FYS3110 – Home Exam

Candidate Number 15020

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Problem 1

a)

We use the generalized coordinate ϕ and it's derivative $\dot{\phi}$, which, together with time, uniquely defines the system.

The rod has negligible mass, meaning the system's kinetic energy comes from the wheel. The center of mass, B, rotates around the point A with a velocity $v = b\dot{\phi}$. This gives a kinetic energy

$$K_1 = \frac{1}{2}mv^2 = \frac{1}{2}mb^2\dot{\phi}^2$$

In addition, we have the kinetic energy from the rotation of the wheel around it's own axis, given as

$$K_2 = \frac{1}{2}I\omega^2 = \frac{1}{2}I(\dot{\phi} + \alpha t)^2$$

This gives a total kinetic energy of

$$K = K_1 + K_2 = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2$$

Since we are in a gravitational field, we have a potential energy from the height of the center of mass B (here, in relation to the point A):

$$V = -mgh = -mgb\cos\phi$$

This gives a total Lagrangian of

$$\begin{aligned} L = K - V &= \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi \\ &= \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I\dot{\phi}^2 + I\alpha\dot{\phi}t + \frac{1}{2}I\alpha^2t^2 + mgb\cos\phi \end{aligned}$$

$$L(\phi, \dot{\phi}, t) = \frac{1}{2}\big(mb^2 + I\big)\dot{\phi}^2 + I\alpha\dot{\phi}t + mgb\cos\phi + \frac{1}{2}I\alpha^2t^2 \tag{1}$$

b)

We have that

$$\frac{\partial L}{\partial \phi} = -mgb\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = \big(mb^2 + I\big)\dot{\phi} + I\alpha t$$

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{\phi}} = \big(mb^2 + I\big)\ddot{\phi} + I\alpha$$

This gives Lagrange's equation

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\big(mb^2 + I\big)\ddot{\phi} + I\alpha + mgb\sin\phi = 0$$

c)

We have that

$$\begin{aligned}\frac{\mathrm{d}f(\phi,t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left[I\alpha\phi t + \frac{1}{6}I\alpha^2 t^3\right] \\ &= I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2\end{aligned}$$

We recognize the first and last term from the Lagrangian (1), and include the middle term by adding and subtracting it from the Lagrangian, giving

$$\begin{aligned}L &= \left[\frac{1}{2}(mb^2{+}I)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi\right] + \left[I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2\right] \\ &= L'(\phi,\dot{\phi}) + \frac{\mathrm{d}f(\phi,t)}{\mathrm{d}t}\end{aligned}$$

where

$$L'(\phi,\dot{\phi}) = \frac{1}{2}(mb^2{+}I)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi \tag{2}$$

We have successfully split our Lagrangian up into a Lagrangian explicitly independent of time, and a total time derivative.

d)

We know that adding a total time derivative to our Lagrangian does not change the equations of motion for the system. We therefore expect the e.o.m. of L and L' to be the same.

We have that

$$\frac{\partial L'}{\partial\phi} = -mgb\sin\phi - I\alpha$$

$$\frac{\partial L'}{\partial\dot{\phi}} = (mb^2+I)\dot{\phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial\dot{\phi}} = (mb^2+I)\ddot{\phi}$$

This gives Lagrange's equation

$$(mb^2+I)\ddot{\phi} + mgb\sin\phi + I\alpha = 0$$

which confirms our expectations.

e)

The canonical momentum p'_ϕ of the coordinate ϕ is already calculated as

$$p'_\phi = \frac{\partial L'}{\partial\dot{\phi}} = (mb^2+I)\dot{\phi}$$

The Hamiltonian is defined as (where q_i are the generalized coordinates)

$$\begin{aligned}H'(\phi,\dot{\phi}) &= \sum_i p_i\dot{q}_i - L = p'_\phi\dot{\phi} - L' \\ &= (mb^2+I)\dot{\phi}^2 - \frac{1}{2}(mb^2{+}I)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi \\ &= \frac{1}{2}(mb^2{+}I)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi\end{aligned}$$

To get H' as a function of ϕ and p'_ϕ only, we use to relation

$$p'_\phi = (mb^2 + I)\dot{\phi} \qquad \Rightarrow \qquad \dot{\phi} = \frac{1}{mb^2 + I} p'_\phi$$

to rewrite H' as

$$\begin{aligned} H'(\phi, p'_\phi) &= \frac{1}{2}\left(mb^2 + I\right)\frac{1}{(mb^2 + I)^2} {p'_\phi}^2 - mgb\cos\phi + I\alpha\phi \\ &= \frac{1}{2}\frac{{p'_\phi}^2}{mb^2 + I} - mgb\cos\phi + I\alpha\phi \end{aligned}$$

f)

H' being a constant of motion involves it having no time-dependence. This is fulfilled if it's corresponding Lagrangian, L', has no *explicit* time dependence. This is captured in the relation

$$\frac{\mathrm{d}H'}{\mathrm{d}t} = -\frac{\partial L'}{\partial t}$$

where $\frac{\partial L'}{\partial t} = 0$ as $L'(\phi, \dot{\phi})$ has no explicit time dependence.

This was our reason for replacing our original Lagrangian with L'. It's corresponding Hamiltonian H' no longer represents the total energy of the system, but it is a constant of motion. This does among other things allow us to make a phase space plot of the motion.

g)

Substituting $\alpha = \frac{mgb}{I}\lambda$, where λ is a dimensionless variable for the angular acceleration of the wheel, gives

$$H'(\phi, p'_\phi) = \frac{1}{2}\frac{{p'_\phi}^2}{mb^2 + I} - mgb\cos\phi + mgb\lambda\phi$$

For simplicity we set the parameters $m = 1$, $b = 1$, $I = 1$, $g = 9.81$. With a simple Python script we can produce the contour plots in figure (1), outlining the movement of the pendulum.

At $\lambda = 0$ (no angular acceleration on the wheel), we retrieve the regular pendulum phase plots, without energy loss. The circular solutions represents pendulums oscillating in the common way, while the wobbly "free" lines represent pendulums with enough kinetic energy to take full turns.

For $0 < \lambda \leq 1$ the circular solutions remain (although slightly shifted in angle, no longer oscillating around the lowest gravitational potential). The "free" solutions however, will now at some point either slow down and turn, or keep accelerating, depending on its initial direction. We can observe this by following a "free" contour line close to the circles, and observe it eventually turn. As λ increases, more and more initial conditions result in free solutions.

When we reach $\lambda = 1$ and above, the pendulum will start turning, no matter it's initial conditions. In the high-density plot (2), we observe that the last circular solutions disappear at $\lambda = 1$, and the pendulum will always accelerate.

It it obvious from there observations that the accelerating wheel applies a force on the pendulum, accelerating it's rotation around the point A.

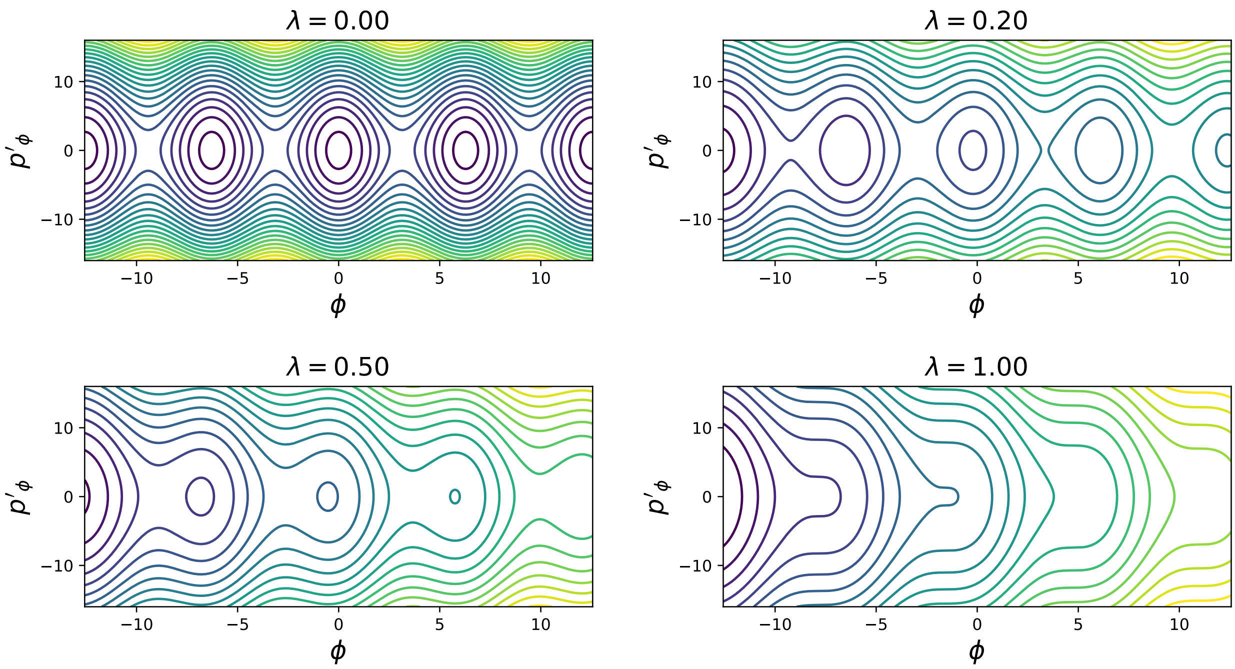

Figure 1: Contour plots of $H'(\phi, p'_\phi)$

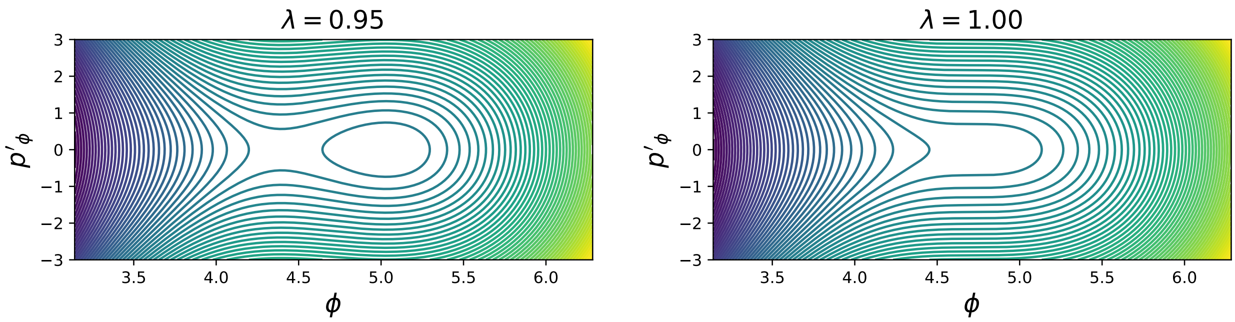

Figure 2: Higher density contour plots of $H'(\phi, p'_\phi)$

Question 2

a)

The invariant mass m_{ab} does not change between reference frames because it is a Lorentz invariant quantity.

The sum of two momentum-energy four-vectors are also a four vector[1], meaning that $p_a + p_b$ is a four-vector. We also know that the length of a four-vector (defined as $x^2 = x^\mu x_\mu$) is a Lorentz invariant quantity. The quantity

$$m_{ab}^2 c^2 = (p_a + p_b)^2 = (p_a + p_b)^\mu (p_a + o_b)_\mu$$

must therefore be invariant. The invariant mass m_{ab} is also invariant, as neither dividing by a constant c^2, nor squaring the solution, breaks Lorentz invariance.

[1]see, for instance, `http://hyperphysics.phy-astr.gsu.edu/hbase/Relativ/vec4.html`

b)

Conservation of energy gives that particle A and a must have a combined energy equal to that of particle B before the decay.

$$E_B = E_A + E_a$$
$$\sqrt{p_B^2 c^2 + m_B^2 c^4} = \sqrt{p_A^2 c^2 + m_A^2 c^4} + \sqrt{p_a^2 c^2 + m_a^2 c^4}$$

(p_A, p_a ... now refer to the magnitude of the relativistic momenta, $p_A = |\vec{p_A}|$ ...)

In the rest frame of A, $p_A = 0$, and conservation of momentum gives that $p_a^2 = p_B^2$ (total momentum before decay must equal total momentum after decay). This gives

$$\sqrt{p_a^2 + m_B^2 c^2} = \sqrt{m_A^2 c^2} + \sqrt{p_a^2 + m_a^2 c^2}$$
$$\left(\sqrt{p_a^2 + m_B^2 c^2}\right)^2 = \left(\sqrt{m_A^2 c^2} + \sqrt{p_a^2 + m_a^2 c^2}\right)^2$$
$$p_a^2 + m_B^2 c^2 = m_A^2 c^2 + p_a^2 + m_a^2 c^2 + 2\sqrt{m_A^2 c^2}\sqrt{p_a^2 + m_a^2 c^2}$$
$$\sqrt{p_a^2 + m_a^2 c^2} = \frac{m_B^2 c^2 - m_a^2 c^2 - m_A^2 c^2}{2 m_A c}$$
$$p_a^2 + m_a^2 c^2 = \frac{\left(m_B^2 c^2 - m_a^2 c^2 - m_A^2 c^2\right)^2}{4 m_A^2 c^2}$$
$$p_a^2 + m_a^2 c^2 = \frac{m_B^4 c^4 + m_a^4 c^4 + m_A^4 c^4 - 2m_B^2 m_a^2 c^4 - 2m_B^2 m_A^2 c^4 + 2m_a^2 m_A^2 c^4}{4 m_A^2 c^2}$$
$$p_a = \frac{\sqrt{m_B^4 c^4 + m_a^4 c^4 + m_A^4 c^4 - 2m_B^2 m_a^2 c^4 - 2m_B^2 m_A^2 c^4 + 2m_a^2 m_A^2 c^4 - 4m_a^2 m_A^2 c^4}}{2 m_A c}$$

$$p_a = c\frac{\sqrt{m_B^4 + m_a^4 + m_A^4 - 2m_B^2 m_a^2 - 2m_B^2 m_A^2 - 2m_a^2 m_A^2}}{2 m_A} \tag{3}$$

c)

We have the invariant mass given as

$$m_{ab}^2 c^2 = (p_a + p_b)^2 = (p_a + p_b)^\mu (p_a + p_b)_\mu$$

We write out the 4-vector with energy and regular momentum components

$$(p_a + p_b)^\mu = \left(\frac{E_a + E_b}{c},\ [\vec{p_a} + \vec{p_b}]\right)$$
$$(p_a + p_b)_\mu = \left(\frac{E_a + E_b}{c},\ -[\vec{p_a} + \vec{p_b}]\right)$$

which gives the invariant mass

$$m_{ab}^2 c^2 = \frac{(E_a + E_b)^2}{c^2} - [\vec{p_a} + \vec{p_b}]^2$$

Inserting the relativistic energy (with $m_a = m_b = 0$)

$$E_a = \sqrt{p_a^2 c^2 + m_a^2 c^4} = p_a c$$
$$E_b = \sqrt{p_b^2 c^2 + m_b^2 c^4} = p_b c$$

gives

$$m_{ab}^2 c^2 = \frac{(p_a c + p_b c)^2}{c^2} - [\vec{p_a} + \vec{p_b}]^2$$
$$= p_a^2 + p_b^2 + 2p_a p_b - \vec{p_a}^2 - \vec{p_b}^2 - 2\vec{p_a} \cdot \vec{p_b}$$

The squared vectors simply become their magnitudes squared ($\vec{p}^2 = p^2$), while the dot product can be written $\vec{p_a} \cdot \vec{p_b} = p_a p_b \cos\theta_{ab}$, where θ_{ab} is the angle between the vectors. This gives

$$m_{ab}^2 c^2 = 2p_a p_b - 2p_a p_b \cos\theta_{ab} = 2p_a p_b (1 - \cos\theta_{ab})$$

$$m_{ab}^2 = \frac{2}{c^2} p_a p_b (1 - \cos\theta_{ab}) \tag{4}$$

We observe that the decay $C \to bB$ from the reference frame of B is identical to the decay $B \to aA$ from the reference frame of A, solved in the last exercise. We therefore borrow equation (3), substituting (B, a, A) with (C, b, B). Setting $m_b = 0$ gives

$$p_b = c\frac{\sqrt{m_C^4 + m_B^4 - 2m_C^2 m_B^2}}{2m_B} = c\frac{\sqrt{(m_C^2 - m_B^2)^2}}{2m_B} = c\frac{m_C^2 - m_B^2}{2m_B} \tag{5}$$

We now need an expression for p_a.

Looking at the second decay from the reference frame of B, we get the slightly different energy conservation, with $p_A^2 = p_a^2$ and $p_B = 0$. We also set $m_a = 0$.

$$\begin{aligned}
E_B &= E_A + E_a \\
\sqrt{p_B^2 c^2 + m_B^2 c^4} &= \sqrt{p_A^2 c^2 + m_A^2 c^4} + \sqrt{p_a^2 c^2 + m_a^2 c^4} \\
\sqrt{p_B^2 + m_B^2 c^2} &= \sqrt{p_A^2 + m_A^2 c^2} + \sqrt{p_a^2 + m_a^2 c^2} \\
m_B c &= \sqrt{p_a^2 + m_A^2 c^2} + p_a \\
(m_B c - p_a)^2 &= p_a^2 + m_A^2 c^2 \\
m_B^2 c^2 + p_a^2 - 2m_B p_a c &= p_a^2 + m_A^2 c^2
\end{aligned}$$

$$p_a = c\frac{m_B^2 - m_A^2}{2m_B} \tag{6}$$

Inserting this into equation (4) gives

$$m_{ab}^2 = \frac{2}{c^2} c\frac{m_B^2 - m_A^2}{2m_B} c\frac{m_C^2 - m_B^2}{2m_B}(1 - \cos\theta_{ab}) = \frac{(m_B^2 - m_A^2)(m_C^2 - m_B^2)}{2m_B^2}(1 - \cos\theta_{ab})$$

d)

The idea is to solve for the invariant masses m_{ab}, m_{bc}... by using our earlier definitions of the invariant mass:

$$\begin{aligned}
m_{ab}^2 &= \frac{(p_a + p_b)^2}{c^2} \\
m_{ac}^2 &= \frac{(p_a + p_c)^2}{c^2} \\
m_{bc}^2 &= \dots
\end{aligned}$$

To achieve this, we need the magnitudes of the four-momenta in the same reference frame. We choose that of the particle E for this.

We start by solving for p_a in reference frame of B (now with particles a and b having mass). This involves $p_B = 0$, and the momenta of a and A to be the same $p_A = p_a$.

$$\begin{aligned}
E_B &= E_A + E_a \\
\sqrt{m_B^2 c^4} &= \sqrt{p_a^2 c^2 + m_A^2 c^4} + \sqrt{p_a^2 c^2 + m_a^2 c^4}
\end{aligned}$$

We let WolframAlpha do this one. The solution is

$$p_a = \frac{\sqrt{c^4 m_a^4 - 2c^4 m_a^2 m_B^2 + c^4 m_B^4 - 2c^2 m_A^2 m_a^2 - 2c^2 m_A^2 m_B^2 + m_A^4}}{2c m_B} \tag{7}$$

Following the exact same logic, we find p_b in reference frame of C.

$$p_b = \frac{\sqrt{c^4m_b^4 - 2c^4m_b^2m_C^2 + c^4m_C^4 - 2c^2m_B^2m_b^2 - 2c^2m_B^2m_C^2 + m_B^4}}{2cm_C}$$

The solutions of p_c and p_d in the reference frames of D and E are similar, but two of the mass terms are zero: $m_c = m_d = 0$. The solutions becomes[2]

$$p_c = \frac{\sqrt{c^4m_D^4 - 2c^2m_C^2m_D^2 + m_C^4}}{2cm_D} = c\frac{m_D^2 - m_C^2}{2m_D}$$
$$p_d = \frac{\sqrt{c^4m_E^4 - 2c^2m_D^2m_E^2 + m_D^4}}{2cm_E} = c\frac{m_E^2 - m_C^2}{2m_E}$$

The idea is now to Lorentz transform these four-momenta into the E reference frame. To use a Lorentz transformation between the $A - B - C - D - E$ frames, we need their relative velocities.

We solve for v_A in the reference frame of B, using the definition $p = \gamma mv$ of the relativistic momentum, and employing that $p_a = p_A$ in the frame B.

$$p_a = p_A = \gamma_{AB}m_Av_A = \frac{1}{\sqrt{1-\frac{v_A^2}{c^2}}}m_Av_A$$
$$p_a^2 = \frac{m_A^2v_A^2}{1-\frac{v_A^2}{c^2}}$$
$$p_a^2 - p_a^2\frac{v_A^2}{c^2} = m_A^2v_A^2$$
$$v_A^2(m_A^2 + \frac{p_a^2}{c^2}) = p_a^2$$

$$v_A = \frac{p_a}{\sqrt{m_A^2 + \frac{p_a^2}{c^2}}} \tag{8}$$

The same logic can be applied to v_B in the frame of C, and so forth:

$$v_B = \frac{p_b}{\sqrt{m_B^2 + \frac{p_b^2}{c^2}}} \qquad v_C = \frac{p_c}{\sqrt{m_C^2 + \frac{p_c^2}{c^2}}} \qquad v_D = \frac{p_d}{\sqrt{m_D^2 + \frac{p_d^2}{c^2}}} \tag{9}$$

We now introduce the general Lorentz matrix

$$L = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_y\beta_x}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_z\beta_x}{\beta^2} & (\gamma-1)\frac{\beta_z\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{pmatrix}$$

to transform the four-momenta between reference frames. Here we have the identities

$$\beta_x = \frac{v_x}{c} \qquad \beta_y = \frac{v_y}{c} \qquad \beta_z = \frac{v_z}{c} \qquad \beta^2 = \frac{|v|^2}{c^2} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

We will now attempt to transform our information about the angles of the particles into Cartesian vector-form. We denote the angles between a and B (in RF B) as θ_a and ϕ_a, and the angels between A and B as θ_A and θ_B. Since the known (uniformly generated) angels are those of a, we transform these to A as

$$\theta_A = \pi - \theta_a \qquad \phi_A = \pi + \phi_a$$

This is due to A and a always having velocities in opposite directions in the RF of B (due to momentum conservation). From here, we get the 3D vector components as

[2] $m_D > m_C$ and $m_E > m_C$, so we keep them in this order to get a positive absolute momentum.

$$\vec{v_A} = v_A[\sin\theta_A \cos\phi_A,\ \sin\theta_A \sin\phi_A,\ \cos\theta]$$

This can all easily be translated to the velocities of v_B, v_C...

We wish the direction of decaying particles to uniformly distributed on a sphere. This does not, contrary to popular belief, correspond to a uniform distribution of the angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. This is because an angular interval does not represent a fixed area on the sphere, and a uniform angular distribution would favor the poles. To create a spherically uniform distribution, we will employ the angles

$$\phi = 2\pi u \qquad \theta = \arccos(2v - 1)$$

where u and v are uniform distributions in $[0, 1]$. This ensures an isotropic decay process.

We use a Python script to randomly generate angles for all 4 decays, calculate the four-momenta and relative velocities, and finally Lorentz-transform everything to reference frame E before calculating the invariant masses.

Figure 3 show the result of randomly generating one million random particle systems and calculating their invariant masses. Figure 4 shows them separately, as histograms.

As we can see, the invariant masses of the one-step decays m_{ab}, m_{bc}, m_{cd} have the same shape. This makes sense, as they all represent invariant masses of isotropic one-decay problems. For some reason, it seems like m_{bd} and m_{ad} also has the same shape, while m_{ac} is unique. I have no good explanation for this.

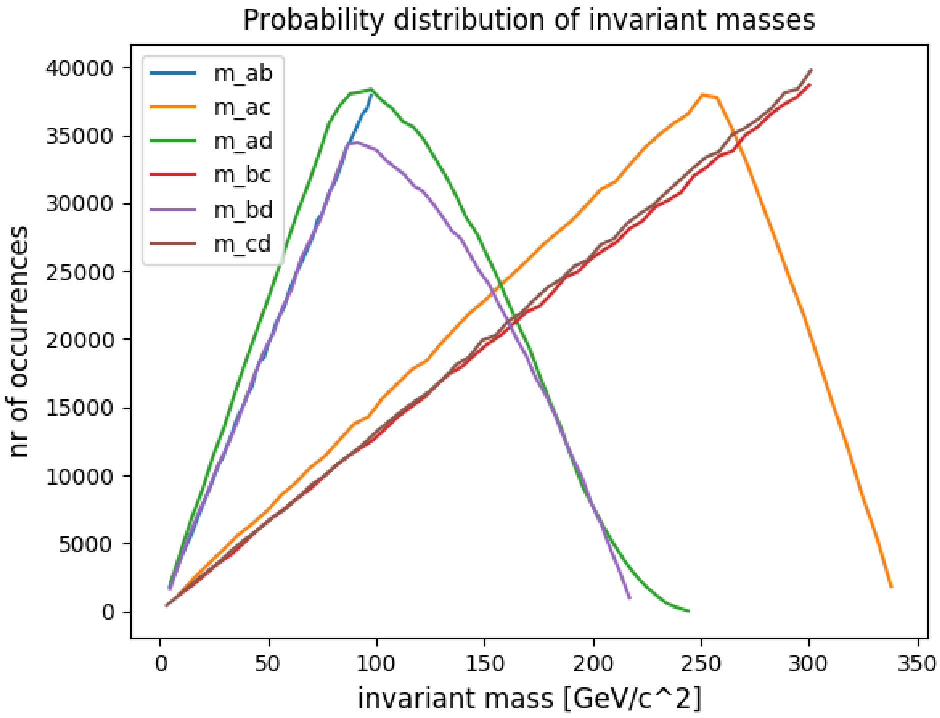

Figure 3: Combined invariant mass distributions

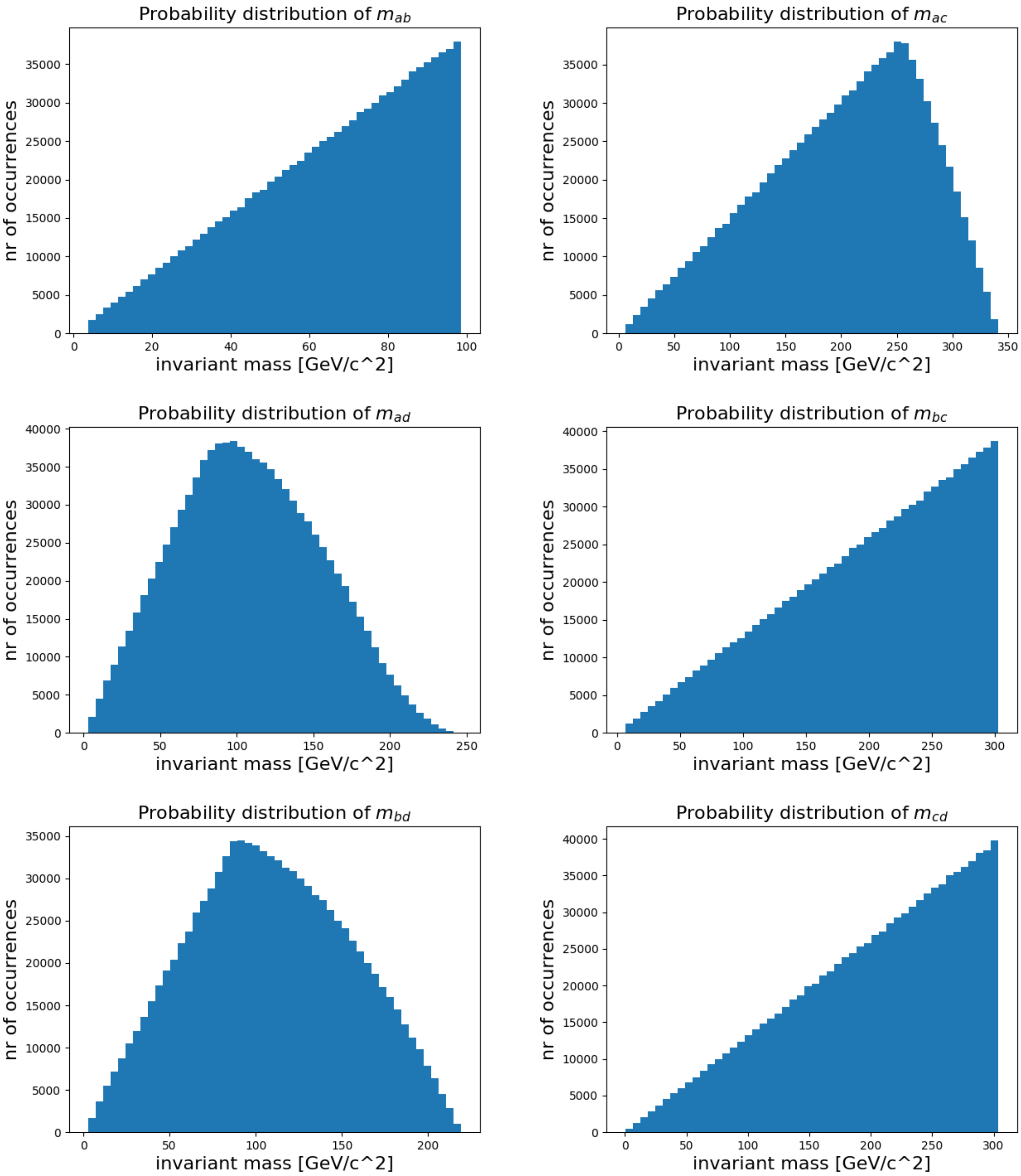

Figure 4: Histogram of invariant mass distributions