The Singapore Mathematical Olympiad Compendium

https://asdia.dev/projects/compendium

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Contents

1. Introduction

About the SMO

The Singapore Mathematical Olympiad (SMO) is an annual mathematical competition organized by the Singapore Mathematical Society. The Olympiad is split into three categories, namely Junior, Senior and Open. Within each category, there are two rounds of papers (the second being an invitational round).

About the Compendium

This compendium was written with the goal to provide clear and concise solutions to each and every problem that has appeared in the SMO. The compendium has been split into two parts: Problems and Solutions. Within each part, SMO papers are further categorized by year, category and round.

Other Resources

Links to relevant resources, such as *Way Tan*'s reviews of Round 1 papers, and *Art of Problem Solving* threads for Round 2 questions, are provided at the beginning of solutions. A less well-known but equally enriching resource for olympiad preparation is the *SIMO Retiree Blog*.

Contributing

The source code for this compendium can be found on GitHub at `asdia0/Compendium`. Contributions are more than welcome.

Acknowledgements

Not all solutions presented in this compendium are original; credits will be displayed at the start of relevant solutions. The template for this compendium is adapted from Evan Chen's LaTeX style file `evan.sty`.

Part I.

Problems

2. 2020 SMO

2.1. Open Section

2.1.1. Round 1 Problems

Solutions can be found in Section 7.1.1.

1. If S is the sum of all the *real* roots of the equation $x^2 + \dfrac{1}{x^2} = 2020^2 + \dfrac{1}{2020^2}$ find $\lfloor S \rfloor$.

2. Find the largest positive integer x that satisfies the equation
$$(\lfloor x \rfloor - 2020)^2 + (\lceil x \rceil - 2030)^2 = (\lfloor x \rfloor - \lceil x \rceil + 10)^2.$$
(Note: If you think that the above equation has no solution in the positive integers, enter your answer as "0".)

3. Let $S_n = \dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n-1)\times(2n+1)}$. Find the value of n such that S_n takes the value of 0.48.

4. Given that the three planes in the Cartesian space with equations $2x + 4y + 6z = 5$, $3x + 5y + 2z = 6$ and $8x + 14y + az = b$ have a common line of intersection, find the value of $a + b$.

5. Let i be the complex number $\sqrt{-1}$, and n be the smallest positive integer such that $(\sqrt{3} + i)^n = a$, where a is a real number. Find the value of $\lfloor n - a \rfloor$.

6. In the three-dimensional Cartesian space, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along three mutually perpendicular x, y and z-axes respectively. Three straight lines l_1, l_2 and l_3 have equations defined by
$$\begin{aligned} l_1 &: \mathbf{r} = (4+\lambda)\mathbf{i} + (5+\lambda)\mathbf{j} + (6+\lambda)\mathbf{k}, \\ l_2 &: \mathbf{r} = (4+3\mu)\mathbf{i} + (5-\mu)\mathbf{j} + (6-2\mu)\mathbf{k}, \\ l_3 &: \mathbf{r} = (1+6\nu)\mathbf{i} + (2+2\nu)\mathbf{j} + (3+\nu)\mathbf{k}, \end{aligned}$$
where μ, λ and ν are real numbers. If the area of the triangle enclosed by the three lines l_1, l_2 and l_3 is denoted by S, find the value of $10S^2$.

7. Given that $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(a^2 - b^2) = (a - b)(f(a) + f(b))$$
for all real numbers a and b, and that $f(1) = \dfrac{1}{101}$, find the value of $\displaystyle\sum_{k=1}^{100} f(k)$.

8. Find the sum of all the positive integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.
(Note: If you think that there are infinitely many such positive integers n that satisfy that above conditions, enter your answer as "9999".)

9. Assume that

$$(x+2+m)^{2019} = a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots + a_{2019}(x+1)^{2019}.$$

Find the largest possible integer m such that

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \le 2020^{2019}.$$

10. Given that $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n+k)}}$, find the value of $\lfloor (S+2)^2 \rfloor$.

11. Let $A = \{1, 2, \cdots, 10\}$. Count the number of ordered pairs (S_1, S_2), where S_1 and S_2 are non-intersecting and non-empty subsets of A such that the largest number in S_1 is smaller than the smallest number in S_2. For example, if $S_1 = \{1, 4\}$ and $S_2 = \{5, 6, 7\}$, then (S_1, S_2) is such an ordered pair.

12. Each cell of a 2020×2020 table is filled with a number which is either 1 or -1. For $u - 1, \ldots, 2020$, let R_i be the product of all the numbers in the ith row and let C_i be the product of all the numbers in the ith column. Suppose $R_i + C_i = 0$ for all $i = 1, \ldots, 2020$. What is the least number of -1's in the table?

13. Assume that the sequence $\{a_k\}_{k=1}^{\infty}$ follows an arithmetic progression with $a_2 + a_4 + a_9 = 24$. Find the maximum value of $S_8 \times S_{10}$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$.

14. Consider all functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the conditions that

    a) $|g(a) - g(b)| \le |a - b|$ for any $a, b \in \mathbb{R}$;

    b) $g(g(g(0))) = 0$.

    Find the *largest* possible value of $g(0)$.

15. A sequence $\{a_i\}_{i=1}^{\infty}$ is called a *good* sequence if $\frac{S_{2n}}{S_n}$ is a constant for all $n \ge 1$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$. Suppose it is known that the sequence $\{a_i\}_{i=1}^{\infty}$ is a *good* sequence that follows an arithmetic progression. Determine a_{2020} if $a_1 = 1 \ne a_2$.

16. Determine the smallest positive integer p such that the system

$$\begin{cases} 6x + 4y + 3z = 0 \\ 4xy + 2yz + pxz = 0 \end{cases}$$

has more the one set of real solutions in x, y, z.

17. Let ABC be a triangle with $a = BC$, $b = AC$ and $c = AB$. It is given that $c = 100$ and

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}.$$

Let P be a point on the inscribed circle of $\triangle ABC$. Find the maximum value of

$$PA^2 + PB^2 + PC^2.$$

18. Find the largest positive integer n less than 2020 such that $\binom{n-1}{k} - (-1)^k$ is divisible by n for $k = 0, 1, \ldots, n-1$.

19. Assume that $\{a_k\}_{k=1}^{\infty}$ is a sequence with the property that for any distinct positive integers m, n, p, q with $m+n=p+q$, the following equality always holds:

$$\frac{a_m+a_n}{(a_m+1)(a_n+1)}=\frac{a_p+a_q}{(a_p+1)(a_q+1)}.$$

Given $a_1=0$ and $a_2=\frac{1}{2}$, determine $\frac{1}{1-a_5}$.

20. In the triangle ABC, the incircle touches the sides BC, CA, AB at D, E, F respectively. The line segments ED and AB are extended to intersect at P such that $AB=BP=PD$. Suppose $CA=9$. Find the value of $[ABC]^2$, where $[ABC]$ is the area of the triangle ABC.

21. In an acute-angled triangle ABC, $AB=75$, $AC=53$, the external bisector of $\angle A$ on CA produced meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Find the value of $AF\times FB$.

22. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence with $a_k<a_{k+1}$ for all $k=1,2,3,\cdots$ formed by arranging all the terms in the set $\left\{2^r+2^s+2^t : 0\leq r<s<t\right\}$ in increasing order. Find the largest value of the integer n such that $a_n\leq 2020$.

23. Let n be a positive integer and S be the set of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1,\ldots,a_k$ being positive integers that sum to n. Suppose the average value of all the numbers in S is 88199. Determine n.

24. Let x, y, z and w be real numbers such that $x+y+z+w=5$. Find the minimum value of $(x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2$.

25. Let p and q be positive integers satisfying the equation $p^2+q^2=3994(p-q)$. Determine the largest possible value of q.

3. 2021 SMO

3.1. Open Section

3.1.1. Round 1 Problems

Solutions can be found in Section 8.1.1.

1. It is given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha - \beta) = \frac{12}{13}$ and $\sin(\alpha + \beta) = -\frac{3}{5}$. Find $\lfloor |2021 \sin(2\alpha)| \rfloor$.

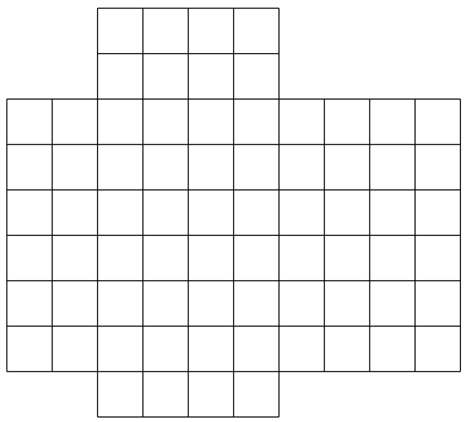
2. Find the number of solutions of the equation $|x - 3| + |x - 5| = 2$.

   (Note: If you think that there are infinitely many solutions, enter your answer as "99999".)

3. Evaluate $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + 10 \times 11 \times 12$.

4. It is given that the solution of the inequality $\sqrt{81 - x^4} \le kx + 1$ is $a \le x \le b$ with $b - a = 2$, where $k > 0$. Determine $\lfloor k \rfloor$.

5. The figure below shows a cross that is cut out from a 10×9 rectangular board.



   Find the total number of rectangles in the above figure.

   (Note: A square is a rectangle.)

6. Consider all polynomials $P(x, y)$ in two variables such that $P(0, 0) = 2020$ and for all x and y, $P(x, y) = P(x + y, y - x)$. Find the largest possible value of $P(1, 1)$.

7. In the three-dimensional Cartesian space with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denoting the unit vectors along three perpendicular directions in a clockwise manner, the line l with equation given by $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}$ intersects the plane Π with equation $x + y + z = 16$ at the point (a, b, c). Find the value of $a + b + c$.

8. Find the minimum value of $(x + 7)^2 + (y + 2)^2$ subject to the constraint $(x - 5)^2 + (y - 7)^2 = 4$.

9. Find the largest possible value $\alpha^4+\beta^4+\gamma^4$ among all possible sets of numbers (α,β,γ) that satisfy the equations

$$\begin{aligned}\alpha+\beta+\gamma&=2\\ \alpha^2+\beta^2+\gamma^2&=14\\ \alpha^3+\beta^3+\gamma^3&=20.\end{aligned}$$

10. If p is the product of all the non-zero real roots of the equation

$$\sqrt[9]{x^7+30x^5}=\sqrt[7]{x^9-30x^5},$$

find $\lfloor |p| \rfloor$.

11. Let S be the sum of a convergent geometric series with first term 1. If the third term of the series is the arithmetic mean of the first two terms, find $\lfloor 3S+4\rfloor$.

12. Given that $\sin\alpha+\sin\beta=\dfrac{1}{10}$, and $\cos\alpha+cos\beta=\dfrac{1}{9}$, find $\left\lfloor \tan^2\alpha+\beta\right\rfloor$.

13. Determine the number of positive integers that are divisible by 2021 and has exactly 2021 divisors (including 1 and itself).

14. Let $S=\displaystyle\sum_{k=0}^{25}\binom{100}{4k}-2^{98}$. Find $\left\lfloor\left|\dfrac{S}{2^{48}}\right|\right\rfloor$.

15. Assume that ABC is an acute triangle with $\sin(A+B)=\dfrac{3}{5}$ and $\sin(A-B)=\dfrac{1}{5}$. If $AB=2022(\sqrt{6}-2)$, determine $\lfloor h\rfloor$, where h is the height of the triangle from C on AB.

16. Let $a_1,a_2,\cdots$ be a sequence with $a_1=1$ and $a_{n+1}=\dfrac{n+2}{n}S_n$ for all $n=1,2,\cdots$, where $S_n=a_1+a_2+\cdots+a_n$. Determine the minimum integer n such that $a_n\geq 2021$.

17. Each card of a stack of 101 cards has one side coloured red and the other coloured blue. Initially all cards have the red side facing up and stacked together in a deck. On each turn, Ah Meng takes 8 cards on the top, flip them over, and place them to the bottom deck. Determine the minimum number of turns required so that all the cards have the red sides facing up again.

18. Let ABC be a triangle with $AB=10$ and $\dfrac{\cos A}{\cos B}=\dfrac{AC}{BC}=\dfrac{4}{3}$. Let P be a point on the inscribed circle of triangle ABC. Find the largest possible value of $PA^2+PB^2+PC^2$.

19. A basket contains 19 apples labelled by the numbers $2,3,\ldots,20$, and 19 bananas labelled by the numbers $2,3,\ldots,20$. Ah Beng picks m apples and n bananas from the basket. However, he needs to ensure that for any apple labelled a and any banana labelled b that he picks, a and b are relatively prime. Determine the largest possible value of mn.

20. Let $p(x)=ax^2-bx+c$ be a polynomial where a, b, c are positive integers and $p(x)$ has two distinct roots in $(0,1)$. Determine the least possible value of abc.

21. In the triangle ABC, $\angle A>90^\circ$, the incircle touches the side BC and AC at A_1 and B_1 respectively. The line A_1B_1 meets the extension of BA at X such that $CXB=90^\circ$. Suppose $BC^2=AB^2+BC\cdot AC$. Find the size of $\angle A$ in degrees.

22. Find the number of positive integers n such that $7n - 16$ divides $n \cdot 13^{2019}$.

23. In the acute triangle ABC, P is a point on AB, Q is a point on AC such that $BP + CQ = PQ$. The bisector of $\angle A$ meets the circumcircle of the triangle ABC at the point R distinct from A. Suppose $\angle PRQ = 52.5°$. Find the size of $\angle BAC$ in degrees.

24. Let $S = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x$. Determine the value of $\lfloor S^2 \rfloor$.

25. Let p, q, r be positive numbers with $p - r = 4q$ and $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two sequences defined by $a_1 = p$, $b_1 = q$ and for $n \geq 2$,

$$a_n = pa_{n-1}, \quad b_n = qa_{n-1} + rb_{n-1}.$$

Find the value of $\displaystyle\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n}$.

4. 2022 SMO

4.1. Open Section

4.1.1. Round 1 Problems

Solutions can be found in Section 9.1.1.

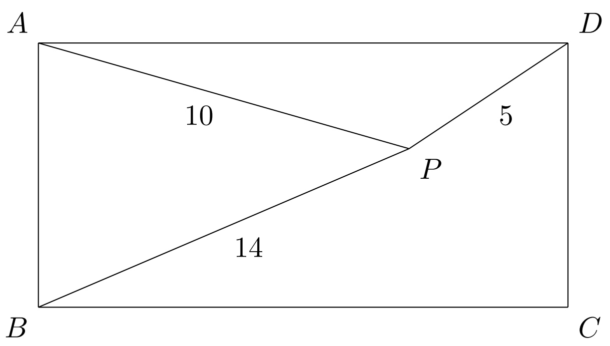
1. If $S = \sum_{k=-2021}^{2021} \frac{1}{10^k + 1}$, find $\lfloor 2S \rfloor$.

2. All the positive integers $1, 2, 3, 4, \cdots$, are grouped in the following way: $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5, 6\}$, $G_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$, that is, the set G_n contains the next 2^n positive integers listed in ascending order after the set G_{n-1}, $n > 1$. If S is the sum of all the positive integers from G_1 to G_8, find $\left\lfloor \frac{S}{100} \right\rfloor$.

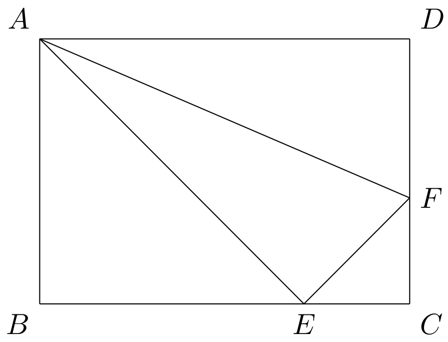
3. A sequence of one hundred positive integers $x_1, x_2, x_3, \cdots, x_{100}$ are such that
$$(x_1)^2 + (2x_2)^2 + (3x_3)^2 + (4x_4)^2 + \cdots + (100x_{100})^2 = 338350.$$
Find the largest possible value of $x_1 + x_2 + x_3 + \cdots + x_{100}$.

4. Let a and b be two real numbers satisfying $a < b$, and such that for each real number m satisfying $a < m < b$, the circle $x^2 + (y - m)^2 = 25$ meets the parabola $4y = x^2$ at four distinct points in the Cartesian plane. Let S be the maximum possible value of $b - a$. Find $\lfloor 4S \rfloor$.

5. Let P be a point within a rectangle $ABCD$ such that $PA = 10$, $PB = 14$ and $PD = 5$, as shown below. Find $\lfloor PC \rfloor$.



6. In the diagram below, the rectangle $ABCD$ has area 180 and both triangles ABE and ADF have areas 60. Find the area of triangle AEF.

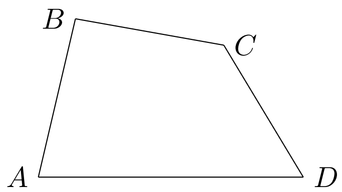


7. A tetrahedron in $\mathbb{R}^3$ has one vertex at the origin O and other vertices at the points $A(6,0,0)$, $B(4,2,4)$ and $C(3,2,6)$. If x is the height of the tetrahedron from O to the plane ABC, find $\lfloor 5x^2 \rfloor$.

8. Let x and y be real numbers such that $(x-2)^2 + (y-3)^2 = 4$. If S is the largest possible value of $x^2 + y^2$, find $\lfloor (S-17)^2 \rfloor$.

9. Let S be the maximum value of $w^3 - 3w$ subject to the condition that $w^4 + 9 \leq 10w^2$. Find $\lfloor S \rfloor$.

10. In the quadrilateral $ABCD$ below, it is given that $AB = BC = CD$ and $\angle ABC = 80^\circ$ and $\angle BCD = 160^\circ$. Suppose $\angle ADC = x^\circ$. Find the value of x.



11. Let a, b, c be integers with $ab + c = 49$ and $a + bc = 50$. Find the largest possible value of abc.

12. Find the largest possible value of $|a| + |b|$, where a and b are coprime integers (i.e., a and b are integers which have no common factors larger than 1) such that $\frac{a}{b}$ is a solution of the equation below:

$$\sqrt{4x + 5 - 4\sqrt{x+1}} + \sqrt{x + 2 - 2\sqrt{x+1}} = 1.$$

13. Let S be the set of real solutions (x, y, z) of the following system of equations:

$$\begin{cases} \dfrac{4x^2}{1+4x^2} = y, \\ \dfrac{4y^2}{1+4y^2} = z, \\ \dfrac{4z^2}{1+4z^2} = x. \end{cases}$$

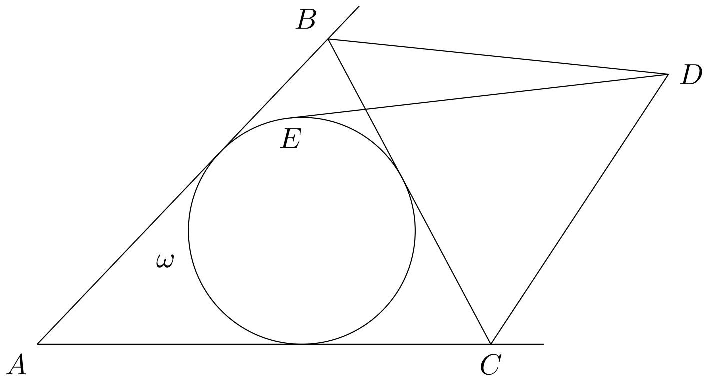
For each $(x, y, z) \in S$, define $m(x, y, z) = 2000(|x| + |y| + |z|)$. Determine the maximum value of $m(x, y, z)$ over all $(x, y, z) \in S$.

14. Assume that t is a positive solution to the equation

$$t = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + t}}}}.$$

Determine the value of $t^4 - t^3 - t + 10$.

15. In the triangle ABC shown in the diagram below, the external angle bisectors of $\angle B$ and $\angle C$ meet at the point D. The tangent from D to the incircle ω of the triangle ABC touches ω at E, where E and B are on the same side of the line AD. Suppose $\angle BEC = 112°$. Find the size of $\angle A$ in degrees.



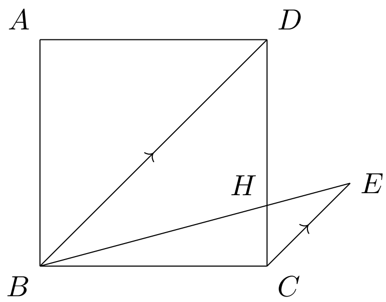
16. Find the largest integer n such that $n^2 + 5n - 9486 = 10s(n)$, where $s(n)$ is the product of all digits of n in the decimal representation of n.

    (For example, $s(481) = 4 \times 8 \times 1 = 32$.)

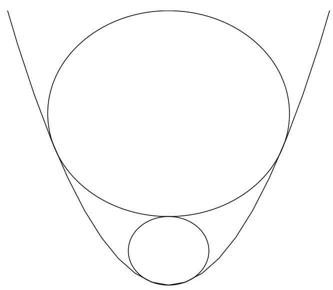
17. Find the number of integer solutions to the equation $19x + 93y = 4xy$.

18. Find the number of integer solutions to the equation $x_1 + x_2 - x_3 = 20$ with $x_1 \geq x_2 \geq x_3 \geq 0$.

19. In the diagram below, E is a point outside a square $ABCD$ such that CE is parallel to BD, $BE = BD$, and BE intersects CD at H. Given $BE = \sqrt{6} + \sqrt{2}$, find the length of DH.



20. The diagram below shows the region $R = \left\{(x, y) \in \mathbb{R}^2 | y \geq \frac{1}{2}x^2\right\}$ on the xy-plane bounded by the parabola $y = \frac{1}{2}x^2$. Let C_1 be the largest circle lying inside R with its lowest point at the origin. Let C_2 be the largest circle lying inside R and resting on top of C_1. Find the sum of radii of C_1 and C_2.



21. Find the smallest positive integer x such that $3x^2 + x = 4y^2 + y$ for some positive integer y.

22. A group of students participates in some sports activities among 6 different types of sports. It is known that for each sport activity there are exactly 100 students in the group participating in it; and the union of all the sports activities participated by any two students is NOT the entire set of 6 sports activities. Determine the minimum number of students in the group.

23. Let p and q be positive prime integers such that $p^3 - 5p^2 - 18p = q^9 - 7q$. Determine the smallest value of p.

24. Given that a, b, c are positive real numbers such that $a+b+c = 9$, find the maximum value of $a^2b^3c^4$.

25. Let $\mathbb{R}^+$ be the set of all positive real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying
$$xyf(x)\big(f(y) - f(yf(x))\big) = 1$$
for all $x, y \in \mathbb{R}^+$. Find $f(\frac{1}{2022})$.

4.1.2. Round 2 Problems

Solutions can be found in Section 9.1.2.

1. For $\triangle ABC$ and its circumcircle ω, draw the tangents at B, C to ω meeting at D. let the line AD meet the circle with centre D and radius DB at E inside $\triangle ABC$. Let F be the point on the extension of EB and G be the point on the segment EC such that $\angle AFB = \angle AGE = \angle A$. Prove that the tangent at A to the circumcircle of $\triangle AFG$ is parallel to BC.

2. Prove that if the length and breadth of a rectangle are both odd integers, then there does not exist a point P inside the rectangle such that each of the distances from P to the 4 corners of the rectangle is an integer.

3. Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying

   $$m!! + n!! \mid f(m)!! + f(n)!!$$

   for each $m, n \in \mathbb{Z}^+$, where $n!! = (n!)!$ for all $n \in \mathbb{Z}^+$.

4. Let n, k, $1 \leq k \leq n$ be fixed integers. Alice has n cards in a row, where the card in position i has the label $i+k$ (or $i+k-n$ if $i+k > n$). Alice starts by colouring each card either red or blue. Afterwards, she is allowed to make several moves, where each move consists of choosing two cards of different colours and swapping them. Find the minimum number of moves she has to make (given that she chooses the colouring optimally) to put the cards in order (i.e. card i is at position i).

5. Let $n \geq 2$ be a positive integer. For any integer a, let $P_a(x)$ denote the polynomial $x^n + ax$. Let p be a prime number and define the set S_a as the set of residues mod p that $P_a(x)$ attains. That is,

   $$S_a = \{b \mid 0 \leq b \leq p-1, \text{ and there is } c \text{ such that } P_a(x) \equiv p \pmod{p}\}.$$

   Show that the expression $\frac{1}{p-1}\sum_{a=1}^{p-1} |S_a|$ is an integer.

5. 2023 SMO

5.1. Open Section

5.1.1. Round 1 Problems

Solutions can be found in Section 10.1.1.

1. The graph C with equation $y = \dfrac{ax^2+bx+c}{x+2}$ has an oblique asymptote with equation $y = 4x - 6$ and one of the stationary points at $x = -4$. Find the value of $a + b + c$.

2. If $x = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+\cdots+100}$, find the value of $\lfloor 1010x \rfloor$.

3. The set of all possible values of x for which the sum of the infinite series
$$1 + \frac{1}{6}\left(x^2-5x\right) + \frac{1}{6^2}\left(x^2-5x\right)^2 + \frac{1}{6^3}\left(x^2-5x\right)^3 + \cdots$$
exists can be expressed as $(a, b) \cup (c, d)$, where $a < b < c < d$. Find $d - a$.

4. Find the value of $\lfloor y \rfloor$, where $y = \displaystyle\sum_{k=0}^{\infty}(2k+1)(0.5)^{2k}$.

   (Hint: Consider the series expansion of $(1-x)^{-2}$)

5. The solution of the inequality $|x-1| + |x+1| < ax + b$ is $-1 < x < 2$. Find the value of $\lfloor a + b \rfloor$.

6. The equation $x^4 - 4x^2 + qx - r = 0$ has three equal roots. Find the value of $\left\lfloor \dfrac{3q^2}{r^2} \right\rfloor$.

7. The parabolas $y = x^2 - 16x + 50$ and $x = y^2$ intersect at 4 distinct points which lie on a circle centred at (a, b). Find $|a - b|$.

8. In the 3-dimensional Euclidean space with origin O and three mutually perpendicular x-, y- and z-axes, two planes $x + y + 3z = 4$ and $2x - z = 6$ intersect at the line $\mathbf{r} \times \begin{pmatrix} -1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$. Find the value of $|a + b + c + d|$.

9. Let x, y, z be real numbers with $3x + 4y + 5z = 100$. Find the minimum value of $x^2 + y^2 + z^2$.

10. Find the area of the region represented by the equation $\lfloor x \rfloor + \lfloor y \rfloor = 1$ in the region $0 \le x < 2$.

    (Note: If you think that there is no area defined by the graph, enter "0"; if you think that the area is infinite, enter "9999".)

11. Let ABC be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$, and $\dfrac{1}{\cos A} + \dfrac{1}{\cos C} = \dfrac{-\sqrt{2}}{\cos B}$. Determine the value of $\dfrac{2022\cos(\frac{A-C}{2})}{\sqrt{2}}$.

12. Find x which satisfies the following equation

$$\frac{x-2019}{1} + \frac{x-2018}{2} + \frac{x-2017}{3} + \cdots + \frac{x+2}{2022} + \frac{x+3}{2023} = 2023.$$

13. Assume that x is a positive number such that $x - \frac{1}{x} = 3$ and

$$\frac{x^{10}+x^8+x^2+1}{x^{10}+x^6+x^4+1} = \frac{m}{n},$$

where m and n are positive integers without common factors larger than 1. Determine the value of $m+n$.

14. Consider the set of all possible pairs (x, y) of real numbers that satisfy $(x-4)^2 + (y-3)^2 = 9$. If S is the largest possible value of $\dfrac{y}{x}$, find the value of $\lfloor 7S \rfloor$.

15. Let x, y be positive integers with $16x^2 + y^2 + 7xy \leq 2023$. Find the maximum value of $4x + y$.

16. Let x be the largest real number such that

$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$

Determine the value of $(2x-1)^4$.

17. Two positive integers m and n differ by 10 and the digits in the decimal representation of mn are all equal to 9. Determine $m+n$.

18. Let $\{a_n\}$ be a sequence of positive numbers, and let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. For any positive integer n, let $b_n = \dfrac{1}{2}\left(\dfrac{a_{n+1}}{a_n} + \dfrac{a_n}{a_{n+1}}\right)$. Given that $\dfrac{a_n+2}{2} = \sqrt{2S_n}$ holds for all positive integers n, determine the limit $\lim_{n\to\infty}(b_1 + b_2 + \cdots + b_n - n)$.

19. Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \dfrac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B - A) = \cos C$ and that the area of triangle $ABC = 3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

20. **[VOID]** Let $g \colon \mathbb{R} \to \mathbb{R}$, $g(0) = 4$ and that

$$g(xy+1) = g(x)g(y) - g(y) - x + 2023.$$

Find the value of $g(2023)$.

21. In the triangle ABC, D is the midpoint of AC, E is the midpoint of BD, and the lines BA and CE are tangent to the circumcircle of the triangle ADE at A and E respectively. Suppose the circumradius of the triangle AED is $(\frac{64}{7})^{1/4}$. Find the area of the triangle ABC.

22. $ABCD$ is a parallelogram such that $\angle ABC < 90^\circ$ and $\sin \angle ABC = \frac{4}{5}$. The point K is on the extension of BC such that $DC = DK$; the point L is on the extension of DC such that $BC = BL$. The bisector of $\angle CDK$ intersects the bisector of $\angle LBC$ at Q. Suppose the circumradius of the triangle ABD is 25. Find the length of KL.

23. A group of 200 monkeys is given the task of picking up all 3000 peanuts on the ground. Determine the maximum number k such that there must be k monkeys picking up the same number of peanuts. (It is possible that some lazy monkeys may not pick up any peanuts at all).

24. A chain of n identical circles $C_1, C_2, \ldots, C_n$ of equal radii and centres on the x-axis lie inside the ellipse $E : \frac{x^2}{2023} + \frac{y^2}{333} = 1$ such that C_1 is tangent to E internally at $(-\sqrt{2023}, 0)$, C_n is tangent to E internally at $(\sqrt{2023}, 0)$, and C_i is tangent to C_{i+1} externally for $i = 1, \ldots, n-1$. Determine the smallest possible value of n.

25. Let $p > 2023$ be a prime. Determine the number of positive integers n such that $(n-p)^2 + 2023(2023 - 2n - 2p)$ is a perfect square.

5.1.2. Round 2 Problems

Solutions can be found in Section 10.1.2.

1. In a scalene triangle ABC with centroid G and circumcircle ω centred at O, the extension of AG meets ω at M; lines AB and CM intersect at P; and lines AC and BM intersect at Q. Suppose the circumcentre S of the triangle APQ lies on ω and A, O, S are collinear. Prove that $\angle AGO = 90°$.

2. A grid of cells is tiled with dominoes such that every cell is covered by exactly one domino. A subset S of dominoes is chosen. Is it true that at least one of the following two statements is false?

   a) There are 2022 more horizontal dominoes than vertical dominoes in S.

   b) The cells covered by the dominoes in S can be tiled completely and exactly by L-shaped tetrominoes.

3. Let $n \geq 2$ be a positive integer. For a positive integer a, let $Q_a(x) = x^n + ax$. Let p be a prime and let $S_a = \{b \mid 0 \leq b \leq p-1, \exists c \in \mathbb{Z}, Q_a(c) \equiv b \pmod{p}\}$. Show that $\frac{1}{p-1}\sum_{a=1}^{p-1} |S_a|$ is an integer.

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that

$$f(x+y)((f(x)-f(y))^2 + f(xy)) = f(x^3) + f(y^3)$$

   for all integers x, y.

5. Determine all real numbers x between 0 and 180 such that it is possible to partition an equilateral triangle into finitely many triangles, each of which has an angle of $x°$.

6. 2024 SMO

6.1. Open Section

6.1.1. Round 1 Problems

Solutions can be found in Section 11.1.1.

1. Let $S_k = 1+2+3+\cdots+k$ for any positive integer k. Find $S_1+S_2+S_3+\cdots+S_{20}$.
2. Let $S=\sum_{r=1}^{64} r\binom{64}{r}$, where $\binom{n}{r}=\frac{n!}{r!(n-r)!}$ and $0!=1$. Find $\log_2 S$.
3. Let x be the largest number in the interval $[0, 2\pi]$ such that $(\sin x)^{2024}-(\cos x)^{2024}=1$. Find $\lfloor x \rfloor$.

   (Note: If you think that such a number x does not exist, enter your answer "99999".)
4. Find the number of real numbers x that satisfies the equation $|x-2|+|x-3|=|2x-5|$.

   (Note: If you think that there are no such numbers, enter "0"; if you think that there are infinitely many such numbers, enter "99999".)
5. Among all the real numbers that satisfies the inequality $e^x \geq 1+2e^{-x}$, find the minimum value of $\lceil e^x+e^{-x} \rceil$.
6. Find the smallest positive integer C greater than 2024 such that the sets $A=\left\{2x^2+2x+C : x\in\mathbb{Z}\right\}$ and $B=\left\{x^2+2024x+2 : x\in\mathbb{Z}\right\}$ are disjoint.
7. Let $ABCD$ be a convex quadrilateral inscribed in a circle ω. The bisector of $\angle BAC$ meets ω at E $(\neq A)$, the bisector of $\angle ABD$ meets ω at F $(\neq B)$, AE intersects BF at P and CF intersects DE at Q. Suppose $EF=20$, $PQ=11$. Find the area of the quadrilateral $PEQF$.
8. Let $f(x)=\sqrt{x^2+1}+\sqrt{(4-x)^2+4}$. Find the minimum value of $f(x)$.
9. It is known that $a\geq 0$ satisfies $\sqrt{4+\sqrt{4+\sqrt{4+\sqrt{4+a}}}}=a$. Find the value of $(2a-1)^2$.
10. A rectangle with sides parallel to the horizontal and vertical axes is inscribed in the region bounded by the graph of $y=60-x^2$ and the x-axis. If the area of the largest such rectangle has area $k\sqrt{5}$, find the value of k.
11. Let x be a real number satisfying the equation $x^{x^5}=100$. Find the value of $\lfloor x^5 \rfloor$.
12. Let a, b, c, d, e be distinct integers with $a+b+c+d+e=9$. If m is an integer such that
$$(m-a)(m-b)(m-c)(m-d)(m-e)=2009,$$
determine the value of m.
13. Let $\{x\}$ be the fractional part of the number x, i.e., $\{x\}=x-\lfloor x \rfloor$. If $S=\int_0^9 \{x\}^2\,\mathrm{d}x$, find $\lfloor S \rfloor$.

14. The solution of the inequality $|(x+1)(x-6)| > |(x+4)(x-2)|$ can be expressed as $x < a$ or $b < x < c$. If $S = |a| + |b| + |c|$, find $\lfloor 14S \rfloor$.

15. Given that $x, y > 0$ and $x\sqrt{2-y^2} + y\sqrt{2-x^2} = 2$, find the value of $x^2 + y^2$.

16. A convex polygon has n sides such that no three diagonals are concurrent. It is known that all its diagonals divide the polygon into 2500 regions. Determine n.

17. Find the number of integers n between -2029 and 2029 inclusive such that $(n+2)^2 + n^2$ is divisible by 2029.

18. Let f be a function such that for any real number x, we have $f(x)+2f(2-x) = x+x^2$. Find the value of $f(1) + f(2) + f(3) + \cdots + f(34)$.

19. Find the largest possible positive prime integer p such that p divides

$$S(p) = 1^{p-2} + 2^{p-2} + 3^{p-2} + 4^{p-2} + 5^{p-2} + 6^{p-2} + 7^{p-2} + 8^{p-2}.$$

20. Let f be a function such that $f(x) + f(\frac{1}{1-x}) = 1 + \frac{1}{x}$ for all $x \notin \{0, 1\}$. Find the value of $\lfloor 180 \cdot f(10) \rfloor$.

21. Let C be a circle with equation $(x-a)^2 + (y-b)^2 = r^2$, where at least one of the a and b are irrational numbers. Find the maximum possible numbers of points (p, q) on C where both p and q are rational numbers.

22. On the plane there are 2024 points coloured either red or blue such that each red point is the centre of a circle passing through 3 blue points. Determine the least number of blue points.

23. It is given that the positive real numbers $x_1, \ldots, x_{2026}$ satisfy $\dfrac{x_1^2}{x_1^2+1} + \cdots + \dfrac{x_{2026}^2}{x_{2026}^2+1} = 2025$. Find the maximum value of $\dfrac{x_1}{x_1^2+1} + \cdots + \dfrac{x_{2026}}{x_{2026}^2+1}$.

24. Let n denote the number of ways of arranging all the letters of the word MATHEMATICS in one row such that

- both M's precede both T's; and
- neither the two M's nor the two T's are next to each other.

Determine the value of $\frac{n}{6!}$.

25. The incircle of the triangle ABC centred at I touches the sides BC, CA, AB at D, E, F respectively. Let D' be the intersection of the extension of ID with the circle through B, I, C; E' be the intersection of the extension of IE with the circle through A, I, C; and F' the intersection of the extension of IF with the circle through A, I, B. Suppose $AB = 52$, $BC = 56$, $CA = 60$. Find $DD' + EE' + FF'$.

6.1.2. Round 2 Problems

Solutions can be found in Section 11.1.2.

1. In triangle ABC, $\angle B = 90°$, $AB > BC$, and P is the point such that $BP = BC$ and $\angle APB = 90°$, where P and C lie on the same side of AB. Let Q be the point on AB such that $AP = AQ$, and let M be the midpoint of QC. Prove that the line through M parallel to AP passes through the midpoint of AB.

2. Let n be a fixed positive integer. Find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

where $x_1, x_2, \ldots, x_n$ are distinct positive integers.

3. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

4. Alice and Bob play a game. Bob starts by picking a set S consisting of M vectors of length n with entries either 0 or 1. Alice picks a sequence of numbers $y_1 \leq y_2 \leq \cdots \leq y_n$ from the interval $[0, 1]$, and a choice of real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Bob wins if he can pick a vector $(z_1, z_2, \ldots, z_n) \in S$ such that

$$\sum_{i=1}^{n} x_i y_i \leq \sum_{i=1}^{n} x_i z_i,$$

otherwise Alice wins. Determine the minimum value of M so that Bob can guarantee a win.

5. Let p be a prime number. Determine the largest possible n such that the following holds: it is possible to fill an $n \times n$ table with integers a_{ik} in the ith row and kth column, for $1 \leq i, k \leq n$, such that for any quadruple i, j, k, l with $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$, the number $a_{ik}a_{jl} - a_{il}a_{jk}$ is not divisible by p.

Part II.

Solutions

7. 2020 SMO

7.1. Open Section

7.1.1. Round 1 Solutions

Review by *Way Tan*

Question 1 [Ans: 0]

If S is the sum of all the *real* roots of the equation $x^2 + \dfrac{1}{x^2} = 2020^2 + \dfrac{1}{2020^2}$ find $\lfloor S \rfloor$.

Observe that $x^2 + \frac{1}{x^2}$ is even. Hence, the sum of roots is 0.

Question 2 [Ans: 2030]

Find the largest positive integer x that satisfies the equation

$$(\lfloor x \rfloor - 2020)^2 + (\lceil x \rceil - 2030)^2 = (\lfloor x \rfloor - \lceil x \rceil + 10)^2.$$

(Note: If you think that the above equation has no solution in the positive integers, enter your answer as "0".)

Since x is an integer, we obviously have $x = \lfloor x \rfloor = \lceil x \rceil$. We are hence left with the equation $(x - 2020)^2 + (x - 2030)^2 = 10^2$, of which 2020 and 2030 are clearly solutions to. Thus, $x = 2030$.

Question 3 [Ans: 12]

Let $S_n = \dfrac{1}{1 \times 3} + \dfrac{1}{3 \times 5} + \dfrac{1}{5 \times 7} + \cdots + \dfrac{1}{(2n-1) \times (2n+1)}$. Find the value of n such that S_n takes the value of 0.48.

Using partial fraction decomposition, we see that

$$S_n = \sum_{i=1}^{n} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2} \sum_{i=1}^{n} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right).$$

This sum clearly telescopes, giving $S_n = \frac{1}{2}(1 - \frac{1}{2n+1})$ Setting $S_n = 0.48$ yields $n = 12$.

Question 4 [Ans: 27]

Given that the three planes in the Cartesian space with equations $2x + 4y + 6z = 5$, $3x + 5y + 2z = 6$ and $8x + 14y + az = b$ have a common line of intersection, find the value of $a + b$.

Solving $2x + 4y + 6z = 5$ and $3x + 5y + 2z = 6$ simultaneously, we see that the line of intersection has equation $x = 11t - \frac{1}{2}$, $y = \frac{3}{2} - 7t$ and $z = t$. Substituting this into

$8x + 14y + az = b$, we get $(17 - b) + t(a - 10) = 0$. Since this must hold for all real t, we immediately have $a = 10$ and $b = 17$, whence $a + b = 27$.

Question 5 [Ans: 70]

Let i be the complex number $\sqrt{-1}$, and n be the smallest positive integer such that $(\sqrt{3} + i)^n = a$, where a is a real number. Find the value of $\lfloor n - a \rfloor$.

Observe that $(\sqrt{3} + i)^n = 2^n e^{in\pi/6}$. Since we want this to be real, its argument must be an integer multiple of π. Thus, $\frac{n}{6} \in \mathbb{Z}$. This immediately gives us $n = 6$, whence $a = 2^6 e^{i\pi} = -64$. Hence, $n - a = 70$.

Question 6 [Ans: 945]

In the three-dimensional Cartesian space, let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along three mutually perpendicular x, y and z-axes respectively. Three straight lines l_1, l_2 and l_3 have equations defined by

$$\begin{aligned} l_1 &: \mathbf{r} = (4 + \lambda)\mathbf{i} + (5 + \lambda)\mathbf{j} + (6 + \lambda)\mathbf{k}, \\ l_2 &: \mathbf{r} = (4 + 3\mu)\mathbf{i} + (5 - \mu)\mathbf{j} + (6 - 2\mu)\mathbf{k}, \\ l_3 &: \mathbf{r} = (1 + 6\nu)\mathbf{i} + (2 + 2\nu)\mathbf{j} + (3 + \nu)\mathbf{k}, \end{aligned}$$

where μ, λ and ν are real numbers. If the area of the triangle enclosed by the three lines l_1, l_2 and l_3 is denoted by S, find the value of $10S^2$.

Rewriting the equations of the three lines in vector form, we get

$$l_1 : \mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad l_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}, \quad l_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \nu \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix}.$$

It is simple to find the pairwise intersections of the planes: l_1 and l_2 intersect at $A(4, 5, 6)$, l_2 and l_3 intersect at $B(7, 4, 4)$, while l_3 and l_1 intersect at $C(1, 2, 3)$. S, the area of $\triangle ABC$, can hence be calculated as

$$S = [ABC] = \frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right| = \frac{3\sqrt{42}}{2}.$$

Thus, $10S^2 = 945$.

Question 7 [Ans: 50]

Given that $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(a^2 - b^2) = (a - b)(f(a) + f(b))$$

for all real numbers a and b, and that $f(1) = \dfrac{1}{101}$, find the value of $\displaystyle\sum_{k=1}^{100} f(k)$.

By inspection, we have $f(x) = kx$. Since $f(1) = \frac{1}{101}$, we get $k = \frac{1}{101}$. Thus, the desired sum evaluates to $\frac{1}{101} \cdot \frac{100(101)}{2} = 50$.

Question 8 [Ans: 4]

Find the sum of all the positive integers n such that $n^4 - 4n^3 + 22n^2 - 36n + 18$ is a perfect square.

(Note: If you think that there are infinitely many such positive integers n that satisfy that above conditions, enter your answer as "9999".)

Observe that $n^4 - 4n^3 + 22n^2 - 36n + 18 = (n^2 - 2n + 9)^2 - 63$. Let this be m^2, where m is some integer. By the difference of two squares identity, we get

$$63 = (n^2 - 2n + 9 + m)(n^2 - 2n + 9 - m).$$

Let $A = n^2 - 2n + 9 + m$ and $B = n^2 - 2n + 9 - m$. We clearly have that $AB = 63$ and $\frac{1}{2}(A+B) - 9 = (n-1)^2$, a perfect square. Going through all factors of 63, we see that the only pairs of factors satisfying this condition are $(A, B) = (9, 7)$ and $(21, 3)$, which give 0^2 and 2^2 respectively. Hence, $n = 1$ or 3, thus the sum desired is 4.

Question 9 [Ans: 43]

Assume that

$$(x + 2 + m)^{2019} = a_0 + a_1(x + 1) + a_2(x + 1)^2 + \cdots + a_{2019}(x + 1)^{2019}.$$

Find the largest possible integer m such that

$$(a_0 + a_2 + a_4 + \cdots + a_{2018})^2 - (a_1 + a_3 + a_5 + \cdots + a_{2019})^2 \leq 2020^{2019}.$$

Substituting $x = 0$ yields $a_0 + a_1 + \cdots + a_{2018} + a_{2019} = (m + 2)^{2019}$, while substituting $x = -2$ yields $a_0 - a_1 + \cdots + a_{2018} - a_{2019} = m^{2019}$. Hence,

$$a_0 + a_2 + \cdots + a_{2018} = \frac{(m + 2)^{2019} + m^{2019}}{2},$$

whence

$$a_1 + a_3 + \cdots + a_{2019} = (m + 2)^{2019} - \frac{(m + 2)^{2019} + m^{2019}}{2}.$$

We thus want

$$\left(\frac{(m + 2)^{2019} + m^{2019}}{2}\right)^2 - \left((m + 2)^{2019} - \frac{(m + 2)^{2019} + m^{2019}}{2}\right)^2 \leq 2020^{2019}.$$

The LHS simplifies to $(m(m + 2))^{2019}$. Thus, $m(m + 2) \leq 2020$, whence $m \leq 43$.

Question 10 [Ans: 8]

Given that $S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n(n + k)}}$, find the value of $\lfloor (S + 2)^2 \rfloor$.

Dividing through by n yields

$$S = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{1 + k/n}},$$

which is very clearly a Riemann sum. In the limit, we get

$$S = \int_0^1 \frac{1}{\sqrt{1 + x}}\,\mathrm{d}x = 2\sqrt{2} - 2.$$

Thus, $(S + 2)^2 = 8$.

Question 11 [Ans: 4097]

Let $A = \{1, 2, \cdots, 10\}$. Count the number of ordered pairs (S_1, S_2), where S_1 and S_2 are non-intersecting and non-empty subsets of A such that the largest number in S_1 is smaller than the smallest number in S_2. For example, if $S_1 = \{1, 4\}$ and $S_2 = \{5, 6, 7\}$, then (S_1, S_2) is such an ordered pair.

Let the largest element of S_1 be k. There are 2^{k-1} ways to choose S_1, and there are $2^{10-k} - 1$ ways to choose S_2 (note that we subtract 1 since $S_2 \neq \varnothing$). There are hence $2^{k-1}\left(2^{10-k} - 1\right)$ possibilities. Summing over $k = 1, \ldots, 9$, we get a total of

$$\sum_{k=1}^{9} 2^{k-1}\left(2^{10-k} - 1\right) = \sum_{k=1}^{9}\left(2^9 - 2^{k-1}\right) = 9 \cdot 2^9 - \frac{1-2^9}{1-2} = 4097$$

ordered pairs.

Question 12 [Ans: 1010]

Each cell of a 2020×2020 table is filled with a number which is either 1 or -1. For $u - 1, \ldots, 2020$, let R_i be the product of all the numbers in the ith row and let C_i be the product of all the numbers in the ith column. Suppose $R_i + C_i = 0$ for all $i = 1, \ldots, 2020$. What is the least number of -1's in the table?

Let a_{ij} represent the number in the cell in the ith row and jth column. Observe that $R_i = \prod_{n=1}^{2020} a_{in}$ and $C_i = \prod_{n=1}^{2020} a_{ni}$. Then

$$R_i + C_i = \prod_{n=1}^{2020} a_{in} + \prod_{n=1}^{2020} a_{ni} = a_{ii}\left(\prod_{\substack{n=1\\ n\neq i}}^{2020} a_{in} + \prod_{\substack{n=1\\ n\neq i}}^{2020} a_{ni}\right) = 0.$$

Since we want to minimize the number of -1s, we take $a_{ii} = 1$ for all $i = 1, \ldots, 2020$. Furthermore, we have

$$\prod_{\substack{n=1\\ n\neq i}}^{2020} a_{in} = -\prod_{\substack{n=1\\ n\neq i}}^{2020} a_{ni}.$$

The number of -1s in the two products hence differ by an odd number. In the optimal case, one product has no -1s, while the other has one -1. We now construct a grid with such a property. Let $a_{ij} = -1$ if $(i, j) \in \{(1, 2), (3, 4), \ldots, (2019, 2020)\}$, and $a_{ij} = 1$ otherwise. It is quite clear that

$$\prod_{\substack{n=1\\ n\neq i}}^{2020} a_{in} = \begin{cases} 1, & i \text{ odd} \\ -1, & i \text{ even} \end{cases}, \qquad \prod_{\substack{n=1\\ n\neq i}}^{2020} a_{ni} = \begin{cases} -1, & i \text{ odd} \\ 1, & i \text{ even} \end{cases},$$

whence $R_i + C_i = 0$ for $i = 1, \ldots, 2020$ as desired. Thus, the least number of -1s is $|\{(1, 2), (3, 4), \ldots, (2019, 2020)\}| = 1010$.

Question 13 [Ans: 5120]

Assume that the sequence $\{a_k\}_{k=1}^{\infty}$ follows an arithmetic progression with $a_2+a_4+a_9 = 24$. Find the maximum value of $S_8 \times S_{10}$, where S_k denotes the sum $a_1+a_2+\cdots+a_k$.

Let $a_k = a_1 + (k-1)d$. From the given equation, we immediately have $3a_1 + 12d = 24$, whence $2a_1 + 8a_1 = 16$. Since $S_k = ka_1 + \frac{(k-1)k}{2} \cdot d$, we have

$$S_8 \cdot S_{10} = (8a_1 + 28d)(10a_1 + 45d) = 20(16-d)(16+d) = 20\left(16^2 - d^2\right).$$

Thus, the maximum value of $S_8 \cdot S_{10}$ is $20 \cdot 16^2 = 5120$, when $d = 0$.

Question 14 [Ans: 0]

Consider all functions $g : \mathbb{R} \to \mathbb{R}$ satisfying the conditions that

1. $|g(a) - g(b)| \leq |a - b|$ for any $a, b \in \mathbb{R}$;
2. $g(g(g(0))) = 0$.

Find the *largest* possible value of $g(0)$.

Let $a = x + h$ and $b = x$. From the first condition, we get

$$\frac{|g(x+h) - g(x)|}{h} \leq 1 \implies \left|g'(x)\right| \leq 1.$$

Now consider the fixed point iteration $x_{n+1} = g(x_n)$, which must converge to the root of $g(x) = x$ since $|g(x)| \leq 1$. The second condition states that if $x_n = 0$, then $x_{n+3} = x_n = 0$. This immediately implies that $x = 0$ is a root of $g(x) = x$, whence $g(0) = 0$.

Question 15 [Ans: 4039]

A sequence $\{a_i\}_{i=1}^{\infty}$ is called a *good* sequence if $\dfrac{S_{2n}}{S_n}$ is a constant for all $n \geq 1$, where S_k denotes the sum $a_1 + a_2 + \cdots + a_k$. Suppose it is known that the sequence $\{a_i\}_{i=1}^{\infty}$ is a *good* sequence that follows an arithmetic progression. Determine a_{2020} if $a_1 = 1 \neq a_2$.

Let $a_i = 1 + (i-1)k$. Then $S_n = n + \frac{k(n-1)n}{2}$. Since $\frac{S_{2n}}{S_n}$ is constant, we have $\frac{S_2}{S_1} = \frac{S_4}{S_2}$. This gives $S_2^2 = S_1 S_4$, whence $(2+k)^2 = 4 + 6k$. This gives us $k = 2$ (note that $k \neq 0$ since $a_1 \neq a_2$). Thus, $a_{2020} = 1 + 2(2020 - 1) = 4039$.

Question 16 [Ans: 12]

Determine the smallest positive integer p such that the system

$$\begin{cases} 6x + 4y + 3z = 0 \\ 4xy + 2yz + pxz = 0 \end{cases}$$

has more the one set of real solutions in x, y, z.

From the first equation, we have $6x = -4y - 3z$. Multiplying the second equation by 6 and substituting $6x$, we get

$$16y^2 + 4pyz + 3pz^2 = 0. \tag{1}$$

Taking the discriminant with respect to $4y$, we have $z^2(p^2 - 12p)$, which must be greater than or equal to 0 to admit multiple solutions. Hence, $p^2 - 12p \geq 0$, whence $p \geq 12$. Thus, $\min p = 12$. Indeed, when $p = 12$, we get $4y + 6z = 0$ from (1), whence $(x, y, z) = (t, -3t, 2t)$ for all real t.

Question 17 [Ans: 8800]

Let ABC be a triangle with $a = BC$, $b = AC$ and $c = AB$. It is given that $c = 100$ and

$$\frac{\cos A}{\cos B} = \frac{b}{a} = \frac{4}{3}.$$

Let P be a point on the inscribed circle of $\triangle ABC$. Find the maximum value of

$$PA^2 + PB^2 + PC^2.$$

Since $\frac{\cos A}{\cos B} = \frac{4}{3}$, $\triangle ABC$ is congruent to a 3-4-5 right triangle, where $\angle C = 90°$. Since $c = 100 = 20 \cdot 5$, we have $a = 20 \cdot 3 = 60$ and $b = 20 \cdot 4 = 80$. Let $C(0, 0)$. Then $A(80, 0)$ and $B(0, 60)$. Note that the incircle of $\triangle ABC$ has radius $20 \cdot 1 = 20$ and centre $(20, 20)$. Let (x, y) be a point on the incircle, i.e.

$$(x - 20)^2 + (y - 20)^2 = 20^2. \implies x^2 + y^2 = 40x + 40y - 20^2. \tag{1}$$

We thus aim to maximize

$$PA^2 + PB^2 + PC^2 = \left[(x - 80)^2 + y^2\right] + \left[x^2 + (y - 60)^2\right] + \left[x^2 + y^2\right].$$

Expanding and using (1), we get

$$PA^2 + PB^2 + PC^2 = -40x + 8800.$$

Since $x \geq 0$, the maximum value of $PA^2 + PB^2 + PC^2$ is 8800.

Question 18 [Ans: 2017]

Find the largest positive integer n less than 2020 such that $\binom{n-1}{k} - (-1)^k$ is divisible by n for $k = 0, 1, \ldots, n - 1$.

Claim 1. *If n is prime, then $\binom{n-1}{k} - (-1)^k \equiv 0 \pmod{n}$.*

Proof. Observe that

$$\binom{n-1}{k} = \frac{(n-1)(n-2)\cdots(n-k)}{k!}.$$

Since n is prime, $k!$ has a multiplicative inverse in $\mathbb{F}_n$. We can thus take congruences in the numerator without any problem:

$$\frac{(n-1)(n-2)\cdots(n-k)}{k!} \equiv \frac{(-1)(-2)\cdots(-k)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k \pmod{n}.$$

Thus,

$$\binom{n-1}{k} - (-1)^k \equiv (-1)^k - (-1)^k = 0 \pmod{n}.$$

□

Observe that the largest prime less than 2020 is 2017. To finish, we show that $n = 2018$ and $n = 2019$ do not work.

Case 1. Suppose $n = 2018$. When $k = 2$, we have

$$\binom{2017}{2} - 1 = 2017 \cdot 1008 - 1 \equiv -1 \cdot 1008 - 1 \equiv 1009 \neq 0 \pmod{2018}.$$

Case 2. Suppose $n = 2019$. When $k = 3$, we have

$$\binom{2018}{3} - 1 = 2018 \cdot 2017 \cdot 336 - 1 \equiv -1 \cdot -2 \cdot 336 - 1 \equiv 672 \neq 0 \pmod{2019}.$$

Thus, the largest n is 2017.

Question 19 [Ans: 41]

Assume that $\{a_k\}_{k=1}^{\infty}$ is a sequence with the property that for any distinct positive integers m, n, p, q with $m + n = p + q$, the following equality always holds:

$$\frac{a_m + a_n}{(a_m + 1)(a_n + 1)} = \frac{a_p + a_q}{(a_p + 1)(a_q + 1)}.$$

Given $a_1 = 0$ and $a_2 = \dfrac{1}{2}$, determine $\dfrac{1}{1 - a_5}$.

Let $m = 1$, $n = 3$, and $p = q = 2$. Using the given equation and conditions, we get

$$\frac{a_1 + a_3}{(a_1 + 1)(a_3 + 1)} = \frac{a_2 + a_2}{(a_2 + 1)(a_2 + 1)} \implies a_3 = \frac{4}{5}.$$

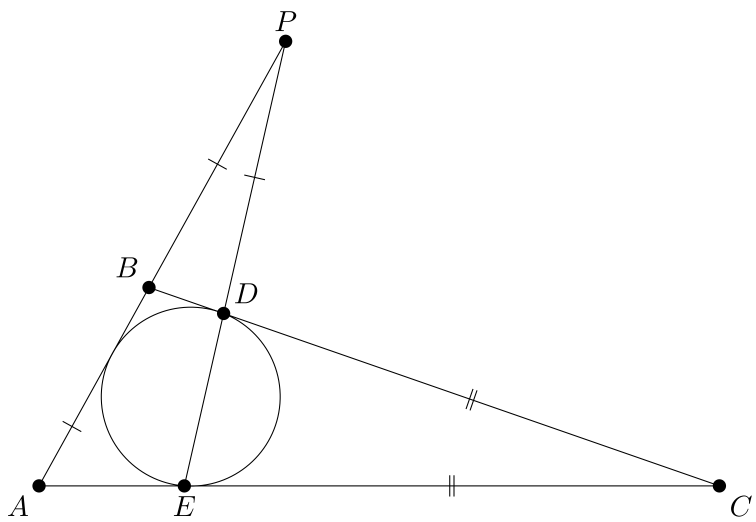
Let $m = 1$, $n = 5$, $p = q = 3$. Once again, using the given equation and conditions, we get

$$\frac{a_1 + a_5}{(a_1 + 1)(a_5 + 1)} + \frac{a_3 + a_3}{(a_3 + 1)(a_3 + 1)} \implies a_5 = \frac{40}{41}.$$

Thus, $\frac{1}{1-a_5} = 41$.

Question 20 [Ans: 140]

In the triangle ABC, the incircle touches the sides BC, CA, AB at D, E, F respectively. The line segments ED and AB are extended to intersect at P such that $AB = BP = PD$. Suppose $CA = 9$. Find the value of $[ABC]^2$, where $[ABC]$ is the area of the triangle ABC.



Let $a = BC$, $b = CA = 9$, $c = AB$ and $s = \frac{1}{2}(a+b+c)$. We clearly have $AF = AE = s - a$, $BF = BD = s - 9$ and $CD = CE = s - c$.

Using Menalaus' theorem with respect to $\triangle ABC$, we have

$$\frac{AP}{PB}\frac{BD}{DC}\frac{CE}{EA} = 1 \implies 2 \cdot \frac{s-9}{s-c} \cdot \frac{s-c}{s-a} = 1.$$

Rearranging, we get $3a + c = 27$.

Using Menalaus' theorem with respect to $\triangle APE$, we have

$$\frac{AC}{CE}\frac{ED}{DP}\frac{PB}{BA} = 1 \implies \frac{9}{s-c} \cdot \frac{ED}{c} \cdot 1 = 1.$$

Rearranging, we get $ED = \frac{c(a+9-c)}{18}$. Since $CE = \frac{1}{2}(a+9-c)$, we have

$$\frac{ED}{CE} = \frac{c}{9}.$$

Since $\triangle BPD$ and $\triangle ECD$ are isosceles, and $\angle BDP = \angle ECD$, it follows that $\angle BDP = \angle C$. Using the cosine rule on $\angle BDP$ in $\triangle BPD$, we have

$$1 - \cos C = \frac{BD^2}{2 \cdot BP^2} = \frac{(s-b)^2}{2c^2}.$$

Using the cosine rule on $\angle C$ in $\triangle CED$, we have

$$1 - \cos C = \frac{ED^2}{2 \cdot CE^2} = \frac{c^2}{2 \cdot 81}.$$

Hence,

$$\frac{(s-b)^2}{2c^2} = \frac{c^2}{2 \cdot 81} \implies \frac{(a+c-9)^2}{4c^2} = \frac{c^2}{81}.$$

Using the substitution $3a + c = 27$ yields

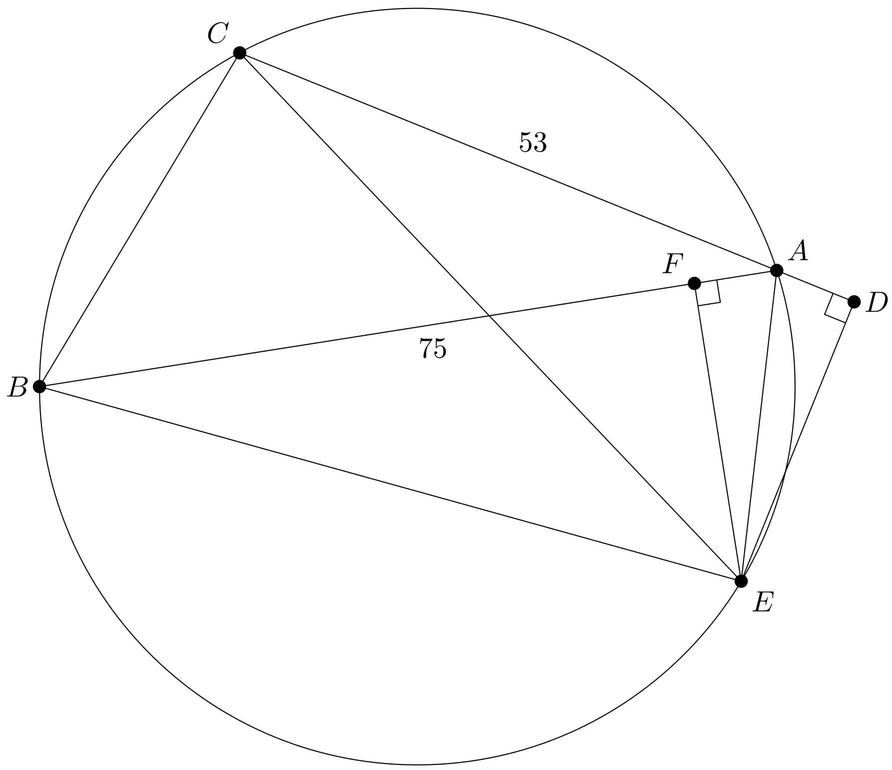
$$\frac{4(9-a)^2}{4 \cdot 9(9-a)^2} = \frac{9(9-a)^2}{81},$$

whence $a = 8$ and $c = 3$. Finally, by Heron's formula, we get

$$[ABC]^2 = s(s-a)(s-b)(s-c) = 10(10-8)(10-9)(10-8) = 140.$$

Question 21 [Ans: 704]

In an acute-angled triangle ABC, $AB = 75$, $AC = 53$, the external bisector of $\angle A$ on CA produced meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Find the value of $AF \times FB$.



We first introduce a new point D on CA produced such that $\angle ADE = 90°$. Clearly $\triangle AFE \equiv \triangle ADE$ by AAS, hence $FE = DE$ and $AF = AD$. Additionally, it is a well-known fact that E is equidistant from B and C. Hence, by RHS, $\triangle CDE \equiv \triangle BFE$. Thus, $CD = BF$. Since $CD = 53 + AD$ and $BF = 75 - AF$, we get $AF = 11$. Thus, $AF \times FB = 11 \times (75 - 11) = 704$.

Question 22 [Ans: 165]

Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence with $a_k < a_{k+1}$ for all $k = 1, 2, 3, \cdots$ formed by arranging all the terms in the set $\left\{2^r + 2^s + 2^t : 0 \le r < s < t\right\}$ in increasing order. Find the largest value of the integer n such that $a_n \le 2020$.

Observe that an integer is in the set if and only if its binary expansion has exactly 3 ones. Since $2020_{10} = 11111100100_2$, the largest a_n less than 2020 is 11100000000_2. Notice that this is also the largest a_n where a_n has 11 digits in binary. Thus, there are a total of $\binom{11}{3} = 165$ integers in the sequence before this a_n, thus $\max n = 165$.

Question 23 [Ans: 838]

Let n be a positive integer and S be the set of all numbers that can be written in the form $\sum_{i=2}^{k} a_{i-1}a_i$ with $a_1, \ldots, a_k$ being positive integers that sum to n. Suppose the average value of all the numbers in S is 88199. Determine n.

Testing small values of n, we see that

$$S = \left\{ s \in \mathbb{Z} \mid n-1 \leq \left(\frac{n}{2}\right)^2 \right\}$$

when n is even, and

$$S = \left\{ s \in \mathbb{Z} \mid n-1 \leq \left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right) \right\}$$

when n is odd.

Case 1. If n is even, we get $\frac{1}{2}\left[(n-1)+\left(\frac{n}{2}\right)^2\right] = 88199$, whence $n = 838$.

Case 2. If n is odd, we get $\frac{1}{2}\left[(n-1)+\left(\frac{n-1}{2}\right)\left(\frac{n+1}{2}\right)\right] = 88199$, which has no integer solutions.

Thus, $n = 838$.

Question 24 [Ans: 1600]

Let x, y, z and w be real numbers such that $x+y+z+w=5$. Find the minimum value of $(x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2$.

By the Cauchy-Schwarz inequality, one has

$$\begin{aligned}&[(x+5)+(y+10)+(z+20)+(w+40)]^2 \\ &\qquad \leq 4\left[(x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2\right]\end{aligned}$$

Hence, the minimum value of $(x+5)^2+(y+10)^2+(z+20)^2+(w+40)^2$ is $\frac{80^2}{4} = 1600$.

Question 25 [Ans: 290]

Let p and q be positive integers satisfying the equation $p^2+q^2 = 3994(p-q)$. Determine the largest possible value of q.

Completing the square gives

$$(p-1997)^2+(q+1997)^2 = 2\cdot 1997^2. \tag{1}$$

Let $P = p-1997$ and $Q = q+1997$. Multiplying (1) by 2 gives

$$2P^2+2Q^2 = 3994^2.$$

We now recognize the LHS to be a sum of two squares:

$$(P+Q)^2+(P-Q)^2 = 3994^2.$$

We hence get a Pythagorean triple. Using the standard parameterization of such triples, we have $P+Q = m^2-n^2$, $P-Q = 2mn$ and $3994 = m^2+n^2$ for some positive integers m and n. Without loss of generality, suppose $m \geq n$. Then $m = 63$ and $\pm n = 5$ is the only pair that gives $m^2+n^2 = 3994$. Hence, $Q = \frac{1}{2}(m^2-n^2-2mn) = 2287$ (note we reject $Q = 1657$ since $Q \geq 1997$), whence $q = 2287-1997 = 290$.

8. 2021 SMO

8.1. Open Section

8.1.1. Round 1 Solutions

Review by *Way Tan*

Question 1 [Ans: 1741]

It is given that $\frac{\pi}{2} < \beta < \alpha < \frac{3\pi}{4}$, $\cos(\alpha - \beta) = \frac{12}{13}$ and $\sin(\alpha + \beta) = -\frac{3}{5}$. Find $\lfloor |2021 \sin(2\alpha)| \rfloor$.

Note that $\alpha - \beta$ is in the first quadrant, while $\alpha + \beta$ is in the third quadrant. Hence, $\sin(\alpha - \beta) = \frac{5}{13}$, while $\cos(\alpha + \beta) = -\frac{4}{5}$. Thus,

$$\sin(2\alpha) = \sin(\alpha + \beta)\cos(\alpha - \beta) + \cos(\alpha + \beta)\sin(\alpha - \beta) = -\frac{56}{65}.$$

The required answer is hence 1741.

Question 2 [Ans: 99999]

Find the number of solutions of the equation $|x - 3| + |x - 5| = 2$.
(Note: If you think that there are infinitely many solutions, enter your answer as "99999".)

Observe that for all $x \in [3, 5]$, we have $|x - 3| + |x - 5| = 2$. There are hence infinitely many solutions.

Question 3 [Ans: 4290]

Evaluate $1 \times 2 \times 3 + 2 \times 3 \times 4 + 3 \times 4 \times 5 + \cdots + 10 \times 11 \times 12$.

We are tasked with evaluating $\sum_{k=1}^{10} k(k+1)(k+2)$. Expanding, we have $\sum_{k=1}^{10} k^3 + 3k^2 + 2k$. Using the standard formulae

$$\sum_{k=1}^{n} k = \frac{k(k+1)}{2}, \qquad \sum_{k=1}^{n} k^2 = \frac{k(k+1)(2k+1)}{6}, \qquad \sum_{k=1}^{n} k^3 = \left(\frac{k(k+1)}{2}\right)^2,$$

we arrive at 4290.

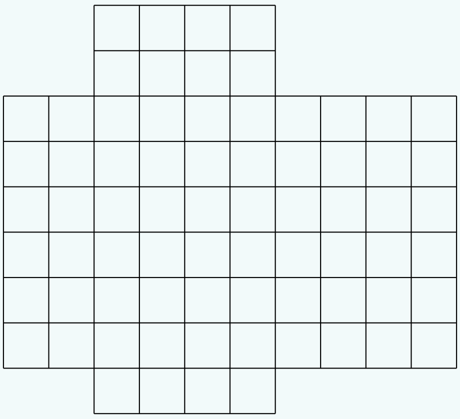
Question 4 [Ans: 7]

It is given that the solution of the inequality $\sqrt{81-x^4} \le kx+1$ is $a \le x \le b$ with $b-a=2$, where $k>0$. Determine $\lfloor k \rfloor$.

Observe that $\sqrt{81-x^4}$ is even, is defined only on $[-3,3]$, and is decreasing for $x>0$. It follows that $b=3$, whence $a=1$. However, at $x=a$, we have equality. Hence, $\sqrt{81-1^2}=k+1$, immediately implying $\lfloor k \rfloor = \lfloor \sqrt{80}-1 \rfloor = 7$.

Question 5 [Ans: 1395]

The figure below shows a cross that is cut out from a 10×9 rectangular board.



Find the total number of rectangles in the above figure.
(Note: A square is a rectangle.)

Consider an $m \times n$ rectangular grid. Choosing a rectangle is equivalent to choosing 2 horizontal lines and two vertical lines (the four lines uniquely outline a rectangle). Since there are a total of $m+1$ horizontal lines and $n+1$ vertical lines, the number of rectangles in such a grid can be calculated as $\binom{m+1}{2}\binom{n+1}{2}$.

Returning to our problem, the total number of rectangles in the figure is hence $\binom{11}{2}\binom{7}{2}+\binom{5}{2}\binom{10}{2}-\binom{5}{2}\binom{7}{2}=1395$. Note that we subtracted $\binom{5}{2}\binom{7}{2}$ to account for the double-counting in the middle of the grid.

Question 6 [Ans: 2020]

Consider all polynomials $P(x,y)$ in two variables such that $P(0,0)=2020$ and for all x and y, $P(x,y)=P(x+y,y-x)$. Find the largest possible value of $P(1,1)$.

Solution 1. Setting $x=0$ and relabelling y as x, we get

$$P(0,x)=P(x,x). \tag{1}$$

Setting $y=x$, we get

$$P(x,x)=P(2x,0).$$

Setting $y=0$, we get

$$P(x,0)=P(x,-x).$$

Setting $y=-x$, we get

$$P(x,-x)=P(0,-2x).$$

We hence have

$$P(0,x) = P(x,x) = P(2x,0) = P(2x,-2x) = P(0,-4x),$$

implying

$$P(0,x) = P(0,-\tfrac{1}{4}x). \tag{2}$$

From (1), we have $P(1,1) = P(0,1)$. Using (2) repeatedly, we have

$$P(0,1) = P(0,-\tfrac{1}{4}) = P(0,\tfrac{1}{16}) = \cdots = P(0,0) = 2020.$$

Thus, $P(1,1) = 2020$.

Solution 2. *(Abusing uniqueness)* Suppose $k > 2020$ is the largest possible value of $P(1,1)$. Then k can be as big as we wish it to be. However, by the nature of the problem, k should be unique. Hence, $k > 2020$ is impossible, implying that 2020 is indeed the largest possible value of $P(1,1)$ (occurring when $P(x,y) = 2020$ for all x, y).

Question 7 [Ans: 16]

In the three-dimensional Cartesian space with $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denoting the unit vectors along three perpendicular directions in a clockwise manner, the line l with equation given by $\mathbf{r} \times (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 5\mathbf{i} - 13\mathbf{j} + 7\mathbf{k}$ intersects the plane Π with equation $x + y + z = 16$ at the point (a,b,c). Find the value of $a+b+c$.

(a,b,c) lies on Π and hence satisfies the equation $x+y+z = 16$. Hence, $a+b+c = 16$.

Question 8 [Ans: 169]

Find the minimum value of $(x+7)^2 + (y+2)^2$ subject to the constraint $(x-5)^2 + (y-7)^2 = 4$.

Let $(x+7)^2 + (y+2)^2 = r^2$, which describes a circle with centre $(-7,-2)$ and radius r. Meanwhile, $(x-5)^2 + (y-7)^2 = 4$ describes a circle with centre $(5,7)$ and radius 2. The smallest r occurs when the two circles are externally tangent. This implies that the sum of the radii is equal to the distance between their centres: $r + 2 = \sqrt{12^2 + 9^2}$. Hence, the minimum value is $r^2 = 169$.

Question 9 [Ans: 98]

Find the largest possible value $\alpha^4+\beta^4+\gamma^4$ among all possible sets of numbers (α,β,γ) that satisfy the equations

$$\begin{aligned} \alpha + \beta + \gamma &= 2 \\ \alpha^2 + \beta^2 + \gamma^2 &= 14 \\ \alpha^3 + \beta^3 + \gamma^3 &= 20. \end{aligned}$$

Newton's identities state that

$$n \geq k \geq 1: \qquad ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$$

$$k > n \geq 1: \qquad 0 = \sum_{i=k-n}^{k} (-1)^{i-1} e_{k-i} p_i$$

where e_k is the kth elementary symmetric polynomial of n variables, and p_k is the kth power sum of n variables.

In our case, we have $n = 3$, along with

$$e_0 = 1, \quad p_0 = 3, \quad p_1 = e_1 = 2, \quad p_2 = 14, \quad p_3 = 20,$$

and we wish to find p_4. Evaluating the above sums at $k = 2, 3, 4$, we get

$$\begin{aligned}
k = 2: &\quad & 2e_2 = e_1p_1 - e_0p_2 &\implies e_2 = -5\\
k = 3: &\quad & 3e_3 = e_2p_1 - e_1p_2 + e_0p_3 &\implies e_3 = -6\\
k = 4: &\quad & 0 = e_3p_1 - e_2p_2 + e_1p_3 - e_0p_4 &\implies p_4 = 98
\end{aligned}$$

Hence, $\alpha^4 + \beta^4 + \gamma^4 = 98$.

Question 10 [Ans: 6]

If p is the product of all the non-zero real roots of the equation

$$\sqrt[9]{x^7 + 30x^5} = \sqrt[7]{x^9 - 30x^5},$$

find $\lfloor |p| \rfloor$.

Dividing through by x and substituting $y = x^2$ yields

$$\sqrt[9]{y^{-1} + 30y^{-2}} = \sqrt[7]{y - 30y^{-1}}.$$

Observe that the LHS is decreasing while the RHS is increasing. There is hence at most one real root. Indeed, by inspection, $y = 6$ satisfies the equation. We hence have $x = \pm\sqrt{6}$, whence $\lfloor |p| \rfloor = 6$.

Question 11 [Ans: 6]

Let S be the sum of a convergent geometric series with first term 1. If the third term of the series is the arithmetic mean of the first two terms, find $\lfloor 3S + 4 \rfloor$.

Let r be the common ratio of the geometric series. Then we have $r^2 = (1+r)/2$, whence $r = -1/2$ (note that we reject $r = 1$ since the series is convergent). Hence,

$$\lfloor 3S + 4 \rfloor = \left\lfloor 3 \cdot \frac{1}{1 - (-1/2)} + 4 \right\rfloor = 6.$$

Question 12 [Ans: 89]

Given that $\sin\alpha + \sin\beta = \dfrac{1}{10}$, and $\cos\alpha + \cos\beta = \dfrac{1}{9}$, find $\left\lfloor \tan^2(\alpha + \beta) \right\rfloor$.

By the sum-to-product identities, we have

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$
$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right).$$

Hence, $\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{1/10}{1/9} = \frac{9}{10}$. By the tangent double-angle identity, we finally get

$$\left\lfloor \tan^2(\alpha + \beta) \right\rfloor = \left\lfloor \left(\frac{2 \cdot 9/10}{1 - (9/10)^2} \right)^2 \right\rfloor = 89.$$

Question 13 [Ans: 2]

Determine the number of positive integers that are divisible by 2021 and has exactly 2021 divisors (including 1 and itself).

Let $n = p_1^{d_1} \cdot p_2^{d_2} \cdots$, where p_i are primes and d_i are non-negative integers. The number of divisors of n can be calculated as $(d_1+1)(d_2+1)\cdots$. Since 2021 only has two prime factors (43 and 47), there are only two possibilities, namely $(d_1, d_2) = (42, 46)$ or $(46, 42)$. Hence, n has only two prime factors, which must be 43 and 47 (since $2021 \mid n$). Thus, there are two possibilities for n, namely $n = 43^{42} \cdot 47^{46}$ or $43^{46} \cdot 47^{42}$.

Question 14 [Ans: 2]

Let $S = \displaystyle\sum_{k=0}^{25} \binom{100}{4k} - 2^{98}$. Find $\left\lfloor \left| \dfrac{S}{2^{48}} \right| \right\rfloor$.

Let

$$S_0 = \sum_{k=0}^{25} \binom{100}{4k}, \quad S_1 = \sum_{k=0}^{24} \binom{100}{4k+1}, \quad S_2 = \sum_{k=0}^{24} \binom{100}{4k+2}, \quad S_3 = \sum_{k=0}^{24} \binom{100}{4k+3}.$$

Consider $(1+x)^{100} = \displaystyle\sum_{k=0}^{100} \binom{100}{k} x^k$. Evaluating the binomial at $x = 1, i, -1, -i$, we have the following system of equations:

$$\begin{cases} (1+1)^{100} = S_0 + S_1 + S_2 + S_3 \\ (1+i)^{100} = S_0 + iS_1 - S_2 - iS_3 \\ (1-1)^{100} = S_0 - S_1 + S_2 - S_3 \\ (1-i)^{100} = S_0 - iS_1 - S_2 + iS_3 \end{cases}$$

Solving, one gets

$$4S_0 = 2^{100} + (1+i)^{100} + (1-i)^{100} = 2^{100} + 2 \cdot 2^{100/2} \cos\left(100 \cdot \tfrac{\pi}{4}\right) = 2^{100} - 2^{51}.$$

Hence,

$$\left\lfloor \left| \frac{S}{2^{48}} \right| \right\rfloor = \left\lfloor \left| \frac{(2^{100} - 2^{51})/4 - 2^{98}}{2^{48}} \right| \right\rfloor = 2.$$

Question 15 [Ans: 1348]

Assume that ABC is an acute triangle with $\sin(A+B) = \dfrac{3}{5}$ and $\sin(A-B) = \dfrac{1}{5}$. If $AB = 2022(\sqrt{6}-2)$, determine $\lfloor h \rfloor$, where h is the height of the triangle from C on AB.

It is trivial to see that $AB = h(\cot A + \cot B)$. Since both A and B are acute, we have $\cos(A+B) = \frac{4}{5}$ and $\cos(A-B) = \frac{2\sqrt{6}}{5}$. We thus have

$$\begin{aligned} 2\cos A \cos B &= \cos(A+B) + \cos(A-B) = \frac{4+2\sqrt{6}}{5} \\ 2\sin A \cos B &= \sin(A+B) + \sin(A-B) = \frac{4}{5} \\ 2\sin A \sin B &= \cos(A-B) - \cos(A+B) = \frac{2\sqrt{6}-4}{5} \end{aligned}$$

It follows that $\cot A = \frac{2+\sqrt{6}}{2}$ and $\cot B = \frac{2}{\sqrt{6}-2}$. Hence,

$$2022(\sqrt{6}-2) = h\left(\frac{2+\sqrt{6}}{2} + \frac{2}{\sqrt{6}-2}\right) \implies h = 1348.$$

Question 16 [Ans: 10]

Let $a_1, a_2, \cdots$ be a sequence with $a_1 = 1$ and $a_{n+1} = \dfrac{n+2}{n} S_n$ for all $n = 1, 2, \cdots$, where $S_n = a_1 + a_2 + \cdots + a_n$. Determine the minimum integer n such that $a_n \geq 2021$.

We claim that $a_n = 2^{n-2}(n+1)$. We prove this via induction. Consider a_2 as the base case: from the given equation, $a_2 = \frac{1+2}{1}(a_1) = 3$, which clearly satisfies our claim. Suppose that $a_k = 2^{k-2}(k+1)$ for some positive integer k. We first show that $S_k = k2^{k-1}$. Observe that

$$\sum_{i=1}^{k} x^i = \frac{x(1-x^k)}{1-x} \overset{\text{d/dx}}{\implies} \sum_{i=1}^{k} ix^{i-1} = \frac{kx^{k+1} - (k+1)x^k + 1}{(1-x)^2}.$$

Evaluating at $x = 2$, we have

$$\sum_{i=1}^{k} i2^{i-1} = k2^{k+1} - (k+1)2^k + 1.$$

Thus,

$$S_k = \sum_{i=1}^{k} \left(\frac{1}{2} \cdot i2^{i-1} + 2^{i-2}\right) = \frac{1}{2}\left(k2^{k+1} - (k+1)2^k + 1\right) + \frac{1}{2}\left(2^k - 1\right) = k2^{k-1}.$$

This immediately implies

$$a_{k+1} = \frac{k+2}{k} \cdot k2^{k-1} = (k+2)2^{k-1},$$

closing the induction. It is hence easy to see that $\min n = 10$.

Question 17 [Ans: 101]

Each card of a stack of 101 cards has one side coloured red and the other coloured blue. Initially all cards have the red side facing up and stacked together in a deck. On each turn, Ah Meng takes 8 cards on the top, flip them over, and place them to the bottom deck. Determine the minimum number of turns required so that all the cards have the red sides facing up again.

Notice that the stack of cards can be represented by an array of numbers. Initially, all entries are 0. When a card is flipped over, the entry associated with that card is incremented by 1. The "incrementer" starts at the first entry and makes its way sequentially across the array, jumping back to the start once it reaches the end.

The figure below shows the array after one turn:

$$\underbrace{1, 1, 1, 1, 1, 1, 1, 1}_{\text{the first 8 entries}}, \underbrace{0, 0, 0, 0, 0, 0, 0, \ldots, 0}_{\text{the remaining 93 entries}}$$

Let k be the sum of the entries. At any given turn, we clearly have $8 \mid k$ (since eight cards are flipped over in a single turn). Furthermore, by the construction of the "incrementer", there are at most two distinct numbers in the array, and their difference must be 1. However, when all the cards have the red sides facing up again, each entry must be even. Hence, all 101 entries have the same number, whence $101 \mid k$. The smallest k is thus $8 \cdot 101 = 808$, which occurs after $808/8 = 101$ turns.

Question 18 [Ans: 88]

Let ABC be a triangle with $AB = 10$ and $\dfrac{\cos A}{\cos B} = \dfrac{AC}{BC} = \dfrac{4}{3}$. Let P be a point on the inscribed circle of triangle ABC. Find the largest possible value of $PA^2 + PB^2 + PC^2$.

By inspection, it is obvious that $\triangle ABC$ is a $6-8-10$ right triangle, with $a = 6$, $b = 8$ and $c = 10$ (where a, b and c denote the sides opposite A, B and C respectively). From $[ABC] = rs$, we see that the inradius of $\triangle ABC$ is 2. Let $A(8,0)$, $B(0,6)$ and $C(0,0)$. Then the incircle has equation

$$(x-2)^2 + (y-2)^2 = 2^2. \tag{1}$$

Our goal now is to maximize $PA^2 + PB^2 + PC^2$, which can be expressed as

$$\left[(x-8)^2 + y^2\right] + \left[x^2 + (y-6)^2\right] + \left[x^2 + y^2\right],$$

where (x, y) are subject to (1). We can rewrite the above expression as

$$3\left[(x-2)^2 + (y-2)^2\right] + 76 - 4x.$$

Using (1), this simplifies to $88 - 4x$. The largest possible value of $PA^2 + PB^2 + PC^2$ is hence 88, where $P(0,2)$.

Question 19 [Ans: 65]

A basket contains 19 apples labelled by the numbers $2, 3, \ldots, 20$, and 19 bananas labelled by the numbers $2, 3, \ldots, 20$. Ah Beng picks m apples and n bananas from the basket. However, he needs to ensure that for any apple labelled a and any banana labelled b that he picks, a and b are relatively prime. Determine the largest possible value of mn.

Let $[n]$ be the set of prime factors of n. For instance, $[18] = \{2, 3\}$. The following table arranges all integers $n \in [2, 20]$ according to $[n]$:

$[n]$	n	$[n]$	n	$[n]$	n
(2)	2, 4, 8, 16	**(11)**	11	**(2,3)**	6, 12, 18
(3)	3, 9	**(13)**	13	**(2,5)**	10, 20
(5)	5	**(17)**	17	**(2,7)**	14
(7)	7	**(19)**	19	**(3,5)**	15

We now create two sets by combining the above sets. It seems clear that the largest set can be formed by grouping (2), (3), (5), $(2,3)$, $(2,5)$ and $(3,5)$ together. This gives a set of size 13. By grouping the remaining sets (except for $(2,7)$, since it would be in conflict with (2)), we get another set of size 5. Hence, $\max mn = 13 \cdot 5 = 65$.

Question 20 [Ans: 25]

Let $p(x) = ax^2 - bx + c$ be a polynomial where a, b, c are positive integers and $p(x)$ has two distinct roots in $(0, 1)$. Determine the least possible value of abc.

Since $p(x)$ has two distinct roots, its discriminant must be positive:

$$b^2 - 4ac > 0. \tag{1}$$

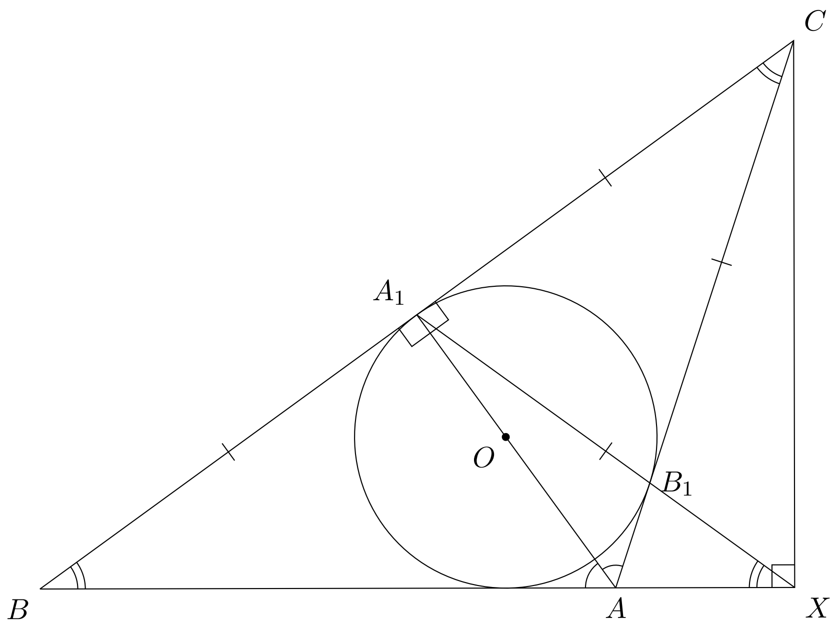
Furthermore, the two roots are in $(0, 1)$. By the quadratic formula, we have $\frac{b\pm\sqrt{b^2-4ac}}{2a} \in (0, 1)$, implying $\sqrt{b^2 - 4ac} \in [0, 2a - b)$. Squaring, we get $b^2 - 4ac < (2a - b)^2$, whence

$$a > b - c. \tag{2}$$

Since we wish to find the smallest value of abc, we fix $c = 1$. From (2), we see that $a = b$. (1) thus implies that $a = b = 5$, whence $abc = 25$.

Question 21 [Ans: 108]

In the triangle ABC, $\angle A > 90°$, the incircle touches the side BC and AC at A_1 and B_1 respectively. The line A_1B_1 meets the extension of BA at X such that $CXB = 90°$. Suppose $BC^2 = AB^2 + BC \cdot AC$. Find the size of $\angle A$ in degrees.



Let $\theta = \frac{1}{2}\angle A$, and let O be the incentre of $\triangle ABC$. Since OA bisects $\angle A$, we have $\angle BAO = \angle CAO = \theta$. Since BC and AC are tangent to the circle, we get several equalities: $CA_1 = CB_1$, and $\angle BA_1O = \angle CA_1O = 90°$. This immediately implies that AOA_1 is a straight line, thus $\triangle BA_1A \equiv \triangle CA_1A$, whence $BA_1 = CA_1 = CB_1$. Since A_1 is the midpoint of BC and $\angle BXC = 90°$, it follows that $XA_1 = CA_1$. We thus have two isosceles triangles, namely $\triangle BA_1X$ and $\triangle A_1CB_1$.

We now find two different expressions for $\angle A_1B_1C$. Firstly, we know that $\angle B_1AX = 180° - 2\theta$, while $\angle A_1XB = \angle A_1BX = 90° - \theta$ (using $\triangle BA_1X$ isosceles), thus $\angle A_1B_1C = \angle AB_1X = 3\theta - 90°$.

Secondly, we have $\angle ACB = \angle ABC = 90° - \theta$ (using $\triangle BA_1A \equiv \triangle CA_1A$). Since $\triangle A_1CB_1$ is isosceles, we have that $\angle A_1B_1C = 45° + \frac{1}{2}\theta$.

Thus, $3\theta - 90° = 45° + \frac{1}{2}\theta$, whence $\angle A = 2\theta = 108°$.

Question 22 [Ans: 2021]

Find the number of positive integers n such that $7n - 16$ divides $n \cdot 13^{2019}$.

Observe that $7 \nmid 7n - 16$. Hence, $7n - 16 \mid 7n \cdot 13^{2019} = (7n-16) \cdot 13^{2019} + 16 \cdot 13^{2019}$. This gives $7n - 16 \mid 2^4 \cdot 13^{2019}$.

Case 1. Suppose $7n - 16 \le 0$. Then $n = 1, 2$. Testing, we see that $n = 1$ fails ($-9 \nmid 13^{2019}$) while $n = 2$ works ($-2 \mid 2 \cdot 13^{2019}$).

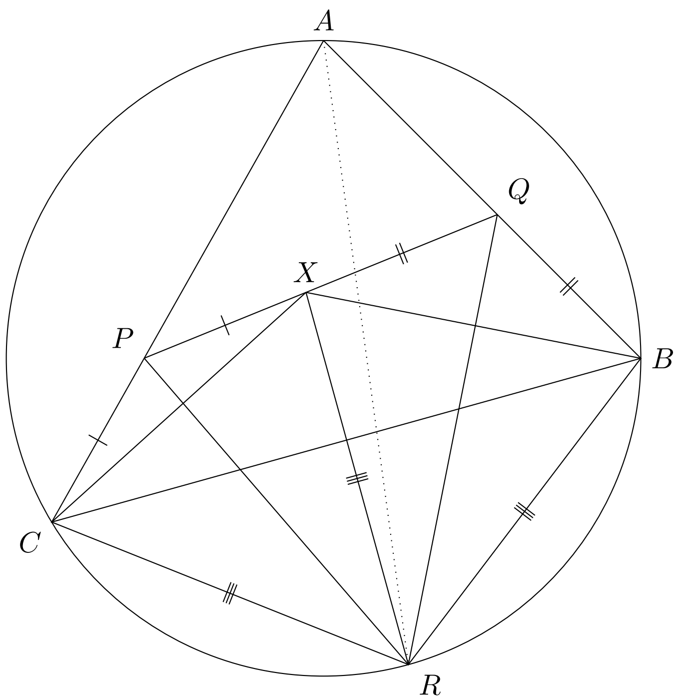
Case 2. Suppose $7n - 16 > 0$. Then $7n - 16$ is a factor of $2^4 \cdot 13^{2019}$. However, observe that $7n - 16 \equiv 5 \pmod 7$. It hence suffices to find the number of factors of $2^4 \cdot 13^{2019}$ that have a residue of 5 modulo 7.

Let $7n - 16 = 2^a \cdot 13^b$, where $a \le 4$ and $b \le 2019$ are non-negative integers. Note that $2^a \cdot 13^b \equiv 5 \pmod 7$ if and only if $a = 1, 4$ and b is odd. This gives $2 \cdot 2020/2 = 2020$ solutions in this case.

Thus, there are a total of $2020 + 1 = 2021$ positive integers n.

Question 23 [Ans: 75]

In the acute triangle ABC, P is a point on AB, Q is a point on AC such that $BP + CQ = PQ$. The bisector of $\angle A$ meets the circumcircle of the triangle ABC at the point R distinct from A. Suppose $\angle PRQ = 52.5°$. Find the size of $\angle BAC$ in degrees.



By the incenter-excenter lemma, we have $CR = RB$. Let X be the point on PQ such that $CP = PX$ and $XQ = QB$. Note that $\angle XPC + \angle XQB = 180° + \angle A$. Since $\triangle CXP$ and $\triangle XBQ$ are both isosceles, it follows that $\angle PXC + \angle QXB = 90° - \frac{1}{2}\angle A$. Hence, $\angle CXB = 90° + \frac{1}{2}\angle A$. Also, since $ABRC$ is cyclic, $\angle CRB = 180° - \angle A$, whence reflex $\angle CRB = 180° + \angle A = 2\angle CXB$. Hence, R is the circumcentre of $\triangle CXB$, implying that $RX = CR = RB$. Thus, $PCXR$ and $QXBR$ are kites, hence $\angle PRX = \angle\angle PRC$, and $\angle QRX = \angle QRB$. Thus, $\angle CRB = 2\angle PRQ = 105°$, whence $\angle A = 180° - 105° = 75°$.

Question 24 [Ans: 6]

Let $S = \displaystyle\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,\mathrm{d}x$. Determine the value of $\lfloor S^2 \rfloor$.

The generalized Gaussian integral $\int_{-\infty}^{\infty} e^{-kx^2}\,\mathrm{d}x$ evaluates to $\sqrt{\pi/k}$. Hence, $\lfloor S^2 \rfloor = \lfloor 2\pi \rfloor = 6$.

Question 25 [Ans: 5]

Let p, q, r be positive numbers with $p - r = 4q$ and $a_1, a_2, \cdots$ and $b_1, b_2, \cdots$ be two sequences defined by $a_1 = p$, $b_1 = q$ and for $n \geq 2$,

$$a_n = pa_{n-1}, \quad b_n = qa_{n-1} + rb_{n-1}.$$

Find the value of $\displaystyle\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n}$.

Observe that a_n is simply a geometric series, with $a_n = p^n$. Hence, $b_n = qp^{n-1} + rb_{n-1}$. We now claim that $b_n = \frac{1}{4}(p^n - r^n)$. We prove this via induction. Observe that the base case $b_1 = q = \frac{1}{4}(4q) = \frac{1}{4}\left(p^1 - r^1\right)$ holds. Suppose that $b_k = \frac{1}{4}\left(p^k - r^k\right)$ for some positive integer k. Then

$$\begin{aligned} b_{k+1} &= qp^k + r\left[\frac{1}{4}\left(p^k - r^k\right)\right] = \frac{1}{4}(p-r)\,p^k + \frac{1}{4}rp^k - \frac{1}{4}r^{k+1} \\ &= \frac{1}{4}\left(p^{k+1} - rp^k + rp^k - r^{k+1}\right) = \frac{1}{4}\left(p^{k+1} - r^{k+1}\right), \end{aligned}$$

closing the induction. The limit hence evaluates to

$$\lim_{n\to\infty} \frac{\sqrt{a_n^2 + (3b_n)^2}}{b_n} = \lim_{n\to\infty} \sqrt{\left(\frac{a_n}{b_n}\right)^2 + 9} = \lim_{n\to\infty} \sqrt{\left(\frac{p^n}{(p^n - r^n)/4}\right)^2 + 9} = \sqrt{4^2 + 9} = 5.$$

9. 2022 SMO

9.1. Open Section

9.1.1. Round 1 Solutions

Review by *Way Tan*

Question 1 [Ans: 4043]

If $S = \displaystyle\sum_{k=-2021}^{2021} \frac{1}{10^k + 1}$, find $\lfloor 2S \rfloor$.

Observe that $\frac{1}{10^{-k}+1} = \frac{10^k}{10^k+1}$. The $-k$th term can thus be paired with the kth term to form $\frac{10^k+1}{10^k+1} = 1$, with the 0th term being the only unpaired term. The sum hence evaluates to

$$S = 2021 + \frac{1}{10^0 + 1} \implies \lfloor 2S \rfloor = 4043.$$

Question 2 [Ans: 1303]

All the positive integers $1, 2, 3, 4, \cdots$, are grouped in the following way: $G_1 = \{1, 2\}$, $G_2 = \{3, 4, 5, 6\}$, $G_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$, that is, the set G_n contains the next 2^n positive integers listed in ascending order after the set G_{n-1}, $n > 1$. If S is the sum of all the positive integers from G_1 to G_8, find $\left\lfloor \frac{S}{100} \right\rfloor$.

It is not too hard to show that the last term in G_n is $2^{n+1} - 2$. The last term of G_8 is hence $2^9 - 2 = 510$, whence S evaluates to $510(511)/2$. Thus, $\lfloor S/100 \rfloor = 1303$.

Question 3 [Ans: 100]

A sequence of one hundred positive integers $x_1, x_2, x_3, \cdots, x_{100}$ are such that

$$(x_1)^2 + (2x_2)^2 + (3x_3)^2 + (4x_4)^2 + \cdots + (100x_{100})^2 = 338350.$$

Find the largest possible value of $x_1 + x_2 + x_3 + \cdots + x_{100}$.

Observe that $1^2 + 2^2 + 3^2 + \cdots + 100^2 = 338350$. Since x_i are positive integers, they must all be 1 (to prevent the sum from exceeding 338350). The desired value is hence 100.

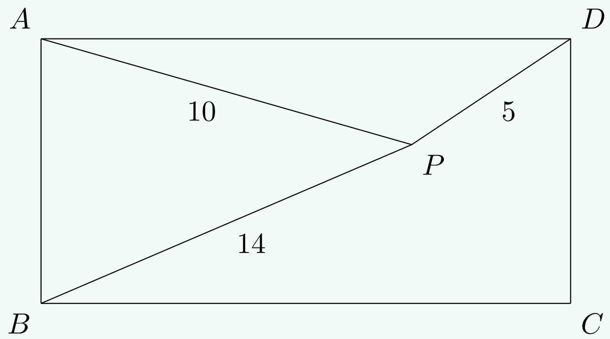
Question 4 [Ans: 9]

Let a and b be two real numbers satisfying $a < b$, and such that for each real number m satisfying $a < m < b$, the circle $x^2 + (y - m)^2 = 25$ meets the parabola $4y = x^2$ at four distinct points in the Cartesian plane. Let S be the maximum possible value of $b - a$. Find $\lfloor 4S \rfloor$.

Note that a is clearly 5. Now consider the extreme case where $m > 5$ and the circle is tangent to the parabola. The discriminant of the quadratic $4y + (y - m)^2 = 25$ must be 0, whence $m = 29/4$. Hence, $S = b - a = 29/4 - 5 = 9/4$. Thus, $\lfloor 4S \rfloor = 9$.

Question 5 [Ans: 11]

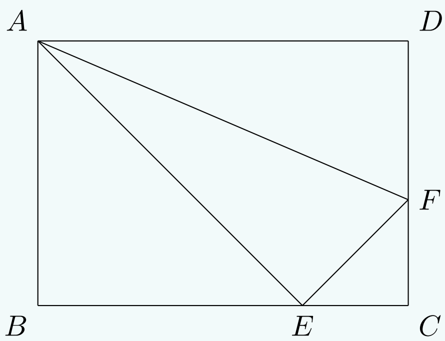
Let P be a point within a rectangle $ABCD$ such that $PA = 10$, $PB = 14$ and $PD = 5$, as shown below. Find $\lfloor PC \rfloor$.



By the British flag theorem, one has $10^2 + PC^2 = 5^2 + 14^2$, whence $\lfloor PC \rfloor = 11$.

Question 6 [Ans: 50]

In the diagram below, the rectangle $ABCD$ has area 180 and both triangles ABE and ADF have areas 60. Find the area of triangle AEF.



Let $BE = x$ and $DF = y$. Observe that $x/(x + EC) = [ABE]/[ABC] = 60/90 = 2/3$. Hence, $EC = x/2$. Similarly, $y/(y + FC) = [ADF]/[ADC] = 60/90 = 2/3$, whence $FC = y/2$. Thus, $[CEF] = [CDE]/3 = [ABE]/6 = 10$. Finally, $[AEF] = 180 - 60 - 60 - 10 = 50$.

Question 7 [Ans: 144]

A tetrahedron in $\mathbb{R}^3$ has one vertex at the origin O and other vertices at the points $A(6,0,0)$, $B(4,2,4)$ and $C(3,2,6)$. If x is the height of the tetrahedron from O to the plane ABC, find $\lfloor 5x^2 \rfloor$.

The plane ABC is given by the vector equation $\mathbf{r} \cdot (2,0,1) = 12$. Since x is the perpendicular distance from O to ABC, we have $x = 12/\sqrt{2^2+0^2+1^2} = 12/\sqrt{5}$. Thus, $\lfloor 5x^2 \rfloor = 144$.

Question 8 [Ans: 208]

Let x and y be real numbers such that $(x-2)^2+(y-3)^2 = 4$. If S is the largest possible value of x^2+y^2, find $\lfloor (S-17)^2 \rfloor$.

Observe that $(x-2)^2+(y-3)^2 = 4$ describes a circle with centre $(2,3)$ and radius 2, while S is the square of the distance from some point P on the circle to the origin. The largest distance clearly occurs when the origin, the centre $(2,3)$, and P are collinear. This distance can be calculated as $\sqrt{2^2+3^2}+2 = \sqrt{13}+2$. Hence,

$$\lfloor (S-17)^2 \rfloor = \left\lfloor \left((\sqrt{13}+2)^2 - 17 \right)^2 \right\rfloor = 208.$$

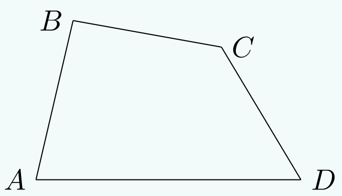
Question 9 [Ans: 18]

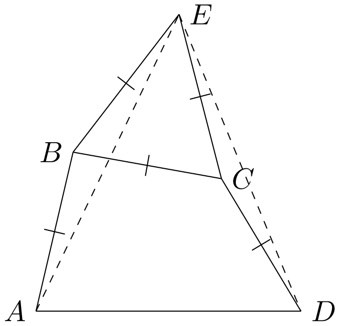
Let S be the maximum value of $w^3 - 3w$ subject to the condition that $w^4 + 9 \le 10w^2$. Find $\lfloor S \rfloor$.

Consider $w^4+9 \le 10w^2$. Solving, we have $(w^2-9)(w^2-1) \le 0$, whence $w \in [-3,-1] \cup [1,3]$. Now notice that w^3-3w is odd and is increasing when $w < -1$ or $w > 1$. The maximum value thus occurs either at $w = -1$ or 3. Comparing values, we see that $\lfloor S \rfloor = \lfloor 3^3 - 3 \cdot 3 \rfloor = 18$.

Question 10 [Ans: 40]

In the quadrilateral $ABCD$ below, it is given that $AB = BC = CD$ and $\angle ABC = 80^\circ$ and $\angle BCD = 160^\circ$. Suppose $\angle ADC = x^\circ$. Find the value of x.





Let E be such that $\triangle BCE$ is equilateral, as shown above.

Note that $\triangle ABE$ is isosceles. Since $\angle ABE = 80^\circ + 60^\circ = 140^\circ$, we have $\angle BAE = \angle BEA = 20^\circ$. Hence, $\angle AED = 60^\circ - 20^\circ = 40^\circ$. However, notice that $\triangle DCE$ is also isosceles, and reflex $\angle CDE = 360^\circ - 60^\circ - 160^\circ = 140^\circ$. Thus, $\angle CED = \angle CDE = 20^\circ$, whence $\triangle ABE \equiv \triangle DCE$, implying $AE = DE$. Because $\angle AED = 40^\circ + 20^\circ = 60^\circ$, it follows that $\triangle AED$ is equilateral, thus giving $x^\circ = 60^\circ - 20^\circ = 40^\circ$.

Question 11 [Ans: 544]

Let a, b, c be integers with $ab + c = 49$ and $a + bc = 50$. Find the largest possible value of abc.

Subtracting the two equations, we obtain $a + bc - ab - c = 1$. This can be factorized as $(b-1)(c-a) = 1$. Since a, b and c are integers, we are left with two cases:

Case 1: $b - 1 = c - a = -1$. We have $b = 0$, whence $abc = 0$.

Case 2: $b - 1 = c - a = 1$. We have $b = 2$ and $c = 1 + a$. Substituting this back into one of the original equations, we get $a = 16$ and $c = 17$, whence $abc = 544$.

The largest possible value of abc is thus 544.

Question 12 [Ans: 17]

Find the largest possible value of $|a| + |b|$, where a and b are coprime integers (i.e., a and b are integers which have no common factors larger than 1) such that $\frac{a}{b}$ is a solution of the equation below:

$$\sqrt{4x + 5 - 4\sqrt{x+1}} + \sqrt{x + 2 - 2\sqrt{x+1}} = 1.$$

Note that

$$4x + 5 - 4\sqrt{x+1} = \left(2\sqrt{x+1}\right)^2 - 2 \cdot \sqrt{x+1} \cdot 1 + 1^2 = \left(2\sqrt{x+1} - 1\right)^2$$

and

$$x+2-2\sqrt{x+1}=\left(\sqrt{x+1}\right)^2-2\cdot\sqrt{x+1}\cdot 1+1^2=\left(\sqrt{x+1}-1\right)^2.$$

We hence have

$$\pm\left(2\sqrt{x+1}-1\right)\pm\left(\sqrt{x+1}-1\right)=1.$$

Case 1: $\left(2\sqrt{x+1}-1\right)+\left(\sqrt{x+1}-1\right)=1$. We have $3\sqrt{x+1}=3$, whence $x=0$. Hence, $a=0$ and $b=1$, whence $|a|+|b|=1$.

Case 2: $\left(2\sqrt{x+1}-1\right)-\left(\sqrt{x+1}-1\right)=1$. We have $\sqrt{x+1}=3$, whence $x=8$. Hence, $a=8$ and $b=1$, whence $|a|+|b|=9$.

Case 3: $-\left(2\sqrt{x+1}-1\right)+\left(\sqrt{x+1}-1\right)=1$. We have $\sqrt{x+1}=-1$, a contradiction.

Case 4: $-\left(2\sqrt{x+1}-1\right)-\left(\sqrt{x+1}-1\right)=1$. We have $3\sqrt{x+1}=1$, whence $x=-8/9$. Hence, $a=-8$ and $b=9$, whence $|a|+|b|=17$.

The maximum value of $|a|+|b|$ is 17.

Question 13 [Ans: 3000]

Let S be the set of real solutions (x,y,z) of the following system of equations:

$$\begin{cases}\dfrac{4x^2}{1+4x^2}=y,\\ \dfrac{4y^2}{1+4y^2}=z,\\ \dfrac{4z^2}{1+4z^2}=x.\end{cases}$$

For each $(x,y,z)\in S$, define $m(x,y,z)=2000(|x|+|y|+|z|)$. Determine the maximum value of $m(x,y,z)$ over all $(x,y,z)\in S$.

Taking reciprocals, we have

$$\begin{cases}1+\dfrac{1}{4x^2}=\dfrac{1}{y},\\ 1+\dfrac{1}{4y^2}=\dfrac{1}{z},\\ 1+\dfrac{1}{4z^2}=\dfrac{1}{x}.\end{cases}$$

Summing, we obtain

$$\left[\frac{1}{(2x)^2}-\frac{1}{x}+1\right]+\left[\frac{1}{(2y)^2}-\frac{1}{y}+1\right]+\left[\frac{1}{(2z)^2}-\frac{1}{z}+1\right]=0.$$

This clearly factors as

$$\left(\frac{1}{2x}-1\right)^2+\left(\frac{1}{2y}-1\right)^2+\left(\frac{1}{2z}-1\right)^2=0,$$

whence $x=y=z=1/2$, giving $\max m(x,y,z)=3000$.

Question 14 [Ans: 11]

Assume that t is a positive solution to the equation

$$t = \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+t}}}}.$$

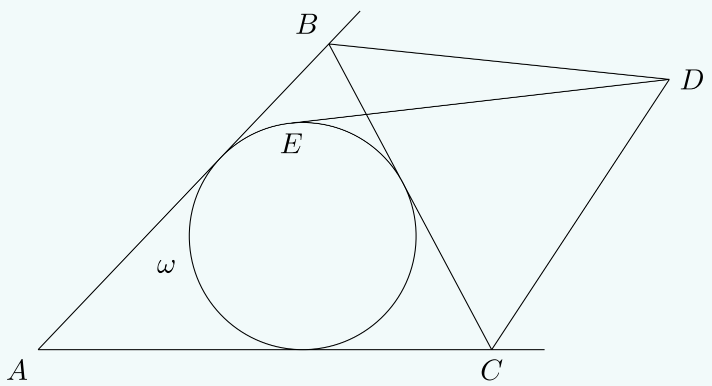
Determine the value of $t^4 - t^3 - t + 10$.

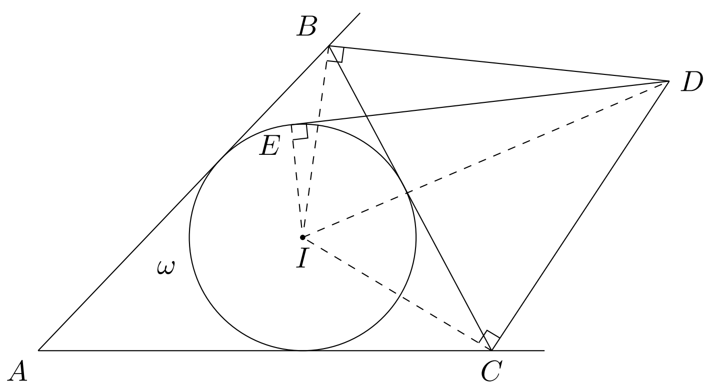
Observe that $t = \sqrt{1+t}$. It follows that $t^2 - t - 1 = 0$, whence t is the golden ratio φ, which has the property that $\varphi^n = \varphi^{n-1} + \varphi^{n-2}$ for all integers n. The desired value is hence

$$t^4 - t^3 - t + 10 = t^2 - t + 10 = 1 + 10 = 11.$$

Question 15 [Ans: 44]

In the triangle ABC shown in the diagram below, the external angle bisectors of $\angle B$ and $\angle C$ meet at the point D. The tangent from D to the incircle ω of the triangle ABC touches ω at E, where E and B are on the same side of the line AD. Suppose $\angle BEC = 112°$. Find the size of $\angle A$ in degrees.





Let I be the incentre of $\triangle ABC$. Observe that BI and IC bisect $\angle B$ and $\angle C$ respectively. Hence, $\angle IBD = \angle ICD = 90°$. Furthermore, since ED is tangent to ω, we have $\angle IED = 90°$. Thus, B, C, D, E and I are concyclic, with ID as the diameter. Hence, $\angle BIC = \angle BEC = 112°$. Since IB and IC are angle bisectors, we have

$$\angle B + \angle C = 2\,(\angle IBC + \angle ICB) = 2\,(180° - 112°) = 136°.$$

It immediately follows that $\angle A = 180° - 136° = 44°$.

Question 16 [Ans: 99]

Find the largest integer n such that $n^2+5n-9486 = 10s(n)$, where $s(n)$ is the product of all digits of n in the decimal representation of n.
(For example, $s(481) = 4 \times 8 \times 1 = 32$.)

Observe that $s(n) \leq n$ for all n. We thus have the inequality $n^2 + 5n - 9486 \leq 10n$, whence it is clear that the largest possible n is 99.

Question 17 [Ans: 8]

Find the number of integer solutions to the equation $19x + 93y = 4xy$.

Note that $(ax+b)(cy+d) = adx+acxy+bcy+bd$, where a, b, c and d are real numbers. Comparing this to the given equation, we have $ad = 19$, $ac = -4$ and $bc = 93$. Taking $a = 1$, $c = -4$, $b = -93/4$ and $d = 19$, we have

$$\left(x - \frac{93}{4}\right)(-4y + 19) = -\frac{19 \cdot 93}{4}.$$

Upon simplification, one gets

$$(4x - 93)(4y - 19) = -1 \cdot 3 \cdot 19 \cdot 31.$$

Observe that both $4x - 93$ and $19 - 4y$ are congruent to 1 modulo 4. On the other hand, all four factors (-1, 3, 19 and 31) are also congruent to 3 modulo 4. We must hence have an even number of factors contributing to both terms. This narrows the possibilities down to a few cases.

Case 1: Both terms have two factors. The number of possibilities in this case is ${}^4C_2 = 6$.

Case 2: One term has four factors, the other has none. The number of possibilities in this case is clearly 2.

Altogether, there are $6 + 2 = 8$ possibilities, thus there are 8 integer solutions to the given equation.

Question 18 [Ans: 121]

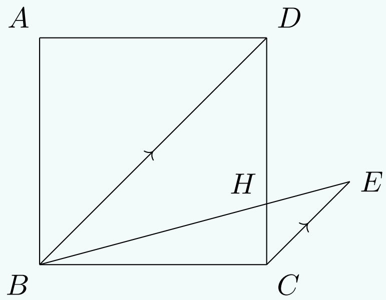
Find the number of integer solutions to the equation $x_1 + x_2 - x_3 = 20$ with $x_1 \geq x_2 \geq x_3 \geq 0$.

Rewriting the equation, we have $x_1 + x_2 = 20 + x_3$. Suppose x_3 is fixed (and admits possible values of x_2 and x_1). Observe that $x_1 \leq 20$, with equality only when $x_2 = x_3$. The only possible values of x_1 are hence $\{20, 19, 18, \cdots, x_2\}$. However, since $x_1 \geq x_2$, we have the condition $x_2 \leq \left\lfloor \frac{1}{2}(20 + x_3) \right\rfloor$. The number of solutions for a given x_3 is hence $11 - \left\lfloor \frac{1}{2} x_3 \right\rfloor$. The total number of integer solutions is thus

$$11 + 10 + 10 + 9 + 9 + \cdots + 1 + 1 = 11 + 2 \cdot \frac{10 \cdot 11}{2} = 121.$$

Question 19 [Ans: 2]

In the diagram below, E is a point outside a square $ABCD$ such that CE is parallel to BD, $BE = BD$, and BE intersects CD at H. Given $BE = \sqrt{6}+\sqrt{2}$, find the length of DH.



Solution 1. Observe that $\triangle HDB$ is similar to $\triangle HCE$. Hence,

$$\frac{EH}{BH} = \frac{CH}{DH} \implies \frac{BE-BH}{BH} = \frac{DC-DH}{DH} \implies \frac{BE}{BH} = \frac{DC}{DH}.$$

Note that the side length of the square is $\frac{1}{\sqrt{2}}BD = \sqrt{3}+1$. Thus,

$$\frac{\sqrt{6}+\sqrt{2}}{BH} = \frac{\sqrt{3}+1}{DH} \implies BH^2 = 2DH^2.$$

Using Pythagoras' theorem on $\triangle BCH$, one obtains

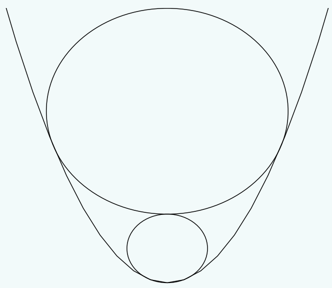
$$BC^2 + CH^2 = BH^2 \implies \left(\sqrt{3}+1\right)^2 + \left(\sqrt{3}+1-DH\right)^2 = 2DH^2.$$

Solving, we have $DH = 2$.

Solution 2. *(Abusing integers)* Note that the side length of the square is $\frac{1}{\sqrt{2}}BD = \sqrt{3}+1 \approx 2.73$. Since DH is an integer and $CH < DH < BD$, it must be 2.

Question 20 [Ans: 4]

The diagram below shows the region $R = \left\{(x,y) \in \mathbb{R}^2 | y \geq \frac{1}{2}x^2\right\}$ on the xy-plane bounded by the parabola $y = \frac{1}{2}x^2$. Let C_1 be the largest circle lying inside R with its lowest point at the origin. Let C_2 be the largest circle lying inside R and resting on top of C_1. Find the sum of radii of C_1 and C_2.



Let the radius of C_1 and C_2 be r_1 and r_2 respectively. The equations of C_1 and C_2 are hence

$$C_1 : x^2 + (y-r_1)^2 = r_1^2,$$
$$C_2 : x^2 + (y-(2r_1+r_2))^2 = r_2^2.$$

Consider the intersection between C_1 and the parabola:

$$\begin{cases} y = \dfrac{1}{2}x^2 \\ x^2 + (y - r_1)^2 = r_1^2 \end{cases}$$

This gives $y^2 + 2y(1 - r_1) = 0$. Since the two curves only intersect at the origin, we have $r_1 = 1$.

Now consider the intersection between C_2 and the parabola:

$$\begin{cases} y = \dfrac{1}{2}x^2 \\ x^2 + (y - (2r_1 + r_2))^2 = r_2^2 \end{cases}$$

This gives $y^2 - 2y(1 + r_2) + 4(1 + r_2) = 0$. By symmetry, the two curves intersect at a unique y-value, hence the discriminant is 0. We hence obtain $4(1 + r_2)^2 - 16(1 + r_2) = 0$, whence $r_2 = 3$. The required sum is thus $1 + 3 = 4$.

Question 21 [Ans: 30]

Find the smallest positive integer x such that $3x^2 + x = 4y^2 + y$ for some positive integer y.

Completing the square, one gets

$$4(6x + 1)^2 - 4 = 3(8y + 1)^2 - 3$$

after simplification. This can be rewritten as

$$(12x + 2)^2 - 3(8y + 1)^2 = 1,$$

which one may recognize as a case of Pell's equation. We hence consider the equation $X^2 - 3Y^2 = 1$. The fundamental solution is clearly $X = 2$ and $Y = 1$. We now have the following standard recurrence relations for X and Y:

$$X_{k+1} = 2X_k + 3Y_k, \quad Y_{k+1} = 2Y_k + X_k.$$

Keeping in mind that X is of the form $12x + 2$ and Y is of the form $8y + 1$, the first valid solution occurs when $k = 5$, where $X = 362$ and $Y = 209$, which corresponds to $x = 30$ and $y = 26$.

Question 22 [Ans: 200]

A group of students participates in some sports activities among 6 different types of sports. It is known that for each sport activity there are exactly 100 students in the group participating in it; and the union of all the sports activities participated by any two students is NOT the entire set of 6 sports activities. Determine the minimum number of students in the group.

Let m be the maximum number of sports a student can take at once. By symmetry, we only need to consider $m \geq 3$ (any $m < 3$ will lead to a less-than-optimal allocation of students). Furthermore, it is obvious that $m \geq 5$ is impossible, since it would immediately violate the given restriction. We hence analyse only the $m = 3$ and $m = 4$ case.

Case 1: $m = 4$. To adhere to the given restriction, there is only one possible allocation of students: 100 to Sport A, 100 to Sport B and 100 taking the other four sports. This gives a total of 300 students.

Case 2: $m = 3$. Note that the absolute minimum number of students is given by $100 \cdot 6/3 = 200$. We now construct an allocation that uses exactly 200 students.

Let the sports be labelled A through F. Consider the following allocation of students:

	A	B	C	D	E	F
Student 1	X	X	X			
Student 2	X			X	X	
Student 3		X		X		X
Student 4			X		X	X

Repeating the above allocation 50 times, we will have 100 students per sport. Hence, the minimum number of students is $4 \cdot 50 = 200$ as desired.

Question 23 [Ans: 29]

Let p and q be positive prime integers such that $p^3 - 5p^2 - 18p = q^9 - 7q$. Determine the smallest value of p.

Observe that the RHS will grow incredibly fast as compared to the LHS. We hence test small values of q. Comparing leading terms, we also note that $p \approx q^3$. Furthermore, we require $p^3 - 5p^2 - 18p > 0$, whence $p \geq 11$.

Case 1: $q = 2$. We have $p\left(p^2 - 5p - 18\right) = 2 \cdot 3 \cdot 83$. It can be easily shown that there are no solutions in this case.

Case 2: $q = 3$. We have $p\left(p^2 - 5p - 18\right) = 2 \cdot 3 \cdot 29 \cdot 113$. Testing $p = 29$, we see that it is indeed a solution.

Hence, the smallest value of p is 29.

Question 24 [Ans: 27648]

Given that a, b, c are positive real numbers such that $a+b+c = 9$, find the maximum value of $a^2b^3c^4$.

Note that

$$9 = a + b + c = \frac{a}{2} + \frac{a}{2} + \frac{b}{3} + \frac{b}{3} + \frac{b}{3} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4} + \frac{c}{4}.$$

By the Cauchy-Schwarz inequality, we obtain

$$\frac{1}{9} \cdot 9 \geq \sqrt[9]{\frac{a^2b^3c^4}{2^2 3^3 4^4}}.$$

Hence, the maximum value of $a^2b^3c^4$ is $2^2 3^3 4^4 = 27648$.

Question 25 [Ans: 2023]

Let $\mathbb{R}^+$ be the set of all positive real numbers. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function satisfying

$$xyf(x)\big(f(y) - f(yf(x))\big) = 1$$

for all $x, y \in \mathbb{R}^+$. Find $f(\frac{1}{2022})$.

Letting $y = 1$, we get the following expression for $f^2(x)$:

$$f^2(x) = f(1) - \frac{1}{xf(x)}. \tag{1}$$

Replacing y with $f(y)$ in the original equation gives

$$f^2(y) = \frac{1}{xf(y)f(x)} + f(f(y)f(x)). \tag{2}$$

Substituting (1) into (2) yields

$$f(1) - \frac{1}{yf(y)} = \frac{1}{xf(y)f(x)} + f(f(y)f(x)). \tag{3}$$

Swapping x and y gives a similar equation:

$$f(1) - \frac{1}{xf(x)} = \frac{1}{yf(x)f(y)} + f(f(x)f(y)). \tag{4}$$

Subtracting (4) from (3) and simplifying, we obtain

$$yf(y) - y = xf(x) - x,$$

from which it is clear that for all $x \in \mathbb{R}^+$, we have $xf(x) - x = c$ for some constant c. This immediately gives $f(x) = 1 + \frac{c}{x}$. Substituting this into (1), we have

$$1 + \frac{c}{1 + \frac{c}{x}} = \left(1 + \frac{c}{1}\right) - \frac{1}{x\left(1 + \frac{c}{x}\right)},$$

whence $c^2 = 1$. Since the range of f is $\mathbb{R}^+$, we take $c = 1$, thus $f(x) = 1 + \frac{1}{x}$ and $f(\frac{1}{2022}) = 2023$.

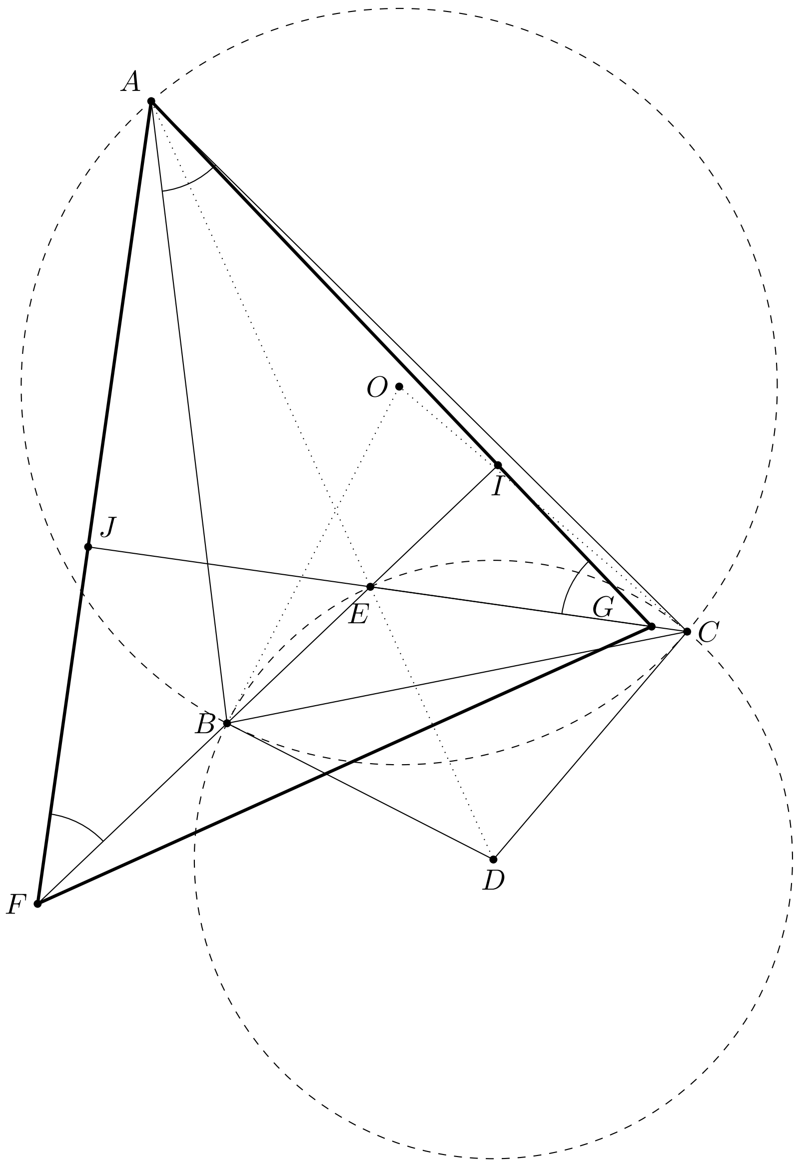
9.1.2. Round 2 Solutions

Question 1

For $\triangle ABC$ and its circumcircle ω, draw the tangents at B, C to ω meeting at D. let the line AD meet the circle with centre D and radius DB at E inside $\triangle ABC$. Let F be the point on the extension of EB and G be the point on the segment EC such that $\angle AFB = \angle AGE = \angle A$. Prove that the tangent at A to the circumcircle of $\triangle AFG$ is parallel to BC.

AoPS thread

Solution by gghx**.**



Claim 1. *E is the orthocentre of $\triangle AFG$.*

Proof. Let O be the circumcentre of $\triangle ABC$. Let $I = AG \cap FE$ and $J = AF \cap GE$.

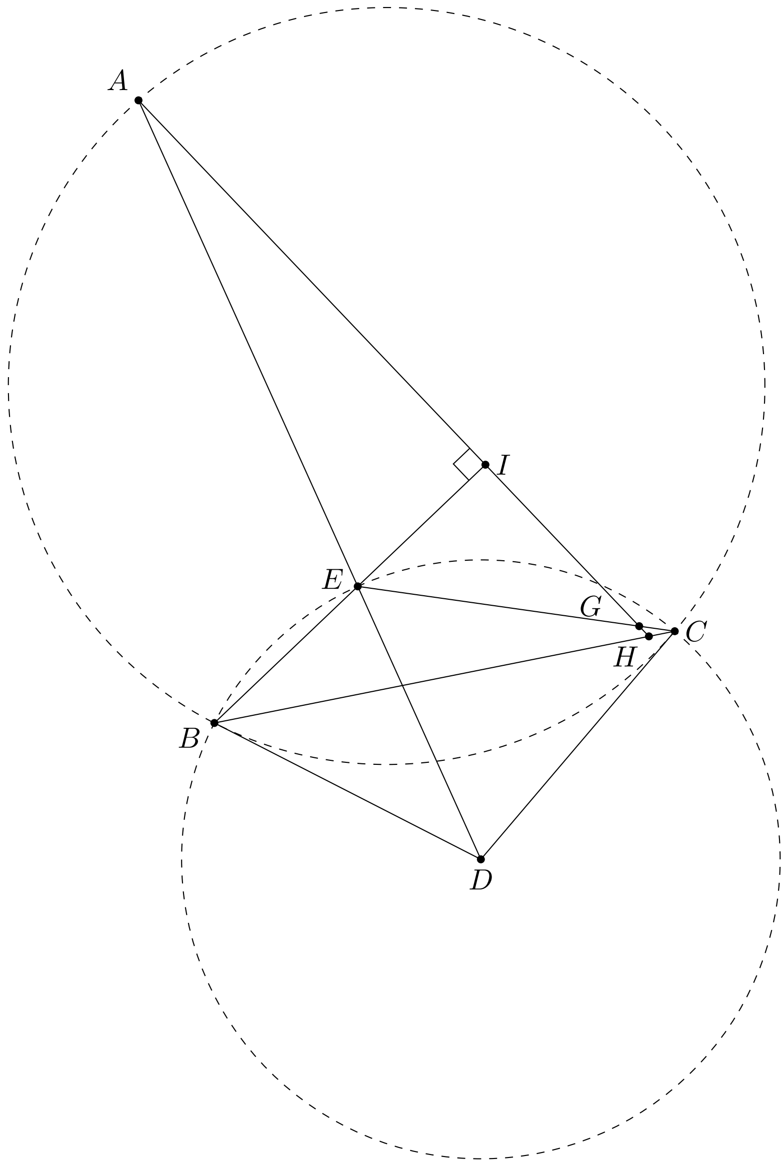
Since $\angle BOC = 2\angle A$ and $\angle OBD = \angle OCD = 90^\circ$, we have $\angle BDC = 180^\circ - 2\angle A$, whence reflex $\angle BDC = 180^\circ + 2\angle A$. Hence, $\angle BEC = 90^\circ + \angle A$. This immediately gives

$$\angle EIC + \angle AGE = 90^\circ + \angle A \implies \angle EIC = 90^\circ$$

and

$$\angle EJF + \angle AFB = 90^\circ + \angle A \implies \angle EJF = 90^\circ.$$

Thus, E is the orthocentre of $\triangle AFG$. □



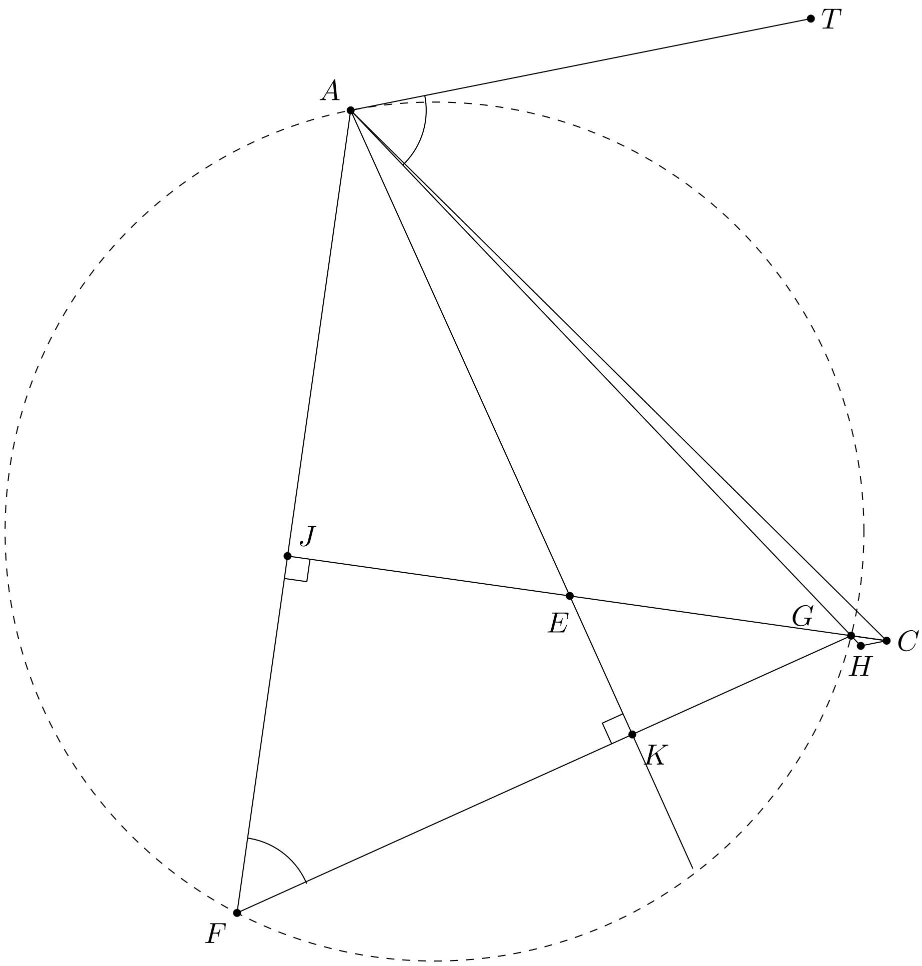
Let $H = AG \cap BC$.

Claim 2. *AEHC is cyclic.*

Proof. Note that $\triangle BDE$ is isosceles, with $\angle EBD = \angle BED$. Hence, $\angle BDE = 180^\circ - 2\angle BED$. Thus,

$$\angle GCH = \frac{1}{2}\angle BDE = 90^\circ - \angle BED = 90^\circ - \angle AEI = \angle GAE.$$

Thus, by the converse of angles in same segment, $AEHC$ is cyclic. □



Claim 3. *The tangent at A to (AFG) is parallel to HC (i.e. BC).*

Proof. Let $K = AE \cap GF$. Let T be a point on the line parallel to HC through A, such that G and T are on the same side of AE. It suffices to show that AT is tangent to (AFG) at A.

Observe that

$$\angle TAG = 180^\circ - \angle GHC = 180^\circ - \angle GEA = 180^\circ - \angle KEJ = \angle GFA.$$

Thus, by the converse of the alternate segment theorem, AT is tangent to (AFG) at A. □

Question 2

Prove that if the length and breadth of a rectangle are both odd integers, then there does not exist a point P inside the rectangle such that each of the distances from P to the 4 corners of the rectangle is an integer.

AoPS thread

Solution by sarjinius. Let m and n be positive odd integers. Let $A(0,0)$, $B(m,0)$, $C(m,n)$ and $D(0,n)$. Let the perpendicular distance from P to AB, BC, CD and DA be a, b, c and d respectively, whence $P(d,a)$. Also, we have $b+d=m$ and $a+c=n$. Seeking a contradiction, suppose PA, PB, PC and PD are all integers.

By Pythagoras' theorem, we have $PA^2 = a^2 + d^2$ and $PB^2 = a^2 + (m-d)^2$. Thus, both $a^2 + d^2$ and $a^2 + d^2 - 2md + m^2$ are perfect squares. This immediately gives

$$d = \frac{m^2 + PA^2 - PB^2}{2m},$$

whence d is rational with a denominator not divisible by 4 (when expressed in lowest terms). Similarly, we also get

$$a = \frac{n^2 + PB^2 - PC^2}{2n}, \quad b = \frac{m^2 + PC^2 - PD^2}{2m}, \quad c = \frac{n^2 + PD^2 - PA^2}{2n},$$

thus a, b and c are also rational with denominator not divisible by 4.

By the British Flag theorem, we have $PA^2 + PC^2 = PB^2 + PD^2$. Thus, $PA^2 - PB^2 = -\left(PC^2 - PD^2\right)$ and $PB^2 - PC^2 = -\left(PD^2 - PA^2\right)$. It follows that a and c share the same denominator. Likewise, b and d share the same denominator. Let k be the denominator of a and c, and l be the denominator of b and d. Note that $k \mid 2n$ and $l \mid 2m$.

We now scale $ABCD$ with respect to A by a factor of $\operatorname{lcm}(k,l)$ (note that this is not divisible by 4). Let X_* represent the point X after scaling and z_* represent the distance z after scaling.

Case 1. Suppose $\operatorname{lcm}(k,l)$ is odd. Thus, both m_* and n_* are odd. Since $a_*, b_*, c_*, d_* \in \mathbb{Z}$, with $b_* + d_* = m_*$ and $a_* + c_* = n_*$, it follows that one of a and c is odd, and one of b and d is odd. Hence, one of $PA_*^2 = a_*^2 + d_*^2$, $PB_*^2 = a_*^2 + b_*^2$, $PC_*^2 = c_*^2 + b_*^2$ and $PD_*^2 = c_*^2 + d_*^2$ must be 2 mod 4, which cannot be perfect square, a contradiction.

Case 1. Suppose $\operatorname{lcm}(k,l)$ is even. Then at least one of k and l is even (and also 2 mod 4).

Subcase 2A. Suppose both k and l are even. This implies that a_*, b_*, c_* and d_* are all odd. This is a clear contradiction since the sum of two odd squares will always be 2 mod 4, which cannot be a perfect square.

Subcase 2B. Without loss of generality, suppose k is odd and l is even. Hence, a_* and c_* are even, while b_* and d_* are odd. However, because $a_* + c_* = n_* \equiv 2 \pmod 4$, without loss of generality, we must have $a_* \equiv 0 \pmod 4$ and $c_* \equiv 2 \pmod 4$. Thus, $a_*^2 \equiv 0 \pmod 8$ while $c_*^2 \equiv 4 \pmod 8$. We also have $b_*^2 \equiv d_*^2 \equiv 1 \pmod 8$. Thus, $PC_*^2 = c_*^2 + b_*^2$ and $PD_*^2 = c_*^2 + d_*^2$ must be 5 mod 8, which cannot be a perfect square, a contradiction. $\square$

Question 3 [Ans: $f(m) \equiv m$]

Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying

$$m!! + n!! \mid f(m)!! + f(n)!!$$

for each $m, n \in \mathbb{Z}^+$, where $n!! = (n!)!$ for all $n \in \mathbb{Z}^+$.

AoPS thread

Solution by DVDthe1st. When $m = n$, we see that $m!! \mid f(m)!!$. This immediately gives us $f(m) \geq m$. Repeatedly chaining the given relation gives

$$m!! + n!! \mid f(m)!! + f(n)!! \mid f^2(m)!! + f^2(n!!) \mid \cdots \mid f^k(m)!! + f^{k+1}(n)!!, \tag{1}$$

where k is a non-negative integer. We now consider the sequence $\left\{m, f(m), f^2(m), \ldots\right\}$.

Case 1. Suppose the sequence is unbounded. Let $k > 0$ be the smallest integer such that $m!! + f(m)!! \mid f^{k+1}(m)!!$ (k must exist due to the unboundedness of the sequence). To not violate the minimality of k, we must have $m!! + f(m)!! \nmid f^k(m)!!$. Thus, $m!! + f(m)!! \nmid f^k(m)!! + f^{k+1}(m)!!$, contradicting (1).

Case 2. Suppose the sequence is bounded. Let $k \geq 0$ be the smallest integer such that $f^k(m) = \max\{m, f(m), f^2(m), \dots\}$. If $k > 0$, we have

$$f^k(m)!! + f^{k+1}(m)!! \leq 2f^k(m)!! < 2\left(f^{k-1}(m)!! + f^k(m)!!\right).$$

However, from (1), we see that

$$f^{k-1}(m)!! + f^k(m)!! \mid f^k(m)!! + f^{k+1}(m).$$

This immediately gives $f^{k-1}(m)!! + f^k(m)!! = f^k(m)!! + f^{k+1}(m)$, whence

$$f^{k-1}(m) = f^{k+1}(m) \geq f^k(m),$$

which violates the minimality of k. Thus, $k = 0$, whence $f(m) \equiv m$ for any initial choice of $m \in \mathbb{Z}^+$, which clearly satisfies the given relation.

Question 4 [Ans: $n - \gcd(n, k)$]

Let n, k, $1 \leq k \leq n$ be fixed integers. Alice has n cards in a row, where the card in position i has the label $i + k$ (or $i + k - n$ if $i + k > n$). Alice starts by colouring each card either red or blue. Afterwards, she is allowed to make several moves, where each move consists of choosing two cards of different colours and swapping them. Find the minimum number of moves she has to make (given that she chooses the colouring optimally) to put the cards in order (i.e. card i is at position i).

AoPS thread

Solution by gghx. We claim that Alice requires a minimum of $n - \gcd(n, k)$ moves.

Consider a graph over vertices labelled 1 through n. If the card at position i has label j, draw a directed edge from i to j. Since each vertex has indegree 1 and outdegree 1, the graph is composed of disjoint cycles. We now consider the effect of swapping two cards (say, at positions i and j) on our graph.

Claim 1. *If the cards at i and j were initially in the same cycle, then the cycle will split into two cycles upon swapping the two cards.*

Proof. Without loss of generality, let the cycle that i and j are in be

$$(i', i, i'', \underbrace{\dots}_{I}, j', j, j'', \underbrace{\dots}_{J}).$$

Consider the effect of swapping i and j on the cycle. Since the label on the card at position i' is still i, we see that i' still maps to i. Similarly, $j' \to j$. However, the card at position j (originally at position i) now has the label i'', hence we now have $j \to i''$. Similarly, $i \to j''$. It is hence easy to see that the cycle now splits as

$$(i', i, j'', \underbrace{\dots}_{J})(j', j, i'', \underbrace{\dots}_{I}).$$

$\square$

Claim 2. *If the cards at i and j were initially in different cycles, then the two cycles will merge upon swapping the two cards.*

Proof. Using a similar argument as Claim 1, the cycles

$$(i', i, i'', \underbrace{\dots}_{I})$$

and

$$(j', j, j'', \underbrace{\dots}_{J})$$

will merge into a single cycle

$$(i', i, j'', \underbrace{\dots}_{J}, j', j, i'', \underbrace{\dots}_{I})$$

upon swapping positions i and j. □

From Claims 1 and 2, it follows that every move, the number of cycles increases by at most one. We now show that the initial number of cycles is $\gcd(n, k)$.

Claim 3. *The number of cycles is initially* $\gcd(n, k)$.

Proof. Let $D = \{d \in \mathbb{Z}_n \mid 1 \leq d \leq \gcd(n, k)\}$. Let $C(m)$ be the cycle starting from some integer m. Then $C(m)$ is clearly of the form

$$(m, m + k, m + 2k, m + 3k, \dots, m + (l - 1)k),$$

where $l = \frac{n}{\gcd(n,k)}$ is the smallest positive integer such that $lk \equiv 0 \pmod{n}$.

Consider $C(m)$ for any choice of m. Since $k \equiv 0 \pmod{\gcd(n, k)}$, it follows that each member of $C(m)$ is congruent to $m \pmod{\gcd(n, k)}$. In addition, from the minimality of l, all elements of $C(m)$ are distinct. Thus, there is exactly one $d \in D$ that is also in $C(m)$ (namely, $d \equiv m \pmod{\gcd(n, k)}$). Conversely, each $d \in D$ has a unique cycle m that it is a member of. Hence, the number of cycles is $\gcd(n, k)$ as desired. □

Since there are n cycles when all cards are in their correct position, Alice must make at least $n - \gcd(n, k)$ moves. We now construct a strategy that guarantees Alice can indeed win in $n - \gcd(n, k)$ moves.

Claim 4. *Alice can win in* $n - \gcd(n, k)$ *moves.*

Proof. Let all cards with labels $1, 2, \dots, \gcd(n, k)$ be red, and all other cards be blue. By Claim 3, each cycle initially contains exactly one red card. For each cycle, keep swapping the red card with the card that is pointing towards it. Doing so removes one blue card every move. Since the cycle has length $\frac{n}{\gcd(n,k)}$, each cycle requires $\frac{n}{\gcd(n,k)} - 1$ moves to completely sort it. Since there are $\gcd(n, k)$ cycles to sort, Alice can sort all cycles within $n - \gcd(n, k)$ moves, as desired. □

As an example, consider the case where $n = 6$ and $k = 4$. We start with the following deck, which has been coloured using the strategy in Claim 4:

5	6	1	2	3	4

The current cycles are $(1, 5, 3)(2, 6, 4)$. We first focus on the $(1, 5, 3)$ cycle. Since card 3 is pointing to card 1, we swap them:

5	6	3	2	1	4

The cycles are now $(1,5)(2,6,4)(3)$. Since card 5 is pointing to card 1, we swap them:

1	6	3	2	5	4

At this point, the $(1,5,3)$ cycle has been completely sorted, taking $\frac{6}{\gcd(6,4)} - 1 = 2$ moves as expected. We now focus on the $(2,6,4)$ cycle. Since card 4 is pointing at card 2, we swap them:

1	6	3	4	5	2

Finally, we swap cards 2 and 6, completing the game in $6 - \gcd(6,4) = 4$ moves as expected.

1	2	3	4	5	6

Question 5

Let $n \geq 2$ be a positive integer. For any integer a, let $P_a(x)$ denote the polynomial $x^n + ax$. Let p be a prime number and define the set S_a as the set of residues mod p that $P_a(x)$ attains. That is,

$$S_a = \{b \mid 0 \leq b \leq p-1, \text{ and there is } c \text{ such that } P_a(x) \equiv p \pmod{p}\}.$$

Show that the expression $\frac{1}{p-1}\sum_{a=1}^{p-1} |S_a|$ is an integer.

AoPS thread

Solution by *Evan Chen*. Note that $0 \in S_a$ for all integers $a \in [1, p-1]$. We thus consider only non-zero elements of S_a. Observe that

$$|S_1 \setminus \{0\}| + |S_2 \setminus \{0\}| + |S_3 \setminus \{0\}| + \cdots + |S_{p-1} \setminus \{0\}|$$

counts the number of elements in

$$\mathcal{T} = \left\{(y,a) \in \mathbb{F}_p^\times \times \mathbb{F}_p^\times \mid \exists x \in \mathbb{F}_p, y = x^n + ax\right\}.$$

Because $y = x^n + ax$ implies that $\lambda^n y = (\lambda x)^n + (\lambda^{n-1}a)(\lambda x)$ for any $\lambda \in \mathbb{F}_p^\times$, we have the equivalence relation

$$(y,a) \sim (\lambda^n y, \lambda^{n-1}a).$$

Now observe that the pairs $(\lambda^n y, \lambda^{n-1}a)$ are pairwise distinct:

$$(\lambda_1^n y, \lambda_1^{n-1}a) = (\lambda_2^n y, \lambda_2^{n-1}a) \implies \lambda_1^n \equiv \lambda_2^n \text{ and } \lambda_1^{n-1} \equiv \lambda_2^{n-1} \implies \lambda_1 = \lambda_2.$$

Since there are $p-1$ choices of λ, each equivalence class generated by the above equivalence relation has exactly $p-1$ elements, whence $|\mathcal{T}|$ is divisible by $p-1$. $\square$

Remark. The condition $n \geq 2$ ensures that we can split λ^n into $\lambda^{n-1} \cdot \lambda$ to form the equivalence relation.

Remark. This question is identical to 2023 Open Round 2 Question 3.

10. 2023 SMO

10.1. Open Section

10.1.1. Round 1 Solutions

Review by *Way Tan*

Question 1 [Ans: 10]

The graph C with equation $y = \dfrac{ax^2 + bx + c}{x+2}$ has an oblique asymptote with equation $y = 4x - 6$ and one of the stationary points at $x = -4$. Find the value of $a + b + c$.

Since C has an oblique asymptote with equation $y = 4x - 6$, its equation can be written as

$$C : y = 4x - 6 + \frac{d}{x+2}$$

for some $d \in \mathbb{R}$. Multiplying throughout by $x+2$ and comparing coefficients, we get $a = 4$, $b = 2$ and $c = -12+d$. Differentiating and using the fact that $\mathrm{d}y/\mathrm{d}x = 0$ at $x = -4$ yields $d = 16$. Thus, $a + b + c = 10$.

Question 2 [Ans: 2000]

If $x = \dfrac{1}{1} + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + \dfrac{1}{1+2+3+4} + \cdots + \dfrac{1}{1+2+3+\cdots+100}$, find the value of $\lfloor 1010x \rfloor$.

Observe that the denominators are the triangular numbers. It is well known that the nth triangular number is given by $n(n+1)/2$. Thus, each term of x is of the form $2/[n(n+1)]$, which can be written as $2/n - 2/(n+1)$ via partial fraction decomposition. x is hence a telescoping sum, which evaluates to $2 - 2/101$, whence $1010x = 2020 - 20 = 2000$.

Question 3 [Ans: 7]

The set of all possible values of x for which the sum of the infinite series

$$1 + \frac{1}{6}\left(x^2 - 5x\right) + \frac{1}{6^2}\left(x^2 - 5x\right)^2 + \frac{1}{6^3}\left(x^2 - 5x\right)^3 + \cdots$$

exists can be expressed as $(a, b) \cup (c, d)$, where $a < b < c < d$. Find $d - a$.

Observe that the common ratio of the given infinite series is $r = (x^2 - 5x)/6$. For the sum to exist, $|r| < 1$, whence $-6 < x^2 - 5x < 6$. Since we are interested in the extreme values of x, we consider only the equality case. We thus obtain $x^2 - 5x = -6$ or $x^2 - 5x = 6$, which clearly has solutions $x = -1, 2, 3, 6$, giving $d - a = 6 - (-1) = 7$.

Question 4 [Ans: 2]

Find the value of $\lfloor y \rfloor$, where $y = \sum_{k=0}^{\infty}(2k+1)(0.5)^{2k}$.
(Hint: Consider the series expansion of $(1-x)^{-2}$*)*

Note that

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k.$$

Differentiating yields

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} kx^{k-1} = x^{-1}\sum_{k=0}^{\infty} kx^k.$$

Now observe y can be rewritten in terms of the above series:

$$y = 2\sum_{k=0}^{\infty} k(1/4)^k + \sum_{k=0}^{\infty}(1/4)^k,$$

from which it clearly follows that

$$\lfloor y \rfloor = \left\lfloor \frac{2 \cdot 1/4}{(1-1/4)^2} + \frac{1}{1-1/4} \right\rfloor = 2.$$

Question 5 [Ans: 3]

The solution of the inequality $|x-1|+|x+1| < ax+b$ is $-1 < x < 2$. Find the value of $\lfloor a+b \rfloor$.

At the extreme ends of the solution interval, equality is achieved. This yields $2 = -a+b$ and $4 = 2a+b$ upon substituting $x = -1$ and $x = 2$ into the two expressions. Solving, we get $a = 2/3$ and $b = 8/3$, whence $\lfloor a+b \rfloor = 3$.

Question 6 [Ans: 32]

The equation $x^4 - 4x^2 + qx - r = 0$ has three equal roots. Find the value of $\left\lfloor \frac{3q^2}{r^2} \right\rfloor$.

Let α be the root of multiplicity 3 and β be the remaining root. By Vieta's formulas, we have the following system of equations:

$$\begin{cases} -r = \alpha^3\beta \\ -q = \alpha^3 + 3\alpha^2\beta \\ -4 = 3\alpha^2 + 3\alpha\beta \\ \quad 0 = 3\alpha + \beta \end{cases}$$

From the third and fourth equations, we have $\alpha\beta = -2$. From the second and third equations, we have $-4\alpha + q = 2\alpha^3$. However, $q = -\alpha^3 - 3\alpha(\alpha\beta) = -\alpha^3 + 6\alpha$. Combining equations gives $3\alpha^3 - 2\alpha = 0$, whence $\alpha = \sqrt{2/3}$, since α is clearly non-zero. Thus,

$$\begin{aligned} \left\lfloor \frac{3q^2}{r^2} \right\rfloor &= \left\lfloor \frac{3(-\alpha^3+6\alpha)^2}{(-2\alpha^2)^2} \right\rfloor = \left\lfloor \frac{3(\alpha^6 - 12\alpha^4 + 36\alpha^2)}{4\alpha^4} \right\rfloor \\ &= \left\lfloor \frac{3((2/3)^3 - 12(2/3)^2 + 36(2/3))}{4(2/3)^2} \right\rfloor = 32 \end{aligned}$$

Question 7 [Ans: 8]

The parabolas $y = x^2 - 16x + 50$ and $x = y^2$ intersect at 4 distinct points which lie on a circle centred at (a, b). Find $|a - b|$.

From the first equation, we have $x^2 - 17x + x - y = -50$. By the second equation, this is equivalent to $x^2 - 17x + y^2 - y = -50$. Completing the square, we obtain

$$\left(x - \frac{17}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = -50 + \left(\frac{17}{2}\right)^2 + \left(\frac{1}{2}\right)^2.$$

It is thus clear that $a = 17/2$ and $b = 1/2$, giving $|a - b| = 8$.

Question 8 [Ans: 33]

In the 3-dimensional Euclidean space with origin O and three mutually perpendicular x-, y- and z-axes, two planes $x + y + 3z = 4$ and $2x - z = 6$ intersect at the line $\mathbf{r} \times \begin{pmatrix} -1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$. Find the value of $|a + b + c + d|$.

Solving the Cartesian equations of the two planes simultaneously, we get

$$\mathbf{r} = \frac{1}{7}\begin{pmatrix} 22 - \lambda \\ 7\lambda \\ 2 - 2\lambda \end{pmatrix},$$

where $\lambda \in \mathbb{R}$. Taking the cross product yields

$$\frac{1}{7}\begin{pmatrix} -2a + \lambda(2a + 7b) \\ -2 - 22b + \lambda(2 + b) \\ 22a + \lambda(7 - a) \end{pmatrix} = \begin{pmatrix} -2 \\ c \\ d \end{pmatrix}$$

Since the above equation must hold for all real λ, we immediately get $a = 7$ and $b = -2$. It quickly follows by equating the $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ components of both vectors that $c = 6$ and $d = 22$, giving $|a + b + c + d| = 33$.

Question 9 [Ans: 200]

Let x, y, z be real numbers with $3x + 4y + 5z = 100$. Find the minimum value of $x^2 + y^2 + z^2$.

Observe that $3x + 4y + 5z = 100$ describes a plane π in 3-dimensional Euclidean space with vector equation

$$\pi : \mathbf{r} \cdot \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix} = 100.$$

Observe also that $\min(x^2 + y^2 + z^2)$ is the square of the perpendicular distance between the origin and π. Applying the standard formula for perpendicular distance between a plane and a point, one gets

$$\min(x^2 + y^2 + z^2) = \left(\frac{100}{\sqrt{3^2 + 4^2 + 5^2}}\right)^2 = 200.$$

Question 10 [Ans: 2]

Find the area of the region represented by the equation $\lfloor x \rfloor + \lfloor y \rfloor = 1$ in the region $0 \le x < 2$.
(Note: If you think that there is no area defined by the graph, enter "0"; if you think that the area is infinite, enter "9999".)

When $x \in [0,1)$, we have $\lfloor x \rfloor = 0$. Thus, $\lfloor y \rfloor = 1$, giving $y \in [1,2)$. This is a square of area 1. Similarly, when $x \in [1,2)$, we have $\lfloor x \rfloor = 1$. Thus, $\lfloor y \rfloor = 0$, giving $y \in [0,1)$. This is another square of area 1. Hence, the total area of the region is 2.

Question 11 [Ans: 1011]

Let ABC be a triangle satisfying the following conditions that $\angle A + \angle C = 2\angle B$, and $\dfrac{1}{\cos A} + \dfrac{1}{\cos C} = \dfrac{-\sqrt{2}}{\cos B}$. Determine the value of $\dfrac{2022\cos\left(\frac{A-C}{2}\right)}{\sqrt{2}}$.

Solution 1. Note that $\angle A + \angle B + \angle C = 180°$, whence $\angle B = 60°$. Clearing denominators in the given equation, we have

$$\cos A + \cos B = -2\sqrt{2}\cos A \cos C.$$

Without loss of generality, let $\angle A = 60° + \theta$ and $\angle C = 60° - \theta$. We now aim to find $\cos\left(\frac{A-C}{2}\right) = \cos\theta$. We have

$$\cos(60° + \theta) + \cos(60° - \theta) = -2\sqrt{2}\cos(60° + \theta)\cos(60° - \theta).$$

Expanding using cosine identities yields

$$4\sqrt{2}\cos^2\theta + 2\cos\theta - 3\sqrt{2} = 0,$$

which has the unique solution $\cos\theta = 1/\sqrt{2}$ (keeping in mind $|\cos\theta| \le 1$). The desired expression thus evaluates to

$$\frac{2022\cos\left(\frac{A-C}{2}\right)}{\sqrt{2}} = \frac{2022/\sqrt{2}}{\sqrt{2}} = 1011.$$

Solution 2. *(Abusing integers)* In order for $\frac{2022\cos\left(\frac{A-C}{2}\right)}{\sqrt{2}}$ to be an integer, we need $\cos\left(\frac{A-C}{2}\right)$ to be of the form $k\sqrt{2}$, where k is a positive rational such that $2022k$ is an integer. It is exceedingly likely that $\frac{2022\cos\left(\frac{A-C}{2}\right)}{\sqrt{2}} = \frac{\sqrt{2}}{2}$, as it is a special value of the cosine function. The required answer is thus $\frac{2022\cdot\sqrt{2}/2}{\sqrt{2}} = 1011$.

Question 12 [Ans: 2020]

Find x which satisfies the following equation

$$\frac{x-2019}{1} + \frac{x-2018}{2} + \frac{x-2017}{3} + \cdots + \frac{x+2}{2022} + \frac{x+3}{2023} = 2023.$$

By inspection, 2020 is clearly a solution, as there are 2023 terms and each term evaluates to 1, giving a sum of 2023 as desired.

Question 13 [Ans: 229]

Assume that x is a positive number such that $x - \frac{1}{x} = 3$ and

$$\frac{x^{10}+x^8+x^2+1}{x^{10}+x^6+x^4+1} = \frac{m}{n},$$

where m and n are positive integers without common factors larger than 1. Determine the value of $m+n$.

Observe that

$$\frac{x^{10}+x^8+x^2+1}{x^{10}+x^6+x^4+1} = \frac{x^8+1}{x^4+1}\cdot\frac{x^2+1}{x^6+1} = \frac{x^4\left(x^4+x^{-4}\right)}{x^2\left(x^2+x^{-2}\right)}\cdot\frac{x\left(x+x^{-1}\right)}{x^3\left(x^3+x^{-3}\right)} = \frac{x^4+x^{-4}}{x^2+x^{-2}}\cdot\frac{x+x^{-1}}{x^3+x^{-3}}$$

Repeatedly squaring $x - x^{-1} = 3$ yields

$$x^2+x^{-2} = 3^2+2 = 11,$$
$$x^4+x^{-4} = 11^2-2 = 119.$$

Now observe that

$$x^3+x^{-3} = \left(x+x^{-1}\right)\left(x^2+x^{-2}-1\right) = 10\left(x+x^{-1}\right).$$

The given expression thus evaluates to

$$\frac{x^4+x^{-4}}{x^2+x^{-2}}\cdot\frac{x+x^{-1}}{x^3+x^{-3}} = \frac{119}{11}\cdot\frac{1}{10} = \frac{119}{110},$$

whence $m+n = 119+110 = 229$.

Question 14 [Ans: 24]

Consider the set of all possible pairs (x, y) of real numbers that satisfy $(x-4)^2+(y-3)^2 = 9$. If S is the largest possible value of $\dfrac{y}{x}$, find the value of $\lfloor 7S \rfloor$.

Observe that $(x-4)^2+(y-3)^2 = 9$ describes a circle with centre $(4, 3)$ and radius 3. Also observe that y/x is the gradient of the line passing through the origin and some point on the circle with coordinates (x, y). The largest possible value of y/x hence occurs when the line in question is tangent to the circle. Consider the simultaneous equations $(x-4)^2+(y-3)^2 = 9$ and $S = y/x$. This combines to give $(x-4)^2+(Sx-3)^2 = 9$. Expanding, we have $(1+S^2)x^2-(8+6S)x+16 = 0$. Since the line is tangent to the circle, there is only one solution. The discriminant of the above quadratic is hence 0, giving $(8+6S)^2-4(1+S^2)(16) = 0$. Solving, we get $S = 24/7$, whence $\lfloor 7S \rfloor = 24$.

Question 15 [Ans: 46]

Let x, y be positive integers with $16x^2+y^2+7xy \le 2023$. Find the maximum value of $4x+y$.

Let k be the maximum value of $4x+y$ without the restriction of x and y being integers. Then the line $4x+y = k$ is tangent to the elliptical region given by $16x^2+y^2+7xy \le 2023$. Equating the two gives

$$16x^2+(k-4x)^2+7x(k-4x) = 2023 \implies 4x^2-kx+(k^2-2023) = 0.$$

Setting the discriminant to 0, we get $k^2-16(k^2-2023) = 0$, whence $k^2 = 2023\cdot 16/15$. Reinstating the integral restriction on x and y, we get $k = 46$, which can indeed be achieved (e.g. $x = 5$, $y = 26$).

Question 16 [Ans: 25]

Let x be the largest real number such that

$$\sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}} = x.$$

Determine the value of $(2x-1)^4$.

Clearing denominators, we have

$$\sqrt{x^2-1} + \sqrt{x-1} = x^{3/2} \implies \sqrt{x^2-1} = x^{3/2} - \sqrt{x-1}.$$

Squaring both sides yields

$$x^2 - 1 = x^3 + x - 1 - 2x^{3/2}\sqrt{x-1} \implies 2x^{1/2}\sqrt{x-1} = x^2 - x + 1.$$

Squaring once again yields

$$4x(x-1) = x^4 - 2x^3 + 3x^2 - 2x + 1 \implies x^4 - 2x^3 - x^2 + 2x + 1 = 0.$$

Now note that

$$\begin{aligned}(2x-1)^4 &= 16x^4 - 32x^3 + 24x^2 - 8x + 1 \\ &= 16(x^4 - 2x^3 - x^2 + 2x + 1) + 10(4x^2 - 4x + 1) + 25 \\ &= 10(2x-1)^2 - 25\end{aligned}$$

whence $(2x-1)^2 = 5$ and thus $(2x-1)^4 = 25$.

Question 17 [Ans: 64]

Two positive integers m and n differ by 10 and the digits in the decimal representation of mn are all equal to 9. Determine $m+n$.

By inspection, $999 = 27 \cdot 37$. Thus, $m + n = 27 + 37 = 64$.

Question 18 [Ans: 1]

Let $\{a_n\}$ be a sequence of positive numbers, and let $S_n = a_1 + a_2 + a_3 + \cdots + a_n$. For any positive integer n, let $b_n = \dfrac{1}{2}\left(\dfrac{a_{n+1}}{a_n} + \dfrac{a_n}{a_{n+1}}\right)$. Given that $\dfrac{a_n + 2}{2} = \sqrt{2S_n}$ holds for all positive integers n, determine the limit $\lim_{n\to\infty}(b_1 + b_2 + \cdots + b_n - n)$.

We claim that $a_n = 4n - 2$. When $n = 1$, $S_1 = a_1$, whence it is clear that

$$\frac{a_1 + 2}{2} = \sqrt{2a_1} \implies a_1 = 2,$$

satisfying our claim. Now assume that $a_k = 4k - 2$ for some $k \in \mathbb{N}$. We have

$$S_k = \sum_{n=1}^{k}(4n-2) = 2k^2.$$

From the given condition,

$$\frac{a_{k+1} + 2}{2} = \sqrt{2\left(2k^2 + a_{k+1}\right)} \implies a_{k+1}^2 - 4a_{k+1} + 4 - 16k^2 = 0,$$

which has the unique positive solution $a_{k+1} = 4k + 2 = 4(k+1) - 2$. This closes the induction.

b_n thus simplifies to

$$b_n = \frac{1}{2}\left(\frac{a_{n+1}}{a_n} + \frac{a_n}{a_{n+1}}\right) = \frac{1}{2} \cdot \frac{a_{n+1}^2 + a_n^2}{a_n a_{n+1}} = \frac{4n^2+1}{4n^2-1} = 1 + \frac{1}{2n-1} - \frac{1}{2n+1}.$$

The limit in question hence telescopes to 1.

Question 19 [Ans: 20]

Let ABC be a triangle with $AB = c$, $AC = b$ and $BC = a$, and satisfies the conditions $\tan C = \dfrac{\sin A + \sin B}{\cos A + \cos B}$, $\sin(B - A) = \cos C$ and that the area of triangle $ABC = 3 + \sqrt{3}$. Determine the value of $a^2 + c^2$.

By the sum-to-product formulae, we have

$$\tan C = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)} = \tan\left(\tfrac{A+B}{2}\right).$$

Hence, $C = (A + B)/2$, implying that $\angle C = 60°$. Let $\angle A = 60° - \theta$ and $\angle B = 60° + \theta$. Then $\sin 2\theta = \cos C = 1/2$, whence $\theta = 15°$, thus $\angle A = 45°$ and $\angle C = 75°$. We now express the area of the triangle in terms of the side lengths:

$$3 + \sqrt{3} = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B.$$

It quickly follows that

$$\frac{a}{c} = \sqrt{\frac{2}{3}}, \quad ac = \frac{2(3+\sqrt{3})}{\sin B}.$$

This immediately gives us a^2 and c^2 in terms of $\sin B$:

$$a^2 = \sqrt{\frac{2}{3}} \cdot \frac{2(3+\sqrt{3})}{\sin B}, \quad c^2 = \sqrt{\frac{3}{2}} \cdot \frac{2(3+\sqrt{3})}{\sin B}.$$

Since $\sin B = \sin(30° + 45°) = \sqrt{2}(1+\sqrt{3})/4$, we finally get

$$a^2 + c^2 = \frac{2(3+\sqrt{3})}{\sqrt{2}(1+\sqrt{3})/4}\left(\sqrt{\frac{2}{3}} + \sqrt{\frac{3}{2}}\right) = 20.$$

Question 20

[VOID] Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(0) = 4$ and that

$$g(xy+1) = g(x)g(y) - g(y) - x + 2023.$$

Find the value of $g(2023)$.

We will show that there are two different expressions for $g(x)$ that result in different answers to $g(2023)$.

Firstly, let $x = 0$. Then

$$g(1) = g(0)g(y) - g(y) - 0 + 2023 \implies g(y) = \frac{g(1) - 2023}{3},$$

which is constant. Letting $y = 1$, we get $g(1) = -2023/2$, whence $g(2023) = -2023/2$.

Secondly, let $y = 0$. Then

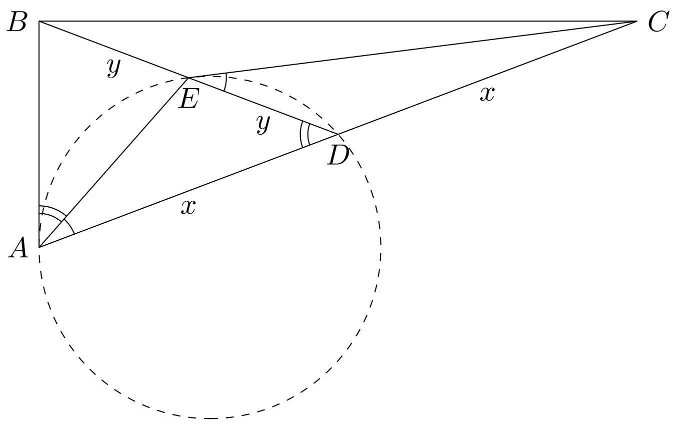
$$g(1) = g(x)g(0) - g(0) - x + 2023 \implies g(x) = \frac{g(1) + x - 2019}{4}.$$

Letting $x = 1$, we get $g(1) = -2018/3$, whence $g(2023) = -1003/6$, a contradiction.

Hence, g does not exist, and the question is void.

Question 21 [Ans: 7]

In the triangle ABC, D is the midpoint of AC, E is the midpoint of BD, and the lines BA and CE are tangent to the circumcircle of the triangle ADE at A and E respectively. Suppose the circumradius of the triangle AED is $(\frac{64}{7})^{1/4}$. Find the area of the triangle ABC.



Let $AD = DC = x$ and $BE = ED = y$.

By the power of a point theorem, we have $BA^2 = BE \cdot BD$ and $CE^2 = CD \cdot CA$, whence

$$BA = \sqrt{2}y, \quad CE = \sqrt{2}x.$$

By the alternate segment theorem, we have $\angle CED = \angle EAD$ and $\angle BAE = \angle BDA$. Hence, $\triangle BAE$ is similar to $\triangle BDA$, and $\triangle CED$ is similar to $\triangle CAE$. Thus,

$$\frac{BE}{BA} = \frac{AE}{DE} \implies AE = \frac{x}{\sqrt{2}}, \quad \frac{CD}{CE} = \frac{ED}{AE} \implies AE = \sqrt{2}y.$$

It follows that $x = 2y$. Using the cosine rule on $\triangle ADE$, we obtain $\cos\angle ADE = 3/4$, whence $\sin\angle ADE = \sqrt{7}/4$. By the extended sine rule,

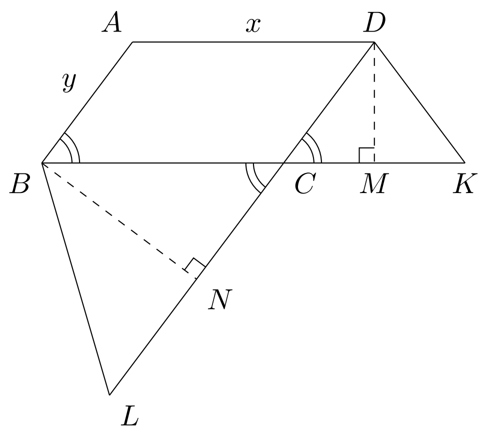
$$\frac{\sqrt{2}y}{\sqrt{7}/4} = 2\left(\frac{64}{7}\right)^{1/4} \implies y = 7^{1/4}.$$

We finally get

$$[ABC] = 2[ABD] = 4[AED] = 4\left(\frac{1}{2} \cdot 2y \cdot y \cdot \sin\angle ADE\right) = 7.$$

Question 22 [Ans: 48]

$ABCD$ is a parallelogram such that $\angle ABC < 90°$ and $\sin\angle ABC = \frac{4}{5}$. The point K is on the extension of BC such that $DC = DK$; the point L is on the extension of DC such that $BC = BL$. The bisector of $\angle CDK$ intersects the bisector of $\angle LBC$ at Q. Suppose the circumradius of the triangle ABD is 25. Find the length of KL.



Let x and y be the lengths AD and BC respectively. Let M be the intersection between the bisector of $\angle CDK$ and BK; let N be the intersection between the bisector of $\angle LBC$ and DL.

Observe that

$$[\triangle ABD] = \frac{xy \cdot BD}{4 \cdot 25} = \frac{1}{2}xy \sin\angle ABC \implies BD = 40.$$

Note that $\cos\angle ABC = 3/5$ and $\cos\angle BCD = -3/5$ (since $\angle BCD = 180° - \angle ABC$). Using the cosine rule on $\triangle BCD$, we obtain

$$x^2 + y^2 + \frac{6}{5}xy = 40^2.$$

Since $\angle BNC = \angle DMC = 90°$, and $\angle BCN = \angle DCM = \angle ABC$, by AAA, we have that $\triangle BNC$ is similar to $\triangle DMC$, $\triangle BNC \equiv \triangle BNL$, $\triangle BMC \equiv \triangle BMK$, and that

$$BN = \frac{4}{5}x, \quad NC = \frac{3}{5}x, \quad DM = \frac{4}{5}y, \quad MC = \frac{3}{5}y.$$

It also follows that $BDMN$ is cyclic. Applying Ptolemy's theorem, we have

$$KL \cdot 40 + xy = \left(x + \frac{6}{5}y\right)\left(y + \frac{6}{5}x\right).$$

Expanding, we finally have

$$KL = \frac{1}{40} \cdot \frac{6}{5}\left(x^2 + y^2 + \frac{6}{5}xy\right) = 48.$$

Question 23 [Ans: 7]

A group of 200 monkeys is given the task of picking up all 3000 peanuts on the ground. Determine the maximum number k such that there must be k monkeys picking up the same number of peanuts. (It is possible that some lazy monkeys may not pick up any peanuts at all).

Consider the worst-case scenario, where there are n monkeys picking 0 peanuts, n monkeys picking 1 peanut, etc. The total number of peanuts picked can be calculated as

$$\underbrace{n\left(0+1+2+\cdots+\left(\left\lfloor\frac{200}{n}\right\rfloor-1\right)\right)}_{n\lfloor 200/n\rfloor \text{ monkeys}}+\underbrace{\left\lfloor\frac{200}{n}\right\rfloor+\left(\left\lfloor\frac{200}{k}\right\rfloor+1\right)+\cdots}_{\text{remaining monkeys}}.$$

The smallest value of n where the above expression is less than 3000 is 7. Hence, $k=7$.

Question 24 [Ans: 7]

A chain of n identical circles $C_1, C_2, \ldots, C_n$ of equal radii and centres on the x-axis lie inside the ellipse $E: \frac{x^2}{2023}+\frac{y^2}{333}=1$ such that C_1 is tangent to E internally at $(-\sqrt{2023}, 0)$, C_n is tangent to E internally at $(\sqrt{2023}, 0)$, and C_i is tangent to C_{i+1} externally for $i=1, \ldots, n-1$. Determine the smallest possible value of n.

The curvature κ of an ellipse $x^2/a^2+y^2/b^2=1$ is given by

$$\kappa(\theta)=\frac{ab}{\left(a^2\sin^2\theta+b^2\cos^2\theta\right)^{3/2}}.$$

Taking $a^2=2023$, $b^2=333$ and $\theta=0$, we have that the curvature of E at $(\sqrt{2023}, 0)$ is $\sqrt{2023}/333$. The maximum radius of C_n is thus $333/\sqrt{2023}$ (since $\kappa=1/R$). It immediately follows that

$$\min n=\left\lceil\frac{2\sqrt{2023}}{2\cdot 333/\sqrt{2023}}\right\rceil=7.$$

Question 25 [Ans: 6]

Let $p>2023$ be a prime. Determine the number of positive integers n such that $(n-p)^2+2023(2023-2n-2p)$ is a perfect square.

Observe that the given expression is nearly a perfect square. Indeed, it can be rewritten as $k^2=(n-p-2023)^2-4\cdot 2023p$. We hence obtain

$$2^2\cdot 7\cdot 17^2\cdot p=(n-p-2023)^2-k^2=(n-p-2023-k)(n-p-2023+k).$$

Observe that the two terms on the right have the same parity, thus both must have a factor of 2. Furthermore, because $n-p-2023+k>n-p-2023-k$, it must be that $n-p-2023+k$ has the factor of p (since $p>2023$). Since there are 3 remaining factors, there are hence ${}^3P_2=6$ possible n.

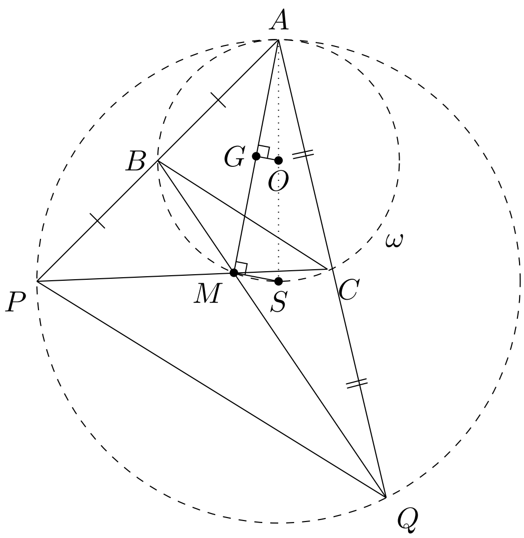
10.1.2. Round 2 Solutions

Review by *Glen Lim*

Question 1

In a scalene triangle ABC with centroid G and circumcircle ω centred at O, the extension of AG meets ω at M; lines AB and CM intersect at P; and lines AC and BM intersect at Q. Suppose the circumcentre S of the triangle APQ lies on ω and A, O, S are collinear. Prove that $\angle AGO = 90°$.

AoPS thread



Consider the homothety H centred at A that sends O to S (i.e. H has a scale factor of $\frac{AS}{AO} = 2$). Since homotheties preserve circumcircles and circumcentres, it follows that H sends $\triangle ABC$ to $\triangle APQ$. This means that B and C are the midpoints of AP and AQ respectively, whence M is the centroid of $\triangle APQ$. Because AS is a diameter of ω, we have $\angle AMS = 90°$. However, because homotheties preserve centroids and angles, we immediately get $\angle AGO = \angle AMS = 90°$ as desired.

Question 2 [Ans: Yes]

A grid of cells is tiled with dominoes such that every cell is covered by exactly one domino. A subset S of dominoes is chosen. Is it true that at least one of the following two statements is false?

1. There are 2022 more horizontal dominoes than vertical dominoes in S.
2. The cells covered by the dominoes in S can be tiled completely and exactly by L-shaped tetrominoes.

AoPS thread

Solution by `bxiao31415`**.** Let (i, j) be coloured in the following manner:

- 0 if both i and j are even;
- 1 if i is odd and j is even;

- 2 if i is even and j is odd; and
- 3 if both i and j are odd.

As an example, the following 4×4 grid shows the colouring scheme:

0	1	0	1
3	2	3	2
0	1	0	1
3	2	3	2

Seeking a contradiction, suppose both (1) and (2) are true. Observe that each horizontal domino covers squares whose sum is 1 mod 4, while each vertical domino covers squares whose sum is -1 mod 4. Hence, the total sum covered by S is $2022 \equiv 2$ mod 4. However, each L-shape tetromino (which contains one horizontal and one vertical domino) covers squares who sum is 0 mod 4, a contradiction. Thus, at least one of the statements is false.

Question 3

Let $n \geq 2$ be a positive integer. For a positive integer a, let $Q_a(x) = x^n + ax$. Let p be a prime and let $S_a = \{b \mid 0 \leq b \leq p-1, \exists c \in \mathbb{Z}, Q_a(c) \equiv b \pmod p\}$. Show that $\frac{1}{p-1}\sum_{a=1}^{p-1} |S_a|$ is an integer.

See 2022 Open Round 2 Question 5.

Question 4

Find all functions $f : \mathbb{Z} \to \mathbb{Z}$, such that

$$f(x+y)((f(x)-f(y))^2 + f(xy)) = f(x^3) + f(y^3)$$

for all integers x, y.

AoPS thread

Solution by Ld_minh4354. Let $P(x, y)$ be the assertion that $f(x+y)((f(x)-f(y))^2 + f(xy)) = f(x^3) + f(y^3)$. From $P(x, x)$, one has

$$f(2x)f(x^2) = 2f(x^3).$$

$P(0, 0)$ hence gives us $f(0)^2 = 2f(0)$, whence $f(0) = 0$ or $f(0) = 2$. This gives us two main cases:

Case 1. Suppose $f(0) = 0$. Then $P(x, 0)$ gives us

$$P(x, 0) : \qquad f(x)^3 = f(x^3).$$

From $P(1, 1)$, we hence have

$$P(1, 1) : \qquad f(2)f(1) = 0.$$

Thus, $f(1) = 0$ or $f(2) = 0$.

Subcase 1A. Suppose $f(1) = 0$. $P(-1, 1)$ clearly gives $f(-1) = 0$. We now prove inductively that $f(x) \equiv 0$. The base case $x = 0$ has already been assumed. Now suppose that $f(k) = 0$ for some $k \in \mathbb{Z}$. Applying this inductive hypothesis to $P(k, 1)$ and $P(-k, 1)$, we immediately get $f(k+1) = f(k-1) = 0$. This closes the induction. One solution is hence $f(x) \equiv 0$.

Subcase 1B. Suppose $f(2) = 2$. Then we have the following:

$$\begin{aligned} P(x,0):& \qquad f(x)^3 &= f(x^3) \\ P(1,0):& \qquad f(1) &\in \{-1,0,1\} \\ P(-1,0):& \qquad f(-1) &\in \{-1,0,1\} \end{aligned}$$

If $f(1) = 0$, then by Subcase 1A, we would have $f(x) \equiv 0$, contradicting $f(2) = 2$. If $f(-1) = 0$, then by $P(-1,-1)$, we would have $f(-2)f(1) = 0$. However, if $f(-2) = 0$, then by $P(-2,2)$, one gets $0 = 8$, a contradiction. Thus, $f(1), f(-1) \neq 0$.

From $P(1,-1)$, we have $f(1) + f(-1) = 0$. Suppose $f(1) = -1$. By $P(2,1)$, we have $f(3) = \frac{8}{11} \notin \mathbb{Z}$, a contradiction. Thus, $f(1) = 1$ and $f(-1) = -1$. We now show that $f(x) \equiv x$ via induction. The base case has already been settled (namely $x = \{-1,0,1\}$). Now suppose that $f(x) = x$ on $[-k,k]$ for some $k \in \mathbb{Z}$. From $P(k,1)$ and $P(-k,1)$, applying the inductive hypothesis yields

$$\begin{aligned} P(k,1):& \qquad f(k+1)(k^2-k+1) = k^3+1 &\implies f(k+1) = k+1 \\ P(-k,1):& \qquad f(-k-1)(k^2-k+1) = -k^3-1 &\implies f(-k-1) = -k-1 \end{aligned}$$

This closes the induction. We hence have a second solution, namely $f(x) \equiv x$.

Case 2. Suppose $f(0) = 2$. From $P(x,0)$ one gets

$$P(x,0): \qquad f(x)(f(x)^2 - 4f(x) + 6) = f(x^3) + 2. \tag{1}$$

Taking $P(1,0)$ hence gives us a cubic in $f(1)$:

$$P(1,0): \qquad f(1)^3 - 4f(1)^2 + 5f(1) - 2 = 0.$$

We thus have $f(1) = 1$ or $f(1) = 2$.

Subcase 2A. Suppose $f(1) = 1$. Taking $P(-1,1)$, one has

$$P(-1,1): \qquad 2f(-1)^2 - 3f(-1) + 1 = 0,$$

whence $f(-1) = 1$. Note that $f(-1)$ is an integer and hence cannot be $\frac{1}{2}$. We now show via induction that $f(x) = 1$ when x is odd, and $f(x) = 2$ when x is even. The base cases (namely $x \in \{-1,0,1\}$) have already been settled. Let k be some integer. Suppose that $f(k) = 1$ when k is odd and $f(k) = 2$ when k is even. From $P(k,1)$, we have

$$P(k,1): \qquad f(k+1) = \frac{f(k^3)+1}{f(k^2) - f(k) + 1},$$

from which it follows that $f(k+1) = 2$ when $k+1$ is even, and $f(k+1) = 1$ when $k+1$ is odd. Also, from $P(k,-1)$, we have

$$P(k,-1): \qquad f(k-1) = \frac{f(k^3)+1}{f(k)^2 - 2f(k) + f(-k) + 1},$$

from which it follows that $f(k-1) = 2$ when $k-1$ is even, and $f(k-1) = 1$ when $k-1$ is odd. This closes the induction. We hence obtain a third solution, namely

$$f(x) \equiv \begin{cases} 1, & x \text{ odd} \\ 2, & x \text{ even} \end{cases}.$$

Subcase 2B. Suppose $f(1) = 2$. From $P(1, -1)$, we obtain a quadratic in $f(-1)$:

$$P(1, -1): \qquad 2f(-1)^2 - 7f(-1) + 6 = 0,$$

which has solutions 2 and $\frac{3}{2}$. Thus, $f(-1) = 2$ (since $f(-1) \in \mathbb{Z}$). We now show that $f(x) \equiv 2$ via induction. The base cases ($x \in \{-1, 0, 1\}$) have already been settled. Now suppose $f(k) = 2$ for some $k \in \mathbb{Z}$. From $P(k, 1)$ and (1), we get

$$P(k, 1): \qquad f(k+1) = \frac{f(k)(f(k)^2 - 4f(k) + 6)}{f(k)^2 - 3f(k) + 4} = 2.$$

Likewise, $P(k, -1)$ gives

$$P(k, -1): \qquad f(k-1) = \frac{f(k)(f(k)^2 - 4f(k) + 6)}{f(k)^2 - 3f(k) + 4} = 2.$$

This closes the induction. We hence get our fourth and final solution: $f(x) \equiv 2$.

To conclude, the following four functions are the only solutions to the given functional equation:

$$f(x) \equiv 0, \quad f(x) \equiv 2, \quad f(x) \equiv x, \quad f(x) \equiv \begin{cases} 1, & x \text{ odd} \\ 2, & x \text{ even} \end{cases}$$

Question 5 [Ans: $x \in (0, 120]$]

Determine all real numbers x between 0 and 180 such that it is possible to partition an equilateral triangle into finitely many triangles, each of which has an angle of x°.

AoPS thread

Solution by oneplusone**.** We claim that $x \in (0, 120]$. We split our proof into 3 cases:

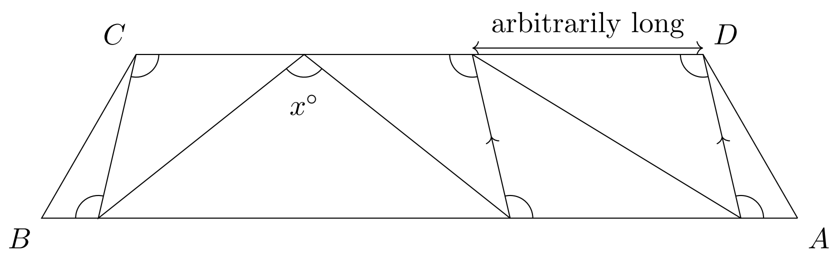
Case 1. Suppose $x > 120$. Let there be a total of n "small triangles" in the partition of the original equilateral triangle. Let A be the set of vertices of the original equilateral triangle. Let B be the set of vertices that lie on an edge. Let C be the remaining vertices. Now observe that angle sum of all n "small triangles" must be $180n^\circ$. Since each vertex in A, B and C contributes 60°, 180° and 360° respectively, we have the equality

$$60 \cdot 3 + 180\,|B| + 360\,|C| = 180n \implies |B| + 2\,|C| = n - 1. \tag{1}$$

Now observe that the vertices in A cannot have an angle x°. On the hand, each vertex in B and C can have at least 1 and 2 such angles respectively. Thus, the total number of x° angles is at least n (by our assumption) and at most $|B| + 2\,|C|$. With (1), we get the contradiction $n \le n - 1$. Thus, $x > 120$ is impossible.

Case 2. Suppose $x = 120^\circ$. This is clearly achievable; Let O be the centre of the equilateral triangle $\triangle ABC$. Then $\triangle AOB$, $\triangle BOC$ and $\triangle COA$ all contain a 120° angle.

Case 3. Suppose $x < 120^\circ$. For $R > 0$, let an R-trapezoid be a trapezoid similar to the trapezium $ABCD$, where $\angle A = \angle D = 60^\circ$ and $\angle B = \angle C = 120^\circ$, with $AB = CD = 1$ and $BC = R$. We call a shape constructible if it can be partitioned into triangles, each of which has an angle x°. Firstly, observe that for R sufficiently large, the R-trapezoid is constructible. This is shown in the figure below:



It thus follows that the 1-trapezoid is constructible: simply slice it horizontally into sufficiently thin R-trapezoids (for sufficiently large R). Since an equilateral triangle can be partitioned into 3 1-trapezoids, it must also be constructible.

Thus, $x \in (0, 120]$.

11. 2024 SMO

11.1. Open Section

11.1.1. Round 1 Solutions

Review by *Way Tan*

Question 1 [Ans: 1540]

Let $S_k = 1+2+3+\cdots+k$ for any positive integer k. Find $S_1+S_2+S_3+\cdots+S_{20}$.

Note that $S_k = \frac{1}{2}\left(k^2+k\right)$. Hence, the required sum is

$$\sum_{k=1}^{20} \frac{1}{2}\left(k^2+k\right) = \frac{1}{2}\left[\frac{20\cdot 21\cdot 41}{6} + \frac{20\cdot 21}{2}\right] = 1540.$$

Question 2 [Ans: 69]

Let $S = \sum_{r=1}^{64} r\binom{64}{r}$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and $0! = 1$. Find $\log_2 S$.

Note that $(1+x)^{64} = \sum_{r=1}^{64}\binom{64}{r}x^r$. Differentiating with respect to x gives $64(1+x)^{63} = \sum_{r=1}^{64} r\binom{64}{r}x^{r-1}$. Evaluating at $x=1$, we have $64\cdot 2^{63} = \sum_{r=1}^{64} r\binom{64}{r} = S$. Hence, $\log_2 S = 69$.

Question 3 [Ans: 4]

Let x be the largest number in the interval $[0, 2\pi]$ such that $(\sin x)^{2024} - (\cos x)^{2024} = 1$. Find $\lfloor x \rfloor$.

(Note: If you think that such a number x does not exist, enter your answer "99999".)

Observe that $(\sin x)^{2024}, (\cos x)^{2024} \in [0,1]$. Hence, the equation holds if and only if $(\sin x)^{2024} = 1$ and $(\cos x)^{2024} = 0$. Thus, $\sin x = \pm 1$ and $\cos x = 0$, whence $x = \frac{3\pi}{2}$ and $\lfloor x \rfloor = 4$.

Question 4 [Ans: 99999]

Find the number of real numbers x that satisfies the equation $|x-2|+|x-3| = |2x-5|$.

(Note: If you think that there are no such numbers, enter "0"; if you think that there are infinitely many such numbers, enter "99999".)

Consider $x < 2$. The given equation simplifies to $(2-x)+(3-x) = 5-2x$, whence x is free. There are hence infinitely many solutions.

Question 5 [Ans: 3]

Among all the real numbers that satisfies the inequality $e^x \geq 1 + 2e^{-x}$, find the minimum value of $\lceil e^x + e^{-x} \rceil$.

Multiplying through by e^x yields a quadratic in e^x: $(e^x)^2 - e^x - 2 \geq 0$. Thus, $e^x \geq 2$ (keeping in mind that $e^x > 0$). Since $e^x + e^{-x}$ is increasing for $x > 0$, the minimum value occurs when $e^x = 2$, whence the desired answer is $\lceil 2 + 1/2 \rceil = 3$.

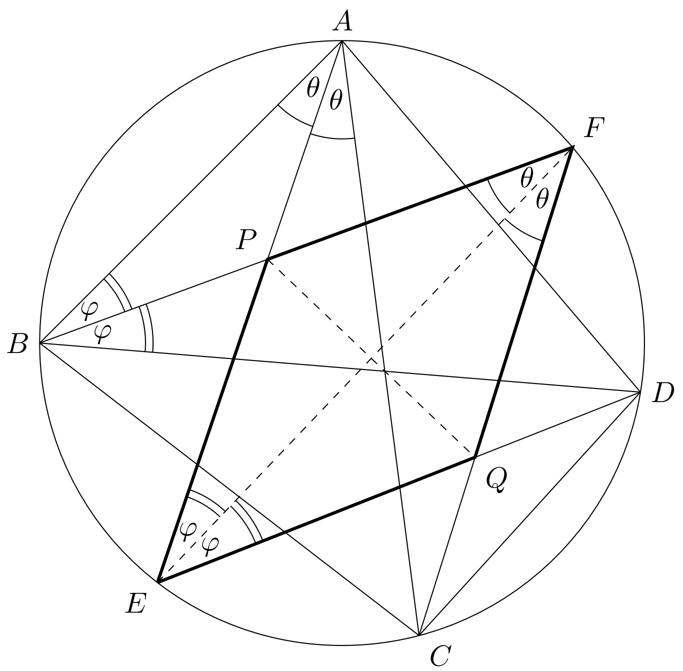
Question 6 [Ans: 2025]

Find the smallest positive integer C greater than 2024 such that the sets $A = \left\{2x^2 + 2x + C : x \in \mathbb{Z}\right\}$ and $B = \left\{x^2 + 2024x + 2 : x \in \mathbb{Z}\right\}$ are disjoint.

Observe that $x^2 + 2024x + 2 \equiv 2, 3 \pmod 4$, while $2x^2 + 2x + C \equiv C \pmod 4$. Thus, so long as $C \not\equiv 2, 3 \pmod 4$, we will have A and B disjoint. The smallest such C is hence 2025, which has a residue of 1 modulo 4.

Question 7 [Ans: 110]

Let $ABCD$ be a convex quadrilateral inscribed in a circle ω. The bisector of $\angle BAC$ meets ω at E ($\neq A$), the bisector of $\angle ABD$ meets ω at F ($\neq B$), AE intersects BF at P and CF intersects DE at Q. Suppose $EF = 20$, $PQ = 11$. Find the area of the quadrilateral $PEQF$.



Let $\angle BAC = 2\theta$ and $\angle ABD = 2\varphi$. Using angles in same segment on $ABEF$, we have $\angle BFE = \angle BAE = \theta$, and $\angle AEF = \angle ABF = \varphi$. Using angles in same segment on $AECF$, we have $\angle CFE = \angle CAE = \theta$. Using angles in same segment on $BEDF$, we have $\angle FBD = \angle FED = \varphi$. By ASA, $\triangle PEF \equiv \triangle QEF$, whence $PEQF$ is a kite. Hence, $[PEQF] = \frac{1}{2} \cdot EF \cdot PQ = 110$.

Question 8 [Ans: 5]

Let $f(x) = \sqrt{x^2+1} + \sqrt{(4-x)^2+4}$. Find the minimum value of $f(x)$.

Let $A(0, \pm 1)$, $B(x, 0)$ and $C(4, \pm 2)$. We can interpret $\sqrt{x^2+1}$ and $\sqrt{(4-x)^2+4}$ geometrically as AB and BC respectively. It follows that $f(x)$, which is $AB + BC$, attains its minimum when ABC is a straight line (crucially, B must be in between A and C). To achieve this, we can set $A(0, -1)$ and $C(4, 2)$. Thus, $\min f(x) = AC = 5$.

Question 9 [Ans: 17]

It is known that $a \geq 0$ satisfies $\sqrt{4+\sqrt{4+\sqrt{4+\sqrt{4+a}}}} = a$. Find the value of $(2a-1)^2$.

Observe that $\sqrt{4+a} = a$, whence $a^2 - a - 4 = 0$. Thus,

$$(2a-1)^2 = 4\left(a^2 - a - 4\right) + 17 = 17.$$

Question 10 [Ans: 160]

A rectangle with sides parallel to the horizontal and vertical axes is inscribed in the region bounded by the graph of $y = 60 - x^2$ and the x-axis. If the area of the largest such rectangle has area $k\sqrt{5}$, find the value of k.

Let A be the area of the rectangle. Let the rectangle have width $2x$. Then $A = 2x\left(60 - x^2\right) = 120x - 2x^3$. Hence, $\mathrm{d}A/\mathrm{d}x = 120 - 6x^2$. The sole stationary point of A (which can easily be verified as a maximum) hence occurs when $x = 2\sqrt{5}$. The area of the rectangle is thus $160\sqrt{5}$, whence $k = 160$.

Question 11 [Ans: 10]

Let x be a real number satisfying the equation $x^{x^5} = 100$. Find the value of $\lfloor x^5 \rfloor$.

Raising the given equation to the 5th power yields $\left(x^5\right)^{x^5} = 10^{10}$, whence $x^5 = 10$.

Question 12 [Ans: 10]

Let a, b, c, d, e be distinct integers with $a+b+c+d+e = 9$. If m is an integer such that

$$(m-a)(m-b)(m-c)(m-d)(m-e) = 2009,$$

determine the value of m.

Note that $2009 = 7^2 \cdot 41$. Since a, b, c, d, e are distinct, the five terms must be 7, -7, 41, 1 and -1. Summing, we get $5m - (a+b+c+d+e) = 41$, whence $m = 10$.

Question 13 [Ans: 3]

Let $\{x\}$ be the fractional part of the number x, i.e., $\{x\} = x - \lfloor x \rfloor$. If $S = \int_0^9 \{x\}^2 \, dx$, find $\lfloor S \rfloor$.

Observe that $\{x\}$ has period 1, and is equivalent to x on the interval $[0, 1)$. Thus,

$$S = \int_0^9 \{x\}^2 \, dx = 9 \int_0^1 x^2 \, dx = 3.$$

Question 14 [Ans: 81]

The solution of the inequality $|(x+1)(x-6)| > |(x+4)(x-2)|$ can be expressed as $x < a$ or $b < x < c$. If $S = |a| + |b| + |c|$, find $\lfloor 14S \rfloor$.

As we are interested in the extreme ends of the solution range, we consider the case of equality, i.e. $|(x+1)(x-6)| = |(x+4)(x-2)|$.

Case 1. Consider $(x+1)(x-6) = (x+4)(x-2)$. Expanding and simplifying, we get $x = 2/7$.

Case 2. Consider $(x+1)(x-6) = -(x+4)(x-2)$. Expanding and simplifying, we get $(2x-7)(x+2) = 0$, whence $x = -2$ and $x = 7/2$.

Together, it stands to reason that $a = -2$, $b = 2/7$ and $c = 7/2$, whence $14S = 81$.

Question 15 [Ans: 2]

Given that $x, y > 0$ and $x\sqrt{2-y^2} + y\sqrt{2-x^2} = 2$, find the value of $x^2 + y^2$.

Solution 1. Let $X = x^2$ and $Y = y^2$. Squaring the given equation, we have

$$X(2-Y) + 2xy\sqrt{(2-X)(2-Y)} + Y(2-X) = 4.$$

This gives

$$xy\sqrt{(2-X)(2-Y)} = 2 - (X+Y) + XY.$$

Squaring once more, we obtain

$$XY(2-X)(2-Y) = [2 - (X+Y) + XY]^2.$$

Upon simplification, one gets

$$(X+Y)^2 - 4(X+Y) + 4 = 0,$$

whence $x^2 + y^2 = X + Y = 2$.

Solution 2. *(Abusing uniqueness)* Suppose $x = y$. We thus get $x\sqrt{2-x^2} = 1$, whence $x = y = 1$. Since $x^2 + y^2$ is a constant (by the way the question is asked), we have $x^2 + y^2 = 2$.

Question 16 [Ans: 17]

A convex polygon has n sides such that no three diagonals are concurrent. It is known that all its diagonals divide the polygon into 2500 regions. Determine n.

Let P be a convex polygon with n sides such that no three diagonals are concurrent. By Euler's formula, we have

$$V - E + F = 1,$$

where V, E and F are the number of vertices, edges and faces of P respectively. Note that we disregard the region "outside" P. Observe that any four of the n vertices of P gives a unique vertex in the interior of P. Hence, $V = n + \binom{n}{4}$. Next, observe that each vertex of P has degree $n-1$, while each vertex in the interior of P has degree 4. By the degree sum formula, $2E = n(n-1) + 4\binom{n}{4}$. We hence obtain the following expression for F:

$$F = \binom{n}{4} + \frac{n(n-3)}{2} + 1.$$

Setting $F = 2500$, we see that $n = 17$.

Question 17 [Ans: 4]

Find the number of integers n between -2029 and 2029 inclusive such that $(n+2)^2+n^2$ is divisible by 2029.

Re-expressing the given condition in the language of modular arithmetic, we get $(n+2)^2 + n^2 \equiv 0 \pmod{2029}$. Simplifying, we obtain

$$(n+1)^2 \equiv -1 \pmod{2029}. \tag{1}$$

Now observe that

$$\left(\frac{-1}{2029}\right) = (-1)^{\frac{2029-1}{2}} = 1,$$

where $\left(\frac{a}{p}\right)$ is the Legendre symbol of a and p. We have hence established that -1 is a quadratic residue modulo 2029 (i.e. there exists some integer m such that $m^2 \equiv -1 \pmod{2029}$). There are hence two integer solutions to (1). However, since the solutions are 2029-periodic, we get a total of 4 solutions.

Question 18 [Ans: 2516]

Let f be a function such that for any real number x, we have $f(x)+2f(2-x) = x+x^2$. Find the value of $f(1) + f(2) + f(3) + \cdots + f(34)$.

Let $S = \sum_{n=1}^{34} f(n)$ and $T = \sum_{n=-32}^{1} f(n)$. From the given equation, we have

$$S + 2T = \sum_{n=1}^{34} \left(n + n^2\right). \tag{1}$$

Now consider the transformation $x \mapsto 2-x$. We get $f(2-x) + 2f(x) = (2-x) + (2-x)^2$. Hence,

$$T + 2S = \sum_{n=1}^{34} \left[(2-n) + (2-n)^2\right]. \tag{2}$$

Simultaneously solving (1) and (2) yields

$$S = \frac{1}{3}\sum_{n=1}^{34}\left(2\left[(2-n)+(2-n)^2\right]-(n+n^2)\right) = 2516.$$

Question 19 [Ans: 761]

Find the largest possible positive prime integer p such that p divides

$$S(p) = 1^{p-2}+2^{p-2}+3^{p-2}+4^{p-2}+5^{p-2}+6^{p-2}+7^{p-2}+8^{p-2}.$$

Multiplying $S(p)$ through by $8!$ yields

$$\frac{8!}{1}\cdot 1^{p-1}+\frac{8!}{2}\cdot 2^{p-1}+\cdots+\frac{8!}{8}\cdot 8^{p-1} \equiv 0 \pmod{p}.$$

However, Fermat's little theorem states that $a^{p-1}\equiv 1 \pmod{p}$ for all natural numbers a such that $p \nmid a$. Assuming that $p > 8$, we have that

$$\frac{8!}{1}+\frac{8!}{2}+\cdots+\frac{8!}{8} \equiv 0 \pmod{p}.$$

The LHS works out to be $109584 = 2^4\cdot 3^2\cdot 761$. Hence, the largest possible p is 761.

Question 20 [Ans: 1009]

Let f be a function such that $f(x)+f(\frac{1}{1-x}) = 1+\frac{1}{x}$ for all $x \notin \{0,1\}$. Find the value of $\lfloor 180\cdot f(10)\rfloor$.

Substituting $x = 10$, we get

$$f(10)+f(-\tfrac{1}{9}) = \frac{11}{10}.$$

Substituting $x = -\frac{1}{9}$, we get

$$f(-\tfrac{1}{9})+f(\tfrac{9}{10}) = -8.$$

Substituting $x = \frac{9}{10}$, we get

$$f(\tfrac{9}{10})+f(10) = \frac{19}{9}.$$

Solving the three equations simultaneously, we get $180\cdot f(10) = 1009$.

Question 21 [Ans: 2]

Let C be a circle with equation $(x-a)^2+(y-b)^2 = r^2$, where at least one of the a and b are irrational numbers. Find the maximum possible numbers of points (p,q) on C where both p and q are rational numbers.

Observe that it is possible for a circle with "irrational centre" to have two rational points. For instance, the circle with centre $(0,\sqrt{2})$ and radius 3 contains the rational points $(-1,0)$ and $(1,0)$.

We now show that three or more rational points is possible only if the coordinates of the centre of the circle are rational. Suppose there exists a circle with three rational points (P, Q and R). Then the gradients of chords PQ and QR are rational. Thus, the gradients of the perpendicular bisector of PQ and QR are also rational. It follows that the equation of the perpendicular bisector of PQ and QR have rational coefficients. However, the perpendicular bisectors of any two chords must meet in the centre, implying that the coordinates of the centre are rational. This concludes the proof.

Question 22 [Ans: 24]

On the plane there are 2024 points coloured either red or blue such that each red point is the centre of a circle passing through 3 blue points. Determine the least number of blue points.

Let there be b blue points. Observe that the absolute maximum number of red points is equal to the number of triplets of blue points (each triplet uniquely defines a red point). That is, there are at most $\binom{b}{3}$ red points. We thus require $\binom{b}{3} + b \geq 2024$. The smallest b that satisfies this is $b = 24$.

Question 23 [Ans: 45]

It is given that the positive real numbers $x_1, \ldots, x_{2026}$ satisfy $\dfrac{x_1^2}{x_1^2+1} + \cdots + \dfrac{x_{2026}^2}{x_{2026}^2+1} = 2025$. Find the maximum value of $\dfrac{x_1}{x_1^2+1} + \cdots + \dfrac{x_{2026}}{x_{2026}^2+1}$.

Solution 1. From the given equation, we see that

$$\sum_{n=1}^{2026}\left(1 - \frac{1}{x_n^2+1}\right) = 2025 \implies \sum_{n=1}^{2026}\frac{1}{x_n^2+1} = 1.$$

By the Cauchy-Schwarz inequality, one thus has

$$\left(\sum_{n=1}^{2026}\frac{x_n}{x_n^2+1}\right)^2 \leq \left[\sum_{n=1}^{2026}\left(\frac{x_n}{\sqrt{x_n^2+1}}\right)^2\right]\left[\sum_{n=1}^{2026}\left(\frac{1}{\sqrt{x_n^2+1}}\right)^2\right] = 2025.$$

The maximum is thus 45.

Solution 2. *(Abusing symmetry)* Observe that the given expressions are all symmetric polynomials. Letting $x_1 = x_2 = x_3 = \cdots = x_{2026}$, one gets that $2026 \cdot \frac{x_i^2}{x_i^2+1} = 2025$, whence $x_i^2 + 1 = 2026$ and $x_i = 45$. Thus, the expression in question evaluates to $2026 \cdot \frac{45}{2026} = 45$.

Question 24 [Ans: 441]

Let n denote the number of ways of arranging all the letters of the word MATHEMATICS in one row such that

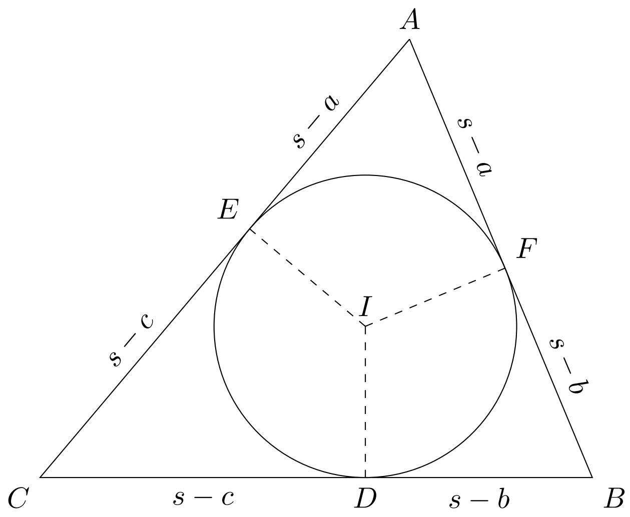
- both M's precede both T's; and
- neither the two M's nor the two T's are next to each other.

Determine the value of $\frac{n}{6!}$.

To ensure that both M's are both T's are not adjacent to each other, we consider the second M and the letter in front of it as one group, and the first T and the letter after it as one group. This gives a total of 9 groups. Since there are 4 groups with M's and T's, we have $\binom{9}{4}$ ways to arrange the M's and T's. Meanwhile, there are $\frac{7!}{2!}$ ways to arrange the remaining letters (note that we divide by 2! to account for the double A). Hence, $n = \binom{9}{4} \cdot \frac{7!}{2!}$, whence $\frac{n}{6!} = 441$.

Question 25 [Ans: 146]

The incircle of the triangle ABC centred at I touches the sides BC, CA, AB at D, E, F respectively. Let D' be the intersection of the extension of ID with the circle through B, I, C; E' be the intersection of the extension of IE with the circle through A, I, C; and F' the intersection of the extension of IF with the circle through A, I, B. Suppose $AB = 52$, $BC = 56$, $CA = 60$. Find $DD' + EE' + FF'$.



It is a well-known property that the points tangents to the incircle divide the triangle into lengths of $s-a$, $s-b$, $s-c$, as shown in the figure above. Here, s is the semiperimeter $\frac{1}{2}(a+b+c)$, $a = BC$, $b = CA$ and $c = AB$.

From Heron's formula, one has $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$. Thus, the inradius r is given by $r = \frac{1}{s}\sqrt{s(s-a)(s-b)(s-c)}$.

We now formulate equations involving DD', EE' and FF'. Invoking the power of a point theorem in the circumcircle of $\triangle BIC$, one has $(s-c)(s-b) = r \cdot DD'$. Likewise, we obtain the formulae $(s-c)(s-a) = r \cdot EE'$ and $(s-a)(s-b) = r \cdot FF'$. Thus,

$$DD' + EE' + FF' = \frac{(s-a)(s-b) + (s-b)(s-c) + (s-c)(s-a)}{r} = 146.$$

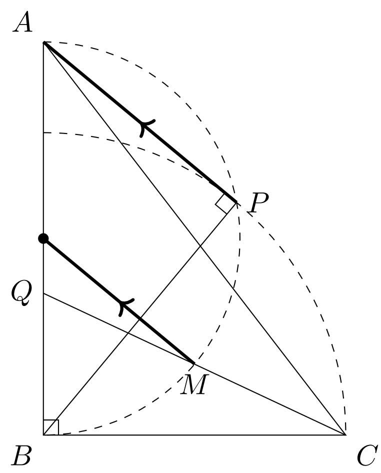
11.1.2. Round 2 Solutions

Review by *Glen Lim*

Question 1

In triangle ABC, $\angle B = 90^\circ$, $AB > BC$, and P is the point such that $BP = BC$ and $\angle APB = 90^\circ$, where P and C lie on the same side of AB. Let Q be the point on AB such that $AP = AQ$, and let M be the midpoint of QC. Prove that the line through M parallel to AP passes through the midpoint of AB.

AoPS thread



We will use coordinate bashing to solve this problem. Firstly, let $A(0, a)$, $B(0, 0)$, and $C(1, 0)$. Note that we fix $C(1, 0)$ as we can always scale the triangle to make $BC = 1$.

Since $BP = BC = 1$, we know that P lies on the unit circle $x^2 + y^2 = 1$. Furthermore, since $\angle APB = 90^\circ$, P also lies on the circle with diameter AB, which has equation $x^2 + (y - \frac{1}{2}a)^2 = (\frac{1}{2}a)^2$. Solving the two equations simultaneously, along with the constraint $x > 0$ (since P and C lie on the same side of AB), we get $P(\frac{1}{a}\sqrt{a^2-1}, \frac{1}{a})$.

Now note that $AQ = AP = \sqrt{(\frac{1}{a}\sqrt{a^2-1})^2 + (\frac{1}{a} - a)^2} = \sqrt{a^2-1}$. Hence, $Q(0, a - \sqrt{a^2-1})$. Thus, $M(\frac{1}{2}, \frac{1}{2}(a - \sqrt{a^2-1}))$.

Note that the gradient of AP is $\frac{a-1/a}{0-\sqrt{a^2-1}/a} = -\sqrt{a^2-1}$. Let l be the line passing through M parallel to AP. By the point-slope formula, l has the equation

$$l : y - \frac{a - \sqrt{a^2-1}}{2} = -\sqrt{a^2-1}\left(x - \frac{1}{2}\right).$$

When $x = 0$, we have $y = \frac{a}{2}$. Hence, l passes through $(0, \frac{1}{2}a)$, which is the midpoint of AB as desired. $\square$

Remark. The condition $AB > BC$ is equivalent to $a > 1$, which is crucial in ensuring that $\sqrt{a^2-1}$ remains real.

Question 2 [Ans: $\frac{1}{2}n(n+1)$]

Let n be a fixed positive integer. Find the minimum value of

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n}$$

where $x_1, x_2, \ldots, x_n$ are distinct positive integers.

AoPS thread

Solution by KHOMNY02**.** Without loss of generality, suppose $x_1 < x_2 < \cdots < x_n$. We begin by showing $\sum_{i=1}^n x_i^3 \geq (\sum_{i=1}^n x_i)^2$ via induction. The case where $n = 1$ is trivial. Suppose that $\sum_{i=1}^k x_i^3 \geq (\sum_{i=1}^k x_i)^2$ for some k. Consider $(\sum_{i=1}^{k+1} x_i)^2$:

$$\begin{aligned}\left(\sum_{i=1}^{k+1} x_i\right)^2 &= \left(\sum_{i=1}^{k} x_i\right)^2 + 2x_{k+1}\sum_{i=1}^{k} x_i + x_{k+1}^2 \\ &\leq \sum_{i=1}^{k} x_i^3 + 2x_{k+1}\sum_{i=1}^{k} x_i + x_{k+1}^2 \\ &= \sum_{i=1}^{k+1} x_i^3 + x_{k+1}\left(2\sum_{i=1}^{k} x_i + x_{k+1} - x_{k+1}^2\right).\end{aligned}$$

It hence suffices to show that $x_{k+1}^2 \geq 2\sum_{i=1}^k x_i + x_{k+1}$. Rearranging, we get the equivalent statement $\frac{1}{2}x_{k+1}(x_{k+1} - 1) \geq \sum_{i=1}^k x_i$. Indeed, we see that

$$\frac{x_{k+1}(x_{k+1}-1)}{2} \geq \frac{x_k(x_k+1)}{2} = 1 + 2 + \cdots + x_k \geq x_1 + x_2 + \cdots + x_k$$

where the last inequality holds because $\{x_1, x_2, \ldots, x_k\} \subseteq \{1, 2, \ldots, x_k\}$. This closes the induction.

Applying our newly-established inequality to the problem at hand, we see that

$$\frac{x_1^3 + \cdots + x_n^3}{x_1 + \cdots + x_n} \geq \frac{(x_1 + \cdots + x_n)^2}{x_1 + \cdots + x_n} \geq x_1 + \cdots + x_n \geq 1 + \cdots + n = \frac{n(n+1)}{2}.$$

Question 3

Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

AoPS thread

We prove via induction that for all $n \in \mathbb{N}$, there must exist some n-digit number a_n divisible by 5^n all of whose digits are odd. When $n = 1$, it is trivial to see that $a_1 = 5$. Now suppose a_k exists for some $k \in \mathbb{N}$. Let $S = \{1, 3, 5, 7, 9\}$. We now show that there exists an $s \in S$ such that $a_{k+1} = s \cdot 10^k + a_k$.

Observe that $s \cdot 10^k + a_k = 5^k \left(s \cdot 2^k + m\right)$. Recall that 2^k can only end with a 2, 4, 6, or 8 in base 10. Hence, the only residues 2^k can take on modulo 5 are 1, 2, 3, and 4. Meanwhile, s can take on any residue modulo 5. It is thus obvious that $s \cdot 2^k$ can take on any residue modulo 5. Hence, by picking $s \equiv -m \pmod 5$, we will have $5^{k+1} \mid 5^k \left(s \cdot 2^k + m\right)$ as desired. This closes the induction. $\square$

Question 4 [Ans: $n+1$]

Alice and Bob play a game. Bob starts by picking a set S consisting of M vectors of length n with entries either 0 or 1. Alice picks a sequence of numbers $y_1 \le y_2 \le \cdots \le y_n$ from the interval $[0,1]$, and a choice of real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Bob wins if he can pick a vector $(z_1, z_2, \ldots, z_n) \in S$ such that

$$\sum_{i=1}^{n} x_i y_i \le \sum_{i=1}^{n} x_i z_i,$$

otherwise Alice wins. Determine the minimum value of M so that Bob can guarantee a win.

AoPS thread

Solution by DVDthe1st. Note that we can rewrite Bob's winning condition as

$$\langle \mathbf{x}, \mathbf{y} \rangle \le \max_{\mathbf{z} \in S} \langle \mathbf{x}, \mathbf{z} \rangle.$$

Geometrically, Bob wins if he can find some $\mathbf{z} \in S$ whose projection on $\mathbf{x}$ is longer than the projection of $\mathbf{y}$ on $\mathbf{x}$. Hence, $\mathbf{y}$ must be in the convex hull of S. However, observe that the space of all possible $\mathbf{y}$ forms a simplex in n-dimensional space. In particular, this simplex has $n+1$ vertices of the form

$$\mathbf{v}_i = (\underbrace{0, \ldots, 0}_{i}, \underbrace{1, \ldots, 1}_{n-i}).$$

Thus, picking $S = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n\}$ will guarantee a win for Bob, whence $M = n+1$. We now show that this is indeed the minimum.

Suppose there exists some $\mathbf{v}_i \notin S$. Alice can capitalize on this by picking

$$\mathbf{x} = (\underbrace{-1, \ldots, -1}_{i}, \underbrace{1, \ldots, 1}_{n-i}) \text{ and } \mathbf{y} = \mathbf{v}_i.$$

This is because $\mathbf{x} \cdot \mathbf{y}$ attains its maximum only when $\mathbf{y} = \mathbf{v}_i$. Hence, Bob can only win by choosing $\mathbf{z} = \mathbf{v}_i$. But because $\mathbf{v}_i \notin S$, he cannot do so and will thus lose.

Question 5 [Ans: $p+1$]

Let p be a prime number. Determine the largest possible n such that the following holds: it is possible to fill an $n \times n$ table with integers a_{ik} in the ith row and kth column, for $1 \le i, k \le n$, such that for any quadruple i, j, k, l with $1 \le i < j \le n$ and $1 \le k < l \le n$, the number $a_{ik}a_{jl} - a_{il}a_{jk}$ is not divisible by p.

AoPS thread

Solution by *Glen Lim*. Observe that the condition $p \nmid a_{ik}a_{jl} - a_{il}a_{jk}$ is equivalent to

$$\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} \not\equiv 0 \pmod{p}$$

That is, (a_{ik}, a_{jk}) and (a_{il}, a_{jl}) are linearly independent. n is thus the maximum number of vectors in $\mathbb{F}_p^2$ that are pairwise linearly independent.

We now show that $n \le p+1$. Take an arbitrary non-zero vector $\mathbf{v} \in \mathbb{F}_p^2$. Then there are clearly $p-1$ non-zero vectors parallel to $\mathbf{v}$, namely $\mathbf{v}, 2\mathbf{v}, \ldots, (p-1)\mathbf{v}$. Since there are $p^2 - 1$ non-zero vectors in $\mathbb{F}_p^2$, the maximum number of pairwise linearly independent vectors is $\frac{p^2-1}{p-1} = p+1$. Hence, $n \le p+1$. Indeed, we can construct the following set with $p+1$ vectors that are pairwise linearly independent:

$$\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}, \ldots, \begin{pmatrix}p-1\\1\end{pmatrix} \right\}.$$

To finish off the problem, we construct a valid $(p+1) \times (p+1)$ grid. We do so as follows:

- $a_{11} = 0$ (the top left corner is 0)
- $a_{1i} = a_{i1} = 1$ for $2 \le i \le p+1$ (all other cells in the first row and first column are 1)
- $a_{ik} = i - k$ for $2 \le i, j \le p+1$ (cells in the remaining $p \times p$ grid are the difference between the row number and the column number)

As an example, the following grid shows the $p = 5$ case.

0	1	1	1	1	1
1	0	1	2	3	4
1	4	0	1	2	3
1	3	4	0	1	2
1	2	3	4	0	1
1	1	2	3	4	0

We now show that $p \nmid a_{ik}a_{jl} - a_{il}a_{jk}$.

Case 1. Consider the case where $i = 1$. Then $a_{1k}a_{jl} - a_{1l}a_{jk} = a_{jl} - a_{jk}$. However, each cell in the jth row has a different number, whence it is clear that $a_{jl} - a_{jk} \not\equiv 0 \pmod p$. A similar argument works for $k = 1$.

Case 2. Consider the case where $i, k \ne 1$. Then $a_{ik}a_{jl} - a_{il}a_{jk} = (i-k)(j-l) - (i-l)(j-k) = (i-j)(k-l)$, which clearly cannot be 0 modulo p.

Hence, $\max n = p + 1$.