

# Pole-Placement Controller Design

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EPD 30.114 ADVANCED FEEDBACK & CONTROL

# Pole-Placement Technique

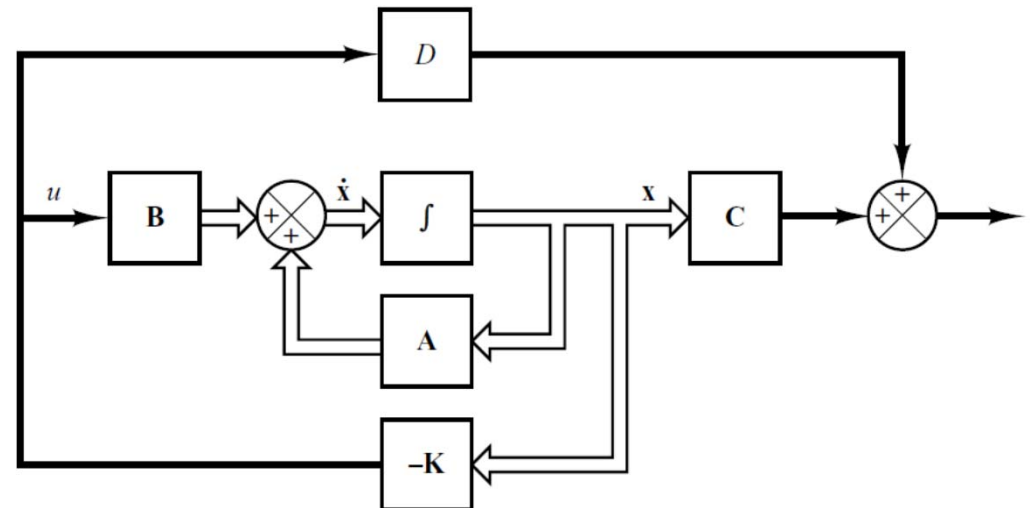
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- The Pole-Placement method is similar to root-locus based design of placing closed-loop poles at desired locations.
  - However, while root-locus method focuses on the dominant poles, pole placement places all closed loop poles at desired locations
- This technique assumes all state variables are measurable and are available for feedback
  - If a system is **completely state controllable**, then poles of the closed-loop system may be placed at any desired location by means of state feedback
- There are 3 ways to approach Pole-Placement design:
  - M1: Using the Controllable Canonical Form as a start
  - M2: Using direct substitution & comparing with desired c.e. (30.101 method)
    - Not recommended for high order systems since it is not easily implemented/scalable on a computer program
  - M3: Using Ackermann's Formula
- Remember it is necessary to check for complete state controllability prior to Pole-Placement design

# Design By Pole Placement

- We have a control system:
 
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + Du$$
- Choose control signal to be a function of all states (**State Feedback**):  $u = -\mathbf{K}\mathbf{x}$
- Here we consider where there is no reference input, where the goal is to maintain zero output
  - Because of disturbances that are present, without control output would deviate from zero
  - Nonzero output will be returned to zero reference input via state feedback
- State-Space System under SF:  $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(-\mathbf{K}\mathbf{x}) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}$ 
  - Solution is:  $\mathbf{x}(t) = e^{(\mathbf{A} - \mathbf{B}\mathbf{K})t} \mathbf{x}(0)$
- Stability and Transient Response determined by eigenvalues of  $\mathbf{A} - \mathbf{B}\mathbf{K}$ 
  - Eigenvalues of  $\mathbf{A} - \mathbf{B}\mathbf{K}$  are regulator poles (closed-loop poles)
  - Problem of placing these poles at the desired location on the LHP is the Pole-Placement problem



# Method 1: Direct Computation from CCF

- Consider the system is already in CCF:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

- We can choose a set of desired eigenvalues (closed-loop poles) at  $-\mu_1, -\mu_2, \dots, -\mu_n$ .

- Desired c.e:

$$(s + \mu_1)(s + \mu_2) \cdots (s + \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

$$y = [b_n - a_n b_0 \mid b_{n-1} - a_{n-1} b_0 \mid \cdots \mid b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

- Define  $\mathbf{K}$ :  $\mathbf{K} = [\beta_n \quad \beta_{n-1} \quad \cdots \quad \beta_1]$

- c.e. of system:  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| s\mathbf{I} - \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} [\beta_n \quad \beta_{n-1} \quad \cdots \quad \beta_1] \right| = \left| \begin{bmatrix} s & -1 & \cdots & 0 \\ 0 & s & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n + \beta_n & a_{n-1} + \beta_{n-1} & \cdots & s + a_1 + \beta_1 \end{bmatrix} \right|$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = s^n + (a_1 + \beta_1)s^{n-1} + \cdots + (a_{n-1} + \beta_{n-1})s + (a_n + \beta_n) = 0$$

Equating coefficients:  $(a_1 + \beta_1) = \alpha_1, \dots, (a_n + \beta_n) = \alpha_n$

$$\mathbf{K} = [\alpha_n - a_n \quad \alpha_{n-1} - a_{n-1} \quad \cdots \quad \alpha_1 - a_1]$$

# Method 2: Direct Substitution Method (30.101)

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- If the system is low order ( $n \leq 3$ ), direct substitution of the gain matrix  $\mathbf{K}$  into the desired characteristic equation may be simpler

- For a  $n=3$  system, the gain may be expressed as:

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

- Substituting directly into desired c.e: (Desired eigenvalues are  $-\mu_1, -\mu_2, -\mu_3$ )

$$\left| s\mathbf{I} - \mathbf{A} + \mathbf{BK} \right| = (s + \mu_1)(s + \mu_2)(s + \mu_3) = 0$$

- Both sides of the c.e. are polynomials in  $s$  and the unknown  $k_1, k_2$  and  $k_3$  can be computed by comparing coefficients
- This method is great for  $n=2,3$  but is very tedious for higher  $n$
- If system is not completely controllable, the gain matrix cannot be determined (no solution exists).

# Method 3: Using Ackermann's Formula

- Starting from the general form:  $\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{BK})\mathbf{x}$

- Defining:  $\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = |s\mathbf{I} - \tilde{\mathbf{A}}| = (s + \mu_1)(s + \mu_2) \cdots (s + \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

- From C-H Theorem:  $\phi(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^n + \alpha_1 \tilde{\mathbf{A}}^{n-1} + \cdots + \alpha_{n-1} \tilde{\mathbf{A}} + \alpha_n \mathbf{I} = 0$

- To assist in understanding, let's consider  $n=3$

- Consider the following identities:

$$\tilde{\mathbf{A}} = \mathbf{A} - \mathbf{BK}$$

$$\begin{aligned} \tilde{\mathbf{A}}^2 &= (\mathbf{A} - \mathbf{BK})(\mathbf{A} - \mathbf{BK}) = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BKA} + (\mathbf{BK})^2 = \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}(\mathbf{A} - \mathbf{BK}) \\ &= \mathbf{A}^2 - \mathbf{ABK} - \mathbf{BK}\tilde{\mathbf{A}} \quad \text{No powers of K} \end{aligned}$$

$$\tilde{\mathbf{A}}^3 = \mathbf{A}^3 - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2 \quad \text{No powers of K}$$

- Inserting into C-H Equation

$$\phi(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^3 + \alpha_1 \tilde{\mathbf{A}}^2 + \alpha_2 \tilde{\mathbf{A}} + \alpha_3 \mathbf{I} = 0$$

$$= \alpha_3 \mathbf{I} + \alpha_2 \mathbf{A} + \alpha_1 \mathbf{A}^2 + \mathbf{A}^3 - \alpha_2 \mathbf{BK} - \alpha_1 \mathbf{ABK} - \alpha_1 \mathbf{BK}\tilde{\mathbf{A}} - \mathbf{A}^2\mathbf{BK} - \mathbf{ABK}\tilde{\mathbf{A}} - \mathbf{BK}\tilde{\mathbf{A}}^2$$

# Method 3: Using Ackermann's Formula

- Remember:  $\phi(\tilde{\mathbf{A}}) = \tilde{\mathbf{A}}^3 + \alpha_1 \tilde{\mathbf{A}}^2 + \alpha_2 \tilde{\mathbf{A}} + \alpha_3 \mathbf{I} = 0$   
 $\phi(\mathbf{A}) = \mathbf{A}^3 + \alpha_1 \mathbf{A}^2 + \alpha_2 \mathbf{A} + \alpha_3 \mathbf{I} \neq 0$
- Continuing:  $\phi(\tilde{\mathbf{A}}) = \phi(\mathbf{A}) - \alpha_2 \mathbf{B}\mathbf{K} - \alpha_1 \mathbf{B}\mathbf{K}\tilde{\mathbf{A}} - \mathbf{B}\mathbf{K}\tilde{\mathbf{A}}^2 - \alpha_1 \mathbf{A}\mathbf{B}\mathbf{K} - \mathbf{A}\mathbf{B}\mathbf{K}\tilde{\mathbf{A}} - \mathbf{A}^2 \mathbf{B}\mathbf{K}$   
 $\phi(\mathbf{A}) = \mathbf{B}(\alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2) + \mathbf{A}\mathbf{B}(\alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}}) + \mathbf{A}^2 \mathbf{B}(\mathbf{K})$

$$= \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2 \mathbf{B} \end{bmatrix} \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix}$$

Controllability Matrix

- Since system is completely state controllable, the inverse of the controllability matrix exists

$$\begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2 \mathbf{B} \end{bmatrix}^{-1} \phi(\mathbf{A}) = \begin{bmatrix} \alpha_2 \mathbf{K} + \alpha_1 \mathbf{K}\tilde{\mathbf{A}} + \mathbf{K}\tilde{\mathbf{A}}^2 \\ \alpha_1 \mathbf{K} + \mathbf{K}\tilde{\mathbf{A}} \\ \mathbf{K} \end{bmatrix} \Rightarrow \mathbf{K} = [0 \ 0 \ 1] \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \mathbf{A}^2 \mathbf{B} \end{bmatrix}^{-1} \phi(\mathbf{A})$$

- In general:

$$\Rightarrow \mathbf{K} = [0 \ 0 \ \dots \ 1] \begin{bmatrix} \mathbf{B} & \mathbf{A}\mathbf{B} & \dots & \mathbf{A}^{n-1} \mathbf{B} \end{bmatrix}^{-1} \phi(\mathbf{A})$$

# Final Comments on Pole-Placement

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- Regulator Systems vs Control Systems
  - Regulator systems: Reference input is constant (including zero)
  - Control Systems : Reference input is time-varying
- Choosing locations of desired closed-loop poles:
  - While it is often desired to place the CL poles far away from the  $j\omega$  axis, to make the system respond fast, in order to do so , the required signals of the system can become very large
    - This may result in the system becoming nonlinear
    - Will require larger and heavier actuators
  - Another alternative to pole-placement is based on the Quadratic Optimal Control approach which places the poles such that it balances between acceptable response and amount of control energy required
- Remember the gain matrix **K** is not unique to a given system and depends on the desired closed-loop poles locations
- In higher order systems, it is desirable to examine by computer simulations the response characteristics of the system for different **K** matrices and choose the one that gives the best overall performance



# Exercise

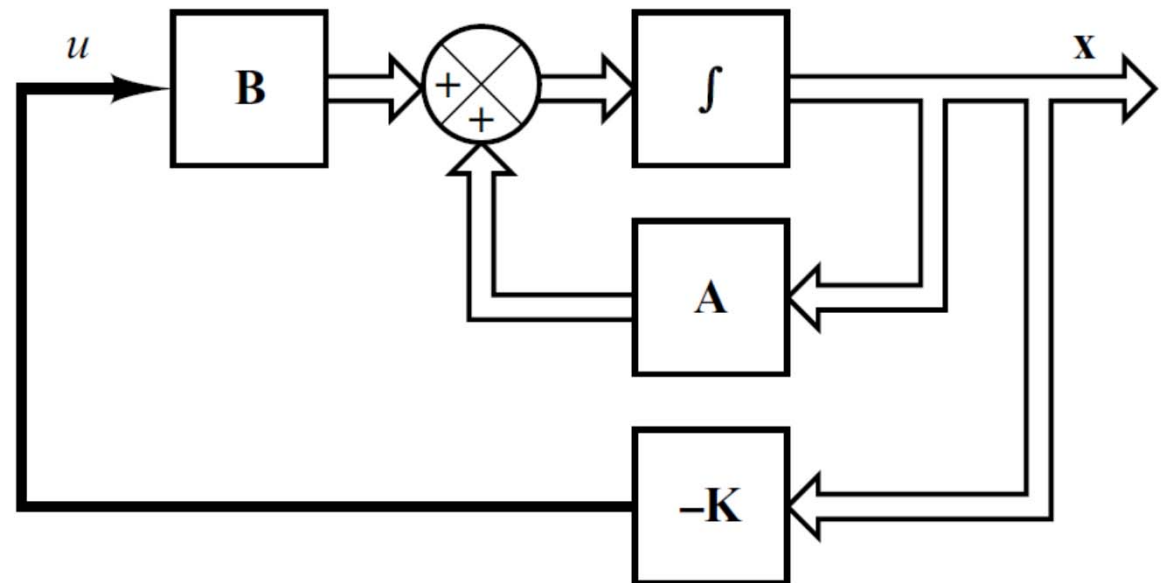
- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Desired CL poles:  $s = -2 + j4$ ,  $s = -2 - j4$ ,  $s = -10$

Check state controllability:

$$\mathbf{C}_o = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$



Rank of controllability matrix is 3 (full rank). System is completely state controllable

# Exercise (Using Direct CCF)

- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Desired CL poles:  $s = -2 + j4$ ,  $s = -2 - j4$ ,  $s = -10$

Desired c.e.:

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$a_1 = 6, a_2 = 5, a_3 = 1$$

$$\alpha_1 = 14, \alpha_2 = 60, \alpha_3 = 200$$

$$\begin{aligned} \mathbf{K} &= [\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1] \\ &= [200 - 1 \quad 60 - 5 \quad 14 - 6] \\ &= [199 \quad 55 \quad 8] \end{aligned}$$

# Exercise (Using Direct Substitution)

- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Desired CL poles:  $s = -2 + j4$ ,  $s = -2 - j4$ ,  $s = -10$

Desired c.e.:

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

$$|s\mathbf{I} - \mathbf{A} + \mathbf{BK}| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} [k_1 \quad k_2 \quad k_3] \right| = \left| \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 1 + k_1 & 5 + k_2 & s + 6 + k_3 \end{bmatrix} \right|$$

$$= s^3 + (6 + k_3)s^2 + (5 + k_2)s + 1 + k_1$$

$$= s^3 + 14s^2 + 60s + 200$$

$$\Rightarrow k_1 = 199, \quad k_2 = 55, \quad k_3 = 8$$

$$\mathbf{K} = [199 \quad 55 \quad 8]$$

# Exercise (Using Ackermann's Formula)

- For the regulator system shown, design a state feedback controller **K** using all 3 methods of Pole-Placement techniques

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Desired CL poles:  $s = -2 + j4$ ,  $s = -2 - j4$ ,  $s = -10$

Desired c.e.:

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$\mathbf{K} = [0 \ 0 \ 1] [\mathbf{B} \ \mathbf{AB} \ \mathbf{A}^2\mathbf{B}]^{-1} \phi(\mathbf{A})$$

$$\phi(\mathbf{A}) = \mathbf{A}^3 + 14\mathbf{A}^2 + 60\mathbf{A} + 200\mathbf{I}$$

$$[\mathbf{B} \ \mathbf{AB} \ \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^3 + 14 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}^2 + 60 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} + 200\mathbf{I}$$

$$[\mathbf{B} \ \mathbf{AB} \ \mathbf{A}^2\mathbf{B}]^{-1} = \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$\mathbf{K} = [0 \ 0 \ 1] \begin{bmatrix} 5 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 199 & 55 & 8 \\ -8 & 159 & 7 \\ -7 & -43 & 117 \end{bmatrix}$$

$$= [199 \ 55 \ 8]$$

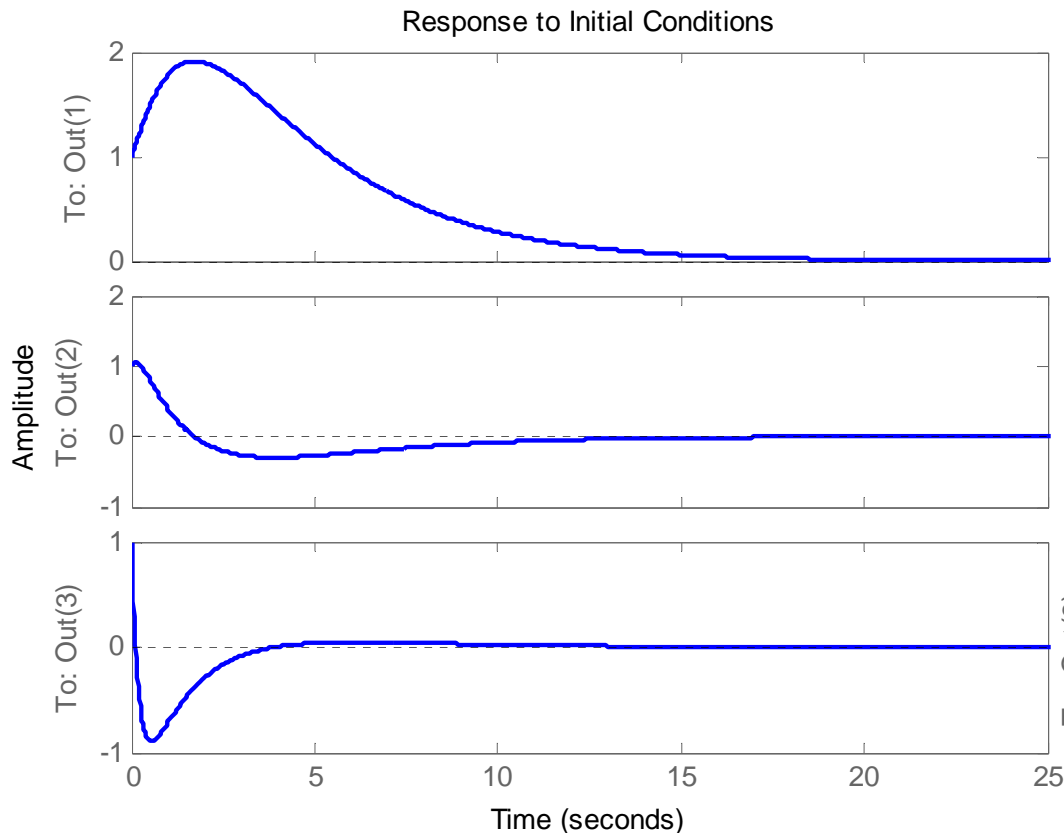
# Exercise (Visualization)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Let's see how introduction of state feedback affects the system!
  - As a comparison, let's see how the system responds without a controller due to non-zero initial conditions.

$$u = -\mathbf{K}\mathbf{x}$$

**No Control**



**Full State Feedback Control**  $\mathbf{K} = [199 \quad 55 \quad 8]$

