Problem 1.

True or False? Explain your answers briefly.

(a) $\sum\limits_{r=1}^{n}(2r+3) = \sum\limits_{k=1}^{n}(2k+3)$

(b) $\sum\limits_{r=1}^{n}\left(\frac{1}{r}+5\right) = \sum\limits_{r=1}^{n}\frac{1}{r}+5$

(c) $\sum\limits_{r=1}^{n}\frac{1}{r} = \frac{1}{\sum\limits_{r=1}^{n} r}$

(d) $\sum\limits_{r=1}^{n} c = \sum\limits_{r=0}^{n-1}(c+1)$

Solution

Part (a)

Since both sums differ only by dummy variables, they are equal.

$$\boxed{\text{True}}$$

Part (b)

Summation is distributive. Since $\sum\limits_{r=1}^{n} 5$ is not equal to 5 in general, the equality does not hold.

$$\boxed{\text{False}}$$

Part (c)

In general, $\sum \frac{a}{b} \neq \frac{\sum a}{\sum b}$.

$$\boxed{\text{False}}$$

Part (d)

Since c is a constant with respect to r, $\sum\limits_{r=1}^{n} c = nc \neq n(c+1) = \sum\limits_{r=0}^{n-1}(c+1)$.

$$\boxed{\text{False}}$$

Problem 2.

Write the following series in Σ notation twice, with $r = 1$ as the lower limit in the first and $r = 0$ as the lower limit in the second.

(a) $-2 + 1 + 4 + \ldots + 40$

(b) $a^2 + a^4 + a^6 + \ldots + a^{50}$

(c) $\dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \ldots + n\text{th term}$

(d) $1 - \dfrac{1}{2} + \dfrac{1}{4} - \dfrac{1}{8} + \ldots \text{ to } n \text{ terms}$

(e) $\dfrac{1}{2 \cdot 4} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{4 \cdot 6} + \ldots + \dfrac{1}{28 \cdot 30}$

Solution

Part (a)

Observe that $-2 + 1 + 4 + \ldots + 40$ is in arithmetic progression with a common difference of 3.

$$\begin{aligned} -2 + 1 + 4 + \ldots + 40 &= \sum_{r=1}^{15} (3r - 5) \\ &= \sum_{r=0}^{14} (3(r + 1) - 5) \\ &= \sum_{r=0}^{14} (3r - 2) \end{aligned}$$

$$\boxed{-2 + 1 + 4 + \ldots + 40 = \sum_{r=1}^{15} (3r - 5) = \sum_{r=0}^{14} (3r - 2)}$$

Part (b)

Observe that $a^2 + a^4 + a^6 + \ldots + a^{50}$ is in geometric progression with a common ratio of a^2.

$$\begin{aligned} a^2 + a^4 + a^6 + \ldots + a^{50} &= \sum_{r=1}^{25} a^{2r} \\ &= \sum_{r=0}^{24} a^{2(r+1)} \\ &= \sum_{r=0}^{24} a^{2r+2} \end{aligned}$$

$$\boxed{a^2 + a^4 + a^6 + \ldots + a^{50} = \sum_{r=1}^{25} a^{2r} = \sum_{r=0}^{24} a^{2r+2}}$$

Part (c)

$$\begin{aligned}\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term} &= \sum_{r=1}^{n}\frac{1}{2r+1}\\ &= \sum_{r=0}^{n-1}\frac{1}{2(r+1)+1}\\ &= \sum_{r=0}^{n-1}\frac{1}{2r+3}\end{aligned}$$

$$\boxed{\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term} = \sum_{r=1}^{n}\frac{1}{2r+1} = \sum_{r=0}^{n-1}\frac{1}{2r+3}}$$

Part (d)

$$\begin{aligned}1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\text{ to } n \text{ terms} &= \sum_{r=1}^{n}\left(-\frac{1}{2}\right)^{r-1}\\ &= \sum_{r=0}^{n-1}\left(-\frac{1}{2}\right)^{(r+1)-1}\\ &= \sum_{r=0}^{n-1}\left(-\frac{1}{2}\right)^{r}\end{aligned}$$

$$\boxed{1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots\text{ to } n \text{ terms} = \sum_{r=1}^{n}\left(-\frac{1}{2}\right)^{r-1} = \sum_{r=0}^{n-1}\left(-\frac{1}{2}\right)^{r}}$$

Part (e)

$$\begin{aligned}\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30} &= \sum_{r=1}^{27}\frac{1}{(r+1)(r+3)}\\ &= \sum_{r=0}^{26}\frac{1}{((r+1)+1)((r+1)+3)}\\ &= \sum_{r=0}^{26}\frac{1}{(r+2)(r+4)}\end{aligned}$$

$$\boxed{\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30} = \sum_{r=1}^{27}\frac{1}{(r+1)(r+3)} = \sum_{r=0}^{26}\frac{1}{(r+2)(r+4)}}$$

Problem 3.

Without using the GC, evaluate the following sums.

(a) $\sum_{r=1}^{50}(2r-7)$

(b) $\sum_{r=1}^{a}(1-a-r)$

(c) $\sum_{r=2}^{n}(\ln r+3^r)$

(d) $\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right)$

Solution

Part (a)

$$\begin{aligned}\sum_{r=1}^{50}(2r-7) &= 2\sum_{r=1}^{50} r - 7\sum_{r=1}^{50} 1 \\ &= 2\cdot\frac{50\cdot 51}{2} - 7\cdot 50 \\ &= 2200\end{aligned}$$

$$\boxed{\sum_{r=1}^{50}(2r-7) = 2200}$$

Part (b)

$$\begin{aligned}\sum_{r=1}^{a}(1-a-r) &= (1-a)\sum_{r=1}^{a} 1 - \sum_{r=1}^{a} r \\ &= (1-a)\cdot a - \frac{a(a+1)}{2} \\ &= \frac{a}{2}\cdot 2(1-a) - \frac{a}{2}\cdot(a+1) \\ &= \frac{a}{2}(2-2a-(a+1)) \\ &= \frac{a}{2}(1-3a)\end{aligned}$$

$$\boxed{\sum_{r=1}^{a}(1-a-r) = \frac{a}{2}(1-3a)}$$

Part (c)

$$
\begin{aligned}
\sum_{r=2}^{n}(\ln r + 3^r) &= \sum_{r=2}^{n}\ln r + \sum_{r=2}^{n}3^r \\
&= \ln n! + \sum_{r=1}^{n-1}3^{r+1} \\
&= \ln n! + 3\sum_{r=1}^{n-1}3^r \\
&= \ln n! + 3\cdot\frac{3(3^{n-1}-1)}{3-1} \\
&= \ln n! + \frac{9}{2}\left(3^{n-1}-1\right)
\end{aligned}
$$

$$\boxed{\sum_{r=2}^{n}(\ln r + 3^r) = \ln n! + \frac{9}{2}\left(3^{n-1}-1\right)}$$

Part (d)

$$
\begin{aligned}
\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right) &= \sum_{r=1}^{\infty}\frac{2^r}{3^r} - \sum_{r=1}^{\infty}\frac{1}{3^r} \\
&= \sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^r - \sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r \\
&= \frac{\frac{2}{3}}{1-\frac{2}{3}} - \frac{\frac{1}{3}}{1-\frac{1}{3}} \\
&= \frac{3}{2}
\end{aligned}
$$

$$\boxed{\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right) = \frac{3}{2}}$$

Problem 4.

The nth term of a series is $2^{n-2} + 3n$. Find the sum of the first N terms.

Solution

$$\begin{aligned}
\sum_{r=1}^{N} \left(2^{r-2} + 3r\right) &= \sum_{r=1}^{N} 2^{r-2} + 3 \sum_{r=1}^{N} r \\
&= \frac{2^{1-2}\left(2^N - 1\right)}{2-1} + \frac{3N(N+1)}{2} \\
&= \frac{2^N - 1}{2} + \frac{3N(N+1)}{2} \\
&= \frac{1}{2}\left(2^N + 3N^2 + 3N - 1\right)
\end{aligned}$$

$$\boxed{\text{The sum of the first } N \text{ terms is } \frac{1}{2}\left(2^N + 3N^2 + 3N - 1\right)}$$

Problem 5.

The rth term, u_r, of a series is given by $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$. Express $\sum_{r=1}^{n} u_r$ in the form $A\left(1 - \frac{B}{27^n}\right)$, where A and B are constants. Deduce the sum to infinity of the series.

Solution

$$\begin{aligned}
\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} \left(\left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}\right) \\
&= \sum_{r=1}^{n} \left(\frac{1}{3}\right)^{3r-2} + \sum_{r=1}^{n} \left(\frac{1}{3}\right)^{3r-1} \\
&= 9\sum_{r=1}^{n} \left(\frac{1}{3}\right)^{3r} + 3\sum_{r=1}^{n} \left(\frac{1}{3}\right)^{3r} \\
&= 12\sum_{r=1}^{n} \left(\frac{1}{3}\right)^{3r} \\
&= 12\sum_{r=1}^{n} \left(\frac{1}{27}\right)^{r} \\
&= 12 \cdot \frac{\frac{1}{27}(1 - (\frac{1}{27})^n)}{1 - \frac{1}{27}} \\
&= 12 \cdot \frac{1 - (\frac{1}{27})^n}{27 - 1} \\
&= \frac{6}{13}\left(1 - \frac{1}{27^n}\right)
\end{aligned}$$

$$\boxed{\sum_{r=1}^{n} u_r = \frac{6}{13}\left(1 - \frac{1}{27^n}\right)}$$

$$\begin{aligned}
\sum_{r=1}^{\infty} u_r &= \lim_{n\to\infty} \sum_{r=1}^{n} u_r \\
&= \lim_{n\to\infty} \frac{6}{13}\left(1 - \frac{1}{27^n}\right) \\
&= \frac{6}{13}(1 - 0) \\
&= \frac{6}{13}
\end{aligned}$$

$$\boxed{\sum_{r=1}^{\infty} u_r = \frac{6}{13}}$$

Problem 6.

The rth term, u_r, of a series is given by $u_r = \ln \frac{r}{r+1}$. Find $\sum_{r=1}^{n} u_r$ in terms of n. Comment on whether the series converges.

Solution

$$\begin{aligned}
\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} \ln \frac{r}{r+1} \\
&= \sum_{r=1}^{n} (\ln r - \ln(r+1)) \\
&= \sum_{r=1}^{n} \ln r - \sum_{r=1}^{n} \ln(r+1) \\
&= \sum_{r=1}^{n} \ln r - \sum_{r=2}^{n+1} \ln r \\
&= \ln 1 + \sum_{r=2}^{n+1} \ln r - \ln(n+1) - \sum_{r=2}^{n+1} \ln r \\
&= \ln 1 - \ln(n+1) \\
&= \ln \frac{1}{n+1}
\end{aligned}$$

$$\boxed{\sum_{r=1}^{n} u_r = \ln \frac{1}{n+1}}$$

Observe that as $n \to \infty$, $\ln \frac{1}{n+1} \to \ln 0$. Hence, $\sum_{r=1}^{n} u_r$ diverges to infinity. Thus, u_n does not converge.

Problem 7.

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, without using the GC, find the following sums.

(a) $\sum_{r=0}^{n}(r(r+4)+n)$

(b) $\sum_{r=n+1}^{2n}(2r-1)^2$

(c) $\sum_{r=-15}^{20} r(r-2)$

Solution

Part (a)

$$\begin{aligned}
\sum_{r=0}^{n}(r(r+4)+n) &= \sum_{r=0}^{n}\left(r^2+4r+n\right)\\
&= \sum_{r=0}^{n} r^2 + 4\sum_{r=0}^{n} r + n\sum_{r=0}^{n} 1\\
&= \sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r + n\sum_{r=0}^{n} 1\\
&= \frac{n}{6}(n+1)(2n+1) + 4\cdot\frac{n(n+1)}{2} + n(n+1)\\
&= \frac{n}{6}(n+1)(2n+1) + 2n(n+1) + n(n+1)\\
&= (n+1)\left(\frac{n}{6}(2n+1) + 2n + n\right)\\
&= \frac{n}{6}(n+1)(2n+1+12+6)\\
&= \frac{n}{6}(n+1)(2n+19)
\end{aligned}$$

$$\boxed{\sum_{r=0}^{n}(r(r+4)+n) = \frac{n}{6}(n+1)(2n+19)}$$

Part (b)

$$\begin{aligned}
\sum_{r=n+1}^{2n} (2r-1)^2 &= \sum_{r=1}^{n}(2(r+n)-1)^2 \\
&= \sum_{r=1}^{n}(2r+2n-1)^2 \\
&= \sum_{r=1}^{n}\left(4r^2+2(2r)(2n-1)+(2n-1)^2\right) \\
&= \sum_{r=1}^{n}\left(4r^2+4(2n-1)r+(2n-1)^2\right) \\
&= 4\sum_{r=1}^{n} r^2 + 4(2n-1)\sum_{r=1}^{n} r + (2n-1)^2\sum_{r=1}^{n} 1 \\
&= 4\cdot\frac{n}{6}(n+1)(2n+1) + 4(2n-1)\frac{n(n+1)}{2} + n(2n-1)^2 \\
&= \frac{2}{3}\cdot n(n+1)(2n+1) + 2n(2n-1)(n+1) + n(2n-1)^2 \\
&= \frac{1}{3}n\left(2(n+1)(2n+1)+6(2n-1)(n+1)+3(2n-1)^2\right) \\
&= \frac{1}{3}n\left(4n^2+4n+2n+2+12n^2-6n+12n-6+12n^2-12n+3\right) \\
&= \frac{1}{3}n\left(28n^2-1\right)
\end{aligned}$$

$$\boxed{\sum_{r=n+1}^{2n}(2r-1)^2 = \frac{1}{3}n\left(28n^2-1\right)}$$

Part (c)

$$\begin{aligned}
\sum_{r=-15}^{20} r(r-2) &= \sum_{r=1}^{36}(r-16)((r-16)-2) \\
&= \sum_{r=1}^{36}(r-16)(r-18) \\
&= \sum_{r=1}^{36}\left(r^2-34r+288\right) \\
&= \sum_{r=1}^{36} r^2 - 34\sum_{r=1}^{36} r + 288\sum_{r=1}^{36} 1 \\
&= \frac{36}{6}\cdot(36+1)(2\cdot 36+1) - 34\cdot\frac{36\cdot 37}{2} + 288\cdot 36 \\
&= 3930
\end{aligned}$$

$$\boxed{\sum_{r=-15}^{20} r(r-2) = 3930}$$

Problem 8.

Let $S = \sum_{r=0}^{\infty} \frac{(x-2)^r}{3^r}$ where $x \neq 2$. Find the range of values of x such that the series S converges. Given that $x = 1$, find

(a) the value of S

(b) S_n, in terms of n, where $S_n = \sum_{r=0}^{n-1} \frac{(x-2)^r}{3^r}$

(c) the least value of n for which $|S_n - S|$ is less than 0.001% of S

Solution

$$\begin{aligned} S &= \sum_{r=0}^{\infty} \frac{(x-2)^r}{3^r} \\ &= \sum_{r=0}^{\infty} \left(\frac{x-2}{3}\right)^r \end{aligned}$$

For S to converge, we must have $\left|\frac{x-2}{3}\right| < 1$.

Case 1: $\frac{x-2}{3} < 1 \implies x - 2 < 3 \implies x < 5$

Case 2: $-\frac{x-2}{3} < 1 \implies \frac{x-2}{3} > -1 \implies x - 2 > -3 \implies x > -1$

Putting both inequalities together, we see that $-1 < x < 5$.

$$\boxed{\text{For } S \text{ to converge, we must have } -1 < x < 5,\ x \neq 2.}$$

Part (a)

When $x = 1$,

$$\begin{aligned} S &= \sum_{r=0}^{\infty} \left(-\frac{1}{3}\right)^r \\ &= \frac{1}{1 - (-\frac{1}{3})} \\ &= \frac{3}{4} \end{aligned}$$

$$\boxed{S = \frac{3}{4}}$$

Part (b)

$$\begin{aligned} S_n &= \sum_{r=0}^{n-1} \frac{(-1)^r}{3^r} \\ &= \sum_{r=0}^{n-1} \left(-\frac{1}{3}\right)^r \\ &= \sum_{r=1}^{n} \left(-\frac{1}{3}\right)^{r-1} \\ &= -3 \sum_{r=1}^{n} \left(-\frac{1}{3}\right)^r \\ &= -3 \cdot \frac{-\frac{1}{3}(1-(-\frac{1}{3})^n)}{1-(-\frac{1}{3})} \\ &= \frac{3}{4}\left(1-\left(-\frac{1}{3}\right)^n\right) \end{aligned}$$

$$\boxed{S_n = \frac{3}{4}\left(1-\left(-\frac{1}{3}\right)^n\right)}$$

Part (c)

Note that $S_n < S$ for even n, and $S_n > S$ for odd n.
Case 1: $n = 2k$, $k \in \mathbb{Z}^+$

$$\begin{aligned} & |S_n - S| < 0.001\% S \\ \implies\quad & S - S_n < \frac{0.001}{100} S \\ \implies\quad & \frac{3}{4} - S_n < \frac{1}{100000} \cdot \frac{3}{4} \\ \implies\quad & S_n > \frac{3}{4}\left(1 - \frac{1}{100000}\right) \\ \implies\quad & \frac{3}{4}\left(1-\left(-\frac{1}{3}\right)^n\right) > \frac{3}{4}\left(1 - \frac{1}{100000}\right) \\ \implies\quad & -\left(-\frac{1}{3}\right)^n > -\frac{1}{100000} \\ \implies\quad & \left(-\frac{1}{3}\right)^n < \frac{1}{100000} \\ \implies\quad & \left(-\frac{1}{3}\right)^{2k} < \frac{1}{100000} \\ \implies\quad & \left(\frac{1}{9}\right)^k < \frac{1}{100000} \\ \implies\quad & 9^k > 100000 \\ \implies\quad & k > \log_9 100000 = 5.24 \text{ (3 s.f.)} \\ \implies\quad & n > 10.5 \end{aligned}$$

Hence, if n is even, then the least value of n is 12.

Case 2: $n = 2k - 1$, $k \in \mathbb{Z}^+$

$$
\begin{aligned}
& & |S_n - S| &< 0.001\% S \\
&\implies & S_n - S &< \frac{0.001}{100} S \\
&\implies & S_n - \frac{3}{4} &< \frac{1}{100000} \cdot \frac{3}{4} \\
&\implies & S_n &< \frac{3}{4}\left(1 + \frac{1}{100000}\right) \\
&\implies & \frac{3}{4}\left(1 - \left(-\frac{1}{3}\right)^n\right) &< \frac{3}{4}\left(1 + \frac{1}{100000}\right) \\
&\implies & -\left(-\frac{1}{3}\right)^n &< \frac{1}{100000} \\
&\implies & -\left(-\frac{1}{3}\right)^{2k-1} &< \frac{1}{100000} \\
&\implies & 3\left(\frac{1}{9}\right)^k &< \frac{1}{100000} \\
&\implies & \frac{1}{9^k} &< \frac{1}{300000} \\
&\implies & 9^k &> 300000 \\
&\implies & k &> \log_9 300000 = 5.74 \text{ (3 s.f.)} \\
&\implies & n &> 10.5
\end{aligned}
$$

Hence, if n is odd, then the least value of n is 11.

$$\boxed{\text{The least value of } n \text{ is } 11.}$$

Problem 9.

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$,

(a) write down $\sum_{r=1}^{2k} r^2$ in terms of k

(b) find $2^2 + 4^2 + 6^2 + \ldots + (2k)^2$.

Hence, show that $\sum_{r=1}^{k} (2r-1)^2 = \frac{k}{3}(2k+1)(2k-1)$.

Solution

Part (a)

$$\begin{aligned}\sum_{r=1}^{2k} r^2 &= \frac{2k}{6}(2k+1)(2(2k)+1) \\ &= \frac{k}{3}(2k+1)(4k+1)\end{aligned}$$

$$\boxed{\sum_{r=1}^{2k} r^2 = \frac{k}{3}(2k+1)(4k+1)}$$

Part (b)

$$\begin{aligned}2^2 + 4^2 + 6^2 + \ldots + (2k)^2 &= \sum_{r=1}^{k} (2r)^2 \\ &= \sum_{r=1}^{k} 4r^2 \\ &= 4\sum_{r=1}^{k} r^2 \\ &= 4 \cdot \frac{k}{6}(k+1)(2k+1) \\ &= \frac{2k}{3}(k+1)(2k+1)\end{aligned}$$

$$\boxed{2^2 + 4^2 + 6^2 + \ldots + (2k)^2 = \frac{2k}{3}(k+1)(2k+1)}$$

$$\begin{aligned}
\sum_{r=1}^{k}(2r-1)^2 &= \sum_{r=1}^{2k} r^2 - \sum_{r=1}^{k}(2r)^2 \\
&= \frac{k}{3}(2k+1)(4k+1) - \frac{2k}{3}(k+1)(2k+1) \\
&= \frac{k}{3}(2k+1)((4k+1) - 2(k+1)) \\
&= \frac{k}{3}(2k+1)(2k-1)
\end{aligned}$$

Problem 10.

Given that $u_n = e^{nx} - e^{(n+1)x}$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. Hence determine the set of values of x for which the infinite series $u_1 + u_2 + u_3 + \ldots$ is convergent and give the sum to infinity for cases where this exists.

Solution

$$\begin{aligned}
\sum_{n=1}^{N} u_n &= \sum_{n=1}^{N} \left(e^{nx} - e^{(n+1)x}\right) \\
&= \sum_{n=1}^{N} e^{nx} - \sum_{n=1}^{N} e^{(n+1)x} \\
&= \sum_{n=1}^{N} e^{nx} - \sum_{n=2}^{N+1} e^{nx} \\
&= \left(e^{x} + \sum_{n=2}^{N} e^{nx}\right) - \left(\sum_{n=2}^{N} e^{nx} + e^{(N+1)x}\right) \\
&= e^{x} - e^{(N+1)x} \\
&= e^{x}(1 - e^{Nx})
\end{aligned}$$

$$\boxed{\sum_{n=1}^{N} u_n = e^{x}(1 - e^{Nx})}$$

For the infinite series $u_1 + u_2 + u_3 + \ldots$ to converge, we require e^x to converge. Hence, $x \leq 0$. Equivalently, $x \in \mathbb{R}_0^-$.

$$\boxed{\mathbb{R}_0^-}$$

Case 1: $x = 0$

$$\begin{aligned}
\lim_{N\to\infty} \sum_{n=1}^{N} u_n &= \lim_{N\to\infty} e^{x}(1 - e^{Nx}) \\
&= \lim_{N\to\infty} e^{0}(1 - e^{N\cdot 0}) \\
&= \lim_{N\to\infty} 1(1 - 1) \\
&= 0
\end{aligned}$$

$$\boxed{\text{When } x = 0\text{, the sum to infinity is } 0.}$$

Case 2: $x < 0$

$$\begin{aligned}
\lim_{N\to\infty} \sum_{n=1}^{N} u_n &= \lim_{N\to\infty} e^{x}(1 - e^{Nx}) \\
&= e^{x}(1 - 0) \\
&= e^{x}
\end{aligned}$$

$$\boxed{\text{When } x < 0\text{, the sum to infinity is } e^{x}.}$$

Problem 11.

Given that r is a positive integer and $f(r) = \frac{1}{r^2}$, express $f(r) - f(r+1)$ as a single fraction. Hence prove that $\sum_{r=1}^{4n} \left(\frac{2r+1}{r^2(r+1)^2} \right) = 1 - \frac{1}{(4n+1)^2}$. Give a reason why the series is convergent and state the sum to infinity. Find $\sum_{r=2}^{4n} \left(\frac{2r-1}{r^2(r-1)^2} \right)$.

Solution

$$\begin{aligned} f(r) - f(r+1) &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\ &= \frac{(r+1)^2 - r^2}{r^2(r+1)^2} \\ &= \frac{2r+1}{r^2(r+1)^2} \end{aligned}$$

$$\boxed{f(r) - f(r+1) = \frac{2r+1}{r^2(r+1)^2}}$$

$$\begin{aligned} \sum_{r=1}^{4n} \left(\frac{2r+1}{r^2(r+1)^2} \right) &= \sum_{r=1}^{4n} (f(r) - f(r+1)) \\ &= \sum_{r=1}^{4n} f(r) - \sum_{r=1}^{4n} f(r+1) \\ &= \sum_{r=1}^{4n} f(r) - \sum_{r=2}^{4n+1} f(r) \\ &= \left(f(1) + \sum_{r=2}^{4n} f(r) \right) - \left(\sum_{r=2}^{4n} f(r) + f(4n+1) \right) \\ &= f(1) - f(4n+1) \\ &= 1 - \frac{1}{(4n+1)^2} \end{aligned}$$

As $r \to \infty$, $\sum_{r=1}^{4n} \left(\frac{2r+1}{r^2(r+1)^2} \right) = 1 - \frac{1}{(4n+1)^2} \to 1$. Hence, the series is convergent and converges to 1.

$$\boxed{\text{The sum to infinity is 1.}}$$

$$\begin{aligned}
\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right) &= \sum_{r=1}^{4n-1}\left(\frac{2(r+1)-1}{(r+1)^2\,r^2}\right)\\
&= \sum_{r=1}^{4n-1}\left(\frac{2r+1}{r^2(r+1)^2}\right)\\
&= \sum_{r=1}^{4n-1}\left(f(r)-f(r+1)\right)\\
&= \sum_{r=1}^{4n}(f(r)-f(r+1)) - (f(4n)-f(4n+1))\\
&= 1-f(4n+1)-(f(4n)-f(4n+1))\\
&= 1-f(4n)\\
&= 1-\frac{1}{(4n)^2}\\
&= 1-\frac{1}{16n^2}
\end{aligned}$$

$$\boxed{\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right) = 1-\frac{1}{16n^2}}$$

Problem 12.

(a) Express $\dfrac{1}{(2x+1)(2x+3)(2x+5)}$ in partial fractions.

(b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$.

(c) Deduce the sum of $\dfrac{1}{1\cdot 3\cdot 5} + \dfrac{1}{3\cdot 5\cdot 7} + \dfrac{1}{3\cdot 5\cdot 7\cdot 9} + \ldots + \dfrac{1}{41\cdot 43\cdot 45}$.

Solution

Part (a)

Let $u = 2x+3$. Then $\dfrac{1}{(2x+1)(2x+3)(2x+5)} = \dfrac{1}{(u-2)u(u+2)}$.

$$\begin{aligned}
& \frac{1}{(u-2)u(u+2)} = \frac{A}{u-2} + \frac{B}{u} + \frac{C}{u+2} \\
\implies \quad & 1 = A(u)(u+2) + B(u-2)(u+2) + C(u-2)u \\
& = Au^2 + 2Au + Bu^2 - 4B + Cu^2 - 2Cu \\
& = (A+B+C)u^2 + (A-C)u - 4B
\end{aligned}$$

Comparing the coefficients of u^2, u and constant terms, we have the following system of equations.

$$\begin{cases} A+B+C &= 0 \\ A-C &= 0 \\ -4B &= 1 \end{cases}$$

Solving, we obtain $A = \dfrac{1}{8}$, $B = -\dfrac{1}{4}$ and $C = \dfrac{1}{8}$. Hence,

$$\begin{aligned}
& \frac{1}{(u-2)u(u+2)} = \frac{1}{8(u-2)} - \frac{1}{4u} + \frac{1}{8(u+2)} \\
\implies \quad & \frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}
\end{aligned}$$

$$\boxed{\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}}$$

Part (b)

$$\begin{aligned}
&\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)(2r+5)} \\
&= \sum_{r=1}^{n} \left(\frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)} \right) \\
&= \frac{1}{8} \sum_{r=1}^{n} \frac{1}{2r+1} - \frac{1}{4} \sum_{r=1}^{n} \frac{1}{2r+3} + \frac{1}{8} \sum_{r=1}^{n} \frac{1}{2r+5} \\
&= \left(\frac{1}{8} \sum_{r=1}^{n} \frac{1}{2r+1} - \frac{1}{8} \sum_{r=1}^{n} \frac{1}{2r+3} \right) + \left(\frac{1}{8} \sum_{r=1}^{n} \frac{1}{2r+5} - \frac{1}{8} \sum_{r=1}^{n} \frac{1}{2r+3} \right) \\
&= \frac{1}{8} \left(\left(\sum_{r=1}^{n} \frac{1}{2r+1} - \sum_{r=1}^{n} \frac{1}{2r+3} \right) + \left(\sum_{r=1}^{n} \frac{1}{2r+5} - \sum_{r=1}^{n} \frac{1}{2r+3} \right) \right) \qquad (12.1)
\end{aligned}$$

Consider $\sum_{r=1}^{n} \frac{1}{2r+1} - \sum_{r=1}^{n} \frac{1}{2r+3}$.

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{2r+1} - \sum_{r=1}^{n} \frac{1}{2r+3} &= \sum_{r=1}^{n} \frac{1}{2r+1} - \sum_{r=2}^{n+1} \frac{1}{2r+1} \\
&= \left(\frac{1}{3} + \sum_{r=2}^{n} \frac{1}{2r+1} \right) - \left(\sum_{r=2}^{n} \frac{1}{2r+1} + \frac{1}{2(n+1)+1} \right) \\
&= \frac{1}{3} - \frac{1}{2n+3} \qquad (12.2)
\end{aligned}$$

Consider $\sum_{r=1}^{n} \frac{1}{2r+5} - \sum_{r=1}^{n} \frac{1}{2r+3}$.

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{2r+5} - \sum_{r=1}^{n} \frac{1}{2r+3} &= \sum_{r=1}^{n} \frac{1}{2r+5} - \sum_{r=0}^{n-1} \frac{1}{2r+5} \\
&= \left(\sum_{r=0}^{n} \frac{1}{2r+5} - \frac{1}{5} \right) - \left(\sum_{r=0}^{n} \frac{1}{2r+5} - \frac{1}{2n+5} \right) \\
&= \frac{1}{2n+5} - \frac{1}{5} \qquad (12.3)
\end{aligned}$$

Substituting Equations 12.2 and 12.3 into Equation 12.1, we have

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{(2r+1)(2r+3)(2r+5)} &= \frac{1}{8} \left(\frac{1}{3} - \frac{1}{2n+3} + \frac{1}{2n+5} - \frac{1}{5} \right) \\
&= \frac{1}{8} \left(\frac{2}{15} - \frac{1}{2n+3} + \frac{1}{2n+5} \right) \\
&= \frac{1}{8} \left(\frac{2}{15} - \frac{2}{(2n+3)(2n+5)} \right) \\
&= \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}
\end{aligned}$$

Part (c)

$$
\begin{aligned}
&\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{3\cdot5\cdot7\cdot9}+\ldots+\frac{1}{41\cdot43\cdot45}\\
&=\sum_{r=0}^{20}\frac{1}{(2r+1)(2r+3)(2r+5)}\\
&=\frac{1}{1\cdot3\cdot5}+\sum_{r=1}^{20}\frac{1}{(2r+1)(2r+3)(2r+5)}\\
&=\frac{1}{15}+\frac{1}{60}-\frac{1}{4(2\cdot20+3)(2\cdot20+5)}\\
&=\frac{161}{1935}
\end{aligned}
$$

$$
\boxed{\frac{1}{1\cdot3\cdot5}+\frac{1}{3\cdot5\cdot7}+\frac{1}{3\cdot5\cdot7\cdot9}+\ldots+\frac{1}{41\cdot43\cdot45}=\frac{161}{1935}}
$$