FYS3110 – Home Exam

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March 21, 2018

Problem 1

a)

We use the generalized coordinate ϕ and it's derivative $\dot{\phi}$, which, together with time and an initial condition, uniquely defines the system.

The rod has negligible mass, meaning the system's kinetic energy comes from the wheel. The center of mass oscilates with a velocity $v = b\dot{\phi}$. This gives a kinetic energi

$$K_1 = \frac{1}{2}mv^2 = \frac{1}{2}mb^2\dot{\phi}^2$$

In addition, we have the kinetic energy from the rotation of the wheel around it's own axis, given as

$$K_2 = \frac{1}{2}I\omega^2 = \frac{1}{2}I(\dot{\phi} + \alpha t)^2$$

This gives a total kinetic energy of

$$K = K_1 + K_2 = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2$$

Since we are in a gravitational field, we have a potential energy from the height of the center of mass B (here, in relation to the point A):

$$V = -mgh = -mgb\cos\phi$$

This gives a total lagrangian of

$$L = K - V = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi$$
$$= \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I\dot{\phi}^2 + I\alpha\dot{\phi}t + \frac{1}{2}I\alpha^2 t^2 + mgb\cos\phi$$

$$L(\phi, \dot{\phi}, t) = \frac{1}{2}\left(mb^2 + I\right)\dot{\phi}^2 + I\alpha\dot{\phi}t + mgb\cos\phi + \frac{1}{2}I\alpha^2 t^2 \tag{1}$$

b)

We have that

$$\frac{\partial L}{\partial \phi} = -mgb\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = \left(mb^2 + I\right)\dot{\phi} + I\alpha t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = \left(mb^2 + I\right)\ddot{\phi} + I\alpha$$

This gives Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\left(mb^2 + I\right)\ddot{\phi} + I\alpha + mgb\sin\phi = 0$$

c)

We have that

$$\frac{\mathrm{d}f(\phi, t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[I\alpha\phi t + \frac{1}{6}I\alpha^2 t^3\right]$$

$$= I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2$$

We reconize the first and last term from the Lagrangian (1), and include the middle term by adding and subtracting it from the Lagrangian, giving

$$L = \left[\frac{1}{2}\left(mb^2 + I\right)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi\right] + \left[I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2\right]$$

$$= L'(\phi, \dot{\phi}) + \frac{\partial f(\phi, t)}{\partial t}$$

where

$$L'(\phi, \dot{\phi}) = \frac{1}{2}\left(mb^2 + I\right)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi$$

We have successfully split our Lagrangian up into a Lagrangian explicitly independent of time, and a total time derivative.

d)

We know that adding a total time derivative to our Lagrangian does not change the equations of motion for the system. We therefore expect the e.o.m. of L and L' to be the same.

We have that

$$\frac{\partial L'}{\partial \phi} = -mgb\sin\phi - I\alpha$$

$$\frac{\partial L'}{\partial \dot{\phi}} = (mb^2 + I)\dot{\phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{\phi}} = (mb^2 + I)\ddot{\phi}$$

This gives Lagrange's equation

$$(mb^2 + I)\ddot{\phi} + mgb\sin\phi + I\alpha = 0$$

which confirms our expectations.

e)

The canonical momentum p'_ϕ of the coordinate ϕ is already calculated as

$$p'_\phi = \frac{\partial L'}{\partial \dot{\phi}} = (mb^2 + I)\dot{\phi}$$

The Hamiltonian is defined as (where q_i are the generalized coordinates)

$$H'(\phi, \dot{\phi}) = \sum_i p_i \dot{q}_i - L = p'_\phi \dot{\phi} - L'$$

$$= (mb^2 + I)\dot{\phi}^2 - \frac{1}{2}\left(mb^2 + I\right)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi$$

$$= \frac{1}{2}\left(mb^2 + I\right)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi$$

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To get H' as a function of ϕ and p'_ϕ only, we use to relation

$$p'_\phi = (mb^2 + I)\dot{\phi} \qquad \Rightarrow \qquad \dot{\phi} = \frac{1}{mb^2 + I}p'_\phi$$

to rewrite H' as

$$H'(\phi, p'_\phi) = \frac{1}{2}\left(mb^2 + I\right)\frac{1}{(mb^2 + I)^2}p'^2_\phi - mgb\cos\phi + I\alpha\phi$$

$$= \frac{1}{2}\frac{p'^2_\phi}{mb^2 + I} - mgb\cos\phi + I\alpha\phi$$

f)

g)

Substituting $\alpha = \frac{mgb}{I}\lambda$, where λ is a dimensionless variable for the angular acceleration of the wheel, gives

$$H'(\phi, p'_\phi) = \frac{1}{2}\frac{p'^2_\phi}{mb^2 + I} - mgb\cos\phi + mgb\lambda\phi$$

For simplicity we set the parameters $m = 1$, $b = 1$, $I = 1$, $g = 9.81$. With a simple Python script we can produce the countour plots in figure (1), outlining the movement of the pendulum.

At $\lambda = 0$ (no angular acceleration on the wheel), we retrieve the regular pendulum phase space, without energy loss. The circular solutions represents pendulums oscilating in the common way, while the wobbly lines represent pendulums with enough kinetic energy to take full turns.

For $0 < \lambda \leq 1$ the circular solutions remain (although slightly shifted in angle, no longer oscilating around $\phi = 2\pi n$). The "free" solutions however, will now at some point either slow down and turn, or keep accelerating, depending on its initial direction. We can observe this by following a "free" contour line close to the circles, and observe it eventually turn.

When we reach $\lambda = 1$ and above, the pendulum will turn, no matter it's initial conditions. In the high-accuracy plot (2), we observe that the last circular solutions dissapear at $\lambda = 1$, and the pendulum will always accelerate.

It it obvious from there observations that the accelerating wheel applies a force on the pendulum, accelerating the pendulum's rotation around its fixed point.
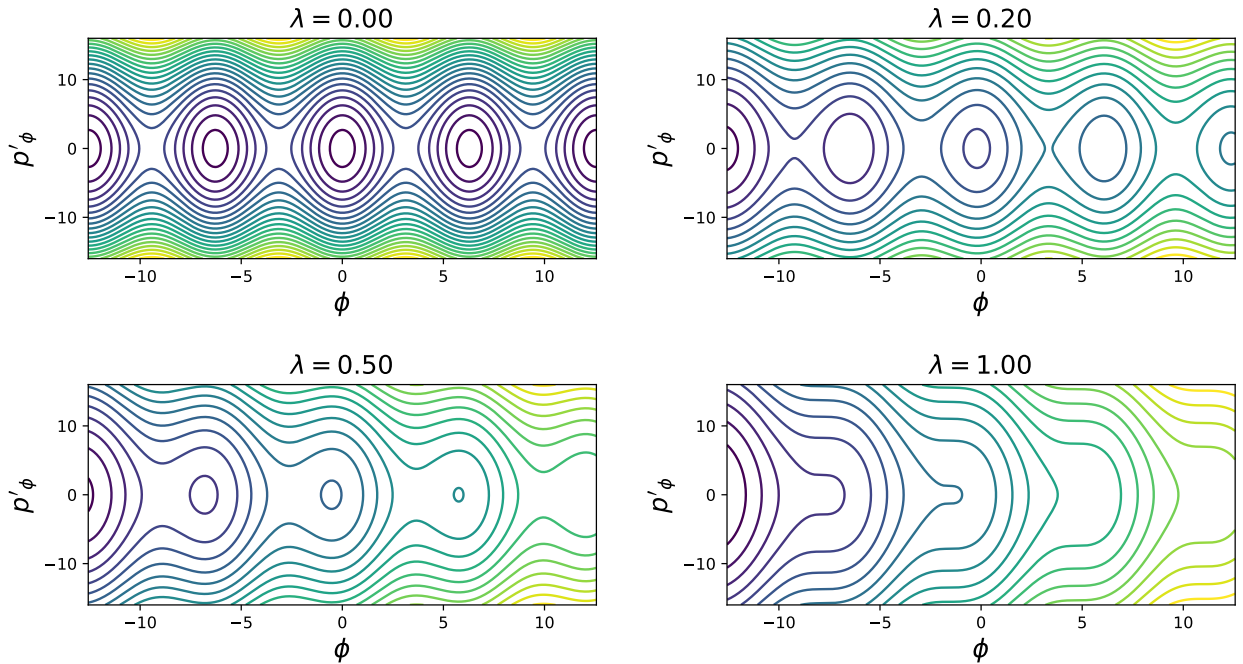

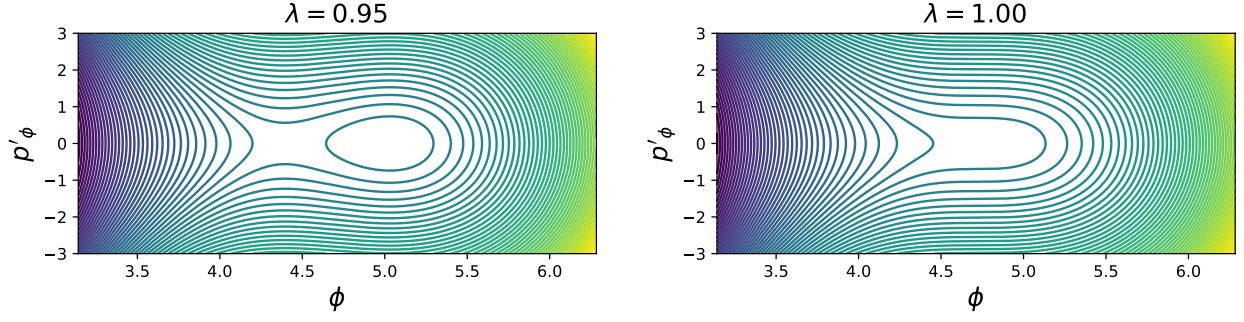Figure 1: Contour plots of $H'(\phi, p'_\phi)$

Figure 2: Higher accuracy contour plots of $H'(\phi, p'_\phi)$

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a)

b)

Conservation of energy gives that particle A and a must have a combined energy equal to that of particle B before the decay.

$$E_B = E_A + E_a$$

$$\sqrt{p_B^2 c^2 + m_B^2 c^4} = \sqrt{p_A^2 c^2 + m_A^2 c^4} + \sqrt{p_a^2 c^2 + m_a^2 c^4}$$

In the rest frame of A, $p_A = 0$, and conservation of momentum gives that $p_a^2 = p_B^2$ (total momentum before decay must equal total momentum after decay). This gives

$$\sqrt{p_a^2 + m_B^2 c^2} = \sqrt{m_A^2 c^2} + \sqrt{p_a^2 + m_a^2 c^2}$$

$$\left(\sqrt{p_a^2 + m_B^2 c^2}\right)^2 = \left(\sqrt{m_A^2 c^2} + \sqrt{p_a^2 + m_a^2 c^2}\right)^2$$

$$p_a^2 + m_B^2 c^2 = m_A^2 c^2 + p_a^2 + m_a^2 c^2 + 2\sqrt{m_A^2 c^2}\sqrt{p_a^2 + m_a^2 c^2}$$

$$\sqrt{p_a^2 + m_a^2 c^2} = \frac{m_B^2 c^2 - m_a^2 c^2 - m_A^2 c^2}{2 m_A c}$$

$$p_a^2 + m_a^2 c^2 = \frac{\left(m_B^2 c^2 - m_a^2 c^2 - m_A^2 c^2\right)^2}{4 m_A^2 c^2}$$

$$p_a^2 + m_a^2 c^2 = \frac{m_B^4 c^4 + m_a^4 c^4 + m_A^4 c^4 - 2 m_B^2 m_a^2 c^4 - 2 m_B^2 m_A^2 c^4 + 2 m_a^2 m_A^2 c^4}{4 m_A^2 c^2}$$

$$p_a = \frac{\sqrt{m_B^4 c^4 + m_a^4 c^4 + m_A^4 c^4 - 2 m_B^2 m_a^2 c^4 - 2 m_B^2 m_A^2 c^4 + 2 m_a^2 m_A^2 c^4 - 4 m_a^2 m_A^2 c^4}}{2 m_A c}$$

$$p_a = c \frac{\sqrt{m_B^4 + m_a^4 + m_A^4 - 2 m_B^2 m_a^2 - 2 m_B^2 m_A^2 - 2 m_a^2 m_A^2}}{2 m_A} \tag{2}$$

c)

We have the invariant mass given as

$$m_{ab}^2 c^2 = (p_a + p_b)^2 = (p_a + p_b)^\mu (p_a + p_b)_\mu$$

We write out the 4-vector with energy and regular momentum components

$$(p_a + p_b)^\mu = \left(\frac{E_a + E_b}{c}, \; [\vec{p_a} + \vec{p_b}] \right)$$

$$(p_a + p_b)_\mu = \left(\frac{E_a + E_b}{c}, \; -[\vec{p_a} + \vec{p_b}] \right)$$

which gives the invariant mass

$$m_{ab}^2 c^2 = \frac{(E_a + E_b)^2}{c^2} - [\vec{p_a} + \vec{p_b}]^2$$

Inserting the relativistic energy (with $m_a = m_b = 0$)

$$E_a = \sqrt{p_a^2 c^2 + m_a^2 c^4} = p_a c$$

$$E_b = \sqrt{p_b^2 c^2 + m_b^2 c^4} = p_b c$$

gives

$$m_{ab}^2 c^2 = \frac{(p_a c + p_b c)^2}{c^2} - [\vec{p_a} + \vec{p_b}]^2$$
$$= p_a^2 + p_b^2 + 2p_a p_b - \vec{p_a}^2 - \vec{p_b}^2 - 2\vec{p_a} \cdot \vec{p_b}$$

The squared vectors simply become their magnitudes squared ($\vec{p}^2 = p^2$), while the dot product can be written $\vec{p_a} \cdot \vec{p_b} = p_a p_b \cos\theta_{ab}$, where θ_{ab} is the angle between the vectors. This gives

$$m_{ab}^2 c^2 = 2p_a p_b - 2p_a p_b \cos\theta_{ab} = 2p_a p_b (1 - \cos\theta_{ab})$$

$$m_{ab}^2 = \frac{2}{c^2} p_a p_b (1 - \cos\theta_{ab}) \tag{3}$$

We observe that the decay $C \to bB$ from the refrence frame of B is identical to the decay $B \to aA$ from the frence frame of A, solved in the last exercise. We therefore borrow equation (2), substituting (B, a, A) with (C, b, B). Setting $m_b = 0$ gives

$$p_b = c \frac{\sqrt{m_C^4 + m_B^4 - 2m_C^2 m_B^2}}{2m_B} = c \frac{\sqrt{(m_C^2 - m_B^2)^2}}{2m_B} = c \frac{m_C^2 - m_B^2}{2m_B} \tag{4}$$

We now need an expression for p_a.

Looking at the second decay from the refrence frame of B, we get the slightly different energy conservation, with $p_A^2 = p_a^2$ and $p_B = 0$. We also set $m_a = 0$.

$$E_B = E_A + E_a$$
$$\sqrt{p_B^2 c^2 + m_B^2 c^4} = \sqrt{p_A^2 c^2 + m_A^2 c^4} + \sqrt{p_a^2 c^2 + m_a^2 c^4}$$
$$\sqrt{p_B^2 + m_B^2 c^2} = \sqrt{p_A^2 + m_A^2 c^2} + \sqrt{p_a^2 + m_a^2 c^2}$$
$$m_B c = \sqrt{p_a^2 + m_A^2 c^2} + p_a$$
$$(m_B c - p_a)^2 = p_a^2 + m_A^2 c^2$$
$$m_B^2 c^2 + p_a^2 - 2m_B p_a c = p_a^2 + m_A^2 c^2$$

$$p_a = c \frac{m_B^2 - m_A^2}{2m_B} \tag{5}$$

Inserting this into equation (3) gives

$$m_{ab}^2 = \frac{2}{c^2} c \frac{m_B^2 - m_A^2}{2m_B} c \frac{m_C^2 - m_B^2}{2m_B} (1 - \cos\theta_{ab}) = \frac{(m_B^2 - m_A^2)(m_C^2 - m_B^2)}{2m_B^2} (1 - \cos\theta_{ab})$$