Discrete-Time Control System Design

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Designing Discrete-Time Controllers

- Recall that for LTI continuous-time systems, there are a number of approaches to analyzing and designing control systems
  - Root Locus Method
  - Frequency Response Method (Using Bode Diagrams)
  - **Analytical Design Method** (Placing poles at specific location on the s-plane)
    - Response Specifications
      - 1st Order system: Time Constant & Steady-State Gain
      - 2nd Order system: Damping Ratio, Natural Frequency & Steady-State Gain

- The approach for LTI discrete-time system is also similar!
- Remember that the pole in the s-plane is: $\boxed{s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}}$
  - It is described in terms of damping ratio and natural frequency

- If you desire a similar performance in the z-plane (discrete domain), you just need to find the corresponding pole location in the z-plane

Using this relationship! $z = e^{sT}$

$$\boxed{z = e^{T\left(-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}\right)}}$$

Locating Poles in the z Plane

- While the *s*-plane is more convenient described using Cartesian coordinates, the Polar coordinate system is more natural in the z-plane

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = \sigma \pm jw$$

- Complex Exponential Refresher!
  - A complex number can be express in complex exponent form
    - Magnitude
    - Angle

$$z = re^{j\theta}$$

- Let's take a look at an *s* plane pole in *z* plane: (we consider the +ve *j* pole)

$$z = e^{T\left(-\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}\right)}$$

$$= e^{-\zeta\omega_n T} e^{j\left[T\omega_n\sqrt{1-\zeta^2}\right]}$$

Magnitude: $|z| = r = e^{-\zeta\omega_n T}$

Angle: $\angle z = \theta = T\omega_n\sqrt{1-\zeta^2}$

Quick Exercise

- It was found that one of pole of a second order discrete-time system was at *z = 0.4098 +j 0.6623*. what is the damping ratio and natural frequency of the system? The sampling period is T=1 sec.

$$z = 0.4098 + j0.6623$$

$$|z| = \sqrt{0.4098^2 + 0.6623^2} = 0.7788$$

$$|z| = 0.7788 = e^{-\zeta\omega_n T} \quad \Rightarrow \zeta\omega_n T = 0.25$$

$$\angle z = T\omega_n\sqrt{1-\zeta^2} = \tan^{-1}\frac{0.6623}{0.4098}$$

$$T\omega_n\sqrt{1-\zeta^2} = 58.25° = 1.0167$$

Combining: $\dfrac{\zeta\omega_n T}{T\omega_n\sqrt{1-\zeta^2}} = \dfrac{0.25}{1.0167}$

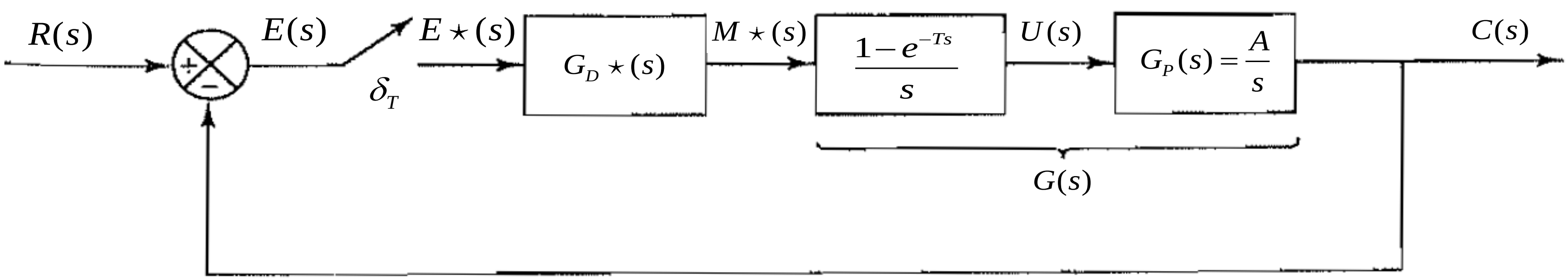
$$\zeta = 0.2388$$

$$\omega_n = \frac{0.25}{\zeta T} = 1.0469 \text{ rad/sec}$$

Remember the opposite is possible as well. If you are given the desired natural frequency and damping ration, you can compute the desired poles in the s-plane and z-plane

Integrative Exercise

- Consider the following control system. Find C(z)/R(z). T=1sec. *A*=1
  - If the discrete controller was a P controller, what is the required proportional gain such that the pole is at z=0.5?
  - If the discrete controller was a PI controller, what is the required proportional and integral gain such that the closed loop system is critically damped an undamped natural frequency of 1 rad/s



$$G(s)=\frac{1-e^{-Ts}}{s}\frac{1}{s}=\frac{1-e^{-Ts}}{s^2}\quad\Rightarrow G(z)=\left(1-z^{-1}\right)\mathcal{Z}\left[\frac{G_p(s)}{s}\right]=\left(1-z^{-1}\right)\mathcal{Z}\left[\frac{1}{s^2}\right]$$

$$G(z)=\left(1-z^{-1}\right)\frac{z^{-1}}{\left(1-z^{-1}\right)^2}=\frac{z^{-1}}{1-z^{-1}}$$

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$$G_D(z) = K_p$$

$$\frac{C(z)}{R(z)} = \frac{K_p G(z)}{1+K_p G(z)} = \frac{K_p \dfrac{z^{-1}}{1-z^{-1}}}{1+K_p \dfrac{z^{-1}}{1-z^{-1}}} = \frac{K_p z^{-1}}{1-z^{-1}+K_p z^{-1}} = \frac{K_p z^{-1}}{1+(K_p-1)z^{-1}}$$

$$\frac{C(z)}{R(z)} = \frac{K_p z^{-1}}{1+(K_p-1)z^{-1}} = \frac{K_p}{z+\left(K_p-1\right)}$$

Pole: $z = -\left(K_p - 1\right) = -K_p + 1$

$$0.5 = -K_p + 1$$

$$K_p = 0.5$$

Integrative Exercise

- Consider the following control system. Find C(z)/R(z). T=1sec. A=1
  - If the discrete controller was a P controller, what is the required proportional gain such that the pole is at z=0.5?
  - If the discrete controller was a PI controller, what is the required proportional and integral gain such that the closed loop system is critically damped an undamped natural frequency of 1 rad/s

$$G_D(z)=K_P+\frac{K_I}{1-z^{-1}}$$

$$\frac{C(z)}{R(z)}=\frac{G_D(z)G(z)}{1+G_D(z)G(z)}=\frac{\left(K_P+\frac{K_I}{1-z^{-1}}\right)\frac{z^{-1}}{1-z^{-1}}}{1+\left(K_P+\frac{K_I}{1-z^{-1}}\right)\frac{z^{-1}}{1-z^{-1}}}=\frac{\left(K_P+\frac{K_I}{1-z^{-1}}\right)z^{-1}}{(1-z^{-1})+\left(K_P+\frac{K_I}{1-z^{-1}}\right)z^{-1}}$$

$$=\frac{\left(K_P(1-z^{-1})+K_I\right)z^{-1}}{(1-z^{-1})^2+\left(K_P(1-z^{-1})+K_I\right)z^{-1}}=\frac{\left(K_P(1-z^{-1})+K_I\right)}{(1-z^{-1})^2 z+\left(K_P(1-z^{-1})+K_I\right)}$$

$$=\frac{\left(K_P(1-z^{-1})z+K_I z\right)}{(1-z^{-1})^2 z^2+\left(K_P(1-z^{-1})z+K_I z\right)}=\frac{\left(K_P(z-1)+K_I z\right)}{(z-1)^2+\left(K_P(z-1)+K_I z\right)}=\frac{(K_P+K_I)z-K_P}{z^2+(-2+K_P+K_I)z+1-K_P}$$

Integrative Exercise

- Consider the following control system. Find C(z)/R(z). T=1sec. *A*=1
  - If the discrete controller was a P controller, what is the required proportional gain such that the pole is at z=0.5?
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$$G_D(z) = K_P + \frac{K_I}{1 - z^{-1}}$$

$$\frac{C(z)}{R(z)} = \frac{(K_P + K_I)z - K_P}{z^2 + (-2 + K_P + K_I)z + 1 - K_P}$$

For critically damped response and undamped natural frequency of 1 rad/s,

$$s = -1, -1$$

$$\boxed{\Rightarrow z = e^{-1} = 0.368}$$

Hence desired characteristic equation:

$$(z - 0.368)(z - 0.368) = z^2 - 0.736z + 0.1354$$

$$\boxed{\begin{aligned} &\Rightarrow 1 - K_P = 0.1354 \\ &K_P = 0.8646 \end{aligned}}$$

$$\boxed{\begin{aligned} &\Rightarrow -2 + K_P + K_I = -0.736 \\ &K_I = -0.736 + 2 - 0.8646 = 0.3994 \end{aligned}}$$

Do reflect on the location of the poles in the s-plane and z-plane!

Minimum Settling Time Response

- Because of the transformation from s plane to z plane, there is a very interesting and peculiar that occurs only in discrete domain
- Let's start in the continuous-time domain, if we wanted to a system to respond as quickly as possible? Where would we like to place the poles on the *s* plane?

Place poles as far left on the s plane as possible! $\boxed{s = -\infty}$

- Of course, this is physically not realizable. But if we were to consider the same requirement on the discrete domain, where would we like to place the poles on the *z* plane?

Place poles on the origin! $\boxed{z = 0}$

$$z = e^{sT} = e^{-\infty} = 0$$

- Verify this using the geometrical relationship between s and z plane!
- What this means is that a control scheme which would have been impossible with analog controls is somehow possible with digital controls!

Dead-Beat Response & Control

- If a response of a closed-loop control system to a step input exhibits the **minimum possible settling time**, **zero steady-state error** and no ripples between sampling instants, this response is called a *Dead-Beat Response*.
- How is this achieved?
  - Minimum possible settling time is achieved if **ALL** the poles of the discrete-time system are strategically **placed at the origin**
  - Zero steady-state error is achieved by ensuring the system is at least a type 1 system (if it is not, add an integrator into the system).
- For an n^{th} order system, the minimum number of discrete steps required is *n.* This assumes of course the system is completely state controllable.
  - For a 2^{nd} order system under dead-beat control, only 2 time steps or samples are required to achieve dead-beat response