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Tham Jit 1005036AFC HW1

1)

$$u + b(0 - \dot{x}) - k(x - 0) = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + kx = u$$

2

$$G(s) = C(sI - A)^{-1}B + D$$

$$\begin{aligned}
G(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2k & -b \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} s & 1 \\ -2k & s+b \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s+b & 1 \\ -2k & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}
\end{aligned}$$

CCF

$$\begin{aligned}
G(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -b & -2k \\ 1 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s+b & -2k \\ 1 & s \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s & -2k \\ 1 & s+b \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}
\end{aligned}$$

CCF

$$\begin{aligned}
G(s) &= \begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -2k \\ 1 & -b \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix}\left(\begin{bmatrix} s & -2k \\ 1 & s+b \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+b & -2k \\ 1 & s \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} s+b \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}
\end{aligned}$$

OCF

$$\begin{aligned}
G(s) &= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -b & 1 \\ -2k & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix}\left(\begin{bmatrix} s+b & 1 \\ -2k & s \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} s & 1 \\ -2k & s+b \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ s+b \end{bmatrix} \\
&= \frac{1}{s^2 + bs + 2k}
\end{aligned}$$

OCF

2)

$$G(s) = \frac{s+4}{s^4 + 3s^3 + 2s^2}$$

$$= \frac{s+4}{s^2(s^2 + 3s + 2)}$$

$$= \frac{s+4}{s^2(s+2)(s+1)}$$

$$= 0 + \frac{2}{s^2} - \frac{\frac{5}{2}}{s} - \frac{\frac{1}{2}}{s+2} + \frac{3}{s+1}$$

State space representation:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&-2&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}0\\1\\1\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}2&-\frac{5}{2}&-\frac{1}{2}&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$\dot{x}_1=x_2\,,\dot{x}_2=u,\dot{x}_3=-2x_3+u,\dot{x}_4=-x_4+u$$

$$y=2x_1-\frac{5}{2}x_2-\frac{1}{2}x_3+3x_4$$

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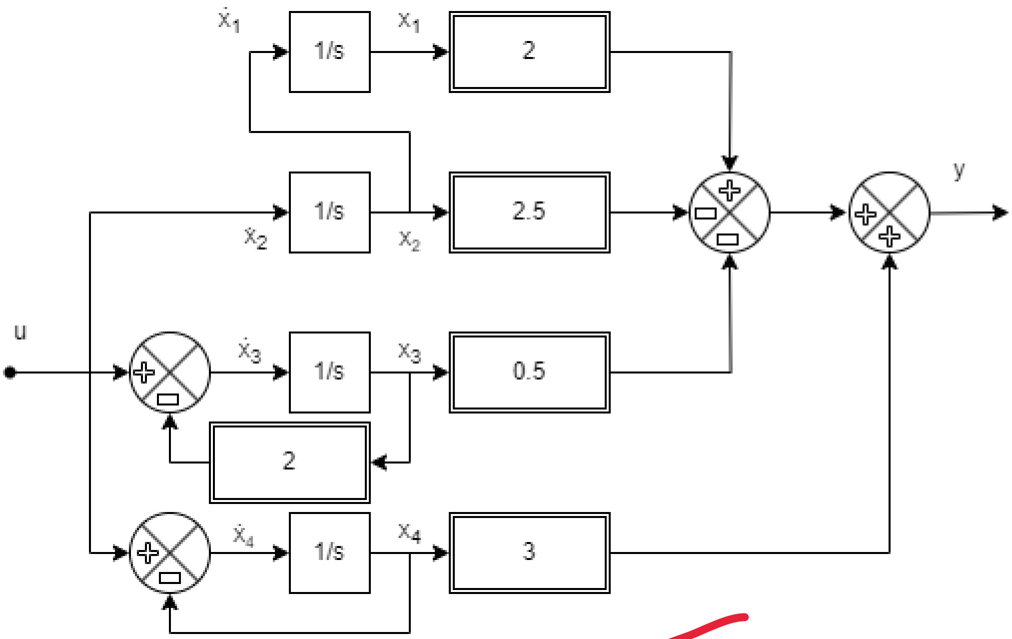

3)

$$O_B = \begin{bmatrix} H \\ HF \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$O_B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$t_1 = O_B^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T = [t_1 \quad Ft_1] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

1

$$T^{-1} = -1\begin{bmatrix} -1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$OCF: \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

$$= \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u$$

$$y = [1 \quad 0]\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [0 \quad 1]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4)

$$m_1\ddot{z}_1 = \sum F_1 = -k_1z_1 + b(\dot{z_2} - \dot{z_1}) + k_2(z_2 - z_1)$$

$$m_2\ddot{z}_2 = \sum F_2 = -k_2(z_2 - z_1) - b(\dot{z_2} - \dot{z_1}) - f$$

$$\begin{bmatrix} \dot{z_1} \\ \dot{z_2} \\ \ddot{z}_1 \\ \ddot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\dfrac{k_1 + k_2}{m_1} & -\dfrac{b}{m_1} & \dfrac{k_2}{m_1} & \dfrac{b}{m_1} \\ \dfrac{k_2}{m_2} & \dfrac{b}{m_2} & -\dfrac{k_2}{m_2} & -\dfrac{b}{m_2} \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \dot{z_1} \\ \dot{z_2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{1}{m_2} \end{bmatrix}f$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ \dot{z_1} \\ \dot{z_2} \end{bmatrix}$$

3

5a)

$$|\lambda I - A| = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{bmatrix}$$

$$= \lambda * \lambda * (\lambda + 6) + (-1) * (-1) * 6 + 0 - 0 - 11 * (-1) * \lambda - 0$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda + 1)(\lambda + 2)(\lambda + 3)$$

$$\lambda_1, \lambda_2, \lambda_3 = -1, -2, -3$$

$$(A - \lambda_1 I)v_1 = 0$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) v_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$

$$v_{11} + v_{12} = 0$$

$$v_{12} + v_{13} = 0$$

$$v_{11} = v_{13} = -v_{12}$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(A - \lambda_1 I)v_2 = 0$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right) v_2 = 0$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -6 & -11 & -4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$2v_{21} + v_{22} = 0$$

$$2v_{22} + v_{23} = 0$$

$$v_{21} = -\frac{1}{2}v_{22} = \frac{1}{4}v_{23}$$

$$v_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$(A - \lambda_1 I)v_3 = 0$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right) v_3 = 0$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -6 & -11 & -3 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0$$

$$3v_{31} + v_{32} = 0$$

$$3v_{32} + v_{33} = 0$$

$$v_{31} = -\frac{1}{3}v_{32} = \frac{1}{9}v_{33}$$

$$v_3 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

5b)

$$P = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

3

Mechanical values = -1,-2,-3

5c)

$$P^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} -1 & -2 & -3 \\ 1 & 4 & 9 \\ -1 & -8 & -27 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

(shown)

6a)

$$J\ddot{\theta} + mgl\sin\theta = u$$

$$J = ml^2$$

$$ml^2\ddot{\theta} + mgl\sin\theta = u$$

$$\ddot{\theta} + \frac{g\sin\theta}{l} - \frac{1}{ml^2}u = 0$$

2

6b)

For small angle, $\sin\theta = \theta$

$$\ddot{\theta} + \frac{g\theta}{l} - \frac{1}{ml^2}u = 0$$

$$\mathcal{L}\left(\ddot{\theta} + \frac{g\theta}{l}\right) = \mathcal{L}\left(\frac{1}{ml^2}u\right)$$

$$s^2\ddot{\theta} + \frac{g}{l}\theta(s) = mlu(s)$$

$$\frac{\theta(s)}{u(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{g}{l}}$$

CCF	OCF
$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u$ $\theta = \begin{bmatrix}\frac{1}{ml^2} & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$	$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -\frac{g}{l}\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{ml^2}\\0\end{bmatrix}u$ $\theta = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$

7)

$$|A - \lambda I| = 0$$

$$\left\|\begin{bmatrix}2-\lambda & 1 & 0\\ 0 & 2-\lambda & 1\\ 0 & 0 & 2-\lambda\end{bmatrix}\right\| = 0$$

$$(2-\lambda)^3 = 0$$

A is already in Jordan form

$$e^{At} = \begin{bmatrix}e^{2t} & te^{2t} & \frac{1}{2}t^2e^{2t}\\ 0 & e^{2t} & te^{2t}\\ 0 & 0 & e^{2t}\end{bmatrix}$$

2

8)

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1(t)$$

$$x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = e^{At}x(0) + A^{-1}(e^{At} - I)Bk$$

Since $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$x(t) = A^{-1}(e^{At} - I)Bk$$

$$e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$$

$$= \mathcal{L}^{-1}\left[\begin{bmatrix} s+1 & 0.5 \\ -1 & s \end{bmatrix}^{-1}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2 - s + 0.5}\begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}\right]$$

$$= \mathcal{L}^{-1}\left[\begin{bmatrix} \frac{s+0.5}{(s+0.5)^2+0.5^2} - \frac{0.5}{(s+0.5)^2+0.5^2} & \frac{0.5}{(s+0.5)^2+0.5^2} \\ \frac{1}{(s+0.5)^2+0.5^2} & \frac{s+0.5}{(s+0.5)^2+0.5^2} + \frac{0.5}{(s+0.5)^2+0.5^2} \end{bmatrix}\right]$$

$$= \begin{bmatrix} e^{-0.5t}(cos0.5t - sin0.5t) & e^{-0.5t}sin0.5t \\ 2e^{-0.5t}sin0.5t & e^{-0.5t}(cos0.5 + sin0.5t) \end{bmatrix}$$

$$x(t) = \frac{1}{0.5}\begin{bmatrix} 0 & 0.5 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} e^{-0.5t}(cos0.5t - sin0.5t) - 1 & e^{-0.5t}sin0.5t \\ 2e^{-0.5t}sin0.5t & e^{-0.5t}(cos0.5 + sin0.5t) - 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} e^{-0.5t}sin0.5t \\ e^{-0.5t}(cos0.5 + sin0.5t) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.5t}(cos0.5 + sin0.5t) - 1 \\ -2(e^{-0.5t}sin0.5t - e^{-0.5t}(cos0.5 + sin0.5t) + 1) \end{bmatrix}$$

$$= \begin{bmatrix} e^{-0.5t}(cos0.5 + sin0.5t) - 1 \\ -2(1 - e^{-0.5t}cos0.5t) \end{bmatrix}$$

$$y(t) = e^{-0.5t}(cos0.5 + sin0.5t) - 1$$