Inference in Bayesian Networks
Variable Elimination Algorithm

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Lecture 9

Readings: RN 13.4. PM 8.4.

Outline

Learning Goals

By the end of the lecture, you should be able to

- Explain how we can perform probabilistic inference more efficiently using the variable elimination algorithm.
- Define factors. Manipulate factors using operations restrict, sum out, multiply and normalize.
- Describe/trace/implement the variable elimination algorithm for calculating a prior or a posterior probability given a Bayesian network.
- Explain how the elimination ordering affects the complexity of the variable elimination algorithm.
- Identify the variables that are irrelevant to a query.

Learning Goals

Why Use the Variable Elimination Algorithm

The Variable Elimination Algorithm

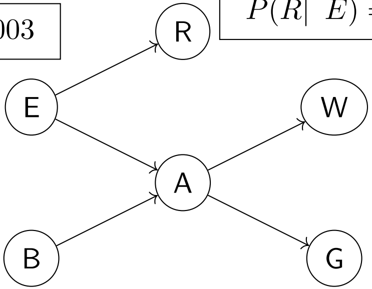
Factors Affecting the Complexity of VEA

Revisiting the Learning goals

A Bayesian Network for the Holmes Scenario

$$P(E) = 0.0003$$

$$P(R|\neg E) = 0.0002$$
$$P(R|\ E) = 0.9$$



$$P(W|\neg A) = 0.4$$
$$P(W|\ A) = 0.8$$

$$P(G|\neg A) = 0.04$$
$$P(G|\ A) = 0.4$$

$$P(B) = 0.0001$$

$$P(A|\neg B \land \neg E) = 0.01$$
$$P(A|\neg B \land \ E) = 0.2$$
$$P(A|\ B \land \neg E) = 0.95$$
$$P(A|\ B \land \ E) = 0.96$$

Answering a Question

What is the probability that a burglary is happening given that Dr. Watson and Mrs. Gibbon both call?

$$P(B|w \wedge g)$$

- Query variables: B
- Evidence variables: W and G
- Hidden variables: E, A, and R.

Notice the new notation: capital letters for random variables and lowercase letters for values.

Answering the query using the joint distribution

Evaluate $P(B|w \wedge g)$ in terms of known distributions from the Bayesian network.

Answering the query using the joint distribution

Evaluate $P(B|w \wedge g)$ in terms of known distributions from the Bayesian network.

$\rightarrow$ Following the approach from Lecture 6:

$$
\begin{aligned}
P(B|w \wedge g) &= \frac{P(B \wedge w \wedge g)}{P(w \wedge g)} \\
&= \frac{P(B \wedge w \wedge g)}{P(b \wedge w \wedge g) + P(\neg b \wedge w \wedge g)} \\
&\propto P(B \wedge w \wedge g) \\
&\propto \sum_e \sum_a \sum_r P(B \wedge e \wedge a \wedge w \wedge g \wedge r) \\
&\propto \sum_e \sum_a \sum_r P(B)P(e)P(a|B \wedge e)P(w|a)P(g|a)P(r|e)
\end{aligned}
$$

Q #1: Number of operations using the joint distribution

How many addition and multiplication operations do we need to calculate the expression below?

$$\sum_{e}\sum_{a}\sum_{r} P(B)P(e)P(a|B \wedge e)P(w|a)P(g|a)P(r|e)$$

(A) ≤ 10

(B) 11-20

(C) 21-40

(D) 41-60

(E) ≥ 61

Q #1: Number of operations using the joint distribution

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(A) ≤ 10

(B) 11-20

(C) 21-40

(D) 41-60

(E) ≥ 61

$\rightarrow$ Correct answer is (D). 47 operations.

Answering the query using variable elimination algorithm

$$\sum_{e}\sum_{a}\sum_{r} P(B)P(e)P(a|B\wedge e)P(w|a)P(g|a)P(r|e)$$

$\rightarrow$ We should move the summations as much to the right of the expression as possible to reduce the overall # of operations.

$$\begin{aligned}
&= \sum_{e}\sum_{a}\sum_{r} P(B)P(e)P(a|B\wedge e)P(w|a)P(g|a)P(r|e) \\
&= P(B)\sum_{e} P(e)\sum_{a} P(a|B\wedge e)P(w|a)P(g|a)\sum_{r} P(r|e) \\
&= P(B)\sum_{e} P(e)\sum_{a} P(a|B\wedge e)P(w|a)P(g|a)
\end{aligned}$$

Q #2: Number of operations via the variable elimination algorithm

How many addition and multiplication operations do we need to calculate the expression below?

$$P(B) \sum_{e} P(e) \sum_{a} P(a|B \wedge e) P(w|a) P(g|a)$$

(A) ≤ 10

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(C) 21-40

(D) 41-60

(E) ≥ 61

Q #2: Number of operations via the variable elimination algorithm

How many addition and multiplication operations do we need to calculate the expression below?

$$P(B) \sum_e P(e) \sum_a P(a|B \wedge e) P(w|a) P(g|a)$$

(A) ≤ 10

(B) 11-20

(C) 21-40

(D) 41-60

(E) ≥ 61

$\rightarrow$ Correct answer is (B) 11-20. 14 operations in total.

Inference by enumeration

We can compute any conditional probability by summing terms from the joint distribution.

- If the value of a variable is observed, calculate the product of its conditional probability multiplied by the rest.
- Otherwise, it must be a hidden variable. Sum it out by calculating the sum of the products of its conditional probability multiplied by the rest.

Inference by Enumeration

Algorithm 1 Enumerate(vars, bn, evidence)

if vars is empty **then return** 1.0
$Y \leftarrow$ first in vars
if Y has value y in evidence **then**
    return $P(y|parents(Y)) \times \text{Enumerate}(REST(\text{vars}), \text{bn}, \text{evidence})$
else
    return $\sum_y P(y|parents(Y)) \times \text{Enumerate}(REST(\text{vars}), \text{bn}, \text{evidence} \cup \{y\})$

Inference by Enumeration

A problem with the Enumeration algorithm:

$P(W|A)P(G|A)$ and $P(W|\neg A)P(G|\neg A)$ are computed twice, once for each value of E.

How can we avoid these wasted computations?

Introducing the Variable Elimination Algorithm

- Performing probabilistic inference is challenging.

  → Computing the posterior distribution of one or more query variables given some evidence is #NP. Estimate the posterior probability in a Bayesian network within an absolute error is already NP-hard. No general efficient implementation.

- Exact and approximate inferences.

  → Compute the probabilities exactly.
  Naive approach: enumerate all the worlds consistent with the evidence. Do better below.

- The variable elimination algorithm uses dynamic programming and exploits the conditional independence.

  → Do the calculations once and save the results for later.

  Factors and operations.

Factors

- A function from some random variables to a number.
- $f(X_1, \ldots, X_j)$: a factor f on variables $X_1, \ldots, X_j$.
- A factor can represent a joint or a conditional distribution. For example, $f(X_1, X_2)$ can represent $P(X_1 \wedge X_2)$, $P(X_1|X_2)$ or $P(X_1 \wedge X_3 = v_3|X_2)$.
- Define a factor for every conditional probability distribution in the Bayes net.

$\rightarrow$ Every conditional probability distribution in the Bayes net is a factor.

$f(B), f(E), f(A, B, E), f(R, E), f(W, A), f(G, A)$

$P(B), P(E), P(A|B \wedge E), P(R|E), P(W|A), P(G|A)$

$$P(B \wedge w \wedge g)$$
$$= P(B) \sum_a P(w|a)P(g|a) \sum_e P(e)P(a|B \wedge e)$$

Restrict a factor

Restrict a factor.

- Assign each evidence variable to its observed value.
- Restricting $f(X_1, X_2, \ldots, X_j)$ to $X_1 = v_1$, produces a new factor $f(X_1 = v_1, X_2, \ldots, X_j)$ on $X_2, \ldots, X_j$.
- $f(X_1 = v_1, X_2 = v_2, \ldots, X_j = v_j)$ is a number.

→ Restrict $f(W, A)$ to $W = t$. Restrict $f(G, A)$ to $G = t$.

$$
\begin{aligned}
&P(B \wedge w \wedge g) \\
&= P(B) \sum_a P(w|a) P(g|a) \sum_e P(e) P(a|B \wedge e)
\end{aligned}
$$

Restrict a factor

$f_1(X,Y,Z)$:

X	Y	Z	val
t	t	t	0.1
t	t	f	0.9
t	f	t	0.2
t	f	f	0.8
f	t	t	0.4
f	t	f	0.6
f	f	t	0.3
f	f	f	0.7

$\rightarrow$

$f_2(Y,Z)$:

Y	Z	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

$f_3(Y)$:

Y	val
t	0.9
f	0.8

$f_4() = 0.8$

- What is $f_2(Y,Z) = f_1(x,Y,Z)$?
- What is $f_3(Y) = f_2(Y,\neg z)$?
- What is $f_4() = f_3(\neg y)$?

Sum out a variable

Sum out a variable.

Summing out X_1 with domain $\{v_1, \ldots, v_k\}$ from factor $f(X_1, \ldots, X_j)$, produces a factor on $X_2, \ldots, X_j$ defined by:

$$(\sum_{X_1} f)(X_2, \ldots, X_j) = f(X_1 = v_1, \ldots, X_j) + \cdots + f(X_1 = v_k, \ldots, X_j)$$

$\rightarrow$ Sum out a and e.

$$\begin{aligned} & P(B \wedge w \wedge g) \\ & = P(B) \sum_a P(w|a) P(g|a) \sum_e P(e) P(a|B \wedge e) \end{aligned}$$

Sum out a variable

$f_1(X, Y, Z)$:

X	Y	Z	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

$\rightarrow$ $f_2(X, Z)$:

X	Z	val
t	t	0.57
t	f	0.43
f	t	0.54
f	f	0.46

What is $f_2(X, Z) = \sum_Y f_1(X, Y, Z)$?

Multiplying factors

Multiply two factors together.

The **product** of factors $f_1(X, Y)$ and $f_2(Y, Z)$, where Y are the variables in common, is the factor $(f_1 \times f_2)(X, Y, Z)$ defined by:

$$(f_1 \times f_2)(X, Y, Z) = f_1(X, Y) * f_2(Y, Z).$$

$\rightarrow$

$$P(B \wedge w \wedge g)$$
$$= P(B) \sum_a P(w|a) P(g|a) \sum_e P(e) P(a|B \wedge e)$$

Multiplying factors

f_1:

X	Y	val
t	t	0.1
t	f	0.9
f	t	0.2
f	f	0.8

f_2:

Y	Z	val
t	t	0.3
t	f	0.7
f	t	0.6
f	f	0.4

$\to f_1 \times f_2$:

X	Y	Z	val
t	t	t	0.03
t	t	f	0.07
t	f	t	0.54
t	f	f	0.36
f	t	t	0.06
f	t	f	0.14
f	f	t	0.48
f	f	f	0.32

What is $f_1(X,Y) \times f_2(Y,Z)$?

Normalize a factor

- Convert it to a probability distribution.
- Divide each value by the sum of all the values.

f_1:

Y	val
t	0.2
f	0.6

Variable elimination algorithm

To compute $P(X_q | X_{o_1} = v_1 \land \ldots \land X_{o_j} = v_j)$:

- **Construct a factor** for each conditional probability distribution.
- **Restrict** the observed variables to their observed values.
- Eliminate each hidden variable X_{h_j}.
  - **Multiply** all the factors that contain X_{h_j} to get new factor g_j.
  - **Sum out** the variable X_{h_j} from the factor g_j.
- **Multiply** the remaining factors.
- **Normalize** the resulting factor.

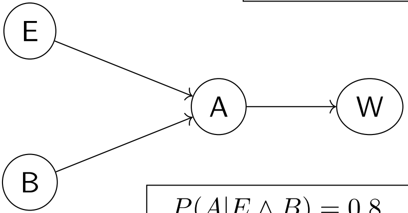
Example of VEA

Given a portion of the Holmes network below,
calculate $P(B|\neg A)$ using the variable elimination algorithm.

Eliminate the hidden variables in reverse alphabetical order.

$$P(E) = 0.1$$

$$P(W|A) = 0.8$$
$$P(W|\neg A) = 0.4$$



$$P(B) = 0.3$$

$$P(A|E \land B) = 0.8$$
$$P(A|E \land \neg B) = 0.2$$
$$P(A|\neg E \land B) = 0.7$$
$$P(A|\neg E \land \neg B) = 0.1$$

Learning Goals

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Factors Affecting the Complexity of VEA

Revisiting the Learning goals

Effect of The Elimination Ordering

In general, VEA is exponential in space and time.

The complexity of VEA depends on:

- The size of the CPT in the Bayesian network.
- The size of the largest factor during algorithm execution.

Effect of the elimination ordering on algorithm complexity:

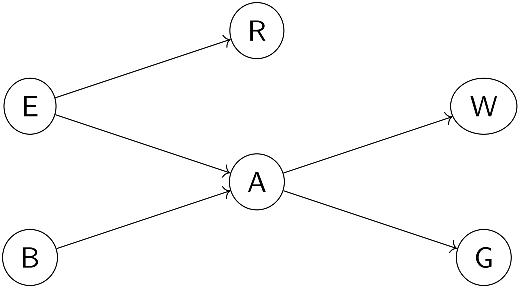
- Every order yields a valid algorithm.
- Different orderings yields different intermediate factors.

Examples of Good and Bad Orderings

Suppose that we want to calculate $P(G)$.
What factors do we produce if we

- Eliminate R first?
- Eliminate A first?

Which ordering leads to worse complexity for VEA?



Choose an Elimination Ordering

How should we choose an order of eliminating the hidden variables?

- Intractable to determine the optimal order.

- A good greedy heuristic: Eliminate the variable that minimizes the size of the next factor.

- For a *polytree:* work outside in.

  $\rightarrow$ Singly-connected network.

  At most one un-directed path between any two nodes.

Complexity of VEA in Polytrees

- A **polytree** is a single-connected network in which there is at most one undirected path between any two nodes.
- The time and space complexity of the exact inference in polytrees is linear in the size of the network (# entries in the conditional probability tables)

Irrelevant Variables

- Certain variables have no impact on the query.
- Every variable that is not an ancestor of a query variable or evidence variable is irrelevant to the query.

Revisiting the Learning Goals

By the end of the lecture, you should be able to

- Explain how we can perform probabilistic inference more efficiently using the variable elimination algorithm.
- Define factors. Manipulate factors using operations restrict, sum out, multiply and normalize.
- Describe/trace/implement the variable elimination algorithm for calculating a prior or a posterior probability given a Bayesian network.
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