STK1100 – Oblig 1

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Kode og figurer ligger til slutt.

OPG 1 a) - Han spiller et gitt antall $n = 20$ ganger.

- Hvert spill har to utfall
- Spillene er uavhengige.
- sannsynligheten for suksess er konstant.

$n = 20$, $p = 18/37$, $P(X) = b(x; 20, 18/37)$, $k = 18$

$E(X) = np = 20 \cdot \frac{18}{37} \approx 9,73..$

$SD(X) = \sqrt{npq} = \sqrt{20 \cdot \frac{18}{37} \cdot (1 - 18/37)} \approx 2,235$

b) $Y = X(\frac{36}{18} - 1) \cdot 100 - 100(20 - X) = 100X + 100X - 2000$

$= 200X - 2000 = 200[X - 10]$

$E(Y) = 200E(X) - 2000 = 200 \cdot 9,73 - 2000 = -54$

$V(Y) = E(Y^2) - E(Y)^2 = E([200X - 2000]^2) - (-54)^2$

$= E(200^2X^2 - 8e5X + 2000^2) - 2916$

$= 4e4 \underbrace{E(X^2)} - 8e5E(X) + 4e6 - 2916$

$= M''_x(t) = n(n-1)p^2 + np = 20(20-1)\frac{18^2}{37^2} + 20\frac{18}{37} = 99.664$

$= 4e4 \cdot 99,7 - 8e5 \cdot 20 \cdot \frac{18}{37} + 4e6 - 2916 = \frac{7999200}{37}$

$SD(Y) = \sqrt{V(x)} \approx \underline{\underline{450}}$

c) $P(Y) = P(X) = b(x; 20, \frac{18}{37})$

$Y = 200x - 2000 \geq 1000 \Rightarrow x \geq \frac{3000}{200} = 15$

Vinne 1000 kr eller mer; $X \geq 15$

$Y = 200x - 2000 \leq -1000 \Rightarrow x \leq \frac{1000}{200} = 5$

Tape 1000 kr eller mer; $X \leq 5$

Python

$P(Y \geq 1000) = P(X \geq 15) = \sum_{x=15}^{20} b(x; 20, \frac{18}{37}) = \underline{\underline{0{,}0154\ldots}}$

$P(Y \leq -1000) = P(X \leq 5) = \sum_{x=0}^{5} b(x; 20, \frac{18}{37}) = \underline{\underline{0{,}0275\ldots}}$

d) $P(Y) = P(X) = b(x; 20, \frac{6}{37})$

$Y = X(\frac{36}{6} - 1) \cdot 100 - 100(20 - X) = 500X + 100x - 2000$

$= 600X - 2000$

$Y = 600x - 2000 \geq 1000 \Rightarrow x \geq \frac{3000}{600} = 5$

Vinne 1000 kr eller mer: $X \geq 5$

$Y = 600x - 2000 \leq -1000 \Rightarrow x \leq \frac{1000}{600} = \frac{5}{3} = 1.666\ldots$

Tape 1000 kr eller mer: $X \leq 1$

Python

$P(Y \geq 1000) = P(X \geq 5) = \sum_{x=5}^{20} b(x; 20, \frac{6}{37}) = \underline{\underline{0.214\ldots}}$

$P(Y \leq -1000) = P(X \leq 1) = \sum_{x=0}^{1} b(x; 20, \frac{6}{37}) = \underline{\underline{0.141\ldots}}$

e) $P(z) = \underbrace{\left(\frac{36}{37}\right)^{z-1}}_{\text{Først taper } z-1 \text{ ganger}} \underbrace{\left(\frac{1}{37}\right)}_{\text{så vinner en gang.}}$

$y = \frac{36}{1} \cdot 100\, x - 100 \cdot z$

$x = 1$. Han må vinne 1 gang for å kunne tjene 1000 kr.

$= 3600 - 100z \geq 1000 \Rightarrow z \geq \frac{3600-1000}{100} = 26$

Python. $P(y \geq 1000) = \sum_{z=1}^{26} P(z) \approx \underline{\underline{0.51}}$

f) $P(y \leq -1000) = 1 - P(y > -1000)$

$y = 3600 - 100z > -1000 \Rightarrow z > \frac{3600+1000}{100} = 46$

Python
$P(y > -1000) = \sum_{z=1}^{45} P(z) = 0.71$

$P(y \leq -1000) = 1 - 0.709 = \underline{\underline{0.29}}$

OPPGAVE 2

a) X = Gjenstående levetid (fra 35 år)

$P(X)$ = Sannsynlighet for X gjenstående år (fra 35)

q_x = Sannsynligheten for å dø i løpet av neste år

Hvis q_x er sannsynligheten for å dø innen et år i en alder x, er $(1-q_x)$ sannsynligheten for å overleve det neste året. Da er $(1-q_{35})(1-q_{36})\cdots(1-q_x) = \prod_{y \leq x}(1-q_y)$

sannsynligheten for å overleve de neste x årene.

Dette tilsvarer $P(x \leq X) = \prod_{y=0}^{x}(1-q_{y+35}) = 1 - P(X \leq x)$

Som gir

$$F(x) = P(X \leq x) = 1 - \prod_{y=0}^{x}(1-q_{35+y})$$

b) $F(x) - F(x-1) = \sum_{y \leq x} P(y) - \sum_{y \leq x-1} P(y) = P(x)$

d) Hvis $X \leq 34$ blir han under 70 år, og får ingen pensjon. Det er da ingen nåverdi ($h(X) = 0$).
Pensjonen utbetalt om k år vil ha en nåverdi på $100'000/1.03^k$. Ettersom utbetalingene går over flere år, fra 35 år fra nå til X år fra nå, vil den totale nåverdien bli:

$$h(X) = \sum_{k=35}^{X} \frac{100\,000}{1.03^k} = 100\,000 \sum_{k=35}^{X} 1.03^{-k}$$

$$h(x) = 100\,000 \left[\frac{1.03^{-34} - 1.03^{-x}}{1 - 1.03} \right]$$

$$= \frac{100\,000}{1.03^{35}} \left[\frac{1.03^{-34} - 1.03^{-x}}{1 - 1.03} \times \frac{1.03^{34}}{1.03^{-1}} \right]$$

$$= \frac{100\,000}{1.03^{35}} \left[\frac{1.03^{34-x} - 1}{1/1.03 - 1} \right] = \frac{100\,000}{1.03^{35}} \left[\frac{1 - (1/1.03)^{x-34}}{1 - 1/1.03} \right]$$

e) $$E[h(X)] = \sum_{x \in X} h(x)\,p(x) = \sum_{x=0}^{71} h(x)\,p(x)$$

$$= \sum_{x=0}^{34} h(x)\,p(x) + \sum_{x=35}^{71} h(x)\,p(x) =$$

$$= \sum_{x=35}^{71} \frac{100\,000}{1.03^{35}} \frac{1 - (1/1.03)^{x-34}}{1 - 1/1.03} \cdot p(x)$$

$$= \frac{100\,000}{1.03^{35}} \frac{1}{1 - 1/1.03} \left[\sum_{x=35}^{71} p(x) - (1/1.03)^{x-34}\, p(x) \right]$$

$$= \frac{100\,000}{1.03^{35}} \frac{P(X \geq 35) - \sum_{x=35}^{71} (1/1.03)^{x-34}\, p(x)}{1 - 1/1.03}$$

f) $E[h(X)] = 387'000$ kr (se kode)

g) Nå-verdien av en sum innbetalt om k år er

$$K \cdot \frac{1}{1.03^k}$$

Ettersom han betaler en summ K hvert år, enten til han dør i en alder < 70, eller i 35 år, blir nå-verdien:

$$\sum_{k=0}^{\min(34,X)} K \cdot \frac{1}{1.03^k} = K \cdot g(X)$$

h) $$E[g(X)] = \sum_{x=0}^{71} g(x) P(x) = \sum_{x=0}^{34} g(x) P(x) + \sum_{x=35}^{71} g(x) P(x)$$

$$= g(X) = \sum_{k=0}^{X} 1.03^{-k} = \left[\frac{1-(1/1.03)^{X+1}}{1-(1/1.03)}\right], \quad X \leq 34$$

$$g(X) = \left[\frac{1-(1/1.03)^{35}}{1-(1/1.03)}\right], \quad X \geq 35$$

$$E[g(X)] = \sum_{x=0}^{34} \left[\frac{1-(1/1.03)^{x+1}}{1-(1/1.03)} P(x)\right] \sum_{x=35}^{71} \left[\frac{1-(1/1.03)^{35}}{1-(1/1.03)} P(x)\right]$$

$$= \frac{P(X<35) - \sum_{x=0}^{34} (1/1.03)^{x+1} P(x) + P(X \geq 35) - \sum_{x=35}^{71} (1/1.03)^{35} P(x)}{1-(1/1.03)}$$

$$= \frac{1 - \sum_{x=0} (1/1.03)^{x+1} P(x) - (1/1.03)^{35} P(X \geq 35)}{1-(1/1.03)}$$

i) $E[g(x)] = 21.56$

j) $K = \frac{E[h(x)]}{E[g(x)]} = 17952$ km

Oppgave 1c and 1d

```
from scipy import misc
import numpy as np

p = 18/37
n = 20

def binom(x, n, p):
    """ Binomial probability b(x;n,p)"""
    nCr = misc.comb(n, x)
    return nCr*p**x*(1-p)**(n-x)

def cum_binom(x_min, x_max, n, p):
    """ Cumulative distribution function F(x_min <= x <= x_max)
    for a binomial function b(x;n,p) """
    P_X = 0
    for x in range(x_min, x_max+1):
        P_X += binom(x, n, p)
    return P_X

#___Exercise c___
print( cum_binom(15,20,n,p) )
print( cum_binom(0,5,n,p) )

#___Exercise d___
p = 6/37
EX = 0
for i in range(0,20):
    EX += (cum_binom(i,i,n,p))*500*i
print( cum_binom(5,20,n,p) )
print( cum_binom(0,1,n,p) )

"""
jonas@ubuntu:~/github/other_courses/STK1100/oblig1/src$ python3 1cd.py
0.015385465001631102
0.027464357940763788
0.21400681498120722
0.14151896666419134
"""
```

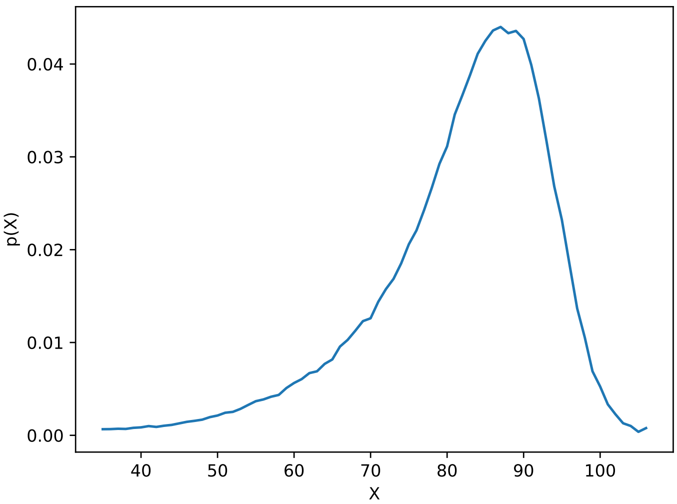
Oppgave 1e og f

```
#___Exercise 1e___
P = 0
for Z in range(1,26+1):
    P += (36/37)**(Z-1) * 1/37
print(P)

#___Exercise 1f___
P = 0
for Z in range(1,45+1):
    P += (36/37)**(Z-1) * 1/37
print(1-P)

"""
jonas@ubuntu:~/github/other_courses/STK1100/oblig1/src$ python3 1ef.py
0.5095212523720186
0.2914304665172236
"""
```

Oppgave 2c

```
import numpy as np
import matplotlib.pyplot as plt

data = np.genfromtxt("dodelighet-felles.txt")[1:]
data = np.transpose(data)
```

```python
ald, q_x = data

def F(x):
    value = 1
    for y in range(x+1):
        value *= (1-q_x[y+35]/1000)
    return 1 - value

p = np.zeros(72)
for x in range(72):
    p[x] = F(x) - F(x-1)

plt.plot(ald[35:], p)
plt.xlabel("X")
plt.ylabel("p(X)")
plt.savefig("../2c.pdf")
```



Oppgave 2f

```python
import numpy as np
import matplotlib.pyplot as plt

def h(X):
    if X < 35:
        return 0
    else:
        return 1e5/1.03**35 * (1-(1/1.03)**(X-34))/(1-1/1.03)

data = np.genfromtxt("dodelighet-felles.txt")[1:]
data = np.transpose(data)
ald, q_x = data

def F(x):
    value = 1
    for y in range(x+1):
        value *= (1-q_x[y+35]/1000)
    return 1 - value

p = np.zeros(72)
for x in range(72):
    p[x] = F(x) - F(x-1)


E_h_X = 0
for x in range(72):
    E_h_X += h(x)*p[x]

print(E_h_X)
```

```
"""
jonas@ubuntu:~/github/other_courses/STK1100/oblig1/src$ python3 2f.py
387141.5341685992
"""
```

Oppgave 2i

```
import numpy as np
import matplotlib.pyplot as plt

def g(X):
    if X > 34:
        X = 34
    return (1-(1/1.03)**(X+1))/(1-(1/1.03))

data = np.genfromtxt("dodelighet-felles.txt")[1:]
data = np.transpose(data)
ald, q_x = data

def F(x):
    value = 1
    for y in range(x+1):
        value *= (1-q_x[y+35]/1000)
    return 1 - value

p = np.zeros(72)
for x in range(72):
    p[x] = F(x) - F(x-1)


E_g_X = 0
for x in range(72):
    E_g_X += g(x)*p[x]

print(E_g_X)

"""
jonas@ubuntu:~/github/other_courses/STK1100/oblig1/src$ python3 2i.py
21.564918895029948
"""
```