1 Three Proofs (26 marks)

(a) Recall that random variables X and Y are conditionally independent, given a third variable Z if and only if:

$$P(X|Y \wedge Z) = P(X|Z) \tag{1}$$

$$P(Y|X \wedge Z) = P(Y|Z) \tag{2}$$

$$P(X \wedge Y|Z) = P(X|Z)P(Y|Z) \tag{3}$$

Show that Equation 3 follows from Equations 1 and 2. Justify each step of your proof.

Hint: you may find some of the probability rules from Lecture 6 to be useful.

Marking Scheme: (6 marks)

- (4 marks) Correct proof
- (2 marks) Each step is justified

Solutions:

$P(X|Y \wedge Z) = P(X \wedge Y \wedge Z)/P(Y \wedge Z) = P(X|Z)$ - Conditional Probability from equation 2

$P(X \wedge Y|Z) = P(X \wedge Y \wedge Z)/P(Z)$ - Conditional Probability

$= P(X \wedge Y \wedge Z)/P(Y \wedge Z) * P(Y \wedge Z)/P(Z)$ - Alegebra

$= P(X|Z) * P(Y \wedge Z)/P(Z)$ - Substitute Equation 2

$= P(X|Z)P(Y|Z)$ - Conditional Probability

(b) The *Markov blanket* of random variable X in a Bayesian network consists of the set of random variables that are parents, children, or parents of children of X. For a Bayesian network with random variables $\mathcal{S}$, let $B = \{B_1, B_2, ..., B_n\} \subset \mathcal{S}$ be the Markov blanket for $X \in \mathcal{S}$. See Figure 1 for an example.

Show that $X \perp\!\!\!\perp Y \,|\, B, \forall\ Y \in \mathcal{S} \setminus B$. In other words, show that X is conditionally independent of every other variable in the network, conditioned on its Markov blanket.

Hint: Consider using the fundamental conditional independence relationships discussed in Lecture 8.

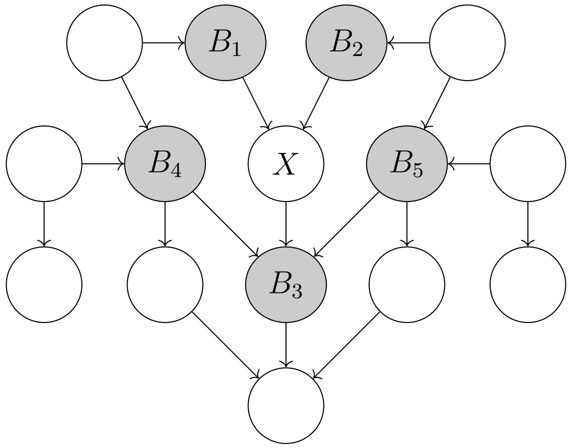

FIGURE 1: An example of a Bayesian network with its Markov blanket shaded grey.

Marking Scheme: (10 marks)

- (8 marks) Correct proof
- (2 marks) Proof is succinct and easy to understand

Solutions: For X to be conditionally independent with any node N given B , there should not be a path from X to N that does not go through B. Since B includes all children and parents of X, this holds for all N and thus X is conditionally independent of every variable in the network,conditioned on its Markov blanket

(c) Not all the random variables in a Bayesian networks are always required to answer a probabilistic query. In fact, all variables that are not ancestors of query variables or evidence variables are irrelevant to the query. Let $Q = \{Q_1, \ldots, Q_m\}$ be the set of query variables and $e = \{e_1, \ldots, e_n\}$ be the set of evidence variables. Prove that the Variable Elimination Algorithm (VEA) returns the same distribution for some query $P(Q|E)$ if all irrelevant variables are pruned from the Bayesian network.

Hint: try a direct proof. Show that a particular ordering of hidden variable elimination results in the same final distribution returned by the normalization operation.

Marking Scheme: (10 marks)

- (8 marks) Correct proof
- (2 marks) Proof is succinct and easy to understand

Solutions: TODO