① Let $S(n)$ the sum of the squares of the first n natural numbers, starting at 1, is $\frac{n(n+1)(2n+1)}{6}$

$$\forall n \in \mathbb{N}, n \geq 1, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Base case $n=1$, $\quad 1^2 = \frac{(1)(1+1)(2(1)+1)}{6} = \frac{6}{6} = 1$

Induction Hypothesis (n=k. holds.)

$$\sum_{i=1}^{n=k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

Inductive step:

check $n=k+1$

$$\sum_{i=1}^{n=k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$= \underbrace{1^2+2^2+\dots+k^2}_{\text{by IH}} + (k+1)^2$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= (k+1)[2k^2+k+6k+6]/6$$

$$= (k+1)(2k+3)(k+2)/6$$

$$= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$\boxed{\text{QED}}$

2. a) We have k odd lines, and 1 new line when $(k+1)$st line is added.
We can counted the number of new triangles as following ways.
There are $\binom{k}{2}$ ways to pick 2 odd lines from k odd lines.
for every 2 odd lines we can match with this 1 new line to form a new triangles.
So in total, $\binom{k}{2}$ new triangles are formed.

$\binom{k}{2}$ or $C(k,2)$

b) We have $(k+1)$ odd lines, and 1 new line when $(k+2)$nd line is added. This is similar logic as in part a), so we will have $\binom{k+1}{2}$ or $C(k+1, 2)$ new triangles.

c) Total number:

c) ~~C(k,3) + C(k+1~~

c) $C(k,3) + C(k,2) + C(k+1,2)$

$$= \frac{k!}{(k-3)!\,3!} + \frac{k!}{(k-2)!\,2!} + \frac{(k+1)!}{(k-1)!\cdot 2!}$$

$$= \frac{(k+1-3)\,k!}{(k+1-3)(k-3)!\,3!} + \frac{3k!}{(k-2)!\,3\cdot 2!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k-2)\,k!}{(k-2)(k-3)!\,3!} + \frac{3k!}{(k-2)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{k\cdot k! - 2k! + 3k!}{(k-2)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{k\cdot k! + k!}{(k-2)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k+1)\,k!}{(k+1-3)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= \frac{(k+1)!}{(k+1-3)!\,3!} + \frac{(k+1)!}{(k-1)!\,2!}$$

$$= C(k+1,3) + C(k+1,2)$$

$\longrightarrow$

Cont^{n}:

$$C(k+1,3)+C(k+1,2)$$

$$=\frac{(k+1)!}{(k+1-3)!\,3!}+\frac{(k+1)!}{(k+1-2)!\,2!}$$

$$=\frac{(k+1-2)(k+1)!}{(k+1-2)(k+1-3)!\,3!}+\frac{3\,(k+1)!}{3\,(k+1-2)!\,2!}$$

$$=\frac{(k+1)(k+1)!-2(k+1)!+3(k+1)!}{(k+1-2)!\,3!}$$

$$=\frac{(k+1)(k+1)!+(k+1)!}{(k+1-2)!\,3!}$$

$$=\frac{(k+1)!\,(k+1+1)}{(k+1-2)!\,3!}$$

$$=\frac{(k+2)!}{(k+2-3)!\,3!}=\frac{(k+2)!}{((k+2)-3)!\,3!}$$

$$=C(k+2,3)$$

we (k+2)nd line added. Total triangle $C(k+2,3)$

3. Let x be the number of ways to choose one pizza.

$$x = \binom{7}{0} + \binom{7}{1} + \binom{7}{2} + \binom{7}{3} + 7 + 7 + 7 \times 6$$

$= 120$ ways.

Let y be the number of ways to choose two pizza.

$$y = \underset{\substack{\uparrow \\ \text{two identitical}}}{120} + \underset{\substack{\uparrow \\ \text{two difference}}}{\binom{120}{2}} = 7260 \text{ ways}$$

~~We can~~ have

7. Let x be the number of ways to choose one pizza.

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Let y be the number of ways to choose two pizza.

$y = 120 + \binom{120}{2} = 7260$ ways

two identitical   two difference

~~We can~~ have

4. We can have following arrangement.

□ X □ X □ X □ X □ X □ X □

If Y are to be added to the row so that they are always apart, they can only be added in the boxes shown above.

There are $C(7,4)$ ways to do that.

$C(7,4) = 35$ ways of arranging six X's and four Y's in a row so that Y's are always apart.