CSCA67 Worksheet – Predicates, Quantifiers, Direct Proofs

1 Sets and Venn Diagrams

Exercise. Which of these mean the same thing?

1. if $x \in A$ then $x \in B$
2. if $x \notin A$ then $x \notin B$
3. if $x \in B$ then $x \in A$
4. if $x \notin B$ then $x \notin A$

Let's draw Venn diagrams to figure it out...

Q. If we have $x \in A \to x \in B$, then what do we call $x \notin B \to x \notin A$?

A.

Q. If we have $x \in A \to x \in B$, then what do we call $x \in B \to x \in A$?

A.

Exercise Which of these have the same meaning?

- $x \notin A \to x \in B$
- $x \in A \to x \notin B$
- $x \notin B \to x \in A$
- $x \in B \to x \notin A$

Exercise. Draw a Venn diagram representing three sets A, B and C. Now shade in the region that represents when the statement

$$A \wedge B \to \neg C$$

is true.

Q. How could we have made this task easier?

A.

2 If and Only If

Sometimes we see the $\leftrightarrow$ symbol. What does it stand for? *If and only if* or *iff*.

Q. If we say $A \leftrightarrow B$ what do we mean in terms of $\to$?

A.

A	B	$A \to B$	$B \to A$	$A \leftrightarrow B$
T	T			
T	F			
F	T			
F	F			

3 Direct Proofs and Quantifiers

Let's try writing a simple direct proof.

Claim. For all real numbers x, if $x > a > 0$, then $x^2 > a^2$.

Proof.

Observation. Notice that here we have an implication of the form *for all* $x \in \mathbb{R}, P(x) \to Q(x)$ or mathematically,

$$\forall x \in D, P(x) \to Q(x)$$

with a variable called x where $P(x)$ means $x > 1$, $Q(x)$ means $x^2 > 1$ and D means $\mathbb{R}$. We call these **predicates**.

Form for such proofs, our proof structure often follows the form:

Direct Proof.

$$\forall x \in D, P(x) \to Q(x)$$

- Let x be arbitrary or any element of the domain.
- Suppose that $P(x)$ is true.
- Use true sentences to derive that $Q(x)$ is true.

Exercise Prove that for all real numbers x and y, if x and y are rational then xy is rational.

Proof.

Exercise Prove that there does not exist a largest natural number.

Let's first try to rewrite this in a way that uses "*for all*" and "*there exists*".

Can we rewrite this expression using the symbols $\forall$ (for all) and $\exists$ (there exists)?

Proof.

Notice that $y > x$ represents a predicate with two variables and that the format of our statement is $\forall x \in D, \exists y \in D, S(x, y)$.

Q. Does the order of the $\forall x$ and $\exists y$ matter? Are these the same?

$$\forall x \in D, \exists y \in D, S(x, y) \quad \overset{?}{\Leftrightarrow} \quad \exists y \in D, \forall x \in D, S(x, y)$$

A.

Q. What do the following statements say in *English* if:

$S(x, y)$: x **scares** y

$D = \{$all people (and superheroes/characters)$\}$?

1. $\exists x \in D, \exists y \in D, S(x, y)$

2. $\exists y \in D, \exists x \in D, S(x, y)$

3. $\forall x \in D, \forall y \in D, S(x, y)$

4. $\forall y \in D, \forall x \in D, S(x, y)$

5. $\forall x \in D, \exists y \in D, S(x, y)$

6. $\exists y \in D, \forall x \in D, S(x, y)$

7. $\exists x \in D, \forall y \in D, S(x, y)$

8. $\forall y \in D, \exists x \in D, S(x, y)$

Q. Which statements have the *same* meaning?

A.

Let's get a little practice writing with *predicates* and *quantifiers* by going back to the statement:

> **For all** real numbers x and y, **if** x and y are rational **then** xy is rational.

Rewrite this using $\forall$:

Q. How can we express x is *rational* using quantifiers?

A.

Q How can we express x is *not* rational using quantifiers?

A.

Let's write this in English:

Q. How can we simplify this mathematically? What is $\neg\forall$?

A.

4 A Proof Using mod

Definition. $a \textbf{ mod } n = b$ means that $a \div n$ has a *remainder* b.

Definition. $a \equiv_n b$ means that $a \textbf{ mod } n = b \textbf{ mod } n$.

Theorem. *For all integers a, b and n with $n \geq 1, a \equiv_n b$ iff n divides $a - b$.*

We will prove this theorem. We split it into two separate proofs. What should they be?

Proof of ($\rightarrow$):

Proof of ($\leftarrow$):