

AST5220 – Milestone I

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1 Theory

1.1 Components of the universe

We consider a flat, expanding universe, governed by the Λ CDM model. Our universe contains some densities of baryonic matter (ρ_b), cold dark matter (ρ_{CDM}), radiation (ρ_r), and dark energy (ρ_Λ). Let $\Omega_i = \frac{\rho_i}{\rho_c}$ be the *relative densities* of each component. Here, ρ_c is the *critical density*, being the total density which would make the universe entirely flat. Since our universe is indeed flat, it follows that

$$\sum_i \rho_i = \rho_c \quad \Rightarrow \quad \sum_i \Omega_i = 1$$

We also define $\rho_{i,o}$ and $\Omega_{i,o}$ to represent the critical and relative densities today.

It can be shown that the density components evolve with time as $\rho_i(a) = \rho_{i,0}a^{-3(1+w_i)}$ where w_i is some constant for each component. For our four components, the densities evolve as

$$\begin{aligned}\rho_b &= \rho_{b,0}a^{-3} \\ \rho_{CDM} &= \rho_{CDM,0}a^{-3} \\ \rho_r &= \rho_{r,0}a^{-4} \\ \rho_\Lambda &= \rho_{\Lambda,0}\end{aligned}$$

where a is the *scale factor* of the universe, describing its relative size to today. Today's value is defined to be $a_0 = 1$.

We wish to avoid the densities where possible, and work directly with the relative densities and the scale factor. The relative densities can be rewritten to exclude ρ_i the following way.

$$\Omega_i = \frac{\rho_i}{\rho_c} = \frac{\rho_{i,0}a^{-3(1+w_i)}}{\rho_c} = \frac{\rho_{c,0}\Omega_{i,0}a^{-3(1+w_i)}}{\rho_c}$$

Knowing that $\rho_c = \frac{3H^2}{8\pi G}$, which gives $\rho_{c,0} = \frac{3H_0^2}{8\pi G}$ we get that

$$\Omega_i = \left(\frac{H_0}{H}\right)^2 \Omega_{i,0}a^{-3(1+w_i)} \quad (1)$$

1.2 The Friedmann equation

The evolution of the scale factor is governed by the (first) Friedmann equation, which for the universe described above reads

$$H(a) = \frac{\dot{a}}{a} = H_0 \sqrt{\Omega_{b,0}a^{-3} + \Omega_{CDM,0}a^{-3} + \Omega_{r,0}a^{-4} + \Omega_{\Lambda,0}} \quad (2)$$

Since the universe takes on scales of many different orders of magnitude, the linear scale factor a is not always well suited for analysis of large time spans. We introduce time (and scale) quantity

$$x = \log a \quad \Rightarrow \quad a = e^x \quad (3)$$

The Friedmann equation now reads

$$H(x) = H_0 \sqrt{\Omega_{b,0} e^{-3x} + \Omega_{\text{CDM},0} e^{-3x} + \Omega_{r,0} e^{-4x} + \Omega_{\Lambda,0}} \quad (4)$$

We also introduce the scaled Hubble parameter $\mathcal{H} = aH = \dot{a}$.

1.3 Conformal Time

As well as a , x , and t for measuring time in the universe, we will introduce the conformal time η . η has units of length, and represents the size of the event horizon at any given time. In other words, η is the distance traversed by undisturbed light since the Big Bang. Conformal time is also commonly used in units of time, gained by dividing our quantity by c , thereby having the interpretation of the time it would take a photon to traverse the universe at that given time (given that the expansion "froze" for the duration of the photons travel).

$$\eta = \int_0^t \frac{c}{a(t)} dt \quad (5)$$

Or, on differential equation form

$$\frac{d\eta}{dt} = \frac{c}{a} \quad (6)$$

This can be written as a differential equation with regards to a .

$$\frac{d\eta}{da} \frac{da}{dt} = \frac{d\eta}{da} \frac{1}{\mathcal{H}} = \frac{c}{a}$$

Resulting in the ODE

$$\frac{d\eta}{da} = \frac{c}{a\mathcal{H}} \quad (7)$$

This, again, can be rewritten as a function of $x = \log(a)$ as

$$\frac{d\eta}{da} = \frac{d\eta}{dx} \frac{dx}{da} = \frac{d\eta}{dx} \frac{1}{a}$$

which, inserting into 7 gives us the ODE

$$\frac{d\eta}{dx} = \frac{c}{\mathcal{H}} \quad (8)$$

1.4 Analytical solution to conformal time

Analytical solutions to the conformal time exist in each of the dominant regimes (radiation, matter, DE). in the radiation dominated regime, η takes the form

$$\eta_r = \frac{c}{aH(a)}. \quad (9)$$

In the matter dominated regime, η takes the form

$$\eta_m = \eta(a_*) + 2c \left[\frac{1}{aH(a)} - \frac{a_*^{1/2}}{a^{3/2}} \right] \quad (10)$$

where a_* is some point where $\Omega_m \approx 1$.

In the DE dominated regime, η takes the form

$$\eta_\Lambda = \eta(a_\Lambda) + \frac{c}{H(a)} \left[\frac{1}{a_\Lambda} - \frac{1}{a} \right] \quad (11)$$

where a_Λ is some point where $\Omega_\Lambda \approx 1$. The two latter equations require known values of η at certain points, preferably deep within their own dominant regime. If only these analytical approximations were available, an extrapolated value of η from the previous era would have to be used, and the solution would suffer as a result.

2 Method and Implementation

2.1 Density parameters and the Friedmann equation

We find the scaled Hubble parameter \mathcal{H} explicitly from the Friedmann equation 4 for x in the range $[-27.6, 4.6]$, using density values roughly corresponding to the best estimates of the Λ CDM model. These are

$$\begin{aligned}\Omega_B &= 0.05 \\ \Omega_{CDM} &= 0.25 \\ \Omega_\Lambda &= 0.7\end{aligned}$$

while Ω_r is directly calculated from today's observed temperature of the CMB, at $T_{CMB} = 2.7255$ K, as

$$\Omega_r = \frac{\pi^2}{15} \frac{(k_b T_{CMB})^4}{\hbar^3 c^5} \frac{8\pi G}{3H_0^2} \approx 5 \times 10^{-5}$$

Using the Hubble parameter, we calculate the relative densities of all components from equation 1.

2.2 Conformal time

Equation 8 is solved using GSLs ODESolver class, which employs the RungeKutta4 solver. η is solved in the interval $x \in [-15, 0]$,

3 Results

Figure 1 shows the distribution of relative densities over time. We observe that the very early universe is completely radiation dominated, which gradually evolves into matter domination, the turnover point being at $x = -8.7$, or $z = 6000$. At $x = -3$, the universe energy content comes almost exclusively from baryons and CDM. This quickly changes into dark energy domination, at $x = -0.28$, or $z = 0.33$, which is the era we're in today. We also observe that the total relative density sums perfectly to 1 at all times.

Figure 3 shows the evolution of conformal time and the Hubble parameter over time. In the two bottom plots, the Hubble parameter shows three regimes of linear behavior (although the last one is subtle in the x -dependent plot), which corresponds to exponential dependence on x , and power-law dependence on z and a . The three linear regimes are split by the matter-radiation equality, and the radiation-dark energy equality, both indicated in the plots. In the radiation dominated regime, $H \propto a^2$, during the matter dominated regime, $H \propto a^{1.5}$, and in the dark energy dominated regime, H is constant.

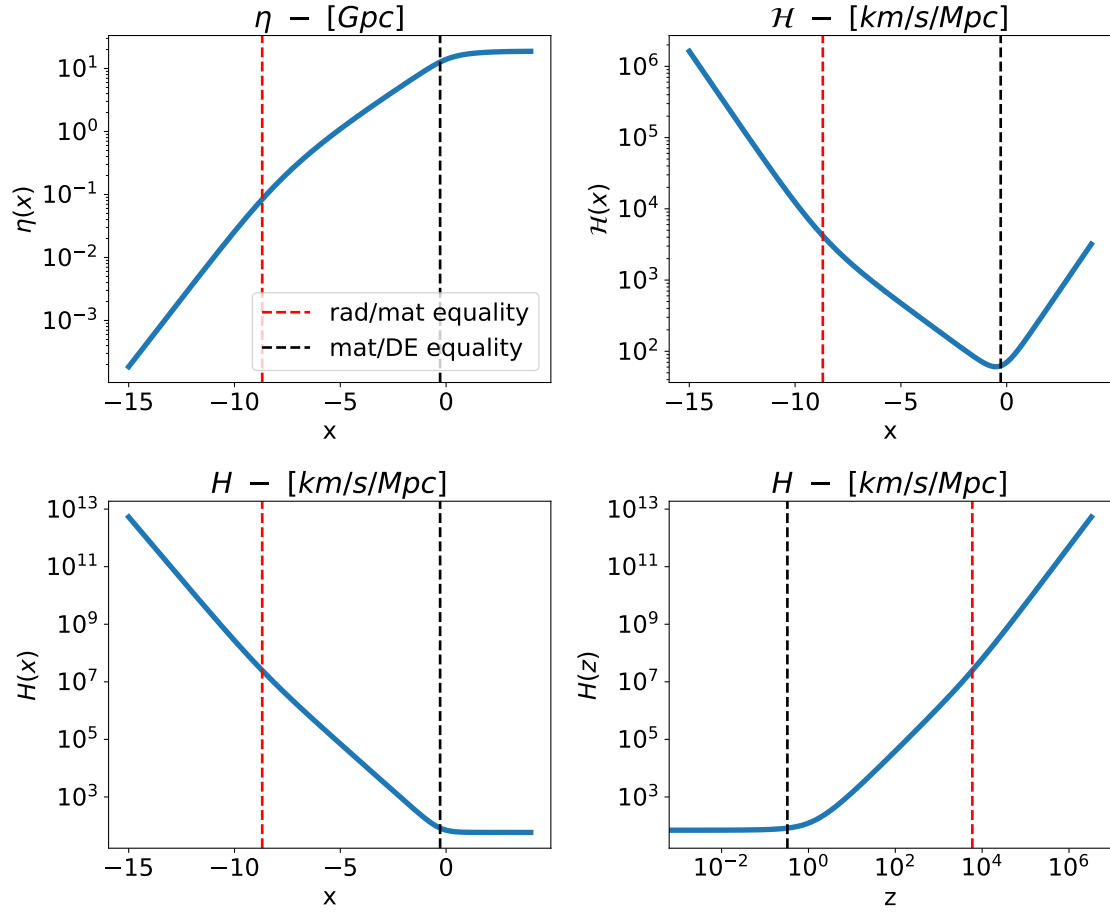


Figure 1: Plots showing the comformal time (top-left), the scaled Hubble parameter (top-right), and the Hubble parameter (bottom-left) as function of $x = \log a$. The bottom-right panel shows the Hubble parameter as function of redshift z . All plots show the matter-radiation equality as a striped red line, and the matter-dark energy equality as a striped black line.

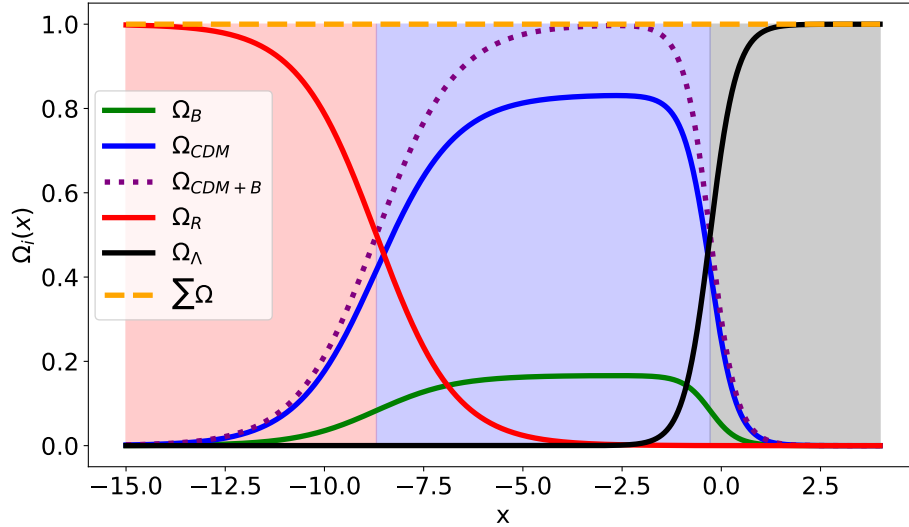


Figure 2: Plot showing the relative density distribution of components as function of $x = \log a$. Radiation, matter, and dark energy dominated eras are highlighted in their respective colors.

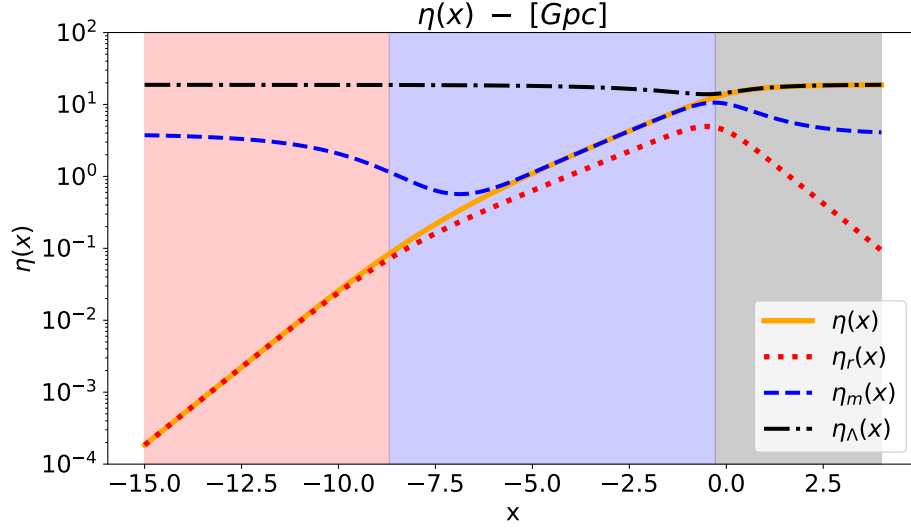


Figure 3: Plot showing the relative density distribution of components as function of $x = \log a$. Radiation, matter, and dark energy dominated eras are highlighted in their respective colors.

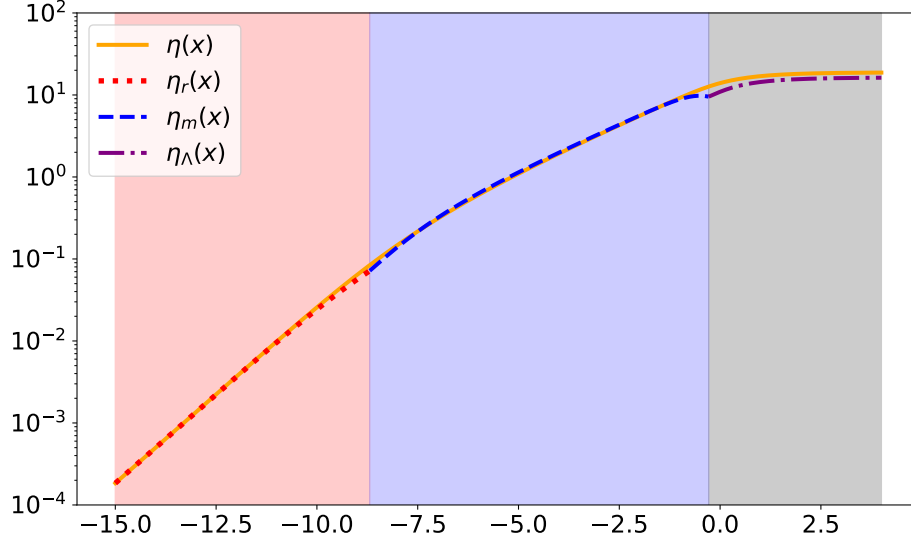


Figure 4: Plot showing the relative density distribution of components as function of $x = \log a$. Radiation, matter, and dark energy dominated eras are highlighted in their respective colors.