Ordinary Differential Equations

First Order, Linear, ODEs - Integrating Factor

$$y'(x) + P(x)y(x) = Q(x)$$

$$y(x)\mu(x) = \int Q(x)\mu(x)\,\mathrm{d}x + C \qquad \text{with} \qquad \mu(x) = e^{\int P(x)\mathrm{d}x}$$

$$\int (uv') = uv - \int (u'v)$$

Second Order, Homogenous, Linear ODEs, with constant coefficients - Particular Equation

$$y''(x) + ay'(x) + by(x) = 0$$

Solve the particular equation

$$\lambda^2 + a\lambda + b = 0$$

for λ_1 and λ_2.

Two, real roots

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

One, real root

$$y(x) = (C_1 + xC_2)e^{\lambda x}$$

Two, complex roots

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = e^{-a/2x}\left[Ae^{i\omega x} + Be^{-i\omega x}\right] = e^{-a/2x}\left[\hat{A}\cos\omega x + \hat{B}\sin\omega x\right]$$

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}$$

Even and Odd functions

If $f(x)$ is **even**:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = 0$$

If $f(x)$ is **odd**:

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Fourier Transforms

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,\mathrm{d}k \qquad F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$$

Odd and even functions

If $f(x)$ is an odd function, $f(x) = -f(-x)$, the Fourier transform can be done using only sine (as cosine is symmetric around 0):

$$f(x) = \sqrt{\frac{2}{\pi}}i\int\limits_0^{\infty} F(k)\sin(kx)\,\mathrm{d}k \quad F(k) = \sqrt{\frac{2}{\pi}}i\int\limits_0^{\infty} f(x)\sin(kx)\,\mathrm{d}x$$

If $f(x)$ is even, $f(x) = f(-x)$, we need only cosine (as sine is anti-symmentric):

$$f(x) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} F(k)\cos(kx)\,\mathrm{d}k \quad F(k) = \sqrt{\frac{2}{\pi}}\int\limits_0^{\infty} f(x)\cos(kx)\,\mathrm{d}x$$

FT of a derivative

$$\mathcal{F}\left[f^{(n)}(x)\right] = (ik)^n\mathcal{F}[f(x)]$$

Partial Differential Equations

Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$$

Solution as linear combination of the the solutions

$$y(x,t) = \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} \times \begin{Bmatrix} \sin(kvt) \\ \cos(kvt) \end{Bmatrix}$$

The end-points are usually fixed at 0, leaving only the $\sin(kt)$ term, and forcing $k = n\pi/L$.

If the velocity is 0 at $t = 0$, we discard the sin-velocity term and get

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)$$

If position is 0 at $t = 0$, we discard the cos-velocity term instead.

Initial position will be on the form

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

where $f(x)$ is the initial position. The coefficients b_n are now Fourier coefficients, given as:

$$b_n = \frac{1}{2L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$