Exercise 6.

1. Let $S(n)$ be the sum of the squares of the first n natural numbers starting at 1, is $\frac{n(n+1)(2n+1)}{6}$.

$$\forall n \in N,\ n \geq 1.\ \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case $n=1$, $1^2 = \frac{1\times(1+1)(2+1)}{6} = \frac{6}{6} = 1.$

I. H. ($n=k$, holds)

$$\sum_{i=1}^{n=k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

I. S. check $n=k+1$.

$$\sum_{i=1}^{n=k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \underbrace{(1^2+2^2+\dots+k^2)}_{\text{by I.H.}} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= (k+1)(2k+3)(k+2)/6$$

$$= (k+1)(k+1+1)[2(k+1)+1]/6 .$$

Proved.

2. a) We have k old lines, l new lines when the $(k+1)^{st}$ line is added.

we can control the number of new triangles like this:

There are $\binom{k}{2}$ ways to pick 2 old lines from k old lines

For every 2 old lines, we can match with this l new lines to form new triangles. So in total, $\binom{k}{2}$ new triangles are formed. [or $C(k,2)$]

2 b) We have $(k+1)$ ~~odd~~ old lines, 1 new lines when $(k+2)^{th}$ line is added.

Here the logic is similar to part a), so we'll have $\binom{k+1}{2}$ or $C(k+1, 2)$ new triangles.

c) Total numbers:

$$C(k,3) + C(k,2) + C(k+1,2)$$

$$= \frac{k!}{(k-3)!\,3!} + \frac{k!}{(k-2)!\,2!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k+1-3)k!}{(k+1-3)(k-3)!\,3!} + \frac{3k!}{(k-2)!\,3\cdot 2!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k-2)k!}{(k-2)(k-3)!\,3!} + \frac{3k!}{(k-2)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k-2)k! + 3k!}{(k-2)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k+1)k!}{(k-2)!\,3!} + C(k+1,2)$$

$$= \frac{(k+1)!\cdot k!}{(k+1-3)!\,3!} + C(k+1,2)$$

$$= C(k+1,3) + C(k+1,2)$$

$$= \frac{(k+1)!}{(k-2)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k+1-2)(k+1)!}{(k+1-2)(k+1-3)!\,3!} + \frac{3(k+1)!}{(k-1)!\,3!}$$

$$= \frac{(k-1)(k+1)! + 3(k+1)!}{(k+1-2)!\,3!}$$

$$= \frac{(k+2)(k+1)!}{(k+1-2)!\,3!}$$

$$= \frac{(k+2)!}{(k+2-3)!\,3!}$$

$$= C(k+2, 3)$$

So when the $(k+2)$ nd line is add,

There're $C(k+2,3)$ triangles in total.

3. let x be the numbers of ways to choose one pizza.

$$x = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + 7 + 7 + 7\times 6$$

$= 120$ ways.

let y be the number of ways to choose 2 pizzas.

$$y = \underset{\substack{\uparrow \\ 2\ \text{same}}}{120} + \underset{\substack{\uparrow \\ 2\ \text{different}}}{\binom{120}{2}} = 7260 \text{ ways.}$$

So there are 7260 ways in total.

4. we have following arrangements.

$\square\ x\ \square\ x\ \square\ x\ \square\ x\ \square\ x\ \square\ x\ \square$.

4 "y" are waiting to be added into the role so that they can be apart from each one. Only one in one box.

There are $\binom{7}{4}$ ways to do that.

$C(7, 4) = 35$

So there are 35 ways to arrange 6 x & 4 y in a row to make each y seperate from each other.