# Problem 1

$$y''(x) + \frac{3}{x}y'(x) - \frac{24}{x^2}y(x) = 56x^6 \tag{1}$$

## Homogeneous solution

The homogenous solution is on the form

$$y'' + \frac{3}{x}y' - \frac{24}{x^2}y = 0 \tag{2}$$
$$x^2y''(x) + 3xy' - 24y = 0 \tag{3}$$

which we recognize as an Euler-Cauchy equation.

We substitute $x = e^z$, and get the equation

$$y''(z) + (3-1)y'(z) - 24y = 0 \tag{4}$$

Which is an ordinary linear first order equation with constant coefficients. The characteristic equation is

$$\lambda^2 + 2\lambda - 24 = 0 \tag{5}$$

The solutions are

$$\lambda_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4-(4\cdot -24)}\right] = \frac{1}{2}\left[-2 \pm \sqrt{100}\right] = -1 \pm 5 \tag{6}$$

giving $\lambda_+ = 4$ and $\lambda_- = -6$.

Since these are two real roots, the solution is on the form

$$y_h(z) = C_1e^{4z} + C_2e^{-6z} \tag{7}$$

inserting for $z = \ln|x|$ gives

$$y_h(x) = C_1e^{4\ln|x|} + C_2e^{-6\ln|x|} = C_1e^{\ln|x^4|} + C_2e^{\ln|x^{-6}|} = C_1x^4 + C_2x^{-6} \tag{8}$$

## Particular solution

We can use variation of parameters to find the particular solution. Firstly, we have two chosen solutions of the homogeneous solution:

$$y_1 = x^4 \qquad y_2 = x^{-6} \tag{9}$$

and the right hand side $R = 56x^6$. We find the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 = x^4(-6x^{-7}) - 4x^3x^{-6} = -10x^{-3} \tag{10}$$

We then have the particular solution

$$y_p(x) = -x^4\int \frac{x^{-6}56x^6}{-10x^{-3}}\,\mathrm{d}x + x^{-6}\int \frac{x^4 56x^6}{-10x^{-3}}\,\mathrm{d}x \tag{11}$$
$$= -x^4\int \frac{28}{5}x^3\,\mathrm{d}x + x^{-6}\int \frac{28}{5}x^{13}\,\mathrm{d}x \tag{12}$$
$$= -x^4\frac{28}{5}\left[\frac{x^4}{4}\right] - x^{-6}\frac{28}{5}\left[\frac{x^{14}}{14}\right] = \frac{7}{5}x^8 - \frac{2}{5}x^8 = x^8 \tag{13}$$

giving a total solution

$$y(x) = y_h(x) + y_p(x) = C_1x^4 + C_2x^{-6} + x^8 \tag{14}$$

# Problem 2: Part A

## a)

$$f(z)=\frac{(z-2i)^4}{(z-[3+i])^3} \tag{15}$$

## b)

The function can be factorized into

$$\frac{(z+2)(z+1+i\sqrt{3})(z-1-i\sqrt{3})}{(z-5)^3(z+2)}=\frac{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}{(z-5)^3} \tag{16}$$

As we see, the suspected singularity at $z=2$ does not actually exist.

The other singularity, at $z=5$, is a pole of 3rd order.

# Problem 2: Part B

## a)

We write out

$$g(z)=\frac{P(z)}{Q(z)}\cdot\pi\cot(\pi z)=\frac{P(z)}{Q(z)}\cdot\frac{\pi\cos(\pi z)}{\pi\sin(\pi z)}=\frac{P(z)\cos(\pi z)}{Q(z)\sin(\pi z)} \tag{17}$$

Equation (6.2) from Boas gives us that

$$Res(g;n)=\frac{P(n)\cos(\pi n)}{[Q(n)\sin(\pi n)]'} \tag{18}$$

which holds as long as $P(n)\cos(\pi n)$ is finite, $Q(n)\sin(\pi n)=0$, and $[Q(n)\sin(n)]'\neq 0$ which all holds. By the product rule, we get

$$Res(g;n)=\frac{P(n)\cos(\pi n)}{Q'(n)\sin(\pi n)+Q(n)\cos(\pi n)} \tag{19}$$

Since $n\in[0,\ \pm1,\ \pm2,...]$, we have that $\sin(\pi n)=0$ and $\cos(\pi n)=1$, giving

$$Res(g;n)=\frac{P(n)\cos(\pi n)}{Q(n)\cos(\pi n)}=\frac{P(n)}{Q(n)}=f(n) \tag{20}$$

## b)

Firstly, we will show that

$$\lim_{N\to\infty}\int\limits_{\Gamma_N}F(z)\,\mathrm{d}z=0 \tag{21}$$

for some rational function $F(z)=P(z)/Q(z)$ when $Q(z)$ is a polynomial of at least 2 degrees higher than $P(z)$.

We have the relation

$$\left|\int\limits_{\Gamma_N}F(z)\,\mathrm{d}z\right|\leq m\cdot L \tag{22}$$

where $m$ is the maximum value of $F(z)$ on $\Gamma$, and $L$ is the length of the contour $\Gamma$.

This means that

$$\lim_{N\to\infty}\left|\int\limits_{\Gamma_N}F(z)\,\mathrm{d}z\right|\leq\lim_{N\to\infty}m\cdot L \tag{23}$$

The length of the contour, $L$, will be go as $\propto N$, since it's a square with sides proportional to $N$. The max of the function, $m$, will go as $\propto \frac{1}{z^2} \propto \frac{1}{N^2}$, since $F(z) = \frac{P(z)}{Q(z)}$, where $degree(Q) \geq degree(P) + 2$. This means that $m \cdot L \propto \frac{1}{N}$, giving that

$$\lim_{N\to\infty} m \cdot L = 0 \tag{24}$$

such that

$$\lim_{N\to\infty} \left| \int\limits_{\Gamma_N} F(z) \right| \leq \lim_{N\to\infty} m \cdot L = 0 \tag{25}$$

meaning the relation 21 holds.

Since $|\pi \cot(\pi z)| \leq M$ over all $z$ for some constant $M$, we have the relation

$$\left| \lim_{N\to\infty} \int\limits_{\Gamma_N} f(z)\pi \cot(\pi z)\, \mathrm{d}z \right| \leq \left| \lim_{N\to\infty} \int\limits_{\Gamma_N} f(z)M\, \mathrm{d}z \right| \tag{26}$$

where $f(z)M$ now is a rational function of the type we defined in 21 (since adding a constant M doesn't change degrees). We therefore have

$$\left| \lim_{N\to\infty} \int\limits_{\Gamma_N} g(z)\, \mathrm{d}z \right| \leq \left| \lim_{N\to\infty} \int\limits_{\Gamma_N} f(z)M\, \mathrm{d}z \right| = 0 \tag{27}$$

or

$$\lim_{N\to\infty} \int\limits_{\Gamma_N} g(z)\, \mathrm{d}z = 0 \tag{28}$$

## c)

The residue theorem states that

$$\int\limits_{\Gamma_N} g(z)\, \mathrm{d}z = 2\pi i \sum Res(g(z); z_i) \tag{29}$$

where $z_i$ are all the singularities of $g(z)$. We can split the residues up into the singularities of $f(z)$ and those of $\pi \cot(\pi z)$, and let $N \to \infty$

$$\lim_{N\to\infty} \int\limits_{\Gamma_N} g(z)\, \mathrm{d}z = \lim_{N\to\infty} 2\pi i \Big[ \sum Res(g(z); n) + \sum Res(g(z);\ \text{poles of } f(z)) \Big] \tag{30}$$

We know from 28 that this equals 0. Also inserting $\sum Res(g(z); n) = f(n)$ gives

$$\lim_{N\to\infty} 2\pi i \left[ \sum_{n=-N}^{N} f(n) + \sum Res(g(z);\ \text{poles of } f(z)) \right] = 0 \tag{31}$$

$$\sum_{n=-N}^{N} f(n) = -\sum Res(g(z);\ \text{poles of } f(z)) \tag{32}$$

## d)

We have the function

$$f(n) = \frac{1}{1+n^2} = \frac{1}{(n+i)(n-1)} \tag{33}$$

with simple poles at $n = \pm i$. Using 32, we have that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{1+n^2} = -\sum (\text{Residues of } g(z) \text{at the poles of } f(z)) \tag{34}$$

where

$$g(n) = \pi \cot(\pi n) \frac{1}{(n+i)(n-i)} \tag{35}$$

The residues can be found as

$$Res(g(n); n) = \lim_{z \to z_0} g(z_0) \tag{36}$$

This gives the residues

$$Res(g(n); i) = \lim_{z \to i} \left[ (n-i)\pi \cot(i\pi) \frac{1}{(n+i)(n-i)} \right] = \frac{\pi \cot(i\pi)}{2i} = -\frac{1}{2}\pi \coth(\pi) \tag{37}$$

where we used the relation $\cot(i\pi) = -i \coth(i)$.

$$Res(g(n); -i) = \lim_{z \to -i} \left[ (n+i)\pi \cot(-i\pi) \frac{1}{(n+i)(n-i)} \right] = \frac{\pi \cot(-i\pi)}{-2i} = -\frac{1}{2}\pi \coth(\pi) \tag{38}$$

This gives

$$\lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{1+n^2} = -\sum (\text{Residues of } g(z) \text{at the poles of } f(z)) = \pi \coth \pi \tag{39}$$

# Problem 3

## a)

We will prove the relation

$$\delta[f(t)] = \sum_i \frac{1}{|f'(t_i)|} \delta[t - t_i] \tag{40}$$

by massaging the left hand side into the right. Since this is an operator, we apply a test-function $g(t)$, and integrate over all $t$. We will remove these at the end.

$$= \int_{-\infty}^{\infty} \delta[f(t)] \cdot g(t) \, \mathrm{d}t \tag{41}$$

$\delta[f(t)]$ will be a series of delta-spikes at the zero-points of $f$, $f(t_i)$, and zero everywher else. We can therefore rewrite the integral to a sum of integrals around each of the zero-points $t_i$.

$$= \sum_i \int_{t_i - \epsilon}^{t_i + \epsilon} \delta[f(t)] \cdot g(t) \, \mathrm{d}t \tag{42}$$

We Taylor-expand $f(t)$ around each zero-point $t_i$:

$$f(t) = \underbrace{f(t_i)}_{=0} + f'(t_i)(t - t_i) + \mathcal{O}(t^2) \approx f'(t_i)(t - t_i) \tag{43}$$

Inserting this gives

$$\sum_i \int_{t_i - \epsilon}^{t_i + \epsilon} \delta[f'(t_i)(t - t_i)] g(t) \, \mathrm{d}t \tag{44}$$

Using the identity $\delta[\alpha x] = \frac{1}{|\alpha|} x$ we get

$$\sum_i \int_{t_i - \epsilon}^{t_i + \epsilon} \frac{1}{|f'(t_i)|} \delta[t - t_i] g(t) \, \mathrm{d}t \tag{45}$$

This is now a sum of integrations over delta-spikes (multiplied by some stuff). We can rewrite this as a integral over the sum of all the delta-spikes by switching the order of the sum and the integral. Since the integrand is zero outside the proximity of the $t_i$'s anyway, we can just let the integral run from $-\infty$ to $\infty$:

$$\int_{-\infty}^{\infty} \sum_i \frac{1}{|f'(t_i)|}\delta[t-t_i]g(t)\,\mathrm{d}t \tag{46}$$

Finally, removing the integral and test-function, we get the left hand side of equation 40

$$\sum_i \frac{1}{|f'(t_i)|}\delta[t-t_i] \tag{47}$$

and our identity is proven.

## b)

### (i)

We have that

$$f(t) = t^2 - a^2 \quad \Rightarrow \quad f'(t_i) = 2t_i \tag{48}$$

The functions has zero-points at

$$t_1 = a \quad \text{and} \quad t_2 = -a \tag{49}$$

We then have that

$$\delta(t^2 - a^2) = \sum_i \frac{1}{|2t_i|}\delta[t-t_i] = \frac{1}{|2\cdot 1|}\delta[t-a] + \frac{1}{|2\cdot -1|}\delta[t--a] \tag{50}$$

$$= \frac{1}{2}\delta[t-a] + \frac{1}{2}\delta[t+a] \tag{51}$$

### (ii)

We have that

$$f(t) = \sin(t) \quad \Rightarrow \quad f'(t_i) = \cos(t_i) \tag{52}$$

The functions has zero-points at

$$t_i = \pi i \quad \text{for} \quad i \in [0, \pm 1, \pm 2, ...] \tag{53}$$

This gives

$$\delta[\sin(t)] = \sum_{i=-\infty}^{\infty} \frac{1}{|\cos(t_i)|}\delta[t-\pi i] = \sum_{i=-\infty}^{\infty} \delta[t-\pi i] \tag{54}$$

since $|\cos(\pi i)| = 1$ for $i \in [0, \pm 1, \pm 2, ...]$

## d)

Inserting $\delta[\sin(t)]$ from equation 54 gives

$$\int_{-\pi/2}^{\pi/2} \cos(t)\delta[\sin(t)]\,\mathrm{d}t = \int_{-\pi/2}^{\pi/2} \left[\cos(t) \sum_{i=-\infty}^{\infty} \delta[t-\pi i]\,\mathrm{d}t\right] \tag{55}$$

Since the integral only runs from $-\pi/2$ to $\pi/2$, the values of the integrand outside these limits are irrelevant. The delta-functions only has values at it's zero-points. The only value of $i$ that makes the delta-function fall inside the limits is $i = 0$.[1] We can therefore replace the sum by only the single delta function $\delta[t]$.

The integral then becomes

$$= \int_{-\pi/2}^{\pi/2} \cos(t)\delta[t]\,\mathrm{d}t = \cos(0) = 1 \tag{56}$$

[1]The two neighbouring values, $i = 1$ and $i = -1$ will make the spikes fall at $t = \pi$ and $t = -\pi$, which are both outside the limits.