Controllability & Observability

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Important Properties of LTI Systems

- Controllability and Observability represent 2 major concepts of Modern Control System theory. They were introduced by R. E. Kalman in 1959/60.
- **CONTROLLABILITY**
  - In order to be able to do whatever we want with the dynamic system under control input, the system must be controllable
- **OBSERVABILITY**
  - In order to see what is going on inside the system under observation (the internal states), the system must be observable
- Concepts of controllability and observability are derived from linear system theory, so knowledge of linear algebra is key
- Controllability and observability are dual concepts and should be understood as a whole rather than separate properties.

R. Kalman, "On the general theory of control systems," *IRE Transactions on Automatic Control*, vol. 4, no. 3, pp. 110-110, Dec 1959.

Complete State Controllability

- Consider the state-space system (scalar input with *n* states):

  A is an *n* x *n* matrix $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

- We know the solution is: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$

- A system is said to be state controllable at $t=t_0$ if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $t_0 \le t \le t_f$. If every state is controllable, the system is said to be **completely state controllable**.

- Without loss of generality, let's assume we are going from an unknown initial state location to the origin (zero) at $t=t_f$

$$\mathbf{x}(t_f) = e^{\mathbf{A}t_f}\mathbf{x}(0) + \int_0^{t_f} e^{\mathbf{A}(t_f-\tau)}\mathbf{B}u(\tau)d\tau = \mathbf{0}$$

$$\mathbf{x}(0) = -e^{-\mathbf{A}t_f}\int_0^{t_f} e^{\mathbf{A}(t_f-\tau)}\mathbf{B}u(\tau)d\tau$$

$$= -\int_0^{t_f} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$$

$$e^{-\mathbf{A}\tau} = \mathbf{I} - \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2!} + \frac{-\mathbf{A}^3\tau^3}{3!} + \frac{\mathbf{A}^4\tau^4}{4!} + \cdots = \sum_{k=0}^{\infty}\frac{(-\mathbf{A})^k\tau^k}{k!} \boxed{= \sum_{k=0}^{\infty}\beta_k(\tau)\mathbf{A}^k} \boxed{= \sum_{k=0}^{n-1}\alpha_k(\tau)\mathbf{A}^k}$$

Cayley-Hamilton Theorem

- The Cayley-Hamilton theorem states that the matrix **A** satisfies it own characteristic equation (**A** is an *n* x *n* matrix)
- c.e: $$|\lambda\mathbf{I}-\mathbf{A}| = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n = 0$$
- By C-H theorem: $$\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = 0$$
- Illustration: $$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$|\lambda\mathbf{I}-\mathbf{A}| = \mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = 0$$

$$\mathbf{A}^2 - 2\mathbf{A} + \mathbf{I} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Cayley-Hamilton Theorem

- So why is $\mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = 0$ helpful?
- Let's consider a 2 x 2 **A** matrix
  - The c.e. is $\lambda^2 + b\lambda + c = 0$
  - By C-H theorem:
  $$\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = 0$$
  $$\mathbf{A}^2 = -b\mathbf{A} - c\mathbf{I}$$
  - Extending this:
  $$\begin{aligned}\mathbf{A}^3 &= \mathbf{A}^2\mathbf{A} = \left(-b\mathbf{A} - c\mathbf{I}\right)\mathbf{A} \\ &= -b\mathbf{A}^2 - c\mathbf{A} \\ &= -b\left(-b\mathbf{A} - c\mathbf{I}\right) - c\mathbf{A} = \left(b^2 - c\right)\mathbf{A} + bc\mathbf{I}\end{aligned}$$
  - And so on…. You can always express higher powers of **A** in terms of **A** and **I**
- For an n x n **A** matrix
  - Powers of A n and above can be expressed as a linear combinations of **I**, **A**, $\mathbf{A}^2$,…, $\mathbf{A}^{n-1}$.
    - This helps for matrix exponentials:
    $$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots = \sum_{k=0}^{\infty}\frac{\mathbf{A}^kt^k}{k!} = \sum_{k=0}^{n-1}\alpha_k(t)\mathbf{A}^k$$

Complete State Controllability

- Back to Controllability: $\mathbf{x}(0) = -\int_0^{t_f} e^{-\mathbf{A}\tau}\mathbf{B}u(\tau)d\tau$
- Using C-H theorem: $e^{-\mathbf{A}\tau} = \sum_{k=0}^{\infty} \frac{(-\mathbf{A})^k \tau^k}{k!} = \sum_{k=0}^{n-1} \alpha_k(\tau)\mathbf{A}^k$
- Combining: $$\mathbf{x}(0) = -\int_0^{t_f} \sum_{k=0}^{n-1} \alpha_k(\tau)\mathbf{A}^k\mathbf{B}u(\tau)d\tau$$ $$= -\sum_{k=0}^{n-1}\left[\mathbf{A}^k\mathbf{B}\int_0^{t_f} \alpha_k(\tau)u(\tau)d\tau\right]$$
- Defining: $\int_0^{t_f} \alpha_k(\tau)u(\tau)d\tau = \beta_k$, $\mathbf{x}(0) = -\sum_{k=0}^{n-1}\mathbf{A}^k\mathbf{B}\beta_k = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}\begin{bmatrix}\beta_0 \\ \beta_1 \\ \vdots \\ \beta_{n-1}\end{bmatrix}$

- If the system is completely state controllable, then given any (non-zero) initial state **x**(0), the above equation must be satisfied
  - This requires the rank of $\begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}$(n x n matrix) to be *n* (full rank)
  - The vectors of **B**, **AB**,…, $\mathbf{A}^{n-1}\mathbf{B}$ are linearly independent

- Result also holds when control input is an *r*-dimension vector $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$

$\mathbf{C}_O = \begin{bmatrix}\mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}_{n\times nr}$ (*n* x *nr* matrix) is called the **Controllability Matrix**

Exercise on Controllability

- For the 2 following systems, comment on state controllability.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$\begin{aligned} \mathbf{C}_O &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$\boldsymbol{C_o}$ is singular. System <u>not</u> completely state controllable.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{aligned} \mathbf{C}_O &= \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned}$$

$\boldsymbol{C_o}$ is non-singular. System completely state controllable.

Extra Exercise on Controllability

- If the state and input matrices of a system are given as follows, can you comment on the state controllability?

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

$$\mathbf{C}_O = \begin{bmatrix} -1 & 2 & -4 \\ 2 & -6 & 18 \\ 0 & 1 & -5 \end{bmatrix}$$

$\boldsymbol{C_o}$ is rank 2 (not full rank =3) . System not completely state controllable.

Output Controllability

- In practical design of control systems, it is usually desired to control the output rather than the internal states of the system
- Complete state controllability is neither necessary nor sufficient for controlling the output of the system. A separate criterion is needed.
- For a system:

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

$\boldsymbol{x}$: n vector
$\boldsymbol{u}$: r-vector
$\mathbf{y}$: m-vector

$\mathbf{A}$: $n \times n$ matrix
$\mathbf{B}$: $n \times r$ matrix
$\mathbf{C}$: $m \times n$ matrix
$\mathbf{D}$: $m \times r$ matrix

- System is said to be completely output controllable if it is possible to construct an unconstrained control vector that will transfer any given output to final output in a finite time interval
- It can be proven that if the following condition is met, the system is completely output controllable

$$\mathbf{O}_C = \begin{bmatrix} \mathbf{CB} & \mathbf{CAB} & \mathbf{CA}^2\mathbf{B} & \cdots & \mathbf{CA}^{n\text{-}1}\mathbf{B} & \mathbf{D} \end{bmatrix}_{m \times (n+1)r}$$

is rank *m*.

Uncontrollable System & Stabilizability

- An uncontrollable system has a subsystem that is physically disconnected from the input
  - No matter that the input and whatever finite time required, there is no way to affect the state(s)

- For partially controllable systems, if the uncontrollable modes are stable and the unstable modes are controllable, the system is said to be **stabilizable**
- Consider: $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
  - The system is not state controllable.
  - Stable mode that corresponds to the pole/eigenvalue at -1 is not controllable
  - Unstable mode that corresponds to the pole/eigenvalue at 1 is controllable
  - System can be made stable by a suitable feedback controller
  - System is said to be **stabilizable**

Complete Observability

- Consider the state-space system:
$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$
- We know the solution is: $$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}$$
- A system is said to be **completely observable** if every state at $t=t_0$ can be determined from the observation of $y(t)$ over a finite time interval
$t_0 \le t \le t_f$
    - Every transition of the state eventually affects every element of the output vector
- Concept of observability is important in practice where there is difficulty during state feedback control when some of the state variables are not accessible for direct measurement
    - Necessary to estimate the unmeasurable states in order to construct control signals

Complete Observability

- From the general solution of $y(t)$:

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}$$

  - **A**,**B**, **C** and **D** are known and so is $\boldsymbol{u}(t)$
  - Last 2 terms of the solution are known and can be accounted for
  - Hence to investigate complete observability, it is sufficient to consider

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$$

- Recall C-H theorem: $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots = \sum_{k=0}^{\infty}\frac{\mathbf{A}^kt^k}{k!} = \sum_{k=0}^{n-1}\alpha_k(t)\mathbf{A}^k$

$$\mathbf{y}(t) = \sum_{k=0}^{n-1}\alpha_k(t)\mathbf{C}\mathbf{A}^k\mathbf{x}(0)$$

  - Or: $\mathbf{y}(t) = \alpha_0(t)\mathbf{C}\mathbf{x}(0) + \alpha_1(t)\mathbf{C}\mathbf{A}\mathbf{x}(0) + \cdots + \alpha_{n-1}(t)\mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(0)$

Complete Observability

- If the system is completely observable, then given the output **y**(t) over time interval $t_0 \leq t \leq t_f$ **x**(0) can be uniquely determined.

$$\mathbf{y}(t) = \begin{bmatrix} \alpha_0(t) & \alpha_1(t) & \cdots & \alpha_{n-1}(t) \end{bmatrix} \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \mathbf{x}(0)$$

- For this to be possible the rank of $\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix}_{nm \times n}$ must be *n*

- This condition can be rewritten. For complete observability, the following matrix must be of rank *n* or has at least *n* linearly independent column vectors.

$$\mathbf{O}_B = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix}_{n \times nm}$$ (*n* x *nm* matrix) is called the **Observability Matrix**

* : Conjugate Transpose $\mathbf{A}^* = \left(\bar{\mathbf{A}}\right)^{\mathrm{T}} = \overline{\mathbf{A}^{\mathrm{T}}}$

Exercise

- For the following system, is the system completely observable?

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 5 & 1 \end{bmatrix}$$

$$\mathbf{O}_B = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & (\mathbf{A}^*)^2\mathbf{C}^* \end{bmatrix}$$

$$\mathbf{A}^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}, \quad \mathbf{C}^* = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$\mathbf{A}^*\mathbf{C}^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix}$$

$$\mathbf{A}^*\mathbf{A}^*\mathbf{C}^* = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}\begin{bmatrix} -6 \\ -7 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$$

$$\mathbf{O}_B = \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\det \mathbf{O}_B = \det \begin{bmatrix} 4 & -6 & 6 \\ 5 & -7 & 5 \\ 1 & -1 & -1 \end{bmatrix} = 0$$

Rank of Observability Matrix is less than 3. System not completely observable

Principle of Duality

- Controllability and Observability share an intricate relationship. The principle of duality was conceived by Kalman to connect the analogies between controllability & observability
- Consider a system S_1 and its dual system S_2:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \qquad \mathbf{y} = \mathbf{C}\mathbf{x}$$

$\boldsymbol{x}$: n vector, $\boldsymbol{u}$: r-vector, $\mathbf{y}$: m-vector

$\mathbf{A}$: $n \times n$ matrix, $\mathbf{B}$: $n \times r$ matrix, $\mathbf{C}$: $m \times n$ matrix

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v} \qquad \mathbf{w} = \mathbf{B}^*\mathbf{z}$$

$\boldsymbol{z}$: n vector, $\boldsymbol{v}$: m-vector, $\mathbf{w}$: r-vector

$\mathbf{A}^*$: $n \times n$ matrix, $\mathbf{B}^*$: $r \times n$ matrix, $\mathbf{C}^*$: $n \times m$ matrix

- The principle of duality states S_1 is completely state controllable (or observable) if and only if system S_2 is completely observable (state-controllable).

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}_{n\times nr} \qquad \mathbf{C}_O = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix}_{n\times nm}$$

$$\mathbf{O}_B = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* & \cdots & (\mathbf{A}^*)^{n-1}\mathbf{C}^* \end{bmatrix}_{n\times nm} \qquad \mathbf{O}_B = \begin{bmatrix} \mathbf{B} & \mathbf{AB} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix}_{n\times nr}$$

- For a partially observable system, if the unobservable modes are stable and the observable modes are unstable, the system is said to be detectable.
  - Concept of detectability is dual to the concept of stabilizability

Illustrative Example

- Consider the following system: $G(s) = \dfrac{1}{s^2 + 2s + 1}$
  - It can be expressed in CCF (S_1), or also in OCF (S_2),

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad \dot{\mathbf{z}} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix}\mathbf{z}$$

  - If we defined: $\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} \end{aligned}$, you will note that: $\begin{aligned} \dot{\mathbf{z}} &= \mathbf{A}^*\mathbf{z} + \mathbf{C}^*u \\ y &= \mathbf{B}^*\mathbf{z} \end{aligned}$
  - In other words, S_1 is completely state controllable (or observable) if and only if system S_2 is completely observable (state-controllable).

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} \qquad \mathbf{C}_O = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix}$$

$$\mathbf{O}_B = \begin{bmatrix} \mathbf{C}^* & \mathbf{A}^*\mathbf{C}^* \end{bmatrix} \qquad \mathbf{O}_B = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix}$$