State-Space System Stability

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Stability Analysis in State-Space

- Stability of a system is a **critical** concern. For any practical purpose, the states of the system (or controlled system) **MUST** be stable.

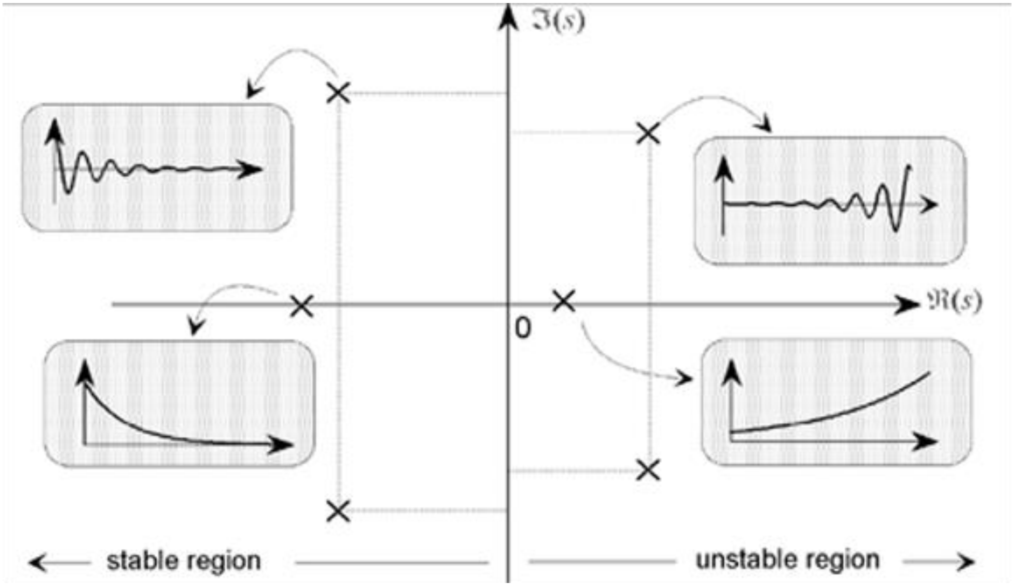
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \dot{\mathbf{x}} = \left(\mathbf{A} - \mathbf{B}\mathbf{K}\right)\mathbf{x}$$

Open-Loop Closed-Loop

- A State-Space LTI system is **STABLE** if **ALL** eigenvalues (aka **Characteristic Equation**) of **A** or **A-BK** have **NEGATIVE real** parts (ALL EIGENVALUES IN THE LEFT HAND S-PLANE). It is unstable otherwise.

c.e.: $|s\mathbf{I} - \mathbf{A}| = 0 \ , \quad |s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}| = 0$

- Types of Stability
  - **<u>Absolute/Internal</u> Stability**
    - **LHP eigenvalues**
  - <u>Neutral</u> Stability
    - Non-repeated $j\omega$ axis eigenvalues
  - Unstable
    - Repeated $j\omega$ axis eigenvalues
    - RHP eigenvalues



Routh-Hurwitz Stability Criterion

- The characteristic equation will be in the form: (*n* is the order of the system)

$$\left|s\mathbf{I}-\mathbf{A}\right|=0 \quad , \quad \left|s\mathbf{I}-\mathbf{A}+\mathbf{BK}\right|=0 \quad \Rightarrow \qquad D(s)=a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

- If *n* is large, it can be challenging to evaluate the roots. A good alternative is to use the Routh-Hurwitz Stability Criterion

- **A System is STABLE <u>IF AND ONLY IF ALL ELEMENTS</u> in the first column of the Routh Array are positive (Necessary and Sufficient Condition for Stability)**

- <u>R-H Stability Criterion Procedure</u>
  1. Write the characteristic equation in descending powers of s.

$$D(s)=a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n = 0$$

  2. If any **coefficients** are **zero or negative** in the presence of at least on positive coefficient, there is a root or roots that are imaginary or have positive real parts. <u>SYSTEM IS NOT STABLE</u>
  3. If all coefficients are positive, construct a Routh Array
  4. The R-H Stability criterion states that the number of roots in the CE with positive real parts (UNSTABLE) is equal to the number **of changes in sign** of the **coefficients in the first column of the Routh array**

The Routh Array

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

- Arrange the coefficients of D(s) in the following pattern:

$$\begin{array}{c|c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}, b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}, b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}, c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}, c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}, d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}, \cdots$$

R-H Criteria:# of roots in *D(s)* with positive real parts = # of changes in sign of the coefficients in the box

For a stable system : ALL COEFFICIENTS IN BOX MUST BE POSITIVE

Example

- Consider the characteristic equation:

$$D(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

What are the conditions in which this C.E is stable? (a_0, a_1, a_2, a_3 are positive numbers

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & b_1 & \\ s^0 & c_1 & \end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$c_1 = \frac{b_1 a_3}{b_1} = a_3$$

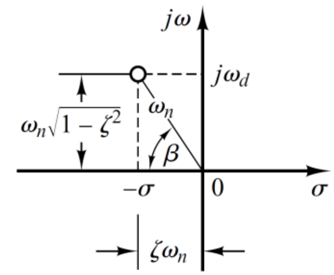
System is stable if:

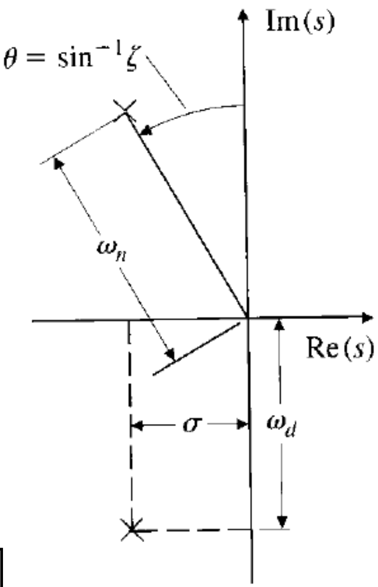
$$a_1 a_2 - a_0 a_3 > 0$$

$$a_1 a_2 > a_0 a_3$$

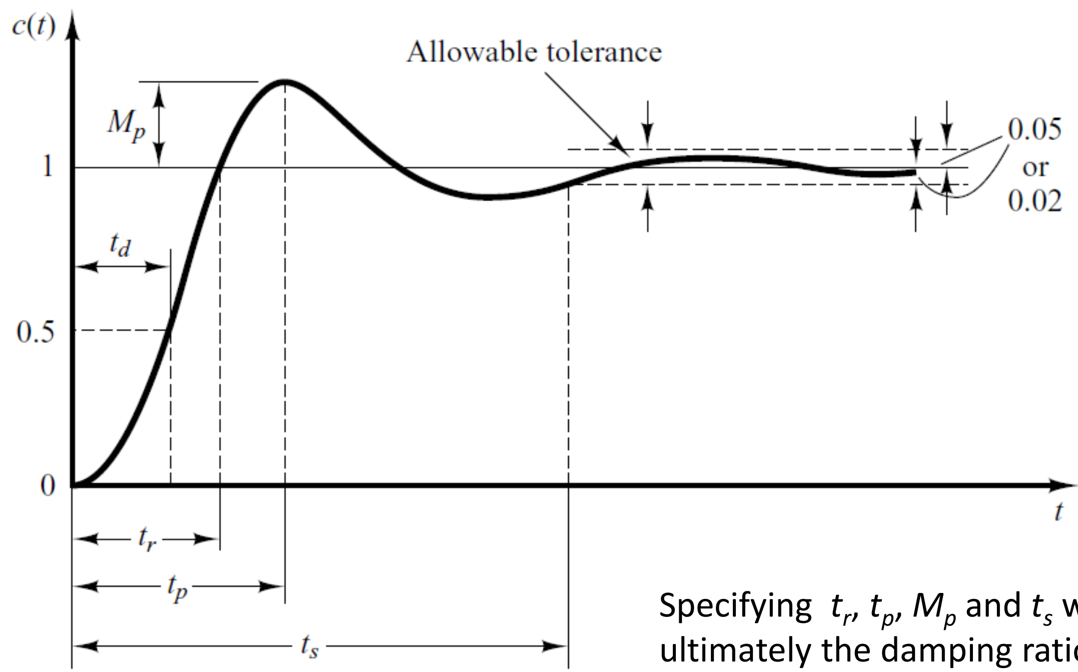
RECAP: 2nd Order System

- Characteristic Equation: $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$
- Unit-Step Response
  - Characterized by damping ratio ζ and natural frequency ω_n





$$\text{poles} = -\zeta\omega_n \pm j\omega_d \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$



Rise Time $\boxed{t_r = \dfrac{\pi - \beta}{\omega_d}}$

Peak Time $\boxed{t_p = \dfrac{\pi}{\omega_d}}$

Maximum Overshoot $\boxed{M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}}$ Depends on ζ only

Settling Time (0.02) $\boxed{t_s = \dfrac{4}{\zeta\omega_n}}$

Specifying t_r, t_p, M_p and t_s will completely define the response and ultimately the damping ratio and natural frequency of the system