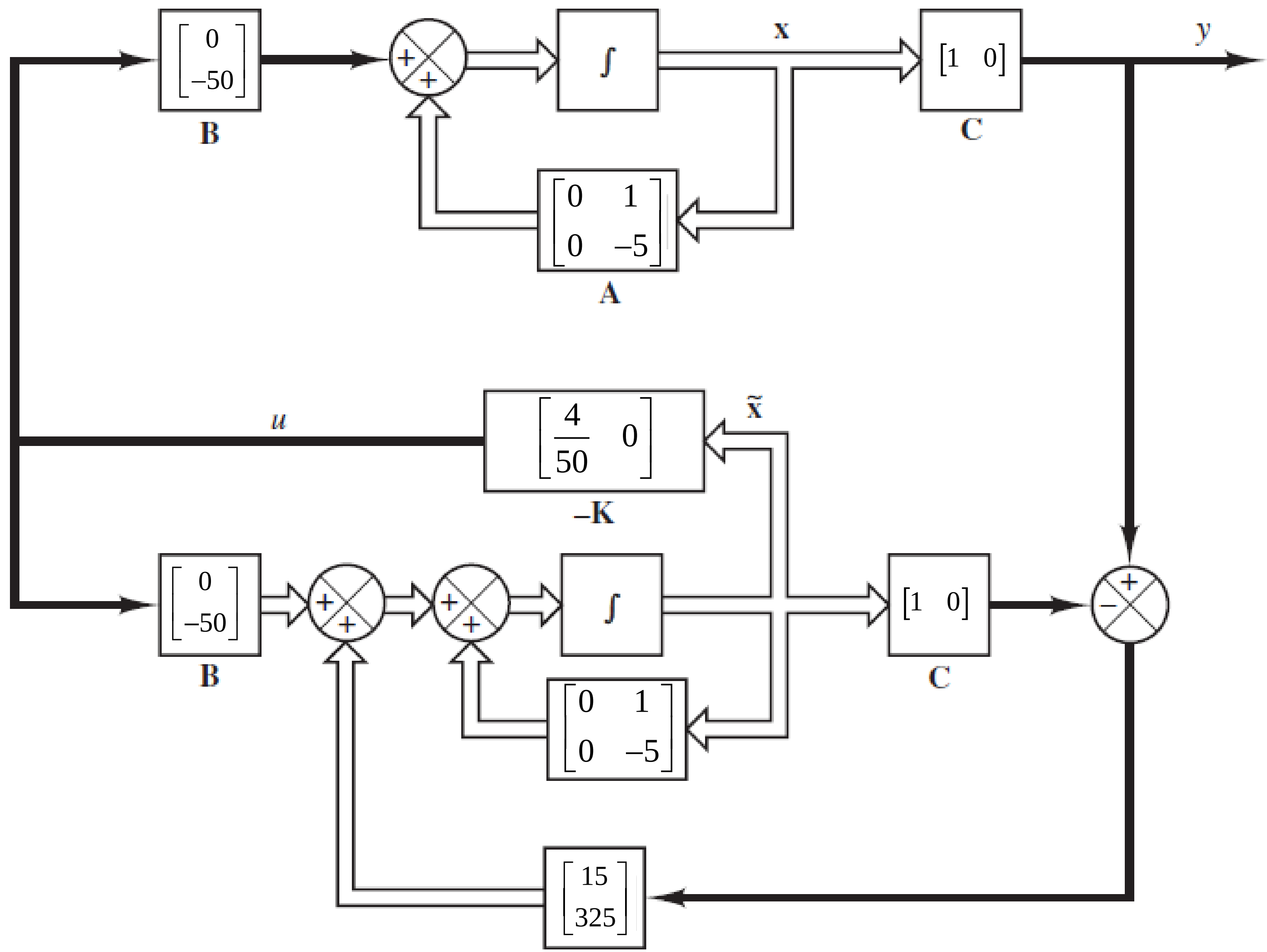
$$\frac{U(s)}{Y(s)} = -\mathbf{K}\left(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK}\right)^{-1}\mathbf{K}_e$$

$$= \frac{1.2s + 32}{s^2 + 20s + 404}$$

**Stable Observer Controller**



# Block Diagram



# Visualization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\tilde{\mathbf{x}}$$

$$\mathbf{K} = \begin{bmatrix} -4/50 & 0 \end{bmatrix}$$

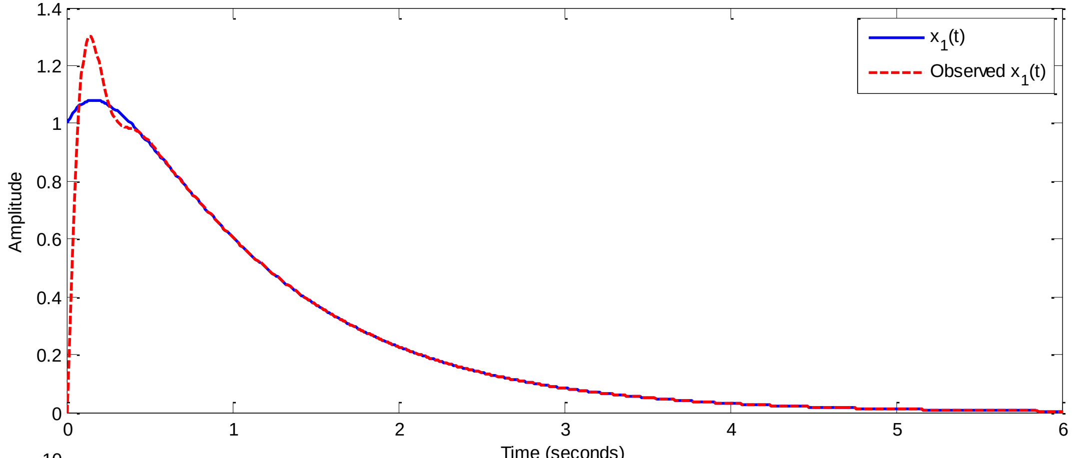
$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ -50 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \mathbf{K}_e = \begin{bmatrix} 15 \\ 325 \end{bmatrix}$$

- Let's see how the system under observed state feedback control responds to an initial condition (which is unknown to the observer). The initial observation value is zero.

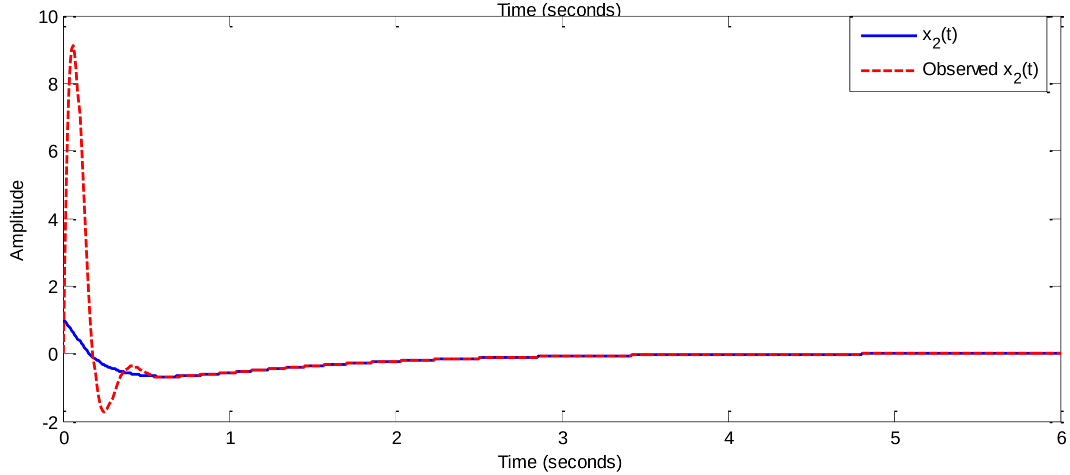
$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\tilde{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1(t)$




$x_2(t)$



# Visualization

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad u = -\mathbf{K}\tilde{\mathbf{x}} \qquad \mathbf{K} = \begin{bmatrix} -4/50 & 0 \end{bmatrix}$$

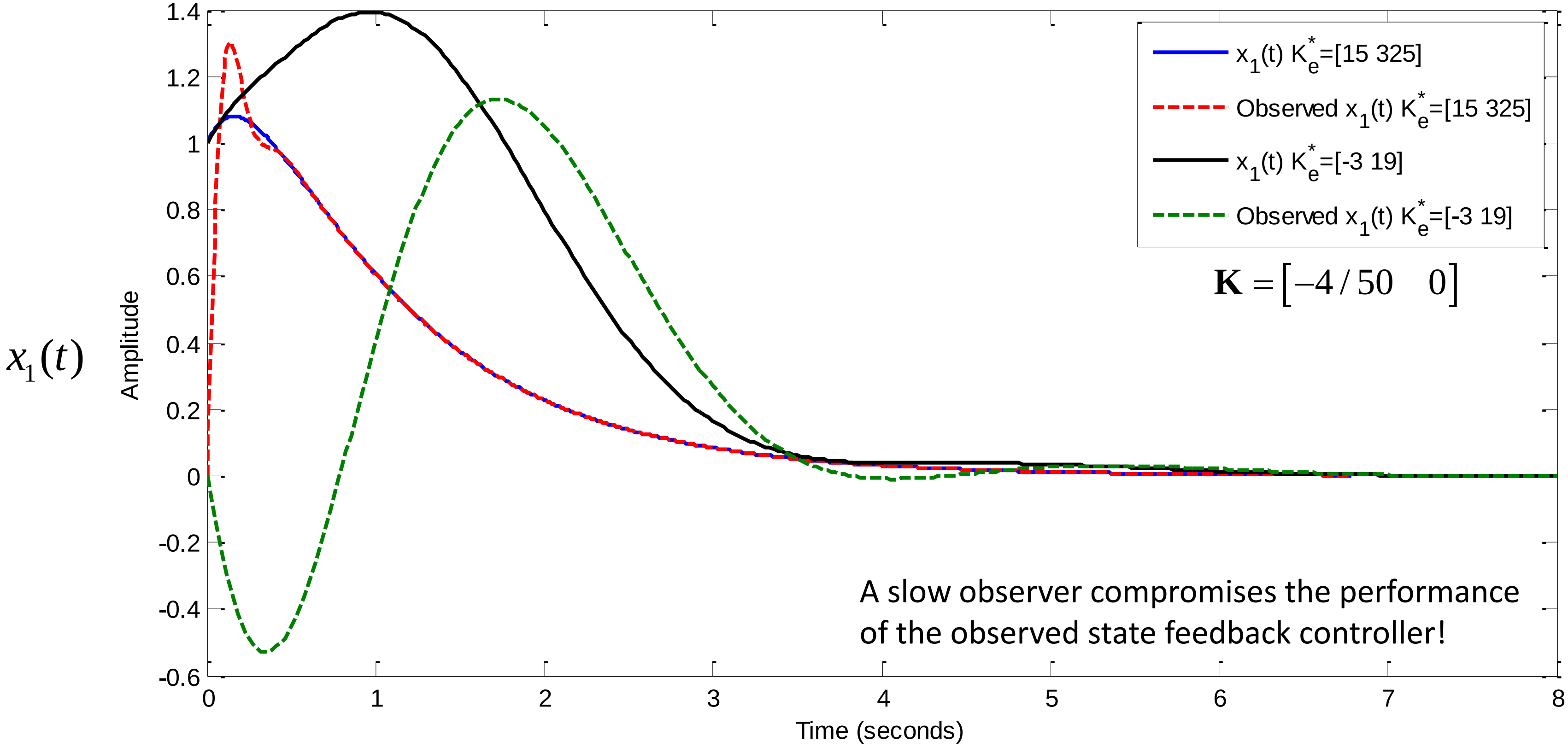
$$y = \mathbf{C}\mathbf{x} \qquad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -5 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 0 \\ -50 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \mathbf{K}_e = \begin{bmatrix} 15 \\ 325 \end{bmatrix}$$

- What if we were to use a much slower observer? Say have the observer poles have a natural frequency of 2 rad/sec and damping ratio of 0.5.

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Desired C.E (Observer): $s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4 = 0 \quad \Rightarrow \mathbf{K}_{e2} = \begin{bmatrix} -3 \\ 19 \end{bmatrix}$

$$\tilde{\mathbf{x}}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



# Supplemental Exercise

- For the following 3rd order system, design a state feedback controller that uses a full state observer for state estimation.
  - Desired closed-loop poles: $s = -1 + 2j,\ -1 - 2j,\ -5$
  - Desired observer poles: $s = -10, -10, -10$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -24 & -10 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 10 \\ -80 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \mathbf{x}$$

Check for state controllability & observability! (Do it yourself!)

Use Pole Placement for Controller and Observer Design:

$$\Rightarrow \mathbf{K} = \begin{bmatrix} 1.25 & 1.25 & 0.1938 \end{bmatrix} \qquad \Rightarrow \mathbf{K}_e = \begin{bmatrix} 20 \\ 76 \\ -240 \end{bmatrix}$$

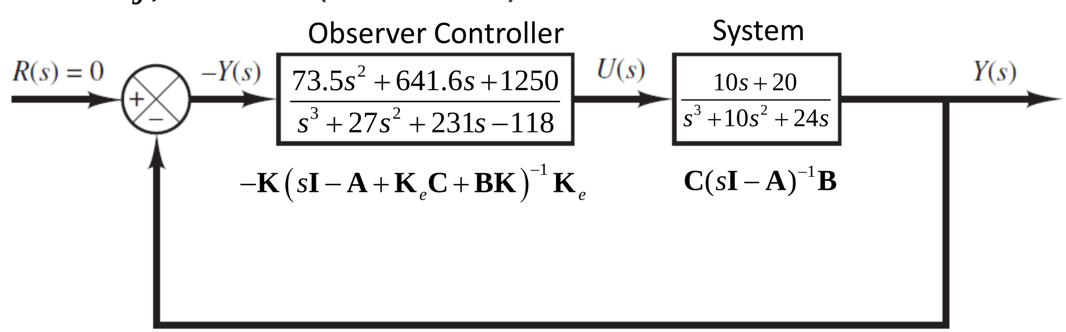
- Combined observer-based state feedback system stable? Check eigenvalues of:

$$\mathbf{A} - \mathbf{K}_e\mathbf{C} - \mathbf{BK}$$

  - Poles @ $s = -13.6 + 7.4j,\ -13.6 - 7.4j,\ 0.483$ (UNSTABLE)

$$\frac{U(s)}{Y(s)} = -\mathbf{K}\left(s\mathbf{I} - \mathbf{A} + \mathbf{K}_e\mathbf{C} + \mathbf{BK}\right)^{-1}\mathbf{K}_e = \frac{73.5s^2 + 641.6s + 1250}{s^3 + 27s^2 + 231s - 118}$$

**Unstable Observer Controller (Not Desirable)**



# Supplemental Exercise (Modify Observer Poles)

- For the following 3rd order system, design a state feedback controller that uses a full state observer for state estimation.
  - Desired closed-loop poles: $s=-1+2j,\ -1-2j,\ -5$
  - Desired **NEW** observer poles: $s=-7,-7,-7$

$$\dot{\mathbf{x}}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & -24 & -10\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\ 10\\ -80\end{bmatrix}u$$

$$y=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\mathbf{x}$$

Use Pole Placement for Controller and Observer Design:

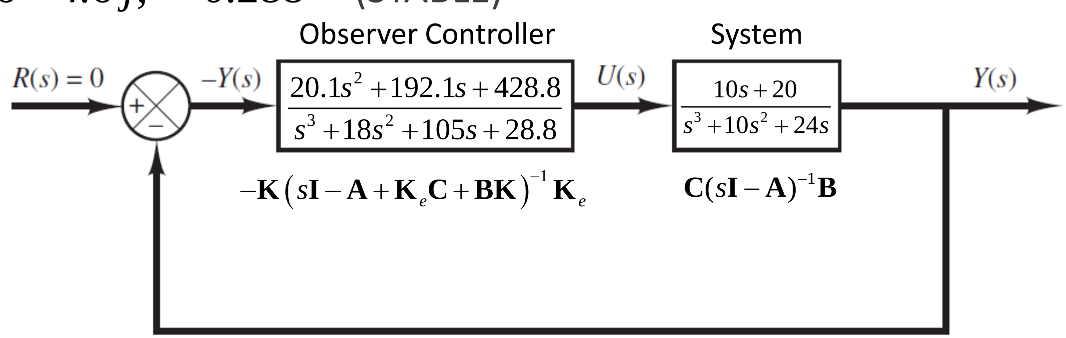
$$\Rightarrow \mathbf{K}=\begin{bmatrix}1.25 & 1.25 & 0.1938\end{bmatrix} \qquad \Rightarrow \mathbf{K}_e=\begin{bmatrix}11\\ 13\\ -51\end{bmatrix}$$

- Combined observer-based state feedback system stable? Check eigenvalues of: $\mathbf{A}-\mathbf{K}_e\mathbf{C}-\mathbf{B}\mathbf{K}$
  - Poles @ $s=-8.86+4.6j,\ -8.86-4.6j,\ -0.288$ (STABLE)

$$\frac{U(s)}{Y(s)}=-\mathbf{K}\left(s\mathbf{I}-\mathbf{A}+\mathbf{K}_e\mathbf{C}+\mathbf{B}\mathbf{K}\right)^{-1}\mathbf{K}_e$$

$$=\frac{20.1s^2+192.1s+428.8}{s^3+18s^2+105s+28.8}$$

**Stable Observer Controller**




Designing observers that are too far 'to the left' (too fast), may result in observer controller becoming unstable even though the entire closed-loop system is stable. An unstable observer controller is not acceptable for deployment.