CSCA67 Tutorial, Week 4*

September 21, 2017

1 Predicate Logic

So far, we have discussed something that we have referred to as "formal logic." However, we have only studied one branch of formal logic, which we will now properly call SENTENTIAL LOGIC (also known as "propositional logic").

Here, we introduce another branch of formal logic: PREDICATE LOGIC.

Predicate logic addresses some of the limitations of sentential logic:

- Sentential logic cannot express quantitities/categories, eg. "*Every* person has a mother," "*Some* horses are white"
- Sentential logic cannot express relationships between properties, eg. "If X is married to Y, then Y is married to X"

Predicate logic uses PREDICATES instead of statements/propositions. Predicates are sentences containing variables which can be assigned values. For example, "$x > 5$" is a predicate, where x is the variable.

Predicates are *not* statements themselves because they have no inherent truth value.
CONSIDER: can we say definitively that "$x > 5$" is true or false?

However, once the variable(s) in the predicate have been assigned values, then the sentence becomes a statement. For example, if we let $x = 3$, then "$x > 5$" is a false statement, and if we let $x = 6$, then "$x > 5$" is a true statement.

1.1 *Syntax*

The language of sentential logic is composed of: sentence symbols, operators/connectives, and parentheses. Since predicate logic uses predicates rather than statements, it uses predicate symbols rather than sentence symbols.

Predicate symbols typically follow mathematical function notation, eg. $\overbrace{P(\underbrace{\ x\ }_{\text{variable}})}^{\text{predicate}}$.

The name of the predicate is typically a single uppercase letter, and the variable is denoted as a single lowercase letter.

NOTE that, as with a mathematical function, the letter chosen to represent the variable is simply a place-holder. $P(x)$ is identical to $P(y)$, $P(z)$, etc.

We can then use the same connectives as in sentential logic to combine predicates and create compound predicates, eg.

*Compiled by G. Singh Cadieux

- $P(x):\ 2x = 6$
  - $P(3)$ is true
  - $P(10)$ is false
- $Q(x, y):$ "x is larger than y"
  - $Q(\text{Earth}, \text{moon})$ is true
  - $Q(\text{apple}, \text{car})$ is false
- $R(x):\ x < 10$
- $P(x) \wedge R(x) \equiv$ "x is between 5 and 10"
  - $P(6) \wedge R(6)$ is true
  - $P(14) \wedge R(14)$ is false

NOTE that, although x is a placeholder, x represents the same value everywhere it is used within a predicate. For example, if we create the compound predicate $P(x) \wedge R(x)$, we cannot assign values such that we have $P(6) \wedge R(14)$.

Conversely, different variables may represent different values. For example, if we create the compound predicate $P(x) \wedge R(y)$, we *can* assign $x = 6$ and $y = 14$ such that we have $P(6) \wedge R(14)$.

- $S(x):\ x < 20$
- $\neg S(x) \equiv$ "x is greater than or equal to 20"
- $R(x) \rightarrow S(x) \equiv$ "if x is less than 10, then it is also less than 20"
  - $R(3) \rightarrow S(3)$ is true
  - $R(25) \rightarrow S(25)$ is (vacuously) true

Predicate logic also adds operators called QUANTIFIERS, which allow us to describe ranges of values for variables, rather than assigning them individual values.

The EXISTENTIAL quantifier is denoted $\exists$ and means "(there) exists."

For example, if $P(x)$: "$x^2 = 5$", then $\exists x(P(x))$ means "There exists (at least) one value of x for which $x^2 = 5$". The statement $\exists x(P(x))$ is true, since $P(\sqrt{5})$ and $P(-\sqrt{5})$ are both true.

If $Q(x)$: "$x^2 = -5$", then $\exists x(Q(x))$ means "There exists (at least) one value of x for which $x^2 = -5$", which is false (assuming that x is restricted to the set of reals).

The UNIVERSAL quantifier is denoted $\forall$ and means "for all/every."

For example, if $P(x)$: "$x^2 = 5$", then $\forall x(P(x))$ means "For every possible value of x, $x^2 = 5$". The statement $\forall x(P(x))$ is false, since only $P(\sqrt{5})$ and $P(-\sqrt{5})$ are true, and $P(x)$ is false for any other value of x.

If $R(x):\ x > 5$ and $S(x):\ x > 7$, then $\forall x(S(x) \rightarrow R(x))$ means "For every possible value of x, if x is greater than 7, x is also greater than 5". We know that this statement is true, since $5 < 7$ and $7 < x$.

As with sentential logic, we may use parentheses to group predicates and operators. When parentheses are omitted, the operators are applied according to the following precedence rules:

$$\begin{array}{c} \text{highest} \\ \text{precedence} \end{array} \; \xleftrightarrow{\quad \neg \quad \wedge \quad \vee \quad \begin{array}{c}\exists \\ \forall\end{array} \quad \begin{array}{c}\rightarrow \\ \leftrightarrow\end{array} \quad} \; \begin{array}{c} \text{lowest} \\ \text{precedence} \end{array}$$

2 Practice problems

Let $P(x,y)$ be the predicate "$x \cdot y = 12$", where x, y are integers.
Q: Which of the following statements is true?

- $P(3,4)$
- $P(3,5)$
- $P(2,6) \vee P(3,7)$
- $\forall x, \forall y (P(x,y) \rightarrow P(y,x))$
- $\forall x, \exists y (P(x,y))$

Given the predicates
$L(x) =$"x is a lion."
$F(x) =$"x is fuzzy."
where x is a mammal,
Q: Translate "All lions are fuzzy" into predicate logic.

Q: Translate "Some lions are fuzzy" into predicate logic.

Consider the three predicates
$P(x)$ symbolizes the statement "x is a prime number"
$E(x)$ symbolizes the statement "x is even"
$D(x,y)$ symbolizes the statement "x evenly divides y"
where x and y represent integers.
Q: Find some values for the variables that make the following logical formulas true, and others making them false.

- $P(x) \wedge E(x)$
- $E(x) \vee D(x,y)$
- $\neg P(x) \wedge D(x,y)$
- $D(x,y) \rightarrow \neg P(x)$

3 Ordering of quantifiers - *Will be covered in lectures during week 4 - try to get students to figure this out in groups.*

When we have a statement containing multiple, different quantifiers, such as $\forall x \exists y (P(x,y))$, the quantified variables are read left to right. For example, if $P(x,y) : x \cdot y = 5$, then $\forall x \exists y (P(x,y))$ means "For every number x, there exists a number y such that $x \cdot y = 5$".

Does the meaning of the sentence change if we reorder the quantifiers?
It depends upon the predicate.

For example, if $P(x,y) : x > y$, then $\forall x \exists y (P(x,y))$ means "For every number x, there exists a number y such that x is greater than y".
We know that this statement is true because the set of integers (or reals, depending upon the allowed values of x) is infinite. Thus, by definition, there is always a number larger than any number we choose from the set.

But $\exists y \forall x (P(x,y))$ means "There exists a number y such that, for every number x, x is greater than y".
We know that this statement is false because it is saying that there is some minimum integer (since every

other integer is larger than it). Since the integers are infinite, this is impossible.

Here, the order of the quantifiers is significant.

3.1 Let's practice.

Determine what each of the following means

1. $\forall x \in \mathbb{N}, \exists y \in \mathbb{N}, (x < y)$.
2. $\exists y \in \mathbb{N}, \forall x \in \mathbb{N}, (x < y)$.
3. $\exists x \in \mathbb{N}, \forall y \in \mathbb{N}, (x < y)$.
4. $\forall y \in \mathbb{N}, \exists x \in \mathbb{N}, (x < y)$.
5. $\exists x \in \mathbb{N}, \exists y \in \mathbb{N}, (x < y)$.
6. $\forall x \in \mathbb{N}, \forall y \in \mathbb{N}, (x < y)$.

4 Negation of quantifiers – *Will be covered in lectures during week 4 - try to get students to figure this out in groups.*

In addition to negating predicates using the negation operator, we want to be able to negate quantifiers - that is, we want to be able to say "There does *not* exist…" and "*It is not the case that* for all…".

For some predicate $P(x)$, "There does *not* exist an x such that $P(x)$ is true" is equivalent to saying "$P(x)$ is false for every value of x", which we can write formally as $\forall x(\neg P(x))$.

For some predicate $P(x)$, "*It is not the case that*, for every value of x, $P(x)$ is true" is equivalent to saying "There is some value of x for which $P(x)$ is false", which we can write formally as $\exists x(\neg P(x))$.

Thus, the negation of $\exists x(P(x))$ is $\forall x(\neg P(x))$ and the negation of $\forall x(P(x))$ is $\exists x(\neg P(x))$.

For example, if $P(x)$: "x is tall", where x is a person, then

- $\exists x(P(x))$ means "There exists a person who is tall"
- $\forall x(\neg(P(x))$ means "Every person is not tall", or "There does not exist a person who is tall"
- $\forall x(P(x))$ means "All people are tall"
- $\exists x(\neg P(x))$ means "There exists a person who is not tall", or "Not every person is tall"

Tarski's World is a computer program that is meant to be an introduction to predicate logic.
You can build two-dimensional worlds of shapes, describe them using predicates, and test whether your predicates are true or false.
Experiment with Tarski's World using the following implementation: Tarski's World (Java Applet)