Solving Nonhomogenous State Equations

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Nonhomogenous Scalar 1st Order Systems

- A Nonhomogenous scalar state equation is

$$\dot{x} = ax + bu$$

$$\dot{x} - ax = bu$$

- Multiplying both sides by the exponential:

$$e^{-at}\left(\dot{x}(t) - ax(t)\right) = e^{-at}bu(t)$$

$$\frac{d}{dt}\left[e^{-at}x(t)\right] = e^{-at}bu(t)$$

- Integrating between 0 and *t*

$$e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)d\tau$$

$$x(t) = e^{at}x(0) + e^{at}\int_0^t e^{-a\tau}bu(\tau)d\tau$$

Response to initial conditions

Response to input

Extending to LTI State-Space Systems

- Now we extend to a vector-matrix differential equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\dot{\mathbf{x}} - \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{u}$$

- Premultiplying both sides by $e^{-\mathbf{A}t}$:

$$e^{-\mathbf{A}t}\left(\dot{\mathbf{x}}(t) - \mathbf{A}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

$$\frac{d}{dt}\left[e^{-\mathbf{A}t}\mathbf{x}(t)\right] = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

- Integrating between 0 and *t*

$$e^{-\mathbf{A}t}\mathbf{x}(t) - \mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

Response to initial conditions

Response to input

LT Approach to Solving LTI Nonhomogenous State Eqns

- Starting from the LTI homogenous state equation: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$
- Taking LT

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

- Recall:

$$\mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots = e^{\mathbf{A}t}$$

$$\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathcal{L}\left[e^{\mathbf{A}t}\right]$$

$$\mathbf{X}(s) = \mathcal{L}\left[e^{\mathbf{A}t}\right]\mathbf{x}(0) + \mathcal{L}\left[e^{\mathbf{A}t}\right]\mathbf{B}\mathbf{U}(s)$$

- Taking ILT:

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau \qquad \mathcal{L}\left[\int_0^t f_1(t-\tau)f_2(\tau)\,d\tau\right] = F_1(s)F_2(s)$$

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Analytical Solution to Unit Step Input

- Recall: $$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

- And the input: $\mathbf{u}(t) = \mathbf{1}(t) = \begin{bmatrix} 1(t) \\ \vdots \\ 1(t) \end{bmatrix} = 1(t)\mathbf{k}$

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{1}{k!}\mathbf{A}^kt^k + \cdots$$

$$e^{-\mathbf{A}t} = \mathbf{I} - \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots + \frac{(-1)^k}{k!}\mathbf{A}^kt^k + \cdots$$

$$\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{1}(\tau)d\tau \\
&= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}\tau}\mathbf{B}1(\tau)\mathbf{k}d\tau \\
&= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\int_0^t\left[\mathbf{I} - \mathbf{A}\tau + \frac{1}{2!}\mathbf{A}^2\tau^2 - \cdots\right]d\tau\right)\mathbf{B}\mathbf{k} \\
&= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\left(\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{1}{3!}\mathbf{A}^2t^3 - \cdots\right)\mathbf{B}\mathbf{k}
\end{aligned}$$

$$\begin{aligned}
\mathbf{I}t - \frac{\mathbf{A}t^2}{2!} + \frac{1}{3!}\mathbf{A}^2t^3 - \cdots &= \mathbf{A}^{-1}\left(\mathbf{A}t - \frac{\mathbf{A}^2t^2}{2!} + \frac{1}{3!}\mathbf{A}^3t^3 - \cdots\right) \\
&= -\mathbf{A}^{-1}\left(-\mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} - \frac{1}{3!}\mathbf{A}^3t^3 - \cdots\right) \\
&= -\mathbf{A}^{-1}\left(\mathbf{I} - \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} - \frac{1}{3!}\mathbf{A}^3t^3 - \cdots - \mathbf{I}\right) \\
&= -\mathbf{A}^{-1}\left(e^{-\mathbf{A}t} - \mathbf{I}\right) = \mathbf{A}^{-1}\left(\mathbf{I} - e^{-\mathbf{A}t}\right)
\end{aligned}$$

$$\boxed{\begin{aligned}
\mathbf{x}(t) &= e^{\mathbf{A}t}\mathbf{x}(0) + e^{\mathbf{A}t}\mathbf{A}^{-1}\left(\mathbf{I} - e^{-\mathbf{A}t}\right)\mathbf{B}\mathbf{k} \\
&= e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}\left(e^{\mathbf{A}t} - \mathbf{I}\right)\mathbf{B}\mathbf{k}
\end{aligned}}$$

Exercise

- Obtain the time response of the following system: *u(t)=1(t)*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u \qquad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}0\\ 1\end{bmatrix} \qquad e^{\mathbf{A}t}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

$$\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)+\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}1(\tau)d\tau$$

$$=\int_0^t\begin{bmatrix}2e^{-(t-\tau)}-e^{-2(t-\tau)} & e^{-(t-\tau)}-e^{-2(t-\tau)}\\ -2e^{-(t-\tau)}+2e^{-2(t-\tau)} & -e^{-(t-\tau)}+2e^{-2(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}[1]d\tau$$

$$=\int_0^t\begin{bmatrix}e^{-(t-\tau)}-e^{-2(t-\tau)}\\ -e^{-(t-\tau)}+2e^{-2(t-\tau)}\end{bmatrix}d\tau=\begin{bmatrix}\left[e^{-(t-\tau)}-0.5e^{-2(t-\tau)}\right]_0^t\\ \left[-e^{-(t-\tau)}+e^{-2(t-\tau)}\right]_0^t\end{bmatrix}=\begin{bmatrix}[1-0.5]-\left[e^{-t}-0.5e^{-2t}\right]\\ [-1+1]-\left[-e^{-t}+e^{-2t}\right]\end{bmatrix}$$

$$=\begin{bmatrix}0.5-e^{-t}+0.5e^{-2t}\\ e^{-t}-e^{-2t}\end{bmatrix}$$

Exercise (Alternative)

- Or you can also use the analytical expression:

$$\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{A}^{-1}\left(e^{\mathbf{A}t} - \mathbf{I}\right)\mathbf{B}u}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{A}^{-1} = \frac{1}{2}\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad e^{\mathbf{A}t} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\begin{aligned}
\mathbf{x}(t) &= \mathbf{A}^{-1}\left(e^{\mathbf{A}t} - \mathbf{I}\right)\mathbf{B}u \\
&= \mathbf{A}^{-1}\left(e^{\mathbf{A}t} - \mathbf{I}\right)\mathbf{B}\left[1(t)\right] = \mathbf{A}^{-1}\left(\begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} - \mathbf{I}\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \mathbf{A}^{-1}\begin{bmatrix} 2e^{-t} - e^{-2t} - 1 & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} - 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} - 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} - 1 \end{bmatrix} = \begin{bmatrix} 0.5\left(-3e^{-t} + 3e^{-2t} + e^{-t} - 2e^{-2t} + 1\right) \\ e^{-t} - e^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}
\end{aligned}$$

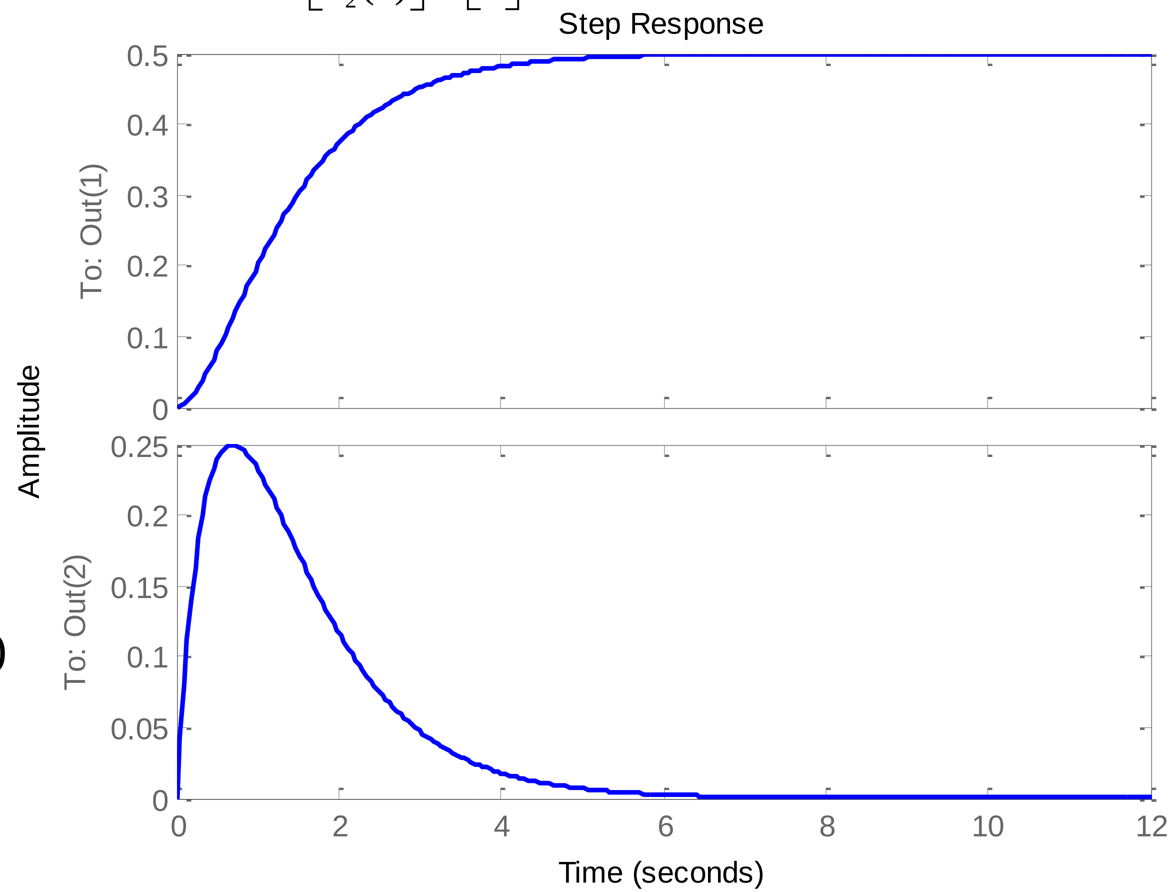
Exercise (Visualization)

- So what does the unit step response look like?
  - All initial conditions zero $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\mathbf{x}(t) = \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$x_1(t)$

$x_2(t)$



Exercise (Extra Visualization!)

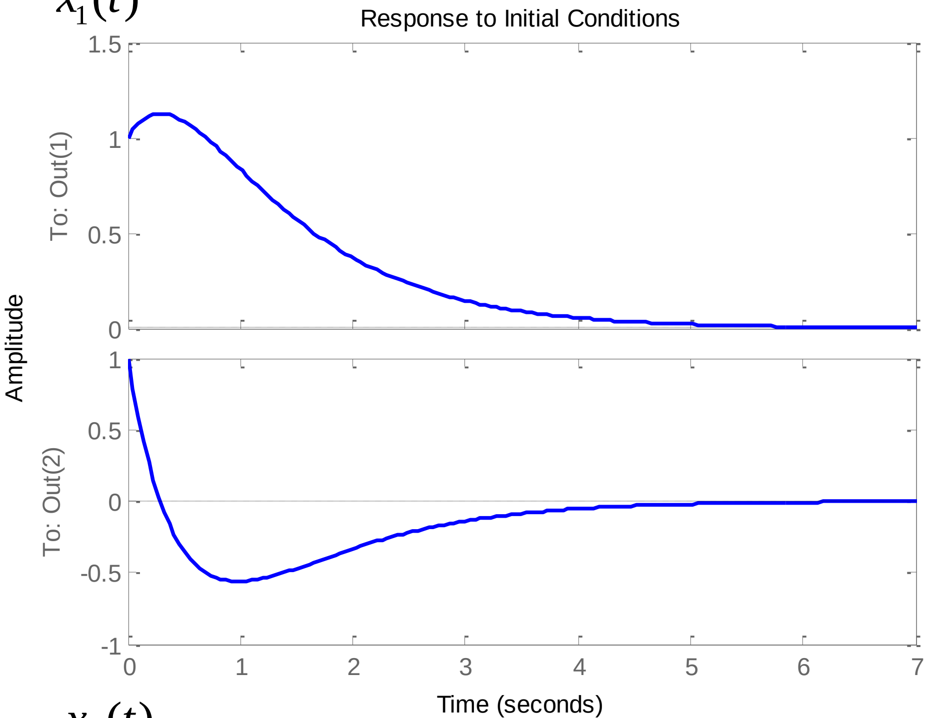
- So what happens if you have non-zero initial conditions and an input?
  - All initial conditions are 1 $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
  - Input is unit step $u(t) = 1(t)$

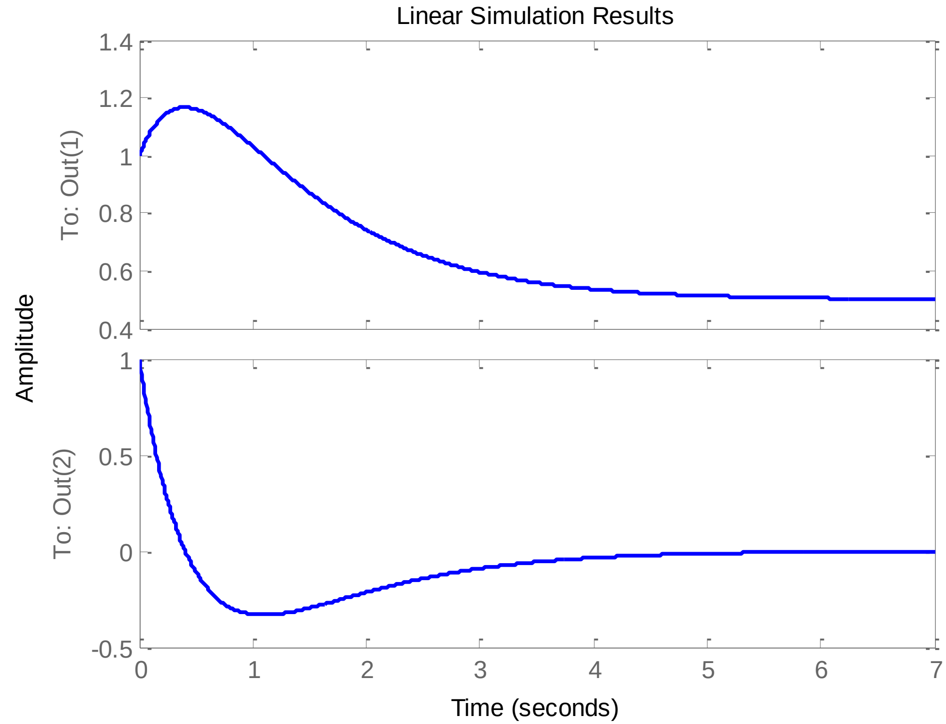
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\boxed{u(t)=0 \qquad x_1(0) = 1, x_2(0) = 1}$$

$$\boxed{u(t)=1 \qquad x_1(0) = 1, x_2(0) = 1}$$

$x_1(t)$





$x_2(t)$

Analytical Solution to Other Inputs?

- What about:
  - If the inputs are of various magnitudes?
  - Impulse inputs? Or Ramp?
- This is an exercise in your next homework!