PHYSICS 141A – Problem Set 4

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Exercise 1 - Simon 9.4

We have defined our vibrational nodes as

$$\delta x_n = Ae^{i\omega t}e^{-ikna} \tag{1}$$

while we have the dispersion relation

$$\omega(k) = \sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{ka}{2}\right)\right| = \omega_{max}\left|\sin\left(\frac{ka}{2}\right)\right|$$

We observe that if we set ω larger than ω_{max}, we will obtain a complex k, as sin doesn't output values larger than 1 for real arguments. Let us set $\omega = \sigma\omega_{max}$, $\sigma > 1$, such that

$$\sigma = \frac{\omega}{\omega_{max}} = \left|\sin\left(\frac{ka}{2}\right)\right| \tag{2}$$

Now, considering a complex $k = k_r + ik_c$, we can write equation 1 as

$$\delta x_n = Ae^{i\omega t}e^{-ik_r na}e^{k_c na}$$

The last term, containing k_c, will either blow up as for large n, if $k_c > 0$, or disappear for large n, if $k_c < 0$.

We can write this as a decay-relation

$$\delta x_n = C(t,k)e^{-qa}, \quad q = -k_c a$$

where $C(t,k)$ is some harmonic planewave solution.

Writing out 2 using the $\sin(x) = \frac{i}{2}\left(e^{-ix} - e^{ix}\right)$, we get

$$\sigma = \sin\left(\frac{ka}{2}\right) = \frac{i}{2}\left(e^{-ika/2} - e^{ika/2}\right)$$

Multiplying each side by $2ie^{ika/2}$, we get

$$2i\sigma e^{ika/2} = e^{ika} - 1$$

which can be written as

$$x^2 - 2i\sigma x - 1 = 0, \qquad x = e^{ika/2}$$

which is a 2nd degree polynomial eqn with solutions

$$e^{ika/2} = x = \sigma i \pm \sqrt{1-\sigma^2}$$

Now, we have defined that $\sigma > 1$, such that $\sigma^2 > 1$, and we can define $\sqrt{1-\sigma^2} = \gamma i$, where γ must be real. This gives

$$\begin{aligned}
e^{ika/2} &= \sigma i \pm \gamma i \\
\frac{ika}{2} &= \ln(\sigma i \pm \gamma i) = \ln(i) + \ln(\sigma \pm \gamma) = \frac{\pi}{2}i + \ln(\sigma \pm \gamma) \\
k &= \frac{\pi}{a} + \frac{2}{a}i\ln(\sigma \pm \gamma) = k_r + ik_c
\end{aligned}$$

where we have that $k_c = \ln(\sigma \pm \gamma)$, as we know both σ and γ must be real. Inserting gives

$$k_c = \ln\left(\sigma \pm \frac{\sqrt{1-\sigma^2}}{i}\right)$$

which is real. Remember that $\sigma = \omega/\omega_{max}$.

Exercise 2 - Simon 9.6

Given the mode

$$\delta x_n = Ae^{i\omega t}e^{q|n|a} \tag{3}$$

We have the double time derivative:

$$\delta\ddot{x}_n = A\omega^2 e^{i\omega t}e^{q|n|a}$$

From Simon eqn (9.1), we have Newtons equation of motion written out for neighboring nodes:

$$m\delta\ddot{x}_n = \kappa(\delta x_{n+1} + \delta x_{n-1} - \ 2\delta x_n)$$

Inserting for equation 3 on the left hand side, and the derivative on the right, we get

$$\begin{aligned} Am\omega^2 e^{i\omega t}e^{q|n|a} &= \kappa A e^{i\omega t}\Big[e^{q|n+1|a} + e^{iq|n-1|a} - 2e^{q|n|a}\Big] \\ \omega^2 m &= \kappa\Big[e^{qa(|n+1|-|n|)} + e^{qa(|n-1|-|n|)} - 2e^{qa(|n|-|n|)}\Big] \end{aligned}$$

Now, for $n \geq 1$, we have $|n| = n$, $|n-1| = n-1$ and $|n+1| = n+1$, giving

$$\omega^2 m = \kappa\big[e^{qa} + e^{-qa} - 2\big], \qquad n \geq 1$$

For $n \leq -1$, we have $|n| = -n$, $|n-1| = -n+1$, and $|n+1| = -n-1$, giving

$$\omega^2 m = \kappa\big[e^{-qa} + e^{qa} - 2\big], \qquad n \leq -1$$

which turns out to be the same thing. At $n = 0$, we end up getting the same story. This could also have been seen due to symetry, as swithcing n-direction only switches $|n-1|$ and $|n+1|$, leaving the same expression.

We regocnize this as

$$m\omega^2 = 2\kappa[1 - \cos(qa)]$$

which we regocnize from Simon, with $q = k$. This has frequencies

$$\omega = \sqrt{\frac{\kappa}{m}}\left|\sin\left(\frac{qa}{2}\right)\right|$$

In 9.4, we solved for k, given $\omega > \omega_{max}$, allowing for a complex k. Since q is supposed to be complex, we apply this solution, giving

$$q = \frac{\pi}{a} + \frac{2}{a}i\ln\left(\frac{\omega}{\omega_{max}} \pm \frac{\sqrt{1-\omega^2/\omega_{max}^2}}{i}\right)$$

We know from 9.4 that these kind of waves fall off for $\omega > \omega_{max}$, which is kindoff what we would expect from the waves of the impurity, if they are not in ressonanse with the frequencies of the other material, and has a higher frequency.