CSC A67/MAT A67 - Discrete Mathematics, Fall 2017

Assignment #1: Proofs

Due: October 29, 2017 at 11:59 p.m. This assignment is worth 10% of your final grade.

Warning: Your electronic submission on MarkUs affirms that this assignment is your own work and no one else's, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters, the Code of Student Conduct, and the guidelines for avoiding plagiarism in CSC A67/MAT A67.

This assignment is due by 11:59 p.m. October 29. **Late assignments will be accepted with a penalty as described in the course syllabus.**

We will use $\mathbb{N}^+$ to refer to the set of all positive natural numbers (ie, starting at 1).

1. (a) Prove [5]

$$\forall n \in \mathbb{N}^+, r \in \mathbb{N}^+, s \in \mathbb{N}^+, rs \leq n \rightarrow r \leq \sqrt{n} \text{ or } s \leq \sqrt{n}.$$

   (b) Prove [10]

$$\forall n \in \mathbb{N}^+, n \textbf{ composite} \rightarrow \exists p \in \mathbb{N}^+, p \text{ is prime and } p \leq \sqrt{n} \text{ and } p \mid n.$$

   (c) One way to test whether a number is prime is the following: [5]

   Given an integer $n > 1$, check if n is divisible by any prime number less than ***equal to*** *its square root. If n is not divisible by any of these primes, then n prime.*

   Write the contrapositive of 1b). Your solution should be written so that it clearly shows that the above test is indeed a valid method.

2. Prove that for all natural numbers $n > 2$ there exists a prime number p such that $n < p < n!$ where $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$. [10]

3. Prove that $\forall k \in \mathbb{N}^+, \forall m \in \mathbb{N}, k = 2m + 1 \rightarrow 11 \mid (10^k + 1)$. [10]

4. Given a positive integer n, we can write the *decimal representation of n* as [10]

$$d_k d_{k-1} d_{k-2} \dots d_1 d_0$$

   where d_i is the i^{th} digit and is an integer from 0 to 9 inclusive, $d_k \neq 0$ (i.e., the left-most digit cannot be 0) for $n > 0, k > 0$ and:

$$n = d_k \cdot 10^k + d_{k-1} \cdot 10^{k-1} + d_{k-2} \cdot 10^{k-2} + \cdots + d_1 \cdot 10^1 + d_0 \cdot 10^0$$

   Define the *alternating sum*

$$s_n = d_0 - d_1 + d_2 - d_3 + \ldots + d_i - d_{i+1} + \ldots d_k.$$

   For example,

$$s_{180928} = 8 - 2 + 9 - 0 + 8 - 1$$

   Prove that if s_n is divisible by 11 then n is divisible by 11. **Hint**: Use what you proved in Question 3.

5. Prove that if $n+1$ integers are selected from $\{1, 2, ..., 2n\}$, then the selection includes integers a and b such that $a \mid b$. You may find it helpful to pick a particular value of n to *"see"* how the proof should work. Your final answer however, should be in terms of general n. [10]

[Total: 60 marks]