Pulse Transfer Function

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Introducing the Pulse Transfer Function

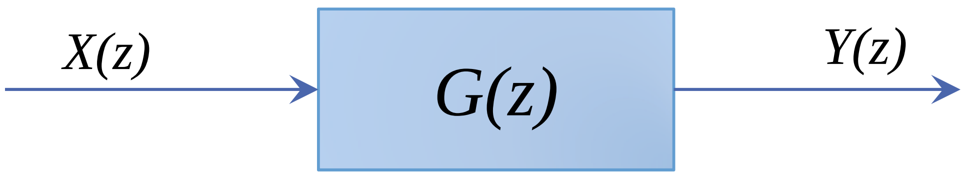
- While the transfer function relates the Laplace Transform of the continuous-time output to that of the continuous-time input, the pulse transfer function relates the z Transform of the sampled output to that of the sampled input

- Similar to the definition of the transfer function, the Pulse Transfer Function is expressed as:

$$G(z) = \frac{Y(z)}{X(z)}$$

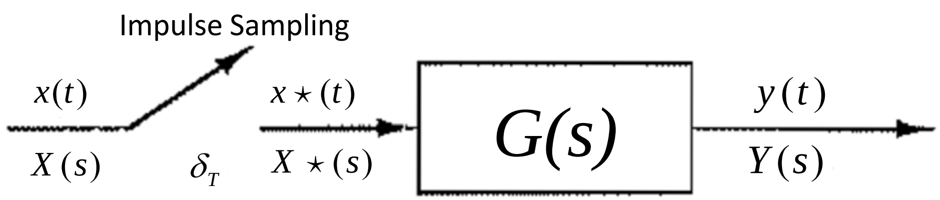
  - where *Y(z)* is the output and *X(z)* is the input.

- If *x(kT)* is the Kronecker delta input, *X(z)=1* and *Y(z)=G(z)*. Hence the Pulse Transfer Function is also defined as the *z* transform of the system's response to the Kronecker delta input
  - This is similar to the fact that the transfer function is also referred to as the Laplace Transform of the impulse-response of the system



Analyzing Mixed Systems (Sampled & Continuous)

- In analyzing discrete-time control systems it is common to find some signals that are impulse-sampled (Starred) and some are not (continuous)
- To obtain the pulse transfer function in these cases, factoring out the impulse-sampled signals is important
- Consider the following impulse-sampled system:



  - The output can be expressed by: $Y(s) = G(s)X\star(s)$
  - If we continue to take **starred LT** on both sides,

$$Y\star(s) = \left[G(s)X\star(s)\right]\star$$

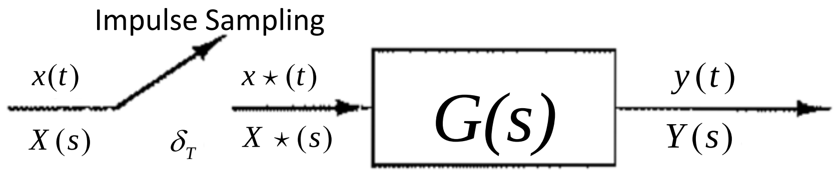
  - We may factor out the $X\star(s)$ Why? This requires going back to the principles of convolution for multiplication of 2 LT terms
  - So that:

$$\boxed{Y\star(s) = \left[G(s)\right]\star X\star(s) = G\star(s)X\star(s)}$$

$$\boxed{Y(z) = G(z)X(z)} \quad \text{with } z = e^{sT}$$

Importance of Impulse Sampling

- The **presence** of the **impulse sampler** is important in deriving pulse transfer functions!
- Compare the following 2 systems:

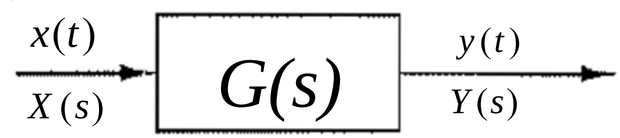


$$\boxed{\frac{Y(z)}{X(z)} = G(z) = \mathcal{Z}\left[G(s)\right]}$$

Only when input to *G(s)* is an impulse-sampled signal!

Hence when deriving pulse transfer functions, the presence of the sampler at the input element is assumed.

The presence of absence of a sampler at the output does not affect the pulse transfer function as even if the sampler is not present, you could also assume a fictitious sampler present at the output
(Even if the output is continuous, you could just collect the sequence)



$$\frac{Y(s)}{X(s)} = G(s)$$

$$\frac{Y(z)}{X(z)} \neq \mathcal{Z}\left[G(s)\right]$$

Why?

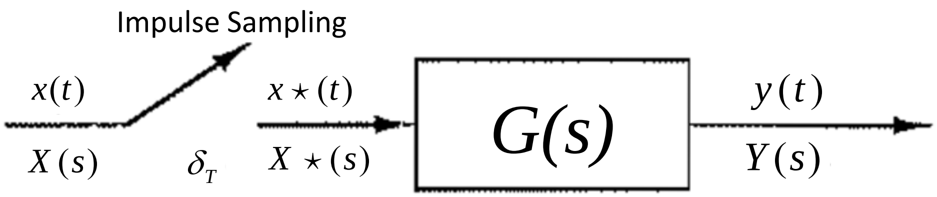
$$Y(s) = G(s)X(s)$$

$$Y\star(s) = \left[G(s)X(s)\right]\star = \left[GX(s)\right]\star$$

$$\boxed{Y(z) = \mathcal{Z}\left[Y(s)\right] = \mathcal{Z}\left[G(s)X(s)\right] = GX(z) \neq G(z)X(z)}$$

Exercise

- Consider the system described in the block diagram and *G(s)=1/(s+a).* What is the pulse transfer function *G(z)?*

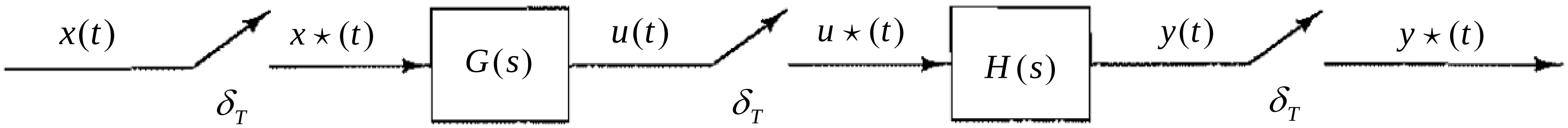


As there is an impulse sampler in the input of *G(s),*

$$G(z) = \mathcal{Z}[G(s)]$$
$$= \mathcal{Z}\left[\frac{1}{s+a}\right] = \frac{1}{1-e^{-at}z^{-1}}$$

Pulse Transfer Function of Cascaded Elements

- Let's consider systems with cascaded elements. Here the samplers are all synchronized and have the same sampling period.



Since there is an impulse sampler in the input of *G(s)*,

$$U(s) = G(s)X\star(s)$$
$$U\star(s) = G\star(s)X\star(s)$$

Since there is an impulse sampler in the input of *H(s)*,

$$Y(s) = H(s)U\star(s)$$
$$Y\star(s) = H\star(s)U\star(s)$$

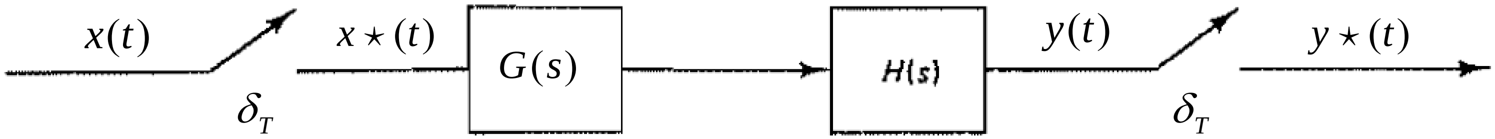
$$\Rightarrow Y\star(s) = H\star(s)G\star(s)X\star(s)$$

$$\boxed{\begin{aligned} &Y(z) = H(z)G(z)X(z) \\ &\frac{Y(z)}{X(z)} = G(z)H(z) \end{aligned}}$$

This is the effective pulse transfer function between output $y\star(t)$ to input $x\star(t)$

Pulse Transfer Function of Cascaded Elements

- Let's consider a system with the middle sampler missing:



$$Y(s) = G(s)H(s)X\star(s)$$

Because there is no impulse sampler in the input of *H(s),*

$$Y(s) = GH(s)X\star(s)$$

$$Y\star(s) = [GH(s)]\star X\star(s)$$

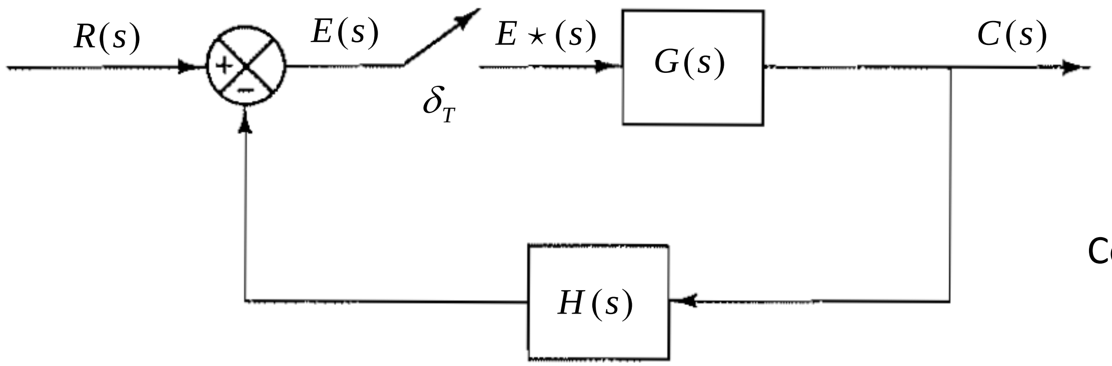
$$Y(z) = GH(z)X(z)$$

$$\boxed{\frac{Y(z)}{X(z)} = GH(z) \neq G(z)H(z)}$$

This is the effective pulse transfer function between output $y\star(t)$ to input $x\star(t)$

Pulse Transfer Function of Closed-Loop Systems

- As with cascaded systems, the existence or nonexistence of impulse samplers affects the pulse transfer function of closed-loop systems
- The most common implementation is when the error signal is sampled:



$$E(s) = R(s) - H(s)C(s)$$
$$C(s) = G(s)E\star(s)$$

Combining:

$$E(s) = R(s) - H(s)G(s)E\star(s)$$
$$E(s) = R(s) - G(s)H(s)E\star(s)$$

Taking starred LT: $E\star(s) = R\star(s) - GH\star(s)E\star(s)$

$$E\star(s) = \frac{R\star(s)}{1+GH\star(s)}$$

$$C\star(s) = \frac{G\star(s)R\star(s)}{1+GH\star(s)}$$

$$\frac{C\star(s)}{R\star(s)} = \frac{G\star(s)}{1+GH\star(s)}$$

$$\boxed{\Rightarrow \frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}}$$

Closed-Loop Pulse Transfer Function (CLPTF)

Pulse Transfer Function of Digital Controllers

- Recall that the input and output of a controller is the error *e(k)* and control output *m(k)* and can be expressed as a difference equation:

$$m(k)+a_1 m(k-1)+a_2 m(k-2)+\cdots+a_n m(k-n)=b_0 e(k)+b_1 e(k-1)+\cdots+b_n e(k-n)$$

- Taking the z transform:

$$M(z)+a_1 z^{-1}M(z)+a_2 z^{-2}M(z)+\cdots+a_n z^{-n}M(z)=b_0 E(z)+b_1 z^{-1}E(z)+\cdots+b_n z^{-n}E(z)$$

- The pulse transfer function of the digital controller is:

$$G_D(z)=\frac{M(z)}{E(z)}=\frac{b_0+b_1 z^{-1}+\cdots b_n z^{-n}}{1+a_1 z^{-1}+\cdots a_n z^{-n}}$$

- What is the pulse transfer function of a Digital PID controller?

$$m(t)=K\left[e(t)+\frac{1}{T_i}\int_0^t e(t)dt+T_d\frac{de(t)}{dt}\right]$$

Proportional Integral Derivative

Pulse Transfer Function of PID Controller

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right] = m_p(t) + m_i(t) + m_d(t)$$

- Let's consider the Proportional term: $m_p(t) = Ke(t)$
  - Discretizing: $m_p(kT) = Ke(kT)$
  - Taking zT: $\boxed{M_p(z) = KE(z)}$

- Let's consider the Derivative term: $m_d(t) = KT_d\frac{de(t)}{d(t)}$
  - Discretizing (using backward approximation): $m_d(kT) = KT_d\left[\frac{e(kT) - e((k-1)T)}{T}\right]$
  - Taking zT: $\boxed{M_d(z) = KT_d\left[\frac{E(z) - z^{-1}E(z)}{T}\right] = K\frac{T_d}{T}\left(1 - z^{-1}\right)E(z)}$

Pulse Transfer Function of PID Controller

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right] = m_p(t) + m_i(t) + m_d(t)$$

- Let's consider the Integral term: $m_i(t) = K\frac{1}{T_i}\int_0^t e(t)dt$
  - Discretizing (using trapezoidal approximation):

$$m_i(kT) = K\frac{T}{T_i}\left[\frac{e(0)+e(T)}{2} + \frac{e(T)+e(2T)}{2} + \cdots + \frac{e((k-1)T)+e(kT)}{2}\right]$$

$$= K\frac{T}{T_i}\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}$$

  - Defining: $f(hT) = \dfrac{e((h-1)T)+e(hT)}{2}$, $f(0) = 0$ $\quad F(z) = \mathcal{Z}[f(hT)] = \dfrac{1+z^{-1}}{2}E(z)$
  - Taking zT: $\mathcal{Z}\left[K\frac{T}{T_i}\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}\right] = K\frac{T}{T_i}\mathcal{Z}\left[\sum_{h=1}^{k}f(hT)\right]$

$$= K\frac{T}{T_i}\left[\frac{1}{1-z^{-1}}\left(F(z)-f(0)\right)\right] = K\frac{T}{T_i}\frac{1}{1-z^{-1}}F(z)$$

Pulse Transfer Function of PID Controller

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_0^t e(t)dt + T_d\frac{de(t)}{dt}\right] = m_p(t) + m_i(t) + m_d(t)$$

- Hence:

$$\mathcal{Z}\left[K\frac{T}{T_i}\sum_{h=1}^{k}\frac{e((h-1)T)+e(hT)}{2}\right] = K\frac{T}{T_i}\frac{1}{1-z^{-1}}\frac{1+z^{-1}}{2}E(z)$$

$$M_i(z) = K\frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}}E(z)$$

- Consolidating: $M(z) = M_p(z) + M_i(z) + M_d(z)$

$$= K\left[1 + \frac{T}{2T_i}\frac{1+z^{-1}}{1-z^{-1}} + \frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$= K\left[1 - \frac{T}{2T_i} + \frac{T}{T_i}\frac{1}{1-z^{-1}} + \frac{T_d}{T}(1-z^{-1})\right]E(z)$$

$$\boxed{\frac{M(z)}{E(z)} = K_P + \frac{K_I}{1-z^{-1}} + K_D\left(1-z^{-1}\right)}$$

$$K_P = K - \frac{KT}{2T_i} = K - \frac{K_I}{2} = \text{proportional gain}$$

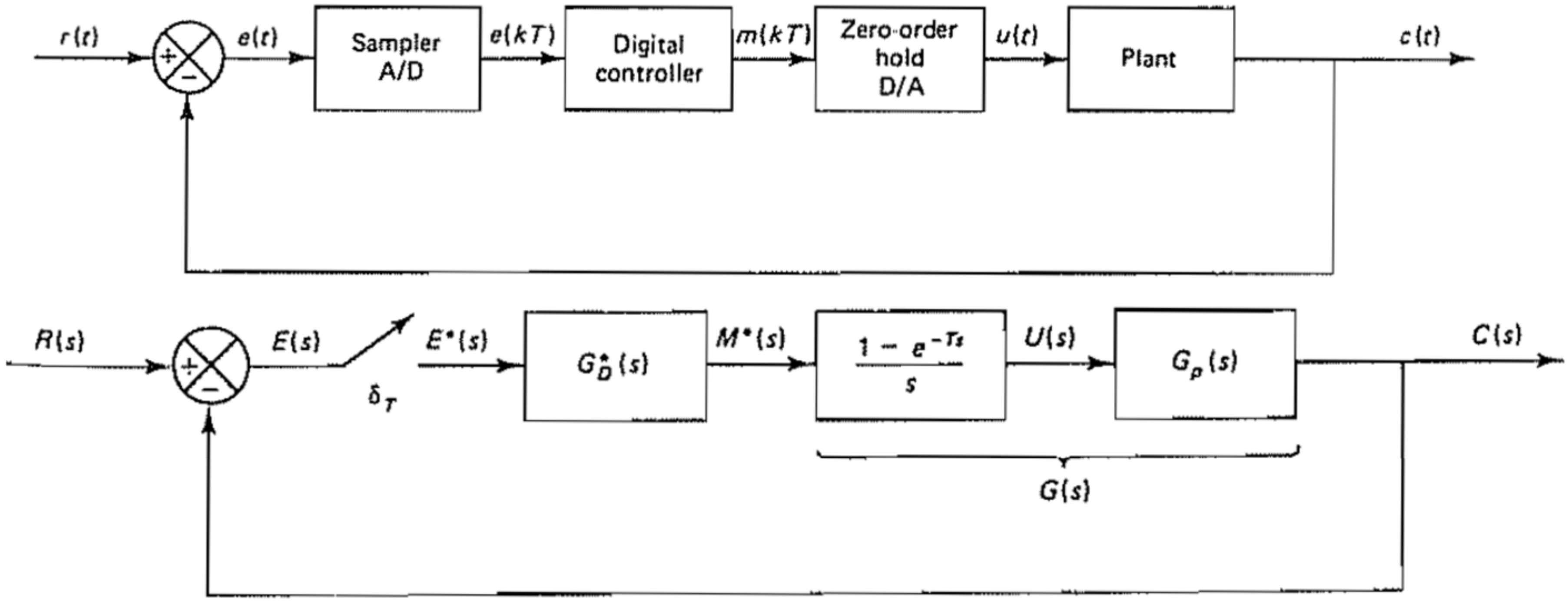
$$K_I = \frac{KT}{T_i} = \text{integral gain}$$

$$K_D = \frac{KT_d}{T} = \text{derivative gain}$$

Note: Proportional gain for **digital** PID controller is different than the **analog** proportional gain due to the numerical integration approximation

CL Pulse Transfer Function of a Digital Control System

- Consider a typical application of a digital control system where a sampler and ADC feeds digital signal into the digital controller which in turn produces control action through a ZOH to the plant



$$G(s) = \frac{1-e^{-sT}}{s} G_p(s)$$

$$C(s) = G(s) G_D \star (s) E \star (s)$$

$$C \star (s) = G \star (s) G_D \star (s) E \star (s)$$

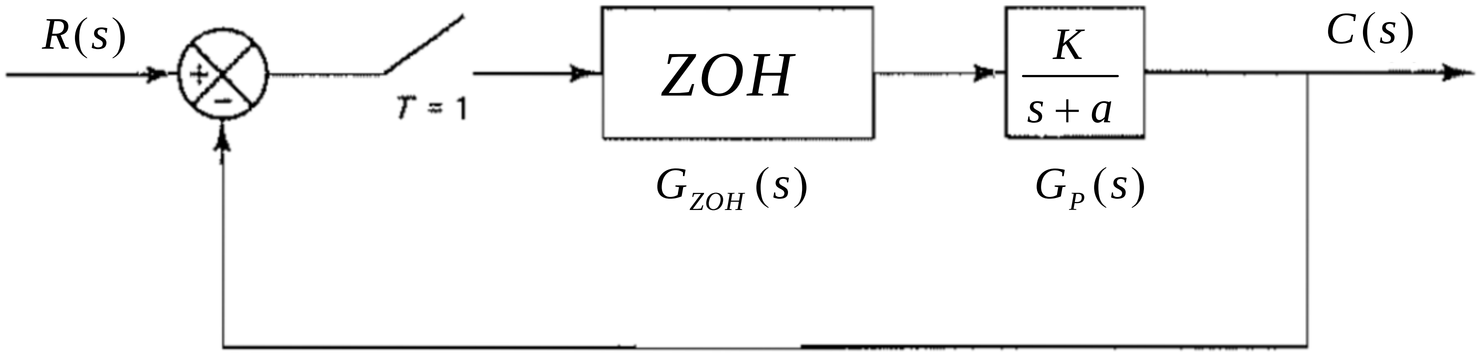
$$\Rightarrow C(z) = G(z) G_D(z) E(z)$$

$$E(z) = R(z) - C(z)$$

$$\boxed{\Rightarrow \frac{C(z)}{R(z)} = \frac{G_D(z) G(z)}{1 + G_D(z) G(z)}}$$

Exercise

- What is the closed-loop pulse transfer function of the system below?



$$G(z) = \mathcal{Z}\left[G_{ZOH}G_P(s)\right] = (1-z^{-1})\mathcal{Z}\left[\frac{G_P(s)}{s}\right]$$

$$= (1-z^{-1})\mathcal{Z}\left[\frac{K}{s(s+a)}\right] = (1-z^{-1})\mathcal{Z}\left[\frac{K}{a}\frac{a}{s(s+a)}\right]$$

$$= (1-z^{-1})\frac{K}{a}\left[\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}\right] = \frac{K}{a}\frac{(1-e^{-aT})z^{-1}}{(1-e^{-aT}z^{-1})}$$

$$T = 1$$

$$G(z) = \frac{K}{a}\frac{(1-e^{-a})z^{-1}}{(1-e^{-a}z^{-1})}$$

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1+G(z)} = \frac{\frac{K}{a}\frac{(1-e^{-a})z^{-1}}{(1-e^{-a}z^{-1})}}{1+\frac{K}{a}\frac{(1-e^{-a})z^{-1}}{(1-e^{-a}z^{-1})}} = \frac{K(1-e^{-a})z^{-1}}{a(1-e^{-a}z^{-1})+K(1-e^{-a})z^{-1}}$$