Adaptive Fuzzy Multi-surface Sliding Mode Control for A Class of Nonlinear Systems

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***Abstract*-An adaptive fuzzy multi-surface sliding mode control is proposed for a class of nonlinear systems with mismatched uncertainties. It is shown that the Mamdani fuzzy logic-based function approximator is used to learn the system uncertainties first, and the iterative multi-surface sliding mode design is then carried out. Using the concept of terminal sliding mode to design the baseline controller, not only strong robustness with respect to approximation errors and nonlinearities can be obtained, but also the tracking error can be proven to converge to a small neighborhood of the origin in finite time. Simulation results are provided to show the effectiveness of the proposed approach.**

I. Introduction

Sliding mode control [1, 2] has been widely applied to various systems since 1960s. As pointed out by Drazenovic [3], the salient feature of the sliding mode control is that it is completely robust to matched uncertainties that lie in the range space of the input matrix, after the system dynamics reaches the sliding mode surface. Although many systems can be classified under this category, there are still many systems with mismatched uncertainties, and the complex for the design of the sliding mode controllers is increased.

Multi-surface sliding mode control (MSMC) has been developed in [4] to control nonlinear systems with mismatched uncertainties. The upper bound information of the uncertainties is required to design the MSMC. However, this upper bound information may not always be available. Therefore, in [5], MSMC and the function approximation technique were combined to control a more general class of nonlinear systems with mismatched uncertainties whose the upper bound information of the uncertainties is not available. It is seen that the finite combination of orthonormal Fourier basis functions were used to design the function approximator. However, because these basis functions are obtained by trial and error approach, the functions selected may not be rich enough to approximate the mismatched uncertainties.

Among many function approximation techniques, neural networks and fuzzy systems are widely used [6-9]. For instance, neural networks were used to approximate a highly nonlinear robot function in [6, 7], fuzzy systems were used to approximate the system uncertainties and the unknown part of the nonlinear system in [8, 9]. Further, it has been proven in [10] that the Mamdani type fuzzy systems are universal approximators.

In this paper, we combine the Mamdani fuzzy logic-based function approximator and the MSMC to design an adaptive fuzzy multi-surface sliding mode control (AFMSMC). It is seen that the Mamdani fuzzy logic-based approximator is first employed to learn the system uncertainties in Lyapunov sense, i.e., the weights are adaptively adjusted so that the output tracking error can asymptotically converge to zero according to the second method of Lyapunov stability. The iterative MSMC design is then carried out. We note here that the time bound for the convergence of the tracking error can be reduced significantly by employing the concept of terminal sliding mode (TSM) [11] to design the baseline controller.

This paper is organized as follows. Section 2 presents the design procedure of MSMC. Section 3 presents the fuzzy systems used for function approximation. Section 4 presents the mathematical proof on the closed-loop stability in detail. Section 5 presents simulation results of a benchmark problem. Section 6 gives concluding remarks.

II. Multi-surface Sliding Mode Control

Consider the nth order SISO nonlinear system:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \Delta_i(x,t), \quad 1 \le i \le n-1, \\ \dot{x}_n &= b(x,t)u(t) + f(x,t) + \Delta_n(x,t), \\ y &= x_1, \end{aligned} \tag{1}$$

where $x = [x_1, ..., x_n]^T \in \Re^n$ is a vector of measurable states, $u(t) \in \Re$ and $y \in \Re$ represent the input and output signals, respectively. The functions $\Delta_i(x,t) \in \Re$, $i = 1,...,n$ and $b(x,t)$ are unknown, the function $b(x,t)$ does not vanish for all feasible x and for all $t \ge 0$.

The goal of this paper is to design an adaptive control law so that the output y can converge to the desired reference signal y_d with a guaranteed transient performance and tracking accuracy. The following steps formulate the design procedure for MSMC.

Step 1

Define n sliding mode surfaces as

$$s_i = x_i - x_{id}, \qquad i = 1,...,n, \tag{2}$$

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where x_{id} represents the desired value of state x_i.

Let $x_{1d} = y_d$, and differentiating s_1 with respect to time t, we have

$$\begin{aligned} \dot{s}_1 &= x_2 + \Delta_1(x,t) - y_d \\ &= s_2 + x_{2d} + \Delta_1(x,t) - \dot{y}_d . \end{aligned} \tag{3}$$

Let $\bar{\Delta}_1(x,t) = \Delta_1(x,t)$ and view x_{2d} as a synthetic control. We can then design a control law for s_1 such that x_1 tracks its desired trajectory x_{1d}.

The control law for the synthetic control is chosen as,

$$\begin{aligned} x_{2d} &= \dot{y}_d - \hat{\bar{\Delta}}_1(x,\Theta_{\Delta_1}) - \alpha_1 s_1 - k_1 s_1 \\ &= \dot{y}_d - \hat{\bar{\Delta}}_1(x,\Theta_{\Delta_1}) - w_1 s_1 , \end{aligned} \tag{4}$$

where $w_1 = \alpha_1 + k_1$, α_1 and k_1 are the design parameters to be determined later, $\hat{\bar{\Delta}}_1(x,\Theta_{\Delta_1})$ is the estimate of $\bar{\Delta}_1(x,t)$, and Θ_{Δ_1} is the collection of the tuning parameters c, φ, and ω.

Substituting (4) into (3), we write the time derivative of the first sliding mode surface as follows

$$\dot{s}_1 = s_2 + \left(\bar{\Delta}_1(x,t) - \hat{\bar{\Delta}}_1(x,\Theta_{\Delta_1})\right) - w_1 s_1 . \tag{5}$$

Step 2

Differentiating x_{2d} with respect to time, we have

$$\dot{x}_{2d} = y_d^2 - \dot{\hat{\bar{\Delta}}}_1(x,\Theta_{\Delta_1}) - w_1\left(x_2 + \Delta_1(x,t) - y_d\right), \tag{6}$$

where $y_d^2 = \dfrac{d^2 y}{dt^2}$.

For clarity and simplicity, we separate (6) into the certain and the uncertain portions. By defining the certain terms as $\tilde{\dot{x}}_{2d}$ and the uncertain terms as $\hat{\dot{x}}_{2d}$, $\dot{x}_{2d}$ is expressed as

$$\dot{x}_{2d} = \tilde{\dot{x}}_{2d} + \hat{\dot{x}}_{2d} , \tag{7}$$

where
$$\tilde{\dot{x}}_{2d} = y_d^2 - w_1\left(x_2 - y_d\right) \tag{8a}$$

and
$$\hat{\dot{x}}_{2d} = -\dot{\hat{\bar{\Delta}}}_1(x,\Theta_{\Delta_1}) - w_1\left(\Delta_1(x,t)\right). \tag{8b}$$

Differentiating s_2 with respect to time, we have

$$\begin{aligned} \dot{s}_2 &= x_3 + \Delta_2(x,t) - \dot{x}_{2d} \\ &= s_3 + x_{3d} + \bar{\Delta}_2(x,t) - \tilde{\dot{x}}_{2d} , \end{aligned} \tag{9}$$

where
$$\bar{\Delta}_2(x,t) = \Delta_2(x,t) - \hat{\dot{x}}_{2d} . \tag{10}$$

Also, the synthetic control input x_{3d} is chosen as

$$\begin{aligned} x_{3d} &= \tilde{\dot{x}}_{2d} - \hat{\bar{\Delta}}_2(x,\Theta_{\Delta_2}) - \alpha_2 s_2 - k_2 s_2 \\ &= \tilde{\dot{x}}_{2d} - \hat{\bar{\Delta}}_2(x,\Theta_{\Delta_2}) - w_2 s_2 , \end{aligned} \tag{11}$$

where $\tilde{\dot{x}}_{2d}$ is the known function from (7), $\hat{\bar{\Delta}}_2(x,\Theta_{\Delta_2})$ is an estimate of $\bar{\Delta}_2(x,t)$, and $w_2 = \alpha_2 + k_2$, $\alpha_2, k_2 > 0$ are the design parameters. Substituting (11) into (9), we obtain the second sliding mode surface as follows

$$\dot{s}_2 = s_3 + \left(\bar{\Delta}_2(x,t) - \hat{\bar{\Delta}}_2(x,\Theta_{\Delta_2})\right) - w_2 s_2 . \tag{12}$$

Step i

Similarly, the synthetic control input $x_{(i+1)d}$, for $i = 1, 2, \ldots, n-1$, can be chosen as [5]:

$$x_{(i+1)d} = \tilde{\dot{x}}_{id} - \hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}) - w_i s_i , \tag{13a}$$

where $w_i = \alpha_i + k_i$, $\alpha_i > 0$, $k_i \ge \dfrac{\left(E_{i+1} + \delta_{\Delta_i}\right)^2}{4e_i}$, $E_i = \sqrt{\dfrac{e_i}{2\alpha_i}}$, and $e_i > 0$, represent the design parameters, $\delta_{\Delta_i} \ge 0$ being the upper bound of the fuzzy approximation error in (26a), then

$$\dot{s}_i = s_{i+1} + \left(\bar{\Delta}_i(x,t) - \hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i})\right) - w_i s_i , \tag{13b}$$

$$\bar{\Delta}_i(x,t) = \Delta_i(x,t) - \hat{\dot{x}}_{id} , \tag{13c}$$

$$\dot{x}_{(i+1)d} = \tilde{\dot{x}}_{(i+1)d} + \hat{\dot{x}}_{(i+1)d} , \tag{13d}$$

$$\tilde{\dot{x}}_{(i+1)d} = y_d^{(i+1)} - \sum_{j=1}^{i} w_j\left(x_{i+1} - \tilde{\Phi}_{ij}\right), \tag{13e}$$

$$\hat{\dot{x}}_{(i+1)d} = -\dot{\hat{\bar{\Delta}}}_i(x,\Theta_{\Delta_i}) - \sum_{j=1}^{i} w_j\left(\Delta_i(x,t) + \bar{\Phi}_{ij}\right), \tag{13f}$$

$$\tilde{\Phi}_{ij} = \begin{cases} y_d^i & for\ j = 1, \\ y_d^i - \sum_{k=1}^{j-1} w_k\left(x_i - \Phi_{(i-1)k}\right) & for\ 2 \le j \le i-1, \\ \tilde{\dot{x}}_{id} & for\ j = i, \end{cases} \tag{13g}$$

$$\bar{\Phi}_{ij} = \begin{cases} 0 & for\ j = 1, \\ \sum_{k=1}^{j-1} w_k\left(\Delta_{i-1}(x,t) + \bar{\Phi}_{(i-1)k}\right) & for\ 2 \le j \le i-1, \\ \bar{\Delta}_i(x,t) - \Delta_i(x,t) & for\ j = i. \end{cases} \tag{13h}$$

Step n

Differentiating the final sliding mode surface s_n with respect to time, we have

$$\begin{aligned} \dot{s}_n &= f(x,t) + b(x,t)u(t) + \Delta_n(x,t) - x_{nd} \\ &= b(x,t)u(t) + \bar{\Delta}_n(x,t) - \tilde{f}(x,t), \end{aligned} \tag{14}$$

where

$$\tilde{\dot{x}}_{nd} = y_d^n - \sum_{j=1}^{n-1} w_j\left(x_n - \tilde{\Phi}_{(n-1)j}\right), \tag{15a}$$

$$\hat{\dot{x}}_{nd} = -\dot{\hat{\bar{\Delta}}}_{n-1}(x,\Theta_{\Delta_{n-1}}) - \sum_{j=1}^{n-1} w_j\left(\Delta_{n-1}(x,t) + \bar{\Phi}_{(n-1)j}\right), \tag{15b}$$

$$\tilde{f}(x,t) = \tilde{\dot{x}}_{nd} - f(x,t), \tag{15c}$$

and
$$\bar{\Delta}_n(x,t) = \Delta_n(x,t) - \hat{\dot{x}}_{nd} . \tag{15d}$$

The baseline input control law is chosen as

$$\begin{aligned} u(t) = \frac{1}{\hat{b}(x,\Theta_b)}\Big[& \tilde{f}(x,t) - \hat{\bar{\Delta}}_n(x,\Theta_{\Delta_n}) \\ & -\alpha_n s_n - \beta\left|s_n\right|^{\gamma} sign(s_n) \Big], \end{aligned} \tag{16}$$

where α_n, $\beta > 0$, $0 < \gamma < 1$, are the design parameters, $\hat{b}(x,\Theta_b)$ and $\hat{\bar{\Delta}}_n(x,\Theta_{\Delta_n})$ are the estimates of $b(x,t)$ and

$\bar{\Delta}_n(x,t)$, respectively.

III. Fuzzy Approximator

A. Mamdani Fuzzy Logic System

The Mamdani fuzzy logic system with m inputs, $I_1,...,I_m$ and the center of average deffuzifier is described as [9]

$$F=\frac{\sum_{k=1}^{g}\prod_{j=1}^{m}\mu_{P_j^k}(I_j)\omega_k}{\sum_{k=1}^{g}\prod_{j=1}^{m}\mu_{P_j^k}(I_j)}, \quad (17)$$

where g is the number of fuzzy rules, P_j^k is the j th fuzzy set corresponding to the k th fuzzy rule, $\mu_{P_j^k}$ is the membership function of the fuzzy set P_j^k, and ω_k is the centroid of the k th fuzzy set corresponding to the controller's output.

To proceed, we represent (17) as

$$F=\sum_{k=1}^{g}\left(\frac{\prod_{j=1}^{m}\mu_{P_j^k}(I_j)}{\sum_{k=1}^{g}\prod_{j=1}^{m}\mu_{P_j^k}(I_j)}\right)\omega_k. \quad (18)$$

Let $\omega=[\omega_1 \quad \omega_2 \quad \cdots \quad \omega_m]^T$,

and $M=[M_1 \quad M_2 \quad \cdots \quad M_m]^T$

$$=\left[\frac{\prod_{j=1}^{m}\mu_{P_j^1}(I_j)}{\sum\sum\prod_{j=1}^{m}\mu_{P_j^k}(I_j)} \quad \cdots \quad \frac{\prod_{j=1}^{m}\mu_{P_j^m}(I_j)}{\sum_{k=1}^{g}\prod_{j=1}^{m}\mu_{P_j^k}(I_j)}\right]^T.$$

Then, we write (18) as

$$F=\omega^T M. \quad (19)$$

In this paper, we use the following Gaussian membership function for P_j^k

$$\mu_{A_j^k}=\exp\left(-\frac{\left(I_j-c_{P_j^k}\right)^2}{2\varphi_{P_j^k}^2}\right), \quad (20)$$

where $c_{P_j^k}$ and $\varphi_{P_j^k}$ represent the centre and the gradient of the Gaussian membership function, respectively.

B. Modeling Uncertainties

In this subsection, we approximate the unknown functions, $\bar{\Delta}_i(x,t)$ and $b(x,t)$ using the Mamdani fuzzy logic systems $\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i})$ and $\hat{b}(x,\Theta_b)$, respectively, where:

$$\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i})=\omega_{\Delta_i}^T M_{\Delta_i}, \quad \hat{b}(x,\Theta_b)=\omega_b^T M_b, \quad (21)$$

with $\Theta_{\Delta_i}=\left\{c_{P_j^k}^{\Delta_i},\varphi_{P_j^k}^{\Delta_i},\omega_k^{\Delta_i}\right\}$ (22a)

and $\Theta_b=\left\{c_{P_j^k}^{b},\varphi_{P_j^k}^{b},\omega_k^{b}\right\}$ for $i=1,...,n$. (22b)

In the following, we use $\Theta_{\Delta_i}^*$ and Θ_b^* to represent the optimal fuzzy parameters such that

$$\Theta_{\Delta_i}^*=\arg\min_{\Theta_{\Delta_i}\in\Phi}\sup_{x,t\in\Omega}\left|\bar{\Delta}_i(x,t)-\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i})\right|, \quad (23a)$$

$$\Theta_b^*=\arg\min_{\Theta_b\in\Phi}\sup_{x,t\in\Omega}\left|b(x,t)-\hat{b}(x,\Theta_b)\right|, \quad (23b)$$

where Φ is a closed constraint set for Θ_{Δ_i} and Θ_b, $\Omega\subseteq\Re^{n+1}$ is a region where the state x and time t are constrained to reside.

For further analysis, we have the following assumptions:

Assumption 3.1: For all x, $t\in\Omega$, the minimum approximation errors satisfy the following bounded conditions:

$$\left|\bar{\Delta}(x,t)-\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}^*)\right|\le d_{\Delta_i} \text{ and } \left|b(x,t)-\hat{b}(x,\Theta_b^*)\right|\le d_b \quad (24)$$

with d_{Δ_i}, $d_b\ge 0$, for $i=1,...,n$.

Assumption 3.2: The optimal fuzzy approximators, $\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}^*)$ and $\hat{b}(x,\Theta_b^*)$ are bounded:

$$\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}^{\min})\le\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}^*)\le\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}^{\max}), \quad (25a)$$

$$0<\hat{b}(x,\Theta_b^{\min})\le\hat{b}(x,\Theta_b^*)\le\hat{b}(x,\Theta_b^{\max}). \quad (25b)$$

Furthermore,

$$\left|\bar{\Delta}_i(x,t)-\hat{\bar{\Delta}}_i(x,\Theta_{\Delta_i}^{\min})\right|\le\delta_{\Delta_i}, \quad (26a)$$

$$\left|b(x,t)-\hat{b}(x,\Theta_b^{\min})\right|\le\delta_b, \quad (26b)$$

and

$$\left|\delta_{\Delta_n}+\delta_b u\right|\le h_n, \quad (26c)$$

where δ_{Δ_i}, δ_b, and $h_n>0$, for $i=1,...,n$.

The above assumptions will be used in the stability analysis of the closed-loop system in the following section.

IV. The Controller Design

In this section, we first discuss the concept on TSM surface, then consider the design of the AFMSMC law and the proof of the closed-loop stability.

Definition 4.1: The fast TSM surface can be described by the following first order nonlinear differential equation [12]:

$$\dot{s}+\alpha s+\beta|s|^\gamma sign(s)=0, \quad (27)$$

where α, $\beta>0$, $0<\gamma<1$, respectively.

Remark 4.1: For any given initial condition s_0, s converges to $s=0$ in finite time, and the finite convergence time T can be computed as follows:

$$T=\frac{1}{\alpha(1-\gamma)}\ln\left(\frac{\alpha|s_0|^{1-\gamma}+\beta}{\beta}\right). \quad (28)$$

In order to obtain the upper bound of s_n, we need to minimize the following cost function:

$$J = \left(\tau \frac{h_n}{\alpha_n} - \left(\frac{(1-\tau) h_n}{\beta_n} \right)^{1/\gamma} \right)^2 . \tag{29}$$

The optimal value of τ is denoted as τ^* where

$$\tau^* = \arg\min_{\tau} \{J\}, \tag{30}$$

with $0 \le \tau \le 1$.

Now, we can approximate the fuzzy systems $\hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i})$ and $\hat{b}(x, \Theta_b)$ by using the truncated Taylor series. Taking the Taylor expansion of both $\hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i})$ and $\hat{b}(x, \Theta_b)$, we obtain

$$\hat{\hat{\Delta}}_i(x, \Theta^*_{\Delta_i}) - \hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i}) = \left(\Theta^*_{\Delta_i} - \Theta_{\Delta_i}\right) \left(\frac{d\hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i})}{d\Theta_{\Delta_i}} \right) + H \tag{31a}$$

$$\hat{b}(x, \Theta^*_b) - \hat{b}(x, \Theta_b) = \left(\Theta^*_b - \Theta_b\right) \left(\frac{d\hat{b}(x, \Theta_b)}{d\Theta_b} \right) + H, \tag{31b}$$

where H represents the higher order terms, which are small enough to be ignored.

Theorem 4.1: Consider the nth-order nonlinear system with mismatched uncertainties (1) under Assumptions 3.1 and 3.2. If we apply the control law (16) for the final sliding mode surface s_n, and the control law (13a) for the i^{th} sliding mode surface s_i where $i = 1, 2, \ldots n-1$, with the following update laws:

$$\dot{c}^{\Delta_i}_{P_j^k} = \frac{s_i}{\rho^{\Delta_i}_{jk}} \left(\frac{d\hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i})}{dc^{\Delta_i}_{P_j^k}} \right), \quad \dot{\varphi}^{\Delta_i}_{P_j^k} = \frac{s_i}{\sigma^{\Delta_i}_{jk}} \left(\frac{d\hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i})}{d\varphi^{\Delta_i}_{P_j^k}} \right),$$

$$\dot{\omega}^{\Delta_i}_k = \frac{s_i}{\eta^{\Delta_i}_k} \left(\frac{d\hat{\hat{\Delta}}_i(x, \Theta_{\Delta_i})}{d\omega^{\Delta_i}_k} \right), \tag{32a}$$

$$\dot{c}^{b}_{P_j^k} = \frac{s_n}{\rho^{b}_{jk}} \left(\frac{d\hat{b}(x, \Theta_b)}{dc^{b}_{P_j^k}} \right), \quad \dot{\varphi}^{b}_{P_j^k} = \frac{s_n}{\sigma^{b}_{jk}} \left(\frac{d\hat{b}(x, \Theta_b)}{d\varphi^{b}_{P_j^k}} \right),$$

$$\dot{\omega}^{b}_k = \frac{s_n}{\eta^{b}_k} \left(\frac{d\hat{b}(x, \Theta_b)}{d\omega^{b}_k} \right), \tag{32b}$$

where $\rho^{\Delta_i}_{jk}, \sigma^{\Delta_i}_{jk}, \eta^{\Delta_i}_k, \rho^b_{jk}, \sigma^b_{jk}, \eta^b_k > 0$ represent the positive weights, then the following properties can be guaranteed under any finite initial conditions: (i) the n^{th} sliding mode surface S_n will first converge to the neighborhood of $s_n = 0$ as

$$|s_n| \le \varepsilon_n \tag{33}$$

in finite time $T \le T_n$, where

$$\varepsilon_n = \begin{cases} \max\left(\dfrac{h_n}{\tau^* \alpha_n}, \left(\dfrac{(\tau^* - 1) h_n}{\tau^* \beta_n} \right)^{1/\gamma} \right) & J \ne 0, \\ \dfrac{h_n}{\tau^* \alpha_n} = \left(\dfrac{(\tau^* - 1) h_n}{\tau^* \beta_n} \right)^{1/\gamma} & J = 0, \end{cases} \tag{34}$$

$$T_n = \frac{1}{2\alpha_n (1-\zeta)} \ln\left(\frac{2\alpha_n |V_{n0}| + 2^{\zeta} \beta}{2^{\zeta} \beta} \right), \tag{35}$$

$$\zeta = \frac{1+\gamma}{2},$$

and next, (ii) the i^{th} sliding mode surface s_i will converge sequentially to the neighborhood of $s_i = 0$ as

$$|s_i| \le \varepsilon_i \tag{36}$$

in finite time $T = T_i$, with

$$T_i = \frac{\ln\left(\dfrac{2V_{i0}}{\varepsilon_i^2 - 2E_i^2} \right)}{2\alpha_i}, \tag{37}$$

where $\varepsilon_i^2 > 2E_i^2$.

Proof:

Case 1: Consider the following Lyapunov function candidate:

$$\begin{aligned} V_n = \frac{1}{2} s_n^2 + \frac{1}{2} \Big(& \sum_{j=1}^{m} \sum_{k=1}^{g} \rho^{\Delta_n}_{jk} \left(\tilde{c}^{\Delta_n}_{P_j^k}\right)^2 + \sum_{j=1}^{m} \sum_{k=1}^{g} \sigma^{\Delta_n}_{jk} \left(\tilde{\varphi}^{\Delta_n}_{P_j^k}\right)^2 \\ & + \sum_{k=1}^{g} \eta^{\Delta_n}_k \left(\tilde{\omega}^{\Delta_n}_k\right)^2 + \sum_{j=1}^{m} \sum_{k=1}^{g} \rho^b_{jk} \left(\tilde{c}^b_{P_j^k}\right)^2 \\ & + \sum_{j=1}^{m} \sum_{k=1}^{g} \sigma^b_{jk} \left(\tilde{\varphi}^b_{P_j^k}\right)^2 + \sum_{k=1}^{g} \eta^b_k \left(\tilde{\omega}^b_k\right)^2 \Big), \end{aligned} \tag{38}$$

where

$$\tilde{c}^{\Delta_n}_{P_j^k} = c^{*\Delta_n}_{P_j^k} - c^{\Delta_n}_{P_j^k}, \quad \tilde{c}^{b}_{P_j^k} = c^{*b}_{P_j^k} - c^{b}_{P_j^k}, \quad \tilde{\varphi}^{\Delta_n}_{P_j^k} = \varphi^{*\Delta_n}_{P_j^k} - \varphi^{\Delta_n}_{P_j^k},$$

$$\tilde{\varphi}^{b}_{P_j^k} = \varphi^{*b}_{P_j^k} - \varphi^{b}_{P_j^k}, \quad \tilde{\omega}^{\Delta_n}_{P_j^k} = \omega^{*\Delta_n}_{P_j^k} - \omega^{\Delta_n}_{P_j^k}, \quad \tilde{\omega}^{b}_{P_j^k} = \omega^{*b}_{P_j^k} - \omega^{b}_{P_j^k}.$$

Differentiating V_n with respect to time, we have

$$\begin{aligned} \dot{V}_n = s_n \dot{s}_n & + \sum_{j=1}^{m} \sum_{k=1}^{g} \rho^{\Delta_n}_{jk} \tilde{c}^{\Delta_n}_{P_j^k} \dot{\tilde{c}}^{\Delta_n}_{P_j^k} + \sum_{j=1}^{m} \sum_{k=1}^{g} \sigma^{\Delta_n}_{jk} \tilde{\varphi}^{\Delta_n}_{P_j^k} \dot{\tilde{\varphi}}^{\Delta_n}_{P_j^k} \\ & + \sum_{k=1}^{g} \eta^{\Delta_n}_k \tilde{\omega}^{\Delta_n}_k \dot{\tilde{\omega}}^{\Delta_n}_k + \sum_{j=1}^{m} \sum_{k=1}^{g} \rho^b_{jk} \tilde{c}^b_{P_j^k} \dot{\tilde{c}}^b_{P_j^k} \\ & + \sum_{j=1}^{m} \sum_{k=1}^{g} \sigma^b_{jk} \tilde{\varphi}^b_{P_j^k} \dot{\tilde{\varphi}}^b_{P_j^k} + \sum_{k=1}^{g} \eta^b_k \tilde{\omega}^b_k \dot{\tilde{\omega}}^b_k . \end{aligned} \tag{39}$$

Substituting (14) and (15) into (39), we have

$$\begin{aligned} \dot{V}_n = s_n \Big(& \left(\hat{b}(x, \Theta^*_b) - \hat{b}(x, \Theta_b) \right) u(t) \\ & + \left(\hat{\hat{\Delta}}_n(x, \Theta^*_{\Delta_n}) - \hat{\hat{\Delta}}_n(x, \Theta_{\Delta_n}) \right) - \alpha_n s_n - \beta |s_n|^{\gamma} sign(s_n) \Big) \\ & + \sum_{j=1}^{m} \sum_{k=1}^{g} \rho^{\Delta_n}_{jk} \tilde{c}^{\Delta_n}_{P_j^k} \dot{\tilde{c}}^{\Delta_n}_{P_j^k} + \sum_{j=1}^{m} \sum_{k=1}^{g} \sigma^{\Delta_n}_{jk} \tilde{\varphi}^{\Delta_n}_{P_j^k} \dot{\tilde{\varphi}}^{\Delta_n}_{P_j^k} \\ & + \sum_{k=1}^{g} \eta^{\Delta_n}_k \tilde{\omega}^{\Delta_n}_k \dot{\tilde{\omega}}^{\Delta_n}_k + \sum_{j=1}^{m} \sum_{k=1}^{g} \rho^b_{jk} \tilde{c}^b_{P_j^k} \dot{\tilde{c}}^b_{P_j^k} \end{aligned}$$

$$+\sum\sum_{j=1}^{n}\sum_{k=1}^{g}\sigma_{jk}^{b}\breve{\varphi}_{P_j^k}^{b}\dot{\breve{\varphi}}_{P_j^k}^{b}+\sum_{k=1}^{g}\eta_k^b\omega_k^b\dot{\omega}_k^b \,. \tag{40}$$

Substituting (31) in (40) and subsequently, using (32) and Assumption 3.2, we have

$$\dot{V}_n \le s_n\left(h_n-\alpha_n s_n-\beta\left|s_n\right|^{\gamma} sign\left(s_n\right)\right),$$

$$= s_n\left(\tau h_n-\alpha_n s_n+\left(1-\tau\right)h_n-\beta\left|s_n\right|^{\gamma} sign\left(s_n\right)\right). \tag{41}$$

It is seen that, if $h_n = 0$,

$$\dot{V}_n = s_n\left(-\alpha_n s_n-\beta\left|s_n\right|^{\gamma} sign\left(s_n\right)\right)$$

$$= -2\alpha_n V_n - 2^{\varsigma}\beta\left|V_n\right|^{\varsigma}. \tag{42}$$

From Definition 4.1 and Remark 4.1, the Lyapunov function candidate (38) converges to zero in finite time

$$T_n = \frac{1}{2\alpha_n\left(1-\zeta\right)}\ln\left(\frac{2\alpha_n\left|V_{n0}\right|^{(1-\zeta)}+2^{\zeta}\beta}{2^{\zeta}\beta}\right). \tag{43}$$

It is noted that, if $h_n \ne 0$, and $\dot{s}_n > 0$, for τ^* obtained with $J=0$, s_n can reach the following region in finite time

$$\left|s_n\right| \le \frac{\tau^* h_n}{\alpha_n} = \left(\frac{\left(1-\tau^*\right)h_n}{\beta_n}\right)^{1/\gamma}. \tag{44}$$

However, for τ^* obtained with $J \ne 0$, s_n can reach the following region in finite time

$$\left|s_n\right| \le \max\left(\frac{\tau^* h_n}{\alpha_n}, \left(\frac{\left(1-\tau^*\right)h_n}{\beta_n}\right)^{1/\gamma}\right). \tag{45}$$

Case 2: Consider the following Lyapunov function candidate

$$V_i = \frac{1}{2}s_i^2+\frac{1}{2}\left(\sum\sum_{j=1}^{n}\sum_{k=1}^{g}\rho_{jk}^{\Delta_i}\left(\breve{c}_{A_j^k}^{\Delta_i}\right)^2\right.$$

$$\left.+\sum\sum_{j=1}^{n}\sum_{k=1}^{g}\sigma_{jk}^{\Delta_i}\left(\breve{\varphi}_{A_j^k}^{\Delta_i}\right)^2+\sum_{k=1}^{g}\eta_k^{\Delta_i}\left(\omega_k^{\Delta_i}\right)^2\right). \tag{46}$$

Differentiating V_i with respect to time, we have

$$\dot{V}_i = s_i\dot{s}_i+\sum\sum_{j=1}^{n}\sum_{k=1}^{g}\rho_{jk}^{\Delta_i}\breve{c}_{A_j^k}^{\Delta_i}\dot{c}_{A_j^k}^{\Delta_i}$$

$$+\sum\sum_{j=1}^{n}\sum_{k=1}^{g}\sigma_{jk}^{\Delta_i}\breve{\varphi}_{A_j^k}^{\Delta_i}\dot{\breve{\varphi}}_{A_j^k}^{\Delta_i}+\sum_{k=1}^{g}\eta_k^{\Delta_i}\omega_k^{\Delta_i}\dot{\omega}_k^{\Delta_i}\,. \tag{47}$$

Using (13a), (31a), and (32a), similar to the prove of the Case 1, we obtain

$$\dot{V}_i \le s_i\left(\left(E_{i+1}+\delta_{\Delta i}\right)-\alpha_i s_i-k_i s_i\right)$$

$$= -2\alpha_i V_i-\left(\left(\frac{\left(E_{i+1}+\delta_{\Delta_i}\right)}{2\sqrt{k_i}}+\sqrt{k_i s_i}\right)^2+\left(\frac{\left(E_{i+1}+\delta_{\Delta_i}\right)}{2\sqrt{k_i}}\right)^2\right)$$

$$\le -2\alpha_i V_i+\left(\frac{\left(E_{i+1}+\delta_{\Delta_i}\right)}{2\sqrt{k_i}}\right)^2$$

$$\le -2\alpha_i V_i+e_i\,. \tag{48}$$

By solving the linear differential equation (48), we get

$$V_i \le \exp\left(-2\alpha_i t\right)V_{i0}+E_i^2\,. \tag{49}$$

Hence, the Lyapunov function candidate (46) converges to $V_i \le \frac{\varepsilon_i^2}{2}$, which implies that the i^{th} sliding mode surface s_i converges to $\left|s_i\right| \le \varepsilon_i$ in finite time (37).

V. Numerical Simulation

The proposed AFMSMC controller is tested on the benchmark system in Freeman and Kokotovic [13] and Won and Hedrick [14]. The dynamic equation is as follows:

$$\dot{x}_1 = x_2+\theta x_1^3,$$

$$\dot{x}_2 = \left(1+\mu\right)u,$$

$$y = x_1, \tag{50}$$

where θ and μ are unknown parameters. In the actual simulation, θ is chosen as 0.95, and μ as 0.09. The bound of the unknown functions are assumed to be unknown to the fuzzy approximator. The desired output y_d is chosen as $y_d = 1+\sin\left(2\pi t\right)$. The initial values of the system states are chosen as $x_1\left(0\right)=x_2\left(0\right)=0$.

The sliding mode surfaces are $s_1 = x_1-y_d$ and $s_2 = x_2-x_{2d}$. Here, we choose $\alpha_1 = 20$, $\alpha_2 = 10$, $\beta = 5$, and $\gamma = 0.5$, respectively. System states, x_1 and x_2 are chosen as the input variables to the fuzzy approximator to model the unknown function, θx_1^3. Three membership functions are assigned to each fuzzy input variable. However, for unknown functions, $\left(1+\mu\right)$, s_1 and s_2, are chosen as the input variables to the fuzzy approximator and similarly, three membership functions are assigned to each fuzzy input variable.

For simplicity, we assigned the initial values to the fuzzy parameters according to human experience where

$$c^{\Delta} = \left[0.3 \quad 1.4 \quad 2.2 \quad -8.9 \quad 1.3 \quad 11.4\right],$$

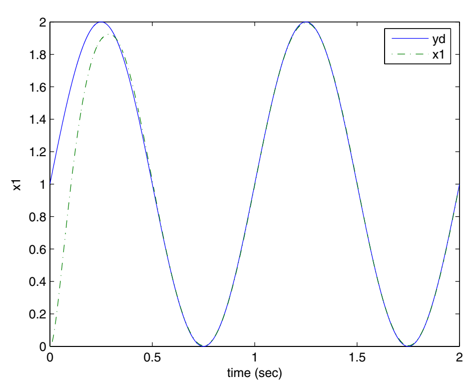

Fig. 1. The output tracking.

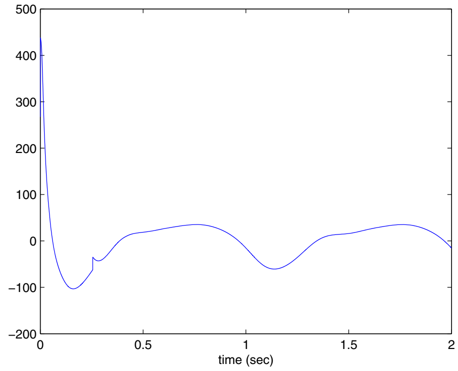

Fig. 2. The control input signal.

$$\varphi^{\Delta} = \begin{bmatrix} 0.7 & 0.6 & 1.2 & 4.9 & 5.1 & 4.9 \end{bmatrix},$$

and $$\omega^{\Delta} = \begin{bmatrix} 0.5 & -5 & -1 & -3 & -14 & 0.1 & 1 & 27 & 7 \end{bmatrix}.$$

Fig. 1 and Fig.2 show the tracking performance and the control input of the AFMSMC, respectively. It is seen that the superior performance of fast and high precision tracking performance with the proposed AFMSMC scheme is obtained.

VI. CONCLUSION

An adaptive fuzzy controller has been developed such that the knowledge and boundary conditions of the uncertainties are not required. The proposed adaptive control scheme has been proven to be able to guarantee the tracking error to converge to a small neighborhood of the origin in finite time. The controller is simple to design, yet gives good tracking performance for a class of nonlinear systems with mismatched uncertainties. Simulation results have been provided to show the effectiveness of the proposed control scheme.

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