Mapping Between *s* & *z* Planes

EPD 30.114 ADVANCED FEEDBACK & CONTROL

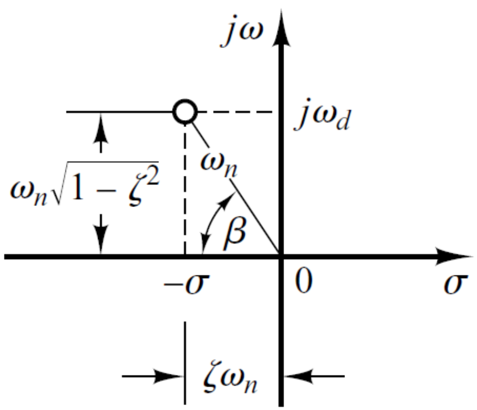
Relationship Between *s* & *z*

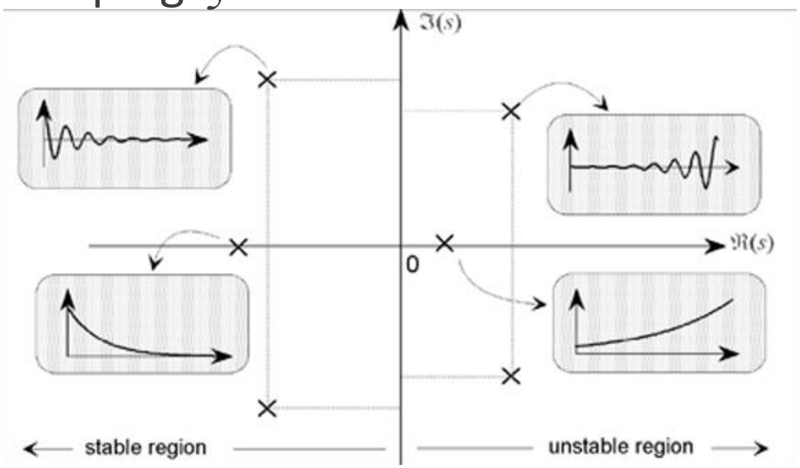
- Both s (in continuous-time) and z (in discrete-time) are complex variables and they share a close relationship
- As you have learnt, transient performance and stability of linear time-invariant continuous-time systems depend on the pole and zero locations in the ***s* plane**
- In the same notion, the transient performance and stability of linear time-invariant discrete-time systems depend on the pole and zero locations in the ***z* plane**
  - A key difference is that for discrete systems, the sampling period T comes into play, affecting both transient performance and stability
  - Change in sampling period modifies pole and zero locations in the z-plane and causes response behavior to change

RECAP: *s* Plane

- Stability boundary in the s plane is the imaginary axis. Any poles to the left of the imaginary axis is asymptotically stable.
- Horizontal lines (parallel to the real axis) are constant damped frequency ω_d
- Vertical lines (parallel to the imaginary axis) are constant settling time/time constant
- Circles about the origin are constant natural frequency ω_n
- Radial lines from the origin are constant damping ζ

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$





Connecting the s and z Planes

- Consider the continuous signal: $f(t) = e^{-at} \qquad t > 0$
- The LT of this signal is: $F(s) = \dfrac{1}{s+a}$
  - This corresponds to a pole at $\boxed{s = -a}$
- The zT of *f(kT)* is $F(z) = \mathcal{Z}\left[e^{-akT}\right]$

$$F(z) = \frac{1}{1 - e^{-aT} z^{-1}} = \frac{z}{z - e^{-aT}}$$

  - This corresponds to a pole at $\boxed{z = e^{-aT}}$
- This means that a pole at $\boxed{s = -a}$ in the *s*-plane corresponds to a pole at $\boxed{z = e^{-aT}}$ in the *z*-plane
- Hence, combining, the relationship between the s and z plane is simply:

$$\boxed{z = e^{sT}}$$

T is the sampling period

Analyzing the Mapping

- Both the complex variables s and z are related via the equation: $\boxed{z = e^{sT}}$
- The complex variable *s* contains a real and imaginary part:

$$s = \sigma + j\omega$$

- Combining: $$z = e^{(\sigma + j\omega)T} = e^{T\sigma} e^{jT\omega}$$
- Because the exponent is periodic:

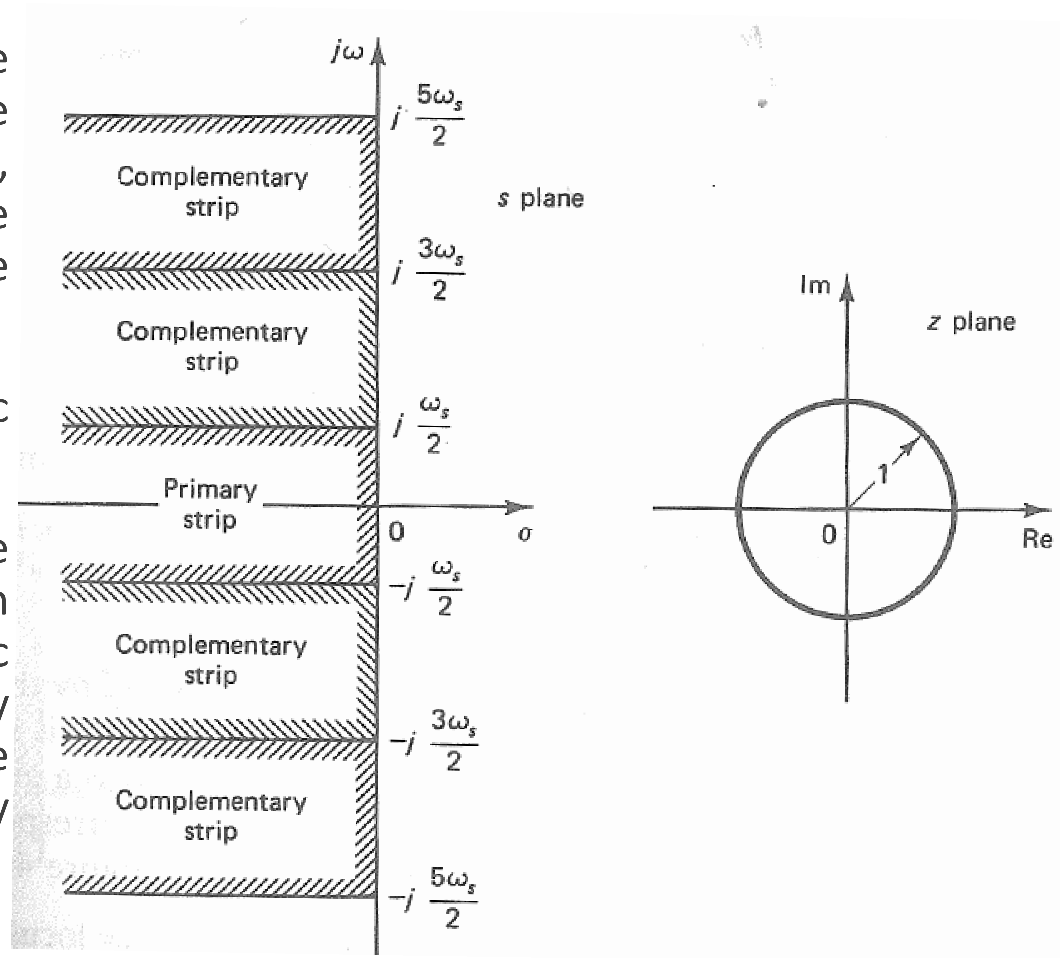
$$z = e^{T\sigma} e^{jT\omega} = e^{T\sigma} e^{j(T\omega + 2\pi k)} = e^{T\sigma} e^{jT\left(\omega + \frac{2\pi}{T}k\right)}$$

k is an integer

- For poles and zeros in the s plane, where frequencies differ in integral multiples of the sampling frequency $2\pi/T$, they are mapped into the **SAME** location on the *z* plane
  - There are infinitely many values of *s* for each value of *z*
- For **STABILITY**, σ must be negative. This corresponds to: $\boxed{|z| = e^{T\sigma} < 1}$
  - The $j\omega$ axis in the s plane corresponds to |*z*|=1. That is, the imaginary axis in the *s* plane corresponds to the unit circle in the z plane and the **INTERIOR** of the unit circle corresponds to the **LEFT HALF PLANE** of the s plane.

Correspondence Between s & z Planes

- Starting from the origin of the s-plane, as you travel along the imaginary axis on the s-plane, you will be tracing a unit circle on the z-plane an infinite number of times
- This motion will be periodic with frequency ω_s
- Hence the left half plane of the s-plane may be divided into an infinite number of periodic strips. There is a primary (main) strip and infinite number of complementary strips

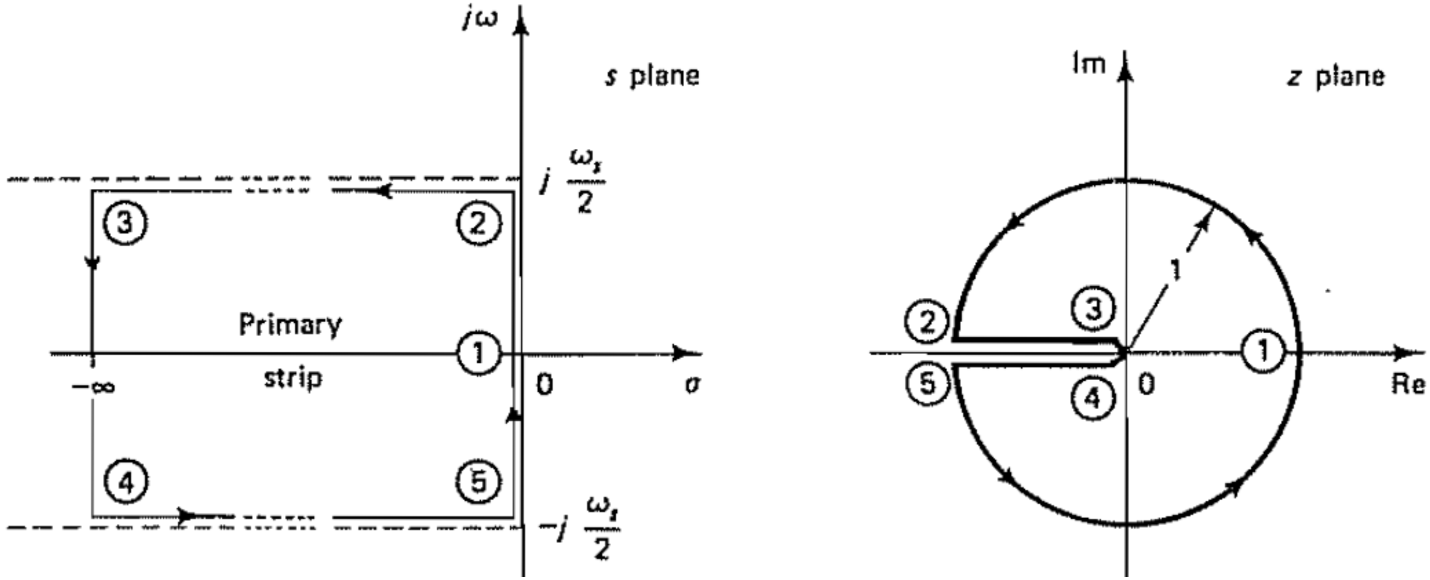


Correspondence Between s & z Planes

- In the primary strip, if you trace the sequence of points 1→2→3→4→5→1 in the s-plane, this path is mapped into the unit circle centered at the origin of the z-plane

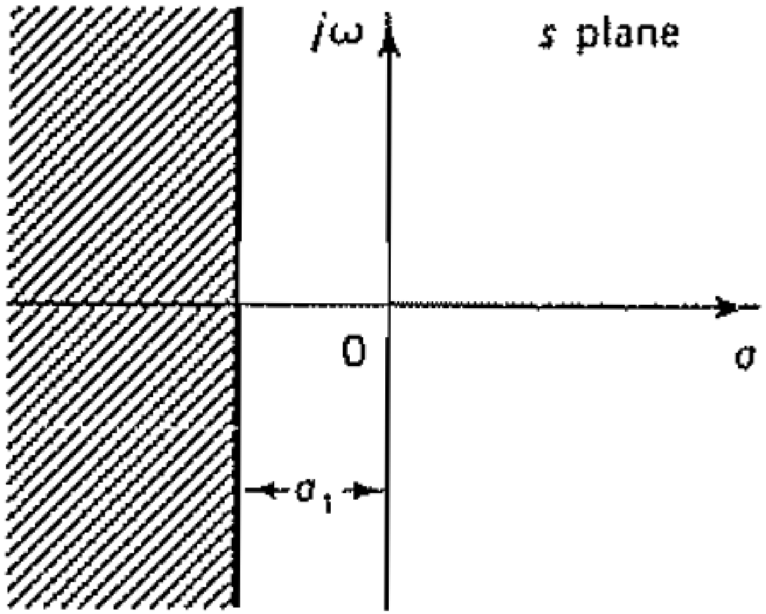
- The area enclosed by any complementary strips is mapped to the same unit circle in the z-plane. A point in the z plane corresponds to an infinite number of points in the s-plane, although a point in the s-plane corresponds to a single point in the z-plane

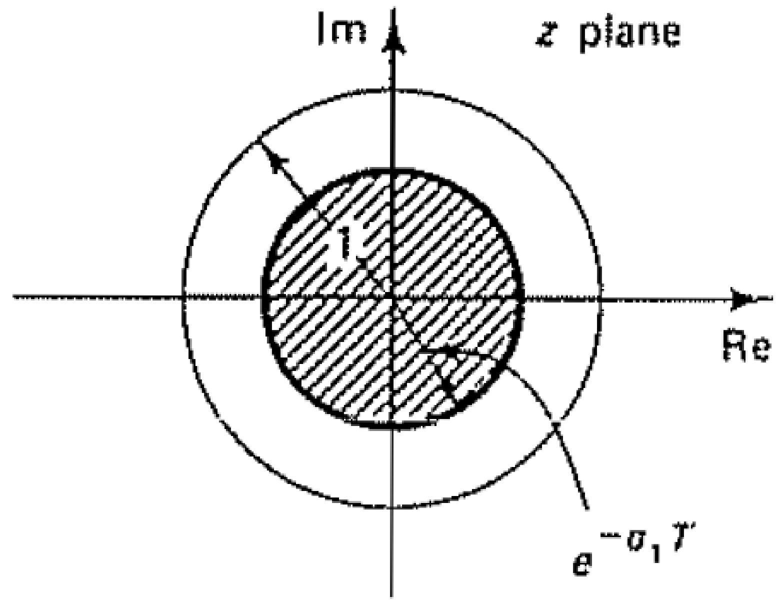
- Left half s-plane is mapped entirely into the interior of the z-plane and the right half s-plane is mapped into the exterior of the unit circle in the z-plane



Mapping of Contours

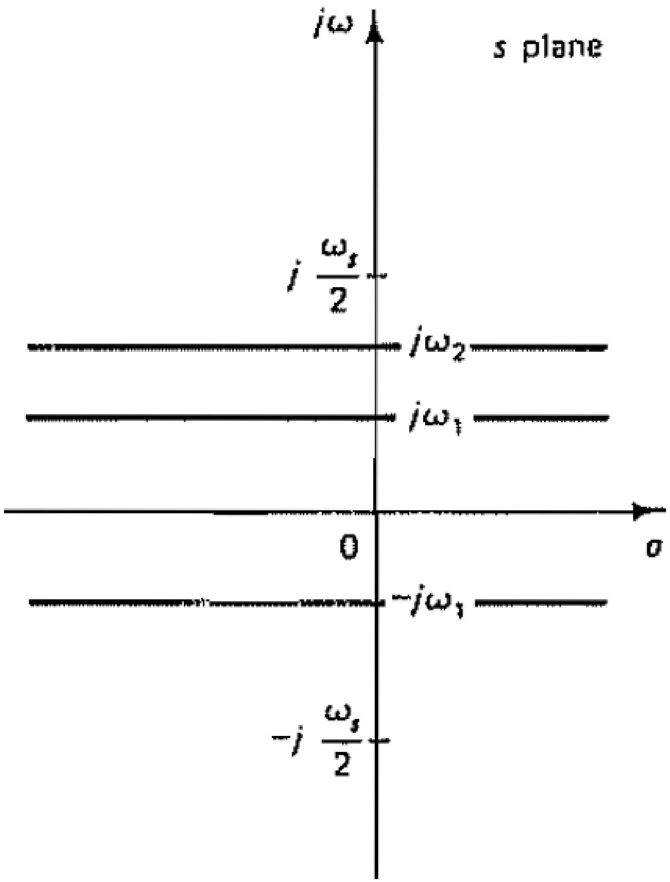
- Constant settling time (less than $4/\sigma_1$)

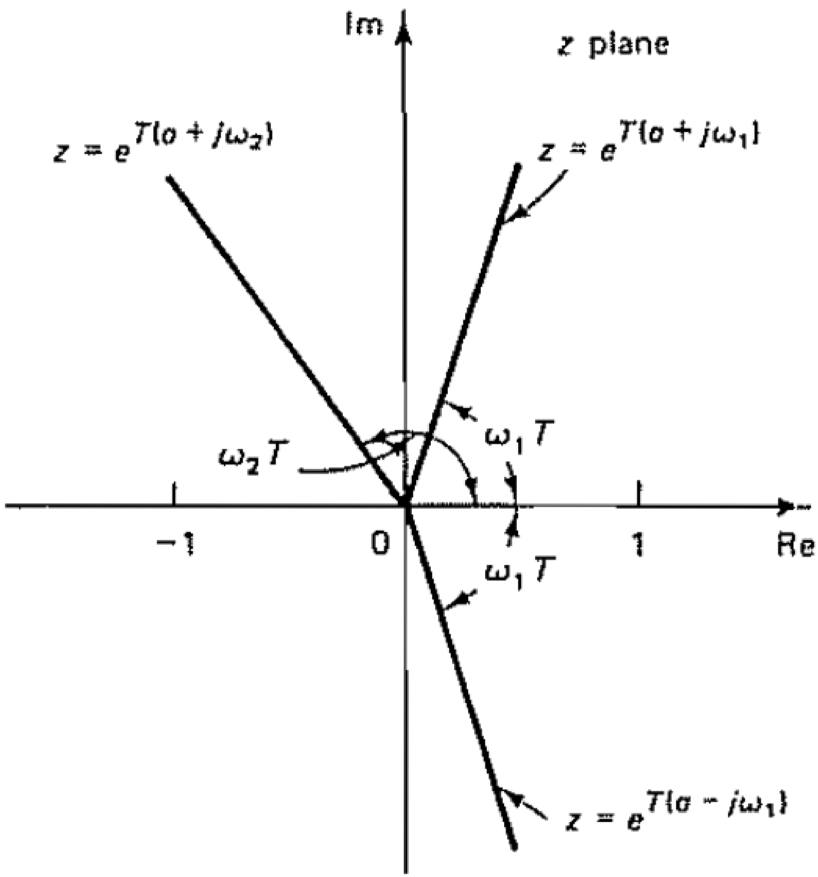




Mapping of Contours

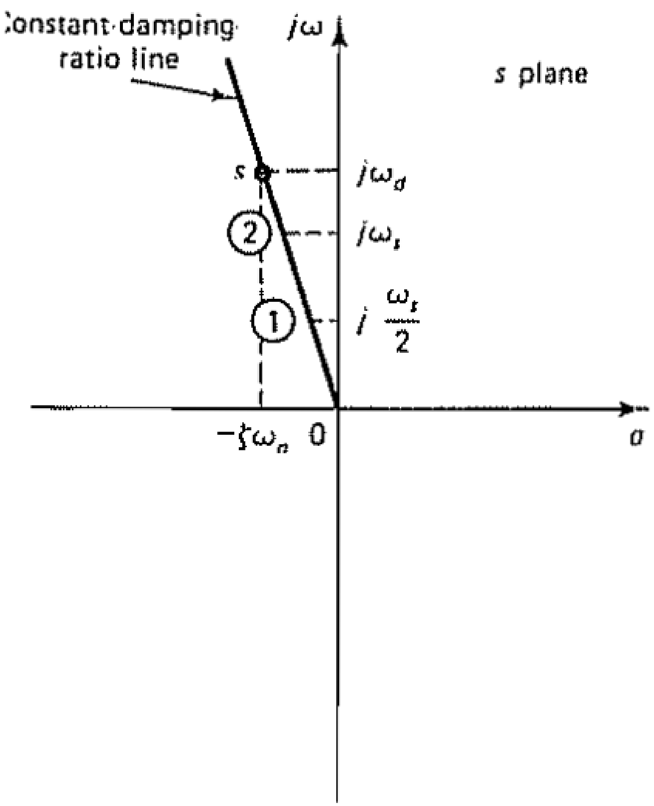
- Constant (damped) frequency

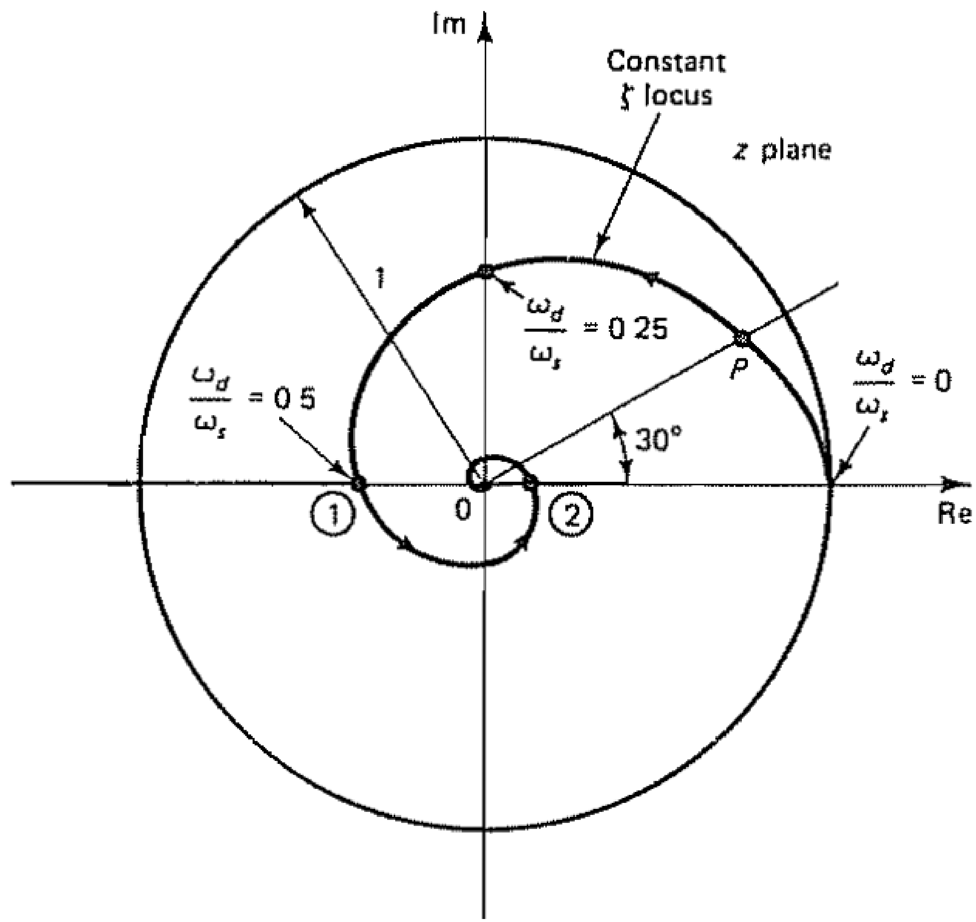




Mapping of Contours

- Constant damping ratio





Visual Comparison