Pole-Placement Controller Design

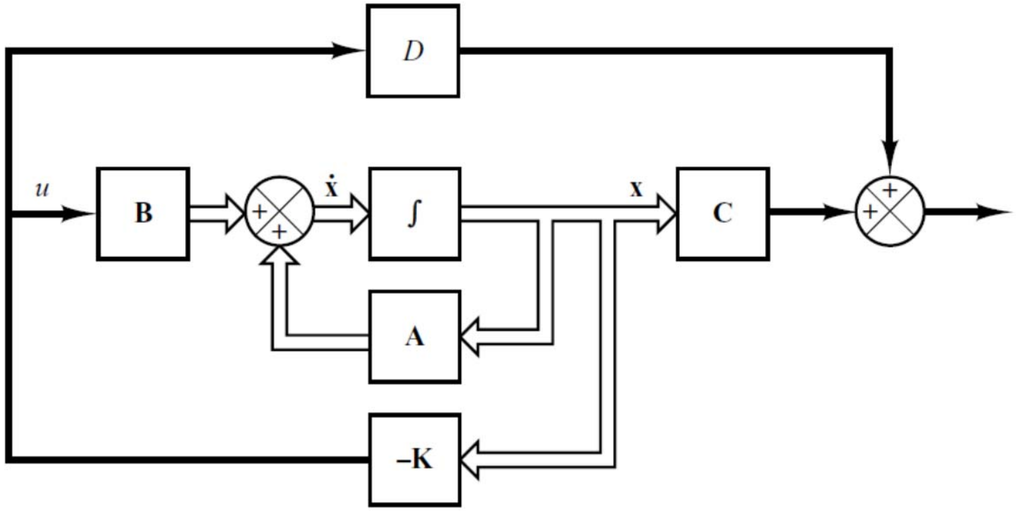
EPD 30.114 ADVANCED FEEDBACK & CONTROL

Pole-Placement Technique

- The Pole-Placement method is similar to root-locus based design of placing closed-loop poles at desired locations.
  - However, while root-locus method focuses on the dominant poles, pole placement places all closed loop poles at desired locations
- This technique assumes all state variables are measurable and are available for feedback
  - If a system is **completely state controllable**, then poles of the closed-loop system may be placed at any desired location by means of state feedback
- There are 3 ways to approach Pole-Placement design:
  - M1: Using the Controllable Canonical Form as a start
  - M2: Using direct substitution & comparing with desired c.e. (30.101 method)
    - Not recommended for high order systems since it is not easily implemented/scalable on a computer program
  - M3: Using Ackermann's Formula
- Remember it is necessary to check for complete state controllability prior to Pole-Placement design

Design By Pole Placement

- We have a control system: $$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$ $$\mathbf{y} = \mathbf{C}\mathbf{x} + Du$$
- Choose control signal to be a function of all states **(State Feedback)**: $u = -\mathbf{K}\mathbf{x}$
- Here we consider where there is no reference input, where the goal is to maintain zero output
  - Because of disturbances that are present, without control output would deviate from zero
  - Nonzero output will be returned to zero reference input via state feedback
- State-Space System under SF: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x}) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$
  - Solution is: $\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{B}\mathbf{K})t}\mathbf{x}(0)$
- Stability and Transient Response determined by eigenvalues of **A-BK**
  - Eigenvalues of A-BK are regulator poles (closed-loop poles)
  - Problem of placing these poles at the desired location on the LHP is the Pole-Placement problem



Method 1: Direct Computation from CCF

- Consider the system is already in CCF:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & \vdots & b_{n-1} - a_{n-1} b_0 & \vdots & \cdots & \vdots & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

- We can choose a set of desired eigenvalues (closed-loop poles) at $-\mu_1, -\mu_2, \ldots, -\mu_n$.

- Desired c.e:

$$(s+\mu_1)(s+\mu_2)\cdots(s+\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

- Define **K**: $\mathbf{K} = \begin{bmatrix} \beta_n & \beta_{n-1} & \cdots & \beta_1 \end{bmatrix}$

- c.e. of system: $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \begin{bmatrix} \beta_n & \beta_{n-1} & \cdots & \beta_1 \end{bmatrix} \right| = \begin{vmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n + \beta_n & a_{n-1} + \beta_{n-1} & & s + a_1 + \beta_1 \end{vmatrix}$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = s^n + (a_1 + \beta_1) s^{n-1} + \cdots + (a_{n-1} + \beta_{n-1}) s + (a_n + \beta_n) = 0$$

Equating coefficients: $(a_1 + \beta_1) = \alpha_1 \,, \; \cdots, \; (a_n + \beta_n) = \alpha_n$

$$\boxed{\mathbf{K} = \begin{bmatrix} \alpha_n - a_n & \alpha_{n-1} - a_{n-1} & \cdots & \alpha_1 - a_1 \end{bmatrix}}$$

Method 2: Direct Substitution Method (30.101)

- If the system is low order ($n \leq 3$), direct substitution of the gain matrix **K** into the desired characteristic equation may be simpler
- For a n=3 system, the gain may be expressed as:

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

- Substituting directly into desired c.e: (Desired eigenvalues are $-\mu_1$, $-\mu_2$, $-\mu_3$)

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| = \left(s + \mu_1 \right)\left(s + \mu_2 \right)\left(s + \mu_3 \right) = 0$$

- Both sides of the c.e. are polynomials in s and the unknown k_1, k_2 and k_3 can be computed by comparing coefficients
- This method is great for n=2,3 but is very tedious for higher n
- If system is not completely controllable, the gain matrix cannot be determined (no solution exists).

Method 3: Using Ackermann's Formula

- Starting from the general form: $\dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{BK}\right)\mathbf{x}$
- Defining: $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$

$$\left|s\mathbf{I} - \mathbf{A} + \mathbf{BK}\right| = \left|s\mathbf{I} - \tilde{\mathbf{A}}\right| = \left(s + \mu_1\right)\left(s + \mu_2\right)\cdots\left(s + \mu_n\right) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

- From C-H Theorem: $\phi\left(\tilde{\mathbf{A}}\right) = \tilde{\mathbf{A}}^n + \alpha_1\tilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1}\tilde{\mathbf{A}} + \alpha_n\mathbf{I} = 0$
- To assist in understanding, let's consider n=3
  - Consider the following identities:

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

$$\tilde{\mathbf{A}}^2 = \left(\mathbf{A} - \mathbf{BK}\right)\left(\mathbf{A} - \mathbf{BK}\right) = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKA} + \left(\mathbf{BK}\right)^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}(\mathbf{A} - \mathbf{BK})$$

$$= \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\tilde{\mathbf{A}}$$ No powers of **K**

$$\tilde{\mathbf{A}}^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2$$ No powers of **K**

  - Inserting into C-H Equation

$$\phi\left(\tilde{\mathbf{A}}\right) = \tilde{\mathbf{A}}^3 + \alpha_1\tilde{\mathbf{A}}^2 + \alpha_2\tilde{\mathbf{A}} + \alpha_3\mathbf{I} = 0$$

$$= \alpha_3\mathbf{I} + \alpha_2\mathbf{A} + \alpha_1\mathbf{A}^2 + \mathbf{A}^3 - \alpha_2\mathbf{BK} - \alpha_1\mathbf{ABK} - \alpha_1\mathbf{BK}\tilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2$$

Method 3: Using Ackermann's Formula

- Remember:

$$\phi\left(\tilde{\mathbf{A}}\right) = \tilde{\mathbf{A}}^3 + \alpha_1\tilde{\mathbf{A}}^2 + \alpha_2\tilde{\mathbf{A}} + \alpha_3\mathbf{I} = 0$$

$$\phi\left(\mathbf{A}\right) = \mathbf{A}^3 + \alpha_1\mathbf{A}^2 + \alpha_2\mathbf{A} + \alpha_3\mathbf{I} \neq 0$$

- Continuing:

$$\phi\left(\tilde{\mathbf{A}}\right) = \phi\left(\mathbf{A}\right) - \alpha_2\mathbf{BK} - \alpha_1\mathbf{BK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 - \alpha_1\mathbf{ABK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK}$$

$$\phi\left(\mathbf{A}\right) = \mathbf{B}\left(\alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2\right) + \mathbf{AB}\left(\alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}\right) + \mathbf{A}^2\mathbf{B}\left(\mathbf{K}\right)$$

$$= \underbrace{\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix}}_{\text{Controllability Matrix}} \begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

- Since system is completely state controllable, the inverse of the controllability matrix exists

$$\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix}^{-1}\phi\left(\mathbf{A}\right) = \begin{bmatrix} \alpha_2\mathbf{K} + \alpha_1\mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1\mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} \qquad \Rightarrow \mathbf{K} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B}\end{bmatrix}^{-1}\phi\left(\mathbf{A}\right)$$

- In general:

$$\boxed{\Rightarrow \mathbf{K} = \begin{bmatrix}0 & 0 & \cdots & 1\end{bmatrix}\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}^{-1}\phi\left(\mathbf{A}\right)}$$

Final Comments on Pole-Placement

- Regulator Systems vs Control Systems
  - Regulator systems: Reference input is constant (including zero)
  - Control Systems : Reference input is time-varying
- Choosing locations of desired closed-loop poles:
  - While it is often desired to place the CL poles far away from the $j\omega$ axis, to make the system respond fast, in order to do so , the required signals of the system can become very large
    - This may result in the system becoming nonlinear
    - Will require larger and heavier actuators
  - Another alternative to pole-placement is based on the Quadratic Optimal Control approach which places the poles such that it balances between acceptable response and amount of control energy required
- Remember the gain matrix **K** is not unique to a given system and depends on the desired closed-loop poles locations
- In higher order systems, it is desirable to examine by computer simulations the response characteristics of the system for different **K** matrices and choose the one that gives the best overall performance

Exercise

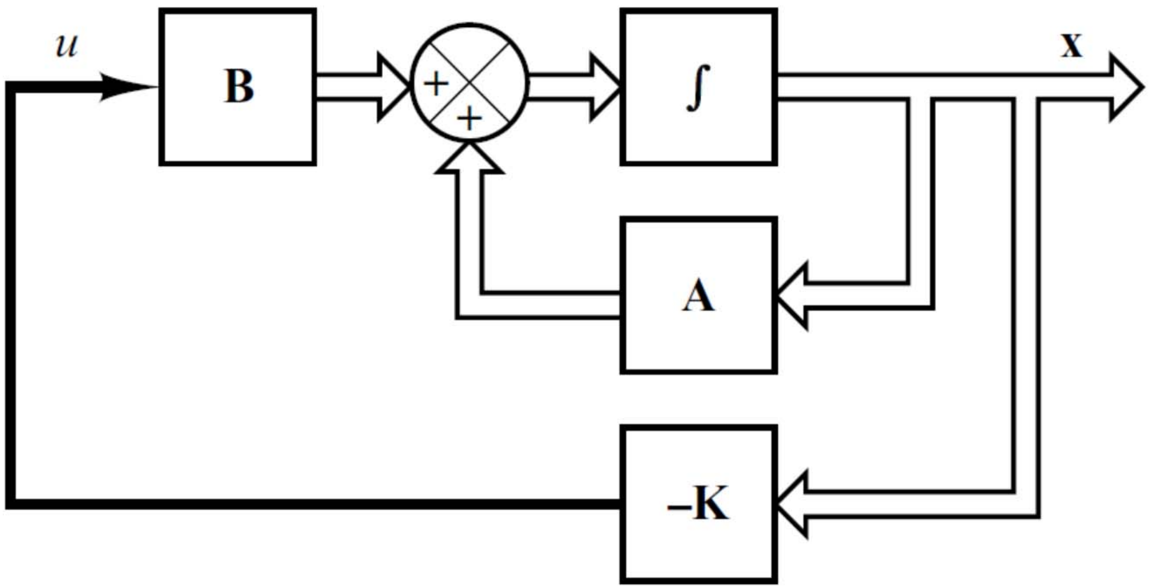
- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

Desired CL poles: $s=-2+j4,\quad s=-2-j4,\quad s=-10$

Check state controllability:

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$



Rank of controllability matrix is 3 (full rank). System is completely state controllable

Exercise (Using Direct CCF)

- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

Desired CL poles: $s=-2+j4,\ \ s=-2-j4,\ \ s=-10$

Desired c.e.:

$$(s+2-j4)(s+2+j4)(s+10) = s^3+14s^2+60s+200$$

$$a_1=6,\ a_2=5,\ a_3=1$$

$$\alpha_1=14,\ \alpha_2=60,\ \alpha_3=200$$

$$\begin{aligned}\mathbf{K} &= \begin{bmatrix}\alpha_3-a_3 & \alpha_2-a_2 & \alpha_1-a_1\end{bmatrix} \\ &= \begin{bmatrix}200-1 & 60-5 & 14-6\end{bmatrix} \\ &= \begin{bmatrix}199 & 55 & 8\end{bmatrix}\end{aligned}$$

Exercise (Using Direct Substitution)

- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

Desired CL poles: $s=-2+j4,\ \ s=-2-j4,\ \ s=-10$

Desired c.e.:

$$(s+2-j4)(s+2+j4)(s+10) = s^3+14s^2+60s+200$$

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{BK}\right| = \left\|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}\right| = \left\|\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1+k_1 & 5+k_2 & s+6+k_3 \end{bmatrix}\right\|$$

$$= s^3+(6+k_3)s^2+(5+k_2)s+1+k_1$$

$$= s^3+14s^2+60s+200$$

$$\Rightarrow k_1 = 199,\ \ k_2 = 55,\ \ k_3 = 8 \qquad \mathbf{K} = \begin{bmatrix} 199 & 55 & 8 \end{bmatrix}$$

Exercise (Using Ackermann's Formula)

- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u$$

Desired CL poles: $s=-2+j4,\quad s=-2-j4,\quad s=-10$

Desired c.e.:

$$(s+2-j4)(s+2+j4)(s+10) = s^3+14s^2+60s+200$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix}^{-1}\phi(\mathbf{A})$$

$$\phi(\mathbf{A}) = \mathbf{A}^3+14\mathbf{A}^2+60\mathbf{A}+200\mathbf{I}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\mathbf{I}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \end{bmatrix}$$

Exercise (Visualization)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Let's see how introduction of state feedback affects the system!
  - As a comparison, let's see how the system responds without a controller due to non-zero initial conditions.

$$u = -\mathbf{K}\mathbf{x}$$

No Control

Full State Feedback Control $\quad \mathbf{K} = \begin{bmatrix} 199 & 55 & 8 \end{bmatrix}$