CHAPTER 4 Vector Spaces

4.1 Coordinate systems and mapping

Consider a vector $\boldsymbol{x}$ living in a vector space V. The vector $\boldsymbol{x}$ is an abstract concept, living in some abstract space V. It may have some physical or gemoetric meaning or whatnot.

We now enforce a *basis* onto V, called $\mathcal{B} = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$. This makes V *behave like* $\mathbb{R}^n$, in the sense that each vector $\boldsymbol{x}$ in V is *mapped onto* a vector $[\boldsymbol{x}]_\mathcal{B}$ in $\mathbb{R}^n$. This is called a *coordinate mapping* $\boldsymbol{x} \mapsto [\boldsymbol{x}]_\mathcal{B}$ "onto" the basis $\mathcal{B}$. The vector space V might be foreign to us, and it can be important to create a mapping onto a more familiar vector space $\mathbb{R}^n$, which we know how behaves. This transformation is "one-to-one", mapping each point in V onto a point in $\mathbb{R}^n$, and vice versa. This relation is called an **isomorphism**, and makes any vector space V with a basis of n vectors indistinguishable from $\mathbb{R}^n$.

Usually, when a vector is written plainly as $\boldsymbol{x}$, we consider it to be written in a *standard basis* $\mathcal{E} = \{\boldsymbol{e}_1, \boldsymbol{e}_2\} = \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$, meaning that $\boldsymbol{x} = [\boldsymbol{x}]_\mathcal{E}$.

The relation between $\boldsymbol{x}$ and $[\boldsymbol{x}]_\mathcal{B}$ is given by a **change-of-basis matrix** $P_\mathcal{B}$, which consists of the basis-vectors of $\mathcal{B}$, written in the basis of $\mathcal{E}$:

$$\boldsymbol{x} = P_\mathcal{B}[\boldsymbol{x}]_\mathcal{B} \qquad P_\mathcal{B} = [\boldsymbol{b}_1\ \boldsymbol{b}_2\ \dots\ \boldsymbol{b}_n]$$

4.2 Change of basis

This change of basis is just a special case of a more general change of basis between two basises $\mathcal{B} = \{\boldsymbol{b}_1, \dots \boldsymbol{b}_n\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \dots \boldsymbol{c}_n\}$, both spanning the same vector space V. The general change of basis is then

$$[\boldsymbol{x}]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\boldsymbol{x}]_\mathcal{B} \qquad \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \Big[[\boldsymbol{b}_1]_\mathcal{C}\ [\boldsymbol{b}_2]_\mathcal{C}\ \dots[\boldsymbol{b}_1]_\mathcal{C}\Big]$$

The change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$ is simply the inverse: $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \left(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\right)^{-1}$

4.3 Linear transformations (mappings) between vector spaces

Consider two vector spaces V and W, with basises $\mathcal{B} = \{\boldsymbol{b}_1, \dots \boldsymbol{b}_n\}$ and $\mathcal{C} = \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_m\}$ in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We introduce a *linear transformation* $T : V \mapsto W$ such that $T(\boldsymbol{x}) = A\boldsymbol{x}$. This is all well and good, but we might only have the vector $\boldsymbol{x}$ represented in the basis $\mathcal{B}$, and usually want it written in the basis $\mathcal{C}$ after the transformation, as $\big[T(\boldsymbol{x})\big]_\mathcal{C}$.

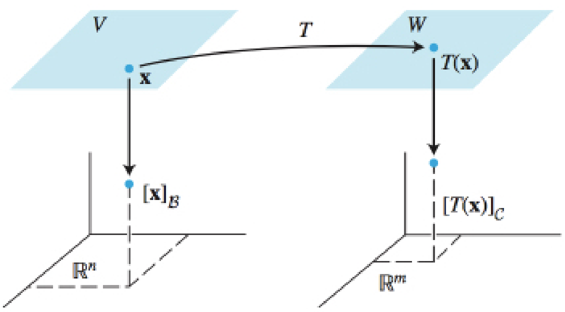

FIGURE 1 A linear transformation from V to W.

What we want is some matrix M that carries us straight from $[\boldsymbol{x}]_\mathcal{B}$ to $\big[T(\boldsymbol{x})\big]_\mathcal{C}$. If we combine the change of basis with T, we get

$$\big[T(\boldsymbol{x})\big]_\mathcal{C} = M[\boldsymbol{x}]_\mathcal{B}$$

where

$$M = \Big[[T(\boldsymbol{b}_1)]_\mathcal{C}\ \dots\ [T(\boldsymbol{b}_n)]_\mathcal{C}\Big] = \Big[[A\boldsymbol{b}_1]_\mathcal{C}\ \dots\ [A\boldsymbol{b}_n]_\mathcal{C}\Big]$$

This matrix is called the **matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.**

CHAPTER 5 Eigenvalues and Eigenvectors

If A has n independent eigenvalues, the eigenvectors of A are linearly independent. If not, we don't know if they are linearly independent or not.

5.1 Diagonalization

If A is a $n \times n$ matrix with with n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$, with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$.

CHAPTER 6 asdf

CHAPTER 7

Orthogonality and least squares

7.1 Projections

7.1.1 Projection of vector onto vector

The projection of a vector $\boldsymbol{y}$ onto another vector $\boldsymbol{x}$ is

$$\hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \cdot \boldsymbol{x}}{\boldsymbol{x} \cdot \boldsymbol{x}} \boldsymbol{x}$$

7.1.2 Projection of vector onto subspace

Let W be subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_p\}$ and $\boldsymbol{y}$ be any vector in $\mathbb{R}^n$. Then the projection of $\boldsymbol{y}$ onto W is simply the projection onto each basis-vector:

$$\hat{\boldsymbol{y}} = \frac{\boldsymbol{y} \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{u}_1} \boldsymbol{u}_1 + \cdots + \frac{\boldsymbol{y} \cdot \boldsymbol{u}_p}{\boldsymbol{u}_p \cdot \boldsymbol{u}_p} \boldsymbol{u}_p$$

7.1.3 The Gram-Schmidt process of orthogonal factorization

The Gram-Schmidt process takes any basis of vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_p\}$ spanning a subspace in $\mathbb{R}^n$, and creates a new *orthogonal* basis of the same space, $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p\}$.

The idea is to create one and one new vector $\boldsymbol{v}_i$ from the corresponding $\boldsymbol{x}_i$, *but* subtract the projection of $\boldsymbol{x}_i$ onto each of the former vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{i-1}$, such that the new vector is orthogonal to all formerly created vectors.

- $\boldsymbol{v}_1 = \boldsymbol{x}_1$
- $\boldsymbol{v}_2 = \boldsymbol{x}_2 - \frac{\boldsymbol{x}_2 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1$
- $\boldsymbol{v}_3 = \boldsymbol{x}_3 - \frac{\boldsymbol{x}_3 \cdot \boldsymbol{v}_1}{\boldsymbol{v}_1 \cdot \boldsymbol{v}_1} \boldsymbol{v}_1 - \frac{\boldsymbol{x}_3 \cdot \boldsymbol{v}_2}{\boldsymbol{v}_2 \cdot \boldsymbol{v}_2} \boldsymbol{v}_2$
- ...