State-Space Method

EPD 30.114 ADVANCED FEEDBACK & CONTROL

State-Space Methodology

- State-variable approach of describing differential equations
  - Often referred to as **Modern Control Theory**, as opposed to **Classical Control Theory** (Transfer Functions, Root Locus, Frequency response)
  - Dynamic system is organized as a set of (only) **first-order** differential equations
  - ODEs do not need to be linear or time-invariant (compared to TF (LTI) approach)
  - Easily extended to MIMO systems (TF is SISO)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_r \end{bmatrix}$$

State Vector, Output Vector, Input Vector

- State-Space Representation
  - Non-linear, Time Varying

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t)$$
$$\mathbf{y} = \mathbf{g}(\mathbf{x}, \mathbf{u}, t)$$

  - Linear, Time-Invariant (LTI)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nr} \end{bmatrix}$$

State Matrix, Input Matrix

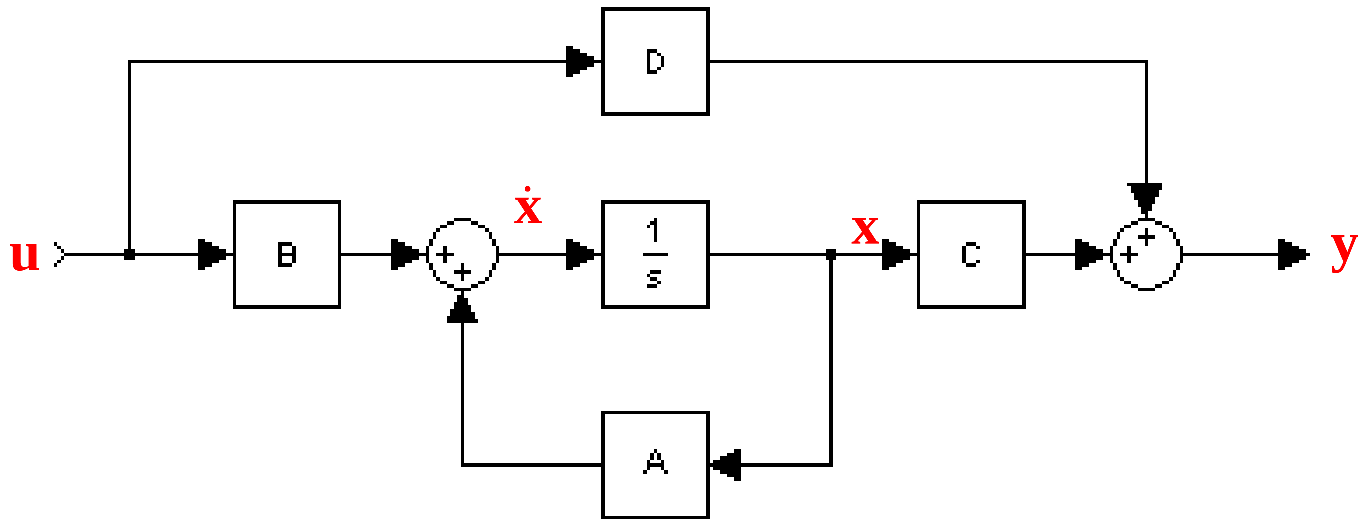
$$\mathbf{C} = \begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{m1} & \cdots & c_{mn} \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} d_{11} & \cdots & d_{1r} \\ \vdots & \ddots & \vdots \\ d_{m1} & \cdots & d_{mr} \end{bmatrix}$$

Output Matrix, Direct Transmission Matrix

Block Diagram Representation & Transfer Matrix

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$



- The Transfer Matrix (extension of the transfer function) is $\mathbf{G}(s) = \dfrac{\mathbf{Y}(s)}{\mathbf{U}(s)}$

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

Why?

Derivation of the Transfer Matrix

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

- Take Laplace Transform:

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s)$$

- Zero initial conditions (Property of Transfer Matrix):

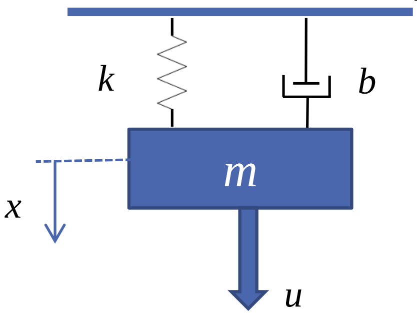
$$s\mathbf{X}(s) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s)$$
$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{X}(s) = \mathbf{B}\mathbf{U}(\mathrm{s})$$
$$\mathbf{X}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(\mathrm{s})$$

- Combining:

$$\mathbf{Y}(s) = \mathbf{C}\left[\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}\mathbf{U}(\mathrm{s})\right] + \mathbf{D}\mathbf{U}(s)$$
$$= \left[\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right]\mathbf{U}(s)$$
$$\frac{\mathbf{Y}(s)}{\mathbf{U}(s)} = \mathbf{G}(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}$$

Constructing State-Space Models

- Define state variables (can be arbitrary)
  - In general, the total order of the system determines the required number of state variables. (number of integrators)
- Consider the Mass-Spring-Damper System under force input, *u*.



Equation of Motion (EOM):

$$m\ddot{x} + b\dot{x} + kx = u$$

- System is 2nd order, so 2 state variables are needed:

$$x_1 = x$$
$$x_2 = \dot{x}_1 = \dot{x}$$

- Combining & Rearranging:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{x} = \frac{1}{m}\left(-kx - b\dot{x}\right) + \frac{1}{m}u$$

$$= \frac{1}{m}\left(-kx_1 - bx_2\right) + \frac{1}{m}u$$

$$x = x_1$$

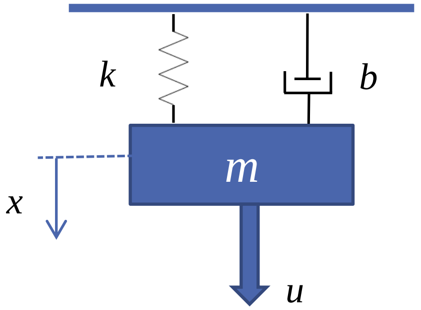
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u$$

$$x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Understanding State-Space Models

$x_1 = x$

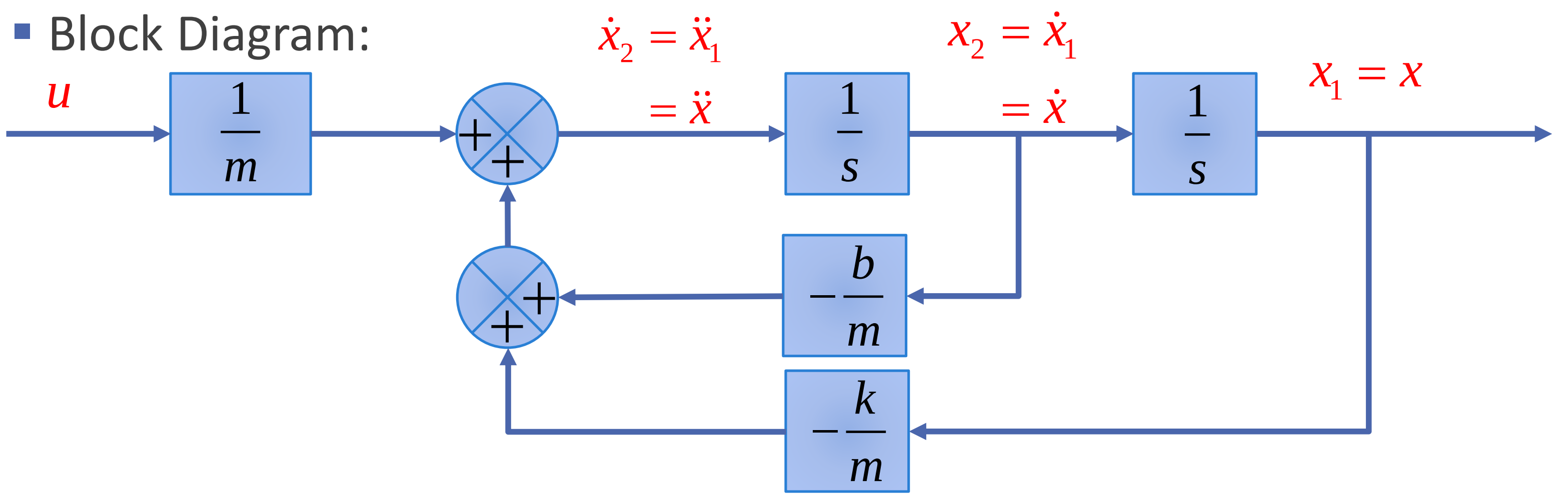
$x_2 = \dot{x}_1 = \dot{x}$



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

$$x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- Here x_1 is displacement and x_2 is velocity
- Also: $\dot{x}_2 = \left(-\dfrac{k}{m}\right) x_1 + \left(-\dfrac{b}{m}\right) x_2 + \dfrac{1}{m} u$ and $x_1 = x,\ x_2 = \dot{x}_1 = \dot{x}$
- Block Diagram:



State-Space Modelling with Input Derivatives

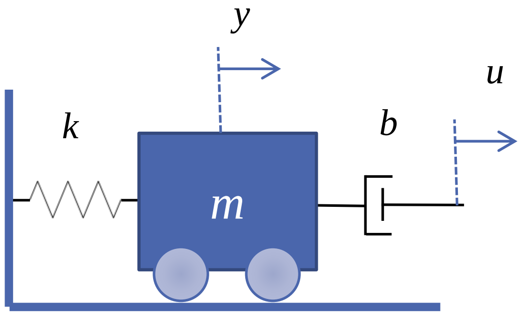
- We have been able to model systems in the form of:

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = bu$$

- But what happens if the system contains derivatives of the input?

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

- What is the equation of motion and SS model of the following system?



$$m\ddot{y} + b\dot{y} + ky = b\dot{u}$$

Choose state variables: $x_1 = y$

$x_2 = \dot{x}_1 = \dot{y}$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{y} = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{u}$$

State-Space Modelling with Input Derivatives

- Okay. Let's try to eliminate the derivative of *u*:
  - Let's try by defining the state variables differently

$$x_1 = y$$

$$x_2 = \dot{y} - \frac{b}{m}u = \dot{x}_1 - \frac{b}{m}u$$

$$\begin{aligned}
\dot{x}_2 &= \ddot{y} - \frac{b}{m}\dot{u} \\
&= -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} - \frac{b}{m}\dot{u} \\
&= -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right) \\
&= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u
\end{aligned}$$

- Now the State-Space Model can be expressed as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{b}{m} \\ -\left(\frac{b}{m}\right)^2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

State-Space Modelling with Input Derivatives

- For a system described by the differential equation:

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

which can also be expressed as a Transfer Function:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

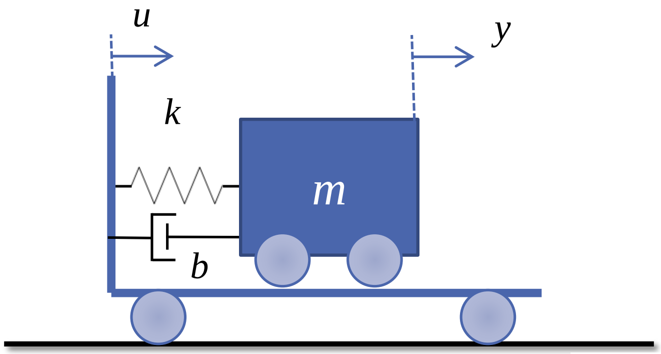
- One possible State-Space Representation (Controllable Canonical Form) is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Exercise!

- Obtain the State Space Representation of the following System:



$$EOM : m\ddot{y} + b\dot{y} + ky = b\dot{u} + ku$$

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

Compare to: $$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

$$a_1 = \frac{b}{m},\ a_2 = \frac{k}{m}$$

$$b_0 = 0,\ b_1 = \frac{b}{m}\ ,\ b_2 = \frac{k}{m}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \ \vdots\ b_{n-1} - a_{n-1} b_0 \ \vdots\ \cdots \ \vdots\ b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \frac{k}{m} & \frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$