CSC/MAT A67H3 F 2016 Midterm
Duration — 90 minutes
Aids allowed: none

Tutorial Number: ____________________

Last Name: ____________________ **First Name:** ____________________

UTORID: ____________________ **Student #:** |_|_|_|_|_|_|_|_|_|

Instructors: Bretscher and Pancer

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*Do **not** turn this page until you have received the signal to start.*
Fill out the identification section above.
Good Luck!

This midterm is double sided and consists of 6 questions on 9 pages (including this one).

Read each question carefully. If you are unsure how to answer, try writing down all the information you have first.

If you use any space for rough work, indicate clearly what you want marked.

1: ______/15

2: ______/15

3: ______/10

4: ______/10

5: ______/10

6: ______/10

TOTAL: ______/70

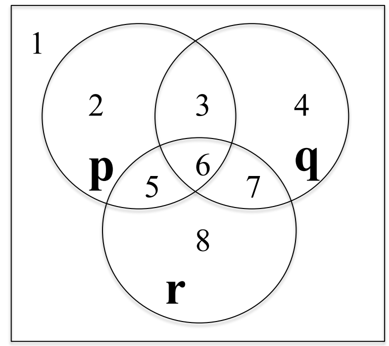
This page is for extra work.

Question 1. [15 MARKS]

Answer the following short answer questions. For each question circle all appropriate answers.

Part (a) [5 MARKS]

Each option below lists regions of the given Venn diagram. Circle all answers whose regions satisfy the statement:



$$p \leftrightarrow (q \wedge r)$$

(a) 1, 4 (b) 2, 4, 5 (c) 1, 8 (d) 6 (e) None

Part (b) [5 MARKS]

Write a propositional statement using $\rightarrow$ and $\neg$ (if necessary) equivalent to regions 3, 5 and 6. For part marks you can leave your solution using $\wedge, \vee, \neg$ and $\rightarrow$. You do **not** need to specify which equivalence laws you use if you need them.

Part (c) [5 MARKS]

Circle every statement equivalent to $\neg a \rightarrow b$.

(a) $\neg b \rightarrow a$

(b) $a \rightarrow \neg b$

(c) a is sufficient for $\neg b$

(d) $\neg(a \wedge \neg b)$

(e) b is sufficient for $\neg a$

(f) a is necessary for $\neg b$

(g) $\neg a$ is necessary for b

(h) b if $\neg a$

(i) $b \vee a$

Question 2. [15 marks]

Part (a) [5 marks]

Give an example to show that $\forall$ does not distribute over $\vee$, in other words,

$$\forall x \in X, p(x) \vee q(x) \not\Longleftrightarrow \forall x \in X, p(x) \vee \forall x \in q(x)$$

Part (b) [5 marks]

It is a fact that $\forall$ does distribute over $\wedge$, i.e., that

$$\forall x \in X, (p(x) \wedge q(x)) \iff (\forall x \in X, p(x)) \wedge (\forall x \in X, q(x)).$$

Show that your previous example satisfies this equivalence law.

Part (c) [5 marks]

Determine whether $\forall$ can be factored from an implication. In other words is

$$\forall x \in X, (p(x) \rightarrow q(x)) \quad \iff \quad (\forall x \in X, p(x)) \rightarrow (\forall x \in X, q(x))$$

true? Explain your reasoning. Marks will only be given for your *explanation*.

Question 3. [10 MARKS]

Answer the following questions to construct a direct proof that $\forall n \in \mathbb{N}, n \geq 2$,

$$\forall a \in \mathbb{N}, \forall b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$$

Part (a) [2 MARKS]

Recall the Division Theorem which says that every natural number can be written in the form of $nq + r$ for some $n, q, r \in \mathbb{N}$ where $0 \leq r < n$. Write a and b in terms of n using the Division Theorem taking into consideration any common variables.

Part (b) [8 MARKS]

Now complete the proof.

Question 4. [10 MARKS]

Prove $\forall n \in \mathbb{N}, (n^2 - 1 \not\equiv_4 0) \rightarrow n$ is even.

Question 5. [10 marks]

In Assignment 1 we proved that for natursal numbers a and n, a and n relatively prime is *sufficient* for there to exist a unique natural number $b < n$ such that $ab \equiv_n 1$ (note that you can still answer this question even if you were unable to do the assignment question).

Part (a) [4 marks]

Write the following claim as an implication using quantifiers and $\rightarrow$. You may use $p(x, y)$ to denote that x and y are relatively prime.

> a and n relatively prime is *necessary* for there to exist a unique natural number $b < n$ such that $ab \equiv_n 1$.

Part (b) [6 marks]

Prove the claim.

Question 6. [10 MARKS]

Prove the following statement $S(n)$ for all $n \in \mathbb{N}, n \geq 8$:

$$S(n) : \exists a \in \mathbb{N}, \exists b \in \mathbb{N}, 3a + 5b = n.$$

You may use *simple* or *strong induction*.

Use this page for rough work.