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Sub-Optimal Extremum Seeking Control ⋆

Christophe LABAR * Emanuele GARONE *
Michel KINNAERT *

* Université Libre de Bruxelles, 1050 Brussels, Belgium (email: {Christophe.Labar,Emanuele.Garone,Michel.Kinnaert}@ulb.ac.be)

Abstract: This paper deals with the problem of steering a system at a certain percentage of the optimal cost, so as to keep a desired amount of reserve. This reserve can be used as a knob to counteract fast changes in the operating conditions of the system. In this paper, we consider the case where the system dynamics, as well as the analytical expression of the cost function, are unknown. We propose a modified version of extremum seeking control that, under certain assumptions, is able to keep a desired margin with respect to the estimated maximum. The theoretical properties of the presented scheme are analysed and the effectiveness of the approach is validated in simulation on a wind turbine example.

Keywords: Extremum Seeking, Sub-Optimization, Reserve, Semi-Regional Practical Stability

1. INTRODUCTION

In several applications, it is quite important to be able to operate within a certain desired margin from the optimal working conditions.

For instance, in the sector of energy, the electrical grid frequency has to be kept constant, implying a balance between the power generated and the power consumed. The power producers have therefore to keep a margin with respect to their rated power to be able to counteract unexpected increases of the consumption or sudden drops of the production (e.g. failure of a plant) [de Vyver et al. 2016]. Nowadays, renewables are more and more present in the network so that they are progressively asked to behave like conventional plants, and to take part to this power reserve [Yingcheng and Nengling 2011]. However, in presence of important uncertainties affecting the energy production, this is not easy to attain. For instance, in the case of wind farms, due to the uncertainty on the wind field distribution and the uncertain mapping between the power produced and the turbine rotational speed, for a given wind speed distribution, the optimal operating conditions are not exactly known a priori [Yang et al. 2013].

Another possible application concerns engine control, and more specifically spark ignition engines. In idle speed control, the engine speed has to be maintained at a given value despite the presence of disturbances (e.g. air conditioning, changes in engine temperature, load, etc.). Due to the delay between the intake of the air/fuel mixture and its combustion, the performances of the so-called throttle loop is limited. One strategy to mitigate this problem consists in creating a spark reserve. Instead of igniting the mixture at the optimal time, the spark is anticipated so as to keep a torque reserve. Whenever needed, such a reserve allows to change almost instantaneously the produced torque. The airflow is then adjusted to recover the spark reserve [Yildiz et al. 2011]. It should be remarked that the higher the reserve, the higher the fuel consumption and hence the lower the engine efficiency. Therefore, it is important to keep this reserve small. This is quite challenging as accurate and simple engine models are difficult to obtain. For normal situations where the fuel is always the same, the tuning can be done off-line. Nevertheless, in advanced scenarios where different mixes of fuels might be used, the system should be able to adapt [Hellström et al. 2013].

Non model-based extremum seeking is a well known control strategy to steer a system to optimize the steady state cost, without requiring the knowledge of the cost function or the system dynamics. One of the first works on extremum seeking is Leblanc [1922]. Since then, numerous works on this subject have been presented in the literature (see e.g. [Dürr et al. 2013], [Krstić and Wang 2000]). A detailled survey is given in [Tan et al. 2010]. Applications of extremum seeking to renewables can be found in [Ghaffari et al. 2012] and [Ghaffari et al. 2014] while applications to spark ignition engines are given in [Hellström et al. 2013] and [Kitazono et al. 2008].

In this paper, we present a black box method, based on extremum seeking, that allows to keep a desired margin with respect to the estimated value of the optimum. The stability properties of the resulting control scheme are proved and the effectiveness of the proposed approach is validated, in simulation, on a simplified wind turbine example.

The rest of this paper is structured as follows. Section 2 introduces the definitions and notations. In Section 3, the problem is formally stated. The proposed approach for addressing the problem is presented in Sections 4 and 5. Finally, simulation results are presented in Section 6.

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2. NOTATIONS AND DEFINITIONS

Notations: $\hat{x}$ denotes the estimated value of x. The derivatives of a function $f(x)$ w.r.t. x are denoted by $f'(x)$,$f''(x)$ and $f^{(n)}(x)$ for the higher orders n. The time derivative of a variable x is denoted $\dot{x}$.

Definition 1. A neighbourhood of a point x is a simply connected set containing a ball centred in that point.

Definition 2. An open set $\mathcal{A}$ is strictly included in an open set $\mathcal{B}$ if the closure of $\mathcal{A}$ is included in $\mathcal{B}$, i.e. $\overline{\mathcal{A}} \subset \mathcal{B}$. This strict inclusion is denoted $\mathcal{A} \Subset \mathcal{B}$.

Definition 3. A function $f(x, \epsilon) \in \mathbb{R}$, with $x \in \mathbb{R}^n$ and $\epsilon \in \mathbb{R}^+$, is $O(\epsilon)$ if, for any compact set $\mathcal{D} \subset \mathbb{R}^n$, there exist positive constants k and ϵ^* such that

$$|f(x, \epsilon)| < k\epsilon \quad \forall \epsilon < \epsilon^*, x \in \mathcal{D}. \tag{1}$$

Definition 4. A maximum u^* of a smooth function $f(u)$ is isolated in a set $\mathcal{D}$ if $f'(u)(u - u^*) < 0, \forall u \in \mathcal{D}\backslash u^*$.

Definition 5. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open neighbourhood of the origin. The origin of the system $\dot{x}(t) = f(t, x, \epsilon)$, with vector of parameters $\epsilon = [\epsilon_1, \epsilon_2, ..., \epsilon_{n_\epsilon}]^T$, is semi-Regionally Practically Asymptotically Stable (RPAS) in $\mathcal{D}$ if, for every open bounded neighbourhood of the origin $\mathcal{V} \Subset \mathcal{D}$ and $\mathcal{B} \Subset \mathcal{D}$, there exist an open bounded set $\mathcal{W} \Subset \mathcal{D}$ containing $\mathcal{V}$, an open bounded neighbourhood $\mathcal{Q} \Subset \mathcal{B}$ of the origin, and scalars $\epsilon_i^* > 0$, such that if $\epsilon_i \in (0, \epsilon_i^*)$ for $i = 1, 2, ..., n_\epsilon$, the following holds:

(1) Boundedness: $\forall x(0) \in \mathcal{V}, \quad x(t) \in \mathcal{W} \quad \forall t \geq 0$;
(2) Stability: $\forall x(0) \in \mathcal{Q}, \quad x(t) \in \mathcal{B} \quad \forall t \geq 0$;
(3) Practical Convergence: $\forall x(0) \in \mathcal{V}, \exists t_0 : x(t) \in \mathcal{B}, \forall t \geq t_0$.

3. PROBLEM STATEMENT

Consider a single-input single-output system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = \quad g(x(t)) \end{cases} \tag{2}$$

with $x(t) \in \mathbb{R}^n$ the state of the system, $u(t) \in \mathbb{R}$ the control input and $y(t) \in \mathbb{R}$ the measurable system output. For the sake of clarity, the time-dependency of the variables is omitted in the sequel. It is assumed that:

Assumption 6. For each value of the control input u, there exists one and only one point of equilibrium. Therefore, there exists a function $l : \mathbb{R} \to \mathbb{R}^n$ such that

$$x = l(u) \Leftrightarrow f(x, u) = 0. \tag{3}$$

Furthermore, the corresponding steady-state input-output function $y_s = g(l(u))$ is of class C^2.

Assumption 7. The points of equilibrium $x = l(u)$ of system (2) are globally asymptotically stable, uniformly in u.

Furthermore, consider a cost function $z = c(y)$ whose analytical expression is not available but whose output $z \in \mathbb{R}$ can be measured continuously. We assume that:

Assumption 8. The function $c(y)$ is of class C^{n_c} with $n_c \geq 4$.

Under Assumption 6, a steady-state cost function can be associated to $c(y)$, namely $z_s = h(u) := c(g(l(u)))$. The following is assumed:

Assumption 9. The steady state cost function $h(u)$ admits at least one maximum, denoted z_s^*.

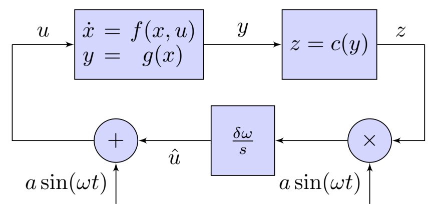

Fig. 1. "Classical" Extremum Seeking

The usual aim of classical extremum seeking is to steer system (2) to reach z_s^*. This paper focuses on a sightly different objective. Namely, we want system (2) to reach a given ratio $0 < \rho \leq 1$ of the optimum z_s^*.

4. PROPOSED APPROACH

The proposed approach starts from the classical extremum seeking scheme, depicted in Figure 1. The corresponding control law is

$$\begin{cases} \dot{\hat{u}} = \delta\omega a \sin(\omega t) z \\ u = \hat{u} + a \sin(\omega t) \end{cases} \tag{4}$$

with δ the integrator gain, ω the dither angular frequency and a the dither amplitude. These parameters are design parameters for the extremum seeking.

The main idea of our approach is the following. Instead of using the extremum seeking to optimize z_s, it will be used to optimize a suitable auxiliary cost function. This cost function is selected so that z_s reaches a certain percentage of its estimated optimal value.

The starting point is to note that, if we want z_s to reach a given value z_d, the auxiliary cost function

$$\gamma(z) = -(z - z_d)^2 \tag{5}$$

can be introduced. Note that γ is nothing but a measure of the distance with respect to the desired value z_d. The associated steady state cost function

$$\gamma_s(u) = -(h(u) - z_d)^2 \tag{6}$$

always admits at least one maximum. In most of the situations, the cost function γ_s will even have at least two maxima (e.g. the cost function h has a strictly negative curvature and z_d is below the maximum). Instead of using the extremum seeking to maximize $h(u)$, we can use it to maximize $\gamma_s(u)$ (Figure 2). Since $\gamma_s(u)$ can admit several maxima, the notion of semi-global practical asymptotic stability used in [Tan et al. 2006] is not valid anymore and the notion of semi-regional practical asymptotic stability has to be used (cf. Definition 5). Consider that the following holds:

Assumption 10. The maximizers $u^*_{\gamma_{s_i}}$ of $\gamma_s(u)$ are isolated.

Let $\mathcal{D}_{\gamma_{s_i}}$ be an open neighbourhood of $u^*_{\gamma_{s_i}}$ in which $u^*_{\gamma_{s_i}}$ is isolated. Under Assumptions 6-8 and 10, we know from the main result of [Labar et al. 2016] that the point $(\hat{u} - u^*_{\gamma_{s_i}} = 0, x - l(u^*_{\gamma_{s_i}}) = 0)$ is RPAS in $(\mathcal{D}_{\gamma_{s_i}} \times \mathbb{R}^n)$ for system (2) with control input (4) and cost function (5) (cf. Figure 2).

Clearly, in the above situation if $z_d = \rho z_s^*$, then the goal of reaching a certain percentage of the optimum is achieved.

However, in most situations, the sub-optimum cannot be known a-priori and has to be estimated online. Two cases can be distinguished. In the first one, a parametrized

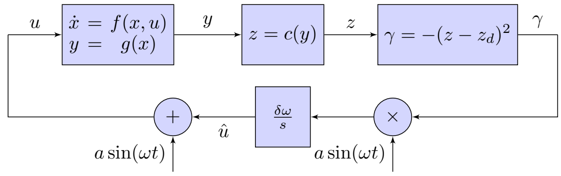

Fig. 2. Modified Extremum Seeking

expression of the cost function can be deduced from the physics governing the problem. From the on-line measurements of y and z, the cost function parameters can be estimated. The corresponding sub-optimal value can then be computed and fed in the previous scheme. In the second case, the physics is unknown or it is too complex to handle the parametrized expression. The cost is then considered as a black box. This is the case we will focus on in the sequel.

Note that, if the cost function is sufficiently smooth, it is well approximated locally by its Taylor expansion. Based on this property, in absence of other knowledge, the idea is to make use of the second order Taylor expansion for estimating, for each value of u, a corresponding sub-optimum $\rho\hat{z}^*(u)$. Clearly, this new function only gives a rough estimate of the sub-optimum. However, the closer to the real maximum, the smaller the estimation error, which is in practice well suited to the aim of keeping a few percent of margin with respect to the optimum.

Consider the following assumption:

Assumption 11. The steady-state input-output function $y_s = g(l(u))$ is strictly monotonic.

Remark 12. Assumption 11 may seem restrictive. However, it applies to a quite wide class of systems. In particular, it is true for all the controlled systems where u represents the reference of the closed-loop system, and thus $g(l(u))$ is a monotonic function of u. This is the case for instance of wind turbines where the output y represents the turbine rotational speed, that is controlled, and u is the reference speed.

Under Assumption 11, the maximum of $h(u)$ is equivalent to the one of $c(y)$ and for each value of y there exists a u such that $c(y) = h(u)$. Accordingly, estimating the maximum of $h(u)$ is equivalent to estimating the maximum of $c(y)$. In the sequel, we will work with $c(y)$ since it allows us to decouple the estimation of the maximum from the system dynamics.

Starting from the second order Taylor expansion of $c(y)$,

$$c(y + \Delta) = c(y) + c'(y)\Delta + 0.5c''(y)\Delta^2, \tag{7}$$

a local estimation of the sub-optimum $\rho\hat{c}^*(y)$ can be computed as follows

$$\rho\widehat{c^*}(y) = \rho c(y) - 0.5\rho c'^2(y)(c''(y))^{-1}. \tag{8}$$

Substituting z_d by the estimated sub-optimum $\rho\widehat{c^*}(y)$ (8) in (5) results in the following approximated cost function

$$\gamma_a(y) = -\left((1-\rho)c(y) + \frac{\rho}{2}\frac{c'^2(y)}{c''(y)}\right)^2, \tag{9}$$

leading to the control scheme depicted in Figure 3.

Remark 13. The estimated optimum $\widehat{c^*}(y)$ in (8) is a maximum only if $c''(y) < 0$. Hence the cost function (9) can only be used in regions where $c''(y) < 0$. In

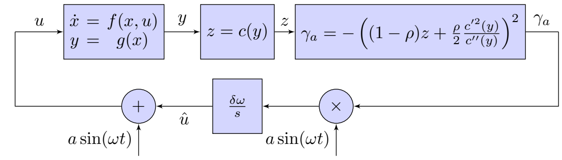

Fig. 3. Extremum Seeking with cost function γ_a

practice, when the initial conditions are located in a region where $c''(y) > 0$, one can proceed as follows. The classical extremum seeking is used to move the state in a region where $c''(y) < 0$ and $|c(y) - \rho\widehat{c^*}(y)| < \epsilon_y$ with $\epsilon_y > 0$ an arbitrarily small parameter. The cost function is then switched to $\gamma_a(y)$.

For the sake of simplicity we will assume in the sequel that

Assumption 14. $c''(y) < 0 \quad \forall y \in \mathbb{R}$.

Assumption 15. The maximizers $u^*_{\gamma_{sa_i}}$ of $\gamma_a(g(l(u)))$ are isolated.

Note that, although the cost z can be measured, one has no access to its first two derivatives $c'(y)$ and $c''(y)$. They have therefore to be estimated from the on-line measurements of (y, z) (Figure 4). Clearly, whatever the selected estimation method, there will be an estimation error. However, the following result can be proved.

Theorem 16. Consider system (2) with control input (4) and cost function (9) (cf. Figure 4). Let $\mathcal{D}_{\gamma_{sa_i}}$ be an open neighbourhood of $u^*_{\gamma_{sa_i}}$ in which $u^*_{\gamma_{sa_i}}$ is isolated. Consider Assumptions 6-8 and 14-15, and assume that for every open bounded neighbourhood $\mathcal{W}_y \subset \mathbb{R}$ of $y^*_i = h(u^*_{\gamma_{sa_i}})$, $\widehat{c'}(y) = c'(y) + O(ap)$ and $\widehat{c''}(y) = c''(y) + O(ap), \forall y \in \mathcal{W}_y$, with a the dither amplitude and p a tunable parameter. Then, it follows that $(\hat{u} - u^*_{\gamma_{sa_i}} = 0, x - l(u^*_{\gamma_{sa_i}}) = 0)$ is RPAS in $\mathcal{D}_{\gamma_{sa_i}} \times \mathbb{R}^n$.

Proof. The proof is divided in two steps. **Step 1.** It will be proved that having a $O(ap)$ estimation error on $c'(y)$ and $c''(y)$ for $y \in \mathcal{W}_y$ is equivalent to having a noise $w = O(ap)$ acting at the output of $\gamma_a(y)$ for $y \in \mathcal{W}_y$. **Step 2.** Starting from the result of Labar et al. [2016] concerning the stability of extremum seeking for noisy cost function output, RPAS will be proved.

Step 1. Consider the expression (9) for γ_a.

- By assumption, $\widehat{c'}(y) = c'(y) + O(ap), \forall y \in \mathcal{W}_y$, hence
$$\widehat{c'}^2(y) = c'^2(y) + 2c'(y)O(ap) + O(a^2p^2), \forall y \in \mathcal{W}_y. \tag{10}$$
Since $c'(y)$ is a continuous function, it is bounded on $\mathcal{W}_y$ and therefore $\widehat{c'}^2(y) = c'^2(y) + O(ap), \forall y \in \mathcal{W}_y$.
- It is also known that:
$$\widehat{c''}(y) = c''(y) + O(ap), \forall y \in \mathcal{W}_y. \tag{11}$$
By Assumption 14, $c''(y) < 0 \quad \forall y \in \mathbb{R}$. From (11), $\widehat{c''}(y)$ is always located in a neighbourhood of $c''(y)$ that can be made as small as desired by reducing the parameters a and p. Therefore, there exist parameters a^*, p^* such that $\forall a \in (0, a^*], p \in (0, p^*]$, the minimum value of $|\widehat{c''}(y)|$ is also strictly positive in $\overline{\mathcal{W}}_y$. Dividing both members of (11) by $c''(y)\widehat{c''}(y)$ results in
$$\frac{1}{\widehat{c''}(y)} = \frac{1}{c''(y)} + \frac{O(ap)}{c''(y)\widehat{c''}(y)}, \forall y \in \mathcal{W}_y. \tag{12}$$

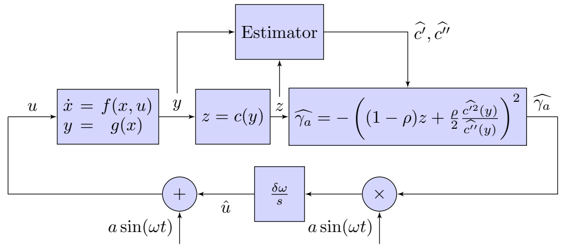

Fig. 4. Closed-loop with estimation in γ_a

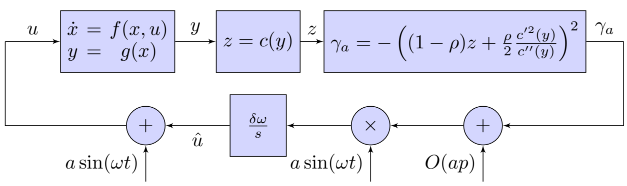

Fig. 5. Closed-loop with noise at the output of γ_a

The denominator of the last term of (12) being lower bounded $\forall y \in \mathcal{W}_y$, the error on $\frac{1}{c''(y)}$ is $O(ap)$ $\forall y \in \mathcal{W}_y$.

From those two conclusions, $\widehat{\gamma}_a$ can be written as

$$\widehat{\gamma_a}(y) = -\left((1-\rho)z + 0.5\rho\frac{\widehat{c'}^2(y)}{\widehat{c''}(y)}\right)^2 \tag{13}$$

$$= -\left((1-\rho)z + 0.5\rho\frac{c'^2(y)}{c''(y)} + O(ap)\right)^2 \tag{14}$$

$$= \gamma_a(y) + O(ap) \quad \forall y \in \mathcal{W}_y. \tag{15}$$

As said, instead of considering explicitly the estimation error on the derivatives in the cost function, we can assume that we have access to the real value of the derivatives and that a noise $w = O(ap)$ is present at the output of the cost function (Figures 4-5).

Step 2. From Labar et al. [2016], we know that if the noise $w(t)$ acting on the cost function output is $O(ap), \forall t \in \mathbb{R}$, then $(\hat{u} = u^*_{\gamma_{sa_i}}, x - l(u^*_{\gamma_{sa_i}}) = 0)$ is RPAS in $\mathcal{D}_{\gamma_{sa_i}} \times \mathbb{R}^n$. From **Step 1** of this proof, we know that for every open bounded neighbourhood $\mathcal{W}_y \subset \mathbb{R}$ of y_i^*, if $y(t) \in \mathcal{W}_y, \forall t$, then the noise acting on the cost function is $O(ap), \forall t$, i.e. $\exists K : |w(t)| < Kap, \forall t$. However, $\mathcal{W}_y$, and hence the value of K, cannot be known a priori as it depends on the set in which $y(t)$, and hence $x(t)$, is confined. Since those sets depend on the amplitude of the noise acting on the cost function output, a loop is created. To break this loop, let us first assume that $|w(t)| < K_t ap, \forall t$ with K_t an arbitrary value. This implies (Labar et al. [2016]) that there exist open bounded neighbourhoods $\mathcal{W}_x \Subset \mathbb{R}^n$ of the origin and $\mathcal{W}_{\hat{u}} \Subset \mathcal{D}_{\gamma_{sa_i}}$ of $u^*_{\gamma_{sa_i}}$ and parameters $a^* > 0, \delta^* > 0$ and $p^* > 0$ such that for all $a \in (0, a^*], \delta \in (0, \delta^*]$ and $p \in (0, p^*]$, there exists an $\omega^*(a, \delta, p) > 0$ such that for all $\omega \in (0, \omega^*]$, $(x(t) - l(u^*_{\gamma_{sa_i}})) \in \mathcal{W}_x$ and $\hat{u}(t) \in \mathcal{W}_{\hat{u}}, \forall t \geq 0$. Accordingly, there exists an open bounded set $\mathcal{W}_y \subset \mathbb{R}$ containing $y(t) \quad \forall t \geq 0$. Let K^* be such that $|w| < K^* ap$ for $y \in \mathcal{W}_y$. Selecting $p_2^* = \min\{p^* K_t / K^*, p^*\}$, we are ensured that for all $a \in (0, a^*], \delta \in (0, \delta^*]$, $p \in (0, p_2^*]$, and $\omega \in (0, \omega^*(a, \delta, p)]$, $(\hat{u} = u^*_{\gamma_{sa_i}}, x - l(u^*_{\gamma_{sa_i}}) = 0)$ fulfils the 3 conditions of RPAS for the sets obtained with the assumption of a global $O(ap)$. This concludes the RPAS of $(\hat{u} = u^*_{\gamma_{sa_i}}, x - l(u^*_{\gamma_{sa_i}}) = 0)$ in $\mathcal{D}_{\gamma_{sa_i}} \times \mathbb{R}^n$. □

Next section presents one approach to get estimations of $c'(y)$ and $c''(y)$ fulfilling the conditions of Theorem 1.

5. ESTIMATION OF $C'(Y)$ AND $C''(Y)$

The following two-steps approach is proposed to estimate $c'(y)$ and $c''(y)$. The variables y and z are continuously measured and couples of data-points (y_i, z_i) are stored so that the n_p last ones, with $n_p > n_o$, satisfy Condition 17.

Condition 17. Let $\overline{y}$ be the mean value of $\{y_i\}_{i=1,...,n_p}$ and y be the current value of the cost input. For pre-selected values of $d > 0$, $\epsilon > 0$ and $r > 0$, we have $\epsilon d \leq |y_i - y_j| \leq n_p d$ and $|\overline{y} - y| \leq rd, \quad \forall i \neq j \in \{1, 2, ..., n_p\}$.

The estimation procedure is the following:

Step 1. Using the last n_p data points (y_i, z_i) recorded on-line, a n_o^{th} order polynomial of the form

$$p(y) = \sum_{i=0}^{n_o} b_i y^i, \tag{16}$$

is fitted to minimize the sum of the squared fitting errors. In other words, the coefficients b_i are solutions of

$$\{b_i\}_{i=0,1,...,n_o} = \underset{\{b_i\}_{i=0,1,...,n_o}}{\arg\min} \sum_{i=1}^{n_p} (z_i - p(\tilde{y}_i))^2, \tag{17}$$

with $\tilde{y}_i = y_i - \overline{y}$. It will be shown that the coefficients of this polynomial allow to get an estimation of $c^{(i)}(\overline{y})$ for $i = 0, 1, ..., n_o$. It is worth to mention that the solution of (17) exists and can be computed analytically.

Remark 18. The higher the number of terms considered in (16), i.e. the higher the n_o, the better the cost function is approximated locally. However, in practice, a compromise has to be set between the accuracy and the computational cost. In addition, since function $c(y)$ belongs to class C^{n_c} (Assumption 8), the maximal order of the polynomial such that the estimation error is bounded is $n_c - 1$ (cf. Taylor-Lagrange's theorem of the remainder [Estep 2002])

Step 2. Based on the obtained values of the derivatives evaluated in $\overline{y}$, estimations at the current y are obtained by performing the interpolation

$$\widehat{c^{(i)}}(y) = \sum_{k=i}^{n_o} \widehat{c^{(k)}}(\overline{y}) \frac{(y - \overline{y})^{k-i}}{(k-i)!} \quad i = 0, 1, ...n_o. \tag{18}$$

The error on $c'(y)$ and $c''(y)$, using the two-steps approach, with data-points satisfying Condition 17, is given by the following lemma. This lemma also shows that, for suitable values of d, the conditions required in Theorem 16 for the error on the derivatives estimation are satisfied.

Lemma 19. Under Assumption 8, and for every open bounded neighbourhood $\mathcal{W}_y \subset \mathbb{R}$ of $h(u^*_{\gamma_{sa_i}})$, consider that the current y is in $\mathcal{W}_y$ and that the n_p last data points $(y_i, z_i)_{i=1,2,...,n_p}$, with $y_i \in \mathcal{W}_y$, fulfil Condition 17. Then,

i) Defining, $\widehat{c^{(i)}}(\overline{y}) = i! b_i$, with b_i the solution of (17), one has $\widehat{c^{(i)}}(\overline{y}) = c^{(i)}(\overline{y}) + O(d^{n_o+1-i})$, for $i = 0, 1, ..., n_o$.
ii) Performing the interpolations (18), it results $\widehat{c^{(i)}}(y) = c^{(i)}(y) + O(d^{n_p+1-i})$ for $i = 0, 1, ..., n_o$.
iii) Selecting $d = \sqrt[n_o-1]{ap}$ leads to

$$\begin{cases} \widehat{c'}(y) = c'(y) + O(ap) \\ \widehat{c''}(y) = c''(y) + O(ap) \end{cases}. \tag{19}$$

Proof. **i)** For each couple of data-points (y_i, z_i), let us perform the n_o^{th} order Taylor expansion of $c(y_i)$ around $\overline{y}$

$$c(y_i) = \sum_{k=0}^{n_o} \frac{c^{(k)}(\overline{y})}{k!}\tilde{y}_i^k + R(y_i) \quad i = 1, 2, ..., n_p \tag{20}$$

with $R(y_i)$ the remainder of the n_o^{th} order expansion for the data-point (y_i, z_i).

The solution of the minimization problem (17) is obtained by solving

$$\begin{bmatrix} \sum_{i=1}^{n_p} 1 & \sum_{i=1}^{n_p} \tilde{y}_i & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o} \\ \sum_{i=1}^{n_p} \tilde{y}_i & \sum_{i=1}^{n_p} \tilde{y}_i^2 & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o+1} \\ \vdots & \iddots & \iddots & \vdots \\ \sum_{i=1}^{n_p} \tilde{y}_i^{n_o} & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o+1} & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{2n_o} \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n_o} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_p} z_i \\ \sum_{i=1}^{n_p} z_i\tilde{y}_i \\ \vdots \\ \sum_{i=1}^{n_p} z_i\tilde{y}_i^{n_o} \end{bmatrix}. \tag{21}$$

Let us decompose the polynomial coefficients in two terms

$$b_j = b_j^0 + \Delta b_j \quad j = 0, 1, ..., n_o, \tag{22}$$

with b_j^0 the polynomial coefficients corresponding to $R(y_i) = 0$, i.e. to the case for which the cost function is a n_o^{th} order polynomial, and Δb_j characterizing the impact of those remainders.

From the n_o^{th} order Taylor expansion given in (20), assuming that $R(y_i) = 0$, one can write

$$\sum_{i=1}^{n_p} z_i\tilde{y}_i^j = \sum_{k=0}^{n_o}\left(\frac{c^{(k)}(\overline{y})}{k!}\sum_{i=1}^{n_p}\tilde{y}_i^{j+k}\right). \tag{23}$$

The right member of (21) can thus be written under the following matrix form

$$\begin{bmatrix} \sum_{i=1}^{n_p} z_i \\ \sum_{i=1}^{n_p} z_i\tilde{y}_i \\ \vdots \\ \sum_{i=1}^{n_p} z_i\tilde{y}_i^{n_o} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_p} 1 & \sum_{i=1}^{n_p} \tilde{y}_i & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o} \\ \sum_{i=1}^{n_p} \tilde{y}_i & \sum_{i=1}^{n_p} \tilde{y}_i^2 & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o+1} \\ \vdots & \iddots & \iddots & \vdots \\ \sum_{i=1}^{n_p} \tilde{y}_i^{n_o} & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o+1} & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{2n_o} \end{bmatrix} \begin{bmatrix} c(\overline{y}) \\ c'(\overline{y}) \\ \vdots \\ \frac{c^{(n_o)}}{n_o!}(\overline{y}) \end{bmatrix}. \tag{24}$$

Comparing (21) and (24), one obtains

$$b_i^0 = \frac{c^{(i)}(\overline{y})}{i!}, \quad i = 0, 1, ..., n_o. \tag{25}$$

Let us now characterize the impact of the remainders on the polynomial coefficients. By definition, the coefficients Δb_i are solutions of

$$\begin{bmatrix} \sum_{i=1}^{n_p} 1 & \sum_{i=1}^{n_p} \tilde{y}_i & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o} \\ \sum_{i=1}^{n_p} \tilde{y}_i & \sum_{i=1}^{n_p} \tilde{y}_i^2 & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o+1} \\ \vdots & \iddots & \iddots & \vdots \\ \sum_{i=1}^{n_p} \tilde{y}_i^{n_o} & \sum_{i=1}^{n_p} \tilde{y}_i^{n_o+1} & \dots & \sum_{i=1}^{n_p} \tilde{y}_i^{2n_o} \end{bmatrix} \begin{bmatrix} \Delta b_0 \\ \Delta b_1 \\ \vdots \\ \Delta b_{n_o} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{n_p} R(y_i) \\ \sum_{i=1}^{n_p} R(y_i)\tilde{y}_i \\ \vdots \\ \sum_{i=1}^{n_p} R(y_i)\tilde{y}_i^{n_o} \end{bmatrix}. \tag{26}$$

From the Taylor-Lagrange's theorem [Estep 2002], $\exists q_i \in (y_i, \overline{y})$ such that

$$R(y_i) = \frac{c^{(n_o+1)}(q_i)}{n_o+1}\tilde{y}_i^{n_o+1}. \tag{27}$$

Therefore one has

$$\sum_{i=1}^{n_p} R(y_i)\tilde{y}_i^k = \sum_{i=1}^{n_p} \frac{c^{(n_o+1)}(q_i)}{n_o+1}\tilde{y}_i^{n_o+1+k}. \tag{28}$$

Since $\epsilon d < |y_i - y_j| \leq n_p d, \forall i \neq j$, one can factorize $\tilde{y}_i = k_i d$ with $k_i \in [-(n_p-1)/2, (n_p-1)/2]$ and $\epsilon < |k_i - k_j| \leq n_p$. Substituting this factorisation in (28), one has

$$\sum_{i=1}^{n_p} R(y_i)\tilde{y}_i^k = d^{n_o+1+k}\sum_{i=1}^{n_p} \frac{c^{(n_o+1)}(q_i)}{n_o+1}k_i^{n_o+1+k}. \tag{29}$$

The number of data-points being finite, the values of k_i being independent of d and $c^{(n_o+1)}$ being a continuous function, it is bounded on $\mathcal{W}_y$ and it results

$$\sum_{i=1}^{n_p} R(y_i)\tilde{y}_i^k = O(d^{n_o+1+k}). \tag{30}$$

Substituting (30) in (26) and using $\tilde{y}_i = k_i d$ leads to

$$\begin{bmatrix} \sum_{i=1}^{n_p} 1 & \sum_{i=1}^{n_p} dk_i & \dots & \sum_{i=1}^{n_p} (dk_i)^{n_o} \\ \sum_{i=1}^{n_p} dk_i & \sum_{i=1}^{n_p} (dk_i)^2 & \dots & \sum_{i=1}^{n_p} (dk_i)^{n_o+1} \\ \vdots & \iddots & \iddots & \vdots \\ \sum_{i=1}^{n_p} (dk_i)^{n_o} & \sum_{i=1}^{n_p} (dk_i)^{n_o+1} & \dots & \sum_{i=1}^{n_p} (dk_i)^{2n_o} \end{bmatrix} \begin{bmatrix} \Delta b_0 \\ \Delta b_1 \\ \vdots \\ \Delta b_{n_o} \end{bmatrix} = \begin{bmatrix} O(d^{n_o+1}) \\ O(d^{n_o+2}) \\ \vdots \\ O(d^{2n_o+1}) \end{bmatrix}. \tag{31}$$

Let us divide each line i by d^{i-1},

$$\underbrace{\begin{bmatrix} \sum_{i=1}^{n_p} 1 & \sum_{i=1}^{n_p} dk_i & \dots & \sum_{i=1}^{n_p} d^{n_o}k_i^{n_o} \\ \sum_{i=1}^{n_p} k_i & \sum_{i=1}^{n_p} dk_i^2 & \dots & \sum_{i=1}^{n_p} d^{n_o}k_i^{n_o+1} \\ \vdots & \iddots & \iddots & \vdots \\ \sum_{i=1}^{n_p} k_i^{n_o} & \sum_{i=1}^{n_p} dk_i^{n_o+1} & \dots & \sum_{i=1}^{n_p} d^{n_o}k_i^{2n_o} \end{bmatrix}}_{A} \begin{bmatrix} \Delta b_0 \\ \Delta b_1 \\ \vdots \\ \Delta b_{n_o} \end{bmatrix} = \begin{bmatrix} O(d^{n_o+1}) \\ O(d^{n_o+1}) \\ \vdots \\ O(d^{n_o+1}) \end{bmatrix}, \tag{32}$$

and factorize the A matrix as

$$A = \underbrace{\begin{bmatrix} \sum_{i=1}^{n_p} 1 & \sum_{i=1}^{n_p} k_i & \dots & \sum_{i=1}^{n_p} k_i^{n_o} \\ \sum_{i=1}^{n_p} k_i & \sum_{i=1}^{n_p} k_i^2 & \dots & \sum_{i=1}^{n_p} k_i^{n_o+1} \\ \vdots & \iddots & \iddots & \vdots \\ \sum_{i=1}^{n_p} k_i^{n_o} & \sum_{i=1}^{n_p} k_i^{n_o+1} & \dots & \sum_{i=1}^{n_p} k_i^{2n_o} \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & d & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d^{n_o} \end{bmatrix}}_{D}. \tag{33}$$

The D matrix is diagonal with $d > 0$ and can therefore be inverted. The K matrix can be written as $K = V'V$ with

$$V = \begin{bmatrix} 1 & k_1 & k_1^2 & \dots & k_1^{n_o} \\ 1 & k_2 & k_2^2 & \dots & k_2^{n_o} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & k_{n_p} & k_{n_p}^2 & \dots & k_{n_p}^{n_o} \end{bmatrix} \tag{34}$$

that is nothing but a Vandermonde matrix that is full column rank given that $k_i \neq k_j \quad \forall i \neq j$. Therefore the K matrix can be inverted.

The polynomial coefficients Δb_i are then given by

$$\begin{bmatrix} \Delta b_0 \\ \Delta b_1 \\ \vdots \\ \Delta b_{n_o} \end{bmatrix} = D^{-1}K^{-1}\begin{bmatrix} O(d^{n_o+1}) \\ O(d^{n_o+1}) \\ \vdots \\ O(d^{n_o+1}) \end{bmatrix}. \tag{35}$$

In particular, we can write

$$\Delta b_i = \frac{1}{d^i} \sum_{k=1}^{n_o+1} (K^{-1})_{(i+1,k)} O(d^{n_o+1}),\ i = 0, 1, ..., n_o. \quad (36)$$

The determinant of the Vandermonde matrix being lower bounded by a strictly positive value and all its elements being bounded, all the elements of its inverse can be bounded. In addition, the elements of K being independent of d, one has finally

$$\Delta b_i = O(d^{n_o+1-i}), \quad i = 0, 1, ..., n_o. \quad (37)$$

Combining (25) and (37) in (22), one obtains

$$b_i = \frac{c^{(i)}(\overline{y})}{i!} + O(d^{n_o+1-i}), \quad i = 0, 1, ..., n_o. \quad (38)$$

Therefore, defining $\widehat{c^{(i)}} = i! b_i$, one has

$$\widehat{c^{(i)}}(\overline{y}) = c^{(i)}(\overline{y}) + O(d^{n_o+1-i}), \quad i = 0, 1, ..., n_o. \quad (39)$$

ii) Substituting (39) in (18) and taking into account the fact that $y - \overline{y} = O(d)$, it results

$$\widehat{c^{(i)}}(y) = \sum_{k=i}^{n_o} c^{(k)}(\overline{y}) \frac{(y-\overline{y})^{k-i}}{(k-i)!} + O(d^{n_o+1-i}),\ i = 0, ..., n_o. \quad (40)$$

On the other hand, performing the Taylor expansion of $c^{(i)}(y)$ around $\overline{y}$ for $i = 0, 1, ..., n_o$ and remembering that both y and $\overline{y}$ are in $\mathcal{W}_y$, one has

$$c^{(i)}(y) = \sum_{k=i}^{n_o} c^{(k)}(\overline{y}) \frac{(y-\overline{y})^{k-i}}{(k-i)!} + O(d^{n_o+1-i}),\ i = 0, .., n_o. \quad (41)$$

Comparing the equations (40) and (41), one has

$$\widehat{c^{(i)}}(y) = c^{(i)}(y) + O(d^{n_o+1-i}) \quad i = 0, 1, ..., n_o. \quad (42)$$

iii) The result is obtained by substituting $d = \sqrt[n_o-1]{ap}$ in (42).

This conclude the proof of Lemma 19 □.

Remark 20. It is worth to mention that the estimation of the derivatives could be directly obtained at the current y with the least square fitting. However, this would have a higher computational cost, as it would require to perform the fitting for each current value of y.

6. SIMULATION RESULTS

For the sake of illustration, let us consider that the dynamics of a wind turbine can be described by the simple first order model

$$\begin{cases} \dot{x} = -100x + 100u \\ y = \quad x \end{cases}, \quad (43)$$

with x the turbine rotational speed and u the reference speed. The cost function $c(y)$ is the power produced by the wind turbine. From Slootweg et al. [2003], a suitable expression for this cost function, considering that the blade pitch angle is zero, is

$$c(y) = \frac{0.73}{2} \rho_{air} \pi R^2 V_w^3 \frac{151 \frac{V_w}{Ry} - 13.653}{\exp\left(18.4(\frac{V_w}{Ry} - 0.003)\right)}, \quad (44)$$

with ρ_{air} the air density (selected equal to 1.225kg/m^3), R the rotor radius (selected equal to 37.5m) and V_w the wind speed (selected equal to 10m/s). This cost function is illustrated in Figure 6. First, let us examine the accuracy

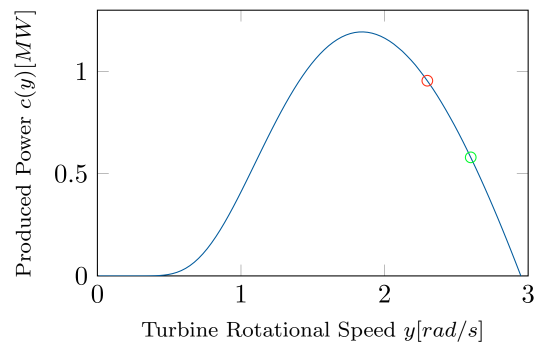

Fig. 6. Cost function

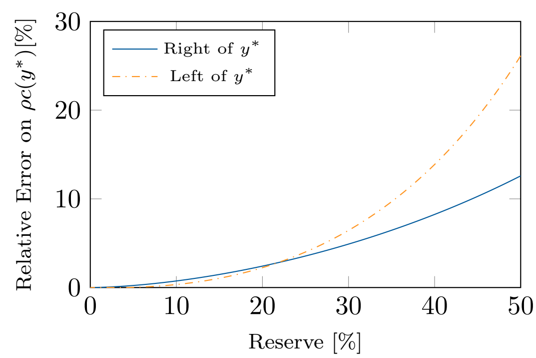

Fig. 7. Error on $\rho c(y^*)$ as a function of the amount of reserve.

of the sub-optimum estimated by using the second order Taylor expansion (i.e. $|\rho c(y^*) - c(y^*_{\gamma_a})|/\rho c(y^*)$, with $y^*_{\gamma_a} = \arg\min \gamma_a(y)$). In Figure 7, we have represented the evolution of this error in function of the percentage of reserve. It can be seen that, as expected, the estimation error grows as the percentage of reserve increases. However, this error stays smaller than 2.5% for reserves up to 20%, confirming the practical validity of our proposed approach. The simulation was performed with the following extremum seeking parameters: $a = 10^{-3} rad/s, \delta = 10 s/rad/kW$ and $\omega = 10 rad/s$. The estimator parameters are $n_o = 3$, $n_p = 10$ and $d = 10^{-4} rad/s$. The initial conditions are $\hat{u}_0 = 2.6 rad/s$ (corresponding to the steady state represented by the green dot in Figure 6) and $x_0 = 2.9 rad/s$. The aim is to reach 80% of the optimum (red dot in Figure 6). The simulation results are shown in Figure 8. From the first two sub-figures, one can notice that both the power produced and the rotor speed (blue curves) are converging close to the real value of the sub-optimum (red dashed curve). The error is smaller than 2.5%. Note that between $0s$ and $0.6s$, the integrator gain δ was set to zero since no previous estimate of $c'(y)$ and $c''(y)$ were available. This can be observed on the second sub-figure. Once the first estimations of $c'(y)$ and $c''(y)$ were available, the integrator gain was set to its rated value. In addition, we can see that, in steady state, the blue curve matches the orange dotted curve, which corresponds to the best that can be achieved with the proposed black box approach (i.e. reaching the maximum of γ_a). Finally, from the last sub-figure, one can see that the sub-optimum $\rho \hat{c^*}(y)$ (8) computed with our two-step approach is very close ($< 1.6\ 10^{-5}\%$) to the sub-optimum obtained from (8) with the actual derivatives.

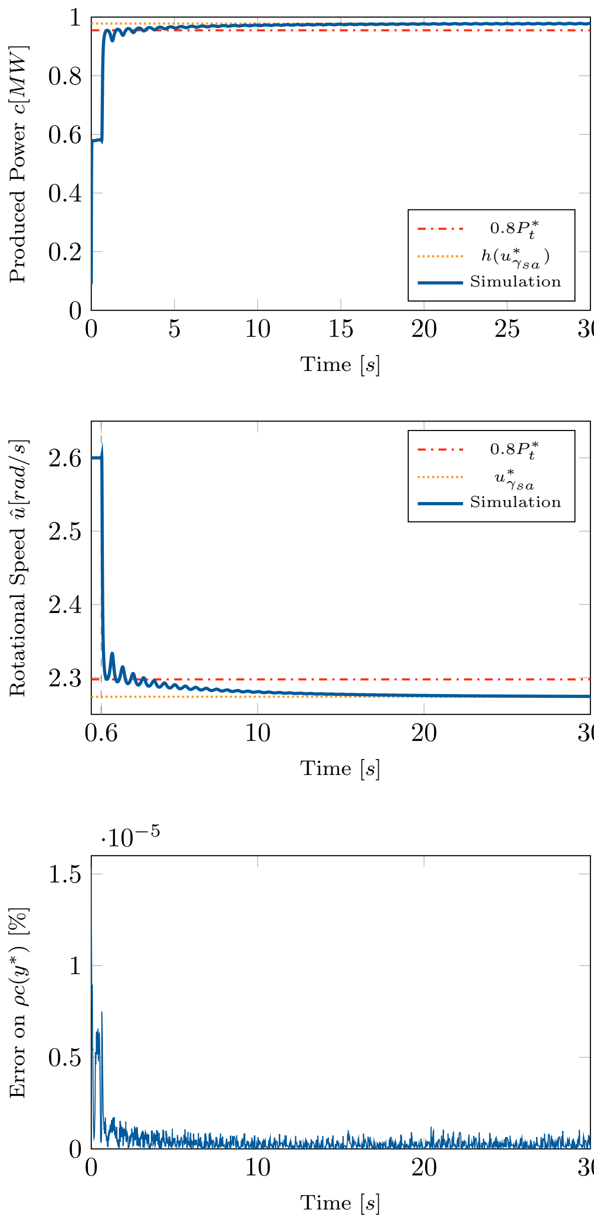

Fig. 8. Simulation results by using the control scheme depicted in Figure 4.

7. CONCLUSION

The aim of this paper was to present a preliminary control scheme able to steer a system to reach a given ratio of the optimal cost. Neither the cost function nor the system dynamics are analytically known. The idea was to introduce auxiliary cost functions and to use extremum seeking to optimize them. The optimum was estimated from the second order Taylor expansion. The values of the required derivatives were determined by a least square polynomial fitting combined with an interpolation.

Future work includes the estimation of the sub-optimal cost by only measuring the cost and the system input. This relaxes the assumption of having a cost function depending only on the measurable system output. Furthermore, the approach will be extended to multi-input systems so as to be able to apply it to systems like wind farms.

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