Independence and Bayesian Networks (Part 1)

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Lecture 7

Readings: RN 12.4, 13.1, & 13.2. PM 8.2 & 8.3.

Outline

Learning Goals

By the end of the lecture, you should be able to

- Given a probabilistic model,
  determine if two variables are unconditionally independent, or conditionally independent given a third variable.
- Give examples of deriving a compact representation of a joint distribution by using independence assumptions.
- Describe components of a Bayesian network.
- Compute a joint probability given a Bayesian network.
- Explain the independence relationships in the three key structures.

(Unconditional) Independence

Definition ((unconditional) independence)

X and Y are (unconditionally) independent iff

$$P(X|Y) = P(X)$$

$$P(Y|X) = P(Y)$$

$$P(X \wedge Y) = P(X)P(Y)$$

Learning Y does NOT influence your belief about X.

$\rightarrow$ Convert between the two equations.

To specify joint probability, it is sufficient to specify the individual probabilities.

Conditional Independence

Definition (conditional independence)

X and Y are conditionally independent given Z if

$$P(X|Y \land Z) = P(X|Z).$$

$$P(Y|X \land Z) = P(Y|Z).$$

$$P(Y \land X|Z) = P(Y|Z)P(X|Z).$$

Learning Y does NOT influence your belief about X
if you already know Z.

$\rightarrow$ X is conditionally independent of Y given Z.

Independence does not imply conditional independence, and vice versa.

Q #1: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

Q #1: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (C) $P(A), P(B|A), P(C|A \land B)$. $1 + 2 + 4 = 7$ probabilities

Draw a graph to prove it to yourself.

Q #2: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. Assume that A, B, and C are independent. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

Q #2: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. Assume that A, B, and C are independent. What is the minimum number of probabilities required to specify the joint distribution?

(A) 3

(B) 7

(C) 8

(D) 16

$\rightarrow$ (A) $P(A), P(B), P(C)$. $1 + 1 + 1 = 3$ probabilities

Draw a graph to prove it to yourself.

Q #3: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. Assume that A and B are conditionally independent given C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 4

(B) 5

(C) 7

(D) 11

Q #3: Deriving a compact representation

Q: Consider a model with three random variables, A, B, C. Assume that A and B are conditionally independent given C. What is the minimum number of probabilities required to specify the joint distribution?

(A) 4

(B) 5

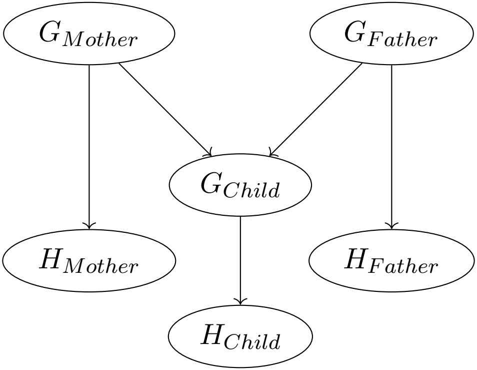
(C) 7

(D) 11

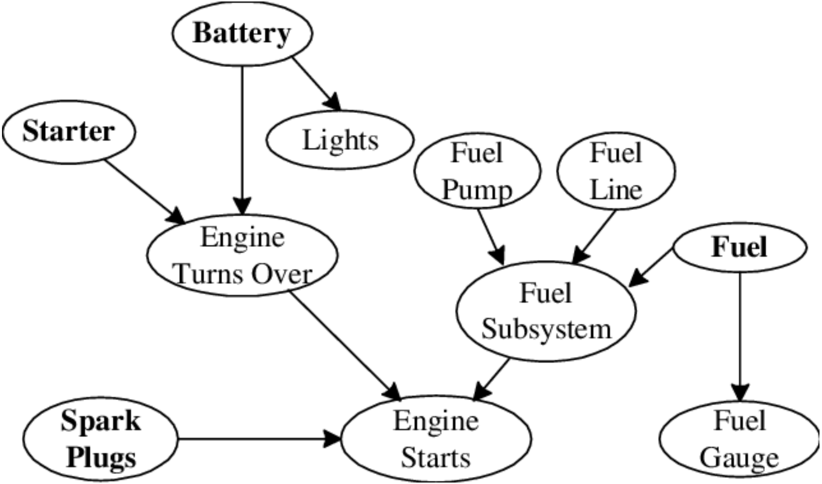
$\rightarrow$ (B) $P(C), P(A|C), P(B|C)$. $1 + 2 + 2 = 5$ probabilities

Draw a graph to prove it to yourself.

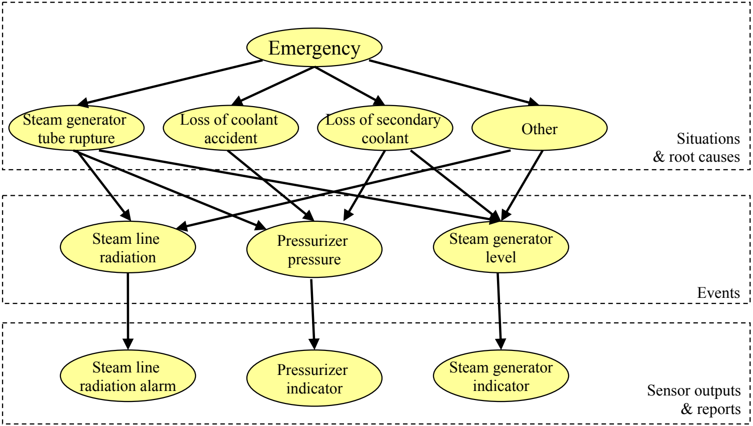
Inheritance of Handedness



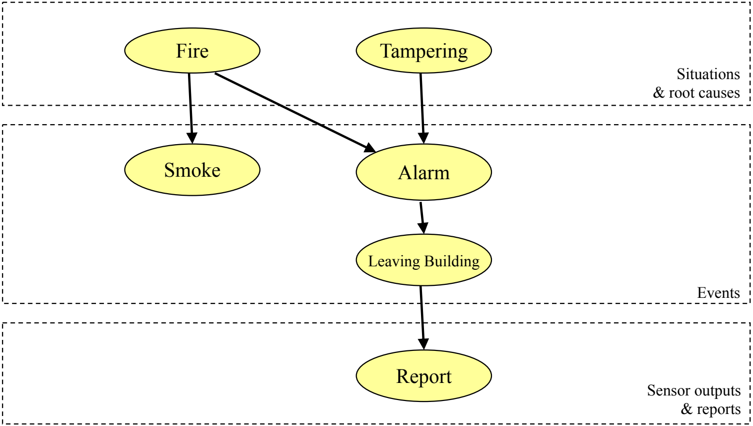
Car Diagnostic Network



Example: Nuclear power plant operations

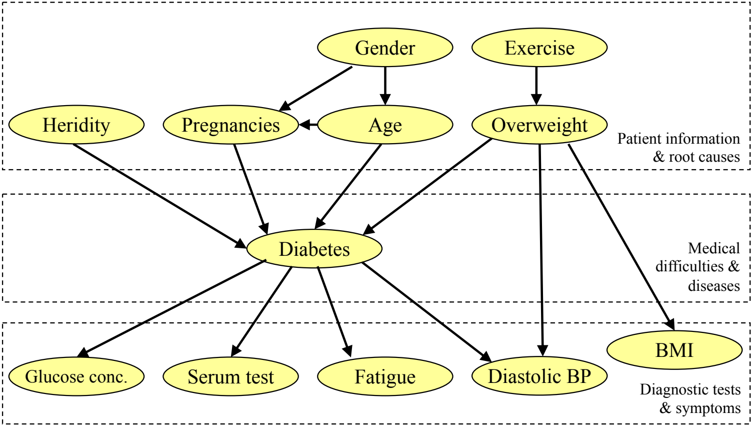


Example: Fire alarms



Report: "report of people leaving building because a fire alarm went off"

Example: Medical diagnosis of diabetes



Why Bayesian Networks?

A probabilistic model of the Holmes scenario:

- The random variables:
  Earthquake, Radio, Burglary, Alarm, Watson, and Gibbon.
- $\#$ of probabilities in the joint distribution: $2^6 = 64$.
- For example,

  $P(E \wedge R \wedge B \wedge A \wedge W \wedge G) =?$

  $P(E \wedge R \wedge B \wedge A \wedge W \wedge \neg G) =?$

  ... etc ...

We can compute any probability using the joint distribution, but

- Quickly become intractable as the number of variables grows.
- Unnatural and tedious to specify all the probabilities.

Why Bayesian Networks?

A Bayesian Network

- is a compact version of the joint distribution
- takes advantage of the unconditional and conditional independence among the variables.

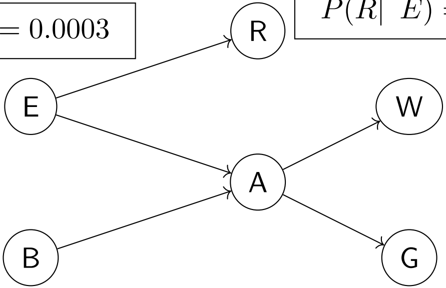
Reminder: Modelling the Holmes Scenario

→ The random variables:

- B: A Burglary is happening.
- A: The alarm is going.
- W: Dr. Watson is calling.
- G: Mrs. Gibbon is calling.
- E: Earthquake is happening.
- R: A report of earthquake is on the radio news.

A Bayesian Network for the Holmes Scenario

$\rightarrow 2^6 = 64 > 1 + 1 + 2 + 4 + 2 + 2 = 12$. Many possible networks.



$P(E) = 0.0003$

$P(R|\neg E) = 0.0002$
$P(R|\ E) = 0.9$

$P(W|\neg A) = 0.4$
$P(W|\ A) = 0.8$

$P(G|\neg A) = 0.04$
$P(G|\ A) = 0.4$

$P(B) = 0.0001$

$P(A|\neg B \wedge \neg E) = 0.01$
$P(A|\neg B \wedge\ E) = 0.2$
$P(A|\ B \wedge \neg E) = 0.95$
$P(A|\ B \wedge\ E) = 0.96$

Bayesian Network

A Bayesian Network is a *directed acyclic graph* (DAG).

- Each node corresponds to a random variable.
- X is a parent of Y if there is an arrow from node X to node Y.

  $\rightarrow$ Like a family tree, there are parents, children, ancestors, descendants.
- Each node X_i has a conditional probability distribution $P(X_i|Parents(X_i))$ that quantifies the effect of the parents on the node.

The Semantics of Bayesian Networks

Two ways to understand Bayesian Networks:

- A representation of the joint probability distribution
- An encoding of the conditional independence assumptions

Representing the joint distribution

We can compute each joint probability using the following formula.

$$P(X_n \wedge \cdots \wedge X_1) = \prod_{i=1}^{n} P(X_i | Parents(X_i))$$

Representing the joint distribution

Example: What is the probability that all of the following occur?

- The alarm has sounded
- Neither a burglary nor an earthquake has occurred
- Both Watson and Gibbon call and say they hear the alarm
- There is no radio report of an earthquake

Representing the joint distribution

Example: What is the probability that all of the following occur?

- The alarm has sounded
- Neither a burglary nor an earthquake has occurred
- Both Watson and Gibbon call and say they hear the alarm
- There is no radio report of an earthquake

$\rightarrow$ Formulate as a joint probability:

$$
\begin{aligned}
&P(\neg B \land \neg E \land A \land \neg R \land G \land W) \\
&= P(\neg B)P(\neg E)P(A|\neg B \land \neg E)P(\neg R|\neg E)P(G|A)P(W|A) \\
&= (1 - 0.0001)(1 - 0.0003)(0.01)(1 - 0.0002)(0.4)(0.8) \\
&= 3.2 \times 10^{-3}
\end{aligned}
$$

Q #4: Calculating the joint probability

Q: What is the probability that all of the following occur?

- NEITHER a burglary NOR an earthquake has occurred,
- The alarm has NOT sounded,
- NEITHER of Watson and Gibbon is calling, and
- There is NO radio report of an earthquake?

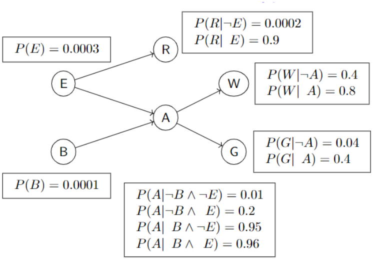
(A) 0.5699

(B) 0.6699

(C) 0.7699

(D) 0.8699

(E) 0.9699



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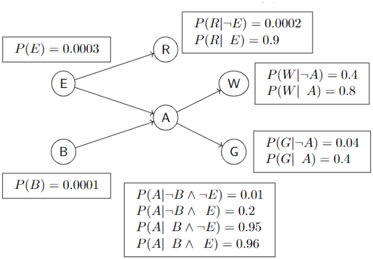
(C) 0.7699

(D) 0.8699

(E) 0.9699



$\rightarrow$ (A)

$$(1 - 0.0001)(1 - 0.0003)(1 - 0.01)(1 - 0.4)(1 - 0.04)(1 - 0.0002) = 0.5699$$

Burglary, Alarm and Watson

Q #5: Unconditional Independence

Q: Are Burglary and Watson independent?

(A) Yes

(B) No

(C) Can't tell.

Q #5: Unconditional Independence

Q: Are Burglary and Watson independent?

(A) Yes

(B) No

(C) Can't tell.

→ Correct answer is *No*.

If you learned the value of B, would your belief of W change? If B is true, then Alarm is more likely to be true, and W is more likely to be true.

Q #6: Conditional Independence

Q: Are Burglary and Watson conditionally independent given Alarm?

(A) Yes

(B) No

(C) Can't tell

Q #6: Conditional Independence

Q: Are Burglary and Watson conditionally independent given Alarm?

(A) Yes

(B) No

(C) Can't tell

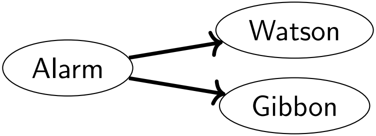
$\rightarrow$ Correct answer is *Yes*.

Assume that W does not observe B directly. W only observes A.

B and W could only influence each other through A.

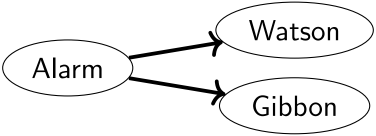
If A is known, then B and W do not affect each other.

Alarm, Watson and Gibbon



Q #7: Unconditional Independence

Q: Are Watson and Gibbon independent?



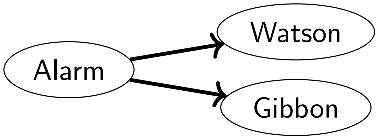
(A) Yes

(B) No

(C) Can't tell

Q #7: Unconditional Independence

Q: Are Watson and Gibbon independent?



(A) Yes

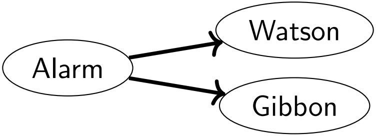
(B) No

(C) Can't tell

$\rightarrow$ Correct answer is No. If Watson is more likely to call, then Alarm is more likely to go off, which means that Gibbon is more likely to call.

Q #8 Conditional Independence

Q: Are Watson and Gibbon conditionally independent given Alarm?



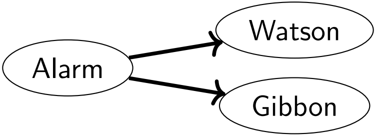
(A) Yes

(B) No

(C) Can't tell

Q #8 Conditional Independence

Q: Are Watson and Gibbon conditionally independent given Alarm?



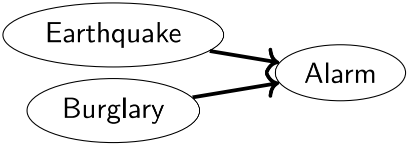
(A) Yes

(B) No

(C) Can't tell

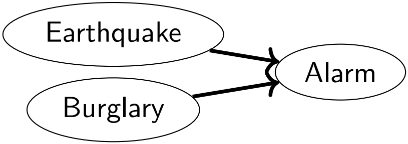
$\rightarrow$ Correct answer is Yes. Watson and Gibbon are both unreliable sensors for Alarm. If Alarm is known, then Watson and Gibbon do not affect each other.

Earthquake, Burglary, and Alarm



Q #9 Unconditional Independence

Q: Are Earthquake and Burglary independent?



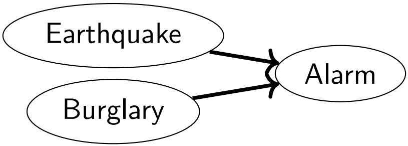
(A) Yes

(B) No

(C) Can't tell

Q #9 Unconditional Independence

Q: Are Earthquake and Burglary independent?



(A) Yes

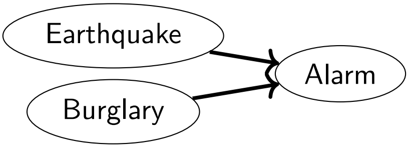
(B) No

(C) Can't tell

→ Correct answer is Yes. Assume that looting is not more frequent during an earthquake.

Q #10: Conditional Independence

Q: Are Earthquake and Burglary conditionally independent given Alarm?



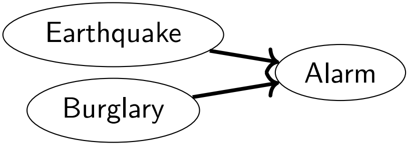
(A) Yes

(B) No

(C) Can't tell

Q #10: Conditional Independence

Q: Are Earthquake and Burglary conditionally independent given Alarm?



(A) Yes

(B) No

(C) Can't tell

$\rightarrow$ Correct answer is No. Suppose that the Alarm is going. If there is an Earthquake, then it is less likely that the Alarm is caused by Burglary. If there is a Burglary, it is less likely that the Alarm is caused by Earthquake.

Revisiting the Learning Goals

By the end of the lecture, you should be able to

- Given a probabilistic model,
  determine if two variables are unconditionally independent, or conditionally independent given a third variable.
- Give examples of deriving a compact representation of a joint distribution by using independence assumptions.
- Describe components of a Bayesian network.
- Compute a joint probability given a Bayesian network.
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