Solving Difference Equations

EPD 30.114 ADVANCED FEEDBACK & CONTROL

Analytical Solution to Difference Equations

- Difference equations can be solved easily using a digital computer since it is already in an iterative form
- However closed form expressions for *x(k)* cannot be obtained from computer solutions
- The *z* transform methods enables us to obtain a closed form expression for *x(k)*!
- Consider a linear time-invariant discrete-time system, which is characterized by a constant coefficient difference equation (CCDE):

$$x(k)+a_1x(k-1)+\cdots+a_nx(k-n)=b_0u(k)+b_1u(k-1)+\cdots+b_nu(k-n)$$

  - *u(k)* and *x(k)* are the system input and output

- To solve the CCDE, the terms needs to be transformed into the *z* domain using z Transform

$$\mathcal{Z}\left[x(k)\right]=X(z)$$

$$\mathcal{Z}\left[u(k)\right]=U(z)$$

Z Transform of CCDE Terms

- What about the other terms of the DE?

Discrete Function	z Transform
$x(k+4)$	$z^4 X(z) - z^4 x(0) - z^3 x(1) - z^2 x(2) - zx(3)$
$x(k+3)$	$z^3 X(z) - z^3 x(0) - z^2 x(1) - zx(2)$
$x(k+2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1} X(z)$
$x(k-2)$	$z^{-2} X(z)$
$x(k-3)$	$z^{-3} X(z)$
$x(k-4)$	$z^{-4} X(z)$

Exercise

- Solve the difference equation using *z* Transform.

$$x(k+2)+3x(k+1)+2x(k)=0 \qquad x(0)=0,\ \ x(1)=1$$

$$\mathcal{Z}\left[x(k+2)\right]=z^2X(z)-z^2x(0)-zx(1)$$

$$\mathcal{Z}\left[x(k+1)\right]=zX(z)-zx(0)$$

$$\mathcal{Z}\left[x(k)\right]=X(z)$$

$$z^2X(z)-z^2x(0)-zx(1)+3\left[zX(z)-zx(0)\right]+2X(z)=0$$

$$X(z)=\frac{z}{z^2+3z+2}$$

$$\frac{X(z)}{z}=\frac{1}{z^2+3z+2}=\frac{1}{z+1}-\frac{1}{z+2}$$

$$X(z)=\frac{z}{z+1}-\frac{z}{z+2}=\frac{1}{1+z^{-1}}-\frac{1}{1+2z^{-1}}$$

$$\mathcal{Z}^{-1}\left[\frac{1}{1+z^{-1}}\right]=(-1)^k \qquad \mathcal{Z}^{-1}\left[\frac{1}{1+2z^{-1}}\right]=(-2)^k$$

$$x(k)=(-1)^k-(-2)^k \qquad k=0,1,2,\cdots$$

Exercise

- Consider the difference equation:

$$x(k+2) = x(k+1) + x(k) \qquad x(0) = 0, \;\; x(1) = 1$$

  - Obtain a closed form expression for *x(k),*
  - Compute the limiting value of *x(k+1)/x(k)* as *k* approaches infinity.

Taking zT: $$z^2 X(z) - z^2 x(0) - zx(1) = zX(z) - zx(0) + X(z)$$

$$X(z) = \frac{z^2 x(0) + zx(1) - zx(0)}{z^2 - z - 1} = \frac{z}{z^2 - z - 1}$$

$$X(z) = \frac{1}{\sqrt{5}} \left(\frac{z}{z - \frac{1+\sqrt{5}}{2}} - \frac{z}{z - \frac{1-\sqrt{5}}{2}} \right) = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - \frac{1+\sqrt{5}}{2} z^{-1}} - \frac{1}{1 - \frac{1-\sqrt{5}}{2} z^{-1}} \right)$$

$$x(k) = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right] \qquad k = 0, 1, 2, \cdots$$

Exercise

- Consider the difference equation:

$$x(k+2) = x(k+1) + x(k) \qquad x(0)=0, \;\; x(1)=1$$

  - Obtain a closed form expression for *x(k),*
  - Compute the limiting value of *x(k+1)/x(k)* as *k* approaches infinity.

$$x(k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] \qquad k = 0,1,2,\cdots$$

$$\lim_{k\to\infty}\frac{x(k+1)}{x(k)} = \lim_{k\to\infty}\frac{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]}{\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]}$$

Note that: $\lim_{k\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^k = 0$

$$\lim_{k\to\infty}\frac{x(k+1)}{x(k)} = \lim_{k\to\infty}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{k}} = \frac{1+\sqrt{5}}{2} = 1.6180$$

GOLDEN RATIO

More Exercise!

- Obtain the solution of the following difference equation in terms of *x(0)* and *x(1)*, where *a* and *b* are constants and *k*=0,1,2,...

$$x(k+2)+(a+b)x(k+1)+abx(k)=0$$

Taking zT:
$$\left[z^2X(z)-z^2x(0)-zx(1)\right]+(a+b)\left[zX(z)-zx(0)\right]+abX(z)=0$$

$$\left[z^2+(a+b)z+ab\right]X(z)=\left[z^2+(a+b)z\right]x(0)+zx(1)$$

$$X(z)=\frac{\left[z^2+(a+b)z\right]x(0)+zx(1)}{z^2+(a+b)z+ab}=\frac{\left[z^2+(a+b)z\right]x(0)+zx(1)}{(z+a)(z+b)}$$

Two possible cases: $a \neq b$ and $a=b$

More Exercise!

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$$x(k+2)+(a+b)x(k+1)+abx(k)=0$$

$$X(z)=\frac{\left[z^2+(a+b)z\right]x(0)+zx(1)}{(z+a)(z+b)}$$

$\boxed{a\neq b}$

$$\frac{X(z)}{z}=\frac{bx(0)+x(1)}{b-a}\frac{1}{(z+a)}+\frac{ax(0)+x(1)}{a-b}\frac{1}{(z+b)}$$

$$X(z)=\frac{bx(0)+x(1)}{b-a}\frac{1}{(1+az^{-1})}+\frac{ax(0)+x(1)}{a-b}\frac{1}{(1+bz^{-1})}$$

$$x(k)=\frac{bx(0)+x(1)}{b-a}(-a)^k+\frac{ax(0)+x(1)}{a-b}(-b)^k \qquad k=0,1,2,\ldots$$

More Exercise!

- Obtain the solution of the following difference equation in terms of *x(0)* and *x(1)*, where *a* and *b* are constants and *k*=0,1,2,…

$$x(k+2)+(a+b)x(k+1)+abx(k)=0$$

$$X(z)=\frac{\left[z^2+(a+b)z\right]x(0)+zx(1)}{(z+a)(z+b)}$$

$\boxed{a=b}$

$$X(z)=\frac{\left[z^2+2az\right]x(0)+zx(1)}{(z+a)^2}=\frac{zx(0)}{z+a}+\frac{z\left[ax(0)+x(1)\right]}{(z+a)^2}$$

$$X(z)=\frac{x(0)}{1+az^{-1}}+\frac{\left[ax(0)+x(1)\right]z^{-1}}{\left(1+az^{-1}\right)^2}$$

$$x(k)=x(0)(-a)^k+\left[ax(0)+x(1)\right]k(-a)^{k-1} \qquad k=0,1,2,\ldots$$

Even More!

- Solve the following difference equation: $2x(k)-2x(k-1)+x(k-2)=u(k)$

where *x(k)=0* for *k*<0 and $u(k)=\begin{cases}1 & k=0,1,2,\ldots\\ 0 & k<0\end{cases}$

Taking zT: $$2X(z)-2z^{-1}X(z)+z^{-2}X(z)=U(z)=\frac{1}{1-z^{-1}}$$

$$\left[2-2z^{-1}+z^{-2}\right]X(z)=\frac{1}{1-z^{-1}}$$

$$X(z)=\frac{1}{\left(2-2z^{-1}+z^{-2}\right)\left(1-z^{-1}\right)}=\frac{z^3}{\left(2z^2-2z+1\right)\left(z-1\right)}$$

$$X(z)=\frac{z}{z-1}+\frac{-z^2+z}{2z^2-2z+1}=\frac{1}{1-z^{-1}}+\frac{-1+z^{-1}}{2-2z^{-1}+z^{-2}}$$

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where *x(k)=0* for *k*<0 and $u(k)=\begin{cases}1 & k=0,1,2,\ldots\\ 0 & k<0\end{cases}$

$$X(z)=\frac{z}{z-1}+\frac{-z^2+z}{2z^2-2z+1}=\frac{1}{1-z^{-1}}+\frac{-1+z^{-1}}{2-2z^{-1}+z^{-2}}$$

$$=\frac{1}{1-z^{-1}}+\frac{1}{2}\left(\frac{-1+z^{-1}}{1-z^{-1}+0.5z^{-2}}\right)$$

Comparing to quadratic term in zT Table: $e^{-2aT}=0.5$

$$\cos\omega T=1/\sqrt{2} \qquad \Rightarrow \omega T=\pi/4 \qquad \Rightarrow \sin\omega T=1/\sqrt{2}$$

$$X(z)=\frac{1}{1-z^{-1}}-\frac{1}{2}\left(\frac{1-0.5z^{-1}}{1-z^{-1}+0.5z^{-2}}\right)+\frac{1}{2}\left(\frac{0.5z^{-1}}{1-z^{-1}+0.5z^{-2}}\right)$$

$$x(k)=1-\frac{1}{2}e^{-akT}\cos\omega kT+\frac{1}{2}e^{-akT}\sin\omega kT \qquad k=0,1,2,\ldots$$

$$=1-\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^k\cos\frac{k\pi}{4}+\frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^k\sin\frac{k\pi}{4}$$