AST5220
Milestone II – Recombination

Jonas Gahr Sturtzel Lunde   (`jonassl`)

March 8, 2020

Code found at `https://github.com/asdfbat/AST5220/tree/master/Project`

1   Theory

1.1   Dimensionality analysis

We introduce the following notation

- T - Temperature
- t - Time
- M - Mass
- L - Length
- E - Energy ($E = ML^2 t^{-2}$)

In natural units, quantities are scaled in such a way that $c = \hbar = k_B = 1$. The dimensions of these three constants are

- $[c] = Lt^{-1}$
- $[\hbar] = Et = ML^2 t^{-1}$
- $[k_B] = ET^{-1} = ML^2 t^{-2} T^{-1}$

1.2   Saha equation dimensionality analysis

$$\frac{X_e^2}{1 - X_e} = \underbrace{\frac{1}{n_b}\left(\frac{m_e T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/T_b}}_{A} \tag{1}$$

Exponents are not physically allowed to be unitless, and the exponent in the last term in the Saha equation must therefore lacks one or more constants. We quickly see that multiplying the temperature in the divisor with k_B gives the divisor units of energy. This cancels the units of energy in the dividend, making the exponent unitless.

The left-hand side(LHS) of the Saha equation is unitless (as X_e is unitless), meaning the left-hand side(LHS), which we've named A, must be unitless as well. A initially contain dimensions of

$$[A] = [n_b^{-1}][m_e^{3/2}][T_b^{3/2}] = L^3 M^{3/2} T^{3/2}$$

We need a combination of $\hbar$, c and k_B which removes these dimensions. We observe that the dimensions of temperature must be removed by a factor of $k_B^{3/2}$, as no other of the constants contains dimensions of temperature. We're then left with the units of

$$[A][k_B^{3/2}] = L^6 M^3 t^{-3}$$

We immediately recognize this as the units of $\hbar^3$, meaning that multiplying by the factor of $\hbar^{-3}$ will make A a unitless quantity. The Saha expression with all relevant constants then reads

$$\frac{X_e^2}{1 - X_e} = A , \quad A = \frac{1}{n_b \hbar^3}\left(\frac{m_e k_B T_b}{2\pi}\right)^{3/2} e^{-\epsilon_0/k_B T_b} \tag{2}$$

1.3 Peebles equation dimensionality analysis

$$\frac{\mathrm{d}X_e}{\mathrm{d}x} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1 - X_e) - n_H\alpha^{(2)}(T_b)X_e^2\right] \tag{3}$$

$$C_r(T_b) = \frac{\Lambda_{2s\to 1s} + \Lambda_\alpha}{\Lambda_{2s\to 1s} + \Lambda_\alpha + \beta^{(2)}(T_b)} \tag{4}$$

$$\Lambda_{2s\to 1s} = 8.227\mathrm{s}^{-1} \tag{5}$$

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}} \tag{6}$$

$$n_{1s} = (1 - X_e)n_H \tag{7}$$

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4T_b} \tag{8}$$

$$\beta(T_b) = \alpha^{(2)}(T_b)\left(\frac{m_e T_b}{2\pi}\right)^{3/2}e^{-\epsilon_0/T_b} \tag{9}$$

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b) \tag{10}$$

$$\phi_2(T_b) = 0.448\ln\left(\epsilon_0/T_b\right) \tag{11}$$

1.3.1 n_{1s}

Starting with the most trivial case, the relative density of 1s Hydrogen, n_{1s} carries the units of L^{-3} from the hydrogen number density n_H. Since this is the correct units for a number density, we leave it unchanged.

1.3.2 $\phi_2(T_b)$

The logarithmic term in $\phi_2(T_b)$ must produce a unitless quantity. Multiplying the temperature T_b with k_B gives the divisor units of energy, making the logarithm unitless. The correct expression for $\phi_2(T_b)$ is then

$$\phi_2(T_b) = 0.448\ln\left(\epsilon_0/k_b T_b\right)$$

1.3.3 $\Lambda_{2s\to 1s}$

$\Lambda_{2s\to 1s}$ is the transition rate of the $2s \to 1s$ transition in a Hydrogen atom. A transition rate should have dimensions of t^{-1}, which it has.

In expression 4, $\Lambda_{2s\to 1s}$ is added to the quantities Λ_α and $\beta^{(2)}$. These two quantities must therefore also have units of t^{-1}.

1.3.4 Λ_α

Λ_α should have units of t^{-1}. In our initial expression, it has units of

$$[\Lambda_\alpha] = [H][\epsilon_0][n_H^{-1}] = t^{-1}E^3 L^3$$

We need something with units $(EL)^{-3}$ in order to get Λ_α to the right units. We can easily observe that $[c][\hbar] = EL$, such that $[c\hbar^{-3}] = (EL)^{-3}$ The correct expression for Λ_α therefore reads

$$\Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2(c\hbar)^3 n_{1s}}$$

1.3.5 $\beta^{(2)}$

The exponent in $\beta^{(2)}$ needs to be unitless. Multiplying the temperature T_b with k_B will give it units of energy, making the exponent unitless. Apart from that, $\beta^{(2)}$ simply constraints β to have units of t^{-1}. The correct expression for $\beta^{(2)}$ is therefore simply

$$\beta^{(2)}(T_b) = \beta(T_b)e^{3\epsilon_0/4k_B T_b}$$

1.3.6 $C_r(T_b)$

Since β further depends on $\alpha^{(2)}$, the constraints so far would leave ambiguity as to where constants should be placed. We therefore take a look at the Peebles equation 3. The right hand side is unitless, meaning the left hand side must be too. If we write out the brackets, each term must be unitless. The left term has units of

$$[C_r(T_b)][H^{-1}][\beta] = [C_r(T_b)]t \cdot t^{-1} = [C_r(T_b)] = (unitless)$$

The equation for $C_r(T_b)$ can therefore be left unchanged. It is already unitless, as it should.

1.3.7 $\alpha^{(2)}$

Following the same logic as above, the right term on the right hand side of the Peebles equation must be unitless. We can therefore make the following constraint:

$$[H^{-1}][n_H][\alpha^{(2)}] = tL^{-3}[\alpha^{(2)}] = (unitless) \quad \Rightarrow \quad [\alpha^{(2)}] = L^3 t^{-1}$$

We know ϕ_2 and the fine structure constant α to be unitless, meaning our initial expression for $\alpha^{(2)}$ has units of

$$[\alpha^{(2)}] = [m_e^{-2}][\epsilon_0^{1/2}][T_b^{-1/2}] = M^{-2}E^{1/2}T^{-1/2}$$

Our only constant containing temperature is k_B, meaning the dimension of temperature must be removed by a factor of k_B. Multiplying by $k_B^{-1/2}$ removes both the dimensions of energy and temperature, none of which is supposed to be in the final expression. We're then left with

$$[\alpha^{(2)}][k_B^{1/2}] = M^{-2}$$

The remaining work must be done by combinations of c and $\hbar$, as not to reintroduce temperature. It's not hard to see that this can be achived by

$$[\hbar^n][c^m] = L^3 t^{-1} M^2 \quad \Rightarrow \quad n = 2, \ m = -1$$

$\alpha^{(2)}$ now reads

$$\alpha^{(2)}(T_b) = \frac{64\pi}{\sqrt{27\pi}} \frac{\alpha^2 \hbar^2}{m_e^2 c} \sqrt{\frac{\epsilon_0}{k_B T_b}} \phi_2(T_b)$$

Inserting for the Thompson cross-section σ_T, we get our final expression for $\alpha^{(2)}$.

$$\alpha^{(2)}(T_b) = \frac{8}{\sqrt{3\pi}} \sigma_T c \sqrt{\frac{\epsilon_0}{k_B T_b}} \phi_2(T_b), \quad \sigma_T = \frac{8\pi}{3} \frac{\alpha^2 \hbar^2}{m_e^2 c^2}$$

1.3.8 β

Now that the units of $\alpha^{(2)}$ is known, the units of β is no longer ambiguous. As previous stated, β is constrained to have units of t^{-1}. In our initial expression, it has units of

$$[\beta] = [\alpha^{(2)}][m_e^{3/2}][T_b^{3/2}] = L^3 t^{-1} M^{3/2} T^{3/2}$$

As before, the only way of getting rid of temperature is k_B, meaning that β must at least contain a factor of $k_B^{3/2}$, giving new units of

$$[\beta][k_B^{3/2}] = L^6 t^{-4} M^3$$

In order to achive dimensions of t^{-1}, we need a factor which holds units of $L^{-6} t^{-3} M^3 = (L^2 t^{-1} M)^{-3}$. We recognize the units inside the paranthesis to be the units of $\hbar$, meaning that β needs a factor of $\hbar^{-3}$ to have the right units.

The exponential term in β needs to unitless. This is, again, achived by multiplying the temperature term by k_B.

The correct expression for β is then

$$\beta(T_b) = \alpha^{(2)}(T_b) \left(\frac{m_e k_B T_b}{2\pi \hbar^2} \right)^{3/2} e^{-\epsilon_0 / k_B T_b}$$

1.4 Solving the Saha equation

The Saha equation 2 is actually analytically solvable. Multiplying by $(1 - X_e)$ on both sides and reshuffling gives us

$$\frac{X_e^2}{(1 - X_e)} = A \quad \Rightarrow \quad X_e^2 + AX_e - A = 0$$

which is simply a second order equation in X_e, with solutions

$$X_e = -\frac{A}{2} \pm \frac{1}{2}\left(A^2 + 4A\right)^{1/2}$$

The free electron fraction can't physically be negative. The positive solution reads

$$X_e = -\frac{A}{2} + \frac{A}{2}\left(1 + \frac{4}{A}\right)^{1/2} \tag{12}$$

We can also observe from 2 that, as $X_e \to 1$, A will quickly converge towards zero as $A \propto X_e^2$. This will present a problem for equation 12, as the $\frac{4}{A}$ term will diverge to infinity.

At the other end, as $X_e \to 1$, A will diverge to infinity, as $A \propto X_e^{-1}$.

References