# Solving Homogeneous State Equations

EPD 30.114 ADVANCED FEEDBACK & CONTROL

# How Do We Solve Scalar 1st Order Systems?

- Let's review the solution of the scalar differential equation:

$$\dot{x} = ax$$

- To solve this equation, we may assume a solution *x(t)* in the form:

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots$$

- Substituting the solution into the differential equation:

$$b_1 + 2b_2 t + 3b_3 t^2 \cdots + kb_k t^{k-1} + \cdots = a\left(b_0 + b_1 t + b_2 t^2 + \cdots + b_k t^k + \cdots\right)$$

- Equating coefficients:

$$b_1 = ab_0 \quad , \quad b_2 = \frac{1}{2}ab_1 = \frac{1}{2}a^2 b_0, \quad b_3 = \frac{1}{3}ab_2 = \frac{1}{3\times 2}a^3 b_0$$

$$b_k = \frac{1}{k!}a^k b_0$$

- Value of $b_0$ determined from initial conditions (*t=0*): $x(0) = b_0$
- Combining: $\boxed{x(t) = \left(1 + at + \frac{1}{2!}a^2 t^2 + \cdots + \frac{1}{k!}a^k t^k + \cdots\right)x(0) = \mathrm{e}^{at}\, x(0)}$

# Extending to LTI State-Space Systems

- Now we extend to a vector-matrix differential equation:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

- As before, the solution as the form: $\mathbf{x}(t) = \mathbf{b}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \cdots + \mathbf{b}_k t^k + \cdots$
- Equating Coefficients: $\mathbf{b}_1 = \mathbf{A}\mathbf{b}_0 \quad , \quad \mathbf{b}_2 = \frac{1}{2}\mathbf{A}\mathbf{b}_1 = \frac{1}{2}\mathbf{A}^2\mathbf{b}_0, \quad b_3 = \frac{1}{3}\mathbf{A}\mathbf{b}_2 = \frac{1}{3\times 2}\mathbf{A}^3\mathbf{b}_0$

$$\mathbf{b}_k = \frac{1}{k!}\mathbf{A}^k\mathbf{b}_0$$

- Solution is: $\boxed{\mathbf{x}(t) = \left(\mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \cdots\right)\mathbf{x}(0)}$

- Recall the matrix exponential: $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2 t^2 + \cdots + \frac{1}{k!}\mathbf{A}^k t^k + \cdots$
- Which means: $\boxed{\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)}$

# LT Approach to Solving LTI Homogeneous State Eqns

- Starting from the LTI homogenous state equation: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$
- Taking LT $s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s)$

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0)$$

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0)$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s}\left(\mathbf{I} - \frac{\mathbf{A}}{s}\right)^{-1} = \frac{1}{s}\left(\mathbf{I} + \frac{\mathbf{A}}{s} + \frac{\mathbf{A}^2}{s^2} + \frac{\mathbf{A}^3}{s^3} + \cdots\right) = \frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \frac{\mathbf{A}^3}{s^4} + \cdots$$

Binomial Series

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[\frac{\mathbf{I}}{s} + \frac{\mathbf{A}}{s^2} + \frac{\mathbf{A}^2}{s^3} + \frac{\mathbf{A}^3}{s^4} + \cdots\right]\mathbf{x}(0)$$

$$= \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2}{2!}t^2 + \frac{\mathbf{A}^3}{3!}t^3 + \cdots\right]\mathbf{x}(0) = e^{\mathbf{A}t}\mathbf{x}(0)$$

# State-Transition Matrix

- In a more general form of a LTV system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$$

- The solution of the homogeneous state equation is

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0)$$

where $\mathbf{\Phi}(t,t_0)$ is the state-transition matrix at time $t$ due to an initial condition $\mathbf{x}(t_0)$ at time $t_0$

- If the state vector has $n$ states, the state-transition matrix is an $n$ x $n$ matrix and note that if $t=t_0$:

$$\mathbf{x}(t_0) = \mathbf{\Phi}(t_0, t_0)\mathbf{x}(t_0)$$

$$\Rightarrow \mathbf{\Phi}(t_0, t_0) = \mathbf{I}$$

- In many occurrences when $t_0$ is $0$, it can be simplified to:

$$\mathbf{\Phi}(t, 0) = \mathbf{\Phi}(t)$$

# State-Transition Matrix for LTI

- For a LTI system: $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$

The state transition matrix, as determined earlier, is:

$$\mathbf{\Phi}(t, t_0) = e^{\mathbf{A}(t-t_0)} \qquad \mathbf{\Phi}(t, 0) = e^{\mathbf{A}(t-0)}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

- The solution of the state equations at any time *t* is a transformation of the initial conditions (hence the name – State-Transition Matrix)
- The State-Transition Matrix contains all information about the free motions of the system
- Note, because the State Transition Matrix is a matrix exponential,

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

$$\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = e^{\mathbf{A}(-t)} = \mathbf{\Phi}(-t)$$

# Properties of State-Transition Matrix (LTI)

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$$

$$\mathbf{\Phi}^{-1}(t) = e^{-\mathbf{A}t} = e^{\mathbf{A}(-t)} = \mathbf{\Phi}(-t)$$

$$\mathbf{\Phi}(0) = e^{\mathbf{A}0} = \mathbf{I}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \left(e^{-\mathbf{A}t}\right)^{-1} = \left[\mathbf{\Phi}(-t)\right]^{-1}$$

$$\mathbf{\Phi}(t_1 + t_2) = e^{\mathbf{A}(t_1+t_2)} = e^{\mathbf{A}t_1} e^{\mathbf{A}t_2} = \mathbf{\Phi}(t_1)\mathbf{\Phi}(t_2) = \mathbf{\Phi}(t_2)\mathbf{\Phi}(t_1)$$

$$\left[\mathbf{\Phi}(t)\right]^n = \mathbf{\Phi}(nt)$$

$$\mathbf{\Phi}(t_2 - t_1)\mathbf{\Phi}(t_1 - t_0) = \mathbf{\Phi}(t_2 - t_0) = \mathbf{\Phi}(t_1 - t_0)\mathbf{\Phi}(t_2 - t_1)$$

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}(0) = \mathbf{\Phi}(t)\mathbf{x}(0) = e^{\mathbf{A}t}\mathbf{x}(0)$$

$$\boxed{\mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right] = \mathbf{\Phi}(t) = e^{\mathbf{A}t}}$$

# Exercise

- Obtain the State-Transition Matrix for the following system. Also find the inverse of the State-Transition Matrix.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[ (s\mathbf{I} - \mathbf{A})^{-1} \right]$$

$$(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \frac{1}{(s+2)(s+1)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+2)(s+1)} & \dfrac{1}{(s+2)(s+1)} \\ \dfrac{-2}{(s+2)(s+1)} & \dfrac{s}{(s+2)(s+1)} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathcal{L}^{-1} \begin{bmatrix} \dfrac{s+3}{(s+2)(s+1)} & \dfrac{1}{(s+2)(s+1)} \\ \dfrac{-2}{(s+2)(s+1)} & \dfrac{s}{(s+2)(s+1)} \end{bmatrix}$$

$$\mathbf{\Phi}(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\mathbf{\Phi}^{-1}(t) = \mathbf{\Phi}(-t) = \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{2t} \end{bmatrix}$$

# Exercise

- For the same system, use the State-Transition Matrix to derive a closed form expression for $x_1(t)$ and $x_2(t)$ if the initial conditions are $x_1(0)$ and $x_2(0)$.

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad \mathbf{\Phi}(t)=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

$$\mathbf{x}=\mathbf{\Phi}(t)\mathbf{x}(0)$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\mathbf{\Phi}(t)\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\left(2e^{-t}-e^{-2t}\right)x_1(0)+\left(e^{-t}-e^{-2t}\right)x_2(0)\\ \left(-2e^{-t}+2e^{-2t}\right)x_1(0)+\left(-e^{-t}+2e^{-2t}\right)x_2(0)\end{bmatrix}$$

Poles of the system? (aka eigenvalues): **-1, -2**