LIMIT CYCLES IN RELAY SYSTEMS

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Abstract

The paper describes programs written in FORTRAN for the investigation of limit cycles in control systems with relays. The program can be used for plant transfer functions which are unstable or have time delays. Stability of a limit cycle solution is evaluated and the solution waveforms can be displayed if requested.

1. Introduction

The analysis of feedback systems becomes significantly complex when nonlinear effects have to be included. The nonlinearity may be inherent in the process to be controlled or may be intentionally introduced in the design to improve the performance or implement the required control. Relay type elements are often used in the latter case; applications of which include on-off temperature controls and gas jets for satellite attitude control. Apart from being a common form of nonlinearity found in feedback systems the relay is unique in that its output does not depend upon its input at all time instants but only those few instants where the input has a value which causes switching. It is this feature of the relay, with the approximate characteristic shown in figure 1(a), which enables the exact evaluation of limit cycles and their stability in a relay feedback system.

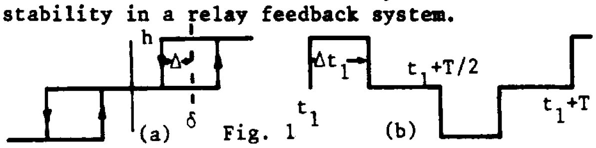

Fig. 1

The question of instability in a control system is usually one of determining whether a limit cycle exists; a method for evaluating limit cycles in relay systems thus provides a stability test applicable to systems of any order. Typical applications of the software described include:- (1) the evaluation of limit cycles in relay systems (2) exact conditions for the disappearance of limit cycles, and thus stability, of relay systems (3) design of compensators to eliminate limit cycles in relay systems or give limit cycles with appropriate parameters and (4) comparison of describing function results with exact solutions for relay systems or approximate ones for other nonlinear elements which may be closely modelled by several relay elements in parallel.

2. Limit cycle evaluation

The starting point in the analysis is an assumed relay output waveform, y(t), of period, T, typically as shown in figure 1(b). Using a Fourier series representation for y(t) it can be shown [1] that the output c(t) of G(s) in figure 2 can be written

$$c(t)=(2h/\pi)\{\mathrm{ImA}_G(-\omega t+\omega t_1,\omega)-\mathrm{ImA}_G(-\omega t+\omega t_1+\omega\Delta t_1,\omega)\} \quad (1.1)$$

and

$$\dot{c}(t)=(2\omega h/\pi)\{\mathrm{ReA}_G(-\omega t+\omega t_1,\omega)-\mathrm{ReA}_G(-\omega t+\omega t_1+\omega\Delta t_1,\omega)\} \quad (1.2)$$

where the A locus [1] is defined by

$$A_G(\theta,\omega)=\mathrm{ReA}_G(\theta,\omega)+j\,\mathrm{ImA}_G(\theta,\omega) \quad (1.3)$$

with

$$\mathrm{ReA}_G(\theta,\omega)=\sum_{n=1}^{\infty} V_G(n\omega)\sin n\theta+U_G(n\omega)\cos n\theta \quad (1.4)$$

$$\mathrm{ImA}_G(\theta,\omega)=\sum_{n=1}^{\infty}(1/n)\{V_G(n\omega)\cos n\theta-U_G(n\omega)\sin n\theta\} \quad (1.5)$$

and $G(j\omega)=U_G(\omega)+jV_G(\omega)$ (1.6)

When the input r(t) = 0 in figure 3, x(t) = -c(t) and to obtain the assumed relay output the switching conditions

$$x(t_1)=\delta+\Delta,\ \dot{x}(t_1)>0 \quad (1.7)$$

$$x(t_1+\Delta t_1)=\delta-\Delta,\ \dot{x}(t_1+\Delta t_1)<0 \quad (1.8)$$

must be satisfied. This leads to the two nonlinear algebraic equations

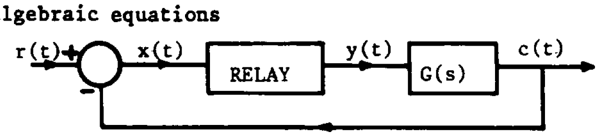

Fig. 2

$A_G(0,\omega)-A_G(\omega\Delta t_1,\omega)$ must have I.P.$=-\pi(\delta+\Delta)/2h$ (1.9)
R.P.<0

and

$A_G(0,\omega)-A_G(-\omega\Delta t_1,\omega)$ must have I.P.$=\pi(\delta-\Delta)/2h$ (1.10)
R.P.<0

which if a limit cycle exists, yield the unknown parameters ω and Δt_1 of the assumed waveform, where R.P. and I.P. denote real and imaginary part, respectively. For the solution to be valid the continuity conditions

$$x(t)>(\delta-\Delta) \text{ for } t_1<t<t_1+\Delta t_1 \quad (1.11)$$

and

$$-(\delta+\Delta)<x(t)<(\delta+\Delta) \text{ for } t_1+\Delta t_1<t<t_1+T/2 \quad (1.12)$$

must also be satisfied. These conditions can be checked by computing the relay input waveform

$-c(t)$, given by equation (1.1), with the solution parameters ω and Δt_1 substituted. More complex multiple pulse oscillations may exist in systems with lightly damped plants and these may be computed using extensions of the above approach [1]. An open question, however, is the determination of limit cycles which involve sliding; fortunately this is only possible when $\lim_{s\to\infty} sG(s) = 0$

3. Limit cycle stability

An important question is the stability of the computed limit cycle solution. A sufficient condition for stability is available [1], which uses an exact equivalent gain for the relay that is dependent on the derivative of the relay input at the switching instants, and the results are then easily obtained using the Hurwitz Routh criterion or a Nyquist plot. A necessary and sufficient condition [2] can also be used which involves rather more computation. The requirement is that the eigenvalues of the product of matrix exponentials lie within the unit circle. The matrices depend upon the parameters of a state space description of G(s) and the derivative of the relay input at the switching instants. This approach has an interesting checking feature since one eigenvalue must always equal -1. No extension of this criterion for systems with a time delay is known and difficulties have also been encountered when the derivative of the relay input is discontinuous at the switching instants.

4. Software description

The software to solve for possible limit cycles is written in FORTRAN and runs in an interactive mode. The program requests the relay parameters and the elements of G(s). Currently the latter input is required as the partial fraction expansion of G(s) in terms of simple transfer functions. Estimated solution parameters, which are most conveniently obtained using the describing function method, are then requested. The program then iterates to an exact limit cycle solution and if requested evaluates its stability by either of the above methods. The solution procedure normally uses the closed form expressions for the A_G loci of the simple transfer functions to compute the limit cycle but term by term summation of the infinite series to a specified accuracy may be selected. To check the continuity conditions (1.11) and (1.12) the solution waveforms for the relay input and its derivative can be plotted if required.

5. Applications

To illustrate applications of the software some examples are briefly discussed. Additional information and examples will be presented at the poster session.

Example 1

The relay is ideal with $\delta=\Delta=0$, $h=1$ and the transfer function $G(s) = (2s+1)/(s^2+1)(s+1)$. Using the describing function method no limit cycle is predicted whereas exact analysis using equation (1.9) gives an infinite number of possible limit cycle frequencies. Evaluation of the limit cycle waveforms for these solutions, however, shows that they

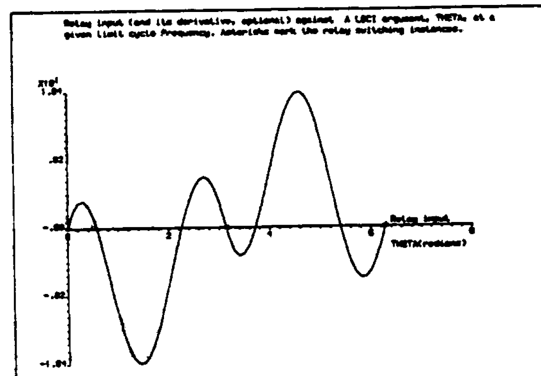

Fig. 3

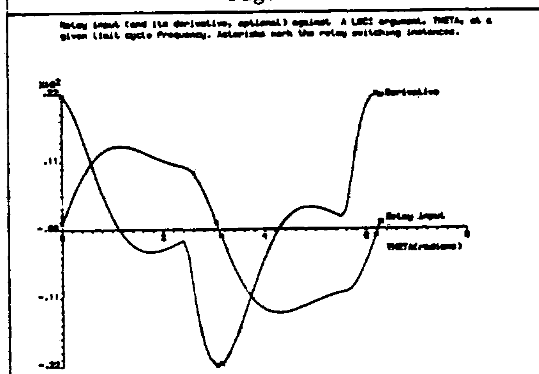

Fig. 4

do not satisfy the continuity conditions. The relay waveform for the solution $\omega=0.4$ is shown in figure 3 where the crosses mark the switching points to produce the assumed output square wave.

Example 2

Obviously the less harmonic content in the limit cycle waveform the fewer the terms that need to be taken in the summations for the A loci to achieve a given accuracy when the closed form expressions are not used. The computation time required for the solution to an accuracy of four decimal places when only ten terms in the series are needed is comparable with the time using the closed form expression. On the other hand figure 4 shows the solution for the stable limit cycle in a system with a relay having $\Delta=0$, $\delta=h=1$ and $G(s)=10e^{-5s}/(s^2+s+1)$. Because of the high distortion, to compute the derivative of the relay input at the switching instants to four decimal place accuracy using term summation required eighteen times as much computation time as using the closed form expressions.

Example 3

For a system with $\Delta=0$, $h=\delta=1$ and $G(s)=2(s^2+0.4s+1)/(s^2+s+1)(s^2-0.2s+1)$ two limit cycle solutions were found with $\omega=1.2416$ and 1.2480 which were stable and unstable, respectively. By the exact stability criterion the eigenvalues for the two cases were -1, -0.3161, $-0.5295 \pm j0.3709$ and -1. -1.2948, $-0.2875 \pm j0.1430$.

6. References

1. Atherton, D.P., 'Stability of Nonlinear Systems', Chapter 6, J. Wiley & Sons, New York, 1981.
2. Balasubramanian, R., 'Stability of limit cycles in Feedback systems containing a relay'., IEE Proc. D. 128, pp 24-29, 1981.