

# Usefull Shit

## Taylor Expansions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

## Random shit I always forget

$$c \ln x = \ln x^c$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \left[ \frac{u(x)}{v(x)} \right] = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

$$\int (uv') = uv - \int (u'v)$$

## Trigonometric Identities

$$e^{\pm ix} = \cos x \pm i \sin x$$

$$\cos x = \frac{1}{2} (e^{ix} + e^{-ix})$$

$$\sin x = \frac{1}{2i} (e^{ix} - e^{-ix})$$

$$\sinh z = \frac{1}{2} (e^z - e^{-z})$$

$$\cosh z = \frac{1}{2} (e^z + e^{-z})$$

## Common ODE solutions

### Harmonic oscilator

$$u''(z) = -\omega^2 u(z)$$

$$u(z) = k_1 \cos(\omega z) + k_2 \sin(\omega z) = c_1 e^{i\omega x} + c_2 e^{-i\omega x}$$

# Tensors

## Random Shit

$$A^{-1}A = \mathcal{I}$$

$$A_T = A_{ij}^T = A_{ji}$$

$$\det\{A\} = \det\{A^T\}$$

$$\det\{AB\} = \det\{A\} \det\{B\}$$

Common tensor product

$$AB = A_{ij}B_{jk}$$

## PLZ SEND HELP

Rotation matrices are orthogonal, such that  $A^T = A^{-1}$ .

Transformational matrices from a coordinate system  $\mathbf{e}'$  to  $\mathbf{e}$  is given as  $A_{ij} = e'_i \cdot e_i$

$$y_i = A_{ij}x_j = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_j = A_{ji}y_i = A^Ty$$

Transformation of a higher order tensor:

$$T'_{\alpha\beta\gamma\delta} = A_{\alpha i}A_{\beta j}A_{\gamma k}A_{\delta l}T_{ijkl}$$

where  $A$  is the transformation matrix.

$$(\mathbf{B} \times \mathbf{C})_i = \epsilon_{ijk}B_jC_k$$

## Dirac Delta & Levi-Civita

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i, j, k \text{ in order} \\ -1 & \text{if } i, j, k \text{ not in order} \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-a)f(x) \, \mathrm{d}x = f(a)$$

# Complex analysis

## Usefull shit

- Positively oriented contour integrals are counter-clockwise.
- $(z - z_0) < R$  means all complex numbers within radius  $R$  of  $z_0$  in the complex field.
- In many functions, the order of it's pole is very obvious. i.e  $1/(z - 3)$  is a first order pole at  $z = 3$ , and  $1/(z + 2i)^3$  is a third order pole at  $z = -2i$ .
- When encountered by a fraction with  $i$  in the denominator, multiply by the complex conjugate to move the  $i$  upstairs. (i.e.  $1/(3 + 2i)$ , multiply by  $(3 - 2i)$ ). In general:

$$(x + iy)(x - iy) = (x^2 + y^2)$$

- When showing that a contour integral is 0, an upper-bound estimate is often usefull.

$$\ln z = \ln |z| + i\theta, \quad \theta \in [-\pi, \pi]$$

## Polar representation and roots

$$z = x + iy = r(\cos(\theta) + i \sin(\theta)) = re^{i\theta}$$

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = \cos(n\theta) + i \sin(n\theta)$$

$$z^{1/n} = r^{1/n} e^{i(\theta + 2\pi k)/n}, \quad k \in 0, 1, 2, \dots, n - 1$$

$z^{1/n}$  has  $n$  roots, spread evenly in a circle in the complex plane.

## Complex Series

The complex sequence

$$\{z_n\} = \{z_1, z_2, z_3, \dots\}$$

converges if both the real and imaginary parts of  $z_n$  approaches zero for large  $n$ .

The complex series

$$s_n = \sum_{k=1}^n z_k$$

converges if  $z_k$  converges.

**Ratio test:** if  $\frac{z_{n+1}}{z_n} \leq 1$  for large  $n$ , then  $z_k$  converges.

## Complex Power Series

$$\sum_{n=0}^\infty a_n(z - z_0)^n$$

Around a point  $z_0$ , series converges for the area of  $z$  where

$$|z - z_0| < \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

where  $R$  is called the *radius of convergence*.

## Analytic Functions

Analytic functions are special in that they treat  $z = x + iy$  as a single unit, i.e. respect the complex structure.

If the output can be expressed solely in  $z$  (without  $x$ ,  $y$  or  $z^*$ ), the function is analytic. Remember that  $x = \frac{1}{2}(z + z^*)$  and  $y = \frac{1}{2i}(z - z^*)$ .

An function analytic in a region always has an unique derivatives of all orders in that region.

*Regular point:* Point where  $f$  is analytic.

*Singular point:* Point where  $f$  is not analytic.

## Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Criteria for a function to be analytic in a region, derived from demanding existence of the derivative.

## Harmonic Functions

Harmonic functions are solutions to the **2D Laplace equation**:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

If  $f(z) = u(x, y) + iv(x, y)$  is analytic in some region, then  $u(x, y)$  and  $v(x, y)$  are harmonic functions.

**Theorem:** Given a harmonic function  $u(x, y)$ , we can always find it's *harmonic conjugate*  $v(x, y)$  such that  $f = u + iv$  is an analytic function.

**Finding Harmonic Conjugates:** Given an harmonic function  $u(x, y)$ , we find it's harmonic conjugate by inserting  $u(x, y)$  and  $v(x, y)$  into the *Cauchy-Riemann Equations*, integrating for  $v$  (remember to include constants, which are only constant in regard to the integrating term), and solve for the constants to get a complete  $v$ .

## Contour Integrals of Complex Functions

$$\int_{C_r} (z - z_0)^n dz = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{else} \end{cases}$$

where  $C_r$  is a circle in positive (counter-clockwise) direction one time around the complex plane.

## Upper Bound Estimate of Contour Integral

$$\left| \int_\Gamma f(z) dz \right| \leq M \cdot L$$

where  $M$  is the maximum value of  $f(z)$  on  $\Gamma$ , and  $L$  is the length of  $\Gamma$ .

Remember the **Triangle Inequalities**:

$$|z_1 + z_2| \leq |z_1| + |z_2| \quad |z_2 - z_1| \geq |z_2| - |z_1|$$

Indepence of Path

If  $\Gamma_1$  and  $\Gamma_2$  are two contours that can be continously deformed into one another (without crossing singularities), then

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz$$

**Cauchy’s Theorem:** As a result, any contour integral that doesn’t enclose a singularity, is 0, as it can be shrunk to a point.

Cauchy’s Integral Formula

Formula for evaluating the contour integral around a  $n + 1$ ’th order pole at  $z_0$ .

$$\int_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

**Note:** Remember to rewrite the expression to *exactly* the form above. If the contour contains several singularities, rewrite to handle each of the singularities seperately. Example, integral around  $z = 4$  singularity of  $\cos z / [(z - 4)(z + 5)]$ , rewrite to  $[\cos z / (z + 5)] / (z - 4) = f(z) / (z - 4)$ .

Taylor Series

$$f(z; z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!}$$

**Theorem:** If  $f(z)$  is analytic in the disk  $|z - z_0| \leq R$ , then the Taylor series converges for all  $z$  *inside* the disk. AKA, Taylor converges around  $z_0$  out to the nearest singularity.

Laurent Series

We combine the *Taylor* series with a *Principal* series of negative powers.

$$f(z; z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z - z_0)^n}$$

- The Taylor series of positive powers converge *inside* some circle  $|z - z_0| < R_2$ .
- The Principal series of negative powers converge *outside* some circle  $R_1 < |z - z_0|$ .
- The Laurent series converges in the donut between the two circles,  $R_1 < |z - z_0| < R_2$ .

**Tip:** If you only need the series to converge outside/inside some circle, you only need one of the series.

The factor  $b_0$  is called the **residue** of  $f$  at  $z_0$ .

Finding Laurent Series

If the Laurent Series should expand from a point  $z_0 \neq 0$ , make a substitution  $w = z - z_0$ , such that the series expands from  $w = 0$ .

**By Geometric Series:** Manipulate the function to the form

$$f(w) = C(w) \cdot \frac{1}{1 - g}$$

where  $g$  is any factor/power of  $w$ , and  $C(w)$  is any function of  $w$ . The Laurent Series is then given as

$$f(w) = C(w) \cdot \frac{1}{1 - g} = \begin{cases} C(w) \sum_{n=0}^{\infty} g^n & \text{(Taylor)} \\ -C(w) \sum_{n=1}^{\infty} \frac{1}{g^n} & \text{(Principal)} \end{cases}$$

**By Taylor Expansion:** If the function has no singularities, you can simply make a Taylor expansion of it. Make sure to do the substitution first.

Singularities and zeros

The **order of a zero or singularity** is the number of times you must derivate the function until the zero or infinity disappears.

Assume  $f(z)$  has an isolated singularity at  $z_0$ , and it’s Larent series is as given above.

- If all  $b_n = 0$ ,  $z_0$  is a *removable* singularity (not actually a singularity).
- If  $b_n \neq 0$  for some  $n$ , but zero for all factors above  $n$  (such that  $(z - z_0)^{-n}$  is the biggest negative power), we say that  $z_0$  is a *pole* of order  $n$ .
- If there are infinite negative terms, we say that  $z_0$  is an *essential* singularity.

Residue Theory

Any integral over a contour  $\Gamma$  can be split up into integrals over only infinitesimally small contours around all singularities in  $\Gamma$ .

An contour integral containing  $N$  singularities  $z_k$  is given as the sum of the residues at all the singularities.

$$\int_{\Gamma} f(z) dz = 2\pi i \sum_{k=1}^N Res(f, z_k)$$

Ways of finding residues

- **Use Laurent Series (always works):** Write out the Laurent Series of the expression around the singularities, and find the  $b_1$  term (the  $1/z$  coefficient).

• **For Simple Poles (alt 1):**

$$Res(f, z_0) = \lim_{z \rightarrow z_0} (z - z_0) f(z)$$

• **For Simple Poles (alt 2):**

If  $f$  is a rational function  $f(z) = \frac{P(z_0)}{Q(z_0)}$ :

$$Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)}$$

• **For Multiple Poles:** If  $f$  has a pole of order  $m$  at  $z_0$ , and  $M \geq m$ , then

$$Res(f, z_0) = \lim_{z \rightarrow z_0} \frac{1}{(M - 1)!} \frac{d^{M-1}}{dz^{M-1}} [(z - z_0)^M f(z)]$$

Naturally, if you know the order of the pole, you pick  $M = m$ .

# Applications to Real Integrals

## Type I: Trigonometric integrals over $[0, 2\pi]$

$$\int_0^{2\pi} u(\cos \theta, \sin \theta) \, \mathrm{d}\theta$$

Substitute for

$$\cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right) \qquad \sin \theta = \frac{1}{2i} \left( z - \frac{1}{z} \right) \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

The integral is now around a circular contour in the complex plane, centered around (0,0) with radius 1. Evaluate the integral by finding singularities inside the circle and solving for residues.

## Type IIa: Rational Functions Over $[-\infty, \infty]$

$$I = \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \, \mathrm{d}x$$

$$I = 2\pi i \sum \operatorname{Res}(f, z_n)$$

where  $z_n$  are the singularities in the *upper* plane.

Works if

- $\operatorname{degree}(Q) \geq \operatorname{degree}(P) + 2$
- $f$  is analytic on and above the complex plane.

## Jordan’s Lemma

If  $m > 0$  is real, and  $P$  and  $Q$  are polynomials such that  $P/Q$  is rational, and  $\operatorname{degree}(Q) \geq \operatorname{degree}(P) + 1$  then:

$$\lim_{\rho \rightarrow \infty} \int_{C_\rho} \frac{P(z)}{Q(z)} e^{imz} \, \mathrm{d}z = 0$$

where  $C_\rho$  is a half-circle contour with radius  $\rho$

Same holds for the lower plane if  $m < 0$ .

## Type IIb: ... with Trigonometric Functions

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(mx) \, \mathrm{d}x \qquad (\text{or } \sin)$$

## Type III: Singularities on the real axis (Principal Value)

When a real integral passes singularities, we say that the integral is not defined, but it’s **principal value** is. It behaves just as an ordinary integral:

$$PV \int_a^b f(x) \, \mathrm{d}x = \lim_{r \rightarrow 0^+} \left[ \int_a^{C-r} f(x) \, \mathrm{d}x + \int_{C+r}^b f(x) \, \mathrm{d}x \right]$$

where  $C$  is a singularity. Using the same logic as in Type II, with an added infinite half-circle on the upper plane, this evaluates to

$$PV \int_{-\infty}^{\infty} f(x) \, \mathrm{d}x = 2\pi i \sum_k \operatorname{Res}(f; z_k) + \pi i \sum_j \operatorname{Res}(f; z_j)$$

where  $z_k$  are singularities in the **upper half plane**, and  $z_j$  are singularities **on the real axis**.

We see that singularities on the exit contribute *half* of those above.

# Ordinary Differential Equations

## First Order, Linear, ODEs - Integrating Factor

$$y'(x) + P(x)y(x) = Q(x)$$

$$y(x)\mu(x) = \int Q(x)\mu(x) \, dx + C \quad \text{with} \quad \mu(x) = e^{\int P(x) \, dx}$$

## Homogenous ODEs

$$y'' + P(x)y' + Q(x)y = 0$$

### Properties

- Linear combination of solutions is also a solution
- General solution on form  $y(x) = c_1y_1(x) + c_2y_2(x)$
- Linearly independent solutions have a Wronskian of 0.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

## Variation of the constant

If you have one of the two linearly indepent solutions, you can find the other as  $y_2(x) = C(x) \cdot y_1(x)$ , where  $C(x)$  is a functions determined by inserting  $y_2(x)$  into the ODE.

When you arrive at a solution for  $C(x)$ , you may discard any constants or coefficients, i.e.  $C(x) = \alpha x^3 + \beta$ .

## Constant coefficients - Particular Equation

$$y''(x) + ay'(x) + by(x) = 0$$

Solve the particular equation

$$\lambda^2 + a\lambda + b = 0$$

for  $\lambda_1$  and  $\lambda_2$ .

### Two, real roots

$$y(x) = C_1e^{\lambda_1x} + C_2e^{\lambda_2x}$$

### One, real root

$$y(x) = (C_1 + xC_2)e^{\lambda x}$$

### Two, complex roots

$$\begin{aligned} y(x) &= Ae^{\lambda_1x} + Be^{\lambda_2x} = e^{-a/2x} [Ae^{i\omega x} + Be^{-i\omega x}] \\ &= e^{-a/2x} [\hat{A} \cos \omega x + \hat{B} \sin \omega x] \end{aligned}$$

## Euler-Cauchy Equations

$$x^2y'' + a_1xy' + a_0y = 0$$

Introducing

$$x = e^z \quad \Rightarrow \quad z = \ln |x|$$

The equation can be rewritten to

$$\frac{\partial^2 y}{\partial z^2} + (a_1 - 1) \frac{\partial y}{\partial z} + a_0y = 0$$

Solve the ODE, and insert for  $z$ .

## Power methods

- Represent  $P(x)$  and  $Q(x)$  as power series (polynomials).
- Assume solution on the form
  - $y(x) = \sum_{n=0}^{\infty} a_nx^n$
  - $y'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$
  - $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$
- Insert back into ODE.
- Split into equations of matching powers of  $x$ .

This will give you one or two undetermined coefficients. The equations maybe give the coefficients as a series depending on each other, like  $a_{s+1} = a_s^2$ . If the series are odd/even, two undetermined coefficients are required to describe them, so the solution is complete.

## Fröbenius method

$$x^2y'' + xb(x)y' + c(x)y = 0$$

Assuming solution on form

$$y_p(x) = \sum_{m=0}^{\infty} a_mx^{m+s}$$

where  $s$  is some real number determined the *indicial equation*

$$s(s-1) + b_0s + c_0 = 0$$

where  $b_0 = b(0)$ , and  $c_0 = c(0)$

### Three possible scenarios:

- Different roots,  $s_1 \neq s_2$ , and  $s_1 - s_2 \neq \text{integer}$ .
  - Two inepedens solutions  $y_i(x) = x^{s_i} \sum_{m=0}^{\infty} a_{0m}x^m$
- Different roots,  $s_1 \neq s_2$ , but  $s_1 - s_2 = \text{integer}$ . ( $s_1 > s_2$ ).
  - Solve for both by Power Series.
  - Often,  $s_2$  gives the complete solution (two undetermined coefficients), so try this first.
  - Sometimes, only  $s_1$  gives a solution. Find the other by variation of the constant.
- Double root,  $s_1 = s_2$ .
  - Find the solution by Power Series.
  - Find the second solution by variation of the constant.

# Inhomogenous ODEs

$$y'' + P(x)y' + Q(x)y = R(x)$$

**Remember** to always rewrite to this form.

## Properties

- Solutions on form  $y(x) = y_h(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$ .
- $y_h(x)$  is the solution to the homogenous equation.
- $y_p(x)$  is *any* solution to the whole ODE.
- Since  $y_h$  contains two arbitrary constants,  $y_p$  should contain none.

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## Inhomo ODEs with constant coefficients

$$y'' + ay' + by = R(x)$$

- Make a guess at  $y_p$  with the same form as  $R(x)$ , with unknown coefficients.
- Insert back into ODE to solve for coefficients.

**Special case:**  $R(x) = Ae^{kx}$ .  
Let  $\alpha$  and  $\beta$  be the roots of  $\lambda^2 + a\lambda + b = 0$ .

1. If  $k \neq \alpha, \beta$ : Try  $y_p = Ce^{\lambda x}$ .
2. If  $k = \alpha$  or  $\beta$ : Try  $y_p = Cxe^{\lambda x}$ .
3. If  $k = \alpha = \beta$ : Try  $y_p = Cx^2e^{\lambda x}$ .

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## Inhomo ODEs with varying coefficients

$$y'' + P(x)y' + Q(x)y = R(x)$$

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### Factorization

If  $u(x)$  is a known solution to the homo-ODE, a particular solution is  $y_p = u(x) \cdot v(x)$  where

$$w' = v \qquad w' + \left[ \frac{2u'}{u} + P \right] w = \frac{R}{u}$$

Solve the ODE for  $w$  with integrating factor.

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### Variation of parameters

$$y_p = -y_1 \int \frac{y_2 R}{W} \, dx + y_2 \int \frac{y_1 R}{W} \, dx$$

where  $y_1$  and  $y_2$  are known linearly independent solutions to the homo-ODE.  
**NOTE:** Remember that  $R(x)$  is the RHS after the ODE is rewritten on the standard form.

# Fourier

## Usefull Shit

- recognize **odd** and **even** integrands. I.e  $\int_{-\infty}^{\infty} \sin x/x^2 = 0$  due to odd, and  $\int_{-\infty}^{\infty} \cos x/(1+x^2) = 2 \int_0^{\infty} \cos x/(1+x^2)$  due to even.

## Orthogonality

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \pi \delta_{mn}$$

## Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \qquad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-in\pi x/L}$$

## Even and Odd functions

If  $f(x)$  is **even** [ $f(x) = f(-x)$ ]:

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \qquad b_n = 0$$

If  $f(x)$  is **odd** [ $f(x) = -f(-x)$ ]:

$$a_n = 0 \qquad b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

## Dirichlet Conditions for Fourier Series

- Finite number of min/max in interval.
- Finitite number of (only) finite discontinuities.

If this holds, then the series will converge to  $f(x)$  at all points. At discontinuities, the series will converge to the mid-point.

## Parseval’s Theorem

$$\int_{-L}^L |f(x)|^2 \, dx = 2L \sum_{n=-\infty}^{\infty} |c_n|^2$$

## Fourier Transforms

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} \, dk$  $F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dx$

## Odd and even functions

If  $f(x)$  is an odd function,  $f(x) = -f(-x)$ , the Fourier transform can be done using only sine (as cosine is symmetric around 0):

$$f(x) = \sqrt{\frac{2}{\pi}} i \int_0^{\infty} F(k) \sin(kx) \, dk \qquad F(k) = \sqrt{\frac{2}{\pi}} i \int_0^{\infty} f(x) \sin(kx) \, dx$$

If  $f(x)$  is even,  $f(x) = f(-x)$ , we need only cosine (as sine is anti-symmetric):

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F(k) \cos(kx) \, dk \qquad F(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(kx) \, dx$$

## FT of a derivative

$$\mathcal{F}\left[f^{(n)}(x)\right] = (ik)^n \mathcal{F}[f(x)]$$

# Partial Differential Equations

## Notes

- In symmetrical systems, it seems you can switch  $x \leftrightarrow y$  if it is required to suit boundary conditions (example: Dirichlet conditions are at  $x=a$  instead of at  $y=b$ ).
- When resulting in cos/sin solutions of frequencies, include  $n=0$  for cos, as it gives a non-zero solution, but not for sin.

## Separation of variables

1) The the solution function as a product of functions of each variable, i.e:

$$u(x,y) = X(x)Y(y) \qquad u(r,\theta) = R(r)T(\theta)$$

2) Insert this into the equation, and separate the equation into parts of only each variable. Each side must then be constant, and equals some *separation constant*.

3) Solve each side of the equation (equaling the separation constant), giving an infinite set of *eigenfunctions*,  $u_n(x,y)$  for the equation.

4) The final solution is a linear combination of the eigenfunctions.

$$u(x,y) = \sum_{n=-\infty}^{\infty} c_n u_n(x,y)$$

## Laplace Equation - 2D Cartesian

$$\nabla u(x,y) = 0$$

Separation of variables,  $u(x,y) = X(x)Y(y)$  gives solutions

$$u(x,y) = X(x)Y(y) = \left\{ \begin{matrix} e^{ky} \\ e^{-ky} \end{matrix} \right\} \times \left\{ \begin{matrix} \sin(kx) \\ \cos(kx) \end{matrix} \right\}$$

**Dirichlet BC:**  $u(x,0) = u(0,y) = u(a,y) = 0$ ,  $u(x,b) = f(x)$   
Eigenfunctions on form

$$u_n(x,y) = A_n F_n(x) G_n(x) = A_n \sin(n\pi x/a) \sinh(n\pi y/a)$$

## Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Separation of variables,  $u(x,t) = F(x)G(t)$  gives equations

$$F''(x) = -k^2 F(x) \qquad \ddot{G}(t) = -k^2 v^2 G(t)$$

where the separation constant,  $-k^2$  must be negative, else the solutions would blow up.

The equations has solutions on the form

$$y(x,t) = \left\{ \begin{matrix} \sin(kx) \\ \cos(kx) \end{matrix} \right\} \times \left\{ \begin{matrix} \sin(kvt) \\ \cos(kvt) \end{matrix} \right\}$$

The end-points are usually fixed at 0, leaving only the  $\sin(kt)$  term, and forcing  $k = n\pi/L$ .

If the velocity is 0 at  $t = 0$ , we discard the sin-velocity term and get

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi vt}{L}\right)$$

If position is 0 at  $t = 0$ , we discard the cos-velocity term instead.

Initial position will be on the form

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

where  $f(x)$  is the initial position or velocity. The coefficients  $b_n$  are now Fourier coefficients, given as:

$$b_n = \frac{1}{2L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

## Orthogonal Functions

Functions on the form

$$p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0$$

on some interval  $[a,b]$  has solutions as linear combinations of eigenfunctions  $y_n(x)$  which are orthogonal with respect to  $r(x)$  such that

$$\int_a^b r(x) y_n(x) y_m(x)^* dx = 0 \qquad \text{for } \lambda_n \neq \lambda_m$$

Any function can be written as a linear combination of these eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

then the set  $\{y_n(x)\}$  is complete. The coefficients  $a_n$  are determined by the orthogonality:

$$a_n = \int_a^b f(x) r(x) y_n(x)^* dx$$