Singapore University of Technology & Design
Engineering Product Development
30.114 Advanced Feedback & Control – Fall 2023

Homework #1

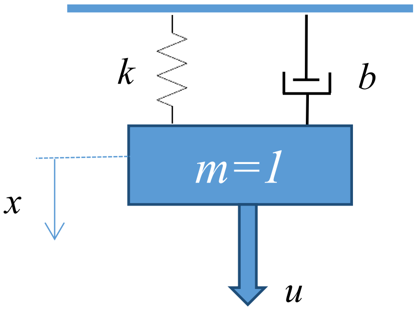
1. Recall the following state space systems that supposedly represented the mass spring damper system (it is slightly different from the example in class). First determine the EOM of the system below and by determining the **Transfer Matrix** for each system, verify that that is the case. Can you also identify the canonical form of each expression, if any? (are they CCF? OCF?)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -k & -b\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$x=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

CCF

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-b & -k\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$
$$x=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Variation of CCF



$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -k\\ 1 & -b\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$
$$x=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

OCF

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-b & 1\\ -k & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$
$$x=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Variation of OCF

Transfer Matrix = $\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$

For all systems, the transfer matrix is the same! (They represent the same system and TF)

$$\mathbf{G}(s)=\mathbf{C}\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}=\begin{bmatrix}1 & 0\end{bmatrix}\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ -k & -b\end{bmatrix}\right)^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}$$
$$=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s & -1\\ k & s+b\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}=\frac{1}{s^2+bs+k}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s+b & 1\\ -k & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}$$
$$=\frac{1}{s^2+bs+k}\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}1\\ s\end{bmatrix}=\frac{1}{s^2+bs+k}$$

2. A car's suspension system can be described with the following Transfer Function. Find the state space representation of the system in modal form and construct the block diagram.

$$G(s)=\frac{s+4}{s^4+3s^3+2s^2}=\frac{s+4}{s^2\left(s^2+3s+2\right)}=\frac{s+4}{s^2\left(s+1\right)\left(s+2\right)}$$
$$=2\left[\frac{1}{s^2}\right]-\frac{5}{2}\left[\frac{1}{s}\right]+3\left[\frac{1}{s+1}\right]-\frac{1}{2}\left[\frac{1}{s+2}\right]$$

There are 2 repeated eigenvalues (poles) and 2 distinct eigenvalues. Will be a Jordan Canonical Form.

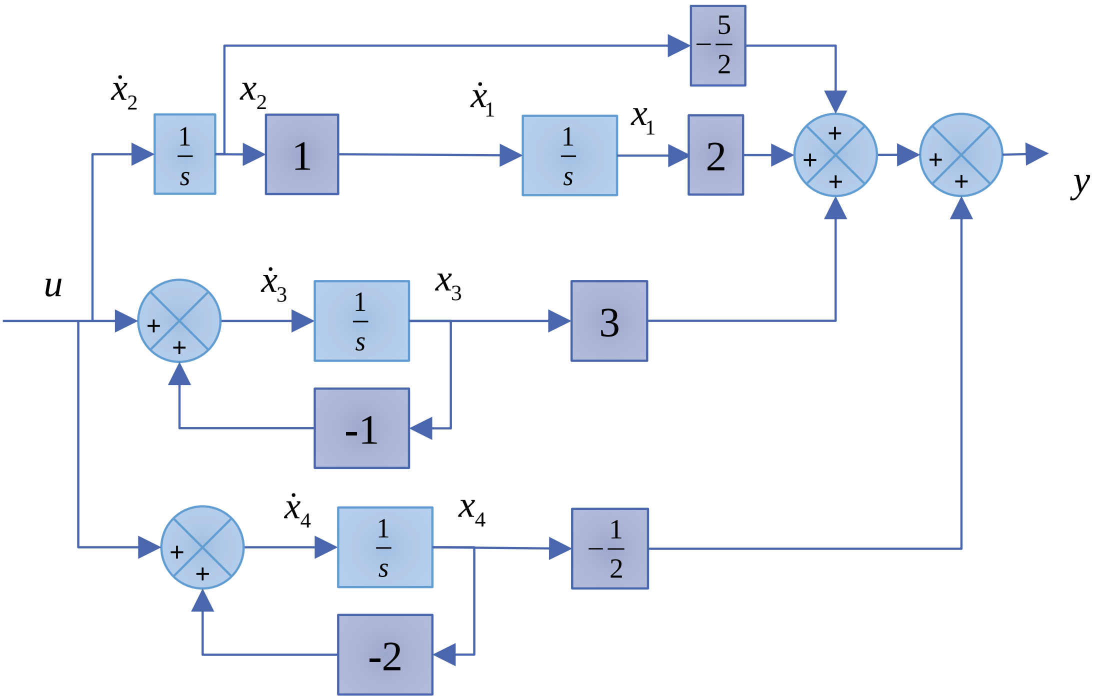
JCF

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\\\dot{x}_4\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}+\begin{bmatrix}0\\1\\1\\1\end{bmatrix}u$$

$$y=\begin{bmatrix}2&-\frac{5}{2}&3&-\frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

$$\dot{x}_1=1x_2$$
$$\dot{x}_2=u$$
$$\dot{x}_3=-x_3+u$$
$$\dot{x}_4=-2x_4+u$$

$$y=2x_1-\frac{5}{2}x_2+3x_3-\frac{1}{2}x_4$$



3. Recall from the second problem of the Quiz. We wish to transform that system to observable canonical form (OCF). Determine the transformation matrix $\mathbf{T}$ to achieve this and the resultant OCF.

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}1&1\\0&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}u$$

$$y=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

To convert to OCF, $\mathbf{O}_B=\begin{bmatrix}\mathbf{H}\\\mathbf{HF}\end{bmatrix}$ needs to be invertible

$\begin{bmatrix}\mathbf{H}\\\mathbf{HF}\end{bmatrix}=\begin{bmatrix}1&0\\1&1\end{bmatrix}$. Observability matrix is full rank. Transformation exists.

$\begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. The Transformation matrix is: $\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2] = [\mathbf{t}_1 \quad \mathbf{Ft}_1]$

Constructing $\mathbf{t}_1 = \mathbf{O}_B{}^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{t}_2 = \mathbf{Ft}_1 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

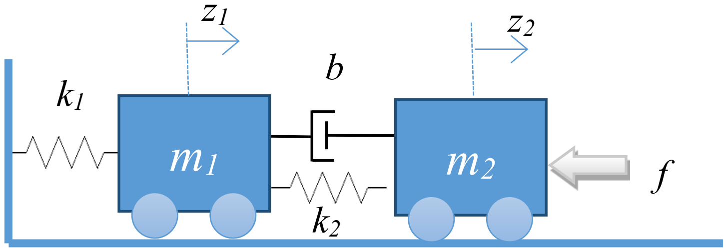
$\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2] = [\mathbf{t}_1 \quad \mathbf{Ft}_1] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ and the inverse is: $\mathbf{T}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Check to see if this is correct:

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{FT} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \mathbf{B} = \mathbf{T}^{-1}\mathbf{G} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{C} = \mathbf{HT} = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

Transformed system indeed is in OCF.

4. Consider the following mechanical system. The displacements z_1 and z_2 are measured from respective equilibrium positions before the input force f is applied. Derive a State-Space representation of the system. The outputs of the system would be $\boldsymbol{y} = [z_1 \ \ z_2]^T$.



EOM:

$$m_1\ddot{z}_1 = -k_1 z_1 - k_2(z_1 - z_2) - b(\dot{z}_1 - \dot{z}_2)$$
$$m_2\ddot{z}_2 = -k_2(z_2 - z_1) - b(\dot{z}_2 - \dot{z}_1) - f$$

Let: $x_1 = z_1, x_2 = \dot{z}_1, x_3 = z_2, x_4 = \dot{z}_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(k_1+k_2)}{m_1} & -\dfrac{b}{m_1} & \dfrac{k_2}{m_1} & \dfrac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_2}{m_2} & \dfrac{b}{m_2} & -\dfrac{k_2}{m_2} & -\dfrac{b}{m_2} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{1}{m_2} \end{bmatrix} f$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

5. For the following state matrix **A**,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

a. Find the eigenvalues and eigenvectors of **A**.

$$\left|\lambda\mathbf{I} - \mathbf{A}\right| = \left\| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right\| = \left\| \begin{matrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{matrix} \right\|$$

$$= \lambda^2(\lambda+6) + 6 + 11\lambda = \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$= (\lambda+1)(\lambda+2)(\lambda+3)$$

$$\lambda = -1, -2, -3$$

The eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ and eigenvectors are $\mathbf{v}_1$,$\mathbf{v}_2$ and $\mathbf{v}_3$.

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = 0$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)\mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$

$$v_{11} + v_{12} = 0$$
$$v_{12} + v_{13} = 0$$
$$-6v_{11} - 11v_{12} - 5v_{13} = 0$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{v}_2 = 0$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \right)\mathbf{v}_2 = 0$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -6 & -11 & -4 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$2v_{21} + v_{22} = 0$$
$$2v_{22} + v_{23} = 0$$
$$-6v_{21} - 11v_{22} - 4v_{23} = 0$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{v}_3 = 0$$

$$\left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right)\mathbf{v}_3 = 0$$

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -6 & -11 & -3 \end{bmatrix}\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = 0$$

$$3v_{31} + v_{32} = 0$$
$$3v_{32} + v_{33} = 0$$
$$-6v_{31} - 11v_{32} - 3v_{33} = 0$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

b. Find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{AP}$ is a diagonal matrix. What are the numerical values for the diagonal elements of the diagonal matrix?

$$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

. They would be the eigenvalues: -1, -2 and -3.

c. Verify your answer in b. that $\mathbf{P}^{-1}\mathbf{AP}$ is diagonal.

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{AP} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

6. A simple pendulum system is shown below. An input torque u is applied to the pendulum (The torque is defined such that positive u results in negative θ). (gravity is acting downwards)

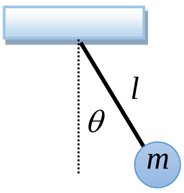
a. Derive the non-linear equation of motion.

$$J\ddot{\theta} + mgl\sin\theta = -u$$

EOM: $$ml^2\ddot{\theta} + mgl\sin\theta = -u$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = -\frac{1}{ml^2}u$$

b. Linearize the system about the equilibrium position and represent the system in state-space (Both in CCF and OCF).



Linearized EOM: $\ddot{\theta} + \frac{g}{l}\theta = -\frac{1}{ml^2}u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} -\frac{1}{ml^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

CCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{g}{l} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{ml^2} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

OCF

7. Compute $e^{\mathbf{A}t}$ where $\mathbf{A} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

A is already in Jordan form: $e^{\mathbf{A}t}=\begin{bmatrix} e^{2t} & te^{2t} & 0.5t^2e^{2t} \\ 0 & e^{2t} & te^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$

8. Obtain the response *y(t)* of the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}=\begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} 0.5 \\ 0 \end{bmatrix}u \qquad \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where *u(t)* is the unit step input.

$$\left(s\mathbf{I}-\mathbf{A}\right)^{-1}=\begin{bmatrix} s+1 & 0.5 \\ -1 & s \end{bmatrix}^{-1}=\frac{1}{s^2+s+0.5}\begin{bmatrix} s & -0.5 \\ 1 & s+1 \end{bmatrix}$$

$$=\begin{bmatrix} \dfrac{s+0.5-0.5}{\left(s+0.5\right)^2+0.5^2} & \dfrac{-0.5}{\left(s+0.5\right)^2+0.5^2} \\ \dfrac{1}{\left(s+0.5\right)^2+0.5^2} & \dfrac{s+0.5+0.5}{\left(s+0.5\right)^2+0.5^2} \end{bmatrix}$$

$$e^{\mathbf{A}t}=\mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\right]=\begin{bmatrix} e^{-0.5t}\left(\cos 0.5t-\sin 0.5t\right) & -e^{-0.5t}\sin 0.5t \\ 2e^{-0.5t}\sin 0.5t & e^{-0.5t}\left(\cos 0.5t+\sin 0.5t\right) \end{bmatrix}$$

$$\mathbf{x}(t)=e^{\mathbf{A}t}\mathbf{x}(0)+\mathbf{A}^{-1}\left(e^{\mathbf{A}t}-\mathbf{I}\right)\mathbf{B}$$

$$=\begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 0.5e^{-0.5t}\left(\cos 0.5t-\sin 0.5t\right)-0.5 \\ e^{-0.5t}\sin 0.5t \end{bmatrix}$$

$$=\begin{bmatrix} e^{-0.5t}\sin 0.5t \\ -e^{-0.5t}\left(\cos 0.5t+\sin 0.5t\right)+1 \end{bmatrix}$$

$$y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=x_1=e^{-0.5t}\sin 0.5t$$