Usefull Shit

Taylor Expansions

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Around 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Random shit I always forget

$$c \ln x = \ln x^c \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ln x = \frac{1}{x}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2}$$

$$\int (uv') = uv - \int (u'v)$$

Trigenometric Identities

$$e^{\pm ix} = \cos x \pm i \sin x$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$\sinh z = \frac{1}{2}\left(e^{z} - e^{-z}\right) \qquad \cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$$

Tensors

$$A^{-1}A = \mathcal{I} \qquad A_T = A_{ij}^T = A_{ji} AB = A_{ij}B_{jk}$$

$$\det\{A\} = \det\{A^T\} \qquad \det\{AB\} = \det\{A\}\det\{B\}$$

Rotation matrices are orthogonal, such that $A^T = A^{-1}$.

Transformational matrices from a coordinate system $\mathbf{e}'$ to $\mathbf{e}$ is given as $A_{ij} = e'_i \cdot e_i$

$$y_i = A_{ij}x_j = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$x_j = A_{ji}y_i = A^T y$$

Transformation of a higher order tensor:

$$T'_{\alpha\beta\gamma\delta} = A_{\alpha i}A_{\beta j}A_{\gamma k}A_{\delta l}T_{ijkl}$$

where A is the transformation matrix.

$$(\mathbf{B} \times \mathbf{C})_i = \epsilon_{ijk}B_jC_k$$

Dirac Delta & Levi-Civita

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{else} \end{cases} \qquad \epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad i,j,k \text{ in order} \\ -1 & \text{if} \quad i,j,k \text{ not in order} \\ 0 & \text{else} \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)\,\mathrm{d}x = f(a)$$

Complex analysis

Usefull shit

- Positively oriented contour integrals are counter-clockwise.
- $(z - z_0) < R$ means all complex numbers within radius R of z_0 in the complex field.
- In many functions, the order of it's pole is very obvious. i.e $1/(z-3)$ is a first order pole at $z = 3$, and $1/(z+2i)^3$ is a third order pole at $z = -2i$.
- When encountered by a fraction with i in the denominator, multiply by the complex conjugate to move the i upstairs. (i.e. $1/(3+2i)$, multiply by $(3-2i)$). In general:

  $$(x+iy)(x-iy) = (x^2+y^2)$$

- When showing that a contour integral is 0, an upper-bound estimate is often usefull.

$$\ln z = \ln|z| + i\theta, \qquad \theta \in [-\pi, \pi]$$

Polar representation and roots

$$z = x + iy = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$$

Powers of z:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

Roots of z:

$$z^{1/n} = r^{1/n}e^{i(\theta+2\pi k)/n}, \qquad k \in 0, 1, 2, ..., n-1$$

$z^{1/n}$ has n roots, spread evenly in a circle in the complex plane.

Complex Series

The complex sequence

$$\{z_n\} = \{z_1, z_2, z_3, ...\}$$

converges if both the real and imaginary parts of z_n approaches zero for large n.

The complex series

$$s_n = \sum_{k=1}^{n} z_k$$

converges if z_k converges.

Ratio test: if $\frac{z_{n+1}}{z_n} \leq 1$ for large n, then z_k converges.

Complex Power Series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Around a point z_0, series converges for the area of z where

$$|z - z_0| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = R$$

where R is called the *radius of convergence*.

Analytic Functions

Analytic functions are special in that they treat $z = x + iy$ as a single unit, i.e. respect the complex structure.

If the output can be expressed solely in z (without x, y or z^*), the function is analytic. Remember that $x = \frac{1}{2}(z + z^*)$ and $y = \frac{1}{2i}(z - z^*)$.

An function analytic in a region always has an unique derivatives of all orders in that region.

Regular point: Point where f is analytic.
Singular point: Point where f is not analytic.

Cauchy-Riemann Equations

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Criteria for a function to be analytic in a region, derived from demanding existence of the derivative.

Harmonic Functions

Harmonic functions are solutions to the **2D Laplace equation**:

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

If $f(z) = u(x, y) + iv(x, y)$ is analytic in some region, then $u(x, y)$ and $v(x, y)$ are harmonic functions.

Theorem: Given a harmonic function $u(x, y)$, we can always find it's *harmonic conjugate* $v(x, y)$ such that $f = u + iv$ is an analytic function.

Finding Harmonic Conjugates: Given an harmonic function $u(x, y)$, we find it's harmonic conjugate by inserting $u(x, y)$ and $v(x, y)$ into the *Cauchy-Riemann Equations*, integrating for v (remember to include constants, which are only constant in regard to the integrating term), and solve for the constants to get a complete v.

Contour Integrals of Complex Functions

$$\int_{C_r} (z - z_0)^n \,\mathrm{d}z = \begin{cases} 2\pi i & \text{if } n = -1 \\ 0 & \text{else} \end{cases}$$

where C_r is a circle in positive (counter-clockwise) direction one time around the complex plane.

Upper Bound Estimate of Contour Integral

$$\left| \int_\Gamma f(z) \,\mathrm{d}z \right| \le M \cdot L$$

where M is the maximum value of $f(z)$ on Γ, and L is the length of Γ.

Remember the **Triangle Inequalities:**

$$|z_1 + z_2| \le |z_1| + |z_2| \qquad |z_2 - z_1| \ge |z_2| - |z_1|$$

Independence of Path

If Γ_1 and Γ_2 are two contours that can be continously deformed into one another (without crossing singularities), then

$$\int_{\Gamma_1} f(z) \,\mathrm{d}z = \int_{\Gamma_2} f(z) \,\mathrm{d}z$$

Cauchy's Theorem: As a result, any contour integral that doesn't enclose a singularity, is 0, as it can be shrinked to a point.

Cauchy's Integral Formula

Formula for evaluating the contour integral around a $n+1$'th order pole at z_0.

$$\int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}} \,\mathrm{d}z = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Note: Remember to rewrite the expression to *exactly* the form above. If the contour contains several singularities, rewrite to handle each of the singularities seperately. Example, integral around $z = 4$ singularity of $\cos z/[(z - 4)(z + 5)]$, rewrite to $[\cos z/(z + 5)]/(z - 4) = f(z)/(z - 4)$.

Taylor Series

$$f(z_0) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}$$

Theorem: If $f(z)$ is analytic in the disk $|z - z_0| \le R$, then the Taylor series converges for all z *inside* the disk.

Laurent Series

We combine the *Taylor* series with a *Principal* series of negative powers.

$$f(z_0) = \sum_{n=0}^{\infty} a_k (z - z_0)^n + \sum_{n=1}^{\infty} b_k \frac{1}{(z - z_0)^n}$$

- The Taylor series of positive powers converge *inside* some circle $|z - z_0| < R_2$.
- The Principal series of negative powers converge *outside* some circle $R_1 < |z - z_0|$.
- The Laurent series converges in the donut between the two circles, $R_1 < |z - z_0| < R_2$.

Tip: If you only need the series to converge outside/inside some circle, you only need one of the series.

The factor b_0 is called the **residue** of f at z_0.

Finding Laurent Series

If the Laurent Series should expand from a point $z_0 \neq 0$, make a substitution $w = z - z_0$, such that the series expands from $w = 0$.

By Geometric Series: Manipulate the function to the form

$$f(w) = \frac{1}{1 - \eta} = \sum_{n=0}^{\infty} \eta^n$$

The series converges for $|\eta| < 1$

- **Taylor**: inside the cirlce $(z - z_0) < R \;\Rightarrow\; \eta = (z - z_0)/R$
- **Principal**: outside the cirlce $(z - z_0) > R \;\Rightarrow\; \eta = R/(z - z_0)$

By Taylor Expansion: If the function has no singularities, you can simply make a Taylor expansion of it. Make sure to do the substitution first.

Singularities and zeros

The **order of a zero or singularity** is the number of times you must derivate the function until the zero or infinity disapears.

Assume $f(z)$ has an isolated singularity at z_0, and it's Larent series is as given above.

- If all $b_n = 0$, z_0 is a *removable* singularity (not actually a singularity).
- If $b_n \neq 0$ for some n, but zero for all factors above n (such that $(z - z_0)^{-n}$ is the biggest negative power), we say that z_0 is a *pole* of order n.
- If there are infinite negative terms, we say that z_0 is an *essential* singularity.

Residue Theory

Any integral over a contour Γ can be split up into integrals over only infinitesimally small contours around all singularities in Γ.

An contour integral containing N singularities z_k is given as the sum of the residues at all the singularities.

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{N} Res(f, z_k)$$

Ways of finding residues

- **Use Laurent Series (always works):** Write out the Laurent Series of the expression around the singularities, and find the b_1 term (the $1/z$ coefficient).
- **For Simple Poles (alt 1):**
$$Res(f, z_0) = \lim_{z\to z_0} (z - z_0) f(z)$$
- **For Simple Poles (alt 2):** If f is a rational function $f(z) = \frac{P(z_0)}{Q(z_0)}$:
$$Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)}$$
- **For Multiple Poles:** If f has a pole of order m at z_0, and $M \geq m$, then
$$Res(f, z_0) = \lim_{z\to z_0} \frac{1}{(M-1)!} \frac{\mathrm{d}^{M-1}}{\mathrm{d}z^{M-1}} \left[(z - z_0)^M f(z)\right]$$
Naturally, if you know the order of the pole, you pick $M = m$.

Applications to Real Integrals

Type I: Trigonometric integrals over $[0, 2\pi]$

$$\int_0^{2\pi} u(\cos\theta, \sin\theta)\,\mathrm{d}\theta$$

Substitute for

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \qquad \mathrm{d}\theta = \frac{\mathrm{d}z}{iz}$$

The integral is now around a circular contour in the complex plane, centered around $(0, 0)$ with radius 1. Evaluate the integral by finding singularities inside the circle and solving for residues.

Type IIa: Rational Functions Over $[-\infty, \infty]$

$$I = \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,\mathrm{d}x$$

$$I = 2\pi i \sum Res(f, z_n)$$

where z_n are the singularities in the *upper* plane.

Works if

- $degree(Q) \geq degree(P) + 2$
- f is analytic on and above the complex plane.

Jordan's Lemma

If $m > 0$ is real, and P and Q are polynomials such that P/Q is rational, and $degree(Q) \geq degree(P) + 1$ then:

$$\lim_{\rho\to\infty} \int_{C_\rho} \frac{P(z)}{Q(z)} e^{imz}\,\mathrm{d}z = 0$$

where C_ρ is a half-circle contour with radius ρ

Same holds for the lower plane if $m < 0$.

Type IIb: ... with Trigonometric Functions

$$I = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(mx)\,\mathrm{d}x \qquad \text{(or sin)}$$

Option 1

Use complex version of sin or cos to split the integral. USE $\frac{1}{2i}$ IF SIN

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{imx}\mathrm{d}x + \frac{1}{2}\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} e^{-imx}\mathrm{d}x = I_1 + I_2$$

Solve I_1 by close contour in upper half plane $y > 0$, I_2 in lower half plane $y < 0$.

Option 2, if $\frac{P(x)}{Q(x)}$ is real!

Replace trig function with e^{imz}, compute the contour integral in upper half plane and take the real or imaginary part at the end.

Type III: Singularities on the real axis (Principal Value)

When a real integral passes singularities, we say that the integral is not defined, but it's **principal value** is. It behaves just as an ordinary integral:

$$PV \int_a^b f(x)\,\mathrm{d}x = \lim_{r\to 0^+} \left[\int_a^{C-r} f(x)\,\mathrm{d}x + \int_{C+r}^{b} f(x)\,\mathrm{d}x\right]$$

where C is a singularity. Using the same logic as in Type II, with an added infinite half-circle on the upper plane, this evaluates to

$$PV \int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 2\pi i \sum_k Res(f; z_k) + \pi i \sum_j Res(f; z_j)$$

where z_k are singularities in the upper half plane, and z_j are singularities *on the real axis.*

We see that singularities on the exit contribute *half* of those above.

Ordinary Differential Equations

First Order, Linear, ODEs - Integrating Factor

$$y'(x) + P(x)y(x) = Q(x)$$

$$y(x)\mu(x) = \int Q(x)\mu(x)\,\mathrm{d}x + C \quad \text{with} \quad \mu(x) = e^{\int P(x)\mathrm{d}x}$$

Homogenous ODEs

$$y'' + P(x)y' + Q(x)y = 0$$

Properties

- Linear combination of solutions is also a solution
- General solution on form $y(x) = c_1 y_1(x) + c_2 y_2(x)$
- Linearly independent solutions have a Wronskian of 0.

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

Variation of the constant

If you have one of the two linearly indepedent solutions, you can find the other as $y_2(x) = C(x) \cdot y_1(x)$, where $C(x)$ is a functions determined by inserting $y_2(x)$ into the ODE.

When you arrive at a solution for $C(x)$, you may discard any constants or coefficients, i.e. $C(x) = \alpha x^3 + \beta$.

Constant coefficients - Particular Equation

$$y''(x) + ay'(x) + by(x) = 0$$

Solve the particular equation

$$\lambda^2 + a\lambda + b = 0$$

for λ_1 and λ_2.

Two, real roots

$$y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

One, real root

$$y(x) = (C_1 + xC_2)e^{\lambda x}$$

Two, complex roots

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} = e^{-a/2x}\left[Ae^{i\omega x} + Be^{-i\omega x}\right]$$

$$= e^{-a/2x}\left[\hat{A}\cos\omega x + \hat{B}\sin\omega x\right]$$

Euler-Cauchy Equations

$$x^2y'' + a_1xy' + a_0y = 0$$

Introducing

$$x = e^z \qquad \Rightarrow \qquad z = \ln|x|$$

The equation can be rewritten to

$$\frac{\partial^2 y}{\partial z^2} + (a_1 - 1)\frac{\partial y}{\partial z} + a_0 y = 0$$

Solve the ODE, and insert for z.

Power methods

- Represent $P(x)$ and $Q(x)$ as power series (polynomials).
- Assume solution on the form
  - $y(x) = \sum_{n=0}^{\infty} a_n x^n$
  - $y'(x) = \sum_{n=1(0)}^{\infty} n a_n x^{n-1}$
  - $y''(x) = \sum_{n=2(0)}^{\infty} n(n-1) a_n x^{n-2}$
- Insert back into ODE.
- Split into equations of matching powers of x.

Since we should only have two undetermined coefficients, we get one of the following:

The coefficients may be linked as a series depending on each other, like $a_{s+1} \ = a_s^2$. If the series are split into odd/even, two undetermined coefficients are required to describe them, so the solution is complete. Otherwise, solve the other by variation of the constant.

They may also come on a form which shows that all but two of the coefficients are zero, which gives the two linearly independent solutions from the above general solution.

Fröbenius method

$$x^2y'' + xb(x)y' + c(x)y = 0$$

Assuming solution on form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+s}$$

where s is some real number determined the *indicial equation*

$$s(s-1) + b_0 s + c_0 = 0$$

where $b_0 = b(0)$, and $c_0 = c(0)$

Three possible scenarios:

- Different roots, $s_1 \neq s_2$, and $s_1 - s_2 \neq$ integer.
  - Two indepedens solutions $y_i(x) = x^{si} \sum_{m=0}^{\infty} a_0 x^m$
- Different roots, $s_1 \neq s_2$, but $s_1 - s_2 =$ integer. ($s_1 > s_2$).
  - Solve for both by Power Series.
  - Often, s_2 gives the complete solution (two undetermined coefficients), so try this first.
  - Sometimes, only s_1 gives a solution. Find the other by variation of the constant.
- Double root, $s_1 = s_2$.
  - Find the solution by Power Series.
  - Find the second solution by variation of the constant.

Inhomogenous ODEs

$$y'' + P(x)y' + Q(x)y = R(x)$$

Remember to always rewrite to this form.

Properties

- Solutions on form $y(x) = y_h(x) + y_p(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$.
- $y_h(x)$ is the solution to the homogenous equation.
- $y_p(x)$ is *any* solution to the whole ODE.
- Since y_h contains two arbitrary constants, y_p should contain none. You can discard any such constants (set them as you wish).

Inhomo ODEs with constant coefficients

$$\boxed{y'' + ay' + by = R(x)}$$

- Works if $R(x)$ has a derivative that resembles itself.
- Make a guess at y_p with the same form as $R(x)$, with unknown coefficients.
- Insert back into ODE to solve for coefficients.

Exponential: $R(x) = Ae^{kx}$.
Let α and β be the roots of $\lambda^2 + a\lambda + b = 0$.

1. If $k \neq \alpha, \beta$: Try $y_p = Ce^{kx}$.
2. If $k = \alpha$ or β: Try $y_p = Cxe^{kx}$.
3. If $k = \alpha = \beta$: Try $y_p = Cx^2e^{kx}$.

Harmonic: $R(x) = Asin(kx)$ or $R(x) = Acos(kx)$: y_p will be of the form $B \cdot sin(kx) + C \cdot cos(kx)$. Efficient to solve for $R(x)$ e^{ikx} and take Re or Im at the end.

Exp times poly: $R(x) = e^{kx} \cdot P_n(x)$: Try above method multiplied by a polynomial of same degree.

Inhomo ODEs with varying coefficients

$$\boxed{y'' + P(x)y' + Q(x)y = R(x)}$$

Factorization

If $u(x)$ is a known solution to the homo-ODE, a particular solution is $\boxed{y_p = u(x) \cdot v(x)}$ where

$$\boxed{w' = v} \qquad \boxed{w' + \left[\frac{2u'}{u} + P\right] w = \frac{R}{u}}$$

Solve the ODE for w with integrating factor.

Variation of parameters

$$\boxed{y_p = -y_1 \int \frac{y_2 R}{W}\,\mathrm{d}x + y_2 \int \frac{y_1 R}{W}\,\mathrm{d}x}$$

where y_1 and y_2 are known linearly indepedendent solutions to the homo-ODE.

NOTE: Remember that $R(x)$ is the RHS after the ODE is rewritten on the standard form.

Greens functions

Let $D = \left[dv[2]x + P(x)\frac{\mathrm{d}}{\mathrm{d}x} + Q(x)\right]$ be the differential operator. Assume BC's for given $y(x)$, $y(a)$ and $y(b)$, then Greens functions will give the full solution including BC's for given D and given BC's, for any $R(x)$.

1. Solution

$$y(x) = \int_a^b G(x,z)R(z)\mathrm{d}z, \quad a \leq x,\, z \leq b$$

Conditions:

2. $D(x)G(x,z) = \delta(x-z)$, original DE with $R(x) \to \delta(x-z)$. Get two separate solutions for $x < z$ and $x > z$. $\delta(x-z) = 0$ at $x = z$.
3. $G(x,z)$ must obey same BC's in x, ex $G(a,z) = G(b,z) = 0$ if $y(a) = y(b) = 0$.
4. $G(x,z)$ is continuous at $x = z$, while $\lim_{\epsilon\to 0} \left|\frac{\mathrm{d}g}{\mathrm{d}z}\right| = 1$

Orthogonal Functions

Functions on the form

$$p(x)y'' + p'(x)y' + [q(x) + \lambda r(x)]y = 0$$

on some interval $[a,b]$ has solutions as linear combinations of eigenfunctions $y_n(x)$ which are orthogonal with respect to $r(x)$ such that

$$\int_a^b r(x)y_n(x)y_m(x)^*\,\mathrm{d}x = 0 \qquad \text{for} \lambda_n \neq \lambda m$$

Any function can be written as a linear combination of these eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x)$$

then the set $\{y_n(x)\}$ is complete. The coefficients a_n are determined by the orthogonality:

$$a_n = \int_a^b f(x)r(x)y_n(x)^*\,\mathrm{d}x$$

Fourier

Usefull Shit

- recognize **odd** and **even** integrands. I.e $\int_{-\infty}^{\infty} \sin x/x^2 = 0$ due to odd, and $\int_{-\infty}^{\infty} \cos x/(1+x^2) = 2\int_0^{\infty} \cos x/(1+x^2)$ due to even.

Orthogonality

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}$$

Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/L} \qquad c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,\mathrm{d}x$$

Even and Odd functions

If $f(x)$ is **even** $[f(x) = f(-x)]$:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \qquad b_n = 0$$

If $f(x)$ is **odd** $[f(x) = -f(-x)]$:

$$a_n = 0 \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Dirichlet Conditions for Fourier Series

1. Finite number of min/max in interval.
2. Finitite number of (only) finite discontinuities.

If this holds, then the series will converge to $f(x)$ at all points. At discontinuities, the series will converge to the mid-point.

Parseval's Theorem

$$\int_{-L}^{L} |f(x)|^2\,\mathrm{d}x = 2L\sum_{-\infty}^{\infty}|c_n|^2$$

Fourier Transforms

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,\mathrm{d}k \qquad F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,\mathrm{d}x$$

Odd and even functions

If $f(x)$ is an odd function, $f(x) = -f(-x)$, the Fourier transform can be done using only sine (as cosine is symmetric around 0):

$$f(x) = \sqrt{\frac{2}{\pi}}i\int_0^\infty F(k)\sin(kx)\,\mathrm{d}k \quad F(k) = \sqrt{\frac{2}{\pi}}i\int_0^\infty f(x)\sin(kx)\,\mathrm{d}x$$

If $f(x)$ is even, $f(x) = f(-x)$, we need only cosine (as sine is anti-symmentric):

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(k)\cos(kx)\,\mathrm{d}k \quad F(k) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(kx)\,\mathrm{d}x$$

FT of a derivative

$$\mathcal{F}\left[f^{(n)}(x)\right] = (ik)^n\mathcal{F}[f(x)] \qquad \mathcal{F}\left[\frac{\partial f}{\partial t}\right] = \frac{\partial}{\partial t}\mathcal{F}[f(x)]$$

Partial Differential Equations

Notes

- In symentrical systems, it seems your can switch x<->y if it is required to suit boundary conditions (example: Dirichlet conditions are at x=a instad of at y=b).
- When resulting in cos/sin solutions of frequencies, include n=0 for cos, as it gives a non-zero solution, but not for sin.

Separation of variables

1) The the solution function as a product of functions of each variable, i.e:

$$u(x,y) = X(x)Y(y) \qquad u(r,\theta) = R(r)T(\theta)$$

2) Insert this into the equation, and separate the equation into parts of only each variable. Each side must then be constant, and equals some *separation constant.*

3) Solve each side of the equation (equaling the seperation constant), giving an infinite set of *eigenfunctions*, $u_n(x,y)$ for the equation.

4) The final solution is a linear combination of the eigenfunctions.

$$u(x,y) = \sum_{n=-\infty}^{\infty} c_n u_n(x,y)$$

Laplace Equation - 2D Cartesian

$$\nabla u(x,y) = 0$$

Separation of variables, $u(x,y) = X(x)Y(y)$ gives solutions

$$u(x,y) = X(x)Y(y) = \begin{Bmatrix} e^{ky} \\ e^{-ky} \end{Bmatrix} \times \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix}$$

Diritchlet BC: $u(x,0) = u(0,y) = u(a,y) = 0$, $u(x,b) = f(x)$ Eigenfunctions on form

$$u_n(x,y) = A_nF_n(x)G_n(x) = A_n\sin(n\pi x/a)\sinh(n\pi y/a)$$

1D Wave Equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Seperation of variables, $u(x,t) = F(x)G(t)$ gives equations

$$F''(x) = -k^2F(x) \qquad \ddot{G}(t) = -k^2v^2G(t)$$

where the seperation constant, $-k^2$ must be negative, else the solutions would blow up.

The equations has solutions on the form

$$y(x,t) = \begin{Bmatrix} \sin(kx) \\ \cos(kx) \end{Bmatrix} \times \begin{Bmatrix} \sin(kvt) \\ \cos(kvt) \end{Bmatrix}$$

The end-points are usually fixed at 0, leaving only the $\sin(kt)$ term, and forcing $k = n\pi/L$.

If the velocity is 0 at $t = 0$, we discard the sin-velocity term and get

$$y(x,t) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)$$

If position is 0 at $t = 0$, we discard the cos-velocity term instead.

Initial position will be on the form

$$y(x,0) = \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right) = f(x)$$

where $f(x)$ is the initial position or velocity. The coefficients b_n are now Fourier coefficients, given as:

$$b_n = \frac{1}{2L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Heat flow equation

General: $\frac{\partial u}{\partial t} = c^2\nabla^2 u$, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, and $u(x,y,z,t)$. In 1D: $\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$. BC: $u(0,t) = u(L,t) = 0$. IC: $u(x,0) = f(x)$
1)Sep. of variables: $\rightarrow u(x,t) = F(x)G(t)$

$$\Rightarrow F \cdot \dot{G} = c^2 F'' \cdot G \Rightarrow \frac{\dot{G}}{c^2 G} = \frac{F''}{F} = k(negative as before)$$
$$\Rightarrow k = -p^2 \Rightarrow F'' + p^2 F = 0 \text{ and } \dot{G} + c^2p^2G = 0$$

2) Impose BCs: Exactly like 1D wave eq.: $F_n(x) = sin(\frac{n\pi x}{L})$, n=1,2,.. Find G(t):

$$\dot{G} + c^2p_n^2 G = 0 \text{ or } \dot{G}_n + \lambda_n^2 G_n = 0, \lambda_n = \frac{cn\pi}{L}$$
$$\Rightarrow G_n(t) = B_n e^{-\lambda_n^2 t}$$
$$\Rightarrow \text{Eigenfunctions: } u_n(x,t) = B_n sin(\frac{n\pi x}{L})e^{-\lambda_n^2 t}$$

3) Full solution, Fourier series, implement ICs:
General: $u(x,t) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L})e^{-\lambda_n^2 t}$
Initial condition: $u(x,0) = \sum_{n=1}^{\infty} B_n sin(\frac{n\pi x}{L}) \stackrel{!}{=} f(x)$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L f(x) sin(\frac{n\pi x}{L})dx$$

2D Wave eq.(Vibrating membrane)

DE: $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$. BCs: $u = 0$ on the boundary at all times. ICs: $u(x,y,0) = f(x,y)$ and $\dot{u}(x,y,0) = g(x,y)$. u(x,y,t): displacement of point (x,y) on the membrane at time t. Rectangular membrane $\rightarrow$ Cartesian. Sep. of variables $\rightarrow$ Double Fourier series. 1)Sep. of variables: 3 ODE's. Seperate out t-dependence:$u(x,y,t) = F(x,y)\cdot G(t)$

$$\Rightarrow F \cdot \ddot{G} = c^2 G\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right)$$
$$\Rightarrow \frac{\ddot{G}}{c^2 G} = \frac{1}{F}\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}\right) = -\nu^2 \text{ (negative constant like before)}$$
$$\Rightarrow \ddot{G} + \lambda^2 G = 0; \lambda = c\cdot\nu \text{ time eq.}$$
$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \nu^2 F = 0 \text{ Spatial eq.}$$

Seperate: $F(x,y) = H(x)Q(y)$. $\frac{d^2H}{dx^2} + k^2 H = 0$ and $\frac{d^2Q}{dy^2} + p^2 H = 0$ with $p^2 + k^2 = \nu^2$.

$$\Rightarrow H(x) = Acos(kx) + Bsin(kx)$$

and

$$Q(y) = c\cdot cos(py) + Dsin(py)$$

2)BC's: F(x,y)=0 on the boundary: $H(0) = H(a) = Q(c) = Q(b) = 0 \Rightarrow A = C = 0$ and $Bsin(ka) = 0 \Rightarrow k = \frac{m\pi}{a}$,m integer,$Dsin(pb) = 0 \Rightarrow p = \frac{n\pi}{b}$,n integer.

$$\Rightarrow F_{nm}(x,y) = sin(\frac{m\pi x}{a})sin(\frac{m\pi y}{b})$$

Eigenvalues: $\lambda = c\nu = c\sqrt{k^2+p^2} \Rightarrow \lambda_{mn} = c\pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}$ From $\ddot{G} + \lambda^2 G = 0 \Rightarrow G_{mn}(t) = \alpha_{mn}cos(\lambda_{mn}t) + \beta_{mn}sin(\lambda_{mn}t)$ Eigenfunctions:

$$u_{mn}(t) = [\alpha_{mn}cos(\lambda_{mn}t) + \beta_{mn}sin(\lambda_{mn}t)]\, sin(\frac{m\pi x}{a})sin(\frac{n\pi y}{b})$$

3) Full solution(with ICs) $\rightarrow$ Double Fourier series. General solution: $u(x,y,t) = \sum_{m,n} = u_n(x,y,t)$

$$\Rightarrow \alpha_{mn} = \frac{2}{a}\cdot\frac{2}{b}\int_0^a\int_0^b f(x,y)sin(\frac{m\pi x}{a})sin(\frac{n\pi y}{b})dydx$$

Find β_{mn} from IC $\dot{u}$

Non-cartesian coordinates

- General strategy for boundary value problem: Use coordinates that match the shape of the boundary.

1) Polar coordinates - Circular membrane

$x = rcos\theta$,$y = rsin\theta$, Laplacian: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$
- 2D wave eq. : $\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right]$ Simplify: Look for radially symmetric solutions, i.e. $\frac{\partial u}{\partial\theta} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right]$$

BC: $u(R,t) = 0$, all $t \geq 0$. Finite at $r = 0$.
IC: $u(r,0) = f(r)$ and $\dot{u}(r,0) = g(r)$ both θ-independent.
1)Sep. of variables $\rightarrow$ Bessel's equation

$u(r,t) = w(r)\cdot G(t)$ Usual procedure
$w'' + \frac{1}{r}w'k^2w = 0$, $\ddot{G} + \lambda^2 G = 0$ ($\lambda = ck$, sep.const. $-k^2$)
Set $s = k\cdot r$, so $\frac{d}{ds} = \frac{1}{k}\frac{d}{dr}, \frac{d^2}{ds^2} = \frac{1}{k^2}\frac{d^2}{dr^2}$

$$\Rightarrow \frac{d^2 w}{ds^2} + \frac{1}{s}\frac{dw}{ds} + w = 0 \text{ Bessel's equation with } \nu = 0$$

General form of Bessel's:

$$s^2w'' + sw + (s^2 - \nu^2)w = 0$$

Solved by Fröbenius, tabulated solutions J_0, Y_0.
Finite solution at origin: $\underbrace{w = J_0(s) = J_0(kr)}_{\text{Bessel function of the first kind}}$
2) BC:
- Finite at $r = 0 \rightarrow$ dismissed Y_0.

$W(R) = J_0(kr) = 0$. J_0 has infinitely many, irregurarly spaced, zeros $\{\alpha_m\}^{m=1,2,\ldots}$

$$\Rightarrow k = k_m = \frac{\alpha_m}{r} \Rightarrow w_m = J_0(k_m r) = J_0(\frac{\alpha_m}{R}r)$$

t-equation: $G_m(t) = a_m cos(\lambda_m t) + b_m sin(\lambda_m t)$, where $\lambda_m = ck_m$

Eigenfunctions:

$$u_m(r,t) = [a_m cos(\lambda_m t) + b_m sin(\lambda_m t)]\, J_0(k_m r) \text{ , m=1,2,..}$$

3)ICs $\rightarrow$ Fourier-Bessel series
General solution:

$$u(r,t) = \sum_{m=1}^{\infty}[a_m cos(\lambda_m t) + b_m sin(\lambda_m t)]\, J_0\left(\frac{\alpha_m}{R}r\right)$$

t=0?

$$u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{R}r\right) = f(r)$$

a_m are coeffs of the Fourier-Bessel series for f(r) in term of $J_0\left(\frac{\alpha_m}{R}r\right)$

$$a_m = \frac{2}{R^2 J_1^2(\alpha_m)}\int_0^R f(r)\cdot J_0\left(\frac{\alpha_m}{R}r\right)dr$$

Find J_{0m} from $\dot{u}(r,0) = g(r)$

Laplace eq. in spherical coordinates, $\nabla^2 u = 0$

$$\nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right] = 0$$

Simplify: Look for ϕ-independent solutions, $\frac{\partial u}{\partial\phi} = 0$

$$\Rightarrow \nabla^2 u = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial u}{\partial\theta}\right)\right] = 0$$

BC: $u(R,\theta) = f(\theta)$. $\lim_{r\to\infty} u(r,\theta) = 0 \to$ Phsyical: finite charge.

Seperate variables: $u(r,\theta) = G(r)H(\theta)$

$$\to \frac{1}{G}\frac{d}{dr}\left(r^2\frac{dG}{dr}\right) = -\frac{1}{Hsin\theta}\frac{d}{d\theta}\left(sin\theta\frac{dH}{d\theta}\right) \equiv k$$

r-equation:

$$\frac{d}{dr}\left(r^2G'(r)\right) = kG$$

$$r^2G'' + 2rG' - kG = 0$$

$$\text{Notation: k} \equiv \text{n(n+1)}$$

$$r^2G'' + 2rG' - n(n+1)G = 0 \text{ Euler-Cauchy eq.}$$

We use $r = e^z$:

$$G''(z) + G(z) - n(n+1)G(z) = 0; G = e^{\lambda z}$$

$$\Rightarrow \lambda^2 + \lambda - n(n+1) = 0$$

$$\Rightarrow \lambda = \begin{cases} n \\ -(n+1) \end{cases}$$

$$\Rightarrow G(r) = r^\lambda = \begin{cases} r^n \\ r^{-(n+1)} \end{cases}$$

So two linearly independent solutions:

$$G_n(r) = r^n, \widetilde{G}_n(r) = \frac{1}{r^{n+1}}$$

θ-equation:

$$\frac{1}{sin\theta}\frac{d}{d\theta}\left(sin\theta\frac{dH}{d\theta}\right) + kH(\theta) = 0$$

Substitute: $w = cos\theta$:

$$\Rightarrow sin^2\theta = 1 - w^2; \frac{d}{d\theta} = \frac{dw}{d\theta}\frac{d}{dw} = -sin\theta\frac{d}{dw}$$

$$\underbrace{\frac{1}{sin\theta}\frac{d}{d\theta}}_{-\frac{d}{dw}}\underbrace{\left(sin\theta\frac{dH}{d\theta}\right)}_{-sin^2\theta\frac{dH}{dw}} + kH(\theta) = 0$$

$$\Rightarrow \frac{d}{dw}\left[(1-w^2)\frac{dH}{dw}\right] + kH = 0$$

$$\Rightarrow \left(1-w^2\right)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + kH = 0$$

$$\text{or } \left(1-w^2\right)\frac{d^2H}{dw^2} - 2w\frac{dH}{dw} + n(n+1)H = 0$$

Solutions finite at $w = cost = \pm 1$ only for integer n. We want this because of physical reasons.

$$\Rightarrow H(\theta) = P_n(w) = P_n(cos\theta) \text{ LEGENDRE POLYNOMIALS}$$

Eigenfunctions:

$$u_n(r,\theta) + \widetilde{u}_n(r,\theta) = A_n r^n P_n(cos\theta) + \frac{B_n}{r^{n+1}}P_n(cos\theta)$$

Full solution:

$$u(r,\theta) = \sum_{n=0}^{\infty}\left[A_n r^n + \frac{B_n}{r^{n+1}}\right]P_n(cos\theta)$$

A_n and B_n determined from BC's, using orthogonality relations for Legendre polynomials:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \int_0^\pi P_n(cos\theta)P_m(cos\theta)sin\theta d\theta = \frac{2}{2n+1}\delta_{nm}$$

Example: Find electrostatic potential $u(r,\theta)$ inside and outside a sphere of radius R, when $u(R,\theta)$ ON the sphere is given.

Outside: $u(r,\theta) \to 0$ as $r \to \infty$ (finite charge) $\Rightarrow A_n = 0$

$$u(r,\theta) = \sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(cos\theta) \text{ , so } u(R,\theta) = \sum_{n=0}^{\infty}\frac{B_n}{R^{n+1}}P_n(cos\theta)$$

Find B_n from orthogonality:

$$\int_0^\pi u(R,\theta)P_m(cos\theta)sin\theta d\theta = \sum_{n=0}^{\infty}\frac{B_n}{R^{n+1}}\cdot\frac{2}{2^{m+1}}\delta_{nm} = \frac{B_m}{2m+1}\cdot\frac{2}{R^{m+1}}$$

$$\Rightarrow B_m = \frac{R^{m+1}(2m+1)}{2}\int_0^\pi u(R,\theta)\cdot P_m(cos\theta)sin\theta d\theta \; (u(R,\theta) \text{ know}$$

Inside: $B_n = 0$ to avoid divergence at $r = 0$ (origin) $\Rightarrow u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(cos\theta)$. Orthogonality gives

$$A_m = \frac{2m+1}{2}\frac{1}{R^m}\int_0^\pi u(R,\theta)P_m(cos\theta)sin\theta d\theta$$

Solving PDE's by Fourier Transform(FT)

- FT DE wrt. one variable $\to$ ordinary DE
- Solve $\to$ find FT of the solution
- FT back

Example: $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ with $u(x,0) = b\cdot\delta(x)$ Use x-FT $\to$ 1st order ODE in t.

$$U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}dx$$

so $\mathcal{F}\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -k^2U(k,t), \mathcal{F}\left\{\frac{\partial u}{\partial t}\right\} = \frac{\partial U(k,t)}{\partial t}$

$$\Rightarrow \text{DE: } a^2(-k^2)U(k,t) - \frac{\partial U(k,t)}{\partial t} = 0 \Leftrightarrow \frac{\partial U}{\partial t} + a^2k^2U = 0$$

$$\Rightarrow U(k,t) = \text{const.}\cdot e^{-k^2a^2t} = U(k,0)e^{-a^2k^2t}$$

IC: $\mathcal{F}\{u(x,0)\} = U(k,0) = \mathcal{F}\{b\delta(x)\} = \frac{b}{\sqrt{2\pi}}$

$$\Rightarrow U(k,t) = \frac{b}{\sqrt{2\pi}}e^{-a^2k^2t}$$

Transform back to get the original variable x:

$$u(x,t) = \frac{b}{\sqrt{2\pi}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a^2k^2t}e^{ikx}dk = \frac{b}{\sqrt{4\pi a^2 t}}e^{-\frac{x^2}{4a^2t}}$$