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Design and Stability Analysis of Extremum Seeking Feedback for General Nonlinear Systems*

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Abstract

While the mainstream methods of adaptive control (both linear and nonlinear) deal only with regulation to *known* set points or reference trajectories, in many applications the set point should be selected to achieve a maximum of an *uncertain* reference-to-output equilibrium map. The techniques of the so-called "extremum control" or "self-optimizing control" developed for this problem in the 1950–1960's have long gone out of fashion in the theoretical control literature because of the difficulties that arise in a rigorous analytical treatment. In this paper we provide the first proof of stability of an extremum seeking feedback scheme by employing the tools of averaging and singular perturbation analysis. Our scheme is much more general that the existing extremum control results which represent the plant as a static nonlinear map possibly cascaded with a linear dynamic block—we allow the plant to be a general nonlinear dynamic system (possibly non-affine in control and open-loop unstable) whose reference-to-output equilibrium map has a maximum, and whose equilibria are locally exponentially stabilizable.

1 Introduction

The mainstream methods of adaptive control for linear [1, 6, 7] and nonlinear [11] systems are applicable only for regulation to *known* set points or reference trajectories. In some applications, the reference-to-output map has an *extremum* (w.l.o.g. we assume that it is a maximum) and the objective is to select the set point to keep the output at the extremum value. The uncertainty in the reference-to-output map makes it necessary to use some sort of adaptation to find the set point which maximizes the output. This problem, called "extremum control" or "self-optimizing control," was popular in the 1950's and 1960's [2, 4, 5, 8, 9, 16, 17, 18], much before the theoretical breakthroughs in adaptive linear control of the 1980's. In fact, the emergence of extremum control dates as far back as the 1922 paper of Leblanc [12], whose scheme may very well have been the first "adaptive" controller reported in the literature. Among the surveys on extremum control, we find the one by Sternby [19] particularly useful, as well as Section 13.3 in Astrom and Wittenmark [1] which puts extremum control among the most promising future areas for adaptive control. Among the many applications of extremum control overviewed in [19] and [1] are combustion processes (for IC engines, steam generating plants, and gas furnaces), grinding processes, solar cell and radio telescope antenna adjustment to maximize the received signal, and blade adjustment in water turbines and wind mills to maximize the generated power. A more recent application of extremum control are anti-lock braking systems where schemes different from that in this paper are currently in use [3]. On the theoretical forefront, the pioneering averaging studies of Meerkov [13, 14, 15] stand out as a precursor to the stability results presented here.

Most of the results available on extremum control consider a plant as a static map. A few references approach problems where the plant is a cascade of a nonlinear static map and a linear dynamic system (the so-called Hammerstein and Wiener models), see [20] and references therein. In this paper we approach the general problem where the nonlinearity with an extremum is a reference-to-output *equilibrium* map for a general nonlinear (non-affine in control) system stabilizable around each of these equilibria by a local feedback controller.

The main contribution of our paper is that it provides the first rigorous proof of stability for an extremum seeking feedback scheme. We employ the tools of averaging and singular perturbations to show that solutions of the closed-loop system converge to a small neighborhood of the extremum of the equilibrium map. The size of the neighborhood is inversely proportional to the adaptation gain and the amplitude and the frequency of a periodic signal used to achieve extremum seeking. Our analysis highlights a fundamentally nonlinear mechanism of stabilization in an extremum seek-

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ing loop. After stating the problem in Section 2 and giving the extremum seeking scheme in Section 3, our proof is presented in Sections 4 and 5.

2 Extremum Seeking—Problem Statement

Consider a general SISO nonlinear model

$$\begin{aligned} \dot{x} &= f(x,u) \qquad (2.1)\\ y &= h(x)\,, \qquad (2.2) \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth. Suppose that we know a smooth control law

$$u = \alpha(x,\theta) \qquad (2.3)$$

parameterized by a scalar parameter θ. The closed-loop system

$$\dot{x} = f\left(x, \alpha(x,\theta)\right) \qquad (2.4)$$

then has equilibria parameterized by θ. We make the following assumptions about the closed-loop system.

Assumption 2.1 *There exists a smooth function $l : \mathbb{R} \to \mathbb{R}^n$ such that*

$$f\left(x, \alpha(x,\theta)\right) = 0 \qquad \textit{if and only if} \qquad x = l(\theta)\,. \qquad (2.5)$$

Assumption 2.2 *For each $\theta \in \mathbb{R}$, the equilibrium $x = l(\theta)$ of the system (2.4) is locally exponentially stable.*

Hence, we assume that we have a control law (2.3) which is robust with respect to its own parameter θ in the sense that it *exponentially stabilizes any of the equilibria that θ may produce.* Except for the requirement that Assumption 2.2 holds *for any* $\theta \in \mathbb{R}$ (which we impose only for notational convenience and can easily relax to an *interval* in $\mathbb{R}$), this assumption is not restrictive. It simply means that we have a control law designed for local stabilization and this control law need not be based on modeling knowledge of either $f(x,u)$ or $l(\theta)$.

The next assumption is central to the problem of peak seeking.

Assumption 2.3 *There exists $\theta^* \in \mathbb{R}$ such that*

$$\begin{aligned} (h \circ l)'\,(\theta^*) &= 0 \qquad (2.6)\\ (h \circ l)''\,(\theta^*) &< 0\,. \qquad (2.7) \end{aligned}$$

Thus, we assume that the output equilibrium map $y = h\left(l(\theta)\right)$ has a *maximum* at $\theta = \theta^*$. Our objective is to develop a feedback mechanism which maximizes the steady state value of y but without requiring the knowledge of either θ^* or the functions h and l. Our assumption that $h \circ l$ has a maximum is without loss of generality—the case with a minimum would be treated identically by replacing y by $-y$ in the subsequent feedback design.

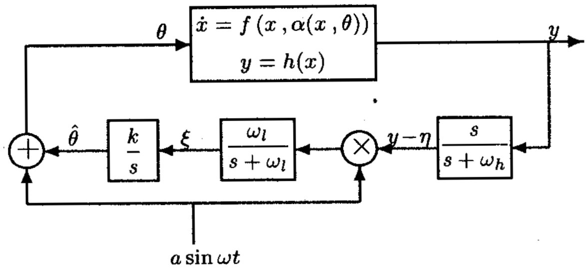

Figure 1: A peak seeking feedback scheme.

3 A Peak Seeking Scheme

Our feedback scheme is shown in Figure 1. It is an extension of a simple method for seeking extrema of *static* nonlinear maps [2]. Before we engage in extensive efforts to prove stability of the scheme, we explain its basic idea. We start by pointing out that it is impossible to conclude that a certain point is a maximum without visiting the neighborhood on both sides of the maximum. For this reason, we employ a *slow* periodic perturbation $a \sin \omega t$ which is added to the signal $\hat{\theta}$, our best estimate of θ^* (the persistent nature of $a \sin \omega t$ may be undesirable but is necessary to maintain a maximum in the face of changes in functions f and h). If the perturbation is slow, then the plant appears as a static map $y = h \circ l(\theta)$ (see Figure 2) and its dynamics do not interfere with the peak seeking scheme. If $\hat{\theta}$ is on either side of θ^*, the perturbation $a \sin \omega t$ will create a periodic response of y which is either in phase or out of phase with $a \sin \omega t$. The high-pass filter $\dfrac{s}{s+\omega_h}$ eliminates the "DC component" of y. Thus, $a \sin \omega t$ and $\dfrac{s}{s+\omega_h} y$ will be (approximately) two sinusoids which are

- in phase for $\hat{\theta} < \theta^*$
- out of phase for $\hat{\theta} > \theta^*$.

In either case, the product of the two sinusoids will have a "DC component" which is extracted by the low-pass filter $\dfrac{\omega_l}{s+\omega_l}$. The sign of the DC component ξ provides the direction to the integrator $\hat{\theta} = \dfrac{k}{s}\xi$ for moving $\hat{\theta}$ towards θ^*.

Despite its apparent simplicity, the proof of "stability" of this feedback scheme has not appeared in the literature even for the static case in Figure 2. As we shall see in the sequel, both the analysis of the scheme and the selection of design parameters are indeed intricate. These parameters are selected as

$$\begin{aligned} \omega_h &= \omega\omega_H = \omega\delta\omega_H' = O(\omega\delta) \qquad (3.1)\\ \omega_l &= \omega\omega_L = \omega\delta\omega_L' = O(\omega\delta) \qquad (3.2)\\ k &= \omega K = \omega\delta K' = O(\omega\delta)\,, \qquad (3.3) \end{aligned}$$

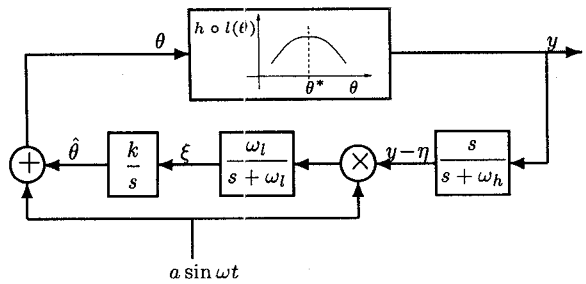

Figure 2: If perturbation $a \sin \omega t$ is slow, the plant can be viewed as a static map.

where ω and δ are small positive constants and ω'_H, ω'_L, and K' are $O(1)$ positive constants. As it will become apparent later, a also needs to be small.

From (3.1) and (3.2) we see that the cut-off frequencies of the filters need to be lower than the frequency of the perturbation signal. In addition, the adaptation gain k needs to be small. Thus, the overall feedback system has three time scales:

- fastest—the plant with the stabilizing controller,
- medium—the periodic perturbation,
- slow—the filters in the peak seeking scheme.

The analysis that follows treats first the static case from Figure 2 using the method of *averaging* (Section 4). Then we use the *singular perturbation* method (Section 5) for the full system in Figure 1.

Before we start our analysis, we summarize the system in Figure 1 as

$$\begin{aligned} \dot{x} &= f\left(x, \alpha(x, \hat{\theta} + a \sin \omega t)\right) && (3.4) \\ \dot{\hat{\theta}} &= k\xi && (3.5) \\ \dot{\xi} &= -\omega_l \xi + \omega_l (y - \eta) a \sin \omega t && (3.6) \\ \dot{\eta} &= -\omega_h \eta + \omega_h y\,. && (3.7) \end{aligned}$$

Let us introduce the new coordinates

$$\begin{aligned} \tilde{\theta} &= \hat{\theta} - \theta^* && (3.8) \\ \tilde{\eta} &= \eta - h \circ l(\theta^*)\,. && (3.9) \end{aligned}$$

Then, in the time scale $\tau = \omega t$, the system (3.4)–(3.7) is rewritten as

$$\omega \frac{dx}{d\tau} = f\left(x, \alpha(x, \theta^* + \tilde{\theta} + a \sin \tau)\right) \qquad (3.10)$$

$$\frac{d}{d\tau}\begin{bmatrix} \tilde{\theta} \\ \xi \\ \tilde{\eta} \end{bmatrix} = \delta \begin{bmatrix} K'\xi \\ -\omega'_L \xi + \omega'_L \left(h(x) - h \circ l(\theta^*) - \tilde{\eta}\right) a \sin \tau \\ -\omega'_H \tilde{\eta} + \omega'_H \left(h(x) - h \circ l(\theta^*)\right) \end{bmatrix} . \qquad (3.11)$$

4 Averaging Analysis

The first step in our analysis is to study the system in Figure 2. We "freeze" x in (3.10) at its "equilibrium" value

$$x = l\left(\theta^* + \tilde{\theta} + a \sin \tau\right) \qquad (4.1)$$

and substitute it into (3.11), getting the "reduced system"

$$\frac{d}{d\tau}\begin{bmatrix} \tilde{\theta}_r \\ \xi_r \\ \tilde{\eta}_r \end{bmatrix} = \delta \begin{bmatrix} K'\xi_r \\ -\omega'_L \xi_r + \omega'_L \left(\nu\left(\tilde{\theta}_r + a \sin \tau\right) - \tilde{\eta}_r\right) a \sin \tau \\ -\omega'_H \tilde{\eta}_r + \omega'_H \nu\left(\tilde{\theta}_r + a \sin \tau\right) \end{bmatrix} , \qquad (4.2)$$

where

$$\nu\left(\tilde{\theta}_r + a \sin \tau\right) = h \circ l\left(\theta^* + \tilde{\theta}_r + a \sin \tau\right) - h \circ l(\theta^*)\,. \qquad (4.3)$$

In view of Assumption 2.3, it is obvious that

$$\begin{aligned} \nu(0) &= 0 && (4.4) \\ \nu'(0) &= (h \circ l)'(\theta^*) = 0 && (4.5) \\ \nu''(0) &= (h \circ l)''(\theta^*) < 0\,. && (4.6) \end{aligned}$$

The system (4.2) is in the form to which the averaging method is applicable. The average model of (4.2) is

$$\frac{d}{d\tau}\begin{bmatrix} \tilde{\theta}_r^a \\ \xi_r^a \\ \tilde{\eta}_r^a \end{bmatrix} = \delta \begin{bmatrix} K'\xi_r^a \\ -\omega'_L \xi_r^a + \frac{\omega'_L}{2\pi} a \int_0^{2\pi} \nu\left(\tilde{\theta}_r^a + a \sin \sigma\right) \sin \sigma d\sigma \\ -\omega'_H \tilde{\eta}_r^a + \frac{\omega'_H}{2\pi} \int_0^{2\pi} \nu\left(\tilde{\theta}_r^a + a \sin \sigma\right) d\sigma \end{bmatrix} . \qquad (4.7)$$

First we need to determine the average equilibrium $\left(\tilde{\theta}_r^{a,e}, \xi_r^{a,e}, \tilde{\eta}_r^{a,e}\right)$ which satisfies

$$\xi_r^{a,e} = 0 \qquad (4.8)$$

$$\int_0^{2\pi} \nu\left(\tilde{\theta}_r^{a,e} + a \sin \sigma\right) \sin \sigma d\sigma = 0 \qquad (4.9)$$

$$\tilde{\eta}_r^{a,e} = \frac{1}{2\pi} \int_0^{2\pi} \nu\left(\tilde{\theta}_r^{a,e} + a \sin \sigma\right) d\sigma\,. \qquad (4.10)$$

By postulating $\tilde{\theta}_r^{a,e}$ in the form

$$\tilde{\theta}_r^{a,e} = b_1 a + b_2 a^2 + O(a^3)\,, \qquad (4.11)$$

substituting in (4.9), using (4.4) and (4.5), integrating, and equating the like powers of a, we get $\nu''(0) b_1 = 0$ and $\nu''(0) b_2 + \frac{1}{8}\nu'''(0) = 0$, which implies that

$$\tilde{\theta}_r^{a,e} = -\frac{\nu'''(0)}{8\nu''(0)} a^2 + O(a^3)\,. \qquad (4.12)$$

Another round of lengthy calculations applied to (4.10) yields

$$\tilde{\eta}_r^{a,e} = \frac{\nu''(0)}{4}a^2 + O(a^3). \tag{4.13}$$

Thus, the equilibrium of the average model (4.7) is

$$\begin{bmatrix} \tilde{\theta}_r^{a,e} \\ \xi_r^{a,e} \\ \tilde{\eta}_r^{a,e} \end{bmatrix} = \begin{bmatrix} -\frac{\nu'''(0)}{8\nu''(0)}a^2 + O(a^3) \\ 0 \\ \frac{\nu''(0)}{4}a^2 + O(a^3) \end{bmatrix}. \tag{4.14}$$

The Jacobian of (4.7) at $\left(\tilde{\theta}, \xi, \tilde{\eta}\right)_r^{a,e}$ is

$$J_r^a = \delta \begin{bmatrix} 0 & K' & 0 \\ \frac{\omega_L'}{2\pi} a \int_0^{2\pi} \nu'\left(\tilde{\theta}_r^{a,e} + a\sin\sigma\right)\sin\sigma d\sigma & -\omega_L' & 0 \\ \frac{\omega_H'}{2\pi}\int_0^{2\pi} \nu'\left(\tilde{\theta}_r^{a,e} + a\sin\sigma\right) d\sigma & 0 & -\omega_H' \end{bmatrix}. \tag{4.15}$$

Since J_r^a is block-lower-triangular we easily see that it will be Hurwitz if and only if

$$\int_0^{2\pi} \nu'\left(\tilde{\theta}_r^{a,e} + a\sin\sigma\right)\sin\sigma d\sigma < 0. \tag{4.16}$$

More calculations that use (4.4) and (4.5) give

$$\int_0^{2\pi} \nu'\left(\tilde{\theta}_r^{a,e} + a\sin\sigma\right)\sin\sigma d\sigma = \pi\nu''(0)a + O(a^2). \tag{4.17}$$

By substituting (4.17) into (4.15) we get

$$\begin{aligned} \det\left(\lambda \mathrm{I} - J_r^a\right) &= \left(\lambda^2 + \delta\omega_L'\lambda - \frac{\delta^2\omega_L' K'}{2}\nu''(0)a^2 \right. \\ &\quad \left. + O(\delta^2 a^3)\right)\left(\lambda + \delta\omega_H'\right), \end{aligned} \tag{4.18}$$

which, in view of (4.6), proves that J_r^a is Hurwitz for sufficiently small a. This, in turn, implies that the equilibrium (4.14) of the average system (4.7) is exponentially stable for a sufficiently small a. Then, according to the Averaging Theorem [10, Theorem 8.3] we have the following result.

Theorem 4.1 *Consider the system (4.2) under Assumption 2.3. There exist $\bar{\delta}$ and $\bar{a}$ such that for all $\delta \in (0, \bar{\delta})$ and $a \in (0, \bar{a})$ the system (4.2) has a unique exponentially stable periodic solution $\left(\tilde{\theta}_r^{2\pi}(\tau), \xi_r^{2\pi}(\tau), \tilde{\eta}_r^{2\pi}(\tau)\right)$ of period 2π and this solution satisfies*

$$\left|\begin{bmatrix} \tilde{\theta}_r^{2\pi}(\tau) + \frac{\nu'''(0)}{8\nu''(0)}a^2 \\ \xi_r^{2\pi}(\tau) \\ \tilde{\eta}_r^{2\pi}(\tau) - \frac{\nu''(0)}{4}a^2 \end{bmatrix}\right| \le O(\delta) + O(a^3), \qquad \forall \tau \ge 0. \tag{4.19}$$

This result implies that all solutions $(\tilde{\theta}_r(\tau), \xi_r(\tau), \tilde{\eta}_r(\tau))$, and, in particular, their $\tilde{\theta}_r(\tau)$-components, converge to an $O(\delta + a^2)$-neighborhood of the origin. It is important to interpret this result in terms of the system in Figure 2. Since $y = h \circ l\left(\theta^* + \tilde{\theta}_r(\tau) + a\sin\tau\right)$ and $(h\circ l)'(\theta^*) = 0$, we have

$$\begin{aligned} y - h\circ l(\theta^*) &= (h\circ l)''(\theta^*)\left(\tilde{\theta}_r + a\sin\tau\right)^2 \\ &\quad + O\left(\left(\tilde{\theta}_r + a\sin\tau\right)^3\right), \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} \tilde{\theta}_r + a\sin\tau &= \left(\tilde{\theta}_r - \tilde{\theta}_r^{2\pi}\right) + \left(\tilde{\theta}_r^{2\pi} + \frac{(h\circ l)'''(\theta^*)}{8(h\circ l)''(\theta^*)}a^2\right) \\ &\quad - \frac{(h\circ l)'''(\theta^*)}{8(h\circ l)''(\theta^*)}a^2 + a\sin\tau. \end{aligned} \tag{4.21}$$

Since the first term converges to zero, the second term is $O\left(\delta + a^3\right)$, the third term is $O\left(a^2\right)$ and the fourth term is $O(a)$, then

$$\limsup_{\tau\to\infty}\left|\tilde{\theta}_r(\tau) + a\sin\tau\right| = O(a + \delta). \tag{4.22}$$

Thus, (4.20) yields

$$\limsup_{\tau\to\infty}\left|y(\tau) - h\circ l(\theta^*)\right| = O\left(a^2 + \delta^2\right). \tag{4.23}$$

The last expression characterizes the asymptotic performance of the peak seeking scheme in Figure 2 and explains why it is not only important that the periodic perturbation be small but also that the cut-off frequencies of the filters and the adaptation gain k be low.

Another important conclusion can be drawn from (4.19). The solution $\tilde{\theta}_r(\tau)$ will converge $O\left(\delta + a^3\right)$-close to $-\frac{(h\circ l)'''(\theta^*)}{8(h\circ l)''(\theta^*)}a^2$. Since $(h\circ l)''(\theta^*) < 0$, the sign of this quantity depends on the sign of $(h\circ l)'''(\theta^*)$. If $(h\circ l)''(\theta^*) > 0$ (respectively, < 0), then the curve $h\circ l(\theta)$ will be more "flat" on the right (respectively, left) side of $\theta = \theta^*$. Since $\tilde{\theta}_r$ will have an offset in the direction of $\operatorname{sgn}\left\{(h\circ l)''(\theta^*)\right\}$, then $\tilde{\theta}_r(t)$ will converge to the "flatter" side of $h\circ l(\theta)$. This is precisely what we want—to be on the side where $h\circ l(\theta)$ is less sensitive to variations in θ and closer to its maximum value.

5 Singular Perturbation Analysis

Now we address the full system in Figure 1 whose state space model is given by (3.10) and (3.11) in the time scale $\tau = \omega t$. To make the notation in our further analysis compact, we write (3.11) as

$$\frac{dz}{d\tau} = \delta G(\tau, x, z), \tag{5.1}$$

where $z = \left(\tilde{\theta}, \xi, \tilde{\eta}\right)$. By Theorem 4.1, there exists an exponentially stable periodic solution $z_r^{2\pi}(\tau)$ such that

$$\frac{dz_r^{2\pi}(\tau)}{d\tau} = \delta G\left(\tau, L\left(\tau, z_r^{2\pi}(\tau)\right), z_r^{2\pi}(\tau)\right), \quad (5.2)$$

where $L(\tau, z) = l\left(\theta^* + \tilde{\theta} + a\sin\tau\right)$. To bring the system (3.10) and (5.1) into the *standard singular perturbation form*, we shift the state z using the transformation

$$\tilde{z} = z - z_r^{2\pi}(\tau) \quad (5.3)$$

and get

$$\frac{d\tilde{z}}{d\tau} = \delta\tilde{G}(\tau, x, \tilde{z}) \quad (5.4)$$
$$\omega\frac{dx}{d\tau} = \tilde{F}(\tau, x, \tilde{z}), \quad (5.5)$$

where

$$\begin{aligned}\tilde{G}(\tau, x, \tilde{z}) &= G\left(\tau, x, \tilde{z} + z_r^{2\pi}(\tau)\right) \\ &\quad - G\left(\tau, L\left(\tau, z_r^{2\pi}(\tau)\right), z_r^{2\pi}(\tau)\right) \quad (5.6)\\ \tilde{F}(\tau, x, \tilde{z}) &= f\left(x, \alpha\left(x, \theta^* + \underbrace{\tilde{\theta} - \tilde{\theta}_r^{2\pi}(\tau)}_{\tilde{z}_1}\right.\right. \\ &\quad \left.\left. + \tilde{\theta}_r^{2\pi}(\tau) + a\sin\tau\right)\right). \quad (5.7)\end{aligned}$$

We note that

$$x = L\left(\tau, \tilde{z} + z_r^{2\pi}(\tau)\right) \quad (5.8)$$

is the *quasi-steady state*, and that the *reduced model*

$$\frac{d\tilde{z}_r}{d\tau} = \delta\tilde{G}\left(\tau, L\left(\tau, \tilde{z}_r + z_r^{2\pi}(\tau)\right), \tilde{z}_r + z_r^{2\pi}(\tau)\right) \quad (5.9)$$

has an equilibrium at the origin $\tilde{z}_r = 0$ (cf. (5.6) with (5.8)). This equilibrium has been shown in Section 4 to be exponentially stable for sufficiently small a.

To complete the singular perturbation analysis, we also study the *boundary layer model* (in the time scale $t = \tau/\omega$):

$$\begin{aligned}\frac{dx_b}{dt} &= \tilde{F}\left(\tau, x_b + L\left(\tau, \tilde{z} + z_r^{2\pi}(\tau)\right), \tilde{z}\right) \\ &= f\left(x_b + l(\theta), \alpha\left(x_b + l(\theta), \theta\right)\right), \quad (5.10)\end{aligned}$$

where $\theta = \theta^* + \tilde{\theta} + a\sin\tau$ should be viewed as a parameter independent from the time variable t. Since $f\left(l(\theta), \alpha\left(l(\theta), \theta\right)\right) \equiv 0$, then $x_b = 0$ is an equilibrium of (5.10). By Assumption 2.2, this equilibrium is exponentially stable.

By combining exponential stability of the reduced model (5.9) with the exponential stability of the boundary layer model (5.10), using Tikhonov's Theorem on the Infinite Interval [10, Theorem 9.4], we conclude the following:

- The solution $z(\tau)$ of (5.1) is $O(\omega)$-close to the solution $z_r(\tau)$ of (5.9), and therefore, it exponentially converges to an $O(\omega)$-neighborhood of the periodic solution $z_r^{2\pi}(\tau)$, which is $O(\delta)$-close to the equilibrium $z_r^{a,e}$. This, in turn, implies that the solution $\tilde{\theta}(\tau)$ of (3.11) exponentially converges to an $O(\omega + \delta)$-neighborhood of $-\frac{(h \circ l)'''(\theta^*)}{8(h \circ l)''(\theta^*)}a^2 + O\left(a^3\right)$. It follows then that $\theta(\tau) = \theta^* + \tilde{\theta}(\tau) + a\sin\tau$ exponentially converges to an $O(\omega + \delta + a)$-neighborhood of θ^*.
- The solution $x(\tau)$ of (5.5) (which is the same as (3.10)) satisfies

$$x(\tau) - l\left(\theta^* + \tilde{\theta}_r(\tau) + a\sin\tau\right) - x_b(t) = O(\omega), \quad (5.11)$$

where $\tilde{\theta}_r(\tau)$ is the solution of the reduced model (4.2) and $x_b(t)$ is the solution of the boundary layer model (5.10). From (5.11) we get

$$\begin{aligned}x(\tau) - l(\theta^*) &= O(\omega) + l\left(\theta^* + \tilde{\theta}_r(\tau) + a\sin\omega\tau\right) \\ &\quad - l(\theta^*) - x_b(t). \quad (5.12)\end{aligned}$$

Since $\tilde{\theta}_r(\tau)$ exponentially converges to the periodic solution $\tilde{\theta}_r^{2\pi}(\tau)$, which is $O(\delta)$-close to the average equilibrium $\frac{(h \circ l)'''(\theta^*)}{8(h \circ l)''(\theta^*)}a^2 + O\left(a^3\right)$, and since the solution $x_b(t)$ of (5.10) is exponentially decaying, then by (5.12), $x(\tau) - l(\theta^*)$ exponentially converges to an $O(\omega + \delta + a)$-neighborhood of zero. Consequently, $y = h(x)$ exponentially converges to an $O(\omega + \delta + a)$-neighborhood of its maximal equilibrium value $h \circ l(\theta^*)$.

We summarize the above conclusions in the following theorem.

Theorem 5.1 *Consider the feedback system (3.4)–(3.7) under Assumptions 2.1–2.3. There exists a ball of initial conditions around the point* $\left(x, \hat{\theta}, \xi, \eta\right) = \left(l(\theta^*), \theta^*, 0, h \circ l(\theta^*)\right)$ *and constants* $\bar{\omega}, \bar{\delta}$, *and* $\bar{a}$ *such that for all* $\omega \in (0, \bar{\omega}), \delta \in (0, \bar{\delta})$, *and* $a \in (0, \bar{a})$, *the solution* $\left(x(t), \hat{\theta}(t), \xi(t), \eta(t)\right)$ *exponentially converges to an* $O(\omega + \delta + a)$*-neighborhood of that point. Furthermore,* $y(t)$ *converges to an* $O(\omega + \delta + a)$*-neighborhood of* $h \circ l(\theta^*)$.

A considerably more elaborate analysis would lead to the following stronger result which we give without proof.

Theorem 5.2 *Under the conditions of Theorem 5.1, there exists a unique exponentially stable periodic solution of (3.4)–(3.7) in an* $O(\omega + \delta + a)$*-neighborhood of the point* $\left(x, \hat{\theta}, \xi, \eta\right) = \left(l(\theta^*), \theta^*, 0, h \circ l(\theta^*)\right)$.

6 Conclusions

We hope that the proof of stability of the extremum seeking scheme will revive interest in this practically important but regrettably neglected area of adaptive control research. Our proof covers only one implementation of extremum control—the method with a periodic perturbation. Other implementations, such as, for example, those where a relay provides self-excitation, would also be worth studying.

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