# CSCA67 – Discrete Mathematics

## Final Exam Study Guide


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## CSCA67 - Lecture 1 - Introduction, Basic Counting

**Counting Pizza Toppings:**

**Q:** A commercial deal offered 2 pizzas, up to 5 toppings on each (no doubling on toppings), 11 toppings to choose from. The commercial claimed there are 1,048,576 possibilities. Why is 1,048,567 incorrect and what is the correct total of possibilities?

**A:** Calculating all possible ways of ordering 1 pizza

1 pizza with 0 toppings → 1 (Only 1 way of order a pizza with no toppings.)

1 pizza with 1 topping → 11 (There are 11 toppings to choose from.)

1 pizza with 2 toppings → 55 (11 toppings to choose from for the first topping, then 10 toppings left to choose from for the second topping gives us 110 by multiplying them. We divide by 2 since order does not matter. Ordering a pizza with pepperoni and cheese is the same as ordering a pizza with cheese and pepperoni.)

> Pepperoni, Cheese
> Cheese, Pepperoni

1 pizza with 3 toppings → 165 (11 toppings to choose from for the first topping, 10 toppings left to choose from for the second topping, and 9 toppings left to choose from for the third option gives us 990 by multiplying them. Again order does not matter so imagine ordering a pizza with 3 different toppings say ham, cheese, and pineapple. Listing out the possible combinations we will get 6 different orders but the same combination. Therefore, we would divide 990 by 6 to get 165.)

| | | |
|---|---|---|
| Ham, Cheese, Pineapple | Cheese, Ham, Pineapple | Pineapple, Ham, Cheese |
| Ham, Pineapple, Cheese | Cheese, Pineapple, Ham | Pineapple, Cheese, Ham |

Pizza with 4 toppings → Same concept as ordering 1 pizza with 3 toppings, but with 4 toppings instead.
(11 x 10 x 9 x 8) / 1 x 2 x 3 x 4 = 330

1 pizza with 5 toppings → Same concept as ordering 1 pizza with 3 toppings, but with 5 toppings instead.

(11 x 10 x 9 x 8 x 7) / 1 x 2 x 3 x 4 x 5 = 462

Adding all these results gives us 1024 (total number of combinations for 1 pizza.)
1 + 11 + 55 + 165 + 330 + 462 = 1024

The commercial got 1,048,576 by 1024 multiplying 1024 since there are 2 pizzas. This is wrong, it is over counting since ordering pizza A first before pizza B is the same as ordering pizza B first then pizza A.

So dividing by 2 may help. 1,048,576/2 = 524,288
Unfortunately this is still incorrect, since now we are actually under counting. This part is tricky to understand, but the problem is there are not two orderings when we order two identical pizzas.

There are 1024 pairs of identical pizzas. Adding half of the identical pizzas which is 512, will correct this issue. 1024 / 2 = 512

524,288 + 512 = 524,800

Therefore, 524,800 is the total number of possibilities of ordering 2 pizzas, up to 5 toppings on each (no doubling on toppings), and 11 toppings to choose from.

## <u>CSCA67 - Lecture 2 - Permutations and Selections</u>

**Counting Pizza Toppings (Different Method):**
**Q:** A commercial deal offered 2 pizzas, up to 5 toppings on each (no doubling on toppings), 11 toppings to choose from. The commercial claimed there are 1,048,576 possibilities. Why is 1,048,567 incorrect and what is the correct total of possibilities? **(Answer: 524,800)**

From last week's notes, we reached a point where there are 1024 ways of ordering a single pizza. Instead of using last week's method, another counting method would be adding the number of ways of ordering 2 different pizzas with the number ways of ordering 2 identical pizzas.

Number of ways of ordering 2 different pizzas + number ways of ordering 2 identical pizzas

$$\frac{1024 \times 1023}{2!} \quad + \quad 1024 \quad = \quad 524{,}800$$

**Q: How many passwords are there with a length of 8 characters using alphanumeric characters?**

**A:** There are a total of 62 alphanumeric characters (a-z, A-Z, 0-9). 26 uppercase, 26 lowercase of the alphabets, and 10 single digits.
26 + 26 + 10 = 62

Since there are 62 possibilities for a single character slot, there would be 62 possibilities per slot. In this case, there are 8 slots so the answer would be 62^8.

**Sum Rule:** $\sum_{i=1}^{n} \quad x_i$
n represents the number of different ways, $x_i$represents possible outcomes, and i is the starting value.
**Product Rule:** $\prod_{i=1}^{k} \quad x_i \ = \ x_1 \cdot x_2 \cdot \ldots \cdot x_k$
k represents the number of different ways, $x_i$represents possible outcomes, and i is the starting value.

**Combinations:** $C(n,r) \ = \ \frac{n!}{r!\,(n-r)!}$ n represents # of objects, and r is amount taken.
(Order does not matter)

**Permutations:** $P(n,r) \ = \ \frac{n!}{(n-r)!}$n represents # of objects, and r is amount taken.
(Order does matter)

**Q:** In a poker game, each player is given 5 cards from a deck of 52 cards. How many possible 5-card hands are there?
**A:** The question is seeking the total amount of possible 5-card hands. (Order does not matter)
Since order is not required, this is a combination question.

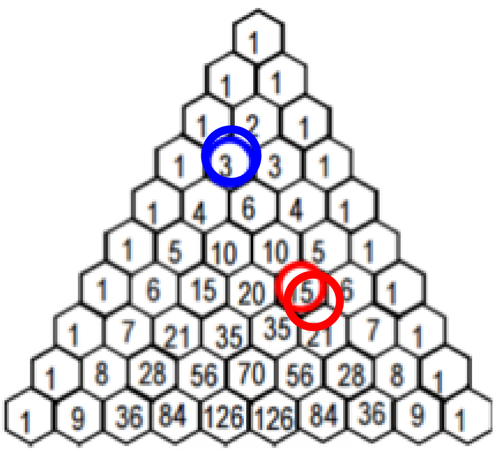
$$C(52,5) \ = \frac{52!}{47! \cdot 5!} = 2{,}598{,}960$$

# <u>CSCA67 - Lecture 3 - Combinations and Pascal's Triangle</u>

**Formula for counting repetitions:**

C(r + (n - 1), r)

Given r objects and n types of objects to choose from, the number of selections with repetitions is defined by this formula.




**Pascal's Triangle:**
Note: counting starts from 0 not 1 from left to right and top to bottom.

Key points: the $n_{th}$ and $r_{th}$ number in that row is C(n,r)
**Ex.** C(3,1) = 3!/1!(3 - 1)! = 6 (Indicated in blue)
C(6,4) = 6!/4!(6 - 4)! = 15(Indicated in red)

**Q.** How is Pascal's Triangle related to binomial expansion?
**A.**

$(a+b)^1 = 1a + 1b$
$(a+b)^2 = 1a^2 + 2ab + 1b^2$
$(a+b)^3 = 1a^3 +3a^2b + 3ab^2 + 1b^3$
$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

Starting from the first row (remember to count from 0), the numbers from left to right represents the binomial leading coefficients.

**Q.** What do each row sum up to?
**A.**
Row 0: 1
Row 1: 2
Row 2: 4
Row 3: 8
Row 4: 16
Row 5: 32
Row 6: 64

Power of 2.

**Q.** Reading each row as a number, what do these number represent?(If a row has two digits carry the tens digit to the left and add)

**A.**1, 11, 121, 1331, 14641

Power of 11.

**<u>CSCA67 - Lecture 4 - Distributing Objects</u>**

**Distributing objects:**

**Theorem.** Given $r$ *objects* and $n$ *types* of objects to choose from the number of *selections* with repetitions is:

$$\binom{r + (n-1)}{r}$$

**Recall:**
**Q.** How many ways are there to distribute 20 identical chocolate bars and 15 identical sticks of gum to 5 children?

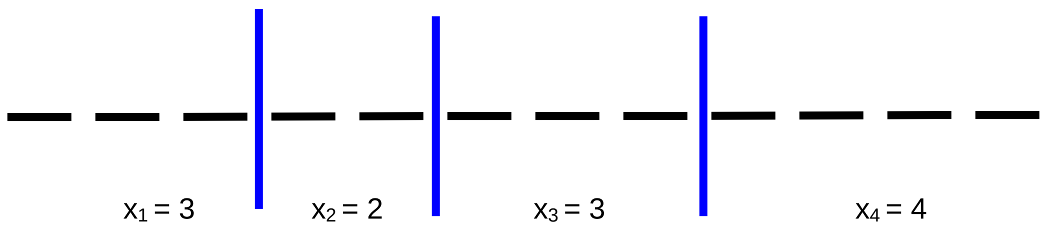
**A.**
C(r + (n-1), r)

Note: Mentioning of “counting” something doesn’t mean counting by 1, 2, 3… It means using counting techniques to count all possibilities (e.g. permutation, combination).

**Case 1**(Chocolate bars):
C(20 + (5 - 1), 4) or C(20(5 - 1), 20)
Using the formula C(r + (n-1), r), we can solve this problem by counting for the chocolate bars or the 4 partitions (the dividers between the children) dividing the chocolate bars between the 5 children.

**Case 2**(Stick of gum):
C(15 + (5-1), 4) or C(15 + (5-1), 15)
Using the formula C(r + (n-1), r), we can solve this problem either counting for the sticks of gum or the 4 partitions between the kids.

**Q.** How many integer solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 12$ with $x_i > 0$?
**A.**




one possible example:
Since there 4 variables that equal to 12 with the condition each variable is equal to 0 or greater. We can count the partition (plus signs dividing one variable from another) or count for 12. Using the formula C(r + (n-1), r), counting the 3 partitions between the variables or counting for 12, we get the total number of possibilities of integer solutions to the equation $x_1 + x_2 + x_3 + x_4 = 12$ with $x_i > 0$.

C(12 + (4-1), 3) or C(12 + (4-1), 12)

## CSCA67 - Lecture 5 - Probability

**Definition:** S represents the sample space of an experiment and E represents an event in S,

$$\text{prob}(E) = \frac{|E|}{|S|}$$

the probability of E is defined as prob(E) = |E|/|S| (Laplace's definition of probability).
**Q.** What is the probability of rolling a 3 on a standard die?
**A.** Prob(rolling a 3) = |1|/|6|

**Q.** Consider rolling 2 dices, what is the probability of the sum being 7?
**A.**

| | | | | | |
|---|---|---|---|---|---|
| **(1, 1)** | **(1, 2)** | **(1, 3)** | **(1, 4)** | **(1, 5)** | **(1, 6)** |
| **(2, 1)** | **(2, 2)** | **(2, 3)** | **(2, 4)** | **(2, 5)** | **(2,6)** |
| **(3, 1)** | **(3, 2)** | **(3, 3)** | **(3, 4)** | **(3, 5)** | **(3, 6)** |

| | | | | | |
|---|---|---|---|---|---|
| **(4, 1)** | **(4, 2)** | **(4, 3)** | **(4, 4)** | **(4, 5)** | **(4, 6)** |
| **(5, 1)** | **(5, 2)** | **(5, 3)** | **(5, 4)** | **(5, 5)** | **(5, 6)** |
| **(6, 1)** | **(6, 2)** | **(6, 3)** | **(6, 4)** | **(6, 5)** | **(6, 6)** |

By listing out all possible sums of 2 dices in this manner. We can make a diagonal (indicted by the yellow highlighting) representing all possible ways of 2 dices adding up to 7.

Prob(Sum of 7) = 6/36 = 1/6

**The Birthday Problem:**

**Q.** What is the probability that in a group of n people at least two have the same birthday (not including leap years)?
**A.** Let's represent the days of the year by the integers 1, 2, …, 365 and E represent the event that at least two people have the same birthday.

|S| = n tuples of the integers (1 to 365)

Listening out all the possibilities would be extremely difficult. Instead, finding the complement of E is the best solution.

Complement formula: 1 - P(complement of E) = E
If n > 365 (n tuples of the integers representing the sample space), and P(E) = 1 (1 since we are accounting for 100% that at least 2 people share the same birthday). However in this case we have a problem, there are not enough unique birthdays for everyone. Since counting over 365 days means we would be restarting the calendar.

|complement E| = 365 x 364 x … x (365 - n + 1)

P(|complement E|) = (365 x 364 x … x (365 - n + 1))/ $365^n$

P(E) = 1 - P(complement E)
= 1 - (365 x 364 x … x (365 - n + 1))/ $365^n$

## CSCA67 - Lecture 6 - Sum and Product Rules

**The Sum Rule:**
If E and F are events in an experiment then the probability that for either E or F to occur is defined by: P(E or F) = P(E) + P(F) - P(E and F)

P(E) represents the probability of event E.
P(F) represents the probability of event F.
P(E and F) represents any overlaps between the 2 events.

**Q.** What is the probability when a pair of dice are rolled that at least one die shows a 5 or the dice sum to 8?

**A.** Let $E_5$ be the event that at least one dice shows a 5 and $E_8$ be the event that the dice sum to 8.

$P(E_5) = 11/36$
$P(E_8) = 5/36$
$P(E_5 \text{ and } E_8) = 2/36$

$$\begin{aligned} P(E_5 \text{ or } E_8) &= P(E_5) + P(E_9) - P(E_5 + E_8) \\ &= (11/36) + (5/36) - (2/36) \\ &= 14/36 \\ &= 7/18 \end{aligned}$$

**The Product Rule:**
if E and F are two events in an experiment then the probability that both E and F occur is defined by: P(E and F) = P(E) x P(F|E)

Note: P(F|E) is read as the probability of event F to occur is given that event E is present. (The bar between F and E is read as "given that.")

**Q.** Suppose there is a noisy communication channel in which either a 0 or a 1 is sent with the following probability:

- Probability a 0 is sent is 0.4.
- Probability a 1 is sent is 0.6.
- Probability that due to noise, a 0 is changed to a 1 during transmission is 0.2.
- Probability that due to noise, a 1 is changed to a 0 during transmission is 0.1.

Suppose that a 1 is received. What is the probability that a 1 was sent?

**A.** Let A denote that A was received and B denoted the event that a 1 was sent. So we are solving for P(B|A).

Rearranging the product rule: P(B|A) = (P(B) x P(A|B))/ P(A)

The results for the below probability are given in the question directly or indirectly:
P(B) = 0.6
P(A|B) = 1 - 0.1
= 0.9
P(A), we don't know.

**Observation:** There are only 2 possible events that result in a 1 being received. Either a 1 is sent and received or a 0 is sent and due to noise a 1 is received.

The probability that A happens is the sum:
P(A) = P(0 is sent) x P(A|0 is sent) + P(1 is sent) x P(A|1 is sent)
= (0.4) x (0.2) + (0.6) x (1 - 0.1)
= 0.62

Now we can solve for P(B|A):

P(B|A) = (0.6 x 0.9)/ 0.62
= 0.87

**Key points:**
From solving the question above, 2 important relationships was used.

**Bayes' rule:** Let A and B be events in the same sample space sample space. If neither

$$P(B|A) = P(A|B)\frac{P(B)}{P(A)}$$

P(A) nor P(B) are zero, then:
**Theorem (Total Probability):** Let a sample space S be disjoint union of events $E_1$, $E_2$,..., $E_n$ with positive probabilities, and let $A \subseteq S$.

$$P(A) = \sum_{i=1}^{n} P(A|E_i) \cdot P(E_i)$$

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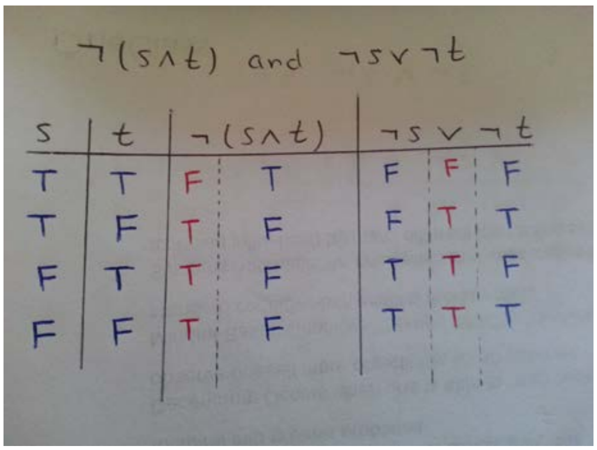
## CSCA67 - Lecture 7 - Logical Connectives, Implication

**Symbols:**

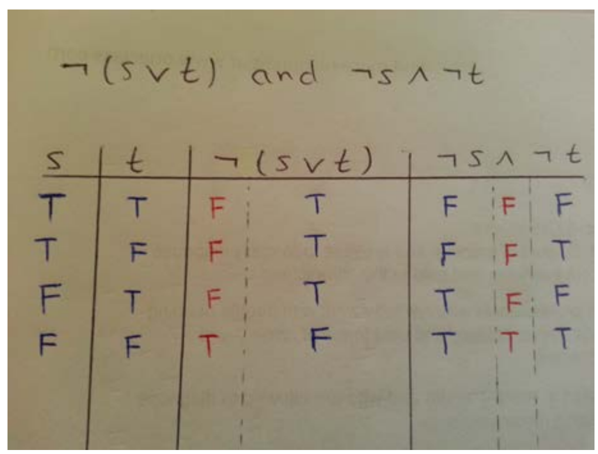
$\land$ and
$\lor$ or
$\neg$ not
$\rightarrow$ implies



$\neg(s \land t)$ and $\neg s \lor \neg t$

| $s$ | $t$ | $\neg(s \land t)$ | | $\neg s \lor \neg t$ | | |
|---|---|---|---|---|---|---|
| T | T | F | T | F | F | F |
| T | F | T | F | F | T | T |
| F | T | T | F | T | T | F |
| F | F | T | F | T | T | T |

**Truth Tables (Equivalent Statements):**



$\neg(s \lor t)$ and $\neg s \land \neg t$

| $s$ | $t$ | $\neg(s \lor t)$ | | $\neg s \land \neg t$ | | |
|---|---|---|---|---|---|---|
| T | T | F | T | F | F | F |
| T | F | F | T | F | F | T |
| F | T | F | T | T | F | F |
| F | F | T | F | T | T | T |

**Q:** Which of the following are equivalent to $t \rightarrow s$?
**A:** The ones highlighted are equivalent to $t \rightarrow s$

| | |
|---|---|
| ==$\neg s \rightarrow \neg t$== | ==$s \lor \neg t$== |
| $\neg t \rightarrow \neg s$ | ==$\neg t \lor s$== |
| $s \rightarrow t$ | ==$s, if\ t$== |
| $if\ s\ then\ t$ | $s\ is\ sufficient\ for\ t$ |
| ==$if\ t\ then\ s$== | ==$t\ is\ sufficient\ for\ s$== |

$t, if\ s$ $s\ is\ necessary\ for\ t$
$\neg s, if\ \neg t$ $t\ is\ necessary\ for\ s$
$\neg s\ \vee\ t$ $\neg(t\ \wedge\ \neg s)$

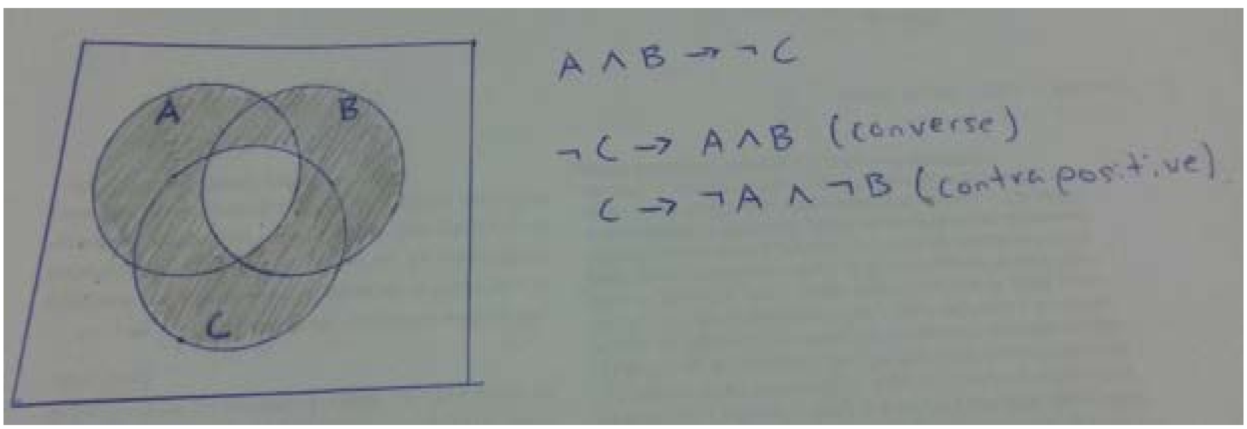
## CSCA67 - Lecture 8 - Implication and Direct Proofs

### Sets and Venn Diagrams:

**Q:** If we have $x \in A \rightarrow x \in B$, then what do we call $x \notin B \rightarrow x \notin A$?
**A:** Contrapositive

**Q:** If we have $x \in A \rightarrow x \in B$, then what do we call $x \in B \rightarrow x \in A$?
**A:** Converse

**Exercise:** Draw a Venn diagram representing 3 sets A, B, and C. Now shade in the region that represents when the statement $A \wedge B \rightarrow \neg C$ is true.




**Q:** How could we make this task easier?
**A:** Find when/ what will make the statement <u>false,</u> and take the opposite.

### If and Only if:

**Q:** If we say $A \leftrightarrow B$what do we mean in terms of $\rightarrow$?
**A:** $(A \leftrightarrow B)\ \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$
The symbol $\Leftrightarrow$is used to indicate how the 2 statements are equivalent.

| $A$ | $B$ | $A \rightarrow B$ | $B \rightarrow A$ | $A \leftrightarrow B$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | F |
| F | T | T | F | F |
| F | F | T | T | T |

**Direct Proofs:**

$$\forall x \in D, P(x) \rightarrow Q(x)$$

- Let x be arbitrary or any element of the domain
- Suppose that $P(x)$ is true
- Using true sentences to derive that $Q(x)$is true

**Q:** For all real numbers $x$, if $x > 1, then\ x^2 > 1$.
**A:** $\forall\ real\ number\ x, if\ x > 1\ \rightarrow x^2 > 1$
Consider any real number $x$and let $x > 1$.
Multiplying both sides by $x$, will give us $x^2 > x$.
We can do this because $x > 1 \rightarrow x^2 > x > 1$, therefore $x^2 > 1$.

**Q:** Prove that for all real numbers $x$ and $y$, if $x$ and $y$ are rational then $xy$ is irrational.
**A:** Let $x$be an arbitrary element in real numbers, and let $y$belong to real numbers.
Assuming $x, y$are rational. Let $x = \frac{a}{b}$and $y = \frac{c}{d}$, where $a, b, d, d$ belong to integers and $b, d\ \neq 0$.So $x \cdot y\ = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, by closure of integers, $ac$, $bd$ belong to integers, since $b \neq 0$and $d \neq 0$.Therefore, by definition of a rational number, $xy$belong to rational.

**<u>CSCA67 - Lecture 9 - Direct Proofs</u>**

**Symbols:**
∀ means "for all"
∃ means "there exist" or "there is a"
$D$ stands for the domain

**Q:** How can we prove that there does not exist a largest natural number?
**A:**

$$\forall x \in natural\ numbers, \exists y \in natural\ numbers, y > x$$

Let $x$ be a natural number.
let $y = x + 1$

$$y > x + 1$$

**Q:** What do the following statements say in English if:

$$S(x, y): x\ scares\ y$$
$$D = \{all\ people(and\ superheros/\ characters)\}$$

**A:**

1. $\exists x \in D, \exists y \in D, S(x, y) \rightarrow$ At least 1 person scares someone.
2. $\exists y \in D, \exists x \in D, S(x, y) \rightarrow$ At least 1 person is scared by someone.
3. $\forall x \in D, \forall y \in D, S(x, y) \rightarrow$ Everyone scares everyone (each person scare everyone).
4. $\forall y \in D, \forall x \in D, S(x, y) \rightarrow$ Everyone is scared of everyone.
5. $\forall x \in D, \exists y \in D, S(x, y) \rightarrow$ Everyone scares someone (not necessarily the same person).
6. $\exists y \in D, \forall x \in D, S(x, y) \rightarrow$ There is someone who is scared of everyone.

7. $\exists x \in D, \forall y \in D, S(x, y)$ → There is someone who scares everyone.
8. $\forall y \in D, \exists x \in D, S(x, y)$ → Everyone is scared of someone.

For all real numbers $x$ and $y$, if $x$ and $y$ are rational then $xy$ is rational.

**Q:** How can we express $x$ is rational using quantifiers?
**A:** $\forall x \in real\ numbers, \forall y \in real\ numbers, x\ is\ rational\ and\ y\ is\ rational\ \rightarrow xy\ is\ rational.$
$x\ is\ rational\ \exists a \in integers, \exists b \in integers, such\ that\ x = \frac{a}{b} \wedge b \neq 0$

**Q:** How can we express x is not rational using quantifies?
**A:** $\neg(\exists a \in integers, \exists b \in integers, such\ that\ x = \frac{a}{b} \wedge b \neq 0)$

**Note:**

$$\neg(\exists) = \forall$$
$$\neg(\forall) = \exists$$

**<u>CSCA67 - Lecture 10 - Indirect Proofs</u>**

**Proof by Contradiction:**

**Q:** Claim $\sqrt{2}$ is irrational
**A:**
Assume that the claim is FALSE. Show that the assumption leads to a “false” statement (a.k.a the contradiction). Therefore the assumption is wrong, and the original claim is correct.

Assume $\sqrt{2}$ is rational
$\exists p, q \in integers$ such that $\sqrt{2} = \frac{p}{q} \wedge q \neq 0$
$2 = \frac{p^2}{q^2}$

$$2q^2 = p^2$$

p is the unique product of k prime numbers.
q is the unique product of m prime numbers.
Therefore, $p^2$is the unique product of 2k (even) prime numbers.
Also, $q^2$ is the unique product of 2m (even) prime numbers.

2 is a prime, so $2 \cdot q^2$is the unique product of an odd number of primes.
The left side ($2q^2$) is a product of odd numbers of primes.
The right side ($p^2$) is the product of an even number.
But by fundamental theorem of arithmetic they must be the same.
This demonstrates that there is a contradiction to our assumption that$\sqrt{2}$ is rational.

**Summary:**

PROOF BY CONTRADICTION.

Claim: $P$

Assume $\neg P$.

Derive a *false* statement.

Conclude the assumption was wrong. Therefore $P$.

**Proof by Contrapositive:**

If we need to prove an implication like $\forall x \in D, p(x) \rightarrow q(x)$ then we have the option of proving $\forall x \in D, \neg q(x) \rightarrow \neg p(x)$instead.

Q: prove $\forall x \in positive\ real\ numbers, \forall y \in real\ numbers, \sqrt{xy} \neq \frac{x+y}{2} \rightarrow x \neq y$

A:

$$\forall x \in positive\ real\ numbers, \forall y \in positive\ real\ numbers, x = y \rightarrow \sqrt{xy} = \frac{x+y}{2}$$

Assume x = y

$\sqrt{xy} = \sqrt{x \cdot x}$, since $x = y$

$= x$

Our goal is to end with $\frac{x+y}{2}$.

In situation like this, it is best go backwards.

$\frac{x+y}{2} = \frac{2x}{2}$, since $x = y$

$= x$

Now the top part equals the bottom part ($x = x$).

Therefore this implication is true by proof by contrapositive.

**Summary:**

PROOF BY CONTRAPOSITIVE.

Claim: $\forall x \in D, p(x) \rightarrow q(x)$

Let $x \in D$ be arbitrary.

Assume $\neg q(x)$.

$\vdots$

Derive $\neg p(x)$, so $\neg q(x) \rightarrow \neg p(x)$.

By the contrapositive, $p(x) \rightarrow q(x)$.

Conclude $\forall x \in D, p(x) \rightarrow q(x)$.

**CSCA67 - Lecture 11 - Proof by Induction**

1. Prove that the base case holds
2. Create an induction hypothesis
3. Perform the Induction

**Induction Proof for the Summation Formula:**

$$\sum_{i=0}^{n} i = \frac{n\,(n\ +\ 1)}{2}$$

**Base Case:**
Show that n = 0 holds.

$$\frac{0\,(0\ +\ 1)}{2} = 0$$

**Induction Hypothesis:**
Assume, $S(k)$ holds for arbitrary $k$ ∈natural numbers, $k \geq 0$.

$$S(k)\colon \sum_{i=0}^{k} i = \frac{k\,(k\ +\ 1)}{2}$$

**Induction Step:**
Prove that $S(k) \rightarrow S(k\ +\ 1)\ holds.$

$$\sum_{i=o}^{k+1} i = (k+1)\ + \sum_{i=0}^{k} i$$

$= (k+1)\ + \frac{k\,(k+1)}{2}\ by\ induction\ hypothesis$

$= \frac{2\,(k+1) + k\,(k+1)}{2}$

$= \frac{k^2+3k+2}{2}$

$= \frac{(k+1)\,(k+2)}{2}\ This\ proves\ that\ the\ induction\ hypothesis\ is\ correct, since\ this\ is\ S(k+1).$

$Therefore, S(k) \rightarrow S(k\ +\ 1)\ holds.$

**Summary:**

**Claim.** $\forall n \in \mathbb{N}, n \geq b \rightarrow S(n)$

**Base Case.** Consider $S(b)$.

**Induction Hypothesis.** Assume that $S(k)$ is true for an arbitrary $k \in \mathbb{N}, k \geq b$.

**Induction Step.** Prove that $S(k) \rightarrow S(k+1)$.

Conclude $\forall n \in \mathbb{N}, n \geq b \rightarrow S(n)$.

**CSCA67 - Lecture 12 - Proof by Strong Induction**

**Strong Induction or Complete Induction:**

- "if $p(i)$ holds for all $i \leq k$then $p(k+1)$holds too, where $p(k)$is dependant on a positive integer k"
- weak induction or simple induction is not the same as strong induction (However, they are equivalent in a sense)
- if weak induction holds, then strong induction will obviously hold as well

**Example (Staple Question):**

**Q:** Given an infinite amount of 4-cents and 7-cents stamps, prove by strong induction that there exists a combination of stamps to make any amount of postage that is 18-cents or more.

**A:**

$P(n): n \geq 18 \rightarrow \exists a \in natural\ numbers, \exists b \in natural\ numbers, n = 4a + 7b.$

Base Cases: (Show that n = 18, 19, 20, and 21 holds)

$$18 = 4(1) + 7(2)$$
$$19 = 4(3) + 7(1)$$
$$20 = 4(5) + 7(0)$$
$$21 = 4(0) + 7(3)$$

Induction Hypothesis:

$$If\ n\ is\ bounded\ by\ 18 \leq n \leq k, such\ that\ k \geq 21\ then\ \exists\ a \in natural\ numbers, \exists\ b \in natural\ numbers, n = 4a + 7b.$$
$$This\ implies\ that\ \exists\ x \in natural\ numbers, \exists\ y \in natural\ numbers, (k+1) = 4x + 7y.$$

Induction Step: (Prove that (k+1) = $4x + 7y$)

$$k + 1 = 4 + (k - 4)$$

We know that $k \geq 21$, so $(k-4) \geq 18$.

$$(k - 4) = 4a = 7b\ by\ induction\ hypothesis.$$

$$\begin{aligned} k + 1 &= 4 + (k - 4) \\ &= 4 + 4a + 7b \\ &= 4(a + 1) + 7b \end{aligned}$$

Therefore (k+1) holds for any amount of 4-cents and 7-cents stamps.