AST5220
Milestone III – Pertubations

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Code found at `https://github.com/asdfbat/AST5220/tree/master/Project`

1   Introduction

We have in previous milestones studied the cosmological evolution of the smooth universe, from the relative density functions and their effect on the curvature and conformal time in Milestone I, to the the free electron fraction and its resulting optical depth in Milestone II. In this milestone we introduce pertubations to our smooth universe, which is vital for the observed anisotropies in the cosmic microwave background radiation. This pertubation is naturally included in the Newtonian gauge, which leaves only two degrees of freedom in the line element, namely the two pertubation parameters Ψ and Φ.

2   Theory

2.1   Gauge and metric

In the newtonian Gauge, the metric takes the form of

$$
\begin{aligned}
g_{00} &= -(1 + 2\Psi(\vec{x}, t)) \\
g_{ij} &= \delta_{ij} a^2 (1 + 2\Phi(\vec{x}, t))
\end{aligned}
\tag{1}
$$

with the off-diagonal elements being zero. Φ and Ψ are known as the newtonian pertubation parameters, and traces the newtonian pertubations of the universe.

2.2   Pertubations and multipoles

We will be working with pertubations in the energy densities of both baryons and CDM, given as δ_b and δ_{CDM}, as well as the velocity of both, as v_b and v_{CDM}.

In addition, we need the pertubations in the photon temperature, which we write as

$$
\Theta = \frac{\delta_T}{\overline{T}}
\tag{2}
$$

where $\overline{T}$ and δ_T are the mean temperature and temperature pertubations in the photons, respectively.

In order to calculate the CMB anisotropies, we need the multipole expansions of Θ, defined as[1]

$$
\Theta_l = \frac{i}{2} \int_{-1}^{1} \mathcal{P}_l(\mu) \Theta(\mu) d\mu
\tag{3}
$$

although we will be solving them as a series of coupled ODEs. Although we will solve for the higher order multipoles required calculating the CMB, we will mainly be focusing on $l = 0$ and $l = 1$ in this milestones results, which can be interpreted as the photon pertubations and velocities, in the same sense as δ_{CDM} and v_{CDM}.

We will be working in Fourier space, such that all these quantities are functions of wavenumber (k), as well as time (x).

2.3 System of ODEs

Having established the coupled quantities of interest as δ_{CDM}, δ_b, v_{CDM}, v_b, Φ, Ψ, and the chain of multipoles Θ_ℓ, we present the coupled ODE system of equations 4 - 12. Ψ comes as an explicit expression of Φ, as eqn 13. Since we are ignoring polarization and neutrinos, we are working with a reduced system of equation. The full system can be found in [1], and a derivation of most of the equations can be found in [2].

The system of ODEs contains quantities calculated in milestone 1, like the Hubble parameter $\mathcal{H}$, the conformal time η, and the relative densities Ω_r, Ω_{CDM}, Ω_b, as well as the derivative of the optical depth τ', from the second milestone.

$$\Theta_0' = -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi' \tag{4}$$

$$\Theta_1' = \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right] \tag{5}$$

$$\Theta_\ell' = \frac{\ell ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell-1} - \frac{(\ell+1)ck}{(2\ell+1)\mathcal{H}}\Theta_{\ell+1} + \tau'\left[\Theta_\ell - \frac{1}{10}\Theta_2\delta_{\ell,2}\right], \quad 2 \leq \ell < \ell_{\max} \tag{6}$$

$$\Theta_\ell' = \frac{ck}{\mathcal{H}}\Theta_{\ell-1} - c\frac{\ell+1}{\mathcal{H}\eta(x)}\Theta_\ell + \tau'\Theta_\ell, \quad \ell = \ell_{\max} \tag{7}$$

$$\delta_{\text{CDM}}' = \frac{ck}{\mathcal{H}}v_{\text{CDM}} - 3\Phi' \tag{8}$$

$$v_{\text{CDM}}' = -v_{\text{CDM}} - \frac{ck}{\mathcal{H}}\Psi \tag{9}$$

$$\delta_b' = \frac{ck}{\mathcal{H}}v_b - 3\Phi' \tag{10}$$

$$v_b' = -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau'R(3\Theta_1 + v_b), \quad R = \frac{4\Omega_r}{3\Omega_b a} \tag{11}$$

$$\Phi' = \Psi - \frac{c^2 k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}\left[\Omega_{\text{CDM}}a^{-1}\delta_{\text{CDM}} + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0\right] \tag{12}$$

$$\Psi = -\Phi - \frac{12H_0^2}{c^2 k^2 a^2}\left[\Omega_r\Theta_2\right] \tag{13}$$

In theory, thousands of multipoles of Θ are required for accurately calculating the CBM power spectrum. Luckily, thanks to a clever trick called line-of-sight integration, we can limit ourselves to calculating the first eight multipoles, and employ some forbidden magic in the next milestone to generate the rest of the multipoles.

2.4 Tight coupling regime

The early universe is in what's called the *tight coupling regime*, meaning that the electrons, photons and baryons are very strongly coupled in a plasma state. In this regime, two effects makes us want to change up the system presented above. Firstly, we dont *need* more than three multipoles of Θ, as the strong coupling means any fluctuation will be very smooth. Secondly, The equations for v_b and Θ_1 are numerically unstable in this regime, due to the extremely high values of τ', and small values of Θ_1 and v_b.

We solve this by replacing the equations for Θ_1 and v_b by

$$q = \frac{-[(1-R)\tau' + (1+R)\tau''](3\Theta_1 + v_b) - \frac{ck}{\mathcal{H}}\Psi + \left(1 - \frac{\mathcal{H}'}{\mathcal{H}}\right)\frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Theta_0'}{(1+R)\tau' + \frac{\mathcal{H}'}{\mathcal{H}} - 1} \tag{14}$$

$$v_b' = \frac{1}{1+R}\left[-v_b - \frac{ck}{\mathcal{H}}\Psi + R\left(q + \frac{ck}{\mathcal{H}}(-\Theta_0 + 2\Theta_2) - \frac{ck}{\mathcal{H}}\Psi\right)\right] \tag{15}$$

$$\Theta_1' = \frac{1}{3}(q - v_b') \tag{16}$$

$$\tag{17}$$

In addition, Θ_2 now takes the explicit form of

$$\Theta_2 = -\frac{4ck}{9\mathcal{H}\tau'}\Theta_1 \tag{18}$$

2.4.1 Initial conditions

The intial conditions of our tight coupling system can be found in [1], and is presented below. The non-tight coupling regime simply takes the initial conditions from the last values in the tight coupling regime.

$$\Psi = -\frac{2}{3} \tag{19}$$

$$\Phi = -\Psi \tag{20}$$

$$\delta_{\text{CDM}} = \delta_b = -\frac{3}{2}\Psi \tag{21}$$

$$v_{\text{CDM}} = v_b = -\frac{ck}{2\mathcal{H}}\Psi \tag{22}$$

$$\Theta_0 = -\frac{1}{2}\Psi \tag{23}$$

$$\Theta_1 = +\frac{ck}{6\mathcal{H}}\Psi \tag{24}$$

$$\Theta_2 = -\frac{4ck}{9\mathcal{H}\tau'}\Theta_1 \tag{25}$$

$$\Theta_\ell = -\frac{\ell}{2\ell+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{\ell-1} \tag{26}$$

3 Implementation

3.1 General Overview

The code used in this report can be found at `https://github.com/asdfbat/AST5220/tree/master/Project`. The simulations are found in the `src/` directory, and is built upon the C++ code template provided by Hans Arnold Winther.

The code builds upon the work from Milestone I [3] and Milestone II [4], and the associated codes, found in `BackgroundCosmology.cpp` and `RecombinationHistory.cpp`, respectively.

The code from this milestone is found in the *Pertubations* class, in the `Pertubations.cpp`, which covers all the coding work done in this milestone. The code, as well as the code from previous milestones, can be compiled and run using the provided `Makefile`, as **make all run**. This will write the datafiles used in the generation of the plots used in this report. The plots are generated by the `Python/m3_plotting.ipynb` Jupyter Notebook file.

The end goal of this milestone is to generate splines of all quantities of interest mentioned in the theory section, which will mostly be functions of both x and k. The *Pertubations::integrate_perturbations* method loop over 200 values of k, logspaced in $k \in [0.00005, 1.0]\,\text{Mpc}$.

3.2 Solving the ODEs

For each k, a 5000 point array of x-values, linearly spaced in $[-16, 0]$, is generated for the system to be solver over. The transition point between tight coupling and non-tight coupling is found as the earliest x which fulfills one of the following:

- $\left|\frac{k}{\mathcal{H}\tau'}\right| = \frac{1}{10}$

- $|\tau'| = 1$

- $X_e = 0.99$ (Recombination is happening)

Generally, one of these conditions are met around the order of $x \approx -7$

The GSL ODEsolver is employed through a wrapper function in `ODESolver.cpp` to solve the system of tight coupling ODEs up to the point of the transition.

Upon the end of the tight coupling regime, the last values of the previous solution is inserted as initial conditions for the non-tight coupling solver, which, employing the GSL ODESolver, solves the system of ODEs for the remainding regime.

Once all the solutions are obtained, they are splined using `Spline.cpp`, a wrapper function for the GSL spline implementation. The splined solutions are written to file, to be plotted.

4 Results

Figure 1 shows the evolution of the pertubation parameter Φ for different modes of k. Since we're ignoring neutrinos and polarization, the other pertubation parameter Ψ, is simply equal to Φ (up to a sign), and we will neglect it. We see that, for smaller pertubations like $k = 0.1Mpc$, there is a quick and sharp fall in Φ to some value close to zero as it enters the horizon.

We know that pertubations undergo a suppression known as the Meszaros effect, if it enters the horizon before the universe has become matter dominated (when the universe becomes non-relativistic). This effect washes out pertubations on the relevant scale at a very high rate, and will as a result suppress the contribution of the relevant scale to the gravitational potential, as we can see from the collapse in Φ.

For larger scales, which enter the horizon later, the effect is reduced. The $k = 0.01Mpc$ mode, which enters the horizon just at the transition between radiation and matter domination, the collapse in Φ is much smaller. At even larger scales, the effect is almost gone altogether, as we can see from the $k = 0.001Mpc$ mode.

Towards the end of the plot, we observe an additional collapse in all modes. This is due to the transition to the dark energy dominated era, where pertubations on all scales gets reduced by the high expansion rate of the universe.
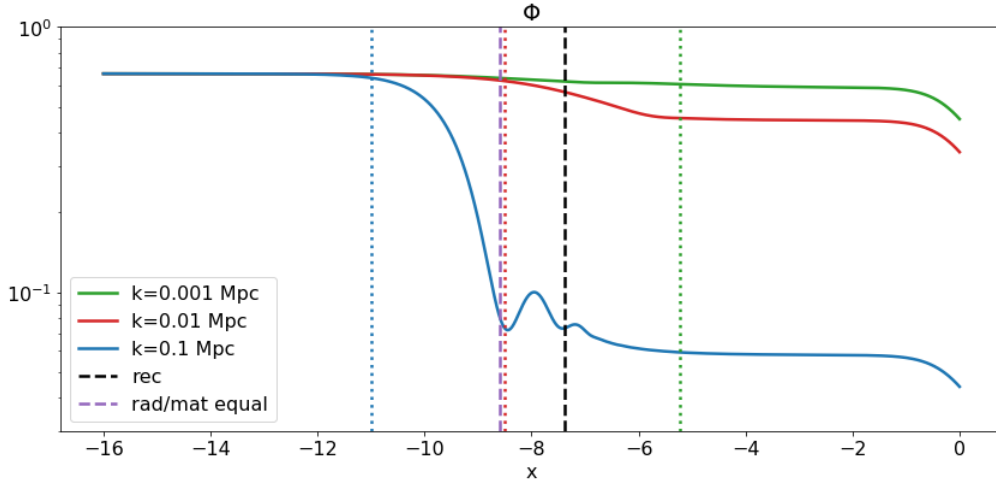
NOTE:

Figure 1 – *Figure showing the evolution of Ψ for different k-modes. Horizon entries for each mode is shown as dotted line of corresponding color. Recombination and radiation/matter equality is also shown as striped lines.*

Figure 2 shows the density pertubations and velocity of the CDM and baryons for the same k-modes as for Φ.

We see that the baryons, before recombination, the baryons oscilate around an overdensity of 1, while the dark matter can form structures. After recombination, the baryons trace the CDM.
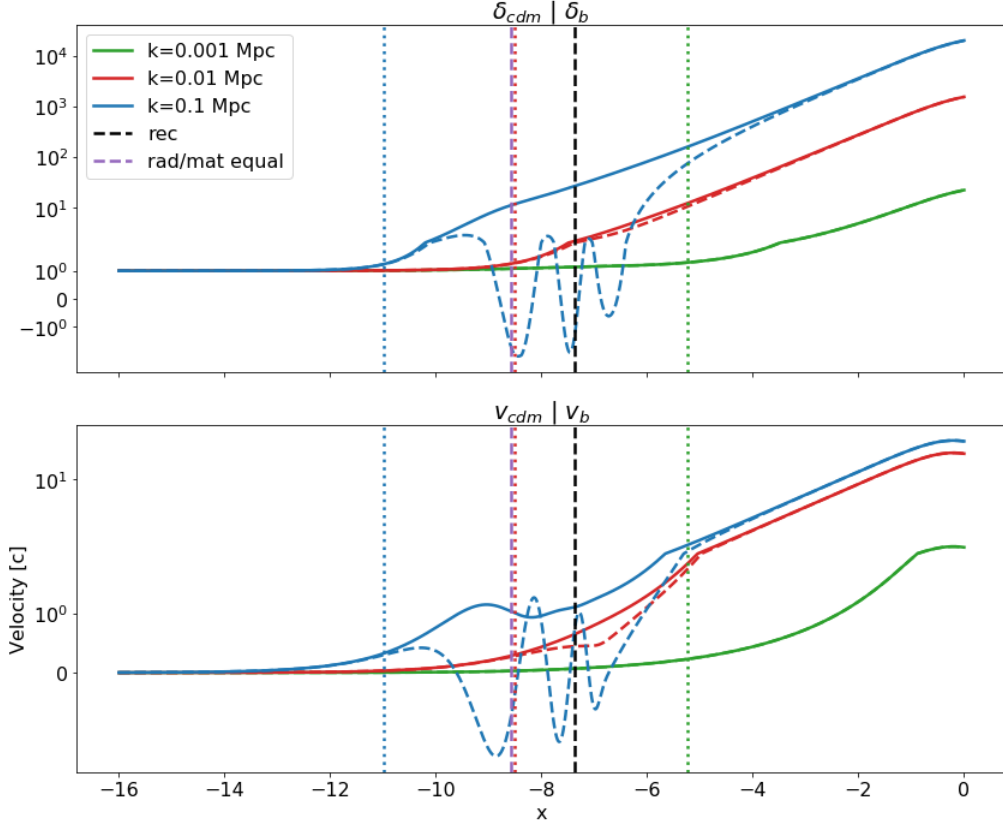
Figure 2 – *Figure showing the evolution of the baryon and CDM density pertubations (top) and the baryon and CDM velocity (bottom) for different k-modes. The CDM is shown as continuous lines, while the baryons are shown as striped lines of the same color. Horizon entries for each mode is shown as vertical dotted lines of corresponding color. Recombination and radiation/matter equality is also shown as vertical striped lines.*
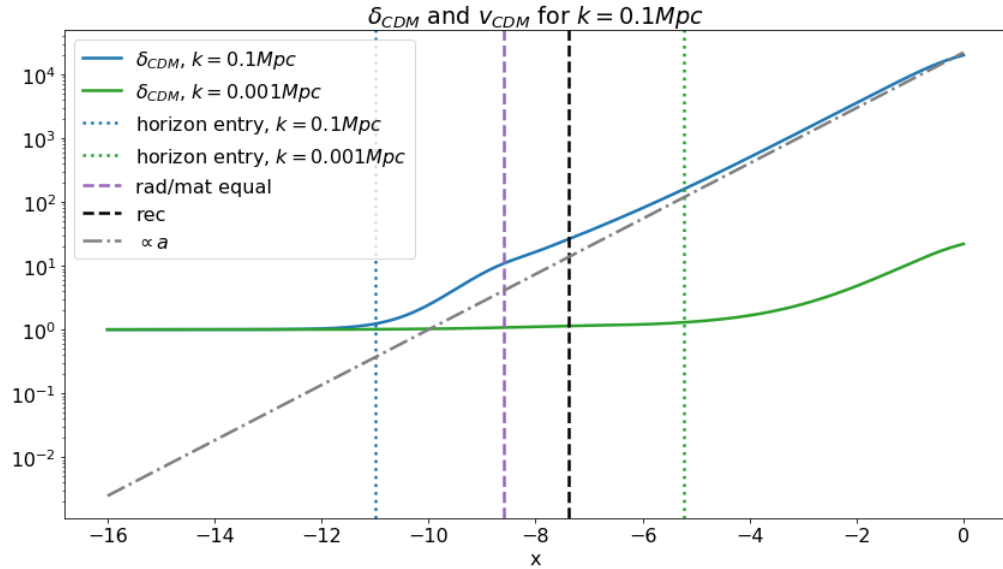

Figure 3 – *Figure showing the $k = 0.1Mpc$ modes of the density pertubations and velocity of CDM from figure 2. A gray line showing a first power "a" proportionality is also shown, for reference.*
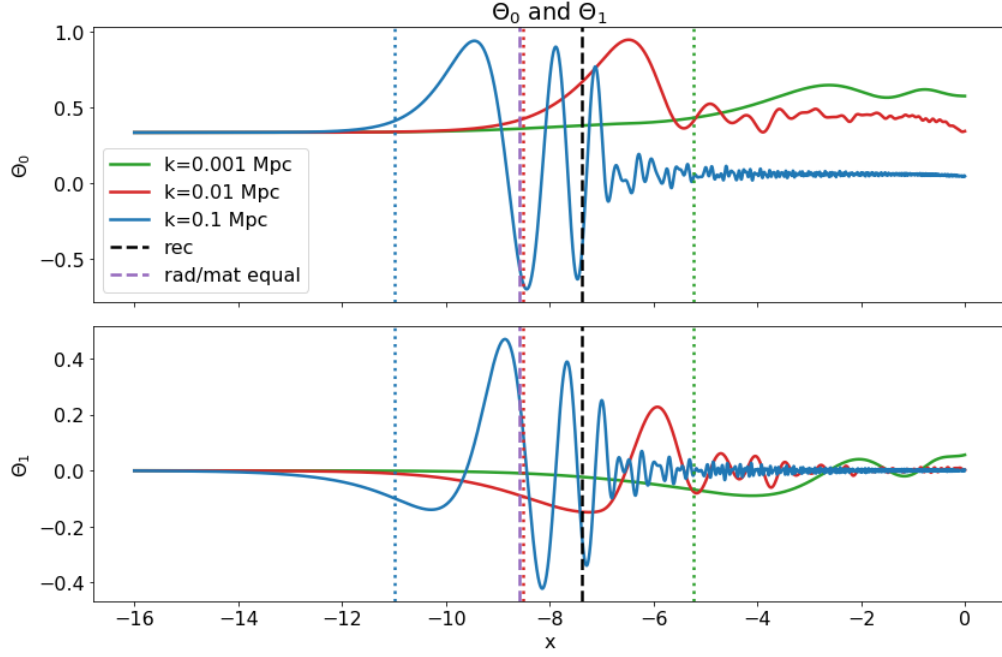
Figure 4 – *Figure showing the evolution of the monopole (top) and dipole (bottom) of the photon temperature pertubations for different k-modes. Horizon entries for each mode is shown as dotted line of corresponding color. Recombination and radiation/matter equality is also shown as striped lines.*
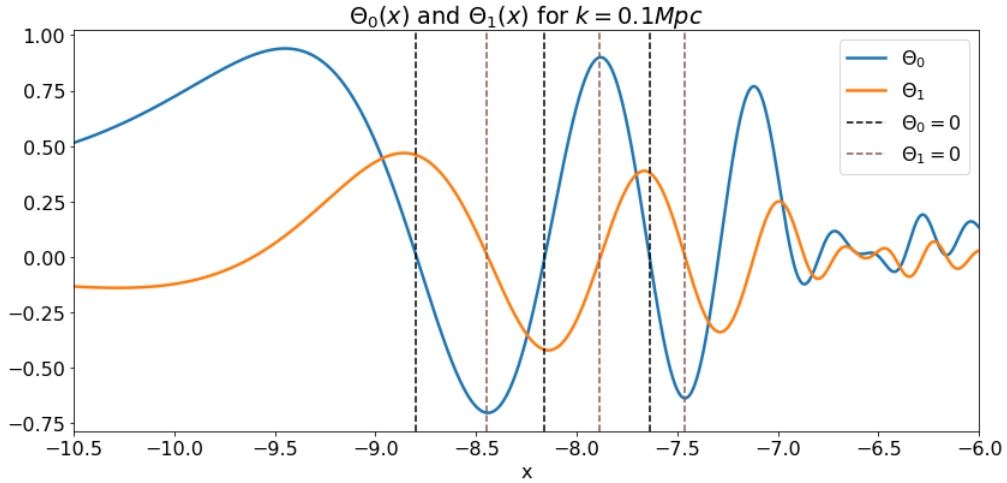

Figure 5 – *Monopole and multipole photon temperature pertubations for $k = 0.1 Mpc$. The first three zero-points of both functions are indicated as striped lines, and seemingly corresponds with an extremum in the other function, indicating a 90 degrees phase shift.*

References

[1] Petter Callin. How to calculate the cmb spectrum, 2006.

[2] Scott Dodelson. *Modern Cosmology*. Academic Press, Elsevier Science, 2003.

[3] Jonas Lunde. Milestone 1 - background cosmology, 2020.

[4] Jonas Lunde. Milestone 2 - recombination, 2020.