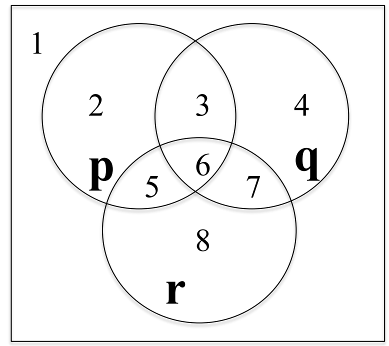
This page is for extra work.

Question 1. [15 marks]

Answer the following short answer questions. For each question circle all appropriate answers.

Part (a) [5 marks]

Each option below lists regions of the given Venn diagram. Circle all answers whose regions satisfy the statement:



$$p \leftrightarrow (q \wedge r)$$

(a) 1, 4 (b) 2, 4, 5 (c) 1, 8 (d) 6 (e) None

Soln. (a), (c), (d) 1 mark off for every missing or extra answer.

Part (b) [5 marks]

Write a propositional statement using $\rightarrow$ and $\neg$ (if necessary) equivalent to regions 3, 5 and 6. For part marks you can leave your solution using $\wedge, \vee, \neg$ and $\rightarrow$. You do **not** need to specify which equivalence laws you use if you need them.

Solution.

$$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r) \iff \neg(p \wedge q) \rightarrow (p \wedge r)$$
$$\iff (\neg p \vee \neg q) \rightarrow (\neg\neg p \wedge \neg\neg r) \iff (\mathbf{p} \rightarrow \neg\mathbf{q}) \rightarrow \neg(\mathbf{p} \rightarrow \neg\mathbf{r})$$
$$\text{Final Answer: } (\mathbf{p} \rightarrow \neg\mathbf{q}) \rightarrow \neg(\mathbf{p} \rightarrow \neg\mathbf{r})$$

Part (c) [5 marks]

Circle every statement equivalent to $\neg a \rightarrow b$.

(a) $\neg b \rightarrow a$ ✓

(b) $a \rightarrow \neg b$

(c) a is sufficient for $\neg b$

(d) $\neg(a \wedge \neg b)$

(e) b is sufficient for $\neg a$

(f) a is necessary for $\neg b$✓

(g) $\neg a$ is necessary for b

(h) b if $\neg a$ ✓

(i) $b \vee a$ ✓

1 mark of for every missing or every extra

Question 2. [15 marks]

Part (a) [5 marks]

Give an example to show that $\forall$ does not distribute over $\vee$, in other words,

$$\forall x \in X, p(x) \vee q(x) \not\Longleftrightarrow \forall x \in X, p(x) \vee \forall x \in q(x)$$

Sample Soln.
Let $X = \{0, 1\}$. Let $p(0) = T, p(1) = F, q(0) = T, q(1) = F$. Then $\forall x \in X, p(x) \vee q(x)$ is True since $x = 0, p(0) = T$ and for $x = 1, q(1) = T$. Notice that the right hand side however, $\forall x \in X, p(x) \vee \forall x \in q(x)$ is not True since neither $p()$ nor $q()$ are True for all values in X.

Part (b) [5 marks]

It is a fact that $\forall$ does distribute over $\wedge$, i.e., that

$$\forall x \in X, (p(x) \wedge q(x)) \iff (\forall x \in X, p(x)) \wedge (\forall x \in X, q(x)).$$

Show that your previous example satisfies this equivalence law.

Sample Soln.

In this case, if $X = \{0, 1\}$. Let $p(0) = T, p(1) = F, q(0) = T, q(1) = F$ then both the left and right sides are False since both p() and q() are not true for all values.

Part (c) [5 marks]

Determine whether $\forall$ can be factored from an implication. In other words is

$$\forall x \in X, (p(x) \to q(x)) \quad \iff \quad (\forall x \in X, p(x)) \to (\forall x \in X, q(x))$$

true? Explain your reasoning. Marks will only be given for your *explanation*.

Sample Soln.

No it cannot.

$$\forall x \in X, (p(x) \to q(x)) \quad \iff \quad \forall x \in X, \neg p(x) \vee q(x)$$

$$\begin{aligned} (\forall x \in X, p(x)) \to (\forall x \in X, q(x)) \quad &\iff \quad \neg \forall x \in X, p(x) \vee \forall x \in X, q(x) \\ &\iff \quad \exists x \in X, \neg p(x) \vee \forall x \in X, q(x) \end{aligned}$$

It is almost enough to say that since $\forall$ does not factor from $\vee$ and $\to$ can be rewritten in terms of $\vee$ that they are not equivalent - however since the negation turns the $\forall$ into $\exists$ we should be a little more careful.

Applying nearly the same domain and predicate definitions as in (a) gives the result, left side not equal to right side.

Let $X = \{0, 1\}$. Let $p(0) = F, p(1) = T, q(0) = T, q(1) = F$. Left hand side is false, right hand side is true.

Question 3. [10 marks]

Answer the following questions to construct a direct proof that $\forall n \in \mathbb{N}, n \geq 2$,

$$\forall a \in \mathbb{N}, \forall b \in \mathbb{N}, a \equiv_n b \rightarrow a^2 \equiv_n b^2$$

Part (a) [2 marks]

Recall the Division Theorem which says that every natural number can be written in the form of $nq + r$ for some $n, q, r \in \mathbb{N}$ where $0 \leq r < n$. Write a and b in terms of n using the Division Theorem taking into consideration any common variables.

Sample Soln.

$$a = nq_1 + r, b = nq_2 + r, 0 \leq r < n$$

Note that r is the same r for both a and b since they are equivalent.

Part (b) [8 marks]

Now complete the proof.

Sample Soln.

From part (a) we have q_1, q_2 and r in the natural numbers such that $a = nq_1 + r$ and $b = nq_2 + r$.

Then $a^2 = (nq_1 + r)(nq_1 + r) = (n^2q_1^2 + 2nq_1) + r^2$

and $b^2 = (nq_2 + r)(nq_2 + r) = (n^2q_2^2 + 2nq_2) + r^2$.

Notice then that $a^2 \mod n = r^2 \mod n$ and $r^2 \mod n = b^2 \mod n$. Therefore $a^2 \equiv_n b^2$.

Question 4. [10 marks]

Prove $\forall n \in \mathbb{N}, (n^2 - 1 \not\equiv_4 0) \to n$ is even.

Sample Solution.

Proof by contrapositive. **4 marks for seeing that an indirect proof method is necessary - contradiction fine too.**

For all natural numbers n, if n is odd then $n^2 - 1 \equiv_4 = 0$. **2 marks for setting it up properly (contradiction or contrapositive**

Since n is odd, we can rewrite as $n = 2k + 1$. Then $n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 4(k^2 + k)$ which is a multiple of 4 so equivalent to 0 mod 4. **4 marks for completing the proof. If they don't complete it all, give 2 marks for half done, 3 for 3/4 done 1 for 1/4 done etc).**

Question 5. [10 marks]

In Assignment 1 we proved that for natursal numbers a and n, a and n relatively prime is *sufficient* for there to exist a unique natural number $b < n$ such that $ab \equiv_n 1$ (note that you can still answer this question even if you were unable to do the assignment question).

Part (a) [4 marks]

Write the following claim as an implication using quantifiers and $\rightarrow$. You may use $p(x, y)$ to denote that x and y are relatively prime.

> a and n relatively prime is *necessary* for there to exist a unique natural number $b < n$ such that $ab \equiv_n 1$.

Sample Soln.

$$\exists x \in \mathbb{N}, b < n \wedge ab \equiv_n 1 \rightarrow p(a, n)$$

Part (b) [6 marks]

Prove the claim. **Sample Soln.**

Suppose there exists $b < n$ such that $ab \equiv_n 1$.

Suppose that a, n are not relatively prime. Then there exists c that divides a and n.
Realizing that an INDIRECT proof is required. 2 marks. Setting up the indirect proof - 2 marks. (so 4 for this sentence).

$ab = nq + 1$

Since c divides both a and b we have that $a = ck$ and $n = cd$.

$ckb = cdq + 1$. Then $c(kb - dq) = 1$ and therefore $kb - dq = \frac{1}{c}$. Notice that k, b, d, q are all natural numbers and so their product and difference must be a natural number, hence c must be 1. Therefore a, n are relatively prime.

Question 6. [10 marks]

Prove the following statement $S(n)$ for all $n \in \mathbb{N}, n \geq 8$:

$$S(n) : \exists a \in \mathbb{N}, \exists b \in \mathbb{N}, 3a + 5b = n.$$

You may use *simple* or *strong induction*.

Sample Solutions.
Base Cases:
n = 8 = 5 + 3
n = 9 = 3(3)
n = 10 = 2(5)

If using simple induction, base case is 8.

I.H. Assume that $S(k)$ holds for $8 \leq k < n$.

I.S. Prove $S(n)$.

Since the base case covers 8, 9 and 10, we can assume that $n \geq 11$. Therefore we have the $n - 3 \geq 8$. Therefore $S(n-3)$ holds by the induction hypothesis. Therefore exists $a \in \mathbb{N}, b \in \mathbb{N}$ such that $n-3 = 3a+5b$ and so $n = 3(a + 1) + 5b$ completing the proof.

Use this page for rough work.