PROBLEM 5

a) $\delta \Gamma^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - \bar{\Gamma}^{\mu}_{\nu\lambda}$

$$= \frac{1}{2} g^{\mu\rho} \left[\partial_\lambda g_{\rho\nu} + \partial_\nu g_{\rho\lambda} - \partial_\rho g_{\lambda\nu} \right]$$

$$- \frac{1}{2} \bar{g}^{\mu\rho} \left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\lambda} - \partial_\rho \bar{g}_{\lambda\nu} \right]$$

$$= \cancel{\frac{1}{2} \bar{g}^{\mu\rho} \left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\nu} - \partial_\rho \bar{g}_{\lambda\nu} \right]}$$

$$+ \frac{1}{2} \bar{g}^{\mu\rho} \left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\nu} - \partial_\rho h_{\lambda\nu} \right]$$

$$\cancel{+ \frac{1}{2} h^{\mu\rho} \left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\nu} - \partial_\rho h_{\lambda\nu} \right]}$$

SECOND ORDER.

$$+ \frac{1}{2} h^{\mu\rho} \left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\nu} - \partial_\rho \bar{g}_{\lambda\nu} \right]$$

$$\cancel{- \frac{1}{2} \bar{g}^{\mu\nu} \left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\nu} - \partial_\rho \bar{g}_{\lambda\nu} \right]}$$

$=$

$$= \frac{1}{2} \bar{g}^{\mu\rho} \left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\nu} - \partial_\rho h_{\lambda\nu} \right]$$

$$+ \frac{1}{2} h^{\mu\rho} \left[\partial_\lambda \bar{g}_{\rho\nu} + \partial_\nu \bar{g}_{\rho\nu} - \partial_\rho g_{\lambda\nu} \right]$$

$$= \left(-\frac{1}{2} \bar{g}^{\mu\rho} h_{\rho\sigma} \bar{g}^{\sigma\rho} \right) \left[\cdots \right] = -\frac{1}{2} \bar{g}^{\mu\rho} h_{\rho\sigma} \cdot 2 \bar{\Gamma}^{\sigma}_{\nu\lambda}$$

$$= \underline{\frac{1}{2} \bar{g}^{\mu\rho} \left[\partial_\lambda h_{\rho\nu} + \partial_\nu h_{\rho\lambda} - \partial_\rho h_{\lambda\nu} - 2 h_{\rho\sigma} \bar{\Gamma}^{\sigma}_{\nu\lambda} \right]}$$

5b) $\delta \Gamma^{i}_{00} =$

$$= \frac{1}{2} g^{i\rho} \left[\partial_0 h_{\rho 0} + \partial_0 h_{\rho 0} - \partial_\rho h_{00} - 2 h_{\rho\sigma} \Gamma^{\sigma}_{00} \right]$$

$\left[\bar{g}^{\mu\nu} = \text{diag} \;\Downarrow\; \rho = i \right]$

$$= \frac{1}{2} \delta^{ij} a^{-2} \left[\partial_0 h_{j0} + \partial_0 h_{j0} - \partial_j h_{00} - \underbrace{\cancel{2 h_{j\sigma} \Gamma^{\sigma}_{00}}}_{\to 0} \right]$$

$(\sigma = 0 \Rightarrow h_{00} = 0)$

$$= \frac{\partial h_{i0}}{a^2}$$

$$\delta \Gamma^{0}_{0i} = \delta \Gamma^{0}_{i0} = \frac{1}{2} g^{0\rho} \left[\partial_0 h_{\rho 0} + \partial_0 h_{\rho i} - \partial_\rho h_{i0} - 2 h_{\rho\sigma} \Gamma^{\sigma}_{0i} \right]$$

$[\rho = 0]$

$$= \frac{1}{2} g^{00} \left[\underbrace{\cancel{\partial_0 h_{00}}}_{=0} + \underbrace{\cancel{\partial_0 h_{0i} - \partial_0 h_{i0}}}_{0 = 0} - 2 h_{0\sigma} \Gamma^{\sigma}_{0i} \right]$$

$$= \frac{1}{2} \cdot 2 h_{0j} \Gamma^{j}_{0i} = \underline{H h_{0i}}$$

$$\delta \Gamma^{i}_{j0} = \frac{1}{2} g^{i\rho} \left[\partial_0 h_{\rho j} + \partial_j h_{\rho 0} - \partial_\rho h_{0j} - 2 h_{\rho\sigma} \Gamma^{\sigma}_{j0} \right]$$

$[\rho = k]$

$$= \frac{1}{2} \bar{g}^{ik} \delta_i^{k} \left[\underbrace{\cancel{\partial_0 h_{kj}}}_{\to 0} + \partial_j h_{k0} - \partial_k h_{0j} - \underbrace{\cancel{2 h_{km} \Gamma^{m}_{j0}}}_{\to 0} \right]$$

$$= \frac{1}{2} a^{-2} \delta_i^{k} \left[\partial_j h_{k0} - \partial_k h_{0j} \right]$$

$$= \frac{1}{2} a^{-2} \left[\partial_j h_{i0} - \partial_i h_{0j} \right]$$

$$\delta \Gamma_{ij}^{\ 0} = \frac{1}{2} \bar{g}^{0\rho} \Big[\partial_j h_{\rho i} + \partial_i h_{\rho j} - \partial_\rho h_{ji} - 2 h_{\rho\sigma} \Gamma_{ij}^{\ \sigma} \Big]$$

$[\rho = 0]$

$$= -\frac{1}{2} \Big[\partial_j h_{0i} + \partial_i h_{0j} - \partial_\rho \cancel{h_{ji}} - 2 \cancel{h_{0\sigma}} \Gamma_{ij}^{\ \sigma} \Big]$$

$$= -\frac{1}{2} \Big[\partial_j h_{0i} + \partial_i h_{0j} \Big]$$

$[\mu = i, \; \nu = 0]$

5c)

$$\delta R_{i0} = \overbrace{\partial_\alpha \delta \Gamma^\alpha_{i0}}^{(1.)} - \overbrace{\partial_0 \delta \Gamma^\alpha_{i\alpha}}^{(2.)} + \overbrace{\delta \Gamma^\beta_{i0} \bar{\Gamma}^\alpha_{\alpha\beta}}^{(3.)}$$

$$+ \underbrace{\delta \Gamma^\alpha_{\alpha\beta} \bar{\Gamma}^\beta_{i0}}_{(4.)} - \underbrace{\delta \Gamma^\beta_{i\alpha} \bar{\Gamma}^\alpha_{0\beta}}_{(5.)} - \underbrace{\delta \Gamma^\alpha_{0\beta} \bar{\Gamma}^\beta_{i\alpha}}_{(6.)}$$

$$= \overbrace{\left(\partial_0 \delta \Gamma^0_{i0} + \partial_j \delta \Gamma^j_{i0}\right)}^{(1.)} - \overbrace{\partial_0 \left(\delta \Gamma^0_{i0} + \delta \Gamma^j_{ij}\right)}^{(2.)}$$

$$+ \underbrace{\left[\delta \Gamma^0_{i0} \cancelto{=0}{\bar{\Gamma}^0_{00}} + \delta \Gamma^0_{i0} \bar{\Gamma}^j_{j0} + \delta \Gamma^j_{i0} \cancelto{0}{\bar{\Gamma}^0_{0j}} + \delta \Gamma^j_{i0} \cancelto{0}{\bar{\Gamma}^k_{kj}}\right]}_{(3.)}$$

$$+ \underbrace{\left[\cancelto{0}{\delta \Gamma^0_{00} \bar{\Gamma}^0_{i0}} + \cancelto{0}{\delta \Gamma^j_{j0} \bar{\Gamma}^0_{i0}} + \delta \Gamma^0_{0j} \bar{\Gamma}^j_{i0} + \delta \Gamma^j_{jk} \bar{\Gamma}^k_{i0}\right]}_{(4.)}$$

$$- \underbrace{\left[\delta \Gamma^0_{i0} \cancelto{0}{\bar{\Gamma}^0_{00}} + \delta \Gamma^0_{ij} \cancelto{0}{\bar{\Gamma}^j_{00}} + \delta \Gamma^j_{i0} \cancelto{0}{\bar{\Gamma}^0_{0j}} + \delta \Gamma^k_{ij} \bar{\Gamma}^j_{0k}\right]}_{(5.)}$$

$$- \underbrace{\left[\cancelto{0}{\delta \Gamma^0_{00}} \bar{\Gamma}^0_{i0} + \delta \Gamma^j_{00} \bar{\Gamma}^0_{ij} + \delta \Gamma^0_{0j} \bar{\Gamma}^j_{i0} + \delta \Gamma^j_{0k} \cancelto{0}{\bar{\Gamma}^k_{ij}}\right]}_{(6.)}$$

$$= \overset{(1.)}{\left[\cancel{\partial_0 (H h_{0i})} + \partial_j \left(\frac{\partial_i h_{j0} - \partial_j h_{0i}}{2a^2}\right)\right]} - \overset{(2.)}{\left[\cancel{\partial_0 (H h_{0i})} + \partial_0 (\delta_{ij} h_{j0} H)\right]}$$

$$+ \overset{(3.)}{\left[H h_{0i} \delta_j^j H\right]} + \overset{(4.)}{\left[H h_{0j} \delta_i^j H - \delta_{jk} h_{j0} H \delta_{ki}\right]}$$

$$- \overset{(5.)}{\left[\delta_{ij} h_{k0} H \delta_{jk} H\right]} - \underset{(6.)}{\left[\frac{\partial_0 h_{j0}}{a^2} \delta_{ij} a^2 H + H h_{0j} \delta_{ji} H\right]}$$

$$= \left[\frac{\partial_j \partial_i h_{j0}}{2a^2} - \frac{\partial_j \partial_j h_{i0}}{2a^2}\right]^{(1.)} + \left[\partial_0 (h_{i0} H)\right]^{(2.)}$$

$$+ \left[3H^2 h_{i0}\right]^{(3.)} + \left[\cancel{H^2 h_{i0} - H^2 h_{i0}} \to 0\right]^{(4.)}$$

$$+ \left[H^2 h_{i0}\right]^{(5.)} - \left[H \partial_0 h_{i0} + h_{i0} H^2\right]^{(6.)}$$

$$= \frac{\partial_j \partial_i h_{j0}}{2a^2} - \frac{\partial_j \partial_j h_{i0}}{2a^2} + 3H^2 h_{i0} + \underbrace{\partial_0 (h_{i0} H)}_{\parallel} - H \partial_0 h_{i0}$$

$$h_{i0}\partial_0(H) + H\partial_0(h_{i0})$$

$$= h_{i0}\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right) + H\partial_0(h_{i0})$$

$$= h_{i0}\frac{\ddot a}{a} - h_{i0}H^2 + H\partial_0 h_{i0}$$

$$= h_{i0}\left[\frac{\ddot a}{a} + 2H^2\right] + \underbrace{\cancel{\frac{\partial_j \partial_i h_{j0}}{2a^2}}}_{= \partial_j \partial_i [a G_i] = 0} - \underbrace{\frac{\partial_j \partial_j h_{i0}}{2a^2}}_{= \frac{\delta^{ij}\partial_i \partial_j h_{i0}}{2a^2} = \frac{\nabla^2 h_{i0}}{2a^2}}$$

$$= h_{0i}\left[\frac{\ddot a}{a} + 2H^2\right] - \frac{\nabla^2 h_{i0}}{2a^2}$$

5d) $\delta S_{i0} = T_{i0} - \underbrace{\bar{T}_{i0}}_{\to 0} - \frac{1}{2} g_{i0} T^{\lambda}{}_{\lambda} + \frac{1}{2} \underbrace{\bar{g}_{i0}}_{\to 0} \bar{T}^{\lambda}{}_{\lambda}$

$$\left[\bar{T}_{i0} = \bar{p} \underbrace{\bar{g}_{i0}}_{\to 0} + (\bar{\rho} + \bar{p}) \underbrace{\bar{u}_i \bar{u}_0}_{\to 0} \right]$$

$$T_{i0} = \bar{p} (\underbrace{\bar{g}_{i0}}_{\to 0} + h_{i0}) + (\bar{\rho} + \bar{p}) u_i u_0 = \bar{p} a G_i - (\bar{\rho} + \bar{p}) u_i$$

$$T^{\lambda}{}_{\lambda} = g^{\lambda\sigma} T_{\sigma\lambda} = g^{\lambda\sigma} \left[\bar{p} g_{\sigma\lambda} + (\bar{\rho} + \bar{p}) u_\sigma u_\lambda \right]$$

$$= \delta^{\lambda}{}_{\lambda} \bar{p} + (\bar{\rho} + \bar{p}) \underbrace{u^{\lambda} u_{\lambda}}_{= 0 \text{ (second order)}}$$

$$= 4 \bar{p}$$

$$g_{i0} = \underbrace{\bar{g}_{i0}}_{\to 0} + h_{i0} = a G_i$$

$$\delta S_{i0} = \bar{p} a G_i - (\bar{\rho} + \bar{p}) u_i - 2 a G_i \bar{p}$$

$$= - (\bar{\rho} + \bar{p}) u_i - a G_i \bar{p}$$

$$R_{i0} = 8\pi G S_{i0}$$

$$\underbrace{\bar{R}_{i0}}_{\to 0} + \delta R_{i0} = 8\pi G \left(\underbrace{\bar{S}_{i0}}_{\to 0} + \delta S_{i0} \right)$$

$$h_{0i} \left(\frac{\ddot{a}}{a} + 2H^2 \right) - \frac{\nabla^2 h_{i0}}{2a^2} = -8\pi G \left([\bar{\rho} + \bar{P}] u_i + a G_i \bar{P} \right)$$

$$\frac{\nabla^2 (a G)}{2a^2} = 8\pi G (\bar{\rho} + \bar{P}) u_i + a G_i \left(\frac{\ddot{a}}{a} + 2H^2 \right) + 8\pi G a G_i \bar{\rho}$$

$$\boxed{\frac{\nabla^2 G}{2a} = 8\pi G (\bar{\rho} + \bar{P}) a u_i} + \underbrace{a^2 G_i \left(\frac{\ddot{a}}{a} + 2H^2 \right) + 8\pi G a^2 G_i \bar{\rho}}$$

Show that this is zero:

$$\frac{\ddot{a}}{a} + \underbrace{2H^2}_{\text{1st Friedmann eqn.}} = -8\pi G \bar{\rho}$$

$$\underbrace{\frac{\ddot{a}}{a} + \frac{4\pi G \bar{\rho}}{3} = -8\pi G \bar{\rho}}_{\text{Second Friedmann eqn.}}$$

$$\underline{\underline{0 = 0}}$$

5e) $\partial_0 \left[(\bar{\rho} + \bar{P}) u_i \right] + \frac{3\dot{a}}{a} \left[(\bar{\rho} + \bar{P}) u_i \right] = 0$

$$\frac{\partial}{\partial t} V_i + \frac{3}{a} \frac{da}{dt} V_i = 0$$

$$\frac{1}{V_i} dV_i = -\frac{3}{a} da$$

$$\log V_i + V_{i0} = -3 \log a$$

$$V_i(x, a) = V_{i0}(x)\, a^{-3} \leftarrow \text{Runs as } \propto a^{-3}.$$

Inserting into Einstein equation:

$$\frac{1}{2} \nabla^2 G_i = 8\pi G \frac{V_{i0}(x)}{a^2}$$

Fourier transforming both sides:

$$G_i k^2 \propto \frac{\tilde{V}_i(x)}{a^2} \Rightarrow \underline{G_i \propto \frac{1}{a^2 k^2} \tilde{V}_i(x)}$$

As we see, vector pertubations decay irregardless of scale, while tensor pertubations have exponential decay for small k, and harmonic oscillator solution for large k.