Stability in z Plane

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Stability in s Plane

- For continuous-time systems, the poles of the system must be on the left half plane on the *s*-plane for the system to be absolutely stable
- If it is on the imaginary axis (non-repeated), it is marginally stable or possesses neutral stability
- If there exists repeated conjugate poles on imaginary axis or any of the pole are on the right half plane of the *s*-plane, the system is unstable
- For this system: $R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$ $\qquad G(s) = \dfrac{C(s)}{R(s)}$

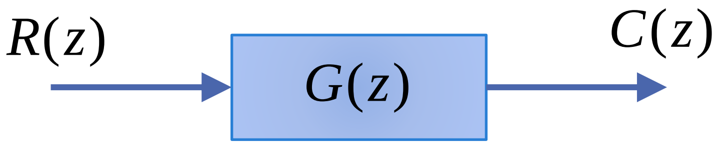
$$G(s) = \frac{C(s)}{R(s)} = K\frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} = K\frac{N(s)}{D(s)}$$

Where *K*, *a* and *b* are constants and $m \leq n$. (proper fraction)

- Roots of the **Characteristic Equation** *D(s)=0* determine stability of the (closed-loop) system
- Recall the Routh-Hurwitz Stability Criterion!

Stability in z Plane

- For discrete-time systems, the poles of the system must lie within the unit circle in the z-plane
- If it is on the unit circle (z=1) (non-repeated), it is marginally stable or possesses neutral stability
- If there exists repeated conjugate poles on the unit circle or any of the pole are outside the unit circle, the system is unstable
- For this system:



$$G(z) = \frac{C(z)}{R(z)}$$

$$G(z) = \frac{C(z)}{R(z)} = K\,\frac{b_0 z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m}{a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n} = K\,\frac{N(z)}{D(z)}$$

Where K, a and b are constants and $m \leq n$. (proper fraction)

- Roots of the **Characteristic Equation** *D(z)=0* determine stability of the (closed-loop) system
- Is there an analogous “Routh-Hurwitz Stability Criterion”?

Jury Stability Criterion

- The Jury Stability Criterion is a method of determining the stability of a linear discrete time system by analysing the coefficients of its characteristic polynomial
- It is the discrete time analogue of the Routh–Hurwitz stability criterion.
- The Jury Stability Criterion requires that the system poles are located inside the unit circle centred at the origin (z-plane), while the Routh-Hurwitz stability criterion requires that the poles are in the left half of the complex plane (s-plane).
- The Jury Criterion is named after Iraqi-born American Eliahu Ibraham Jury who devised it in 1981.
- Like the Routh-Hurwitz Stability Criterion, it provides the necessary and sufficient conditions for stability.

Jury Stability Criterion

- For a given (closed-loop) discrete-time system, the characteristic equation is a polynomial in *z* and given by:

$$\boxed{P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0}$$

  - a_0 MUST be positive. (if it is not, multiply -1 to make it so)

- First check the following 3 conditions:

1
  - $\boxed{|a_n| < a_0}$

2
  - $\boxed{P(z)\big|_{z=1} > 0}$

3
  - $\boxed{P(z)\big|_{z=-1} \begin{cases} > 0 & \text{for } n \text{ even} \\ < 0 & \text{for } n \text{ odd} \end{cases}}$

- If any of these 3 conditions fail, the system is unstable. If not, a **Jury Table** needs to be constructed. (Not enough information to tell if stable or unstable at this point).

General Form of Jury Table

- Construct the Jury Table as follows: $P(z) = a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_{n-1} z + a_n = 0$

Row	z^0	z^1	z^2	z^3	…	z^{n-2}	z^{n-1}	z^n
1	a_n	a_{n-1}	a_{n-2}	a_{n-3}	…	a_2	a_1	a_0
2	a_0	a_1	a_2	a_3	…	a_{n-2}	a_{n-1}	a_n
3	b_{n-1}	b_{n-2}	b_{n-3}	b_{n-4}		b_1	b_0	
4	b_0	b_1	b_2	b_3		b_{n-2}	b_{n-1}	
5	c_{n-2}	c_{n-3}	c_{n-4}	c_{n-5}		c_0		
6	c_0	c_1	c_2	c_3		c_{n-2}		
…								
$2n-5$	p_3	p_2	p_1	p_0				
$2n-4$	p_0	p_1	p_2	p_3				
$2n-3$	q_2	q_1	q_0					

Notice that the elements in the first row consists of the coefficients in P(z) arranged in ascending order of powers of *z* and the 2nd row are arranged in descending order.
Also last row of the table will always consist of 3 elements.

Computation of Jury Elements

- The elements for row 3 through $2n-3$ are given by the following determinants:

$$b_k = \begin{vmatrix} a_n & a_{n-1-k} \\ a_0 & a_{k+1} \end{vmatrix} \qquad k = 0, 1, 2, \ldots, n-1$$

$$c_k = \begin{vmatrix} b_{n-1} & b_{n-2-k} \\ b_0 & b_{k+1} \end{vmatrix} \qquad k = 0, 1, 2, \ldots, n-2$$

$$q_k = \begin{vmatrix} p_3 & p_{2-k} \\ p_0 & p_{k+1} \end{vmatrix} \qquad k = 0, 1, 2$$

- Also notice that the elements in any even-numbered row are the reverse of the preceding odd-numbered row

- With the constructed table, finally check for the **4th and final conditions**:

4 $$\boxed{|b_{n-1}| > |b_0| \;,\; |c_{n-2}| > |c_0|, \;\cdots,\; |q_2| > |q_0|}$$

- If any of these conditions are not met, the system is unstable.

Illustrative Example

- Construct the Jury Stability Table for the following characteristic equation: $a_0 > 0$. What are the conditions for stability?

$$P(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$$

1 $\boxed{|a_4| < a_0}$

2 $\boxed{P(z)\big|_{z=1} = a_0 + a_1 + a_2 + a_3 + a_4 > 0}$

3 $\boxed{P(z)\big|_{z=-1} = a_0 - a_1 + a_2 - a_3 + a_4 > 0}$

4 $\boxed{\begin{array}{l} |b_3| > |b_0| \, , \\ |c_2| > |c_0| \end{array}}$

Illustrative Example

- Using a modified Jury Stability Table:

Row	z^0	z^1	z^2	z^3	z^4	
1	a_4	a_3	a_2	a_1	a_0	
2	a_0	a_1	a_2	a_3	a_4	
	$\begin{vmatrix} a_4 & a_0 \\ a_0 & a_4 \end{vmatrix}$					$= b_3$
	$\begin{vmatrix} a_4 & a_1 \\ a_0 & a_3 \end{vmatrix}$					$= b_2$
	$\begin{vmatrix} a_4 & a_2 \\ a_0 & a_2 \end{vmatrix}$					$= b_1$
	$\begin{vmatrix} a_4 & a_3 \\ a_0 & a_1 \end{vmatrix}$					$= b_0$
3	b_3	b_2	b_1	b_0		
4	b_0	b_1	b_2	b_3		

Illustrative Example

Row	z^0	z^1	z^2	z^3	z^4	
1	a_4	a_3	a_2	a_1	a_0	
2	a_0	a_1	a_2	a_3	a_4	
3	b_3	b_2	b_1	b_0		
4	b_0	b_1	b_2	b_3		
	$\begin{vmatrix} b_3 & b_0 \\ b_0 & b_3 \end{vmatrix}$					$= c_2$
	$\begin{vmatrix} b_3 & b_1 \\ b_0 & b_2 \end{vmatrix}$					$= c_1$
	$\begin{vmatrix} b_3 & b_2 \\ b_0 & b_1 \end{vmatrix}$					$= c_0$
5	c_2	c_1	c_0			

4 $\boxed{\left|b_3\right| > \left|b_0\right|, \quad \left|c_2\right| > \left|c_0\right|}$

Exercise

- Examine the stability of the following characteristic equation:

$$P(z) = z^4 - 1.2z^3 + 0.07z^2 + 0.3z - 0.08$$

$$a_0 = 1,\ a_1 = -1.2,\ a_2 = 0.07,\ a_3 = 0.3,\ a_4 = -0.08$$

1 $|a_4| < a_0$ $\qquad |0.08| < 1$

2 $P(z)\big|_{z=1} = a_0 + a_1 + a_2 + a_3 + a_4 > 0$ $\qquad 1 - 1.2 + 0.07 + 0.3 - 0.08 = 0.09 > 0$

3 $P(z)\big|_{z=-1} = a_0 - a_1 + a_2 - a_3 + a_4 > 0$ $\qquad 1 + 1.2 + 0.07 - 0.3 - 0.08 = 1.89 > 0$

First 3 conditions satisfied. So need to construct Jury Stability Table

Exercise

Row	z^0	z^1	z^2	z^3	z^4	
1	-0.08	0.3	0.07	-1.2	1	
2	1	-1.2	0.07	0.3	-0.08	
	$\begin{vmatrix} -0.08 & 1 \\ 1 & -0.08 \end{vmatrix}$					$= b_3$ = - 0.994
	$\begin{vmatrix} -0.08 & -1.2 \\ 1 & 0.3 \end{vmatrix}$					$= b_2$ = 1.176
	$\begin{vmatrix} -0.08 & 0.07 \\ 1 & 0.07 \end{vmatrix}$					$= b_1$ = - 0.0756
	$\begin{vmatrix} -0.08 & 0.3 \\ 1 & -1.2 \end{vmatrix}$					$= b_0$ = - 0.204
3	b_3	b_2	b_1	b_0		
4	b_0	b_1	b_2	b_3		

Exercise

Row	z^0	z^1	z^2	z^3	z^4	
1	-0.08	0.3	0.07	-1.2	1	
2	1	-1.2	0.07	0.3	-0.08	
3	-0.994	1.176	-0.0756	-0.204		
4	-0.204	-0.0756	1.176	-0.994		
	$\begin{vmatrix} -0.994 & -0.204 \\ -0.204 & -0.994 \end{vmatrix}$					$= c_2$ = 0.946
	$\begin{vmatrix} -0.994 & -0.0756 \\ -0.204 & 1.176 \end{vmatrix}$					$= c_1$ = -1.184
	$\begin{vmatrix} -0.994 & 1.176 \\ -0.204 & -0.0756 \end{vmatrix}$					$= c_0$ = 0.315
5	c_2	c_1	c_0			

4 $\quad |b_3| > |b_0|, \quad |c_2| > |c_0|$

$$|0.994| > |0.204|, \quad |0.946| > |0.315|$$

All conditions satisfied. System is stable.
All poles lie within unit circle.

$$P(z) = (z - 0.8)(z + 0.5)(z - 0.5)(z - 0.4) = 0$$