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Tham Jit 1005036 AFC HW2

1)

$[B \quad AB \quad A^2B] = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & -5 \end{bmatrix}$ Since rank is full, system is completely state controllable $A^* = \begin{bmatrix} -1 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 1 & -1 \end{bmatrix}$ $[C^* \quad A^*C^* \quad {A^*}^2C^*] = \begin{bmatrix} 1 & -1 & 0 \\ 1 & -3 & 5 \\ 0 & -1 & 0 \end{bmatrix}$ Since rank is full, system is completely observable $[CB \quad CAB \quad CA^2B \quad D] = [2 \quad -3 \quad 0 \quad 0]$ Since rank is 1, system is completely output controllable	$[B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 1 & 9 & 1 \end{bmatrix}$ $Reduced = \begin{bmatrix} 1 & 0 & 0 & 2/3 & -2 & 2 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & -1/3 & 3 & -1 \end{bmatrix}$ Since rank is full, system is completely state controllable $A^* = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $[C^* \quad A^*C^* \quad {A^*}^2C^*] = \begin{bmatrix} 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ Since rank is only 2, system is not completely observable $[CB \quad CAB \quad CA^2B \quad D] = \begin{bmatrix} 0 & 1 & 0 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 & 0 & 0 \end{bmatrix}$ Since rank is 1, system is completely output controllable

2)

$$C.E. = |sI - A|$$

$$= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1 & 3 & s-3 \end{vmatrix}$$

$$= s^3 - 3s^2 - 1 + 0 + 0 + 3s + 0$$

$$= s^3 - 3s^2 + 3s - 1$$

$$= (s-1)^3$$

Roots are $s = 1,1,1$

a)

$$(A - \lambda I)v_1 = 0, \qquad \lambda = 1$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$
$$v_{11} = v_{12}$$
$$v_{12} = v_{13}$$
$$v_{11} - 3v_{12} + 2v_{13} = 0$$
$$v_{11} = v_{12} = v_{13} = 1$$
$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$(A-\lambda I)v_2 = v_1, \qquad \lambda \ = 1$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$
$$-v_{21} + v_{22} = 1$$
$$-v_{22} + v_{23} = 1$$
$$v_{21} - 3v_{22} + 2v_{23} = 1$$
$$v_{21} = -2$$
$$v_{22} = -1$$
$$v_{23} = 0$$
$$v_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$(A-\lambda I)v_3 = v_2, \qquad \lambda \ = 1$$
$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}\begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$
$$-v_{31} + v_{32} = -2$$
$$-v_{32} + v_{33} = -1$$
$$v_{31} - 3v_{32} + 2v_{33} = 0$$
$$v_{31} = 0$$
$$v_{32} = -2$$
$$v_{33} = -3$$
$$v_3 = \begin{bmatrix} 0 \\ -2 \\ -3 \end{bmatrix}$$
$$S = [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}$$
$$S^{-1} = \begin{bmatrix} 3 & -6 & 4 \\ 1 & -3 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$
$$J = S^{-1}AS = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{at} = Se^{Jt}S^{-1}$$
$$= \begin{bmatrix} 1 & -2 & 0 \\ 1 & -1 & -2 \\ 1 & 0 & -3 \end{bmatrix}\begin{bmatrix} e^t & te^t & 0.5t^2e^t \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{bmatrix}\begin{bmatrix} 3 & -6 & 4 \\ 1 & -3 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & t-2 & 0.5t^2-2t \\ 1 & t-1 & 0.5t^2-t-2 \\ 1 & t & 0.5t^2-3 \end{bmatrix} e^t \begin{bmatrix} 3 & -6 & 4 \\ 1 & -3 & 2 \\ 1 & -2 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3+t-2+0.5t^2-2t & -6-3t+6-t^2+4t & 4+2t-4+0.5t^2-2t \\ 3+t-1+0.5t^2-t-2 & -6-3t+3-t^2+2t+4 & 4+2t-2+0.5t^2-t-2 \\ 3+t+0.5t^2-3 & -6-3t-t^2+6 & 4+2t+0.5t^2-3 \end{bmatrix} e^t$$
$$= \begin{bmatrix} 0.5t^2-t+1 & t-t^2 & 0.5t^2 \\ 0.5t^2 & 1-t-t^2 & t+0.5t^2 \\ 0.5t^2+t & 3t-t^2 & 1+2t+0.5t^2 \end{bmatrix} e^t$$

b)

$$t = 0, e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3a)

$$A^* = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[C^* \quad A^*C^* \quad A^{*2}C^*] = \begin{bmatrix} 1 & 3 & 9 \\ 1 & 6 & 24 \\ 1 & 2 & 4 \end{bmatrix}$$

$$Reduced = \begin{bmatrix} 1 & 3 & 9 \\ 0 & 3 & 15 \\ 0 & -1 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Since rank is only 2, system is not completely observable

3b)

$$[C^* \quad A^*C^* \quad A^{*2}C^*] = \begin{bmatrix} 1 & 1 & 3 & 3 & 9 & 9 \\ 1 & 2 & 6 & 15 & 24 & 57 \\ 1 & 3 & 2 & 6 & 4 & 12 \end{bmatrix}$$

$$Reduced = \begin{bmatrix} 1 & 1 & 3 & 3 & 9 & 9 \\ 0 & 1 & 3 & 12 & 15 & 48 \\ 0 & 2 & -1 & 3 & 5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -9 & -6 & -39 \\ 0 & 1 & 3 & 12 & 15 & 48 \\ 0 & 0 & -7 & -21 & -25 & -93 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -9 & -6 & -39 \\ 0 & 1 & 0 & 3 & 0 & \frac{57}{7} \\ 0 & 0 & 1 & 3 & 5 & \frac{93}{7} \end{bmatrix}$$

Since full rank, system in completely observable

4)

For solving non-homogenous equations,

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\, d\tau$$

a)

For impulse function $u(t) = \delta(t)\vec{k}$, where $\vec{k}$ contains magnitude of function

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\delta(\tau)\vec{k}\, d\tau$$

$$x(t) = e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}\delta(\tau)\, d\tau\, B\vec{k}$$

$$= e^{At}x(0) + e^{At}B\vec{k}$$

b)

For step function $u(t) = 1(t)\vec{k}$, where $\vec{k}$ contains magnitude of function

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B1(\tau)\vec{k}\, d\tau$$

$$= e^{At}x(0) + e^{At}\int_0^t e^{-A\tau} d\tau\, B\vec{k}$$

$$= e^{At}x(0) + e^{At}\left(It - \frac{AT^2}{2!} + \frac{A^2t^3}{3!} + \cdots\right)B\vec{k}$$

$$= e^{At}x(0) + e^{At}(A^{-1})(I - e^{-At})B\vec{k}$$

$$= e^{At}x(0) + (A^{-1})(e^{-At} - I)B\vec{k}$$

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c)

For step function $u(t) = t\vec{k}$, where $\vec{k}$ contains magnitude of function

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\tau\vec{k}\, d\tau$$

$$= e^{At}x(0) + e^{At}\int_0^t e^{-A\tau}\tau d\tau\, B\vec{k}$$

$$= e^{At}x(0) + e^{At}\left(\frac{It^2}{2} - \frac{AT^3}{3} + \frac{A^2t^4}{2!\,4} + \cdots\right)B\vec{k}$$

$$= e^{At}\, x(0) + \left(I + At + \frac{A^2t^2}{2!} + \cdots\right)\left(\frac{A^2t^2}{2!} - \frac{2A^3t^3}{3!} + \frac{3A^4t^4}{4!} - \cdots\right)\ (A^{-1})^2\ B\,\vec{K}$$

$$= e^{At}\, x(0) + \left(I + At + \frac{A^2t^2}{2!} + \cdots - I - At\right)\ (A^{-1})^2\ B\,\vec{K}$$

$$= e^{At}x(0) + (e^{At} - I - At)\ (A^{-1})^2\ B\,\vec{K}$$

5)

$$G(s) = \frac{10}{(s+1)(s+2)(s+3)}$$

$$= \frac{10}{s^3 + 6s^2 + 11s + 6}$$

$$\mathcal{L}^{-1}(s^3 + 6s^2 + 11s + 6)Y(s) = \mathcal{L}^{-1}10U(s)$$

$$\ddot{y} + 6\ddot{y} + 11y + 6 = 10u$$

Statespace representation:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -60 \\ 10 & -60 & 250 \end{bmatrix}$$

Since rank is full, system is completely state controllable

Desired CE $= \left(s + 2 - j2\sqrt{3}\right)\left(s + 2 + j2\sqrt{3}\right)(s + 10) = s^3 + 14s^2 + 56 + 160$

$$|sI - A + BK| = \left|\begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10k_1 & 10k_2 & 10k_3 \end{bmatrix}\right|$$

$$= \left|\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 10k_1 + 6 & 10k_2 + 11 & s + 10k_3 + 6 \end{bmatrix}\right|$$

$$= s^3 + (10k_3 + 6)s^2 + 10k_1 + 6 + (10k_2 + 11)s$$

$$= s^3 + 14s^2 + 56 + 160$$

$$K = \begin{bmatrix} \frac{160 - 6}{10} & \frac{56 - 11}{10} & \frac{14 - 6}{10} \end{bmatrix} = \begin{bmatrix} 15.4 & 4.5 & 0.8 \end{bmatrix}$$

a)

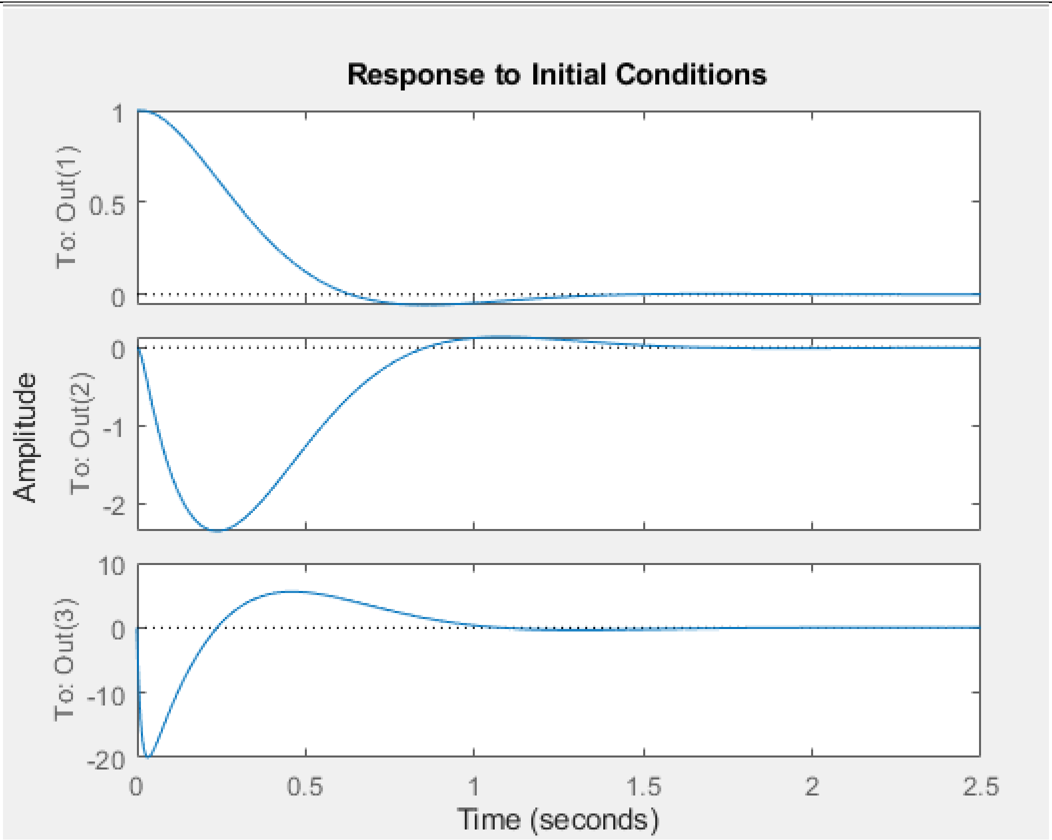
```
A=[ 0 1 0;

    0 0 1;
    -6 -11 -6]
B=[0; 0; 10]
P=[-2+j*2*sqrt(3) -2-j*2*sqrt(3) -10]
K1=place(A,B,P)
K2=acker(A,B,P)
```

Values match

b)

```
sys = ss(A-B*K,eye(3),eye(3),eye(3))
initial(sys,[1;0;0])
```



Steady state values are 0

6)

Routh array:

$$\begin{array}{cccc} s^4 & 1 & 4+K & 10 \\ s^3 & 2 & 9 & \\ s^2 & \dfrac{2K-1}{2} & 10 & \\ s^1 & \dfrac{9K-24.5}{K-0.5} & & \\ s^0 & 10 & & \end{array}$$

For stability, all coefficients in first coloumn > 0

$K - 0.5 > 0$ $K > 0.5$	$\frac{9K-24.5}{K-0.5} > 0$ $9K > 24.5$ $K > \dfrac{49}{18}$

For stability, $K > 0.5$

Hence, for instability, $K \leq 0.5$

7a)

$$u(t) = k_1 r(t) - k_1 x_1(t) - k_2 x_2(t) - k_3 x_3(t)$$

7b)

$$[B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 31 \end{bmatrix}$$

Since rank is full, system is completely state controllable

Desired CE $= (s+2-j4)(s+2+j4)(s+10) = s^3 + 14s^2 + 60s + 200$

$$|sI - A + BK| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -5 & -6 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_1 & k_2 & k_3 \end{bmatrix} \right|$$

$$= \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ k_1 & k_2+5 & s+k_3+6 \end{bmatrix}$$

$$= s^3 + (k_3+6)s^2 + k_1 + (k_2+5)s$$

$$s^3 + 14s^2 + 60s + 200$$

$$K = [k_1 \quad k_2 \quad k_3] = [200 \quad 55 \quad 8]$$

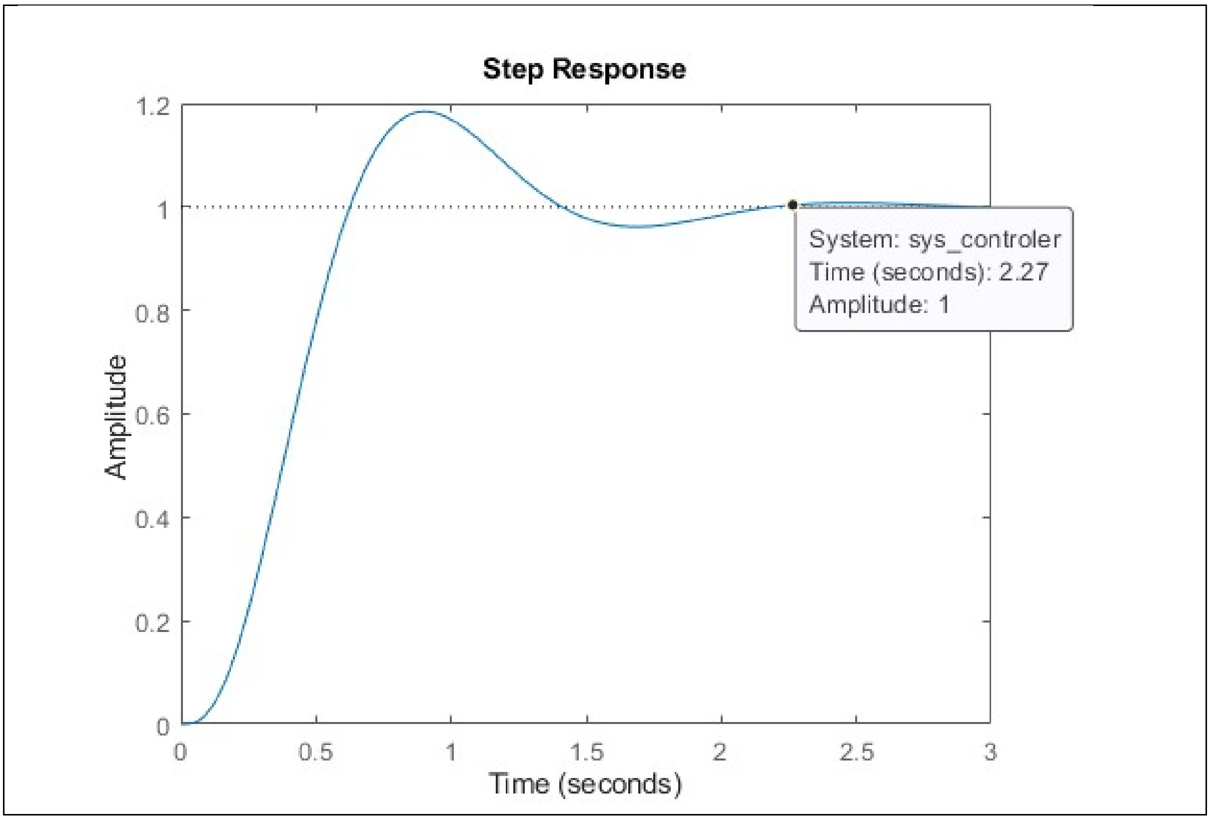
7c)

For a servo control system:

$$u(t) = -Kx(t) + k_1 r(t)$$

$$\dot{x}(t) = (A - BK)x(t) + Bk_1 r(t)$$

```
A=[ 0 1 0;
    0 0 1;
    0 -5 -6]
B=[0; 0; 1]
C=[1 0 0]
D= 0
P=[-2+j*4 -2-j*4 -10]
K=place(A,B,P)
sys_controller=ss(A-B*K,B.*K(1),C,D)
step(sys_controller)
```



8)

Checking for observability:

$$A^{*} = \begin{bmatrix} 0 & 0 & -5 \\ 1 & 0 & -6 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} C^{*} & A^{*}C^{*} & {A^{*}}^{2}C^{*} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since matrix is full rank, system is completely observable

$$\phi(s) = (s+10)(s+15)(s+10) = s^3 + 35s^2 + 400s + 1500$$

$$\phi(A) = A^3 + 35A^2 + 400A + 1500$$

$$= \begin{bmatrix} 1500 & 0 & 0 \\ 0 & 1500 & 0 \\ 0 & 0 & 1500 \end{bmatrix} + \begin{bmatrix} 0 & 400 & 0 \\ 0 & 0 & 400 \\ -2000 & -2400 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 35 \\ -175 & -210 & 0 \\ 0 & -175 & -210 \end{bmatrix} + \begin{bmatrix} -5 & -6 & 0 \\ 0 & -5 & -6 \\ 30 & 36 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1495 & 394 & 35 \\ -175 & 1285 & 394 \\ -1970 & -2539 & 1285 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}^{-1}$$

$$K_e = \phi(A) \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1495 & 394 & 35 \\ -175 & 1285 & 394 \\ -1970 & -2539 & 1285 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 35 \\ 394 \\ 1285 \end{bmatrix}$$