Canonical Forms

EPD 30.114 ADVANCED FEEDBACK & CONTROL

State-Space Representation of TF Systems

- Many techniques exist in describing a Transfer Function systems in state-space: (Non-uniqueness)
  - **Controllable Canonical Form** (Any)
  - **Observable Canonical Form** (Any)
  - **Modal Canonical Form**
    - **Diagonal Canonical Form** (Only for distinct roots)
    - **Jordan Canonical Form** (Only for repeated distinct roots)
- Consider the following general differential equation

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

And transfer function:

$$\boxed{\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}}$$

Controllable Canonical Form (CCF)

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Observable Canonical Form (OCF)

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Notice anything peculiar between **CCF** and **OCF**?

State matrix (**A**) of CCF is the transpose of the state matrix of OCF

Diagonal Canonical Form (DCF)

- If the system contains only **distinct roots**, the transfer function can be expressed as: $\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{(s+p_1)(s+p_2)\cdots(s+p_n)} = b_0 + \frac{c_1}{s+p_1} + \frac{c_2}{s+p_2} + \cdots + \frac{c_n}{s+p_n}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & & 0 \\ & -p_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Jordan Canonical Form (JCF)

- If the system contain repeated roots, the TF can be expressed as:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{(s+p_1)^3 (s+p_4) \cdots (s+p_n)} = b_0 + \frac{c_1}{(s+p_1)^3} + \frac{c_2}{(s+p_1)^2} + \frac{c_3}{s+p_1} + \cdots + \frac{c_n}{s+p_n}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \left[\begin{array}{ccc|ccc} -p_1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -p_1 & 1 & \vdots & & \vdots \\ 0 & 0 & -p_1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & -p_4 & & 0 \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & \\ \cdot & & \cdot & & \cdot & \\ 0 & \cdots & 0 & 0 & & -p_n \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Exercise!

- Consider the following TF: $\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{2}{(s+1)} + \frac{-1}{(s+2)}$
  - How many and where are the poles of the system? **2 poles, s=-2,-1**
  - System Order? **2^{nd} Order System**
  - Number of state variables required for State-Space representation?
  - Express the system in **2 state variables required**
    - Controllable Canonical Form (CCF)
    - Observable Canonical Form (OCF)
    - Diagonal Canonical Form (DCF)
    - Jordan Canonical Form (JCF)

$$a_1 = 3, \quad a_2 = 2$$
$$b_0 = 0, \quad b_1 = 1, \quad b_2 = 3$$

CCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

OCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

DCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Block Diagrams from Canonical Forms (CCF)

- Consider a 3[rd] order dynamic system (y is output): $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$
  - Transfer function: $\dfrac{Y(s)}{U(s)} = \dfrac{6}{s^3 + 6s^2 + 11s + 6}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = -6x_1 - 11x_2 - 6x_3 + u$$

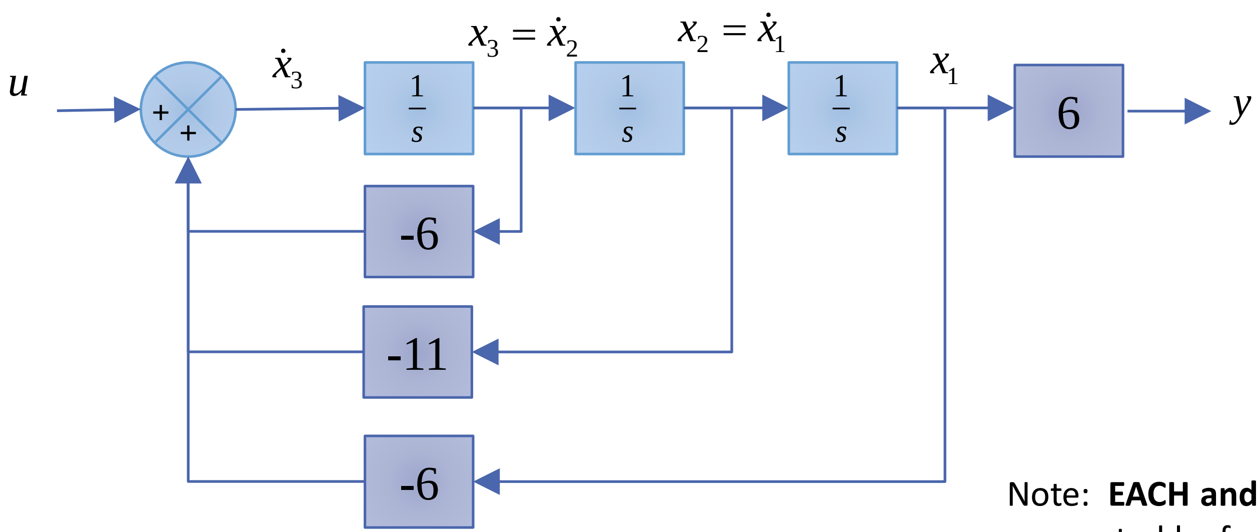
$$y = 6x_1$$

$$a_1 = 6,\ a_2 = 11,\ a_3 = 6$$
$$b_0 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 6$$

CCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 6 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Note: **EACH and EVERY** state-variable connected by feedback to the control input

Block Diagrams from Canonical Forms (OCF)

- Realize OCF Block Diagram from $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

$$\dot{x}_1 = -6x_3 + 6u$$
$$\dot{x}_2 = x_1 - 11x_3$$
$$\dot{x}_3 = x_2 - 6x_3$$
$$y = x_3$$

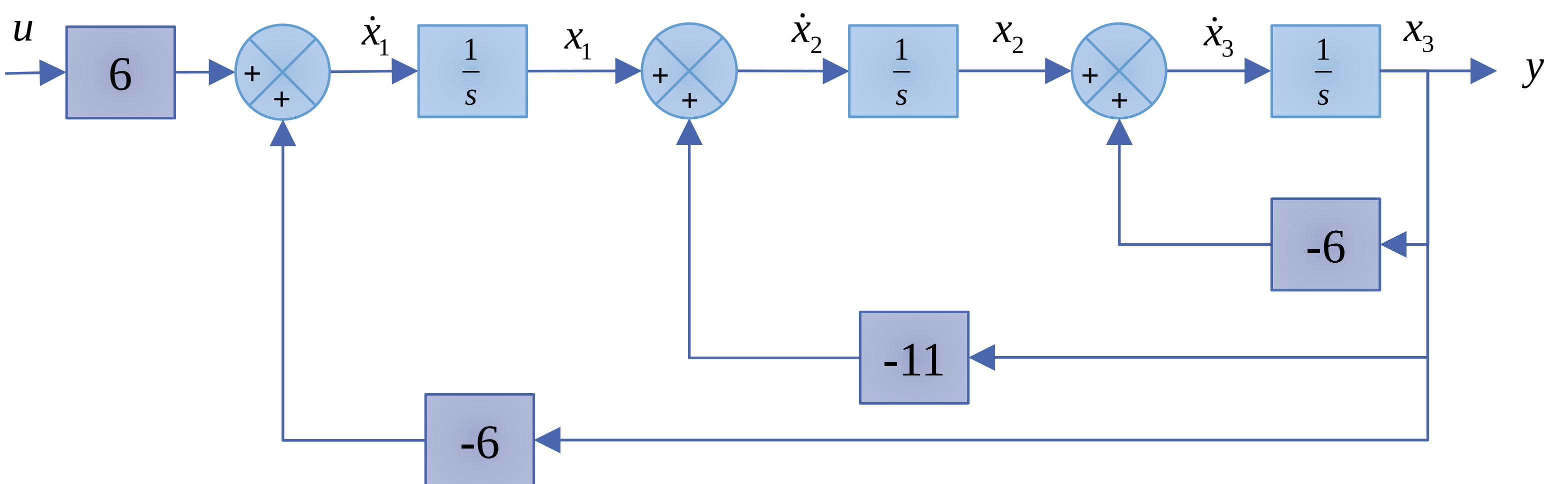
$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}$$

$$a_1 = 6,\ a_2 = 11,\ a_3 = 6$$
$$b_0 = 0,\ b_1 = 0,\ b_2 = 0,\ b_3 = 6$$

OCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$



Note: There is feedback from the output to **EACH and EVERY** state-variable

Block Diagrams from Canonical Forms (DCF)

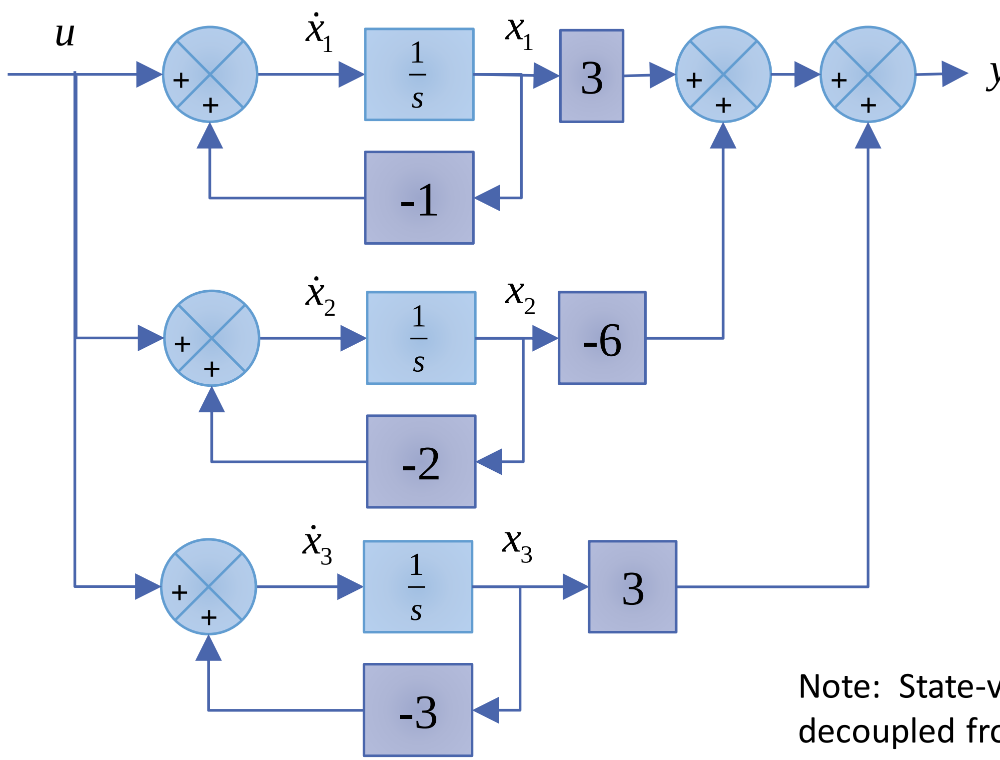
- Realize DCF Block Diagram from $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6} = \frac{6}{(s+1)(s+2)(s+3)} = \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

$$\dot{x}_1 = -1x_1 + u$$
$$\dot{x}_2 = -2x_2 + u$$
$$\dot{x}_3 = -3x_3 + u$$

$$p_1 = 1,\ p_2 = 2,\ p_3 = 3$$
$$c_1 = 3,\ c_2 = -6,\ c_3 = 3$$

$$y = 3x_1 - 6x_2 + 3x_3$$



DCF

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note: State-variables are completely decoupled from one another

Exercise!

- A thermal system is described by the following transfer function. Express system in CCF, OCF and DCF and construct the associated block diagram.

$$G(s)=\frac{s+2}{s^2+7s+12}=\frac{2}{s+4}+\frac{-1}{s+3}$$

CCF

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -12 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}2 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

OCF

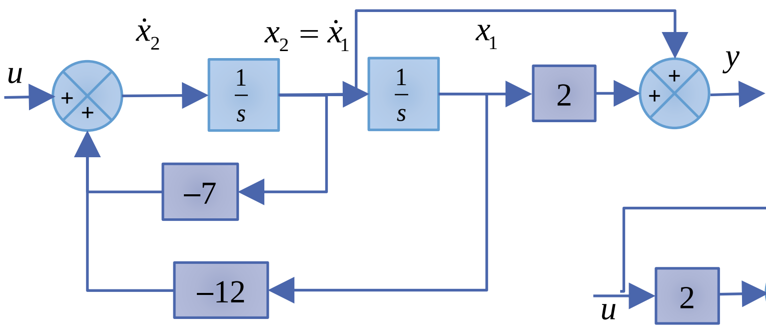
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & -12\\ 1 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}2\\ 1\end{bmatrix}u$$

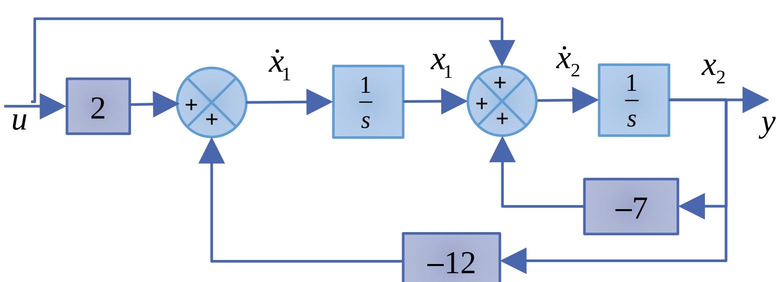
$$y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

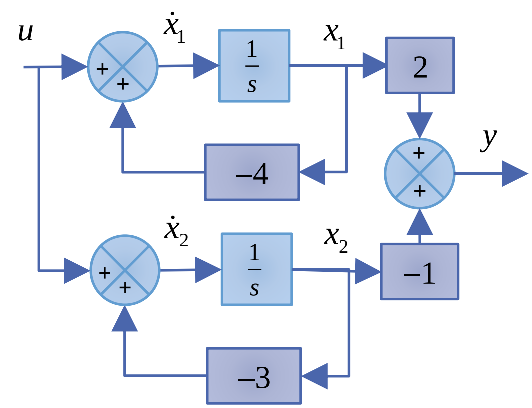
DCF

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-4 & 0\\ 0 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$







Transforming Between Canonical Forms

- From a differential equation or transfer function you can realize the state-space canonical forms
- Is it possible to calculate desired canonical form from any SS representation without obtaining TF first?
- Consider a system described by an arbitrary state equations (SISO)

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u$$
$$y = \mathbf{H}\mathbf{x} + Ju$$

- We seek a transformation to $\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$

$$y = \mathbf{C}\mathbf{z} + Du$$

such that **A**,**B**,**C** and D are in required form for desired canonical form
  - Controllable Canonical Form (CCF)
  - Observable Canonical Form (OCF)
  - Diagonal Canonical Form (DCF)

State Transformation

$$\dot{\mathbf{x}} = \mathbf{F}\mathbf{x} + \mathbf{G}u \qquad\Longrightarrow\qquad \dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

$$y = \mathbf{H}\mathbf{x} + Ju \qquad\qquad y = \mathbf{C}\mathbf{z} + Du$$

- Consider a linear transformation from **x** to **z**:

$$\boxed{\mathbf{x} = \mathbf{T}\mathbf{z}}$$

- Substituting: $\dot{\mathbf{x}} = \mathbf{T}\dot{\mathbf{z}} = \mathbf{F}\mathbf{T}\mathbf{z} + \mathbf{G}u$

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T}\mathbf{z} + \mathbf{T}^{-1}\mathbf{G}u = \mathbf{A}\mathbf{z} + \mathbf{B}u \qquad y = \mathbf{H}\mathbf{T}\mathbf{z} + Ju = \mathbf{C}\mathbf{z} + Ju$$

- Hence:

$$\boxed{\begin{aligned}\mathbf{A} &= \mathbf{T}^{-1}\mathbf{F}\mathbf{T}\\ \mathbf{B} &= \mathbf{T}^{-1}\mathbf{G}\end{aligned}} \qquad \boxed{\begin{aligned}\mathbf{C} &= \mathbf{H}\mathbf{T}\\ D &= J\end{aligned}}$$

- Rewriting:

$$\boxed{\mathbf{A}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{F}}$$

$$\boxed{\mathbf{T}^{-1}\mathbf{G} = \mathbf{B}}$$

- Describe $\mathbf{T}^{-1}$ as matrix with vector rows $\mathbf{t}_1,\ldots,\mathbf{t}_n$

$$\boxed{\mathbf{T}^{-1} = \begin{bmatrix}\mathbf{t}_1\\ \mathbf{t}_2\\ \vdots\\ \mathbf{t}_{n-1}\\ \mathbf{t}_n\end{bmatrix}}$$

State Transformation (to CCF)

- Without loss of generality, a third order case is considered (Goal: to CCF)

$$\boxed{\mathbf{A}\mathbf{T}^{-1} = \mathbf{T}^{-1}\mathbf{F}}$$

$$\boxed{\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \left[b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0 \right] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} \mathbf{t}_1 \\ \mathbf{t}_2 \\ \mathbf{t}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{t}_1\mathbf{F} \\ \mathbf{t}_2\mathbf{F} \\ \mathbf{t}_3\mathbf{F} \end{bmatrix}$$

$$\mathbf{t}_2 = \mathbf{t}_1\mathbf{F}$$

$$\mathbf{t}_3 = \mathbf{t}_2\mathbf{F} = \mathbf{t}_1\mathbf{F}^2$$

$$\boxed{\mathbf{T}^{-1}\mathbf{G} = \mathbf{B}}$$

$$\begin{bmatrix} \mathbf{t}_1\mathbf{G} \\ \mathbf{t}_2\mathbf{G} \\ \mathbf{t}_3\mathbf{G} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{t}_1\mathbf{G} = 0$$

$$\mathbf{t}_2\mathbf{G} = \mathbf{t}_1\mathbf{F}\mathbf{G} = 0$$

$$\mathbf{t}_3\mathbf{G} = \mathbf{t}_1\mathbf{F}^2\mathbf{G} = 1$$

Writing in Matrix Form: $\boxed{\mathbf{t}_1 \begin{bmatrix} \mathbf{G} & \mathbf{F}\mathbf{G} & \mathbf{F}^2\mathbf{G} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}}$

$$\boxed{\mathbf{C}_\mathbf{O} = \begin{bmatrix} \mathbf{G} & \mathbf{F}\mathbf{G} & \mathbf{F}^2\mathbf{G} \end{bmatrix} \quad \text{CONTROLLABILITY MATRIX}}$$

$$\boxed{\mathbf{t}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{F}\mathbf{G} & \mathbf{F}^2\mathbf{G} \end{bmatrix}^{-1}}$$

$$\mathbf{t}_2 = \mathbf{t}_1\mathbf{F} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{F}\mathbf{G} & \mathbf{F}^2\mathbf{G} \end{bmatrix}^{-1} \mathbf{F}$$

$$\mathbf{t}_3 = \mathbf{t}_1\mathbf{F}^2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{G} & \mathbf{F}\mathbf{G} & \mathbf{F}^2\mathbf{G} \end{bmatrix}^{-1} \mathbf{F}^2$$

Note: Transformation is only possible if the CONTROLLABILITY MATRIX is invertible (non-singular, full rank)

State Transformation (to CCF)

- For a system with *n* states, converting any SS representation to CCF is as follows:
  - From **F** & **G**, construct the **CONTROLLABILITY MATRIX**

$$\mathbf{C_O} = \begin{bmatrix} \mathbf{G} & \mathbf{FG} & \cdots & \mathbf{F}^{n-2}\mathbf{G} & \mathbf{F}^{n-1}\mathbf{G} \end{bmatrix}$$

  - Compute the first row of the inverse of the transformation matrix

$$\mathbf{t}_1 = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix} \mathbf{C_O}^{-1}$$

  - Construct the entire transformation matrix as

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_1\mathbf{F}^{n-2} \\ \mathbf{t}_1\mathbf{F}^{n-1} \end{bmatrix}$$

  - The transformed matrices are: $\mathbf{A} = \mathbf{T}^{-1}\mathbf{FT}$ $\quad \mathbf{C} = \mathbf{HT}$

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G} \qquad D = J$$

State Transformation (to OCF)

- Without loss of generality, a third order case is considered (Goal: to OCF)

$$\boxed{\mathbf{TA} = \mathbf{FT}} \qquad \boxed{\mathbf{T} = [\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{t}_3]}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \cdot \\ \cdot \\ \cdot \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = [0 \quad 0 \quad \cdots \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

$$[\mathbf{t}_1 \quad \mathbf{t}_2 \quad \mathbf{t}_3] \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} = [\mathbf{Ft}_1 \quad \mathbf{Ft}_2 \quad \mathbf{Ft}_3]$$

$$\mathbf{t}_2 = \mathbf{Ft}_1$$

$$\mathbf{t}_3 = \mathbf{Ft}_2 = \mathbf{F}^2\mathbf{t}_1$$

Writing in Matrix Form: $\boxed{\begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix} \mathbf{t}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$

$$\boxed{\mathbf{C} = \mathbf{HT}}$$

$$[0 \quad 0 \quad 1] = [\mathbf{Ht}_1 \quad \mathbf{Ht}_2 \quad \mathbf{Ht}_3]$$

$$\mathbf{Ht}_1 = 0$$

$$\mathbf{Ht}_2 = \mathbf{HFt}_1 = 0$$

$$\mathbf{Ht}_3 = \mathbf{HF}^2\mathbf{t}_1 = 1$$

$$\boxed{\mathbf{O_B} = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix} \quad \text{OBSERVABILITY MATRIX}}$$

$$\boxed{\mathbf{t}_1 = \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\mathbf{t}_2 = \mathbf{Ft}_1 = \mathbf{F} \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{t}_3 = \mathbf{F}^2 \begin{bmatrix} \mathbf{H} \\ \mathbf{HF} \\ \mathbf{HF}^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Note: Transformation is only possible if the OBSERVABILITY MATRIX is invertible (non-singular, full rank)

State Transformation (to OCF)

- For a system with *n* states, converting any SS representation to OCF is as follows:
  - From **F** & **H**, construct the **OBSERVABILITY MATRIX**

$$\mathbf{O_B} = \begin{bmatrix} \mathbf{H} \\ \vdots \\ \mathbf{HF}^{n-2} \\ \mathbf{HF}^{n-1} \end{bmatrix}$$

  - Compute the first column of the transformation matrix

$$\mathbf{t}_1 = \mathbf{O_B}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

  - Construct the entire transformation matrix as

$$\mathbf{T} = \begin{bmatrix} \mathbf{t}_1 & \cdots & \mathbf{F}^{n-2}\mathbf{t}_1 & \mathbf{F}^{n-1}\mathbf{t}_1 \end{bmatrix}$$

  - The transformed matrices are:

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T} \qquad \mathbf{C} = \mathbf{H}\mathbf{T}$$

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G} \qquad D = J$$

State Transformation (to DCF)

- Without loss of generality, a third order case is considered (Goal: to DCF)

$$\boxed{\mathbf{TA}=\mathbf{FT}} \qquad \boxed{\mathbf{T}^{-1}\mathbf{G}=\mathbf{B}} \qquad \boxed{\mathbf{T}=\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3\end{bmatrix}}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \cdot\\ \cdot\\ \cdot\\ \dot{x}_n\end{bmatrix}=\begin{bmatrix}-p_1 & & & & & 0\\ & -p_2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ 0 & & & & & -p_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \cdot\\ \cdot\\ \cdot\\ x_n\end{bmatrix}+\begin{bmatrix}1\\ 1\\ \cdot\\ \cdot\\ \cdot\\ 1\end{bmatrix}u$$

$$y=\begin{bmatrix}c_1 & c_2 & \cdots & c_n\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ \cdot\\ \cdot\\ \cdot\\ x_n\end{bmatrix}+b_0u$$

$$\begin{bmatrix}\mathbf{t}_1 & \mathbf{t}_2 & \mathbf{t}_3\end{bmatrix}\begin{bmatrix}-p_1 & 0 & 0\\ 0 & -p_2 & 0\\ 0 & 0 & -p_3\end{bmatrix}=\begin{bmatrix}\mathbf{Ft}_1 & \mathbf{Ft}_2 & \mathbf{Ft}_3\end{bmatrix}$$

$$-p_1\mathbf{t}_1=\mathbf{Ft}_1$$
$$-p_2\mathbf{t}_2=\mathbf{Ft}_2$$
$$-p_3\mathbf{t}_3=\mathbf{Ft}_3$$

This is the famous eigenvector/eigenvalue problem

Eigenvalues of **F** are the poles of the system: p_1, p_2 and p_3
Eigenvectors of **F** are the columns of **T**: $\mathbf{t}_1$, $\mathbf{t}_2$ and $\mathbf{t}_3$
Magnitude of eigenvectors are chosen such that **B**=[1 1 … 1]^T

State Transformation (to DCF)

- For a system with *n* states, converting any SS representation to DCF is as follows:
  - From **F**, compute the <u>eigenvalues</u> and <u>eigenvectors</u>.

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \qquad v_1, v_2, v_3, \ldots, v_n$$

  - The new state matrix is a diagonal matrix with elements made up of the eigenvalues

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

  - Construct the entire transformation matrix as $\mathbf{T} = \begin{bmatrix} \alpha_1 v_1 & \cdots & \alpha_{n-1} v_{n-1} & \alpha_n v_n \end{bmatrix}$
    - where the scaling factors are chosen such that **B** is composed of all 1's.
  - The transformed matrices are:

$$\mathbf{A} = \mathbf{T}^{-1}\mathbf{F}\mathbf{T} \qquad \mathbf{C} = \mathbf{H}\mathbf{T}$$

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{G} \qquad D = J$$

Exercise

- Find the matrix to transform the following system to DCF.

$$\dot{\mathbf{x}} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix}\mathbf{x}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} -p_1 & & & & 0 \\ & -p_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & -p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u$$

Goal is to find a Transformation such that **A** is diagonal.

Need to find Eigenvalues/Eigenvectors of $\mathbf{F} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix}$

Let's take the eigenvectors to be: $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Eigenvectors should satisfy: $\mathbf{Fv} = \lambda_1 \mathbf{v},\ \mathbf{Fu} = \lambda_2 \mathbf{u}$

$$\begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \lambda \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$-7w_1 - 12w_2 = \lambda w_1$$

$$w_1 = \lambda w_2$$

$$\Rightarrow -7\lambda w_2 - 12 w_2 = \lambda^2 w_2$$

$$\Rightarrow \lambda^2 + 7\lambda + 12 = 0$$

$$\Rightarrow (\lambda+4)(\lambda+3) = 0$$

$$\mathbf{v}:\ v_1 = -4v_2\ ,\ \mathbf{v} = \begin{bmatrix} -4v_2 \\ v_2 \end{bmatrix}$$

$$\mathbf{u}:\ u_1 = -3u_2\ ,\ \mathbf{u} = \begin{bmatrix} -3u_2 \\ u_2 \end{bmatrix}$$

$$\mathbf{T} = \begin{bmatrix} -4v_2 & -3u_2 \\ v_2 & u_2 \end{bmatrix} \quad \mathbf{T}^{-1} = \frac{1}{-4v_2u_2 + 3u_2v_2} \begin{bmatrix} u_2 & 3u_2 \\ -v_2 & -4v_2 \end{bmatrix}$$

Need to ensure: $\mathbf{B} = \mathbf{T}^{-1}\mathbf{G} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\frac{1}{-4v_2u_2 + 3u_2v_2} \begin{bmatrix} u_2 & 3u_2 \\ -v_2 & -4v_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u_2 = 1,\ v_2 = -1$$

$$\mathbf{T} = \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \qquad \mathbf{A} = \mathbf{T}^{-1}\mathbf{FT}$$

Check for **A**!

Eigenvalues & Characteristic Equation (c.e.)

- The **A** matrix has a special property whereby the eigenvalues of **A** are the roots of the characteristic equation for the system in absence of input

$$\left|s\mathbf{I}-\mathbf{A}\right|=0$$

- Remember, the roots of the characteristic equations are also the **poles** of the systems.

- The Eigenvalues are also **invariant** (not affected by) to linear transformations applied to **A**
  - Consider **T** as an arbitrary linear Transformation Matrix $\mathbf{A}=\mathbf{T}^{-1}\mathbf{F}\mathbf{T}$

$$\begin{aligned}\left|s\mathbf{I}-\mathbf{T}^{-1}\mathbf{F}\mathbf{T}\right| &= \left|s\mathbf{T}^{-1}\mathbf{T}-\mathbf{T}^{-1}\mathbf{F}\mathbf{T}\right| \\ &= \left|\mathbf{T}^{-1}\left(s\mathbf{I}-\mathbf{F}\right)\mathbf{T}\right| \\ &= \left|\mathbf{T}^{-1}\right|\left|s\mathbf{I}-\mathbf{F}\right|\left|\mathbf{T}\right| = \left|\mathbf{T}^{-1}\mathbf{T}\right|\left|s\mathbf{I}-\mathbf{F}\right| \\ &= \left|s\mathbf{I}-\mathbf{F}\right| = \left|s\mathbf{I}-\mathbf{A}\right|\end{aligned}$$

Eigenvalues of **A** are also the Eigenvalues of **F**