University of Toronto at Scarborough
CSC A67/MAT A67 - Discrete Mathematics, Fall 2016

Exercise #2: Logic and Venn Diagrams – Solutions

Due: September 21, 2016 at 11:59 p.m.
This exercise is worth 3% of your final grade.

Warning: Your electronic submission on MarkUs affirms that this exercise is your own work and no one else's, and is in accordance with the University of Toronto Code of Behaviour on Academic Matters, the Code of Student Conduct, and the guidelines for avoiding plagiarism in CSC A67/MAT A67.

This exercise is due by 11:59 p.m. September 24. Late exercises will not be accepted.

1. (a) Construct a truth table for the following statement: $(p \leftrightarrow q) \rightarrow r$. [10]

p	q	r	$p \leftrightarrow q$	$(p \leftrightarrow q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	F	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	F

(b) Derive from this truth table an equivalent formula to $(p \leftrightarrow q) \rightarrow r$ that only uses connectives from $\{\wedge, \vee, \neg\}$ **and** has exactly 6 propositional variables in it (for example, $(p \wedge q) \vee (\neg q \wedge r)$ has exactly 4 propositional variables).

$$\neg((p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge \neg r))$$

(c) Explain how you derived your formula.

Can create one of two equivalent formulas: read off each line that is true and OR the conjunction (AND) or the **values** of each of p, q, r for that line. For example for lines 1 and 3 and 4 we get $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r)$ and continue in this manner for all true lines. Why is this equivalent? each line represents a set of truth values that make the formula true, so any one of those sets makes the given formula true and similarly any one of those lines makes our constructed formula true. In this case, this gives us a formula that has way too many propositional variables.

Another way we can create this equivalent formula is by considering the false lines. They are false exactly when the formula is false. Therefore the negation of these lines is true when the formula is true. Since there are two false lines this method constructs a formula wiht 6 propositional variables.

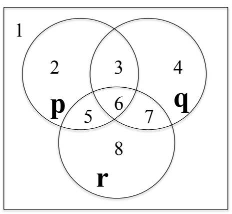
(d) Prove that $(p \leftrightarrow q) \rightarrow r$ is logically equivalent to $(\neg p \wedge q) \vee (\neg q \wedge p) \vee r$ using the logical equivalence laws given in class. You may assume the meaning of $\iff$ that we learned in class, namely that

$$A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$$

and refer to this as the definition of $\leftrightarrow$ or definition of biconditional.

$$\begin{aligned}
&\iff (p \leftrightarrow q) \to r && \\
&\iff \neg(p \leftrightarrow q) \vee r && \text{arrow law}\\
&\iff \neg((p \to q) \wedge (q \to p)) \vee r && \text{definition of biconditional}\\
&\iff (\neg(p \to q) \vee \neg(q \to p)) \vee r && \text{DeMorgan}\\
&\iff (\neg(\neg p \vee q) \vee \neg(\neg q \vee p)) \vee r && \text{arrow law twice}\\
&\iff ((\neg\neg p \wedge \neg q) \vee (\neg\neg q \wedge \neg p)) \vee r && \text{Demorgan twice}\\
&\iff ((p \wedge \neg q) \vee (q \wedge \neg p)) \vee r && \text{Double negation twice}\\
&\iff (p \wedge \neg q) \vee (q \wedge \neg p) \vee r && \text{associativity of } \vee
\end{aligned}$$

2. List the regions of the Venn diagram below where each of the following statements is true: [10]



   (a) $(p \to \neg q) \to \neg r$ **Regions:** 1, 2, 3, 4, 6
   (b) $p \to (\neg q \to \neg r)$ **Regions:** 1, 2, 3, 4, 6, 7, 8
   (c) $(p \leftrightarrow q) \leftrightarrow r)$ **Regions:** 2,4, 6,8
   (d) $p \leftrightarrow (q \leftrightarrow r)$ **Regions:** 2,4,6,8
   (e) $(p \wedge r) \to (q \vee \neg r)$ **Regions:** 1, 2, 3, 4, 6, 7, 8

3. What do you observe from your solutions to 2(a)-(d) above about $\to$ and $\leftrightarrow$? [2]

   That $\to$ is not associative and $\leftrightarrow$ is.

[3]

4. Disprove $(\neg r \to \neg(p \vee q)) \iff (p \to q) \to r$. How can one disprove an equivalence?

   Soln.
   Find a set of truth values for the propositions that make one side true and the other false. In this case: $r = F, p = F, q = T/F$. In this case the left side is of the form $F \to ...$ and evaluates to T and on the right side we have $(F \to T) \to F$ which simplifies to $T \to F$ which is false.

   One can also use equivalence laws to make the two expressions more obviously not equivalent.

[Total: 25 marks]

Logical Equivalences

Commutative	$p \wedge q \iff q \wedge p$	$p \vee q \iff q \vee p$
Associative	$(p \wedge q) \wedge r \iff p \wedge (q \wedge r)$	$(p \vee q) \vee r \iff p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r)$
Identity	$p \wedge T \iff p$	$p \vee F \iff p$
Negation	$p \vee \neg p \iff T$	$p \wedge \neg p \iff F$
Double Negative	$\neg(\neg p) \iff p$	
Idempotent	$p \wedge p \iff p$	$p \vee p \iff p$
Universal Bound	$p \vee T \iff T$	$p \wedge F \iff F$
De Morgan's	$\neg(p \wedge q) \iff (\neg p) \vee (\neg q)$	$\neg(p \vee q) \iff (\neg p) \wedge (\neg q)$
Absorption	$p \vee (p \wedge q) \iff p$	$p \wedge (p \vee q) \iff p$
Conditional or ($\implies$) Law	$(p \to q) \iff (\neg p \vee q)$	$\neg(p \to q) \iff (p \wedge \neg q)$
Biconditional	$p \leftrightarrow q \iff (p \to q) \wedge (q \to p)$	