# Impulse Sampling & Data Hold

EPD 30.114 ADVANCED FEEDBACK & CONTROL

# Impulse Sampling

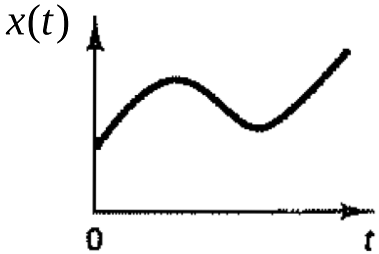
- Discrete-time control systems may operate partly in discrete time and partly in continuous time:
  - Some signals represented as discrete-time functions, *x(k)*
  - Other signals as continuous functions, *y(t)*.
- Impulse sampling is the ideal process of obtaining a sampled (discrete) function from a continuous signal The output of this process is train of **impulses**:
  - That begins with *t=0*, (*x(t)=0* for *t<0*)
  - Sampling period equal to T
  - Strength of each impulse equal to the sampled value of the continuous-time signal at the corresponding sampling instant.
- The impulse-sampled output will be an infinite sequence of impulses and denoted by (referred to as the starred notation):

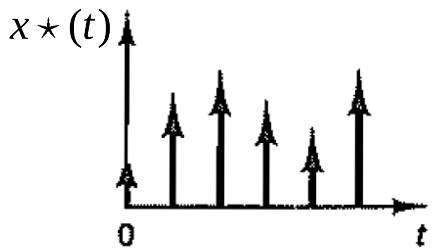
$$x\star(t)=\sum_{k=0}^{\infty}x(kT)\delta(t-kT)$$

$$x\star(t)=x(0)\delta(t)+x(T)\delta(t-T)+\cdots+x(kT)\delta(t-kT)+\cdots$$

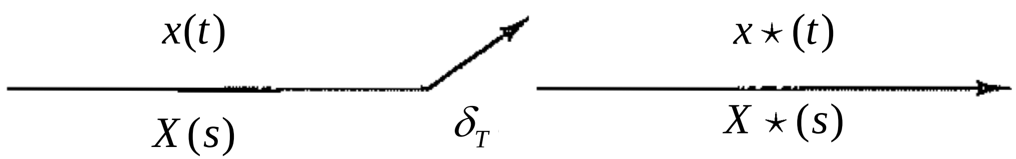
# The Impulse Sampler

- Visually, the process can be illustrated as:




$$x\star(t)=\sum_{k=0}^{\infty}x(kT)\delta(t-kT)$$




$$\delta_T(t)=\sum_{k=0}^{\infty}\delta(t-kT)$$

- Consider the LT of the starred notation:

$$X\star(s)=\mathcal{L}\{x\star(t)\}=\mathcal{L}\left\{\sum_{k=0}^{\infty}x(kT)\delta(t-kT)\right\}=x(0)\mathcal{L}[\delta(t)]+x(T)\mathcal{L}[\delta(t-T)]+x(2T)\mathcal{L}[\delta(t-2T)]+\cdots$$

Recall Time Shift Property of LT: $\mathcal{L}\left[f\left(t-T\right)\right]=e^{-Ts}F(s)$

$$\boxed{\begin{aligned}X\star(s)&=x(0)+x(T)e^{-sT}+x(2T)e^{-2sT}+\cdots\\&=\sum_{k=0}^{\infty}x(kT)e^{-kTs}\end{aligned}}$$

# Impulse Sampling & z Transform

- Does this look familiar? $X\star(s) = \sum_{k=0}^{\infty} x(kT)e^{-kTs}$
- What if we let: $\boxed{e^{Ts} = z} \quad \Rightarrow s = \frac{1}{T}\ln z$
- Then: $$X\star(s)\big|_{s=(1/T)\ln z} = \sum_{k=0}^{\infty} x(kT)z^{-k}$$
- Recall this equation! $\mathcal{Z}\left[x(t)\right] = X(z) = \sum_{k=0}^{\infty} x(kT)z^{-k}$
- WOAH! What this mean? $$\boxed{X\star(s)\big|_{s=(1/T)\ln z} = X(z)}$$
- The Laplace Transform of the **impulse-sampled** signal $x\star(t)$ is the same as the *z* transform of the signal $x(t)$ if: $e^{Ts} = z$

# Data Hold

- Data Hold is the process of generating a continuous-time signal *h(t)* from a discrete-time sequence *x(kT)*
- A discrete-time sequence only provides information at specific instants and governed by the sampling period *T*
- It does not provide any information for the signal between sampled pulses. In other words, the signal in the time-interval $kT \le t \le (k+1)T$ is unknown
- We could approximate the signal in this interval with a polynomial!

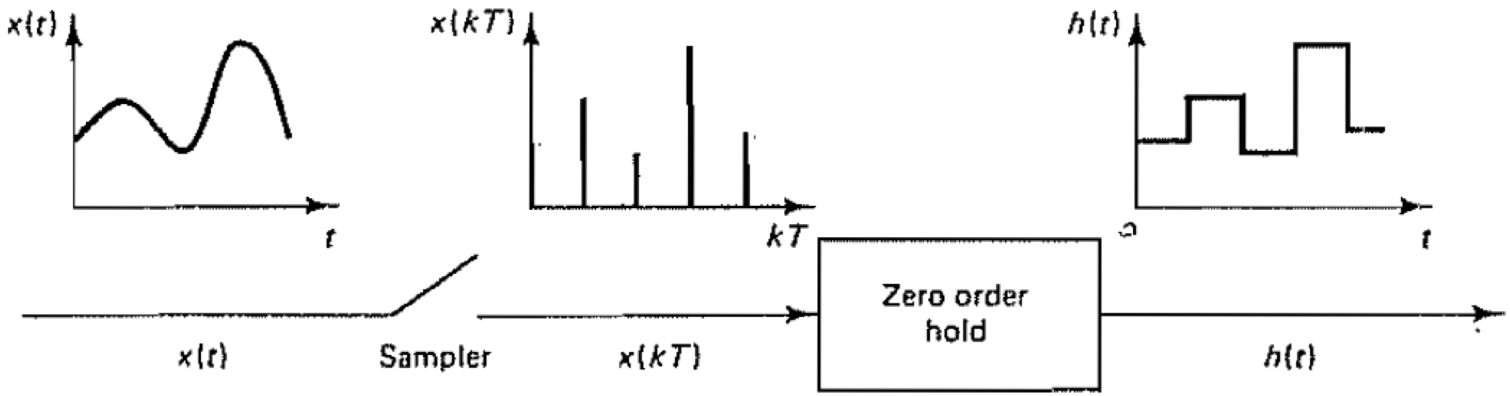
$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(kT) \qquad , 0 \le \tau < T$$

- This expression represents a data hold circuit and is an $n^{th}$ order polynomial extrapolator
- Because a higher-order hold uses more past samples to extrapolate a continuous-time signal (to determine the coefficients), the accuracy improves but comes at a cost of greater time delay from computation

# Zero-Order Hold (ZOH)

- The simplest data hold is the Zero-Order Hold, when *n=0*,

$$\boxed{h(kT+\tau)=x(kT) \qquad ,0\le\tau<T}$$



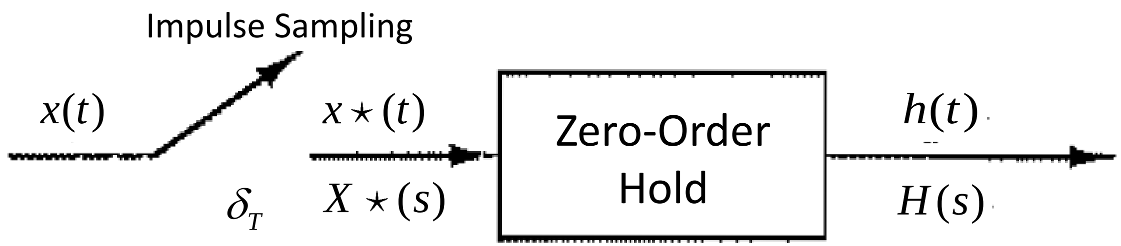

- Let's consider the mathematical expression for *h(t)* [the impulse sampled signal of *x(t)* under zero-order hold]

$$h(t)=x(0)\big[1(t)-1(t-T)\big]+x(T)\big[1(t-T)-1(t-2T)\big]+x(2T)\big[1(t-2T)-1(t-3T)\big]+\cdots$$

$$=\sum_{k=0}^{\infty}x(kT)\big[1(t-kT)-1(t-(k+1)T)\big]$$

# Zero-Order Hold (ZOH)

- With the data hold, the output is now continuous with available signal data! (not zero in between impulse samples)




- As *h(t)* is continuous, Taking LT of: $h(t)=\sum_{k=0}^{\infty}x(kT)\left[1(t-kT)-1(t-(k+1)T)\right]$

Recall Time Shift Property of LT: $\mathcal{L}\left[1\left(t-kT\right)\right]=\dfrac{e^{-kTs}}{s}$

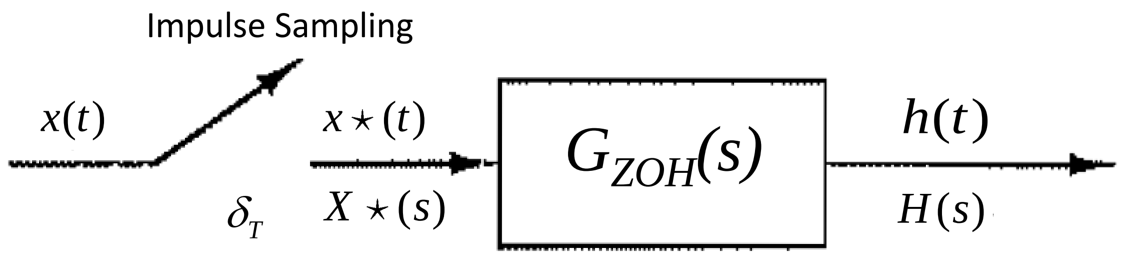
$$H(s)=\sum_{k=0}^{\infty}x(kT)\left[\frac{e^{-kTs}-e^{-(k+1)Ts}}{s}\right]$$

$$=\frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty}x(kT)e^{-kTs}$$

$$=\frac{1-e^{-Ts}}{s}X\star(s)=\frac{1-e^{-Ts}}{s}X(z)$$

# ZOH Transfer Function

- Denote the transfer function of ZOH by $G_{ZOH}(s)$ (with impulse sampling)




- From the previous expression:

$$H(s) = \frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

$$= \frac{1-e^{-Ts}}{s} X\star(s) = G_{h0}(s)X\star(s)$$

$$\boxed{\Rightarrow G_{ZOH}(s) = \frac{1-e^{-Ts}}{s}}$$

# z Transform of Functions with ZOH

- As it will be shown later, obtaining z Transforms of functions involving ZOH will be very important

$$G_{ZOH}(s) = \frac{1-e^{-Ts}}{s}$$

- Given an expression: $X(s) = G_{ZOH}(s)G(s)$
  - The goal here is to derive an expression for *X(z)* (z Transform of such *an X(s))*
- Note that: $X(s) = \frac{1-e^{-Ts}}{s}G(s) = \left(1-e^{-Ts}\right)G_1(s)$ $\qquad G_1(s) = \frac{G(s)}{s}$
- If we consider: $X_1(s) = e^{-Ts}G_1(s)$
  - Then, $X(s) = G_1(s) - e^{-Ts}G_1(s)$
  - By definition of LT: $\mathcal{L}\left[g_1(t)\right] = G_1(s)$ , this also means for zT: $\mathcal{Z}\left[g_1(t)\right] = G_1(z)$

    $\mathcal{L}\left[x_1(t)\right] = X_1(s)$ $\qquad \mathcal{Z}\left[x_1(t)\right] = X_1(z)$
  - Using the relationship between *s* and *z* domains: $z = e^{sT}$

$$X_1(z) = z^{-1}G_1(z)$$

$$X(z) = \mathcal{Z}\left[g_1(t)\right] - \mathcal{Z}\left[x_1(t)\right] = G_1(z) - z^{-1}G_1(z)$$

$$= \left(1-z^{-1}\right)G_1(z)$$

$$\boxed{\Rightarrow X(z) = \left(1-z^{-1}\right)\mathcal{Z}\left[\frac{G(s)}{s}\right]}$$

# Exercise

- Find the z Transform of $X(s) = G_{ZOH}(s)\left(\frac{1}{s+1}\right)$

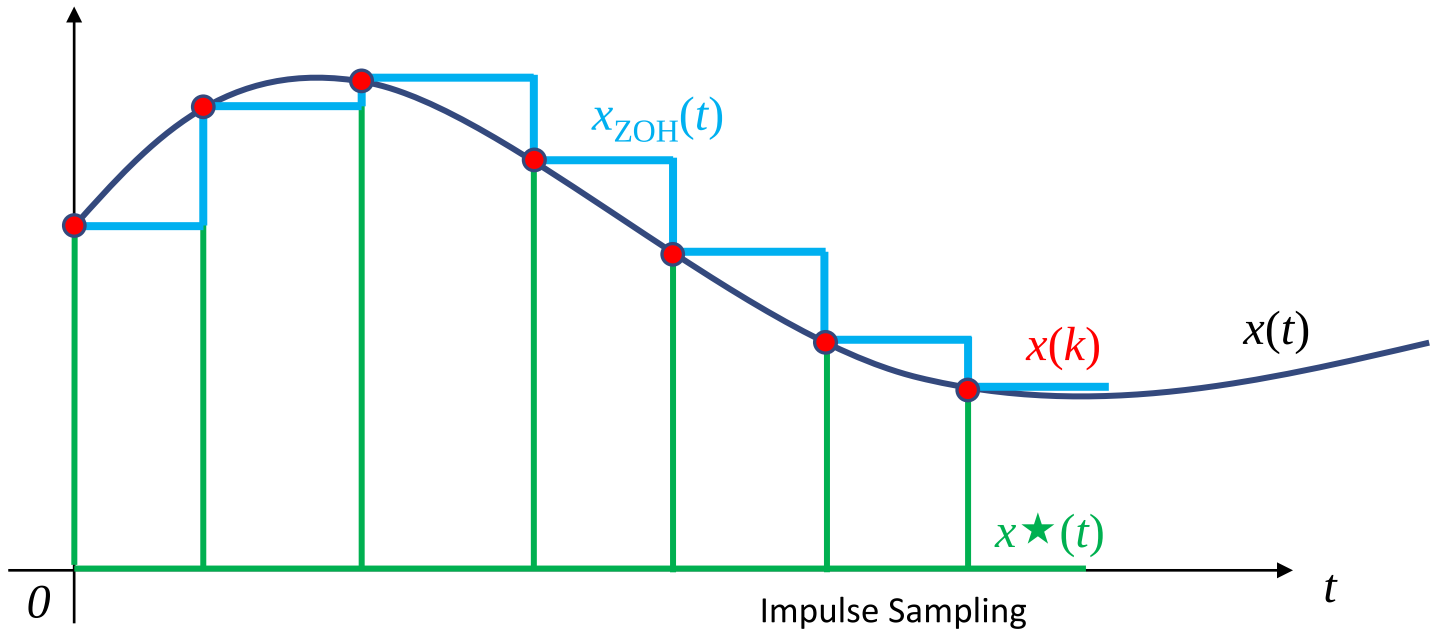
$$X(z) = \mathcal{Z}\left[G_{ZOH}(s)\left(\frac{1}{s+1}\right)\right] = \mathcal{Z}\left[\frac{1-e^{-Ts}}{s}\left(\frac{1}{s+1}\right)\right]$$

$$= \left(1-z^{-1}\right)\mathcal{Z}\left[\frac{1}{s(s+1)}\right]$$

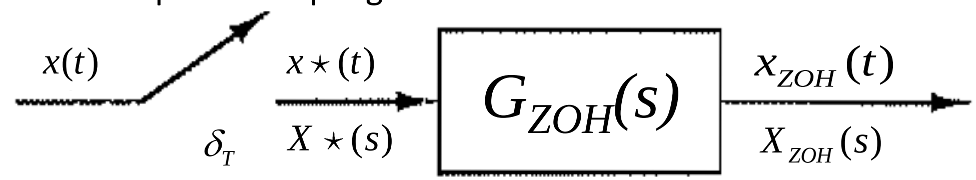
$$= \left(1-z^{-1}\right)\mathcal{Z}\left[\frac{1}{s}-\frac{1}{s+1}\right]$$

$$= \frac{\left(1-e^{-T}\right)z^{-1}}{1-e^{-T}z^{-1}}$$

# Summary of Various Functions




Impulse Sampling




| | **zT** | **LT** |
|---|---|---|
| $x(t)$ | $X(z)$ | $X(s)$ |
| $x(k)$ | $X(z)$ | |
| $x\star(t)$ | $X(z)$ | $X\star(s)$ |
| $x_{ZOH}(t)$ | $X_{ZOH}(z)$ | $X_{ZOH}(s)$ |

$$\boxed{X\star(s)\big|_{s=(1/T)\ln z} = X(z)}$$

$$X_{ZOH}(s) = \frac{1-e^{-Ts}}{s} X\star(s) = \frac{1-e^{-Ts}}{s} X(z)$$

$$X_{ZOH}(z) = \left(1-z^{-1}\right)\mathcal{Z}\left[\frac{X(s)}{s}\right]$$