Problem 1

$$y''(x) + \frac{3}{x}y'(x) - \frac{24}{x^2}y(x) = 56x^6 \tag{1}$$

Homogeneous solution

The homogenous solution is on the form

$$y'' + \frac{3}{x}y' - \frac{24}{x^2}y = 0 \tag{2}$$
$$x^2y''(x) + 3xy' - 24y = 0 \tag{3}$$

which we recognize as an Euler-Cauchy equation.

We substitute $x = e^z$, and get the equation

$$y''(z) + (3-1)y'(z) - 24y = 0 \tag{4}$$

Which is an ordinary linear first order equation with constant coefficients. The characteristic equation is

$$\lambda^2 + 2\lambda - 24 = 0 \tag{5}$$

The solutions are

$$\lambda_{\pm} = \frac{1}{2}\left[-2 \pm \sqrt{4 - (4 \cdot -24)}\right] = \frac{1}{2}\left[-2 \pm \sqrt{100}\right] = -1 \pm 5 \tag{6}$$

giving $\lambda_+ = 4$ and $\lambda_- = -6$.

Since these are two real roots, the solution is on the form

$$y_h(z) = C_1e^{4z} + C_2e^{-6z} \tag{7}$$

inserting for $z = \ln|x|$ gives

$$y_h(x) = C_1e^{4\ln|x|} + C_2e^{-6\ln|x|} = C_1e^{\ln|x^4|} + C_2e^{\ln|x^{-6}|} = C_1x^4 + C_2x^{-6} \tag{8}$$

Particular solution

We can use variation of parameters to find the particular solution. Firstly, we have two chosen solutions of the homogeneous solution:

$$y_1 = x^4 \qquad y_2 = x^{-6} \tag{9}$$

and the right hand side $R = 56x^6$. We find the Wronskian:

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 = x^4(-6x^{-7}) - 4x^3x^{-6} = -10x^{-3} \tag{10}$$

We then have the particular solution

$$y_p(x) = -x^4\int \frac{x^{-6}56x^6}{-10x^{-3}}\,\mathrm{d}x + x^{-6}\int \frac{x^4 56x^6}{-10x^{-3}}\,\mathrm{d}x \tag{11}$$
$$= -x^4\int \frac{28}{5}x^3\,\mathrm{d}x + x^{-6}\int \frac{28}{5}x^{13}\,\mathrm{d}x \tag{12}$$
$$= -x^4\frac{28}{5}\left[\frac{x^4}{4}\right] - x^{-6}\frac{28}{5}\left[\frac{x^{14}}{14}\right] = \frac{7}{5}x^8 - \frac{2}{5}x^8 = x^8 \tag{13}$$

giving a total solution

$$y(x) = y_h(x) + y_p(x) = C_1x^4 + C_2x^{-6} + x^8 \tag{14}$$

Problem 2: Part A

a)

$$f(z) = \frac{1}{(z-[3+i])^3(z-2i)^4} \tag{15}$$

b)

The function can be factorized into

$$\frac{(z+2)(z+1+i\sqrt{3})(z-1-i\sqrt{3})}{(z-5)^3(z+2)} = \frac{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}{(z-5)^3} \tag{16}$$

As we see, the suspected singularity at $z=2$ does not actually exist.

The other singularity, at $z=5$, is a pole of 3rd order.

Problem 2: Part B

a)

We write out

$$g(z) = \frac{P(z)}{Q(z)} \cdot \pi\cot(\pi z) = \frac{P(z)}{Q(z)} \cdot \frac{\pi\cos(\pi z)}{\pi\sin(\pi z)} = \frac{P(z)\cos(\pi z)}{Q(z)\sin(\pi z)} \tag{17}$$

Equation (6.2) from Boas gives us that

$$Res(g;n) = \frac{P(n)\cos(\pi n)}{[Q(n)\sin(\pi n)]'} \tag{18}$$

which holds as long as $P(n)\cos(\pi n)$ is finite, $Q(n)\sin(\pi n) = 0$, and $[Q(n)\sin(n)]' \neq 0$ which all holds. By the product rule, we get

$$Res(g;n) = \frac{P(n)\cos(\pi n)}{Q'(n)\sin(\pi n) + Q(n)\cos(\pi n)} \tag{19}$$

Since $n \in [0,\ \pm1,\ \pm2, ...]$, we have that $\sin(\pi n) = 0$ and $\cos(\pi n) = 1$, giving

$$Res(g;n) = \frac{P(n)\cos(\pi n)}{Q(n)\cos(\pi n)} = \frac{P(n)}{Q(n)} = f(n) \tag{20}$$

b)

Firstly, we will show that

$$\lim_{N\to\infty} \int\limits_{\Gamma_N} F(z)\,\mathrm{d}z = 0 \tag{21}$$

for some rational function $F(z) = P(z)/Q(z)$ when $Q(z)$ is a polynomial of at least 2 degrees higher than $P(z)$.

We have the relation

$$\left| \int\limits_{\Gamma_N} F(z)\,\mathrm{d}z \right| \leq m \cdot L \tag{22}$$

where m is the maximum value of $F(z)$ on Γ, and L is the length of the contour Γ.

This means that

$$\lim_{N\to\infty} \left| \int\limits_{\Gamma_N} F(z)\,\mathrm{d}z \right| \leq \lim_{N\to\infty} m \cdot L \tag{23}$$

The length of the contour, L, will be go as $\propto N$, since it's a square with sides proportional to N. The max of the function, m, will go as $\propto \frac{1}{z^2} \propto \frac{1}{N^2}$, since $F(z) = \frac{P(z)}{Q(z)}$, where $degree(Q) \geq degree(P) + 2$. This means that $m \cdot L \propto \frac{1}{N}$, giving that

$$\lim_{N\to\infty} m \cdot L = 0 \tag{24}$$

such that

$$\lim_{N\to\infty} \left| \int\limits_{\Gamma_N} F(z) \right| \leq \lim_{N\to\infty} m \cdot L = 0 \tag{25}$$

meaning the relation 21 holds.

Since $|\pi \cot(\pi z)| \leq M$ over all z for some constant M, we have the relation

$$\left| \lim_{N\to\infty} \int\limits_{\Gamma_N} f(z)\pi \cot(\pi z)\, \mathrm{d}z \right| \leq \left| \lim_{N\to\infty} \int\limits_{\Gamma_N} f(z)M\, \mathrm{d}z \right| \tag{26}$$

where $f(z)M$ now is a rational function of the type we defined in 21 (since adding a constant M doesn't change degrees). We therefore have

$$\left| \lim_{N\to\infty} \int\limits_{\Gamma_N} g(z)\, \mathrm{d}z \right| \leq \left| \lim_{N\to\infty} \int\limits_{\Gamma_N} f(z)M\, \mathrm{d}z \right| = 0 \tag{27}$$

or

$$\lim_{N\to\infty} \int\limits_{\Gamma_N} g(z)\, \mathrm{d}z = 0 \tag{28}$$

c)

The residue theorem states that

$$\int\limits_{\Gamma_N} g(z)\, \mathrm{d}z = 2\pi i \sum Res(g(z); z_i) \tag{29}$$

where z_i are all the singularities of $g(z)$. We can split the residues up into the singularities of $f(z)$ and those of $\pi \cot(\pi z)$, and let $N \to \infty$

$$\lim_{N\to\infty} \int\limits_{\Gamma_N} g(z)\, \mathrm{d}z = \lim_{N\to\infty} 2\pi i \Big[\sum Res(g(z); n) + \sum Res(g(z);\ \text{poles of } f(z))\Big] \tag{30}$$

We know from 28 that this equals 0. Also inserting $\sum Res(g(z); n) = f(n)$ gives

$$\lim_{N\to\infty} 2\pi i \left[\sum_{n=-N}^{N} f(n) + \sum Res(g(z);\ \text{poles of } f(z))\right] = 0 \tag{31}$$

$$\sum_{n=-N}^{N} f(n) = -\sum Res(g(z);\ \text{poles of } f(z)) \tag{32}$$

d)

We have the function

$$f(n) = \frac{1}{1+n^2} = \frac{1}{(n+i)(n-1)} \tag{33}$$

with simple poles at $n = \pm i$. Using 32, we have that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{1+n^2} = -\sum (\text{Residues of } g(z) \text{at the poles of } f(z)) \tag{34}$$

where

$$g(n) = \pi\cot(\pi n)\frac{1}{(n+i)(n-i)} \tag{35}$$

The residues can be found as

$$Res(g(n);n) = \lim_{z\to z_0} g(z_0) \tag{36}$$

This gives the residues

$$Res(g(n);i) = \lim_{z\to i}\left[(n-i)\pi\cot(i\pi)\frac{1}{(n+i)(n-i)}\right] = \frac{\pi\cot(i\pi)}{2i} = -\frac{1}{2}\pi\coth(\pi) \tag{37}$$

where we used the relation $\cot(i\pi) = -i\coth(i)$.

$$Res(g(n);-i) = \lim_{z\to -i}\left[(n+i)\pi\cot(-i\pi)\frac{1}{(n+i)(n-i)}\right] = \frac{\pi\cot(-i\pi)}{-2i} = -\frac{1}{2}\pi\coth(\pi) \tag{38}$$

This gives

$$\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{1+n^2} = -\sum(\text{Residues of } g(z)\text{at the poles of } f(z)) = \pi\coth\pi \tag{39}$$

Problem 3

a)

We will prove the relation

$$\delta[f(t)] = \sum_i \frac{1}{|f'(t_i)|}\delta[t-t_i] \tag{40}$$

by massaging the left hand side into the right. Since this is an operator, we apply a test-function $g(t)$, and integrate over all t. We will remove these at the end:

$$= \int_{-\infty}^{\infty} \delta[f(t)]\cdot g(t)\,\mathrm{d}t \tag{41}$$

$\delta[f(t)]$ will be a series of delta-spikes at the zero-points of f, $f(t_i)$, and zero everywher else. We can therefore rewrite the integral to a sum of integrals around each of the zero-points t_i.

$$= \sum_i \int_{t_i-\epsilon}^{t_i+\epsilon} \delta[f(t)]\cdot g(t)\,\mathrm{d}t \tag{42}$$

We Taylor-expand $f(t)$ around each zero-point t_i:

$$f(t) = \underbrace{f(t_i)}_{=0} + f'(t_i)(t-t_i) + \mathcal{O}(t^2) \approx f'(t_i)(t-t_i) \tag{43}$$

Inserting this gives

$$\sum_i \int_{t_i-\epsilon}^{t_i+\epsilon} \delta[f'(t_i)(t-t_i)]g(t)\,\mathrm{d}t \tag{44}$$

Using the identity $\delta[\alpha x] = \frac{1}{|\alpha|}x$ we get

$$\sum_i \int_{t_i-\epsilon}^{t_i+\epsilon} \frac{1}{|f'(t_i)|}\delta[t-t_i]g(t)\,\mathrm{d}t \tag{45}$$

This is now a sum of integrations over delta-spikes (multiplied by some stuff). We can rewrite this as a integral over the sum of all the delta-spikes by switching the order of the sum and the integral. Since the integrand is zero outside the proximity of the t_i's anyway, we can just let the integral run from $-\infty$ to ∞:

$$\int_{-\infty}^{\infty} \sum_i \frac{1}{|f'(t_i)|} \delta[t - t_i] g(t) \, \mathrm{d}t \tag{46}$$

Finally, removing the integral and test-function, we get the left hand side of equation 40

$$\sum_i \frac{1}{|f'(t_i)|} \delta[t - t_i] \tag{47}$$

and our identity is proven.