CSCA 67. Assignment 1.

$\neg\left[(rs>n)\vee(r\leq\sqrt{n}\vee s\leq\sqrt{n})\right]$

1. (a). Prove: $\forall n\in N^+, r\in N^+, s\in N^+$. $rs\leq n \Rightarrow r\leq\sqrt{n}$ or $s\leq\sqrt{n}$

Contrasive: $\forall n\in N^+, r\in N^+, s\in N^+$, $(rs\leq n)\wedge(r>\sqrt{n})\wedge(s>\sqrt{n})$.

in the contrasive statement, $r>\sqrt{n}$, $s>\sqrt{n}$,

$\therefore r\cdot s>\sqrt{n}\cdot\sqrt{n}=n$.

which contradicts to $rs\leq n$

Therefore the contrasive statement is wrong

The original statement is proved.

(b). Prove: $\forall n\in N^+$, n composite $\rightarrow \exists p\in N^+$, p is prime and $p\leq\sqrt{n}$ and $p|n$.

Contradiction: $\forall n\in N^+$, n composite $\wedge$ $\forall p\in N^+$, p is not prime or $p>\sqrt{n}$ or $p\nmid n$.

But for $n=50$, $p=5$ p is prime, $p<\sqrt{n}$, $p|n$.

which contradict the contrasive statement.

Therefore the contradiction statement is wrong

The original statement is true.

(c) Contrapositive: $\forall p\in N^+$, p is not prime or $p>\sqrt{n}$ or $p\nmid n$,

$\exists n\in N^+$, n is a prime.

2. For $n=1$, $n=2$, the condition holds.

For every integer x such that $1<x<n!$, we have $x|n!$ and $x\nmid(n!-1)$

So either $(n!-1)$ is prime, or $\exists p$ that $p\geq(n+1)$ and $p|(n!-1)$

So in any case, there exist a p such that $n<p\leq n!+1$

3.

$\forall k \in N^+.\ \forall m \in N,\ k=2m+1 \Rightarrow 11 | (10^k+1)$

let $S(k)$ be. 10^k+1

Base Case: $m=0,\ k=1,\ S(k)=11,\ 11|11.$

$S(1)$ holds.

Case 2: for every $S(k+2)$.

$$S(k+2)-S(k) = 10^{(k+2)}+1-(10^k+1)$$
$$= 10^k \cdot 100 - 10^k$$
$$= 10^k(100-1)$$
$$= 11 \cdot 9 \cdot 10^k$$, which is divisible by 11.

So if $S(k)$ is divisible by 11, $S(k+2)$ will be divisible.

The base case $S(1)$ holds, $k=2m+1$, which increase 2 by each time.

So $S(k)$ holds.

The statement is proved.

for $n = d_k d_{k-1} d_{k-2} \cdots d_1 d_0$,

it can be represented as $n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} \cdots + d_1 \cdot 10 + d_0 \cdot 10^0$

if k is odd, $n = d_k[(10^k+1)-1] + d_{k-1}[(10^{k-1}-1)+1] + \cdots + d_1[(10+1)-1] + d_0$

$$n = [d_k(10^k+1) + d_{k-2}(10^{k-2}+1) + d_{k-4}(10^{k-4}+1) + \cdots + d_1(10+1)] \quad \cdots ①$$
$$+ [d_{k-1}(10^{k-1}-1) + d_{k-3}(10^{k-3}-1) + \cdots + d_2(10^2-1)] \quad \cdots ②$$
$$+ [d_0 - d_1 + d_2 - d_3 + \cdots + d_i - d_{i+1} + \cdots - d_k] \quad \cdots ③$$

We can seperate n into 3 parts as shown above.

①. for the first part, as proved above, $\forall k \in N^+, \forall m \in N, k=2m+1 \Rightarrow 11|(10^k+1)$.
each part of the first part is divisible by 11. Therefore part 1 is divisible by 11

② for $(10^{k-1}-1)$, k is odd. base case $k=1$. $10^{k-1}-1 = 0$. $11|0$. $S(1)$ holds.

Case 2 $k=3$. $10^2-1 = (10+1)(10-1) = 99$. $S(3)$ holds

Case 3. $k=5$. $10^4-1 = (10^2+1)S(2)$. as $S(3)$ holds. $S(5)$ holds

Cas k $10^{k-1}-1 = (10^{\frac{k}{2}}+1)S(\frac{k}{2}-1)$

as $S(\frac{k}{2}-1)$ is divisible by 11, $S(k)$ holds.

$\therefore$ for every part in the second part, they are divisible by 11. So the Second part is divisible by 11

For k is even, the proof is the same.

So if the third part is divisible by 11, n is divisible by 11. The third part is S_n.

$S_n = d_0 - d_1 + d_2 - d_3 + \cdots + d_i - d_{i+1} + \cdots d_k$ QED.

5. Solution: Let the pigeons be the $(n+1)$ integers selected.

Seperate the $2n$ numbers into 2 parts: odd and even.

for the odd numbers: $\{1, 3, 5, 7 \cdots (2k-1) \cdots (2n-1)\}$ $(2k-1) < 2n, k \in N$

for the even numbers: $\{2, 4, 6, 8 \cdots (2k-1)2^r \cdots 2n\}$ $r \geq 0, r \in N$

So the odd numbers can be represented as $(2k-1)$ where there are n of them

the even numbers can be represented as $(2k-1)2^r$ where there are n of them

for all $k = 1, 2, 3, \cdots n$, every element in the set $\{1, 2, 3 \cdots 2n\}$ can be written as $2^a(2b-1)$ while $a \in N$ and $b \in \{1, 2 \cdots n\}$.

To set up the pigeon hole, notice that the largest odd diviser of an integer between 1 and $2n$ is one of the n numbers

$1, 3, 5, 7, \cdots 2n-1$

Set up the pigeon holes by partitioning the numbers between 1 and $2n$ into subsets have the same largest divisor.

Since $(n+1)$ integers are selected, the Pigeon Hole principle guarantees that there are 2 of them have the same largest odd divisor, t. Let the 2 numbers be a and b, $(a<b)$.

Then $a = 2^{k_1}t$, $b = 2^{k_2}t$ $(k_1, k_2 \in N, k_1 < k_2)$

So $a \cdot 2^{(k_2-k_1)} = b$. as $k_1 < k_2$, (k_2-k_1) is a positive integer.

So $a | b$.

The given statement is proved.