# Vector-Matrix Algebra

EPD 30.114 ADVANCED FEEDBACK & CONTROL

# Definitions

- **Square Matrix**
  - Number of rows = Number of columns
  - Also called Matrix of order *n*, where *n* is the number of rows or columns
- **Diagonal Matrix**
  - All elements other than the main diagonal elements of a square matrix are zero
- **Identity / Unity Matrix, I**
  - A square and diagonal matrix where all the elements are equal to unity
- **Singular Matrix**
  - Not all rows or columns of matrix are independent of each other
  - Determinant of the matrix is zero
- **Transpose**
  - Interchanging columns and rows of a matrix
- **Symmetric Matrix**
  - A matrix whose transpose is equal to itself

# Determinant

- The determinant is a value associated with a square matrix **A**. It is denoted by:
  - det(**A**), det **A**
  - |**A**|

- Properties of the determinant:
  - $\det(\mathbf{I}_n)=1$
  - $\det(\mathbf{A}^T)=\det(\mathbf{A})$
  - $\det(\mathbf{A}^{-1})=1/\det(\mathbf{A})=\det(\mathbf{A})^{-1}$
  - $\det(\mathbf{AB})=\det(\mathbf{A})\det(\mathbf{B})$
  - $\det(c\mathbf{A})=c^n\det(\mathbf{A})$, for an $n$ x $n$ matrix

- Computing the determinant:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$\det\left(\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}\right) = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - c_1b_2a_3 - c_2b_3a_1 - c_3b_1a_2$$

# Matrix Algebra

- Multiplication of a matrix with another matrix is not commutative
  - $\mathbf{AB} \neq \mathbf{BA}$

- Power of a square matrix is defined as
  - $\mathbf{A}^k = \mathbf{AAAA....A}$ (k times)
  - If A is diagonal matrix,

$$\mathbf{A} = \mathrm{diag}(a_1, a_2, \ldots, a_n)$$
$$\mathbf{A}^k = \mathrm{diag}(a_1{}^k, a_2{}^k, \ldots, a_n{}^k)$$

- Rank of a Matrix
  - A matrix is said to have rank $m$ if there exist an $m \times m$ submatrix **M** of **A** such that the determinant of **M** is **nonzero** and the determinant of every $r \times r$ submatrix (where $r \geq m + 1$) of **A** is zero
  - Also the number of independent rows/columns or ‘pivots’ in the matrix

# Matrix Inversion

- If a square matrix **A** and a matrix **B** exists such that **BA=AB=I**, then **B** is denoted $\mathbf{A}^{-1}$ and defined as the inverse of **A**
  - $\mathbf{A}\mathbf{A}^{-1}=\mathbf{A}^{-1}\mathbf{A}=\mathbf{I}$
  - $(\mathbf{AC})^{-1}=\mathbf{C}^{-1}\mathbf{A}^{-1}$
  - $(\mathbf{A}^{-1})^{-1}=\mathbf{A}$
- General form for inversion:
  - where $A_{ij}$ is the cofactor of $a_{ij}$ of **A**

$$\mathbf{A}^{-1}=\frac{\operatorname{adj}\mathbf{A}}{|\mathbf{A}|}=\begin{bmatrix} \frac{A_{11}}{|\mathbf{A}|} & \frac{A_{21}}{|\mathbf{A}|} & & \frac{A_{n1}}{|\mathbf{A}|} \\ \frac{A_{12}}{|\mathbf{A}|} & \frac{A_{22}}{|\mathbf{A}|} & \cdots & \frac{A_{n2}}{|\mathbf{A}|} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{A_{1n}}{|\mathbf{A}|} & \frac{A_{2n}}{|\mathbf{A}|} & \cdots & \frac{A_{nn}}{|\mathbf{A}|} \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \mathbf{A}^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad \mathbf{A}^{-1}=\frac{1}{|\mathbf{A}|}\begin{bmatrix} \begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & \begin{vmatrix} b & c \\ e & f \end{vmatrix} \\ -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & \begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} \\ \begin{vmatrix} d & e \\ g & h \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} & \begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{bmatrix}$$

# Quick Exercise

- Compute the inverse of $\mathbf{A}=\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & -2\\ 1 & 0 & -3\end{bmatrix}$

$$|\mathbf{A}|=\det\left(\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & -2\\ 1 & 0 & -3\end{bmatrix}\right)=(-1)(-3)+2(-2)(1)-(-3)(3)(2)=3-4+18=17$$

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 0\\ 3 & -1 & -2\\ 1 & 0 & -3\end{bmatrix},\quad \mathbf{A}^{-1}=\frac{1}{|\mathbf{A}|}\begin{bmatrix}\begin{vmatrix}-1 & -2\\ 0 & -3\end{vmatrix} & -\begin{vmatrix}2 & 0\\ 0 & -3\end{vmatrix} & \begin{vmatrix}2 & 0\\ -1 & -2\end{vmatrix}\\ -\begin{vmatrix}3 & -2\\ 1 & -3\end{vmatrix} & \begin{vmatrix}1 & 0\\ 1 & -3\end{vmatrix} & -\begin{vmatrix}1 & 0\\ 3 & -2\end{vmatrix}\\ \begin{vmatrix}3 & -1\\ 1 & 0\end{vmatrix} & -\begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix} & \begin{vmatrix}1 & 2\\ 3 & -1\end{vmatrix}\end{bmatrix}=\frac{1}{17}\begin{bmatrix}3 & 6 & -4\\ 7 & -3 & 2\\ 1 & 2 & -7\end{bmatrix}$$

# Matrix Exponentials

- Recall the exponential function: $e^{x}$, which can be defined by the power series,

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- Which also means: $$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \frac{a^4x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{a^n x^n}{n!}$$

- This also holds true if *a* is a matrix **A**:

$$\boxed{e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}}$$

Note: it converges absolutely for all finite time t.

# Matrix Exponential Properties

- The differentiation of the series:

$$e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \frac{\mathbf{A}^4t^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k t^k}{k!}$$

$$\begin{aligned}
\frac{d}{dt} e^{\mathbf{A}t} &= \mathbf{A} + \frac{2\mathbf{A}^2t}{2!} + \frac{3\mathbf{A}^3t^2}{3!} + \frac{4\mathbf{A}^4t^3}{4!} + \cdots = \mathbf{A} + \mathbf{A}^2t + \frac{\mathbf{A}^3t^2}{2!} + \frac{\mathbf{A}^4t^3}{3!} + \cdots \\
&= \mathbf{A}\left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots\right] = \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \cdots\right]\mathbf{A} \\
&= \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}
\end{aligned}$$

# Matrix Exponential Properties

- Consider the multiplication: $e^{\mathbf{A}t}e^{\mathbf{A}s}$

$$e^{\mathbf{A}t}e^{\mathbf{A}s} = \left(\sum_{j=0}^{\infty}\frac{\mathbf{A}^j t^j}{j!}\right)\left(\sum_{k=0}^{\infty}\frac{\mathbf{A}^k s^k}{k!}\right)$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{A}^{j+k}\frac{t^j s^k}{j!k!}$$

$n = j+k,\ \ j = n-k$

$$e^{\mathbf{A}t}e^{\mathbf{A}s} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\mathbf{A}^n\frac{t^{n-k}s^k}{(n-k)!k!}$$

$$= \sum_{n=0}^{\infty}\frac{\mathbf{A}^n}{n!}\sum_{k=0}^{\infty}\frac{n!}{(n-k)!k!}t^{n-k}s^k$$

$$= \sum_{n=0}^{\infty}\frac{\mathbf{A}^n(t+s)^n}{n!}$$

From Binomial Theorem

$$= e^{\mathbf{A}(t+s)}$$

# Matrix Exponential Properties

- Since $e^{\mathbf{A}t}e^{\mathbf{A}s} = e^{\mathbf{A}(t+s)}$
- If $s = -t$, $e^{\mathbf{A}t}e^{-\mathbf{A}t} = e^{\mathbf{A}(t-t)} = \mathbf{I}$
- This means the inverse of $e^{\mathbf{A}t}$ is $e^{-\mathbf{A}t}$
- Since the inverse **always** exists $e^{\mathbf{A}t}$ is non-singular

$$e^{(\mathbf{A}+\mathbf{B})t} = e^{\mathbf{A}t}e^{Bt} \qquad \text{if} \quad \mathbf{AB} = \mathbf{BA}$$

$$e^{(\mathbf{A}+\mathbf{B})t} \neq e^{\mathbf{A}t}e^{Bt} \qquad \text{if} \quad \mathbf{AB} \neq \mathbf{BA}$$

# Scalar and Matrix Exponentials

| Scalar exponential: | Matrix exponential: |
|---|---|
| $e^{at} = 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \ldots$ | $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{\mathbf{A}^3t^3}{3!} + \ldots$ |
| $e^{a0} = 1$ | $e^{\mathbf{A}0} = \mathbf{I}$ |
| $e^{-at} = \frac{1}{e^{at}}$ | $e^{-\mathbf{A}t} = \left[e^{\mathbf{A}t}\right]^{-1}$ |
| $e^{a(t_1+t_2)} = e^{at_1}e^{at_2}$ | $e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1}e^{\mathbf{A}t_2}$ |
| $e^{(a_1+a_2)t} = e^{a_1t}e^{a_2t}$ | $e^{(\mathbf{A}_1+\mathbf{A}_2)t} = e^{\mathbf{A}_1t}e^{\mathbf{A}_2t}$ only if $\mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$ |
| $\frac{d}{dt}e^{at} = ae^{at} = e^{at}a$ | $\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$ |
| $\int_0^t e^{at}dt = \frac{1}{a}\left[e^{at} - 1\right]$ | $\int_0^t e^{\mathbf{A}t}dt = \mathbf{A}^{-1}\left[e^{\mathbf{A}t} - \mathbf{I}\right] = \left[e^{\mathbf{A}t} - \mathbf{I}\right]\mathbf{A}^{-1}$ if $\mathbf{A}^{-1}$ exists. Otherwise defined by the series. |