Usefull Shit

Random shit I always forget

$$c\ln x = \ln x^c \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integration

$$\int (uv') = uv - \int (u'v)$$

Trigenometric Identities

$$e^{\pm iz} = \cos z \pm i \sin z$$
$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Common ODE solutions

Harmonic oscilator

$$u''(z) = -\omega^2 u(z)$$
$$u(z) = k_1 \cos(\omega z) + k_2 \sin(\omega z) = c_1 e^{i\omega x} + c_2 e^{-i\omega x}$$

Complex analysis

Ask Julie about principal value thingy.

Ordinary Differential Equations

First Order, Linear, ODEs - Integrating Factor

$$y'(x) + P(x)y(x) = Q(x)$$

$$y(x)\mu(x) = \int Q(x)\mu(x)\,\mathrm{d}x + C \qquad \text{with} \qquad \mu(x) = e^{\int P(x)\mathrm{d}x}$$

Homogenous ODEs

$$y'' + P(x)y' + Q(x)y = 0$$

Properties

- Linear combination of solutions is also a solution
- General solution on form $y(x) = c_1y_1(x) + c_2y_2(x)$
- Linearly independent solutions have a Wronskian of 0.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Variation of the constant

If you have one of the two linearly indepedent solutions, you can find the other as $y_2(x) = C(x) \cdot y_1(x)$, where $C(x)$ is a functions determined by inserting $y_2(x)$ into the ODE.

When you arrive at a solution for $C(x)$, you may discard any constants or coefficients, i.e. $C(x) = \alpha x^3 + \beta$.

Constant coefficients - Particular Equation

$$y''(x) + ay'(x) + by(x) = 0$$

Solve the particular equation

$$\lambda^2 + a\lambda + b = 0$$

for λ_1 and λ_2.

Two, real roots

$$y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$

One, real root

$$y(x) = (C_1 + xC_2)e^{\lambda x}$$

Two, complex roots

$$\begin{aligned} y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} &= e^{-a/2x}\left[Ae^{i\omega x} + Be^{-i\omega x}\right] \\ &= e^{-a/2x}\left[\hat{A}\cos\omega x + \hat{B}\sin\omega x\right] \end{aligned}$$

Euler-Cauchy Equations

$$x^2y'' + a_1xy' + a_0y = 0$$

Introducing

$$x = e^z \qquad \Rightarrow \qquad z = \ln|x|$$

The equation can be rewritten to

$$\frac{\partial^2 y}{\partial z^2} + (a_1 - 1)\frac{\partial y}{\partial z} + a_0y = 0$$

Solve and insert for z.

Power methods

- Represent $P(x)$ and $Q(x)$ as power series.
- Assume solution on the form
  - $y(x) = \sum_{n=0}^{\infty} a_nx^n$
  - $y'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$
  - $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$
- Insert back into ODE.
- Split into equations of matching powers of x.

This will give you one or two undetermined coefficients. The equations maybe give the coefficients as a series depending on each other, like $a_{s+1} = a_s^2$. If the series are odd/even, two undetermined coefficients are required to describe them, so the solution is complete.

Fröbenius method

$$x^2y'' + xb(x)y' + c(x)y = 0$$

Assuming solution on form

$$y_p(x) = \sum_{m=0}^{\infty} a_mx^{m+s}$$

where s is some real number determined the *indicial equation*

$$s(s-1) + b_0s + c_0 = 0$$

where $b_0 = b(0)$, and $c_0 = c(0)$

Three possible scenarios:

- Different roots, $s_1 \neq s_2$, and $s_1 - s_2 \neq$ integer.
  - Two indepedens solutions $y_i(x) = x^{si}\sum_{m=0}^{\infty} a_0x^m$
- Different roots, $s_1 \neq s_2$, but $s_1 - s_2 =$ integer. ($s_1 > s_2$).
  - Solve for both by Power Series.
  - Often, s_2 gives the complete solution (two undetermined coefficients), so try this first.
  - Sometimes, only s_1 gives a solution. Find the other by variation of the constant.
- Double root, $s_1 = s_2$.
  - Find the solution by Power Series.
  - Find the second solution by variation of the constant.

Inhomogenous ODEs

$$y'' + P(x)y' + Q(x)y = R(x)$$

Remember to always rewrite to this form.

Properties

- Solutions on form $y(x) = y_h(x) + y_p(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$.
- $y_h(x)$ is the solution to the homogenous equation.
- $y_p(x)$ is *any* solution to the whole ODE.
- Since y_h contains two arbitrary constants, y_p should contain none.

Inhomo ODEs with constant coefficients

$$y'' + ay' + by = R(x)$$

- Make a guess at y_p with the same form as $R(x)$, with unknown coefficients.
- Insert back into ODE to solve for coefficients.

Special case: $R(x) = Ae^{kx}$.
Let α and β be the roots of $\lambda^2 + a\lambda + b = 0$.

1. If $k \neq \alpha, \beta$: Try $y_p = Ce^{\lambda x}$.
2. If $k = \alpha$ or β: Try $y_p = Cxe^{\lambda x}$.
3. If $k = \alpha = \beta$: Try $y_p = Ce^{\lambda x}$.

Inhomo ODEs with varying coefficients

$$y'' + P(x)y' + Q(x)y = R(x)$$

Factorization

If $u(x)$ is a known solution to the homo-ODE, a particular solution is $y_p = u(x) \cdot v(x)$ where

$$w' = v \qquad w' + \left[\frac{2u'}{u} + P\right]w = \frac{R}{u}$$

Solve the ODE for w with integrating factor.

Variation of parameters

$$y_p = -y_1 \int \frac{y_2R}{W}\,\mathrm{d}x + y_2 \int \frac{y_1R}{W}\,\mathrm{d}x$$

where y_1 and y_2 are known linearly indepedendent solutions to the homo-ODE. **NOTE:** Remember that $R(x)$ is the RHS after the ODE is rewritten on the standard form.

Trigonometric Functions

Orthogonality

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}$$

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}$$

Even and Odd functions

If $f(x)$ is **even** $[f(x) = f(-x)]$:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = 0$$

If $f(x)$ is **odd** $[f(x) = -f(-x)]$:

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Dirichlet Conditions for Fourier Series

1. Finite number of min/max in interval.
2. Finitite number of (only) finite discontinuities.

If this holds, then the series will converge to $f(x)$ at all points. At discontinuities, the series will converge to the mid-point.

Parseval's Theorem

$$\int_{-L}^{L} |f(x)|^2\,\mathrm{d}x = 2L\sum_{-\infty}^{\infty} |c_n|^2$$

Fourier Transforms

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,\mathrm{d}k \qquad F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$$

Odd and even functions

If $f(x)$ is an odd function, $f(x) = -f(-x)$, the Fourier transform can be done using only sine (as cosine is symmetric around 0):

$$f(x) = \sqrt{\frac{2}{\pi}}i\int\limits_0^{\infty} F(k)\sin(kx)\,\mathrm{d}k \qquad F(k) = \sqrt{\frac{2}{\pi}}i\int\limits_0^{\infty} f(x)\sin(kx)\,\mathrm{d}x$$

If $f(x)$ is even, $f(x) = f(-x)$, we need only cosine (as sine is anti-symmentric):

$$f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} F(k)\cos(kx)\,\mathrm{d}k \qquad F(k) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} f(x)\cos(kx)\,\mathrm{d}x$$

FT of a derivative

$$\mathcal{F}\left[f^{(n)}(x)\right] = (ik)^n\mathcal{F}[f(x)]$$

Partial Differential Equations

Notes

- In symentrical systems, it seems your can switch x<->y if it is required to suit boundary conditions (example: Diritchlet conditions are at x=a instad of at y=b).
- When resulting in cos/sin solutions of frequencies, include n=0 for cos, as it gives a non-zero solution, but not for sin.

Separation of variables

1) The the solution function as a product of functions of each variable, i.e:

$$u(x,y) = X(x)Y(y) \qquad u(r,\theta) = R(r)T(\theta)$$

2) Insert this into the equation, and separate the equation into parts of only each variable. Each side must then be constant, and equals some *separation constant*.

3) Solve each side of the equation (equaling the seperation constant), giving an infinite set of *eigenfunctions*, $u_n(x,y)$ for the equation.

4) The final solution is a linear combination of the eigenfunctions.

$$u(x,y) = \sum_{n=-\infty}^{\infty} c_n u_n(x,y)$$

Laplace Equation - 2D Cartesian

$$\nabla u(x,y) = 0$$

Separation of variables, $u(x,y) = X(x)Y(y)$ gives solutions

$$u(x,y) = X(x)Y(y) = \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix} \times \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix}$$

Diritchlet BC: $u(x,0) = u(0,y) = u(a,y) = 0$, $u(x,b) = f(x)$
Eigenfunctions on form

$$u_n(x,y) = A_n F_n(x) G_n(x) = A_n \sin(n\pi x/a)\sinh(n\pi y/a)$$

Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Seperation of variables, $u(x,t) = F(x)G(t)$ gives equations

$$F''(x) = -k^2 F(x) \qquad \ddot{G}(t) = -k^2 c^2 G(t)$$

where the seperation constant, $-k^2$ must be negative, else the solutions would blow up.

The equations has solutions on the form

$$y(x,t) = \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} \times \begin{Bmatrix} \sin(kct) \\ \cos(kct) \end{Bmatrix}$$

The end-points are usually fixed at 0, leaving only the $\sin(kt)$ term, and forcing $k = n\pi/L$.

If the velocity is 0 at $t = 0$, we discard the sin-velocity term and get

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$

If position is 0 at $t = 0$, we discard the cos-velocity term instead.

Initial position will be on the form

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

where $f(x)$ is the initial position or velocity. The coefficients b_n are now Fourier coefficients, given as:

$$b_n = \frac{1}{2L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Orthogonal Functions

Functions on the form

$$p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0$$

on some interval $[a,b]$ has solutions as linear combinations of eigenfunctions $y_n(x)$ which are orthogonal with respect to $r(x)$ such that

$$\int_a^b r(x)y_n(x)y_m(x)^*\,\mathrm{d}x = 0 \qquad \text{for}\lambda_n \neq \lambda m$$

Any function can be written as a linear combination of these eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

then the set $\{y_n(x)\}$ is complete. The coefficients a_n are determined by the orthogonality:

$$a_n = \int_a^b f(x)r(x)y_n(x)^*\,\mathrm{d}x$$