CSCA67 Tutorial, Week 3*

September 14, 2017

0.1 Logical Equivalence

Two statements are LOGICALLY EQUIVALENT if they have the same truth table - that is, given the same combination of truth values for their constituent statements, they both have the same truth value.

For example, using truth tables, we can demonstrate that p and $\neg\neg p$ are logically equivalent:

p
T
F

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

Likewise, we can demonstrate that $\neg y \wedge (y \vee x)$ and $\neg y \wedge x$ are logically equivalent:

x	y	$\neg y$	$y \vee x$	$\neg y \wedge (y \vee x)$	$\neg y \wedge x$
T	T	F	T	F	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	F	F	F

Q: *Show that* $(a \vee b) \wedge \neg(a \wedge b)$ *is logically equivalent to* $a \leftrightarrow \neg b$ *using truth tables.*

Using the method discussed last week, we construct a truth table for each of the statements.
(The truth table shown below merges the two truth tables into one.)

a	b	$a \wedge b$	$\neg(a \wedge b)$	$a \vee b$	$(a \vee b) \wedge \neg(a \wedge b)$	$\neg b$	$a \leftrightarrow \neg b$
T	T	T	F	T	F	F	F
T	F	F	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	F	T	F	F	T	F

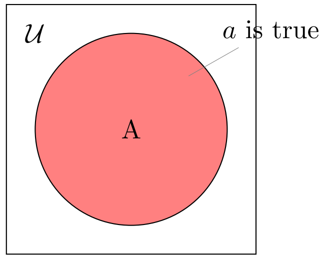
We have shown that the truth table for $(a \vee b) \wedge \neg(a \wedge b)$ and the truth table for $a \leftrightarrow \neg b$ are identical (the truth value of two statements is the same, given the same combination of truth values of a and b). Thus, the two statements are logically equivalent.

0.2 Using Venn Diagrams

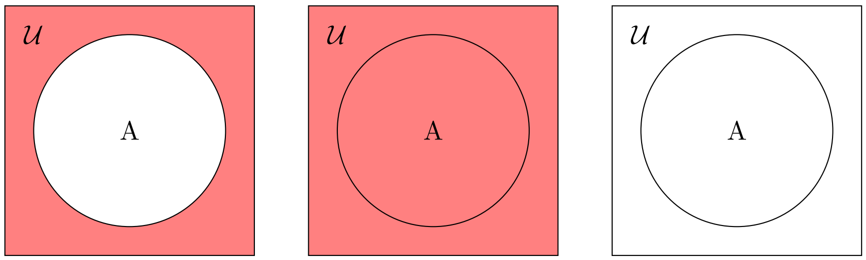
We can consider any statement a to be the statement "$x \in A$", where A is a set and x is an element.

Then, if we draw a Venn diagram containing A, a is true at every location in the diagram where an element x at that location is in A. a is false everywhere else.

*Compiled by G. Singh Cadieux



Q: *Shade the region(s) where each of the following is true:* $\neg a$, $\neg a \vee a$, $\neg a \wedge a$.

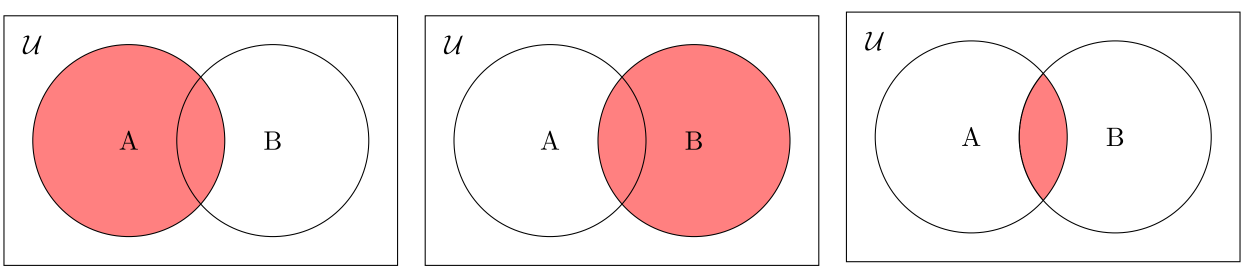


$\neg(x \in A)$ $\qquad$ $\neg(x \in A) \vee (x \in A)$ $\qquad$ $\neg(x \in A) \wedge (x \in A)$

NOTICE that $\neg a \vee a$ is true for every region in the diagram. This is because, regardless of the truth value of a (or of its constituent statements), $\neg a \vee a$ is always true. This type of statement is known as a *tautology*.

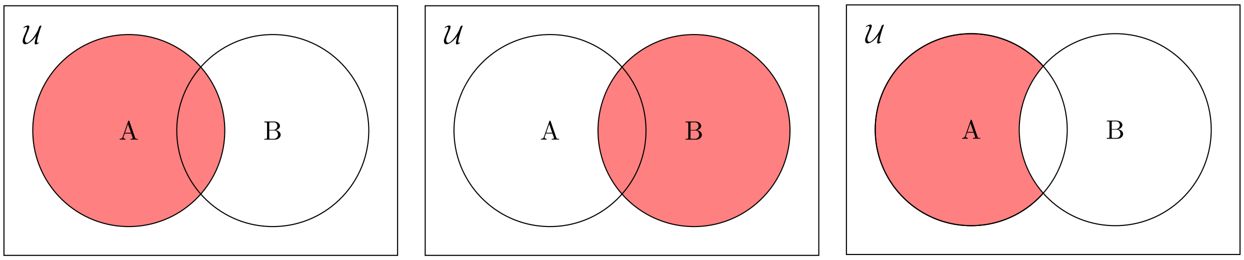
Notice also that $\neg a \wedge a$ is false for every region in the diagram. This is because, regardless of the truth value of a (or of its constituent statements), $\neg a \wedge a$ is always false. This type of statement is known as a *contradiction*.

We follow the same process to represent a statement with multiple constituent statements.
For example, to create a Venn diagram representing $a \wedge b$, we let a be the statement "$x \in A$" and b be the statement "$x \in B$", where A and B are sets and x is an element. Then



$x \in A$ $\qquad$ $x \in B$ $\qquad$ $(x \in A) \wedge (x \in B)$

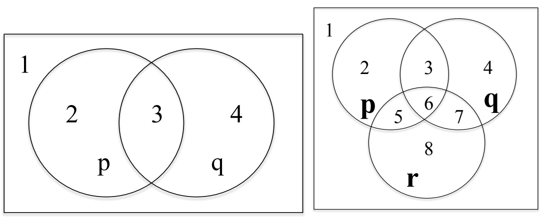
Q: *Shade the region(s) where* $a \to b$ *is false.*



$x \in A$ $\qquad$ $x \in B$ $\qquad$ $x \in A$ but $x \notin B$

NOTICE that this region is the region in which a is true but b is false. Everywhere else (that is, for every other combination of truth values for a and b), the implication is true.

Q. Shade the regions of a Venn diagram where each of the following statements is true: [8]



1. $p \to \neg q$ **Regions:** 1, 2, 4
2. $p \leftrightarrow \neg(q \wedge r)$ **Regions:** 3, 7 **2, 5**
3. $(p \to q) \to r$.

   Regions: 2, 5, 6, 7 **8**
4. Construct an equivalent statement using only $\wedge$, $\vee$, and/or $\neg$ using the equivalence rules from class.

   Soln.

$$
\begin{aligned}
(p \to q) \to r &\Leftrightarrow \neg(p \to q) \vee r \quad (\to \text{ Law}) && (1)\\
&\Leftrightarrow \neg(\neg p \vee q) \vee r \quad (\to \text{ Law}) && (2)\\
&\Leftrightarrow (p \wedge \neg q) \vee r \quad (\text{ DeMorgan}) && (3)\\
& && (4)
\end{aligned}
$$

For fun!

There are 3 boxes A, B, C. Exactly one contains gold.

Each box has a message on top, but only one of the messages is true.

Box A: *Gold is not in this box.* Box B: *Gold is not in this box.* Box C: *Gold is in box A.*

Which box contains the gold?