Usefull Shit

$$c \ln x \; = \ln x^c \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

Random shit I always forget

$$c \ln x \; = \ln x^c \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Integration

$$\int (uv') = uv - \int (u'v)$$

Trigenometric Identities

$$e^{\pm ix} = \cos x \pm i \sin x$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x \; = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right) \qquad \cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

Common ODE solutions

Harmonic oscilator

$$u''(z) = -\omega^2 u(z)$$

$$u(z) = k_1 \cos(\omega z) + k_2 \sin(\omega z) = c_1 e^{i\omega x} + c_2 e^{-i\omega x}$$

Complex analysis

Usefull shit

- $(z - z_0) < R$ means all complex numbers within radius R of z_0 in the complex field.
- In many functions, the order of it's pole is very obvious. i.e $1/(z-3)$ is a first order pole at $z = 3$, and $1/(z+2i)^3$ is a third order pole at $z = -2i$.
- When encountered by a fraction with i in the denominator, multiply by the complex conjugate to move the i upstairs. (i.e. $1/(3+2i)$, multiply by $(3-2i)$). In general:

$$(x+iy)(x-iy) = (x^2+y^2)$$

$$\ln z = \ln |z| + i\theta, \qquad \theta \in [-\pi, \pi]$$

Polar representation and roots

$$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

Powers of z:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Roots of z:

$$z^{1/n} = r^{1/n} e^{i(\theta+2\pi k)/n}, \qquad k \in 0, 1, 2, ..., n-1$$

$z^{1/n}$ has n roots, spread evenly in a circle in the complex plane.

Complex Series

The complex sequence

$$\{z_n\} = \{z_1, z_2, z_3, ...\}$$

converges if both the real and imaginary parts of z_n approaches zero for large n.

The complex series

$$s_n = \sum_{k=1}^{n} z_k$$

converges if z_k converges.

Ratio test: if $\frac{z_{n+1}}{z_n} \leq 1$ for large n, then z_k converges.

Complex Power Series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Around a point z_0, series converges for the area of z where

$$|z - z_0| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

where R is called the *radius of convergence.*

Analytic Functions

Analytic functions are special in that they treat $z = x + iy$ as a single unit, i.e. respect the complex structure.

If the output can be expressed solely in z (without x or y), the function is analytic. Remember that $x = \frac{1}{2}(z + z^*)$ and $= \frac{1}{2i}(z - z^*)$.

An function analytic in a region always has an unique derivative in the region.

Regular point: Point where f is analytic.
Singular point: Point where f is not analytic.

If f is analytic in some region (has a first derivative), it has derivatives of all orders in that region.

Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Criteria for a function to be analytic in a region, derived from demanding existence of the derivative.

Harmonic Functions

Harmonic functions are solutions to the **2D Laplace equation**:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

If $f(z) = u(x, y) + iv(x, y)$ is analytic in some region, then $u(x, y)$ and $v(x, y)$ are harmonic functions.

Theorem: Given a harmonic function $u(x, y)$, we can always find it's *harmonic conjugate* $v(x, y)$ such that $f = u + iv$ is an analytic function.

Contour Integrals of Complex Functions

$$\int_\Gamma f(z)\,\mathrm{d}z$$

Upper Bound Estimate of Contour Integral

$$\left| \int_\Gamma f(z)\,\mathrm{d}z \right| \leq M \cdot L$$

where M is the maximum value of $f(z)$ on Γ, and L is the length of Γ.

Independence of Path

If Γ_1 and Γ_2 are two contours that can be continously deformed into one another (without crossing singularities), then

$$\int_{\Gamma_1} f(z)\,\mathrm{d}z = \int_{\Gamma_2} f(z)\,\mathrm{d}z$$

Cauchy's Theorem: As a result, any contour integral that doesn't enclose a singularity, is 0.

Cauchy's Integral Formula

Formula for evaluating the contour integral around a $n+1$'th order pole at z_0.

$$\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Note: Rewrite the expression until it is on the form above. If the contour contains several singularities, rewrite the above expression to handle each of the poles seperately.

Taylor Series

$$f(z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Theorem: If $f(z)$ is analytic in the disk $|z - z_0| \leq R$, then the Taylor series converges for all z *inside* the disk.

Laurent Series

We combine the *Taylor* series with a *Principal* series of negative powers.

$$f(z_0) = \sum_{n=0}^{\infty} a_k (z - z_0)^n - \sum_{n=1}^{\infty} b_k \frac{1}{(z - z_0)^n}$$

- The Taylor series of positive powers converge *inside* some circle $|z - z_0| < R_2$.
- The Principal series of negative powers converge *outside* some circle $R_1 < |z - z_0|$.
- The Laurent series converges in the donut between the two circles, $R_1 < |z - z_0| < R_2$.

Tip: If you only need the series to converge outside/inside some circle, you only need one of the series.

The factor b_0 is called the **residue** of f at z_0.

Finding Laurent Series

- If the Laurent Series should expand from a point $z_0 \neq 0$, you must make a substitution $w = z - z_0$.
- Manipulate the function to the form

  $$f(w) = C(w) \cdot \frac{1}{1 - g}$$

  where g is any factor/power of w, and $C(w)$ is any function of w.
- The Laurent Series is then given as

$$f(w) = C(w) \cdot \frac{1}{1-g} = \begin{cases} C(w) \sum\limits_{n=0}^{\infty} w^n & \text{(Taylor)} \\ -C(w) \sum\limits_{n=1}^{\infty} \frac{1}{w^n} & \text{(Principal)} \end{cases}$$

Singularities

Assume $f(z)$ has an isolated singularity at z_0, and it's Larent series is as given above.

- If all $b_n = 0$, z_0 is a *removable* singularity (not actually a singularity).
- If $b_n \neq 0$ for some n, but zero for all factors above n (such that $(z - z_0)^{-n}$ is the biggest negative power), we say that z_0 is a *pole* of order n.
- If there are infinite negative terms, we say that z_0 is an *essential* singularity.

Residue Theory

Any integral over a contour Γ can be split up into integrals over only infinitesimally small contours around all singularities in Γ.

An contour integral containing N singularities z_k is given as the sum of the residues at all the singularities.

$$\int_{\Gamma} f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} Res(f, z_k)$$

Ways of finding residues

- **Use Laurent Series (always works):** Write out the Laurent Series of the expression around the singularities, and find the b_1 term (the $1/z$ coefficient).
- **For Simple Poles (alt 1):**

  $$Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

- **For Simple Poles (alt 2):**
  If f is a rational function $f(z) = \frac{P(z_0)}{Q(z_0)}$:

  $$Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)}$$

- **For Multiple Poles:** If f has a pole of order m at z_0, and $M \geq m$, then

  $$Res(f, z_0) = \lim_{z \to z_0} \frac{1}{(M-1)!} \frac{\mathrm{d}^{M-1}}{\mathrm{d}z^{M-1}} \left[(z - z_0)^M f(z) \right]$$

Applications to Real Integrals

Type I: Trigonometric integrals over $[0, 2\pi]$

$$\int_0^{2\pi} u(\cos\theta, \sin\theta)\,\mathrm{d}\theta$$

Substitute for

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

The integral is now around a circular contour in the complex plane, centered around $(0,0)$ with radius 1. Evaluate the integral by finding singularities inside the circle and solving for residues.

Type IIa: Rational Functions Over $[-\infty, \infty]$

$$I = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

$$I = 2\pi i \sum Res(f, z_n)$$

where z_n are the singularities in the *upper* plane.

Works if

- $degree(Q) \geq degree(P) + 2$
- f is analytic on and above the complex plane.

Jordan's Lemma

If $m > 0$ is real, and P and Q are polynomials such that P/Q is rational, then:

$$\lim_{\rho\to\infty} \int\limits_{C_\rho} \frac{P(z)}{Q(z)} e^{imz}\,\mathrm{d}z = 0$$

where C_ρ is a half-circle contour with radius ρ

Type IIb: ... with Trigonometric Functions

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(mx)\,\mathrm{d}x \qquad \text{(or sin)}$$

Type III: Singularities on the real axis (Principal Value)

When a real integral passes singularities, we say that the integral is not defined, but it's **principal value** is. It behaves just as an ordinary integral:

$$PV \int_a^b f(x)\,\mathrm{d}x = \lim_{r\to 0^+} \left[\int\limits_a^{C-r} f(x)\,\mathrm{d}x + \int\limits_{C+r}^{b} f(x)\,\mathrm{d}x \right]$$

where C is a singularity. Using the same logic as in Type II, with an added infinite half-circle on the upper plane, this evaluates to

$$PV \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_k Res(f; z_k) + \pi i \sum_j Res(f; z_j)$$

where z_k are singularities in the upper half plane, and z_j are singularities *on* the real axis.

We see that singularities on the exit contribute *half* of those above.

Ordinary Differential Equations

First Order, Linear, ODEs - Integrating Factor

$$y'(x) + P(x)y(x) = Q(x)$$

$$y(x)\mu(x) = \int Q(x)\mu(x)\,\mathrm{d}x + C \qquad \text{with} \qquad \mu(x) = e^{\int P(x)\mathrm{d}x}$$

Homogenous ODEs

$$y'' + P(x)y' + Q(x)y = 0$$

Properties

- Linear combination of solutions is also a solution
- General solution on form $y(x) = c_1y_1(x) + c_2y_2(x)$
- Linearly independent solutions have a Wronskian of 0.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Variation of the constant

If you have one of the two linearly indepedent solutions, you can find the other as $y_2(x) = C(x) \cdot y_1(x)$, where $C(x)$ is a functions determined by inserting $y_2(x)$ into the ODE.

When you arrive at a solution for $C(x)$, you may discard any constants or coefficients, i.e. $C(x) = \alpha x^3 + \beta$.

Constant coefficients - Particular Equation

$$y''(x) + ay'(x) + by(x) = 0$$

Solve the particular equation

$$\lambda^2 + a\lambda + b = 0$$

for λ_1 and λ_2.

Two, real roots

$$y(x) = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$$

One, real root

$$y(x) = (C_1 + xC_2)e^{\lambda x}$$

Two, complex roots

$$\begin{aligned} y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} &= e^{-a/2x}\left[Ae^{i\omega x} + Be^{-i\omega x}\right] \\ &= e^{-a/2x}\left[\hat{A}\cos\omega x + \hat{B}\sin\omega x\right] \end{aligned}$$

Euler-Cauchy Equations

$$x^2y'' + a_1xy' + a_0y = 0$$

Introducing

$$x = e^z \qquad \Rightarrow \qquad z = \ln|x|$$

The equation can be rewritten to

$$\frac{\partial^2 y}{\partial z^2} + (a_1 - 1)\frac{\partial y}{\partial z} + a_0y = 0$$

Solve and insert for z.

Power methods

- Represent $P(x)$ and $Q(x)$ as power series.
- Assume solution on the form
  - $y(x) = \sum_{n=0}^{\infty} a_nx^n$
  - $y'(x) = \sum_{n=1}^{\infty} na_nx^{n-1}$
  - $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_nx^{n-2}$
- Insert back into ODE.
- Split into equations of matching powers of x.

This will give you one or two undetermined coefficients. The equations maybe give the coefficients as a series depending on each other, like $a_{s+1} = a_s^2$. If the series are odd/even, two undetermined coefficients are required to describe them, so the solution is complete.

Fröbenius method

$$x^2y'' + xb(x)y' + c(x)y = 0$$

Assuming solution on form

$$y_p(x) = \sum_{m=0}^{\infty} a_mx^{m+s}$$

where s is some real number determined the *indicial equation*

$$s(s-1) + b_0s + c_0 = 0$$

where $b_0 = b(0)$, and $c_0 = c(0)$

Three possible scenarios:

- Different roots, $s_1 \neq s_2$, and $s_1 - s_2 \neq$ integer.
  - Two indepedens solutions $y_i(x) = x^{si}\sum_{m=0}^{\infty} a_0x^m$
- Different roots, $s_1 \neq s_2$, but $s_1 - s_2 =$ integer. ($s_1 > s_2$).
  - Solve for both by Power Series.
  - Often, s_2 gives the complete solution (two undetermined coefficients), so try this first.
  - Sometimes, only s_1 gives a solution. Find the other by variation of the constant.
- Double root, $s_1 = s_2$.
  - Find the solution by Power Series.
  - Find the second solution by variation of the constant.

Inhomogenous ODEs

$$y'' + P(x)y' + Q(x)y = R(x)$$

Remember to always rewrite to this form.

Properties

- Solutions on form
  $y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$.
- $y_h(x)$ is the solution to the homogenous equation.
- $y_p(x)$ is *any* solution to the whole ODE.
- Since y_h contains two arbitrary constants, y_p should contain none.

Inhomo ODEs with constant coefficients

$$y'' + ay' + by = R(x)$$

- Make a guess at y_p with the same form as $R(x)$, with unknown coefficients.
- Insert back into ODE to solve for coefficients.

Special case: $R(x) = Ae^{kx}$.
Let α and β be the roots of $\lambda^2 + a\lambda + b = 0$.

1. If $k \neq \alpha, \beta$: Try $y_p = Ce^{\lambda x}$.
2. If $k = \alpha$ or β: Try $y_p = Cxe^{\lambda x}$.
3. If $k = \alpha = \beta$: Try $y_p = Ce^{\lambda x}$.

Inhomo ODEs with varying coefficients

$$y'' + P(x)y' + Q(x)y = R(x)$$

Factorization

If $u(x)$ is a known solution to the homo-ODE, a particular solution is $y_p = u(x) \cdot v(x)$ where

$$w' = v \qquad w' + \left[\frac{2u'}{u} + P\right] w = \frac{R}{u}$$

Solve the ODE for w with integrating factor.

Variation of parameters

$$y_p = -y_1 \int \frac{y_2 R}{W} \,\mathrm{d}x + y_2 \int \frac{y_1 R}{W} \,\mathrm{d}x$$

where y_1 and y_2 are known linearly indepedendent solutions to the homo-ODE. **NOTE:** Remember that $R(x)$ is the RHS after the ODE is rewritten on the standard form.

Trigonometric Functions

Usefull Shit

- recognize **odd** and **even** integrands. I.e $\int_{-\infty}^{\infty} \sin x/x^2 = 0$ due to odd, and $\int_{-\infty}^{\infty} \cos x/(1+x^2) = 2\int_{-\infty}^{\infty} \cos x/(1+x^2)$ due to even.

Orthogonality

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}$$

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}$$

Even and Odd functions

If $f(x)$ is **even** $[f(x) = f(-x)]$:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = 0$$

If $f(x)$ is **odd** $[f(x) = -f(-x)]$:

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Dirichlet Conditions for Fourier Series

1. Finite number of min/max in interval.
2. Finitite number of (only) finite discontinuities.

If this holds, then the series will converge to $f(x)$ at all points. At discontinuities, the series will converge to the mid-point.

Parseval's Theorem

$$\int_{-L}^{L} |f(x)|^2\,\mathrm{d}x = 2L\sum_{-\infty}^{\infty} |c_n|^2$$

Fourier Transforms

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,\mathrm{d}k \qquad F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$$

Odd and even functions

If $f(x)$ is an odd function, $f(x) = -f(-x)$, the Fourier transform can be done using only sine (as cosine is symmetric around 0):

$$f(x) = \sqrt{\frac{2}{\pi}}i\int\limits_0^{\infty} F(k)\sin(kx)\,\mathrm{d}k \quad F(k) = \sqrt{\frac{2}{\pi}}i\int\limits_0^{\infty} f(x)\sin(kx)\,\mathrm{d}x$$

If $f(x)$ is even, $f(x) = f(-x)$, we need only cosine (as sine is anti-symmentric):

$$f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} F(k)\cos(kx)\,\mathrm{d}k \quad F(k) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} f(x)\cos(kx)\,\mathrm{d}x$$

FT of a derivative

$$\mathcal{F}\left[f^{(n)}(x)\right] = (ik)^n\mathcal{F}[f(x)]$$

Partial Differential Equations

Notes

- In symentrical systems, it seems your can switch x<->y if it is required to suit boundary conditions (example: Diritchlet conditions are at x=a instad of at y=b).
- When resulting in cos/sin solutions of frequencies, include n=0 for cos, as it gives a non-zero solution, but not for sin.

Separation of variables

1) The the solution function as a product of functions of each variable, i.e:

$$u(x,y) = X(x)Y(y) \qquad u(r,\theta) = R(r)T(\theta)$$

2) Insert this into the equation, and separate the equation into parts of only each variable. Each side must then be constant, and equals some *separation constant.*

3) Solve each side of the equation (equaling the seperation constant), giving an infinite set of *eigenfunctions*, $u_n(x,y)$ for the equation.

4) The final solution is a linear combination of the eigenfunctions.

$$u(x,y) = \sum_{n=-\infty}^{\infty} c_n u_n(x,y)$$

Laplace Equation - 2D Cartesian

$$\nabla u(x,y) = 0$$

Separation of variables, $u(x,y) = X(x)Y(y)$ gives solutions

$$u(x,y) = X(x)Y(y) = \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix} \times \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix}$$

Diritchlet BC: $u(x,0) = u(0,y) = u(a,y) = 0$, $u(x,b) = f(x)$
Eigenfunctions on form

$$u_n(x,y) = A_n F_n(x) G_n(x) = A_n \sin(n\pi x/a) \sinh(n\pi y/a)$$

Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Seperation of variables, $u(x,t) = F(x)G(t)$ gives equations

$$F''(x) = -k^2 F(x) \qquad \ddot{G}(t) = -k^2 c^2 G(t)$$

where the seperation constant, $-k^2$ must be negative, else the solutions would blow up.

The equations has solutions on the form

$$y(x,t) = \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} \times \begin{Bmatrix} \sin(kct) \\ \cos(kct) \end{Bmatrix}$$

The end-points are usually fixed at 0, leaving only the $\sin(kt)$ term, and forcing $k = n\pi/L$.

If the velocity is 0 at $t = 0$, we discard the sin-velocity term and get

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right)$$

If position is 0 at $t = 0$, we discard the cos-velocity term instead.

Initial position will be on the form

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

where $f(x)$ is the initial position or velocity. The coefficients b_n are now Fourier coefficients, given as:

$$b_n = \frac{1}{2L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

Orthogonal Functions

Functions on the form

$$p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0$$

on some interval $[a,b]$ has solutions as linear combinations of eigenfunctions $y_n(x)$ which are orthogonal with respect to $r(x)$ such that

$$\int_a^b r(x) y_n(x) y_m(x)^* \,\mathrm{d}x = 0 \qquad \text{for} \lambda_n \neq \lambda m$$

Any function can be written as a linear combination of these eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

then the set $\{y_n(x)\}$ is complete. The coefficients a_n are determined by the orthogonality:

$$a_n = \int_a^b f(x) r(x) y_n(x)^* \,\mathrm{d}x$$