Principle of Duality

- Controllability and Observability share an intricate relationship. The principle of duality was conceived by Kalman to connect the analogies between controllability & observability

- Consider a system S_1 and its dual system S_2:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{y} = \mathbf{C}\mathbf{x}$$

$\boldsymbol{x}$: n vector, $\boldsymbol{u}$: r-vector, $\mathbf{y}$: m-vector; $\mathbf{A}$: $n \times n$ matrix, $\mathbf{B}$: $n \times r$ matrix, $\mathbf{C}$: $m \times n$ matrix

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v} \qquad \mathbf{w} = \mathbf{B}^*\mathbf{z}$$

$\boldsymbol{z}$: n vector, $\boldsymbol{v}$: m-vector, $\mathbf{w}$: r-vector; $\mathbf{A}^*$: $n \times n$ matrix, $\mathbf{B}^*$: $r \times n$ matrix, $\mathbf{C}^*$: $n \times m$ matrix

- The principle of duality states S_1 is completely state controllable (or observable) if and only if system S_2 is completely observable (state-controllable).

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}_{n\times nr} \qquad \mathbf{C}_O = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix}_{n\times nm}$$

$$\mathbf{O}_B = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix}_{n\times nm} \qquad \mathbf{O}_B = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}_{n\times nr}$$

- For a partially observable system, if the unobservable modes are stable and the observable modes are unstable, the system is said to be detectable.
    - Concept of detectability is dual to the concept of stabilizability

Illustrative Example

- Consider the following system: $G(s) = \dfrac{1}{s^2 + 2s + 1}$
  - It can be expressed in CCF (S_1), or also in OCF (S_2),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad \dot{\mathbf{z}} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \mathbf{z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x} \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{z}$$

  - If we defined: $\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$, you will note that: $\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}^*\mathbf{z} + \mathbf{C}^*u \\ y &= \mathbf{B}^*\mathbf{z} \end{aligned}$
  - In other words, S_1 is completely state controllable (or observable) if and only if system S_2 is completely observable (state-controllable).

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \qquad \mathbf{C}_O = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix}$$

$$\mathbf{O}_B = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix} \qquad \mathbf{O}_B = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix}$$