FYS3110 – Home Exam

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Problem 1

a)

The pendulum oscilates with a velocity $v = b\dot{\phi}$. This gives a kinetic energi $K_1 = \frac{1}{2}mv^2 = \frac{1}{2}mb^2\dot{\phi}^2$.

In addition, we have the kinetic energy from the rotation of the wheel around it's own axis, given as $K_2 = \frac{1}{2}I\omega^2 = \frac{1}{2}I(\dot{\phi} + \alpha t)^2$. This gives a total kinetic energy of

$$K = K_1 + K_2 = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2$$

The potential energy is purely from gravity, given by the height of the mass mass point B.

$$V = -mgh = -mgb\cos\phi$$

Giving a total lagrangian of

$$L = K - V = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi$$

or, writing it out.

$$L = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I\dot{\phi}^2 + I\alpha\dot{\phi}t + \frac{1}{2}I\alpha^2t^2 + mgb\cos\phi$$

$$L = \frac{1}{2}\big(mb^2 \dot{+} I\big)\dot{\phi}^2 + I\alpha\dot{\phi}t + mgb\cos\phi + \frac{1}{2}I\alpha^2t^2 \tag{1}$$

b)

We have that

$$\frac{\partial L}{\partial \phi} = -mgb\sin\phi$$

$$\frac{\partial L}{\partial \dot{\phi}} = \big(mb^2 + I\big)\dot{\phi} + I\alpha t$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = \big(mb^2 + I\big)\ddot{\phi} + I\alpha$$

Giving a lagrangian equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0$$

$$\big(mb^2 + I\big)\ddot{\phi} + I\alpha + mgb\sin\phi = 0$$

c)

We have that

$$\frac{\mathrm{d}f(\phi,t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[I\alpha\phi t + \frac{1}{6}I\alpha^2 t^3\right]$$
$$= I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2$$

We reconize the first and last term from the Lagrangian(1), and include the middle term by adding and subtracting it from the Lagrangian, giving

$$L = \left[\frac{1}{2}\left(mb^2\dot{+}I\right)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi\right] + \left[I\alpha\dot{\phi}t + I\alpha\phi + \frac{1}{2}I\alpha^2 t^2\right]$$
$$= L' + \frac{\partial f(\phi,t)}{\partial t}$$

where

$$L' = \frac{1}{2}\left(mb^2\dot{+}I\right)\dot{\phi}^2 + mgb\cos\phi - I\alpha\phi$$

d)

We have

$$\frac{\partial L'}{\partial \phi} = -mgb\sin\phi - I\alpha$$

$$\frac{\partial L'}{\partial \dot{\phi}} = (mb^2 + I)\dot{\phi}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{\phi}} = (mb^2 + I)\ddot{\phi}$$

giving a Lagrangian equation

$$(mb^2 + I)\ddot{\phi} + mgb\sin\phi + I\alpha = 0$$

e)

The canonical momentum p'_ϕ of the coordinate ϕ is already calculated as

$$p'_\phi = \frac{\partial L'}{\partial \dot{\phi}} = (mb^2 + I)\dot{\phi}$$

The Hamiltonian is defined as

$$H = \sum_i p_i\dot{q}_i - L = p'_\phi\dot{\phi} - L'$$
$$= (mb^2 + I)\dot{\phi}^2 - \frac{1}{2}\left(mb^2\dot{+}I\right)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi$$
$$= \frac{1}{2}\left(mb^2\dot{+}I\right)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi$$

f)

g)

Firstly, we have

$$p'_\phi = (mb^2 + I)\dot{\phi} \qquad \Rightarrow \qquad \dot{\phi} = \frac{1}{mb^2 + I}p'_\phi$$

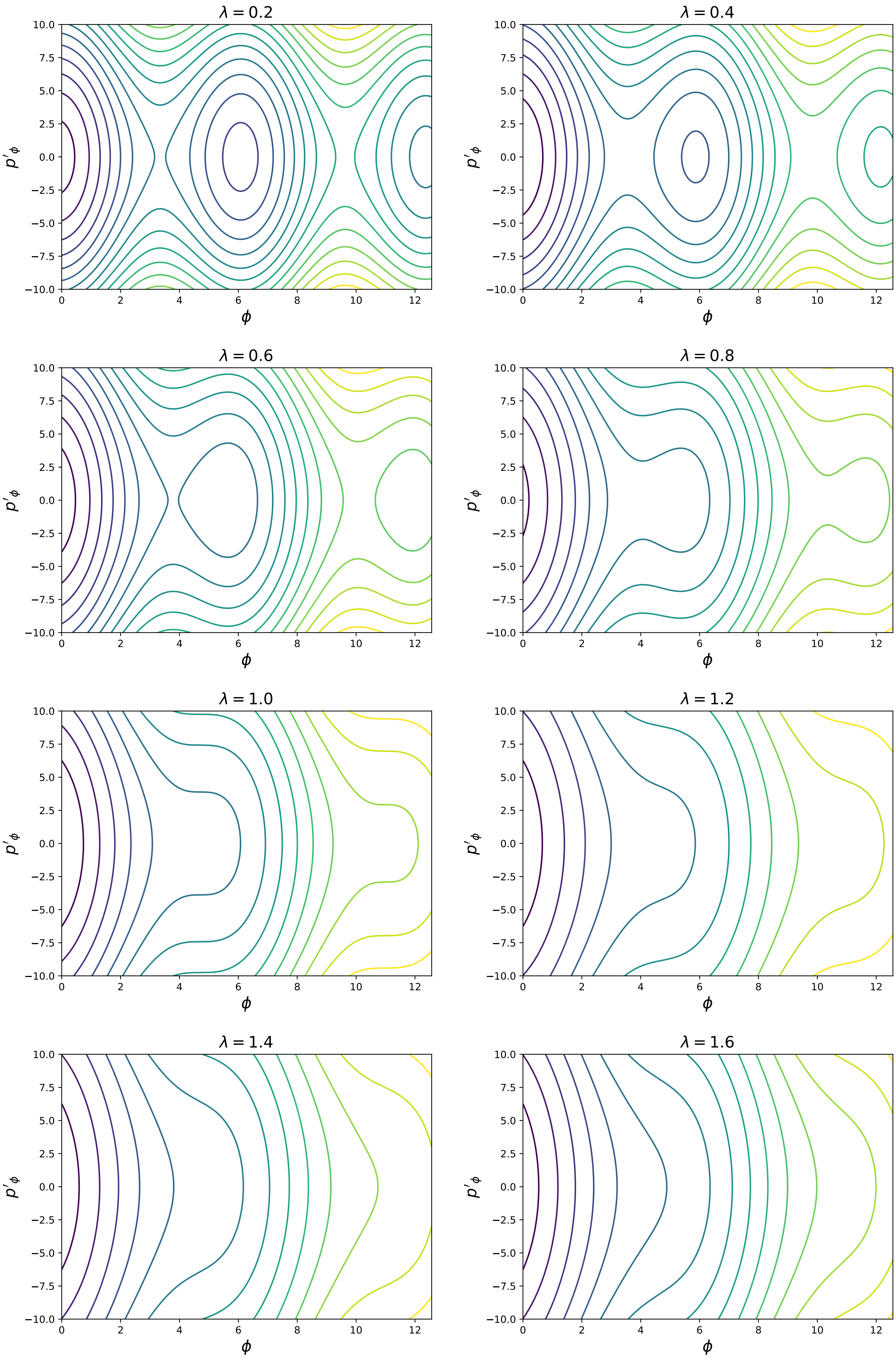

Figure 1: asdf

giving the Hamiltonian as a function of ϕ and p'_ϕ only:

$$\begin{aligned} H &= \frac{1}{2}\left(mb^2 \dot{+} I\right)\dot{\phi}^2 - mgb\cos\phi + I\alpha\phi \\ &= \frac{1}{2}\left(mb^2 \dot{+} I\right)\frac{1}{(mb^2+I)^2}{p'_\phi}^2 - mgb\cos\phi + I\alpha\phi \\ &= \frac{1}{2}\frac{{p'_\phi}^2}{mb^2+I} - mgb\cos\phi + mgb\lambda\phi \end{aligned}$$

Oppgave 2

a)

b)

Conservation of energy gives that particle A and a must have a combined energy equal to that of particle B before the decay.

$$\begin{aligned} E_B &= E_A + E_a \\ \sqrt{P_B^2c^2 + m_B^2c^4} &= \sqrt{P_A^2c^2 + m_A^2c^4} + \sqrt{P_a^2c^2 + m_a^2c^4} \end{aligned}$$

In the rest frame of A, $P_A = 0$, and conservation of momentum gives that $P_a^2 = P_B^2$ (total momentum before decay must equal total momentum after dacay). This gives

$$\begin{aligned} \sqrt{P_a^2 + m_B^2c^2} &= \sqrt{m_A^2c^2} + \sqrt{P_a^2 + m_a^2c^2} \\ \left(\sqrt{P_a^2 + m_B^2c^2} - \sqrt{P_a^2 + m_a^2c^2}\right)^2 &= m_A^2c^2 \\ P_a^2 + m_B^2c^2 - \sqrt{P_a^2 + m_B^2c^2}\sqrt{P_a^2 + m_a^2c^2} + P_a^2 + m_a^2c^2 &= m_A^2c^2 \\ P_a^2 + m_B^2c^2 - \sqrt{P_a^4 + P_a^2m_B^2c^2 + P_a^2m_a^2c^2 + m_a^4c^4} + P_a^2 + m_a^2c^2 &= m_A^2c^2 \end{aligned}$$