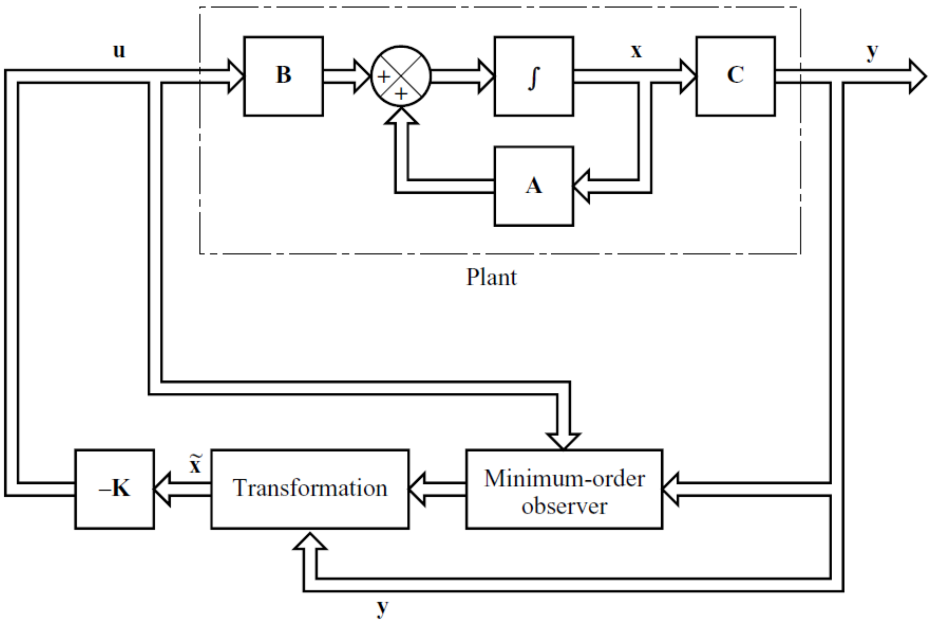
Reduced-Order Observers

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Choosing States to Observer

- Full State Observers are designed to reconstruct all state variables. In practice some of these variables can be accurately measured using sensors (position, speed, temperature, pressure, etc)
  - Such accurately measurable state variables need not be estimated
- For an *n*-dimension state vector, and *m*-dimension output vector, due to formulation of state-space, the output *m* variables are linear combinations of state variables
  - *m* state measurements do not need to be estimated
  - Only need to estimate *n-m* state variables (Reduced-Order Observer)
- If only *n-m* states are estimated, the reduced-order observer becomes an $(n\text{-}m)^{th}$ order observer and in this case is also called a **Minimum-Order Observer (MOO)**



Conceiving Minimum-Order Observer

- Consider a system with *n* state variables (**x**) and 1 output variable (y)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

- We can partition the state vector into 2 parts
  - x_a : (scalar) is the measurable portion of the state vector (related to output)
  - $\mathbf{x}_b$: (*n*-1 vector) is the unmeasurable portion of the state vector

- The partitioned state and output equations are:

$$\begin{bmatrix} \dot{x}_a \\ \dot{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} + \begin{bmatrix} B_a \\ \mathbf{B}_b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix}$$

- Rearranging x_a state equation

$$\dot{x}_a = A_{aa}x_a + \mathbf{A}_{ab}\mathbf{x}_b + B_a u$$

$$\dot{x}_a - A_{aa}x_a - B_a u = \mathbf{A}_{ab}\mathbf{x}_b$$

Can be measured — Cannot be measured

Conceiving Minimum-Order Observer

- The state equation for the unmeasurable state: $\dot{\mathbf{x}}_b = \mathbf{A}_{ba}x_a + \mathbf{A}_{bb}\mathbf{x}_b + \mathbf{B}_b u$
- Design by analogy:

	Full-order observer	Minimum-order observer
State Equation	$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$	$\dot{\mathbf{x}}_b = \mathbf{A}_{bb}\mathbf{x}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u$
Output Equation	$y = \mathbf{C}\mathbf{x}$	$\dot{x}_a - A_{aa}x_a - B_a u = \mathbf{A}_{ab}\mathbf{x}_b$

Full-Order State Observer	Minimum-Order State Observer
$\tilde{\mathbf{x}}$	$\tilde{\mathbf{x}}_b$
$\mathbf{A}$	$\mathbf{A}_{bb}$
$\mathbf{B}u$	$\mathbf{A}_{ba}x_a + \mathbf{B}_b u$
y	$\dot{x}_a - A_{aa}x_a - B_a u$
$\mathbf{C}$	$\mathbf{A}_{ab}$
$\mathbf{K}_e$ ($n \times 1$ matrix)	$\mathbf{K}_e$ [$(n-1) \times 1$ matrix]

Observer Dynamics

$$\dot{\tilde{\mathbf{x}}} = \left(\mathbf{A} - \mathbf{K}_e\mathbf{C}\right)\tilde{\mathbf{x}} + \mathbf{B}u + \mathbf{K}_e y$$

$$\boxed{\dot{\tilde{\mathbf{x}}}_b = \left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)\tilde{\mathbf{x}}_b + \mathbf{A}_{ba}x_a + \mathbf{B}_b u + \mathbf{K}_e\left(\dot{x}_a - A_{aa}x_a - B_a u\right)}$$

Improving Minimum-Order Observer

- The following expression for the minimum-order observer requires the derivative of x_a: $\dot{\tilde{\mathbf{x}}}_b = \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\tilde{\mathbf{x}}_b + \mathbf{A}_{ba} x_a + \mathbf{B}_b u + \mathbf{K}_e \left(\dot{x}_a - A_{aa} x_a - B_a u\right)$
  - This is usually not desirable as differentiation amplifies noise
- Eliminate it by re-expressing into $y = x_a$

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}}_b - \mathbf{K}_e \dot{x}_a &= \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\tilde{\mathbf{x}}_b + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right) y + \left(\mathbf{B}_b - \mathbf{K}_e B_a\right) u \\
&= \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\left(\tilde{\mathbf{x}}_b - \mathbf{K}_e y\right) + \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\mathbf{K}_e y + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right) y + \left(\mathbf{B}_b - \mathbf{K}_e B_a\right) u \\
&= \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\left(\tilde{\mathbf{x}}_b - \mathbf{K}_e y\right) + \left[\left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\mathbf{K}_e + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right)\right] y + \left(\mathbf{B}_b - \mathbf{K}_e B_a\right) u
\end{aligned}$$

- Defining:

$$\boxed{\begin{aligned} \mathbf{x}_b - \mathbf{K}_e y = \mathbf{x}_b - \mathbf{K}_e x_a &= \boldsymbol{\eta} \\ \tilde{\mathbf{x}}_b - \mathbf{K}_e y = \tilde{\mathbf{x}}_b - \mathbf{K}_e x_a &= \tilde{\boldsymbol{\eta}} \end{aligned}}$$

$$\dot{\tilde{\boldsymbol{\eta}}} = \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\tilde{\boldsymbol{\eta}} + \left[\left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}\right)\mathbf{K}_e + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right)\right] y + \left(\mathbf{B}_b - \mathbf{K}_e B_a\right) u$$

- Or:

$$\boxed{\dot{\tilde{\boldsymbol{\eta}}} = \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}} y + \hat{\mathbf{F}} u}$$

$$\boxed{\hat{\mathbf{A}} = \mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab}} \qquad \boxed{\hat{\mathbf{B}} = \hat{\mathbf{A}}\mathbf{K}_e + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right)} \qquad \boxed{\hat{\mathbf{F}} = \mathbf{B}_b - \mathbf{K}_e B_a}$$

Minimum-Order Observer Error Dynamics

- Recalling y and estimate of x:

$$y = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix}$$

$$\tilde{\mathbf{x}} = \begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} y \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-1} \end{bmatrix} \left[\tilde{\mathbf{x}}_b - \mathbf{K}_e y \right] + \begin{bmatrix} 1 \\ \mathbf{K}_e \end{bmatrix} y = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-1} \end{bmatrix} \tilde{\boldsymbol{\eta}} + \begin{bmatrix} 1 \\ \mathbf{K}_e \end{bmatrix} y$$

  - Making some definitions:
    - This describes the relationship between η and x:

$$\tilde{\mathbf{x}} = \hat{\mathbf{C}} \tilde{\boldsymbol{\eta}} + \hat{\mathbf{D}} y \qquad \hat{\mathbf{C}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n-1} \end{bmatrix} \qquad \hat{\mathbf{D}} = \begin{bmatrix} 1 \\ \mathbf{K}_e \end{bmatrix}$$

- Minimum-order error dynamics:
  - Recall these 2 equations and combining: $\dot{x}_a - A_{aa} x_a - B_a u = \mathbf{A}_{ab} \mathbf{x}_b$

$$\dot{\tilde{\mathbf{x}}}_b = \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \right) \tilde{\mathbf{x}}_b + \mathbf{A}_{ba} x_a + \mathbf{B}_b u + \mathbf{K}_e \left(\dot{x}_a - A_{aa} x_a - B_a u \right)$$

$$\boxed{\dot{\tilde{\mathbf{x}}}_b = \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \right) \tilde{\mathbf{x}}_b + \mathbf{A}_{ba} x_a + \mathbf{B}_b u + \mathbf{K}_e \mathbf{A}_{ab} \mathbf{x}_b}$$

  - And the unmeasurable state portion: $\dot{\mathbf{x}}_b = \mathbf{A}_{ba} x_a + \mathbf{A}_{bb} \mathbf{x}_b + \mathbf{B}_b u$
  - Creating the error vector: $\mathbf{e} = \mathbf{x}_b - \tilde{\mathbf{x}}_b = \boldsymbol{\eta} - \tilde{\boldsymbol{\eta}}$

$$\dot{\mathbf{x}}_b - \dot{\tilde{\mathbf{x}}}_b = \mathbf{A}_{ba} x_a + \mathbf{A}_{bb} \mathbf{x}_b + \mathbf{B}_b u - \left(\left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \right) \tilde{\mathbf{x}}_b + \mathbf{A}_{ba} x_a + \mathbf{B}_b u + \mathbf{K}_e \mathbf{A}_{ab} \mathbf{x}_b \right)$$

$$\boxed{\dot{\mathbf{x}}_b - \dot{\tilde{\mathbf{x}}}_b = \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \right) \left(\mathbf{x}_b - \tilde{\mathbf{x}}_b \right)} \qquad \boxed{\dot{\mathbf{e}} = \left(\mathbf{A}_{bb} - \mathbf{K}_e \mathbf{A}_{ab} \right) \mathbf{e}}$$

Minimum-Order Observer Error Dynamics

- The Error Dynamics is: $\boxed{\dot{\mathbf{e}} = \left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)\mathbf{e}}$
  - **e** is a (*n*-1) vector
- Again, similarly to the design of the full-order state observer, the **minimum-order system** needs to be completely observable

$$\left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)^* = \left(\mathbf{A}_{bb}{}^* - \mathbf{A}_{ab}{}^*\mathbf{K}_e{}^*\right)$$

- In order to place poles at arbitrary locations:

$$\mathbf{O}_{BM} = \left[\mathbf{A}_{ab}{}^* \quad \mathbf{A}_{bb}{}^*\mathbf{A}_{ab}{}^* \quad \cdots \quad \left(\mathbf{A}_{bb}{}^*\right)^{n-2}\mathbf{A}_{ab}{}^*\right]$$ should be rank ***n-1***

- Characteristic equation for minimum-order observer:
  - c.e. is now *n*-1 order

$$\left|s\mathbf{I} - \left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)\right| = \left|s\mathbf{I} - \left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)^*\right| = \left|s\mathbf{I} - \left(\mathbf{A}_{bb}{}^* - \mathbf{A}_{ab}{}^*\mathbf{K}_e{}^*\right)\right|$$
$$= s^{n-1} + \hat{\alpha}_1 s^{n-2} + \cdots + \hat{\alpha}_{n-2}s + \hat{\alpha}_{n-1} = 0$$

- Use Pole-Placement Techniques to determine $\mathbf{K}_e$

State Feedback Control with MOO

- Recall the Separation Principle for state feedback control with full-order observer
  - The closed-loop poles of the system consist of the poles of the state feedback system as well as the reduced-order observer
  - State feedback design and reduced observer design are **independent** of each other
  - Can be designed **separately** and combined

- Combined MOO & system characteristic equation:

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{BK}\right|\left|s\mathbf{I}-\mathbf{A}_{bb}+\mathbf{K}_{e}\mathbf{A}_{ab}\right|=0$$

$$\left|s\mathbf{I}-\mathbf{A}+\mathbf{BK}\right|\left|s\mathbf{I}-\mathbf{A}_{bb}{}^{*}+\mathbf{A}_{ab}{}^{*}\mathbf{K}_{e}{}^{*}\right|=0$$

State Feedback Control with MOO

- The state-space equation for a system under MOO state feedback is:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{B}u = \mathbf{Ax} + \mathbf{B}(-\mathbf{K}\tilde{\mathbf{x}}) = \mathbf{Ax} - \mathbf{BK}\begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix} = \mathbf{Ax} - \mathbf{BK}\begin{bmatrix} x_a \\ \mathbf{x}_b - \mathbf{e} \end{bmatrix}$$

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -\begin{bmatrix} K_a & \mathbf{K}_b \end{bmatrix}\begin{bmatrix} x_a \\ \tilde{\mathbf{x}}_b \end{bmatrix}$$

$$= \mathbf{Ax} - \mathbf{BK}\left[\begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix}\right] = \mathbf{Ax} - \mathbf{BK}\left[\mathbf{x} - \begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix}\right]$$

$$= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{B}\begin{bmatrix} K_a & \mathbf{K}_b \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{e} \end{bmatrix}$$

- Remember the error dynamics for MOO: $\dot{\mathbf{e}} = (\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab})\mathbf{e}$
- Combining and augmenting:

$$\boxed{\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BK} & \mathbf{BK}_b \\ \mathbf{0} & \mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab} \end{bmatrix}\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}}$$

  - This describes the system under minimum-order observer based state feedback control and the MOO's error dynamics

Transfer Function of MOO Based Controller

- The minimum-order observer dynamics is given by:

$$\dot{\tilde{\boldsymbol{\eta}}} = \left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)\tilde{\boldsymbol{\eta}} + \left[\left(\mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab}\right)\mathbf{K}_e + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right)\right]y + \left(\mathbf{B}_b - \mathbf{K}_e B_a\right)u$$

- The equations that govern an MOO + SF Control:

$$\boxed{\begin{aligned}\dot{\tilde{\boldsymbol{\eta}}} &= \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}}y + \hat{\mathbf{F}}u \\ \tilde{\boldsymbol{\eta}} &= \tilde{\mathbf{x}}_b - \mathbf{K}_e y \\ u &= -\mathbf{K}\tilde{\mathbf{x}}\end{aligned}}$$

$$\begin{aligned}\hat{\mathbf{A}} &= \mathbf{A}_{bb} - \mathbf{K}_e\mathbf{A}_{ab} \\ \hat{\mathbf{B}} &= \hat{\mathbf{A}}\mathbf{K}_e + \left(\mathbf{A}_{ba} - \mathbf{K}_e A_{aa}\right) \\ \hat{\mathbf{F}} &= \mathbf{B}_b - \mathbf{K}_e B_a\end{aligned}$$

- Expanding the state feedback and expressing in terms of $\boldsymbol{\eta}$

$$u = -\mathbf{K}\tilde{\mathbf{x}} = -\left[K_a \quad \mathbf{K}_b\right]\begin{bmatrix} y \\ \tilde{\mathbf{x}}_b \end{bmatrix} = -K_a y - \mathbf{K}_b\tilde{\mathbf{x}}_b = -\mathbf{K}_b\tilde{\boldsymbol{\eta}} - \left(K_a + \mathbf{K}_b\mathbf{K}_e\right)y$$

- Inserting back to observer dynamics:

$$\dot{\tilde{\boldsymbol{\eta}}} = \hat{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \hat{\mathbf{B}}y + \hat{\mathbf{F}}\left(-\mathbf{K}_b\tilde{\boldsymbol{\eta}} - \left(K_a + \mathbf{K}_b\mathbf{K}_e\right)y\right)$$

$$\dot{\tilde{\boldsymbol{\eta}}} = \left(\hat{\mathbf{A}} - \hat{\mathbf{F}}\mathbf{K}_b\right)\tilde{\boldsymbol{\eta}} + \left[\hat{\mathbf{B}} - \hat{\mathbf{F}}\left(K_a + \mathbf{K}_b\mathbf{K}_e\right)\right]y$$

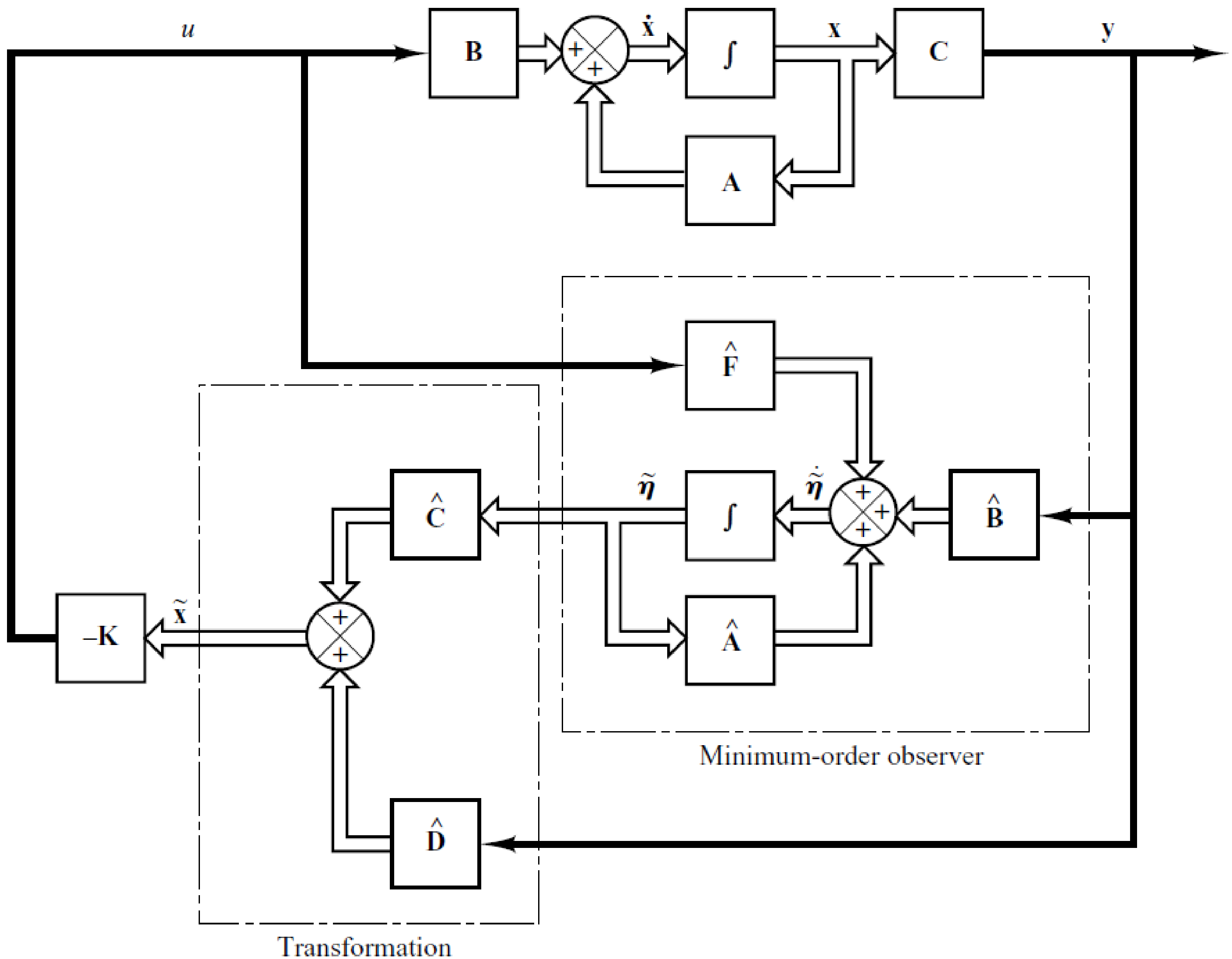
$$\boxed{\begin{aligned}\dot{\tilde{\boldsymbol{\eta}}} &= \tilde{\mathbf{A}}\tilde{\boldsymbol{\eta}} + \tilde{\mathbf{B}}y \\ u &= \tilde{\mathbf{C}}\tilde{\boldsymbol{\eta}} + \tilde{D}y\end{aligned}}$$

'traditional' SS system!

$$\begin{aligned}\tilde{\mathbf{A}} &= \left(\hat{\mathbf{A}} - \hat{\mathbf{F}}\mathbf{K}_b\right) \\ \tilde{\mathbf{B}} &= \left[\hat{\mathbf{B}} - \hat{\mathbf{F}}\left(K_a + \mathbf{K}_b\mathbf{K}_e\right)\right] \\ \tilde{\mathbf{C}} &= -\mathbf{K}_b \\ \tilde{D} &= -\left(K_a + \mathbf{K}_b\mathbf{K}_e\right)\end{aligned}$$

TF of MOO controller: $$\frac{U(s)}{-Y(s)} = -\left[\tilde{\mathbf{C}}\left(s\mathbf{I} - \tilde{\mathbf{A}}\right)^{-1}\tilde{\mathbf{B}} + \tilde{D}\right]$$

Block Diagram of State Feedback with Minimum-Order Observer



Exercise

- For the following system, design a state feedback controller that utilizes a minimum-order observer.
  - Desired observer poles: $s=-10, s=-10$
  - Desired state feedback poles: $s=-2+j2\sqrt{3}, s=-2-j2\sqrt{3}, s=-6$
  - Construct a Block Diagram schematic of the entire system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad y = \mathbf{C}\mathbf{x}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_a \\ \dot{\mathbf{x}}_b \end{bmatrix} = \begin{bmatrix} A_{aa} & \mathbf{A}_{ab} \\ \mathbf{A}_{ba} & \mathbf{A}_{bb} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix} + \begin{bmatrix} B_a \\ \mathbf{B}_b \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & \mathbf{0} \end{bmatrix} \begin{bmatrix} x_a \\ \mathbf{x}_b \end{bmatrix}$$

$$A_{aa}=0,\quad \mathbf{A}_{ab} = \begin{bmatrix} 1 & 0 \end{bmatrix},\quad \mathbf{A}_{ba} = \begin{bmatrix} 0 \\ -6 \end{bmatrix},\quad \mathbf{A}_{bb} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix},\quad B_a = 0,\quad \mathbf{B}_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Check state controllability

$$\mathbf{C}_o = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \mathbf{A}^2\mathbf{B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

Controllability matrix is rank 3.(State vector is 3 dimension). System is completely state controllable

Check minimum observability

$$\mathbf{O}_{BM} = \begin{bmatrix} \mathbf{A}_{ab}{}^* & \mathbf{A}_{bb}{}^* \mathbf{A}_{ab}{}^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Minimum Observability matrix is rank 2.(unmeasurable state vector is 2 dimension). System is completely minimum-order observable

Exercise (Minimum-Order Observer Design)

- Desired c.e: $(s+10)(s+10)=s^2+20s+100=0$ $\quad \left|s\mathbf{I}-\mathbf{A}_{bb}*+\mathbf{A}_{ab}*\mathbf{K}_e*\right|=0$

Using Ackermann's Formula:

$$\mathbf{K}_e*=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\mathbf{A}_{ab}* & \mathbf{A}_{bb}*\mathbf{A}_{ab}*\end{bmatrix}^{-1}\phi(\mathbf{A}_{bb}*)$$

$$\mathbf{K}_e=\phi(\mathbf{A}_{bb})\begin{bmatrix}\mathbf{A}_{ab}\\ \mathbf{A}_{ab}\mathbf{A}_{bb}\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}$$

$$\phi(\mathbf{A}_{bb})=(\mathbf{A}_{bb})^2+20\mathbf{A}_{bb}+100\mathbf{I}=\begin{bmatrix}89 & 14\\ -154 & 5\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{A}_{ab}\\ \mathbf{A}_{ab}\mathbf{A}_{bb}\end{bmatrix}^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\boxed{\mathbf{K}_e=\phi(\mathbf{A}_{bb})\begin{bmatrix}\mathbf{A}_{ab}\\ \mathbf{A}_{ab}\mathbf{A}_{bb}\end{bmatrix}^{-1}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}89 & 14\\ -154 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}14\\5\end{bmatrix}}$$

Exercise (State Feedback Design)

- Desired c.e: $\left(s+2-j2\sqrt{3}\right)\left(s+2+j2\sqrt{3}\right)(s+6)=\left(s^2+4s+16\right)(s+6)=s^3+10s^2+40s+96$

$$\alpha_1 = 10 \quad \alpha_2 = 40 \quad \alpha_3 = 96$$

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \qquad a_1 = 6 \quad a_2 = 11 \quad a_3 = 6$$

$$\boxed{\begin{aligned} \mathbf{K} &= \left[\alpha_3 - a_3 \quad \alpha_2 - a_2 \quad \alpha_1 - a_1\right] \\ &= \left[96-6 \quad 40-11 \quad 10-6\right] \\ &= \left[90 \quad 29 \quad 4\right] \end{aligned}}$$

Block Diagram