Servo Systems & Integral Control

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Designing Servo Systems

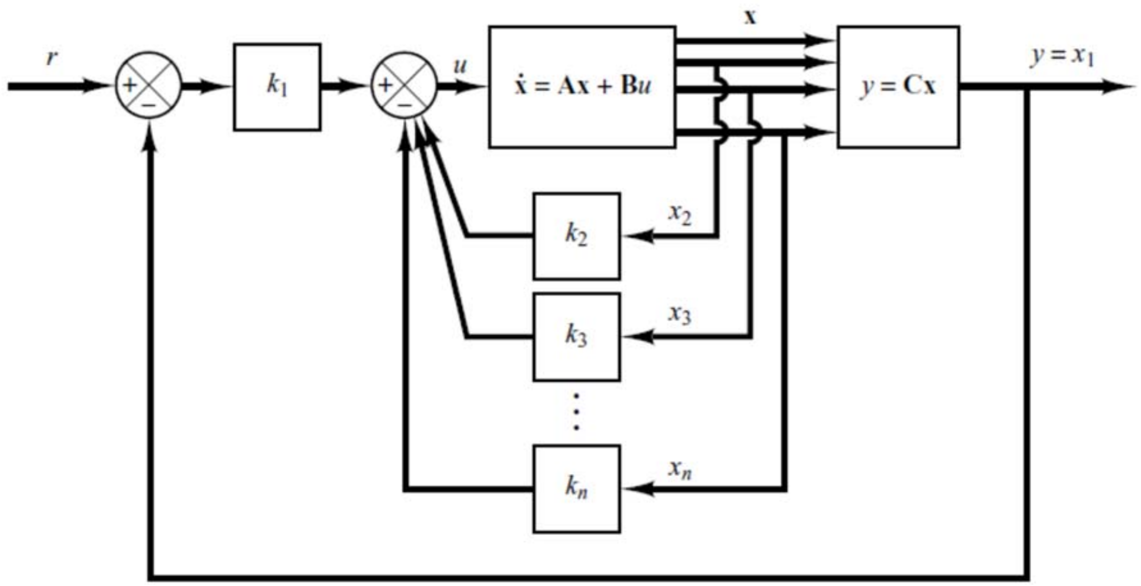
- Recall PID controllers where there were 3 items:
  - Proportional Gain (P)
  - Derivative Gain (D)
  - Integral Gain (I)
- When designing State Feedback Controllers, we could roughly equate that to a PD system. What about Integral action? How could we incorporate this very useful action into State-Space Systems?
- Recall Servo systems (such as a servomotor) where the goal is to control position/speed of a system with **zero steady-state error**
  - If the system is a **Type 1** system (has an embedded integrator), no integral action is needed
    - State Feedback Controller is sufficient
  - If the system is a **Type 0** system (no integrator in plant), an integral action is needed to address the static error constant (steady-state error)
    - State Feedback Controller is insufficient. An integrator is needed to provide integral action

Design of Servo Control System for Type 1 Plant

- Consider the following plant (SISO): $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$ $$y = \mathbf{C}\mathbf{x}$$
  - Choose a SS representation where output is equal to one of the state variables (e.g. CCF). There are *n* states.
  - System already has 1 integrator (<u>one eigenvalue of **A** is zero</u>)
  - For simplicity, assume $y=x_1$
- General configuration for a **Servo Control** System (with internal integrator)
  - State Feedback Control scheme:

$$\boxed{\begin{aligned} u &= -\mathbf{K}\mathbf{x} + k_1 r \\ \mathbf{K} &= \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix} \end{aligned}}$$

$$u = -\begin{bmatrix} 0 & k_2 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + k_1 (r - y)$$



Design of Servo Control System for Type 1 Plant

- Inserting control scheme into state equation, the system dynamics is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\left(-\mathbf{K}\mathbf{x} + k_1 r\right)$$

$$\dot{\mathbf{x}}(t) = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x}(t) + \mathbf{B}k_1 r(t)$$

- The goal of this controller design is to place the closed-loop poles at desired positions such that if *r(t)* is a step input,
  - System is asymptotically stable,
  - $y(\infty)$ approaches the constant value *r,*
  - $u(\infty)$ approach zero
- At steady state: $\dot{\mathbf{x}}\left(\infty\right) = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x}\left(\infty\right) + \mathbf{B}k_1 r\left(\infty\right)$
  - As r(t) is a step input, it is a constant for *t>0, r(∞)=r(t)*

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}\left(\infty\right) = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\left[\mathbf{x}(t) - \mathbf{x}\left(\infty\right)\right]$$

  - Let's define the error vector:

$$\mathbf{e}(t) = \mathbf{x}(t) - \mathbf{x}\left(\infty\right) \quad \Rightarrow \quad \dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{e}$$

Introducing the Error Dynamics

- This equation describes the error dynamics of the system: $\dot{\mathbf{e}} = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{e}$
  - Similar to State Dynamics (State-Space)
  - It a regulator design problem (absence of input)
  - If system is completely state controllable, **K** can be :
    - Computed using Pole-Placement techniques
    - Selected (in the LHP) such that the error approaches zero given any initial conditions (initial error: *e(0)*)

- The steady-state value of the states and input are found by letting $t \to \infty$
  - From the state-equation
    - As eigenvalues **A-BK** are on LHP, inverse of **A-BK** exists

$$\dot{\mathbf{x}}(\infty) = 0 = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}(\infty) + \mathbf{B}k_1 r(\infty)$$

$$\left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}(\infty) = -\mathbf{B}k_1 r$$

$$\mathbf{x}(\infty) = -\left(\mathbf{A} - \mathbf{BK}\right)^{-1}\mathbf{B}k_1 r$$

  - From the input-equation
    - At steady state the control output is zero

$$\begin{aligned} u\left(\infty\right) &= -\mathbf{Kx}\left(\infty\right) + k_1 r \\ &= -k_1 x_1\left(\infty\right) + k_1 r \\ &= 0 \end{aligned}$$

Exercise (Servo Design)

- For the following system, design a servo controller to control the output *y(t)* to an input *u(t)*. It is desired that the closed loop poles are placed at: $s=-2+j2\sqrt{3},\ \ s=-2-j2\sqrt{3},\ \ s=-10$
  - What is the system type of the plant?
  - What is the State-Space Representation of System?

$$\frac{Y(s)}{U(s)}=\frac{1}{s(s+1)(s+2)}$$

$$\frac{Y(s)}{U(s)}=\frac{1}{s^3+3s^2+2s}$$

Since we are doing controller design, it is advantageous to use CCF

$$\dot{\mathbf{x}}=\begin{bmatrix}0&1&0\\0&0&1\\0&-2&-3\end{bmatrix}\mathbf{x}+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}1&0&0\end{bmatrix}\mathbf{x}$$

$$a_1=3,\ a_2=2,\ a_3=0$$

$$u=-\begin{bmatrix}0&k_2&\cdots&k_n\end{bmatrix}\begin{bmatrix}x_1\\x_2\\\vdots\\x_n\end{bmatrix}+k_1(r-y)=-\mathbf{K}\mathbf{x}+k_1r$$

$$\mathbf{K}=\begin{bmatrix}k_1&k_2&k_3\end{bmatrix}$$

Check for controllability:

$$\mathbf{C}_O=\begin{bmatrix}\mathbf{B}&\mathbf{AB}&\mathbf{A}^2\mathbf{B}\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&-3\\1&-3&7\end{bmatrix}$$

Rank of controllability matrix is 3 (full rank). System is completely state controllable. Servo can be designed!

Exercise (Servo Design)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

$$a_1 = 3,\ a_2 = 2,\ a_3 = 0$$

$$u = -\begin{bmatrix} 0 & k_2 & \cdots & k_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + k_1(r - y) = -\mathbf{K}\mathbf{x} + k_1 r$$

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

Design by Pole-Placement: (Direct CCF Method)

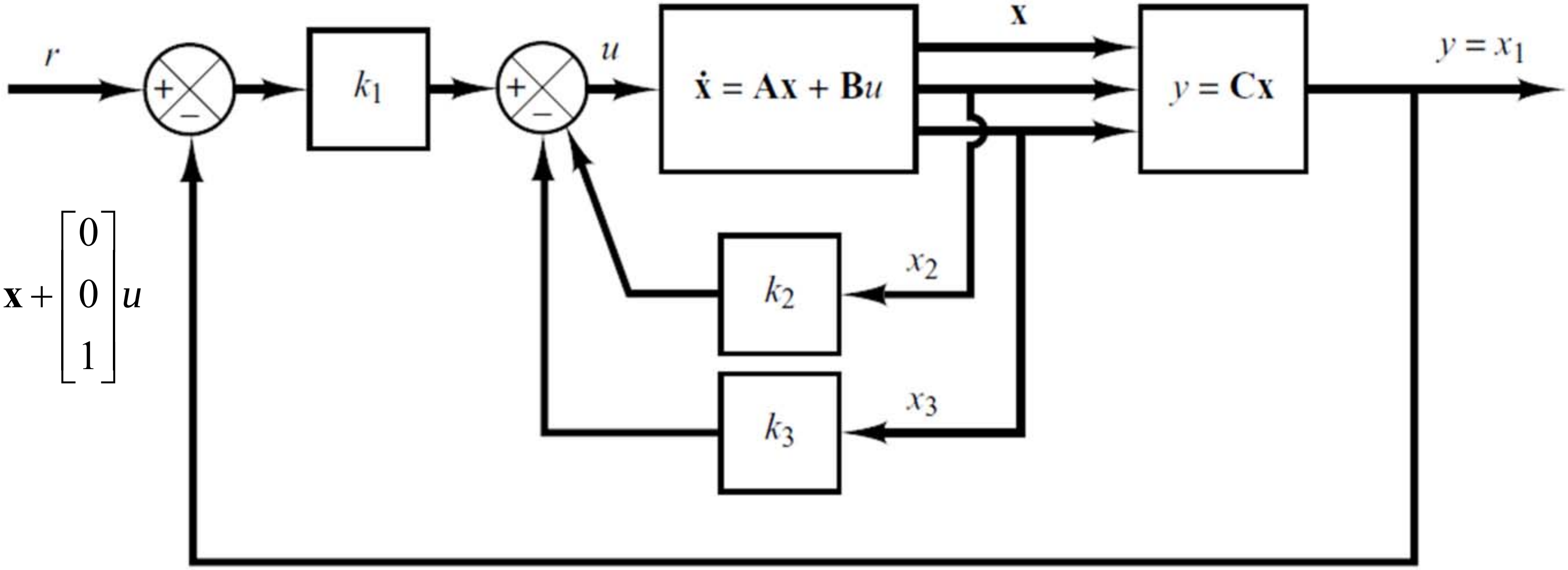
Desired c.e.: $\left(s + 2 - j2\sqrt{3}\right)\left(s + 2 + j2\sqrt{3}\right)(s + 10) = s^3 + 14s^2 + 56s + 160$ $\qquad \alpha_1 = 14,\ \alpha_2 = 56,\ \alpha_3 = 160$

$$\mathbf{K} = \begin{bmatrix} \alpha_3 - a_3 & \alpha_2 - a_2 & \alpha_1 - a_1 \end{bmatrix} = \begin{bmatrix} 160 - 0 & 56 - 2 & 14 - 3 \end{bmatrix}$$

$$= \begin{bmatrix} 160 & 54 & 11 \end{bmatrix}$$

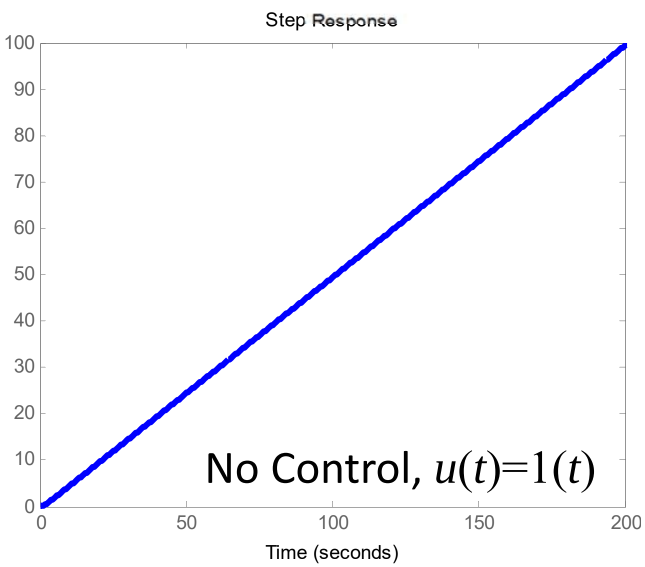
- Try it again but using a different Pole-Placement Technique!

Exercise (Block Diagram Representation)

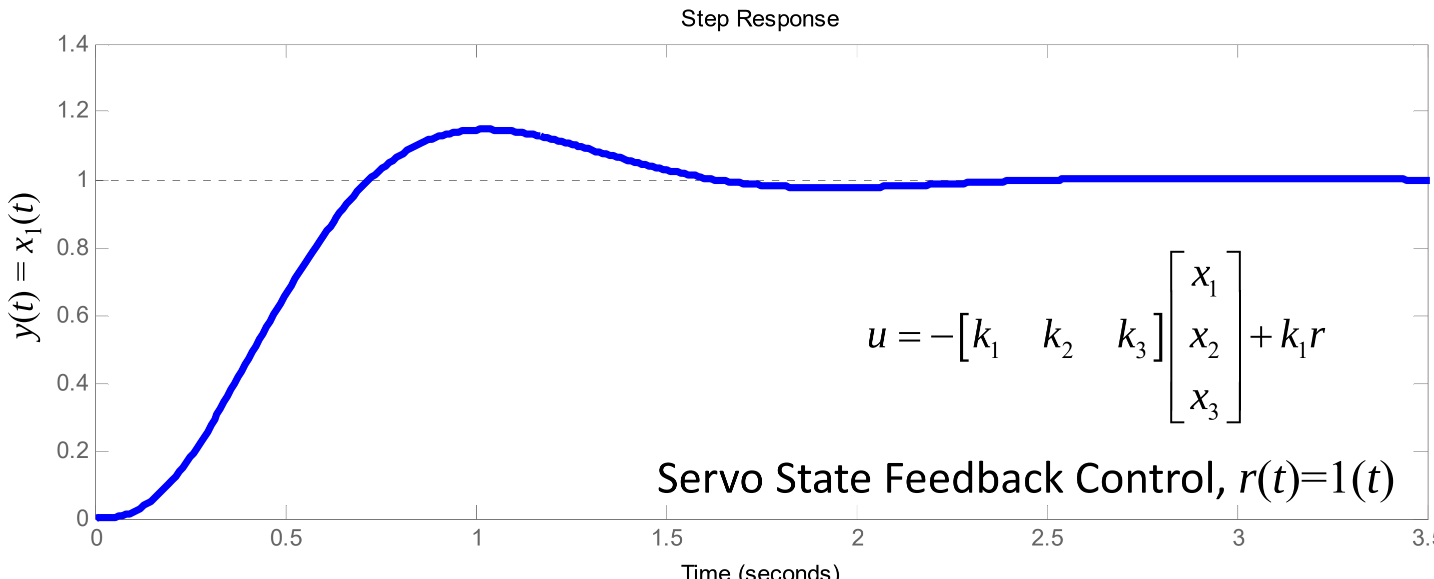


$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$



No Control, $u(t)=1(t)$



$$u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + k_1 r$$

Servo State Feedback Control, $r(t)=1(t)$

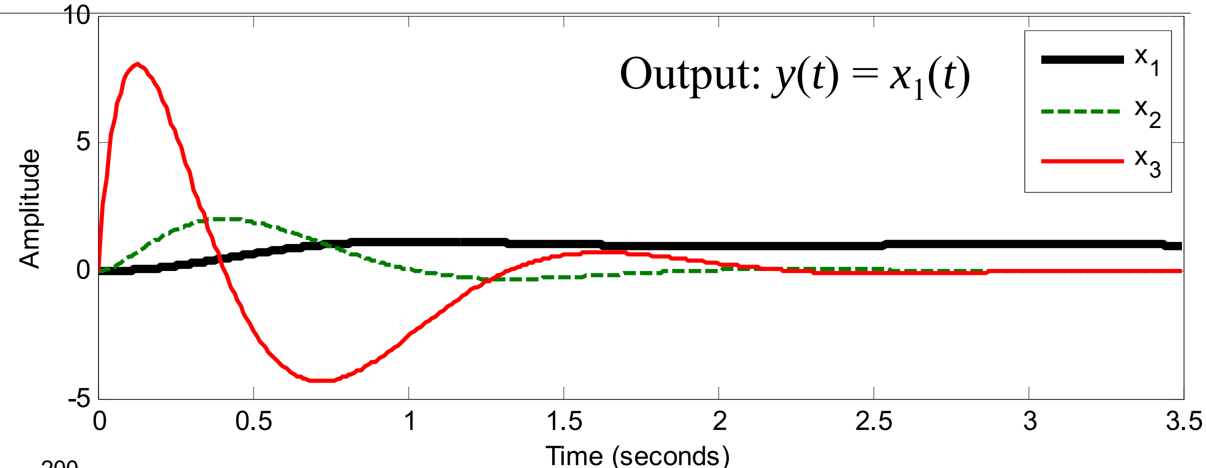
Exercise (More Visualization!)

Servo State Feedback Control, $r(t)=1(t)$
Zero initial conditions

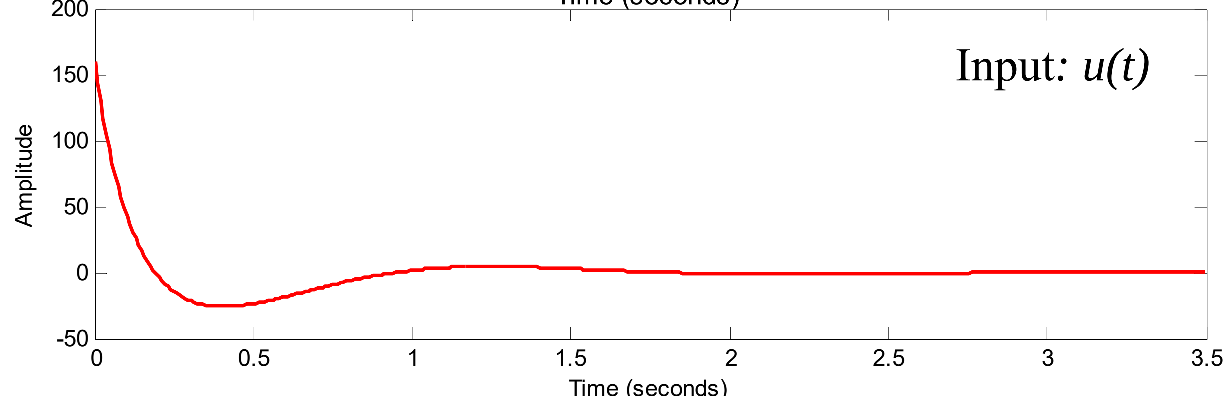
$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}$$

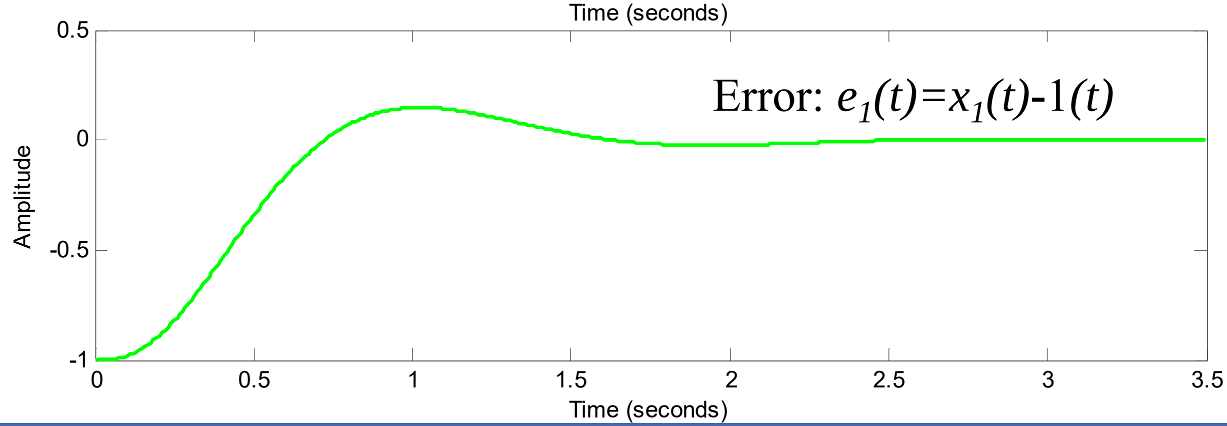
$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r$$



$$u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + k_1 r$$



$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{e}$$

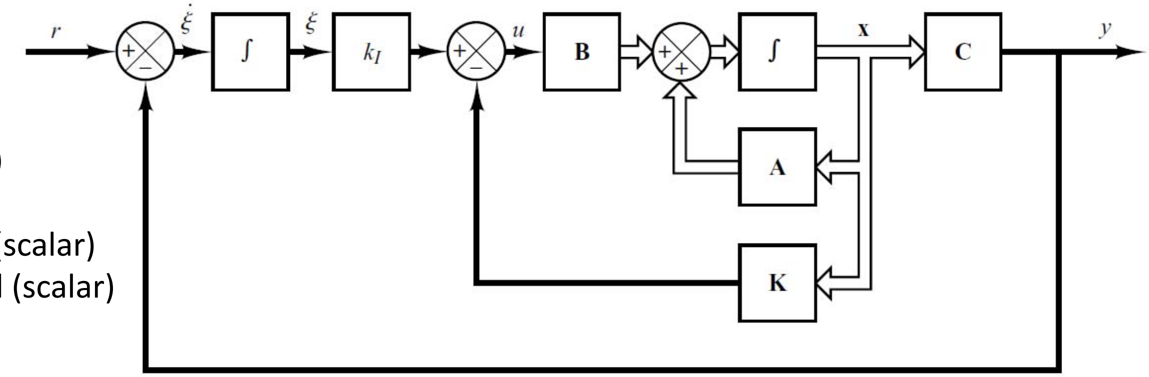


Design of Servo Control System for Type 0 Plant

- For a plant with no integrator (Type 0), an integrator is inserted in the feedforward path between error comparator and plant

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x}$$
$$u = -\mathbf{K}\mathbf{x} + k_I \xi$$
$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

u: control signal (scalar)
y: output signal (scalar)
ξ: output of integrator (scalar)
r: reference input signal (scalar)



  - Plant (**C**[s**I**-**A**]$^{-1}$**B**) should have no zero at origin to prevent cancellation of integrator

- Consolidating (2 LTI State Space systems):

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

System Dynamics

$$\dot{\xi} = -\mathbf{C}\mathbf{x} + r$$

Integrator Dynamics

  - Hmm since they are LTI and in SS form... Let's Combine them further!

$$\boxed{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{-C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \xi \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u + \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix} r}$$

Matrixception! Matrices Within Matrices!

- When r is a step function applied at t=0, the full system dynamics is now compactly expressed:

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\xi}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{-C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)\\ \xi(t)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ 0\end{bmatrix}u(t)+\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(t)$$

- At steady state, an asymptotically designed system, $x(\infty)$, $\xi(\infty)$ and $u(\infty)$ will approach constant values:
  - At steady state: $\dot{\xi}(\infty)=0,\ \ y(\infty)=r$

$$\begin{bmatrix}\dot{\mathbf{x}}(\infty)\\ \dot{\xi}(\infty)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{-C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(\infty)\\ \xi(\infty)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ 0\end{bmatrix}u(\infty)+\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}r(\infty)$$

- Combining via subtraction:

$$\begin{bmatrix}\dot{\mathbf{x}}(t)-\dot{\mathbf{x}}(\infty)\\ \dot{\xi}(t)-\dot{\xi}(\infty)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{-C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(\infty)\\ \xi(t)-\xi(\infty)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ 0\end{bmatrix}\left[u(t)-u(\infty)\right]+\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}\left[r(t)-r(\infty)\right]$$

  - At steady-state: $r(t)=r(\infty)$

$$\begin{bmatrix}\dot{\mathbf{x}}(t)-\dot{\mathbf{x}}(\infty)\\ \dot{\xi}(t)-\dot{\xi}(\infty)\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{-C} & 0\end{bmatrix}\begin{bmatrix}\mathbf{x}(t)-\mathbf{x}(\infty)\\ \xi(t)-\xi(\infty)\end{bmatrix}+\begin{bmatrix}\mathbf{B}\\ 0\end{bmatrix}\left[u(t)-u(\infty)\right]$$

  - Defining: $\mathbf{x}_e(t)=\mathbf{x}(t)-\mathbf{x}(\infty)$ $\quad \xi_e(t)=\xi(t)-\xi(\infty)$ $\quad u_e(t)=u(t)-u(\infty)$

Augmented Error Dynamics

- The full system dynamics in terms of the new terms are:

$$\begin{bmatrix} \dot{\mathbf{x}}_e(t) \\ \dot{\xi}_e(t) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{-C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} u_e(t)$$

  - Recall the control law: $u_e(t) = -\mathbf{K}\mathbf{x}_e(t) + k_I \xi_e(t) = -\begin{bmatrix} \mathbf{K} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix} + k_I \xi_e(t) = -\begin{bmatrix} \mathbf{K} & -k_I \end{bmatrix} \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix}$

- Define the augmented error vector: $\boxed{\mathbf{e}(t) = \begin{bmatrix} \mathbf{x}_e(t) \\ \xi_e(t) \end{bmatrix}}$
  - We get back a very familiar expression:

$$\boxed{\begin{aligned} \dot{\mathbf{e}} &= \hat{\mathbf{A}}\mathbf{e} + \hat{\mathbf{B}}u_e \\ u_e &= -\hat{\mathbf{K}}\mathbf{e} \end{aligned}} \qquad \boxed{\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{-C} & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \quad \hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K} & -k_I \end{bmatrix}}$$

(n+1) x (n+1)

  - To determine the gains, you can use Pole-Placement methods on this system:

$$\boxed{\dot{\mathbf{e}} = \left(\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}\right)\mathbf{e}}$$

    - Here the state feedback gains **K** and the integral gain k_I are determined simultaneously
    - Remember to check for complete state controllability before gain computation

Exercise

- Consider a motor speed system defined by $\dfrac{Y(s)}{U(s)} = \dfrac{1}{s+3}$
  - Design the system to have integral control and 2 poles at s=-5,-5.

$$\text{EOM}: \dot{y} + 3y = u$$

$$\dot{x} = Ax + Bu \qquad \dot{x} = -3x + u$$
$$y = Cx \qquad y = x$$

$$\dot{\mathbf{e}} = \left(\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}\right)\mathbf{e}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \hat{\mathbf{K}} = \begin{bmatrix} K & -k_I \end{bmatrix}$$

$$\hat{\mathbf{A}} = \begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix}, \quad \hat{\mathbf{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{K}} = \begin{bmatrix} K & -k_I \end{bmatrix}$$

$$A = -3$$
$$B = 1$$
$$C = 1$$

Check for controllability:

$$\mathbf{C}_O = \begin{bmatrix} \hat{\mathbf{B}} & \hat{\mathbf{A}}\hat{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & -1 \end{bmatrix}$$

Rank of controllability matrix is 2 (full rank). System is completely state controllable. Integral control can be designed.

Remember the c.e. is

$$\left|s\mathbf{I} - \left(\hat{\mathbf{A}} - \hat{\mathbf{B}}\hat{\mathbf{K}}\right)\right| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} K & -k_I \end{bmatrix}\right)\right|$$

$$= \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -3 & 0 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} K & -k_I \\ 0 & 0 \end{bmatrix}\right)\right|$$

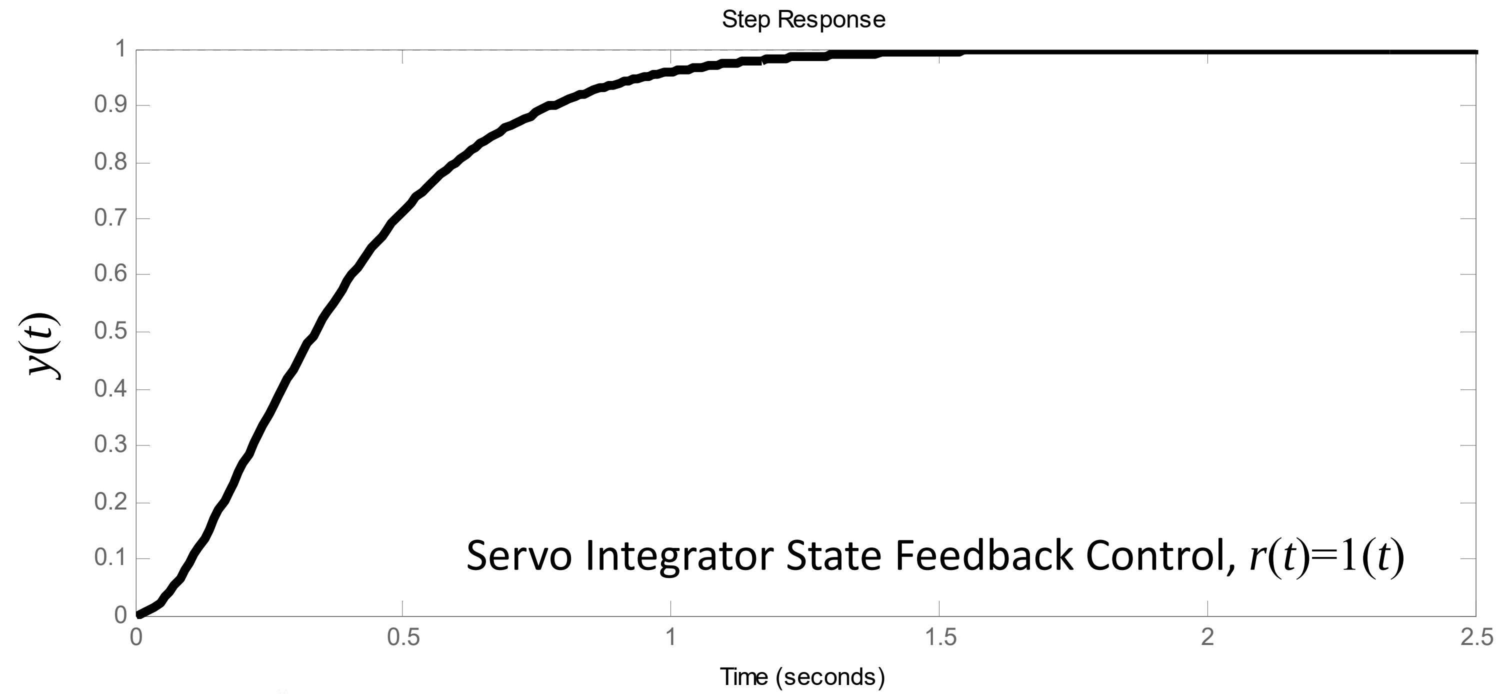
$$= \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -3-K & k_1 \\ -1 & 0 \end{bmatrix}\right| = \left|\begin{bmatrix} s+3+K & -k_I \\ 1 & s \end{bmatrix}\right|$$

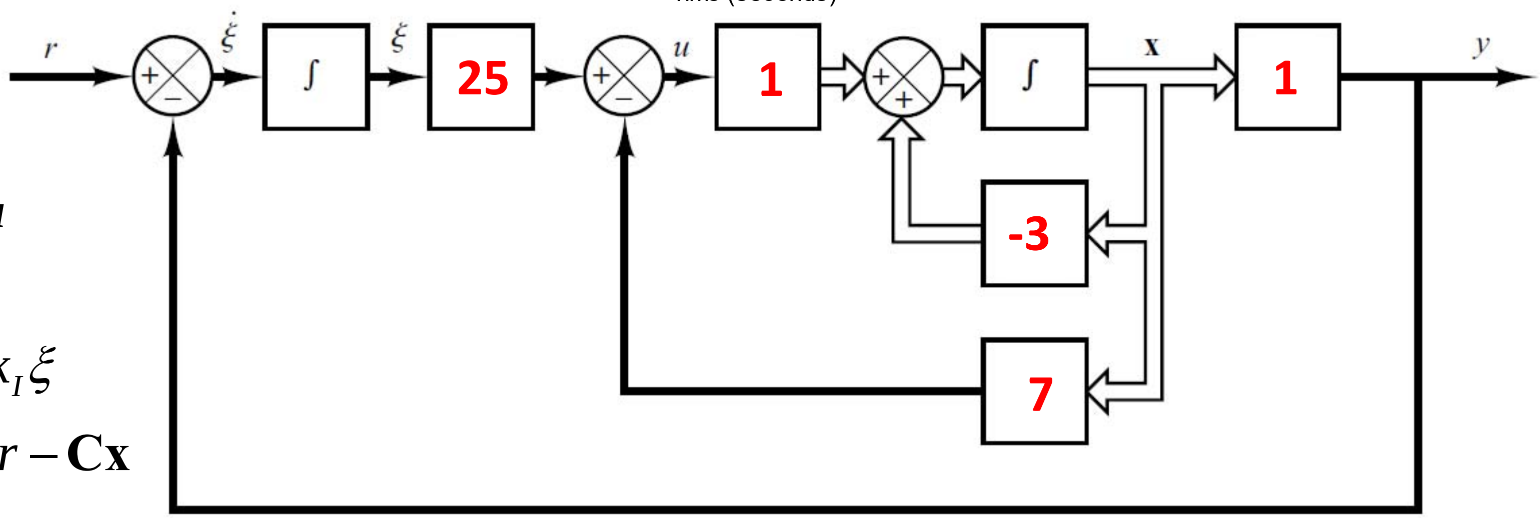
$$= s^2 + \left(3+K\right)s + k_I$$

Desired c.e. is $(s+5)(s+5) = s^2 + 10s + 25$

$$\boxed{\begin{array}{l} 3+K = 10 \Rightarrow K = 7 \\ k_I = 25 \end{array}}$$

Exercise (Block Diagram)





$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

$$u = -\mathbf{K}\mathbf{x} + k_I\xi$$

$$\dot{\xi} = r - y = r - \mathbf{C}\mathbf{x}$$

Extra Exercise

- What if ignored the fact that the system is Type 0 and proceeded with a controller design without an integrator?
  - System is only 1st order so you can only place 1 pole: -5.