

Problem 1. ACJC Prelim 9758/2017/01/Q5

The points O , A and B are on a plane such that relative to the point O , the points A and B have non-parallel position vectors \mathbf{a} and \mathbf{b} respectively.

- (a) The point C with position vector \mathbf{c} is on the plane OAB such that OC bisects the angle AOB . Show that $\left(\frac{\mathbf{a}}{|\mathbf{a}|} - \frac{\mathbf{b}}{|\mathbf{b}|}\right) \cdot \mathbf{c} = 0$.
- (b) The lines AB and OC intersect at P . By first verifying that \overrightarrow{OC} is parallel to $\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$, show that the ratio of $AP : PB = |\mathbf{a}| : |\mathbf{b}|$.

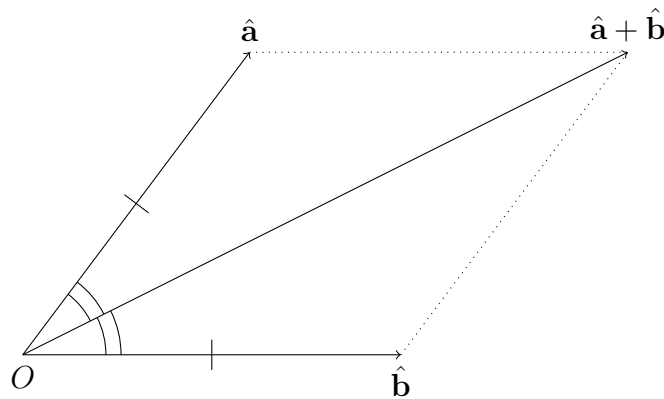
Solution

Part (a)

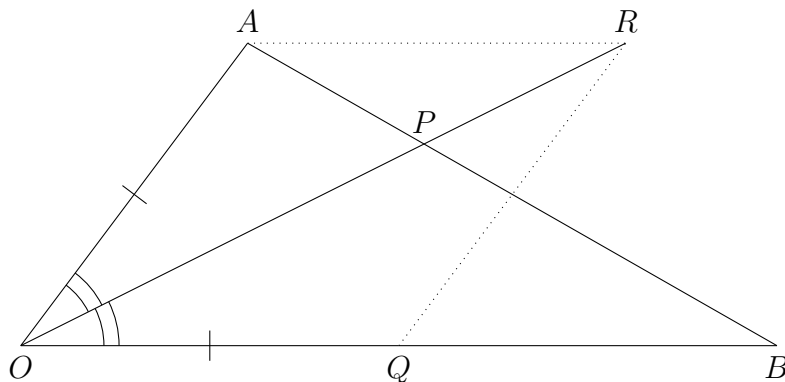
Since OC bisects $\angle AOB$,

$$\begin{aligned} \angle AOC &= \angle COB \\ \Rightarrow \cos \angle AOC &= \cos \angle COB \\ \Rightarrow \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}| |\mathbf{c}|} &= \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}| |\mathbf{c}|} \\ \Rightarrow \left(\frac{\mathbf{a}}{|\mathbf{a}|}\right) \cdot \mathbf{c} &= \left(\frac{\mathbf{b}}{|\mathbf{b}|}\right) \cdot \mathbf{c} \\ \Rightarrow \left(\frac{\mathbf{a}}{|\mathbf{a}|} - \frac{\mathbf{b}}{|\mathbf{b}|}\right) \cdot \mathbf{c} &= 0 \end{aligned}$$

Part (b)



Consider the above diagram. Since $|\hat{\mathbf{a}}| = |\hat{\mathbf{b}}|$, they form a rhombus. Recall that the diagonals of a rhombus bisect opposite angles. Thus, the sum $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ bisects $\angle AOB$ and is hence parallel to \overrightarrow{OC} .



Consider the above diagram. We have Q on OB such that $OA = OQ$. We also have R such that $OA \parallel QR$ and $OA = AR$. From the earlier discussion, P is the intersection of OR and AB .

Now observe that $\triangle OBP$ is similar to $\triangle RAP$. Let λ be the scale factor of $\triangle RAP$ with respect to $\triangle OBP$. We hence have

$$\begin{aligned} |\mathbf{a}| = OA = AR = \lambda OB = \lambda |\mathbf{b}| \text{ and } AP = \lambda BP \\ \Rightarrow \frac{|\mathbf{a}|}{|\mathbf{b}|} = \lambda \text{ and } \frac{AP}{BP} = \lambda \end{aligned}$$

Thus, $\frac{|\mathbf{a}|}{|\mathbf{b}|} = \frac{AP}{BP}$, whence $AP : PB = |\mathbf{a}| : |\mathbf{b}|$.

Problem 2. AJC Prelim 9758/2017/01/Q9

The position vectors of A , B and C with respect to the origin O are \mathbf{a} , \mathbf{b} and \mathbf{c} respectively. It is given that $\overrightarrow{AC} = 4\overrightarrow{CB}$ and $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

- By considering $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$, show that \mathbf{a} and \mathbf{b} are perpendicular.
- Find the length of the projection of \mathbf{c} on \mathbf{a} in terms of $|\mathbf{a}|$.
- Given that F is the foot of the perpendicular from C to OA and \mathbf{f} denotes the position vector \overrightarrow{OF} , state the geometrical meaning of $|\mathbf{c} \times \mathbf{f}|$.
- Two points X and Y move along line segments OA and AB respectively such that

$$\begin{aligned}\overrightarrow{OX} &= (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + \frac{1}{2}\mathbf{k} \\ \overrightarrow{OY} &= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} - 2\mathbf{k}\end{aligned}$$

where t is a real parameter, $0 \leq t \leq 2\pi$. By expressing the scalar product of \overrightarrow{OX} and \overrightarrow{OY} in the form of $p \sin(qt) + r$ where p , q and r are real values to be determined, find the greatest value of the angle XOY .

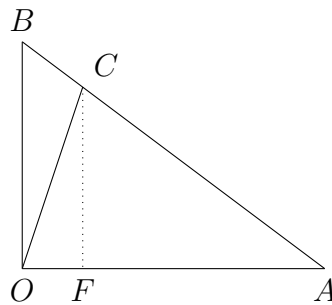
Solution

Part (a)

$$\begin{aligned}(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) &= \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= |\mathbf{a}|^2 + 2\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2 \\ &= |\mathbf{a} + \mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}\end{aligned}$$

Since $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = |\mathbf{a} + \mathbf{b}|^2$, we have that $\mathbf{a} \cdot \mathbf{b} = 0$, whence \mathbf{a} and \mathbf{b} are perpendicular.

Part (b)



By the Ratio Theorem, $\overrightarrow{OC} = \frac{1}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$. Since F lies on OA , it has the direction vector $\frac{1}{5}\mathbf{a}$. Thus, OF , the length of projection of \mathbf{c} on \mathbf{a} , is $\frac{1}{5}|\mathbf{a}|$.

The length of projection of \mathbf{c} on \mathbf{a} is $\frac{1}{5}|\mathbf{a}|$.

Part (c)

$|\mathbf{c} \times \mathbf{f}|$ is the area of a parallelogram defined by \mathbf{c} and \mathbf{f} .

Part (d)

We have $\overrightarrow{OX} = \begin{pmatrix} \cos 3t \\ \sin 3t \\ 1/2 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} \sin t \\ \cos t \\ -2 \end{pmatrix}$. Hence,

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{OY} &= \begin{pmatrix} \cos 3t \\ \sin 3t \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} \sin t \\ \cos t \\ -2 \end{pmatrix} \\ &= \cos 3t \sin t + \sin 3t \cos t - 1 \\ &= \sin 4t - 1 \end{aligned}$$

From the geometric definition of the scalar product, we have

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{OY} &= \left| \begin{pmatrix} \cos 3t \\ \sin 3t \\ 1/2 \end{pmatrix} \right| \left| \begin{pmatrix} \sin t \\ \cos t \\ -2 \end{pmatrix} \right| \cos \angle XOY \\ \implies \sin 4t - 1 &= \sqrt{\cos^2 3t + \sin^2 3t + \left(\frac{1}{2}\right)^2} \sqrt{\sin^2 t + \cos^2 t + (-2)^2} \cos \angle XOY \\ &= \sqrt{1 + \frac{1}{4}} \sqrt{1 + 4} \cos \angle XOY \\ &= \frac{5}{2} \cos \angle XOY \\ \implies \cos \angle XOY &= \frac{2}{5} (\sin 4t - 1) \end{aligned}$$

Observe that $\angle XOY \in [0, \pi)$, where $\cos \angle XOY$ is decreasing. Hence, the maximum value of $\angle XOY$ occurs when $\cos \angle XOY$ is at a minimum. Since the minimum of $\sin 4t$ is -1 , we have

$$\begin{aligned} \min \cos \angle XOY &= \frac{2}{5} (-1 - 1) \\ \implies \max \angle XOY &= \arccos\left(-\frac{4}{5}\right) \\ &= 2.50 \text{ (3 s.f.)} \end{aligned}$$

The greatest value of $\angle XOY$ is 2.50.

Problem 3. CJC Prelim 9758/2017/02/Q2

Referred to the origin O , the points A , B , P and Q have position vectors \mathbf{a} , \mathbf{b} , \mathbf{p} and \mathbf{q} respectively, such that $|\mathbf{a}| = 2$, \mathbf{b} is a unit vector, and the angle between \mathbf{a} and \mathbf{b} is $\frac{\pi}{4}$.

- (a) Give a geometrical interpretation of $|\mathbf{b} \cdot \mathbf{a}|$.
- (b) Find $|\mathbf{a} \times \mathbf{b}|$, leaving your answer in exact form.

It is also given that $\mathbf{p} = 3\mathbf{a} + (\mu + 2)\mathbf{b}$ and $\mathbf{q} = (\mu + 3)\mathbf{a} + \mu\mathbf{b}$, where $\mu \in \mathbb{R}$.

- (c) Show that $\mathbf{p} \times \mathbf{q} = (\mu^2 + 2\mu + 6)(\mathbf{b} \times \mathbf{a})$.
- (d) Hence, find the smallest area of the triangle OPQ as μ varies.

Solution

Part (a)

$|\mathbf{b} \cdot \mathbf{a}|$ is the area of the parallelogram defined by \mathbf{b} and \mathbf{a} .

Part (b)

Let $\theta = \frac{\pi}{4}$ be the angle between \mathbf{a} and \mathbf{b} .

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}| |\mathbf{b}| \sin \theta \\ &= 2 \cdot 1 \cdot \sin \frac{\pi}{4} \\ &= \sqrt{2} \end{aligned}$$

$|\mathbf{a} \times \mathbf{b}| = \sqrt{2}$

Part (c)

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= [3\mathbf{a} + (\mu + 2)\mathbf{b}] \times [(\mu + 3)\mathbf{a} + \mu\mathbf{b}] \\ &= 3(\mu + 3)\mathbf{a} \times \mathbf{a} + 3\mu\mathbf{a} \times \mathbf{b} + (\mu + 2)(\mu + 3)\mathbf{b} \times \mathbf{a} + \mu(\mu + 2)\mathbf{b} \times \mathbf{b} \\ &= 3\mu\mathbf{a} \times \mathbf{b} + (\mu + 2)(\mu + 3)\mathbf{b} \times \mathbf{a} \\ &= -3\mu\mathbf{b} \times \mathbf{a} + (\mu^2 + 5\mu + 6)\mathbf{b} \times \mathbf{a} \\ &= (\mu^2 + 2\mu + 6)\mathbf{b} \times \mathbf{a} \end{aligned}$$

Part (d)

$$\begin{aligned} \min \text{Area } \triangle OPQ &= \min \frac{1}{2} |\mathbf{p} \times \mathbf{q}| \\ &= \min \frac{1}{2} |\mu^2 + 2\mu + 6| |\mathbf{b} \times \mathbf{a}| \\ &= \min \frac{1}{2} |(\mu + 1)^2 + 5| \sqrt{2} \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \cdot 5 \cdot \sqrt{2} \\ &= \frac{5}{\sqrt{2}} \end{aligned}$$

The smallest area of $\triangle OPQ$ is $\frac{5}{\sqrt{2}}$ units².

Problem 4. IJC Prelim 9758/2017/01/Q3

The vectors \mathbf{p} and \mathbf{q} are given by $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + a\mathbf{k}$ and $\mathbf{q} = b\mathbf{i} + \mathbf{j}$, where a and b are non-zero constants.

- (a) Find $(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})$ in terms of a and b .

Given that the \mathbf{i} - and \mathbf{j} - components of the answer to part (a) are equal, find the value of b . Use the value of b you have found to solve parts (b) and (c).

- (b) Given that the magnitude of $(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})$ is 80, find the possible exact values of a .
- (c) Given instead that $2\mathbf{p} - 5\mathbf{q}$ and $2\mathbf{p} + 5\mathbf{q}$ are perpendicular, find the exact value of $|\mathbf{p}|$.

Solution

Part (a)

We have $\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix}$. Hence,

$$\begin{aligned} (2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q}) &= 4\mathbf{p} \times \mathbf{p} + 10\mathbf{p} \times \mathbf{q} - 10\mathbf{q} \times \mathbf{p} - 25\mathbf{q} \times \mathbf{q} \\ &= 10\mathbf{p} \times \mathbf{q} - 10\mathbf{q} \times \mathbf{p} \\ &= 10\mathbf{p} \times \mathbf{q} + 10\mathbf{p} \times \mathbf{q} \\ &= 20\mathbf{p} \times \mathbf{q} \\ &= 20 \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix} \times \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix} \\ &= 20 \begin{pmatrix} -a \\ ab \\ 2 - b \end{pmatrix} \end{aligned}$$

$$(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q}) = 20 \begin{pmatrix} -a \\ ab \\ 2 - b \end{pmatrix}$$

Since the \mathbf{i} - and \mathbf{j} -components are equal, we have

$$\begin{aligned} -a &= ab \\ \implies ab + a &= 0 \\ \implies a(b + 1) &= 0 \end{aligned}$$

We thus have $b = -1$. Note that we reject $a = 0$ since a is non-zero.

$$\boxed{b = -1}$$

Part (b)

$$\begin{aligned}
& |(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})| = 80 \\
\Rightarrow & \left| 20 \begin{pmatrix} -a \\ -a \\ 3 \end{pmatrix} \right| = 80 \\
\Rightarrow & \left| \begin{pmatrix} -a \\ -a \\ 3 \end{pmatrix} \right| = 4 \\
\Rightarrow & \sqrt{(-a)^2 + (-a)^2 + 3^2} = 4 \\
\Rightarrow & 2a^2 + 9 = 16 \\
\Rightarrow & a^2 = \frac{7}{2} \\
\Rightarrow & a = \pm \sqrt{\frac{7}{2}}
\end{aligned}$$

$$a = \pm \sqrt{\frac{7}{2}}$$

Part (c)

Since $2\mathbf{p} - 5\mathbf{q}$ and $2\mathbf{p} + 5\mathbf{q}$, their dot product is 0.

$$\begin{aligned}
& (2\mathbf{p} - 5\mathbf{q}) \cdot (2\mathbf{p} + 5\mathbf{q}) = 0 \\
\Rightarrow & 4\mathbf{p} \cdot \mathbf{p} + 10\mathbf{p} \cdot \mathbf{q} - 10\mathbf{q} \cdot \mathbf{p} - 25\mathbf{q} \cdot \mathbf{q} = 0 \\
\Rightarrow & 4\mathbf{p} \cdot \mathbf{p} - 25\mathbf{q} \cdot \mathbf{q} = 0 \\
\Rightarrow & 4|\mathbf{p}|^2 - 25|\mathbf{q}|^2 = 0 \\
\Rightarrow & 4|\mathbf{p}|^2 - 25 \left| \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right|^2 = 0 \\
\Rightarrow & 4|\mathbf{p}|^2 - 25 \cdot 2 = 0 \\
\Rightarrow & |\mathbf{p}|^2 = \frac{25}{2} \\
\Rightarrow & |\mathbf{p}| = \frac{5}{\sqrt{2}}
\end{aligned}$$

Note that we reject $|\mathbf{p}| = -\frac{5}{\sqrt{2}}$ since $|\mathbf{p}| \geq 0$.

$$|\mathbf{p}| = \frac{5}{\sqrt{2}}$$

Problem 5. JJC Prelim 9758/2017/01/Q6

With respect to the origin O , the position vectors of the points U , V and W are \mathbf{u} , \mathbf{v} and \mathbf{w} respectively. The mid-points of the sides VW , WU and UV of the triangle UVW are M , N and P respectively.

- (a) Show that $\overrightarrow{UM} = \frac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u})$.
- (b) Find the vector equations of the lines UM and VN . Hence, show that the position vector of the point of intersection, G , of UM and VN is $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$.

Solution**Part (a)**

By the Midpoint Theorem,

$$\begin{aligned}\overrightarrow{OM} &= \frac{1}{2}(\mathbf{v} + \mathbf{w}) \\ \Rightarrow \overrightarrow{OU} + \overrightarrow{UM} &= \frac{1}{2}(\mathbf{v} + \mathbf{w}) \\ \Rightarrow \overrightarrow{UM} &= \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \mathbf{u} \\ &= \frac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u})\end{aligned}$$

Part (b)

Note that the line UM has direction vector $\mathbf{v} + \mathbf{w} - 2\mathbf{u}$ and passes through U . Hence,

$$\boxed{l_{UM} : \mathbf{r} = \mathbf{u} + \lambda(\mathbf{v} + \mathbf{w} - 2\mathbf{u}), \lambda \in \mathbb{R}}$$

From the Midpoint Theorem, we have $\overrightarrow{ON} = \frac{1}{2}(\mathbf{w} + \mathbf{u})$. Thus, $\overrightarrow{VN} = \overrightarrow{ON} - \overrightarrow{OV} = \frac{1}{2}(\mathbf{w} + \mathbf{u} - 2\mathbf{v})$. Thus, line VN has direction vector $\mathbf{w} + \mathbf{u} - 2\mathbf{v}$ and passes through V .

$$\boxed{l_{VN} : \mathbf{r} = \mathbf{v} + \mu(\mathbf{w} + \mathbf{u} - 2\mathbf{v}), \mu \in \mathbb{R}}$$

Consider $l_{UM} = l_{VN}$.

$$\begin{aligned}l_{UM} &= l_{VN} \\ \Rightarrow \mathbf{u} + \lambda(\mathbf{v} + \mathbf{w} - 2\mathbf{u}) &= \mathbf{v} + \mu(\mathbf{w} + \mathbf{u} - 2\mathbf{v}) \\ \Rightarrow (1 - 2\lambda)\mathbf{u} + \lambda\mathbf{v} + \lambda\mathbf{w} &= \mu\mathbf{u} + (1 - 2\mu)\mathbf{v} + \mu\mathbf{w}\end{aligned}$$

Comparing coefficients of \mathbf{u} , \mathbf{v} and \mathbf{w} terms, we have the system:

$$\begin{cases} 1 - 2\lambda = \mu \\ \lambda = 1 - 2\mu \\ \lambda = \mu \end{cases}$$

which has solution $\lambda = \mu = \frac{1}{3}$. Thus,

$$\begin{aligned}\overrightarrow{OG} &= \mathbf{v} + \frac{1}{3}(\mathbf{w} + \mathbf{u} - \mathbf{v}) \\ &= \frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})\end{aligned}$$

$$\boxed{\overrightarrow{OG} = \frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})}$$

Problem 6. MI Prelim 9740/2017/01/Q5

A line L passes through the points $(3, -1, 0)$ and $B(11, 11, 4)$.

(a) Find the angle between L and the y -axis.

(b) State the geometrical meaning of $\left| \overrightarrow{OB} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|$.

The point $F(2a + 1, a, a - 1)$ is a point on L , where a is a positive constant. The point P is such that $\overrightarrow{PF} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and the area of the triangle AFP is $\sqrt{\frac{59}{2}}$ units².

(c) Determine the value of a .

(d) The point C on L is such that the ratio of the area of triangle AFP to the area of triangle FCP is $2 : 1$. State the ratio $AF : CF$, justifying your answer.

Solution

Part (a)

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 11 \\ 11 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Since L passes through A , it has the vector equation

$$L : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Observe that the y -axis has vector equation $\mathbf{r} = \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where $\mu \in \mathbb{R}$. Let θ be the angle between L and the y -axis.

$$\begin{aligned} \cos \theta &= \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right|} \\ &= \frac{3}{\sqrt{14}} \\ \Rightarrow \theta &= \arccos \frac{3}{\sqrt{14}} \\ &= 0.641 \text{ (3 s.f.)} \end{aligned}$$

The angle between L and the y -axis is 0.641.

Part (b)

$\left| \overrightarrow{OB} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|$ is the length of projection of \overrightarrow{OB} on the z -axis.

Part (c)

Since F is on the line L , we have that

$$\begin{pmatrix} 2a+1 \\ a \\ a-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. This gives the system

$$\begin{cases} 2a+1 = 3+2\lambda \\ a = -1+3\lambda \\ a-1 = \lambda \end{cases}$$

which has solution $a = 2, \lambda = 1$.

$$\boxed{a = 2}$$

Part (d)

Since $\triangle AFP$ and $\triangle FCP$ have the same height, the length of the bases of both triangles are in the same ratio as their area. Hence, $AF : CF = \text{Area } \triangle AFP : \text{Area } \triangle FCP = 2 : 1$.

$$\boxed{AF : CF = 2 : 1}$$

Problem 7. MJC Prelim 9578/2017/01/Q4

- (a) The points A and B relative to the origin O have position vectors $3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively.
- (i) Find the angle between \overrightarrow{OA} and \overrightarrow{OB} .
- (ii) Hence or otherwise, find the shortest distance from B to line OA .
- (b) The points C , D and E relative to the origin O have non-zero and non-parallel position vectors \mathbf{c} , \mathbf{d} and \mathbf{e} respectively. Given that $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{e} = 0$, state with reason(s) the relationship between O , C , D and E .

Solution

Part (a)

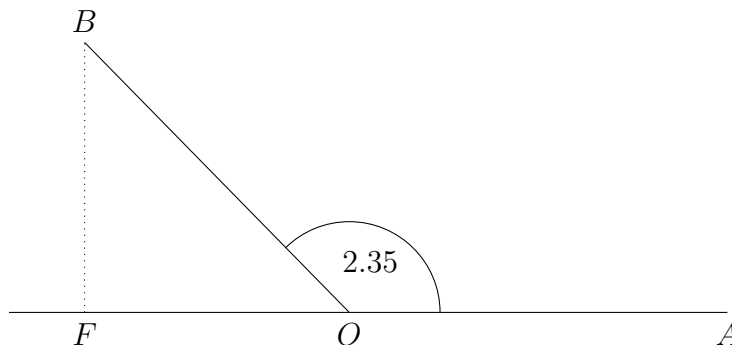
Subpart (i)

We have $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$. Let θ be the angle between \overrightarrow{OA} and \overrightarrow{OB} .

$$\begin{aligned} \cos \theta &= \frac{\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}}{\left| \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \right| \left| \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} \right|} \\ &= -\frac{11}{\sqrt{247}} \\ \Rightarrow \theta &= \arccos\left(-\frac{11}{\sqrt{247}}\right) \\ &= 2.35 \text{ (3 s.f.)} \end{aligned}$$

The angle between \overrightarrow{OA} and \overrightarrow{OB} is 2.35.

Subpart (ii)



Consider the above diagram, where F is the foot of the perpendicular from B to the line OA . Note that $\angle BOF = \pi - \arccos\left(-\frac{11}{\sqrt{247}}\right)$. Hence,

$$\begin{aligned}\sin \angle BOF &= \frac{BF}{OB} \\ \Rightarrow BF &= OB \sin \angle BOF \\ &= \sqrt{13} \sin\left(\pi - \arccos\left(-\frac{11}{\sqrt{247}}\right)\right) \\ &= 2.58 \text{ (3 s.f.)}\end{aligned}$$

The shortest distance from B to the line OA is 2.58 units.

Part (b)

Recall that $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{e}$ is the volume of the parallelepiped defined by \mathbf{c} , \mathbf{d} and \mathbf{e} . Since the volume is 0 and all three vectors are non-zero and non-parallel, they must be coplanar.

Problem 8. NJC Prelim 9758/2017/01/Q1

Given that $\mathbf{p} = 2\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \alpha\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, where α is a real constant and \mathbf{w} is the position vector of a variable point W relative to the origin such that $\mathbf{w} \times \mathbf{p} = \mathbf{q}$.

- (a) Find the value of α .
- (b) Find the set of vectors \mathbf{w} in the form $\{\mathbf{w} : \mathbf{w} = \mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$.

Solution

Part (a)

We have $\mathbf{p} = \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix}$. Since $\mathbf{w} \times \mathbf{p} = \mathbf{q}$, the vectors \mathbf{w} , \mathbf{p} and \mathbf{q} are pairwise orthogonal. Hence,

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= 0 \\ \Rightarrow \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix} &= 0 \\ \Rightarrow 2\alpha + \alpha + 6 &= 0 \\ \Rightarrow \alpha &= -2 \end{aligned}$$

$$\boxed{\alpha = -2}$$

Part (b)

Let $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{aligned} \mathbf{w} \times \mathbf{p} &= \mathbf{q} \\ \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} \\ \Rightarrow \begin{pmatrix} y + 2z \\ 2z - x \\ -2x - 2y \end{pmatrix} &= \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} \end{aligned}$$

This gives the system:

$$\begin{cases} y + 2z = -2 \\ -x + 2z = 1 \\ -2x - 2y = 6 \end{cases}$$

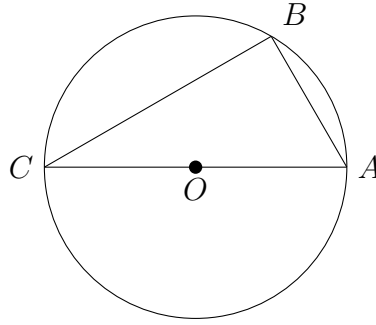
which has solution

$$\begin{cases} x = -1 + 2t \\ y = -2 - 2t \\ z = t \end{cases}$$

Thus, $\mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

$$\left\{ \mathbf{w} : \mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \right\}$$

Problem 9. NJC Prelim 9758/2017/01/Q8



The diagram above shows the cross-section of a sphere containing the centre O of the sphere. The points A , B and C are on the circumference of the cross-section with the line segment AC passing through O . It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Find \overrightarrow{BC} in terms of \mathbf{a} and \mathbf{b} .

(b) D is a point on BC such that $\triangle OCD$ is similar to $\triangle ACB$. Find \overrightarrow{OD} in terms of \mathbf{a} and \mathbf{b} .

Point B lies on the x - z plane and has a positive z -component. It is also given that $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\angle OCB = \frac{\pi}{6}$.

(c) Show that $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$.

(d) Hence, determine whether the line passing through O and B and the line $\frac{x-2}{3} = \frac{y}{3} = z-1$ are skew.

Solution

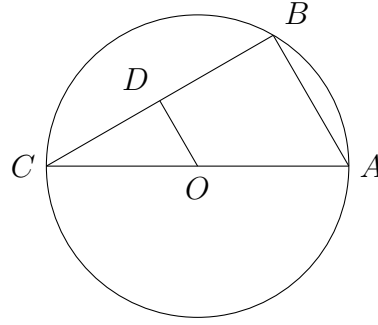
Part (a)

By symmetry, we have $\overrightarrow{OC} = -\overrightarrow{OA}$. Hence,

$$\begin{aligned} \overrightarrow{OC} &= -\overrightarrow{OA} \\ \Rightarrow \overrightarrow{OB} + \overrightarrow{BC} &= -\overrightarrow{OA} \\ \Rightarrow \overrightarrow{BC} &= -\overrightarrow{OA} - \overrightarrow{OB} \\ &= -\mathbf{a} - \mathbf{b} \end{aligned}$$

$\overrightarrow{BC} = -\mathbf{a} - \mathbf{b}$

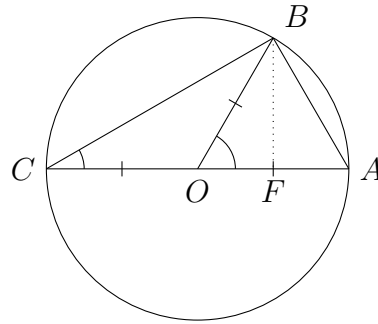
Part (b)



Since $\triangle OCD$ is similar to $\triangle ACB$, we have $\frac{1}{2} = \frac{OC}{AC} = \frac{OD}{AB} \implies OD = \frac{1}{2}AB$. Since $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, we have

$$\boxed{\overrightarrow{OD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})}$$

Part (c)



It is given that $\angle OCB = \frac{\pi}{6}$. Since the angle at the centre is twice the angle at the circumference, we have $\angle AOB = 2\angle OCB = \frac{\pi}{3}$. Since $OB = OA$, it must be that $\triangle OAB$ is equilateral. Let F be the foot of the perpendicular from B to OA . Note that $OB = OC = 2$. Thus, $\cos \angle AOB = \frac{OF}{OB} \implies OF = 1 \implies \overrightarrow{OF} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$. Further,

note that $\sin \angle AOB = \frac{FB}{OB} \implies FB = \sqrt{3}$. Since \overrightarrow{OB} has a positive z -component, we have

$$\begin{aligned} \overrightarrow{OB} &= \overrightarrow{OF} + \overrightarrow{FB} \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \end{aligned}$$

$$\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$$

Part (d)

Observe that the line with Cartesian equation $\frac{x-2}{3} = \frac{y}{3} = z-1$ has vector equation

$$l_C : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Also note that the line OB has equation

$$l_{OB} : \mathbf{r} = \mu \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}, \mu \in \mathbb{R}$$

Consider $l_C = l_{OB}$.

$$\begin{aligned} l_C &= l_{OB} \\ \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} &= \mu \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \\ \Rightarrow \mu \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the system

$$\begin{cases} \mu - 3\lambda = 2 \\ 3\lambda = 0 \\ \sqrt{3}\mu - \lambda = 1 \end{cases}$$

which has no solutions. Hence, the lines are skew.

The lines are skew.

Problem 10. NYJC Prelim 9758/2017/02/Q1

The position vectors of points A and B with respect to the origin O are \mathbf{a} and \mathbf{b} respectively where \mathbf{a} and \mathbf{b} are non-zero vectors. Point C lies on OA produced such that $4OA = AC$ and point D lies on OB produced such that $OB = BD$. The line BC and AD meet at the point M .

- (a) Giving a necessary condition for \mathbf{a} and \mathbf{b} , find the position vector of M in terms of \mathbf{a} and \mathbf{b} .
- (b) If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, find the shortest distance of M from the line OC , giving your answer in the form $k|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be determined.

Solution

Part (a)

\mathbf{a} and \mathbf{b} must be non-parallel.

Note that $\overrightarrow{OC} = 5\mathbf{a}$ and $\overrightarrow{OD} = 2\mathbf{b}$. Hence, $\overrightarrow{AD} = 2\mathbf{b} - \mathbf{a}$ and $\overrightarrow{BC} = 5\mathbf{a} - \mathbf{b}$. Thus, the lines AD and BC have vector equations

$$\begin{aligned} l_{AD} : \mathbf{r} &= \mathbf{a} + \lambda(2\mathbf{b} - \mathbf{a}), \lambda \in \mathbb{R} \\ l_{BC} : \mathbf{r} &= \mathbf{b} + \mu(5\mathbf{a} - \mathbf{b}), \mu \in \mathbb{R} \end{aligned}$$

Consider $l_{AD} = l_{BC}$.

$$\begin{aligned} l_{AD} &= l_{BC} \\ \implies \mathbf{a} + \lambda(2\mathbf{b} - \mathbf{a}) &= \mathbf{b} + \mu(5\mathbf{a} - \mathbf{b}) \end{aligned}$$

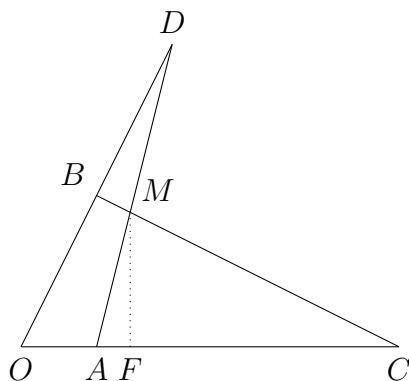
Comparing coefficients of \mathbf{a} and \mathbf{b} , we have the system

$$\begin{cases} 1 - \lambda = 5\mu \\ 2\lambda = 1 - \mu \end{cases}$$

which has solution $\lambda = \frac{4}{9}$ and $\mu = \frac{1}{9}$. Thus,

$$\begin{aligned} \overrightarrow{OM} &= \mathbf{b} + \frac{1}{9}(5\mathbf{a} - \mathbf{b}) \\ &= \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b} \end{aligned}$$

$$\overrightarrow{OM} = \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}$$

Part (b)

Let F be the foot of the perpendicular of M to OC . Observe that $\overrightarrow{OM} = \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b} - \mathbf{a} = -\frac{4}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}$ and $\overrightarrow{AC} = 5\mathbf{a} - \mathbf{a} = 4\mathbf{a}$.

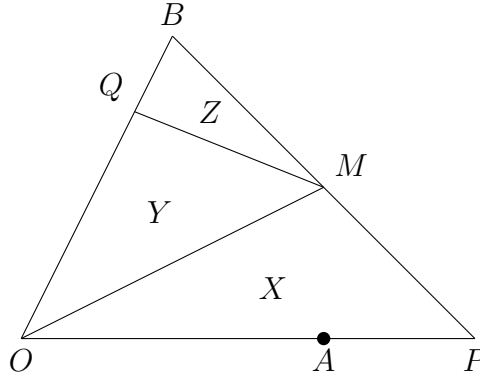
$$\begin{aligned}
 & \left| \overrightarrow{AM} \times \overrightarrow{AC} \right| = 2 \text{ Area } \triangle AMC \\
 \Rightarrow & \left| \left(-\frac{4}{9}\mathbf{a} + \frac{8}{9}\mathbf{b} \right) \times 4\mathbf{a} \right| = 2 \cdot \frac{1}{2} \cdot FM \cdot AC \\
 \Rightarrow & \left| -\frac{16}{9}\mathbf{a} \times \mathbf{a} + 4 \cdot \frac{8}{9}\mathbf{b} \times \mathbf{a} \right| = FM \cdot |4\mathbf{a}| \\
 \Rightarrow & 4 \cdot \frac{8}{9} \cdot |\mathbf{b} \times \mathbf{a}| = 4 \cdot FM \\
 \Rightarrow & FM = \frac{8}{9} |\mathbf{b} \times \mathbf{a}| \\
 & = \frac{8}{9} |\mathbf{a} \times \mathbf{b}|
 \end{aligned}$$

The shortest distance of M from the line OC is $\frac{8}{9} |\mathbf{a} \times \mathbf{b}|$ units.

Problem 11. PJC Prelim 9758/2017/02/Q1

Referred to the origin O , points A and B have position vectors \mathbf{a} and \mathbf{b} respectively. Point P lies on OA produced such that $OA : AP = 1 : \lambda$. Point Q lies on OB , between O and B , such that $OQ : QB = 3 : 1$. The mid-point of PB is M . Show that the ratio of the area of triangle OPM to the area of triangle OQM is independent of λ .

Solution



Let the area of $\triangle OPM$, $\triangle OQM$ and $\triangle BQM$ be X , Y and Z respectively. Since $\triangle OPM$ and $\triangle BOM$ share the same height and $BM = MP$, we have

$$X = Y + Z$$

Similarly, since $\triangle OQM$ and $\triangle BQM$ share the same height and $OQ = 3QM$, we have

$$Y = 3Z$$

Thus, $X = Y + \frac{1}{3}Y$, whence $\frac{\text{Area } \triangle OPM}{\text{Area } \triangle OQM} = \frac{X}{Y} = \frac{4}{3}$. Thus, the required ratio is independent of λ .

Problem 12. RI Prelim 9758/2017/02/Q1

Referred to the origin O , the points A , B and C have position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} respectively such that

$$\begin{aligned}\mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \\ \mathbf{b} &= 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \\ \mathbf{c} &= 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}\end{aligned}$$

- (a) Given that M is the mid-point of AC , use a vector product to find the exact area of triangle ABM .
- (b) Find the position vector of the point N on the line AB such that \overrightarrow{MN} is perpendicular to \overrightarrow{AB} .

Solution

We have $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -2 \\ 3 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$.

Part (a)

By the Midpoint Theorem, $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \begin{pmatrix} 3 \\ 2 \\ -3/2 \end{pmatrix}$. Thus, $\overrightarrow{AM} = \begin{pmatrix} 1 \\ -1 \\ -1/2 \end{pmatrix}$. We

also have $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}$. Hence,

$$\begin{aligned}\text{Area } \triangle ABM &= \frac{1}{2} \left| \overrightarrow{AM} \times \overrightarrow{AB} \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 1 \\ -1 \\ -1/2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \right| \\ &= \frac{1}{4} \left| \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 13 \\ 11 \\ 4 \end{pmatrix} \right| \\ &= \frac{\sqrt{306}}{4} \\ &= \frac{3\sqrt{34}}{4}\end{aligned}$$

$$\text{Area } \triangle ABM = \frac{3\sqrt{34}}{4} \text{ units}^2$$

Part (b)

Note that the line AB has vector equation

$$l_{AB} : \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

Since \overrightarrow{MN} is perpendicular to \overrightarrow{AB} , we have

$$\begin{aligned} & \overrightarrow{MN} \cdot \overrightarrow{AB} = 0 \\ \Rightarrow & (\overrightarrow{ON} - \overrightarrow{OM}) \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = 0 \\ \Rightarrow & \left[\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -3/2 \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = 0 \\ \Rightarrow & \left[\begin{pmatrix} -1 \\ 1 \\ 1/2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = 0 \\ \Rightarrow & \begin{pmatrix} -1 \\ 1 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = 0 \\ \Rightarrow & -6 + 50\lambda = 0 \\ \Rightarrow & \lambda = \frac{3}{25} \end{aligned}$$

Hence, $\overrightarrow{ON} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \frac{3}{25} \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 59 \\ 60 \\ -13 \end{pmatrix}.$

$$\overrightarrow{ON} = \frac{1}{25} \begin{pmatrix} 59 \\ 60 \\ -13 \end{pmatrix}$$