Linear Quadratic Regulator (LQR)

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Introduction to Optimal Control

- Pole-Placement Technique was great at determining required controller gains **K**
  - However, it requires an iterative approach to choose 'good' closed-loop poles
  - Several different sets of desired closed-loop poles need to be considered and the response characteristics compared, and the best one chosen
- The Quadratic Optimal Control method provides a **systematic** way of computing the state feedback control gain **K**
- A cost function, described using a quadratic function, characterizes the relative importance between ***output measurement deviation*** and ***required control input***
  - It is also one of the most fundamental problems in control theory
- The solution that minimizes this cost function is called the **Linear Quadratic Regulator (LQR)** and is a key component of the Linear Quadratic Gaussian (LQG) problem (Kalman Filter + LQR)

The Linear Quadratic Regulator (LQR)

- We consider the regulator problem where given the following state equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

  - It is desired to find an optimal control vector:

$$\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$$

  - that minimizes the following performance index: (Infinite-horizon)

$$J = \int_0^\infty \left(\mathbf{x}^* \mathbf{Q}\mathbf{x} + \mathbf{u}^* \mathbf{R}\mathbf{u} \right) dt$$

  - where **Q** and **R** are positive-definite (or real symmetric) matrices
  - **Q** and **R** matrices determine the relative importance of state vector error (the goal of the regulator controller is to bring **x** to zero) and expenditure of energy due to the required control input **u**.

- The solution, if it can be computed, is the **OPTIMAL** control law
  - It is optimal for any initial state **x**(0)

- Mathematically, we are trying to find **u** to minimize *J* subject to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ (constraint)

Solving the LQR Optimization Problem

- Let's introduce a matrix **P** which is positive-definite (or real symmetric):

$$\mathbf{P} = \mathbf{P}^*$$

  - Now since the cost function comprises of quadratic terms, and we want to somehow infuse the constraint equation into *J*, consider the following:

$$\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right) = \dot{\mathbf{x}}^*\mathbf{P}\mathbf{x} + \mathbf{x}^*\mathbf{P}\dot{\mathbf{x}}$$

  - Because the cost function is in integral form, let's derive the integration:

$$\int_0^\infty \left[\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right)\right] dt = \left[\mathbf{x}^*\mathbf{P}\mathbf{x}\right]_0^\infty = \mathbf{x}(\infty)^*\mathbf{P}\mathbf{x}(\infty) - \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0)$$

  - Since the controlled system will be stable, $\mathbf{x}(\infty) \to 0$

$$\boxed{\int_0^\infty \left[\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right)\right] dt = -\mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0)}$$

  - Collecting them to one side (both expressions sum to zero):

$$\mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left[\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right)\right] dt = 0, \quad \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) - \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) = 0$$

Solving the LQR Optimization Problem

- Recalling the cost function: $J = \int_0^\infty \left(\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right) dt$
- Let's introduce **P** into the cost function using the zero sum expression

$$\boxed{\int_0^\infty \left[\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right)\right] dt = -\mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0)}$$

$$\begin{aligned} J &= 0 + \int_0^\infty \left(\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right) dt \\ &= \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) - \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right) dt \\ &= \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left[\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right)\right] dt + \int_0^\infty \left(\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right) dt \end{aligned}$$

  - Since we already have an expression for: $\frac{d}{dt}\left(\mathbf{x}^*\mathbf{P}\mathbf{x}\right) = \dot{\mathbf{x}}^*\mathbf{P}\mathbf{x} + \mathbf{x}^*\mathbf{P}\dot{\mathbf{x}}$

$$= \left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right)^*\mathbf{P}\mathbf{x} + \mathbf{x}^*\mathbf{P}\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right)$$

  - Substituting back in the cost function and combining the integrals obtains:

$$\begin{aligned} J &= \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right)^*\mathbf{P}\mathbf{x} + \mathbf{x}^*\mathbf{P}\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right)\right) dt + \int_0^\infty \left(\mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right) dt \\ &= \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right)^*\mathbf{P}\mathbf{x} + \mathbf{x}^*\mathbf{P}\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right) + \mathbf{x}^*\mathbf{Q}\mathbf{x} + \mathbf{u}^*\mathbf{R}\mathbf{u}\right) dt \end{aligned}$$

Solving the LQR Optimization Problem

- Let's simplify by grouping the **x***(?)**x** terms together

$$J = \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\left(\mathbf{Ax}+\mathbf{Bu}\right)^*\mathbf{Px} + \mathbf{x}^*\mathbf{P}\left(\mathbf{Ax}+\mathbf{Bu}\right) + \mathbf{x}^*\mathbf{Qx} + \mathbf{u}^*\mathbf{Ru}\right)dt$$

$$= \underbrace{\mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0)}_{\text{Independent of } \mathbf{u}} + \int_0^\infty \left(\underbrace{\mathbf{x}^*(\mathbf{A}^*\mathbf{P}+\mathbf{PA}+\mathbf{Q})\mathbf{x}}_{\text{Independent of } \mathbf{u}} + \mathbf{u}^*\mathbf{Ru} + \mathbf{x}^*\mathbf{PBu} + \mathbf{u}^*\mathbf{B}^*\mathbf{Px}\right)dt$$

- For the terms with **u**, we can 'complete the square': $\boxed{\mathbf{u}^*\mathbf{Ru}+\mathbf{x}^*\mathbf{PBu}+\mathbf{u}^*\mathbf{B}^*\mathbf{Px}}$
- Consider: $\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right)^*\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right)$

$$= \left(\mathbf{u}^* + \left(\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right)^*\right)\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right)$$

$$= \left(\mathbf{u}^* + \mathbf{x}^*\mathbf{P}^*\mathbf{B}\left(\mathbf{R}^{-1}\right)^*\right)\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right)$$

$$= \mathbf{u}^*\mathbf{Ru} + \mathbf{u}^*\mathbf{RR}^{-1}\mathbf{B}^*\mathbf{Px} + \mathbf{x}^*\mathbf{P}^*\mathbf{BR}^{-1*}\mathbf{Ru} + \mathbf{x}^*\mathbf{P}^*\mathbf{BR}^{-1*}\mathbf{RR}^{-1}\mathbf{B}^*\mathbf{Px}$$

$$= \mathbf{u}^*\mathbf{Ru} + \mathbf{u}^*\mathbf{B}^*\mathbf{Px} + \mathbf{x}^*\mathbf{PBu} + \mathbf{x}^*\mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{Px}$$

Remember:

$\mathbf{P} = \mathbf{P}^*$

$\mathbf{R} = \mathbf{R}^*$

$\mathbf{R}^{-1} = \left(\mathbf{R}^{-1}\right)^*$

  - To get the above boxed expression:

$$\boxed{\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right)^*\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{Px}\right) - \mathbf{x}^*\mathbf{PBR}^{-1}\mathbf{B}^*\mathbf{Px}}$$

Solving the LQR Optimization Problem

- Okay, let's now insert this 'complete the square' expression back to *J*:

$$J = \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\mathbf{x}^*(\mathbf{A}^*\mathbf{P}+\mathbf{P}\mathbf{A}+\mathbf{Q})\mathbf{x} + \left[\mathbf{u}^*\mathbf{R}\mathbf{u} + \mathbf{x}^*\mathbf{P}\mathbf{B}\mathbf{u} + \mathbf{u}^*\mathbf{B}^*\mathbf{P}\mathbf{x}\right]\right)dt$$

$$= \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\mathbf{x}^*(\mathbf{A}^*\mathbf{P}+\mathbf{P}\mathbf{A}+\mathbf{Q})\mathbf{x} + \left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)^*\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right) - \mathbf{x}^*\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)dt$$

  - Rearranging the terms and grouping them:

$$J = \underbrace{\mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0)}_{\text{Independent of }\mathbf{u}} + \int_0^\infty \left(\underbrace{\mathbf{x}^*(\mathbf{A}^*\mathbf{P}+\mathbf{P}\mathbf{A}+\mathbf{Q}-\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P})\mathbf{x}}_{\text{Independent of }\mathbf{u}} + \left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)^*\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)\right)dt$$

- How to minimize $\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)^*\mathbf{R}\left(\mathbf{u}+\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)$?
  - Since this is a 'squared' expression, it is easy!
  - The minimum is when the entire expression is zero! Or in other words:

$$\boxed{\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}}$$

  - Recall full state feedback control law! $\mathbf{u} = -\mathbf{K}\mathbf{x}$
  - The optimal controller is a **full state feedback controller** with gain matrix:

$$\boxed{\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}}$$

Solving the LQR Optimization Problem

- We are not quite done yet! What about **P**?? How to calculate it? Let's look at the cost function again!

$$J = \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0) + \int_0^\infty \left(\mathbf{x}^*(\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P})\mathbf{x} + \left(\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)^*\mathbf{R}\left(\mathbf{u} + \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}\right)\right)dt$$

- Since we are 'free' to select **P**, to get the minimum for *J*, we can set the remaining terms in the integral to be zero as well!

$$\boxed{\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} + \mathbf{Q} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} = \mathbf{0}}$$

Continuous-time Algebraic Riccati Equation (CARE)

- This also means the minimum of the cost function is: $J_{\min} = \mathbf{x}(0)^*\mathbf{P}\mathbf{x}(0)$

- **IN SUMMARY:**
  - The optimal control law & gain: $\boxed{\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t) = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}\mathbf{x}(t)}$ $\boxed{\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P}}$
  - The matrix **P** (real symmetric) must satisfy CARE

- **<u>Design Approach:</u>**
  - Choose **Q** and **R**
  - Solve for **P** in the Riccati Equation. Remember it is **positive definite** & **symmetric**!
  - Compute **K** from the expression of the optimal gain matrix.

Note: for P to be positive definite, all the eigenvalues of P must be positive

Exercise 1 (Effect of Q)

- For the following system under state feedback control: $u(t) = -\mathbf{K}\mathbf{x}(t)$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + u^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

  - Design the optimal feedback gain matrix **K** such that the performance index is minimized.
  - Where are the closed loop poles when μ=1?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R = 1$$

Check for complete state controllability:

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Controllability matrix is rank 2. System state vector is a 2-dimension.
System is completely state controllable

Exercise 1 (Effect of Q)

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Step 1: Find **P** using the Riccati equation.

$$\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1][0 \quad 1]\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$1 - p_{12}^2 = 0$$

$$p_{11} - p_{12}p_{22} = 0$$

$$\mu + 2p_{12} - p_{22}^2 = 0$$

$$\boxed{\begin{aligned} &\Rightarrow p_{11} = \sqrt{\mu+2} \\ &\Rightarrow p_{12} = 1 \\ &\Rightarrow p_{22} = \sqrt{\mu+2} \end{aligned}}$$

$$\mathbf{P} = \begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix}$$

Exercise 1 (Effect of Q)

- For the following system under state feedback control: $u(t) = -\mathbf{K}\mathbf{x}(t)$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + u^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

  - Design the optimal feedback gain matrix **K** such that the performance index is minimized.
  - Where are the closed loop poles when μ=1?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R = 1$$

Step 1: Find **P** using the Riccati equation.

$$\mathbf{P} = \begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix}$$

Check **P** for Positive Definite:

Determine eigenvalues of **P**!

$$|\lambda\mathbf{I} - \mathbf{P}| = \left\| \begin{bmatrix} \lambda - \sqrt{\mu+2} & -1 \\ -1 & \lambda - \sqrt{\mu+2} \end{bmatrix} \right\|$$

$$= \left(\lambda - \sqrt{\mu+2}\right)^2 - 1 = \lambda^2 - 2\left(\sqrt{\mu+2}\right)\lambda + \mu + 2 - 1$$

$$= \lambda^2 - 2\left(\sqrt{\mu+2}\right)\lambda + \mu + 1$$

$$\lambda = \frac{2\left(\sqrt{\mu+2}\right) \pm \sqrt{4(\mu+2) - 4(\mu+1)}}{2}$$

$$= \frac{2\left(\sqrt{\mu+2}\right) \pm 2}{2}$$

$$= \left(\sqrt{\mu+2}\right) \pm 1$$

Since μ is a positive number, the eigenvalues of **P** are always positive. **P** is Positive Definite.

Exercise 1 (Effect of Q)

- For the following system under state feedback control: $u(t) = -\mathbf{K}\mathbf{x}(t)$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + u^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}$$

  - Design the optimal feedback gain matrix **K** such that the performance index is minimized.
  - Where are the closed loop poles when μ=1?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R = 1$$

Step 2: Compute **K**.

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} = [1]\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \sqrt{\mu+2} \end{bmatrix}$$

Control Law: $u = -\mathbf{K}\mathbf{x} = -x_1 - \sqrt{\mu+2}\,x_2$

Closed Loop Characteristic Equation:

$$\left|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right| = \left| s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & \sqrt{\mu+2} \end{bmatrix} \right| = 0$$

μ=1 $\quad s^2 + \sqrt{\mu+2}\,s + 1 = 0 \Rightarrow s = -\frac{\sqrt{3}}{2} + j\frac{1}{2}, \quad s = -\frac{\sqrt{3}}{2} - j\frac{1}{2}$

Exercise 2 (Effect of R)

- For the following system under state feedback control: $u(t) = -\mathbf{K}\mathbf{x}(t)$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  - Design the optimal feedback gain matrix **K** such that the performance index is minimized.
  - Determine the settling time of the system?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R = r$$

Check for complete state controllability:

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Controllability matrix is rank 2. System state vector is a 2-dimension.
System is completely state controllable

Exercise 2 (Effect of R)

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$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  - Design the optimal feedback gain matrix **K** such that the performance index is minimized.
  - Determine the settling time of the system?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R = r$$

Step 1: Find **P** using the Riccati equation.

$$\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = \mathbf{0}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} -$$

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} \frac{1}{r} \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \frac{1}{r}\begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow 1 - \frac{1}{r}p_{12}^2 = 0 \quad \Rightarrow p_{11} - \frac{1}{r}p_{12}p_{22} = 0 \quad \Rightarrow 2p_{12} - \frac{1}{r}p_{22}^2 = 0$$

$$\Rightarrow p_{11} = \sqrt{2}r^{\frac{1}{4}} \qquad p_{12} = \sqrt{r} \qquad p_{22} = \sqrt{2}r^{\frac{3}{4}}$$

$$\boxed{\mathbf{P} = \begin{bmatrix} \sqrt{2}r^{\frac{1}{4}} & \sqrt{r} \\ \sqrt{r} & \sqrt{2}r^{\frac{3}{4}} \end{bmatrix}}$$

Remember to check **P** for Positive Definite!

Exercise 2 (Effect of R)

- For the following system under state feedback control: $u(t) = -\mathbf{K}\mathbf{x}(t)$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  - Design the optimal feedback gain matrix **K** such that the performance index is minimized.
  - Determine the settling time of the system?

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad R = r$$

Step 2: Compute **K**.

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^*\mathbf{P} = \begin{bmatrix} \frac{1}{r} \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2}r^{\frac{1}{4}} & \sqrt{r} \\ \sqrt{r} & \sqrt{2}r^{\frac{3}{4}} \end{bmatrix}$$

$$= \frac{1}{r}\begin{bmatrix} \sqrt{r} & \sqrt{2}r^{\frac{3}{4}} \end{bmatrix}$$

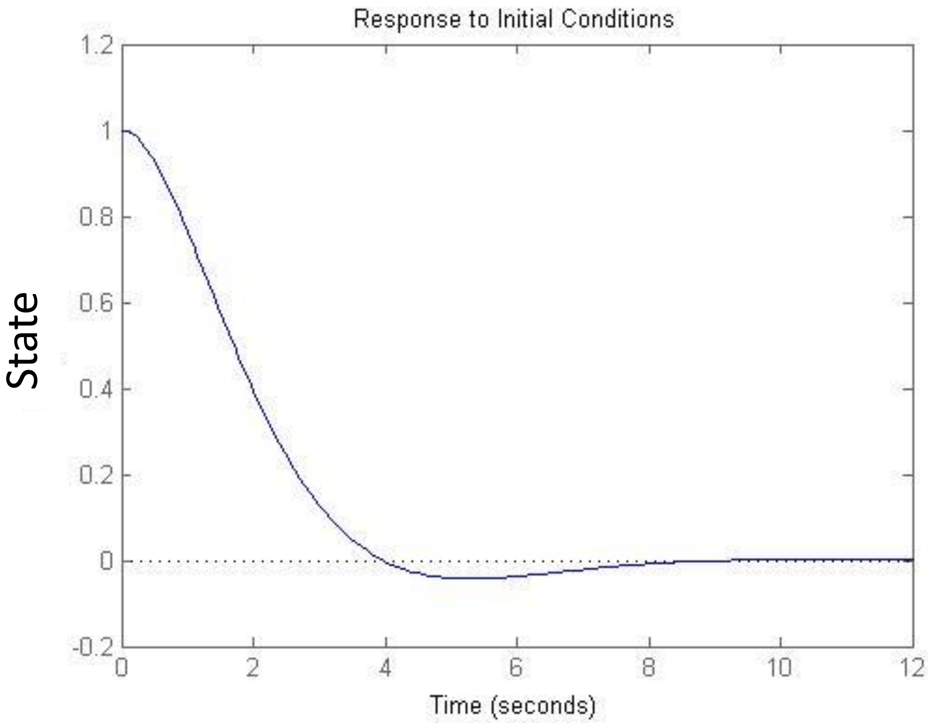
Closed Loop Characteristic Equation:

$$\left|s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{K}\right| = \left| s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{r}\begin{bmatrix} \sqrt{r} & \sqrt{2}r^{\frac{3}{4}} \end{bmatrix} \right| = 0$$

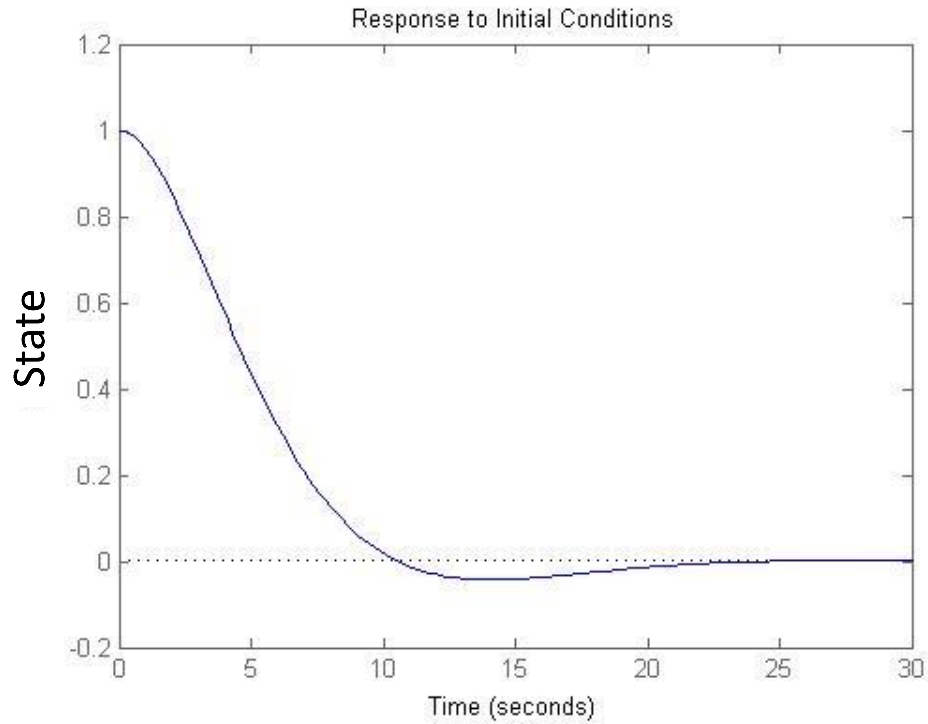
$$s^2 + \sqrt{2}r^{-\frac{1}{4}}s + r^{-\frac{1}{2}} = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad \Rightarrow \quad \zeta = \frac{1}{\sqrt{2}}, \quad \omega_n = r^{-\frac{1}{4}}, \quad t_s = \frac{4}{\zeta\omega_n} = \frac{4\sqrt{2}}{r^{-\frac{1}{4}}}$$

Exercise 2 (Effect of R)



$R = 2$

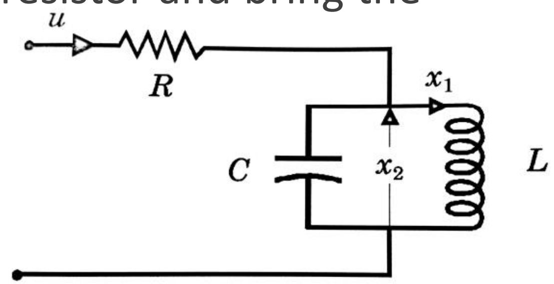


$R = 100$

Exercise (Electrical Circuit)

$$V_L = L\frac{dI_L}{dt} \quad I_C = C\frac{dV_C}{dt}$$

- Consider the RLC circuit as shown. Design an optimal controller that regulates the input current u to minimize the energy wasted in the resistor and bring the system to rest.
  - Find a relationship between u, x_1, x_2, R, L and C.
  - Obtain the state space representation of the system
  - Setting all electrical parameters to 1, $R = L = C = 1$,



Design the optimal feedback gain matrix **K** such that the performance index is minimized.

$$J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2\right) dt$$

Capacitor Element: $u - x_1 = C\dot{x}_2 \quad \Rightarrow \dot{x}_2 = -\frac{1}{C}x_1 + \frac{1}{C}u$

Inductor Element: $x_2 = L\dot{x}_1 \quad \Rightarrow \dot{x}_1 = \frac{1}{L}x_2$

$$\boxed{\Rightarrow \dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/C \end{bmatrix} u}$$

Exercise (Electrical Circuit)

$$V_L = L\frac{dI_L}{dt} \quad I_C = C\frac{dV_C}{dt}$$

- Consider the RLC circuit as shown. Design an optimal controller that regulates the input current *u* to minimize the energy wasted in the resistor and bring the system to rest.
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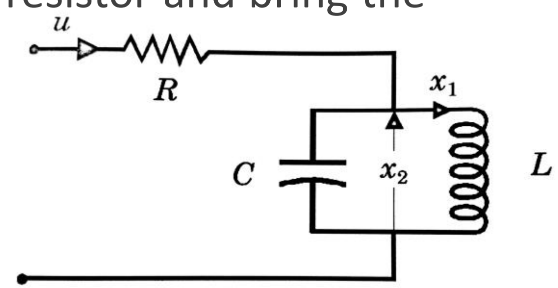

Design the optimal feedback gain matrix **K** such that the performance index is minimized.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2\right)dt$$

Compare to generic form:

$$J = \int_0^\infty \left(\mathbf{x}^T\mathbf{Q}\mathbf{x} + ru^2\right)dt \quad \Rightarrow \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, r = 1$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Check for complete state controllability:

$$\mathbf{C}_O = \begin{bmatrix} \mathbf{B} & \mathbf{AB} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Controllability matrix is rank 2. System state vector is a 2-dimension. System is completely state controllable

Exercise (Electrical Circuit)

$$V_L = L\frac{dI_L}{dt} \qquad I_C = C\frac{dV_C}{dt}$$

- Consider the RLC circuit as shown. Design an optimal controller that regulates the input current *u* to minimize the energy wasted in the resistor and bring the system to rest.
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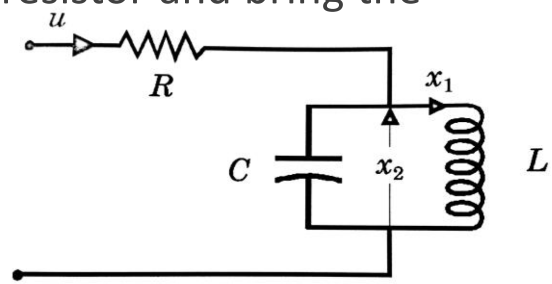

Design the optimal feedback gain matrix **K** such that the performance index is minimized.

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u \qquad J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2\right)dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad r = 1$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Step 1: Find **P** using the Riccati equation.

$$\mathbf{A}^*\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}r^{-1}\mathbf{B}^*\mathbf{P} + \mathbf{Q} = 0$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}[1]\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -p_{12} & -p_{22} \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} -p_{12} & p_{11} \\ -p_{22} & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$1 - 2p_{12} - p_{12}^2 = 0$$
$$-p_{22} + p_{11} - p_{12}p_{22} = 0$$
$$2p_{12} - p_{22}^2 + 1 = 0$$

$$\boxed{\begin{aligned} &\Rightarrow p_{11} = \sqrt{4\sqrt{2} - 2} \\ &\Rightarrow p_{12} = \sqrt{2} - 1 \\ &\Rightarrow p_{22} = \sqrt{2\sqrt{2} - 1} \end{aligned}}$$

Exercise (Electrical Circuit)

$$V_L = L\frac{dI_L}{dt} \qquad I_C = C\frac{dV_C}{dt}$$

- Consider the RLC circuit as shown. Design an optimal controller that regulates the input current *u* to minimize the energy wasted in the resistor and bring the system to rest.
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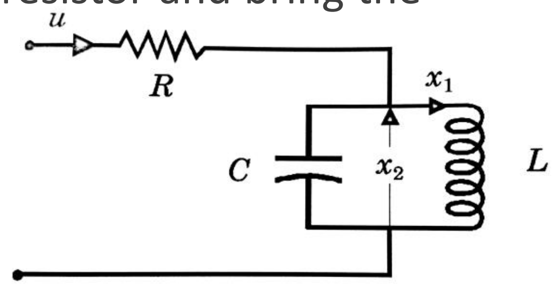

Design the optimal feedback gain matrix **K** such that the performance index is minimized.

$$J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2\right) dt \qquad \mathbf{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad r = 1$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\mathbf{P} = \begin{bmatrix} \sqrt{4\sqrt{2}-2} & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2\sqrt{2}-1} \end{bmatrix}$$

Remember to check **P** for Positive Definite!

Step 2: Compute **K**.

$$\mathbf{K} = r^{-1}\mathbf{B}^*\mathbf{P} = [1][0 \quad 1]\begin{bmatrix} \sqrt{4\sqrt{2}-2} & \sqrt{2}-1 \\ \sqrt{2}-1 & \sqrt{2\sqrt{2}-1} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{2}-1 & \sqrt{2\sqrt{2}-1} \end{bmatrix}$$

Final Notes on LQR

- Sometimes the performance index may be given in terms of the output vector **y** rather than the state vector **x**:

$$J = \int_0^\infty \left(\mathbf{y}^* \mathbf{Q} \mathbf{y} + \mathbf{u}^* \mathbf{R} \mathbf{u} \right) dt$$

- But not to worry, you can manipulate it back to a familiar form using:

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

- So that the performance index is now:

$$J = \int_0^\infty \left(\mathbf{x}^* \mathbf{C}^* \mathbf{Q} \mathbf{C} \mathbf{x} + \mathbf{u}^* \mathbf{R} \mathbf{u} \right) dt$$

$$= \int_0^\infty \left(\mathbf{x}^* \bar{\mathbf{Q}} \mathbf{x} + \mathbf{u}^* \mathbf{R} \mathbf{u} \right) dt \qquad \bar{\mathbf{Q}} = \mathbf{C}^* \mathbf{Q} \mathbf{C}$$

- In cases where **A-BK** cannot be made stable (not stabilizable), there is no positive-definite **P** that can satisfy the Riccati equation. No solution exists.