Computing Matrix Exponentials

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Various Approaches to Compute e^{At}

- As you have seen by now, the matrix exponential plays a crucial role in solving for the state equations
- The general expression is:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \frac{\mathbf{A}^4 t^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$$

- This is an **infinite** series! Unless **A** is well behaved (equals zero for high order terms), it is difficult to use this expression
- However, there are a number of alternative approaches you can use:
  - Method1: When **A** is in diagonal form [Easy]
  - Method2: Using Laplace Transform relationship [Harder]

Method 1

- If **A** is a diagonal matrix, with each diagonal element of $-p_1$, $-p_2$, …, $-p_n$
  - No repeated eigenvalues (if there is, it wouldn't be diagonal: see Jordan form)

$$\mathbf{A} = \begin{bmatrix} -p_1 & 0 & \cdots & 0 \\ 0 & -p_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & -p_n \end{bmatrix}$$

- The matrix exponential is simply:

$$e^{\mathbf{A}t} = \begin{bmatrix} e^{-p_1 t} & 0 & \cdots & 0 \\ 0 & e^{-p_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & e^{-p_n t} \end{bmatrix}$$

- But most of the time **A** is never in diagonal form…
- But it doesn't matter! You can transform it to diagonal form!

Transforming to Diagonal Form

- Let's consider **A** to be a matrix (with distinct eigenvalues) that is not in Diagonal Form
- As we have found out, it is possible to use a linear transformation to convert it to diagonal form

$$\mathbf{D} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

D is the Diagonal Matrix and **T** is the transformation matrix. Remember that **T** is constructed using the eigenvectors of **A** and the diagonal entries of **D** are the eigenvalues of **A**.

- Rearranging: $\mathbf{A} = \mathbf{T}\mathbf{D}\mathbf{T}^{-1}$
- Hence:

$$e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1} = \mathbf{T}\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n t} \end{bmatrix}\mathbf{T}^{-1}$$

$\lambda_1, \lambda_2, \lambda_3, \ldots$ are eigenvalues of **A**

Supplemental Proof

- Why $e^{\mathbf{A}t} = \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$?

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots$$
$$= \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$$

$$\mathbf{A} = \mathbf{TDT}^{-1}$$

$$e^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{\left(\mathbf{TDT}^{-1}\right)^k t^k}{k!}$$
$$= \mathbf{I} + \mathbf{TDT}^{-1}t + \frac{\left(\mathbf{TDT}^{-1}\right)^2 t^2}{2!} + \frac{\left(\mathbf{TDT}^{-1}\right)^3 t^3}{3!} + \cdots$$
$$= \mathbf{TT}^{-1} + \mathbf{TDT}^{-1}t + \frac{\left(\mathbf{TDT}^{-1}\right)\left(\mathbf{TDT}^{-1}\right)t^2}{2!} + \frac{\left(\mathbf{TDT}^{-1}\right)\left(\mathbf{TDT}^{-1}\right)\left(\mathbf{TDT}^{-1}\right)t^3}{3!} + \cdots$$
$$= \mathbf{TT}^{-1} + \mathbf{TDT}^{-1}t + \frac{\mathbf{TD}^2\mathbf{T}^{-1}t^2}{2!} + \frac{\mathbf{TD}^3\mathbf{T}^{-1}t^3}{3!} + \cdots$$
$$= \mathbf{T}\left[\mathbf{I} + \mathbf{D}t + \frac{\mathbf{D}^2t^2}{2!} + \frac{\mathbf{D}^3t^3}{3!} + \frac{\mathbf{D}^4t^4}{4!} + \cdots\right]\mathbf{T}^{-1}$$
$$= \mathbf{T}e^{\mathbf{D}t}\mathbf{T}^{-1}$$

Exercise

Important

- Compute $e^{\mathbf{A}t}$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$$

A is not diagonal: Need to diagonalize:

Find Eigenvalues and Eigenvectors $|\mathbf{A} - \lambda \mathbf{I}| = 0$

Eigenvalues: λ_1=0 and λ_2=-2

Eigenvectors:

$$\mathbf{A}\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \ (\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v}_1 = 0$$

$$\mathbf{A}\mathbf{v}_2 = \lambda_2 \mathbf{v}_2, \ (\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v}_2 = 0$$

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\mathbf{T}^{-1} = \frac{1}{-2} \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 \\ 0 & -0.5 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathbf{T} e^{\mathbf{D}t} \mathbf{T}^{-1} = \mathbf{T} \begin{bmatrix} e^0 & 0 \\ 0 & e^{-2t} \end{bmatrix} \mathbf{T}^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.5\left(1 - e^{-2t}\right) \\ 0 & e^{-2t} \end{bmatrix}$$

Transforming to Jordan Form

- What happens if you can't diagonalize A? (repeated eigenvalues)
- Computation is still possible if the system can be transformed into the Jordan Form, $$\mathbf{J} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$$

where **J** is the matrix in Jordan form and **S** is the transformation matrix required to complete the transformation (constructed using ***generalized*** eigenvectors). λ is the repeating eigenvalue.

- In general, the Jordan form can be expressed as:

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \lambda & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix} = \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \lambda\mathbf{I} + \mathbf{N}$$

- The matrix **N** has a very special property:
  - Consider a 4x4 **N** matrix,

$$\mathbf{N} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{N}^{4,5,..} = 0$$

Transforming to Jordan Form

- With that property, it can be shown that since $\mathbf{A}=\mathbf{SJS}^{-1}$

$$e^{\mathbf{A}t}=\mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}$$

$$=\mathbf{S}\begin{bmatrix} e^{\lambda t} & \frac{1}{1!}te^{\lambda t} & \cdots & \frac{1}{(n-1)!}t^{n-1}e^{\lambda t} \\ 0 & e^{\lambda t} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \frac{1}{1!}te^{\lambda t} \\ 0 & 0 & \cdots & e^{\lambda t} \end{bmatrix}\mathbf{S}^{-1}$$

- E.g n=2, $\mathbf{J}=\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},\ e^{\mathbf{J}t}=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$
- E.g n=3, $\mathbf{J}=\begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix},\ e^{\mathbf{J}t}=\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2}e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}$

Computing Generalized Eigenvectors

- Some matrices are considered *defective* when they have repeated eigenvalues (Jordan form) because they have fewer linearly independent eigenvectors than eigenvalues.
- In such cases, generalized eigenvectors can be constructed
- For a matrix **A** which has repeated eigenvalue of λ, and you already computed the eigenvector v_1, the remaining generalized eigenvectors can be derived using:

$$(\mathbf{A} - \lambda\mathbf{I})v_2 = v_1$$

- Once you have v_2, you can obtain v_3 incrementally:

$$(\mathbf{A} - \lambda\mathbf{I})v_3 = v_2$$

- And so on… (if any)

$$(\mathbf{A} - \lambda\mathbf{I})v_4 = v_3$$

Quick Exercise

- Find the generalized eigenvectors of:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

$$\left|\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right| = 0$$

$$(1-\lambda)^2 = 0$$

$$\lambda = 1,\ 1$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}_1 = 0$$

$$\left[\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\mathbf{v}_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(\mathbf{A} - \mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{S} = [\mathbf{v}_1 \quad \mathbf{v}_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Method 2

- The second method uses the Laplace Transform Approach. Recall when deriving the solution of the state equations,

$$\mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{A}\right)^{-1}\right]=e^{\mathbf{A}t}$$

- This approach does not require **A** to be in any form but can be complicated if **A** is a large matrix as the inverse computation can be very involved.
- Thus to obtain the matrix exponential of **A**,
  - Invert (s**I**-**A**). This results in a matrix whose elements are rational functions of s
  - Take inverse Laplace Transform of each element of the matrix

Exercise

- Compute $e^{\mathbf{A}t}$ using the LT method. $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}$$

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{1}{s\left(s+2\right)} \begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} 1 & 0.5\left(1 - e^{-2t}\right) \\ 0 & e^{-2t} \end{bmatrix}$$

Extra Exercise

- Calculate e^{At} if $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Eigenvalues: λ_1= 2 and λ_2= 2 (repeated)

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

$$\left|\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix}\right| = 0$$

$$(1-\lambda)(3-\lambda)+1=0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda-2)(\lambda-2)=0$$

$$\boxed{e^{\mathbf{A}t} = \mathbf{S}e^{\mathbf{J}t}\mathbf{S}^{-1}}$$

$$\mathbf{J} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix},\ e^{\mathbf{J}t} = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$$

Eigenvectors:

$$\mathbf{A}\mathbf{v}_1 = \lambda\mathbf{v}_1,\ (\mathbf{A}-2\mathbf{I})\mathbf{v}_1 = 0$$

$$\left[\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right]\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$\left[\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\right]\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = 0$$

$$v_{11} = -v_{12}\ ,\ \mathbf{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$(\mathbf{A}-2\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$$

$$\left[\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\right]\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$-v_{21} - v_{22} = -1$$

$$v_{21} + v_{22} = 1$$

$$v_{21} = 1\ , v_{22} = 0,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{S}^{-1} = \begin{bmatrix} -0 & 1 \\ 1 & 1 \end{bmatrix}$$

Check: $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} -0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \mathbf{J}$

$$e^{\mathbf{A}t} = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}\begin{bmatrix} -0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} (1-t)e^{2t} & -te^{2t} \\ te^{2t} & (1+t)e^{2t} \end{bmatrix}$$