Problem 1.

(a) Given that $f(x) = e^{\cos x}$, find $f(0)$, $f'(0)$ and $f''(0)$. Hence, write down the first two non-zero terms in the MacLaurin series for $f(x)$. Give the coefficients in terms of e.

(b) Given that $g(x) = \tan\left(2x + \frac{1}{4}\pi\right)$, find $g(0)$, $g'(0)$ and $g''(0)$. Hence, find the first three terms in the MacLaurin series of $g(x)$.

Solution

Part (a)

$$\begin{aligned} f(x) &= e^{\cos x} \\ \implies f'(x) &= e^{\cos x} \cdot -\sin x \\ &= -\sin x \cdot f(x) \\ \implies f''(x) &= -\cos x \cdot f(x) - \sin x \cdot f'(x) \end{aligned}$$

Evaluating the above derivatives at $x = 0$,

$$\begin{aligned} f(0) &= e \\ f'(0) &= 0 \\ f''(0) &= -e \end{aligned}$$

Hence,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{e}{0!}x^0 + \frac{0}{1!}x^1 + \frac{-e}{2!}x^2 + \dots \\ &= e - \frac{e}{2}x^2 + \dots \end{aligned}$$

$$\boxed{f(x) = e - \frac{e}{2}x^2 + \dots}$$

Part (b)

$$\begin{aligned} g(x) &= \tan\left(2x + \frac{1}{4}\pi\right) \\ \implies g'(x) &= \sec^2\left(2x + \frac{1}{4}\pi\right) \cdot 2 \\ &= 2\left(1 + \tan^2\left(2x + \frac{1}{4}\pi\right)\right) \\ &= 2 + 2g^2(x) \\ \implies g''(x) &= 2 \cdot 2g(x) \cdot g'(x) \\ &= 4g(x)g'(x) \end{aligned}$$

Evaluating the above derivatives at $x = 0$,

$$\begin{aligned} g(x) &= 1 \\ g'(x) &= 4 \\ g''(x) &= 16 \end{aligned}$$

Hence,

$$\begin{aligned} g(x) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n \\ &= \frac{1}{0!} x^0 + \frac{4}{1!} x^1 + \frac{16}{2!} x^2 + \dots \\ &= 1 + 4x + 8x^2 + \dots \end{aligned}$$

$$\boxed{g(x) = 1 + 4x + 8x^2 + \dots}$$

Problem 2.

Find the first three non-zero terms of the MacLaurin series for the following in ascending powers of x. In each case, find the range of values of x for which the series is valid.

(a) $\dfrac{(1+3x)^4}{\sqrt{1+2x}}$

(b) $\dfrac{\sin 2x}{2+3x}$

Solution

Part (a)

$$\begin{aligned} && y &= \frac{(1+3x)^4}{\sqrt{1+2x}} && (2.1)\\ \implies && y^2 &= \frac{(1+3x)^8}{1+2x} \\ \implies && (1+2x)\cdot y^2 &= (1+3x)^8 && (2.2)\end{aligned}$$

Implicitly differentiating Equation 2.2,

$$\begin{aligned} && (1+2x)\cdot 2y\cdot y' + y^2\cdot 2 &= 8(1+3x)^7\cdot 3 \\ \implies && (1+2x)\cdot y\cdot y' + y^2 &= 12(1+3x)^7 \\ \implies && y\left((1+2x)\cdot y' + y\right) &= 12(1+3x)^7 && (2.3)\end{aligned}$$

Implicitly differentiating Equation 2.3,

$$\begin{aligned} && y'\big((1+2x)\cdot y' + y\big) + y\big((1+2x)\cdot y'' + y'\cdot 2 + y'\big) &= 12\cdot 7(1+3x)^6\cdot 3 \\ \implies && (1+2x)\big(y'\big)^2 + (1+2x)y\cdot y'' + 4y\cdot y' &= 252(1+3x)^6 && (2.4)\end{aligned}$$

Evaluating Equations 2.1, 2.3 and 2.4 at $x=0$,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 11 \\ y''(0) &= 87 \end{aligned}$$

Hence,

$$\begin{aligned} \frac{(1+3x)^4}{\sqrt{1+2x}} &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!}x^n \\ &= \frac{1}{0!}x^0 + \frac{11}{1!}x^1 + \frac{87}{2!}x^2 + \dots \\ &= 1 + 11x + \frac{87}{2}x^2 + \dots \end{aligned}$$

$$\boxed{\frac{(1+3x)^4}{\sqrt{1+2x}} = 1 + 11x + \frac{87}{2}x^2 + \dots}$$

Note that the series is valid only when $|2x| < 1 \implies -\frac{1}{2} < x < \frac{1}{2}$.

$$\boxed{-\frac{1}{2} < x < \frac{1}{2}}$$

Part (b)

$$y = \frac{\sin 2x}{2+3x} \tag{2.5}$$
$$\implies (2+3x)y = \sin 2x \tag{2.6}$$

Implicitly differentiating Equation 2.6,

$$\begin{aligned} &(2+3x)y' + y \cdot 3 = \cos 2x \cdot 2 \\ \implies \quad &(2+3x)y' + 3y = 2\cos 2x \end{aligned} \tag{2.7}$$

Implicitly differentiating Equation 2.7,

$$\begin{aligned} (2+3x)y'' + y' \cdot 3 + 3y' &= 2 \cdot -\sin 2x \cdot 2 \\ \implies \quad (2+3x)y'' + 6y' &= -4\sin 2x \end{aligned} \tag{2.8}$$

Implicitly differentiating Equation 2.8,

$$\begin{aligned} (2+3x)y''' + y'' \cdot 3 + 6y'' &= -4 \cdot \cos 2x \cdot 2 \\ \implies \quad (2+3x)y''' + 9y'' &= -8\cos 2x \end{aligned} \tag{2.9}$$

Evaluating Equations 2.5, 2.7, 2.8 and 2.9 at $x = 0$,

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \\ y''(0) &= -3 \\ y'''(0) &= \frac{19}{2} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\sin 2x}{2+3x} &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= \frac{0}{0!}x^0 + \frac{1}{1!}x^1 + \frac{-3}{2!}x^2 + \frac{\frac{19}{2}}{3!}x^3 + \dots \\ &= x - \frac{3}{2}x^2 + \frac{19}{12}x^3 + \dots \end{aligned}$$

$$\boxed{\frac{\sin 2x}{2+3x} = x - \frac{3}{2}x^2 + \frac{19}{12}x^3 + \dots}$$

Note that the denominator can be rewritten as $2\left(1 + \frac{3}{2}x\right)$. Hence, the series is only valid when $\left|\frac{3}{2}x\right| < 1 \implies -\frac{2}{3} < x < \frac{2}{3}$.

Problem 3.

Find the MacLaurin series of $\ln(1+\cos x)$, up to and including the term in x^4.

Solution

Recall that

$$\ln(1+x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

and

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}$$

Hence,

$$\begin{aligned}\ln(1+\cos x)&=\sum_{n=0}^{\infty}(-1)^n\frac{\cos^{n+1}x}{n+1}\\&=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}\right)^{n+1}\end{aligned}$$

Consider $\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}\right)^{n+1}$, which is equivalent to

$$\underbrace{\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)\dots\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)}_{(n+1)\text{ copies}}$$

The constant term is clearly 1. Now consider the coefficient of the x^2 term. The only to obtain an x^2 term is to select a constant term (1) from n copies, and an x^2 term $\left(-\frac{x^2}{2!}\right)$ from the remaining copy. There are $\binom{n+1}{1}=n+1$ ways to do this. Hence, the coefficient of the x^2 term is $(n+1)\cdot 1\cdot -\frac{1}{2!}=-\frac{n+1}{2}$.

Now consider the coefficient of the x^4 term. There are two ways to obtain an x^4 term. The first way is to select a constant term (1) from n copies, and an x^4 term $\left(\frac{x^4}{4!}\right)$ from the remaining copy. There are $\binom{n+1}{1}=n+1$ ways to do this, which contributes $(n+1)\cdot 1\cdot\frac{1}{4!}=\frac{n+1}{24}$ to the coefficient of x^4.

The second way to obtain an x^4 term is to select an x^2 term $\left(-\frac{x^2}{2!}\right)$ from 2 copies and a constant term (1) from the remaining copies. There are $\binom{n+1}{2}=\frac{(n+1)n}{2}$ ways to do this, which further contributes $\frac{(n+1)n}{2}\cdot 1\cdot\left(-\frac{1}{2!}\right)^2=\frac{n(n+1)}{8}$ to the coefficient of x^4.

Hence, the coefficient of x^4 is given by $\frac{n+1}{24}+\frac{n(n+1)}{8}=\frac{(n+1)(3n+1)}{24}$.

Thus, up to and including the term in x^4,

$$\begin{aligned}
\ln(1+\cos x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\left(1 - \frac{n+1}{2}x^2 + \frac{(n+1)(3n+1)}{24} + \dots\right) \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+1} - \frac{1}{2}x^2 + \frac{3n+1}{24}x^4 + \dots\right) \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} - \frac{1}{2}x^2 \sum_{n=0}^{\infty}(-1)^n + \frac{3}{24}x^4 \sum_{n=0}^{\infty} n(-1)^n + \frac{1}{24}x^4 \sum_{n=0}^{\infty}(-1)^n + \dots
\end{aligned}$$

Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} &= \sum_{n=0}^{\infty}(-1)^n \frac{1^{n+1}}{n+1} \\
&= \ln(1+1) \\
&= \ln 2
\end{aligned}$$

Now consider the Abel regularization of $\sum_{n=0}^{\infty}(-1)^n$.

$$\begin{aligned}
\sum_{n=0}^{\infty}(-1)^n &= \lim_{x\to 1^-} \sum_{n=0}^{\infty}(-1)^n x^n \\
&= \lim_{x\to 1^-} \sum_{n=0}^{\infty}(-x)^n \\
&= \lim_{x\to 1^-} \frac{1}{1-(-x)} \\
&= \frac{1}{2}
\end{aligned}$$

Now observe that $\sum_{n=0}^{\infty} x^n$ is absolutely convergent for $|x| < 1$. Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^{n-1} &= \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} x^n \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} x^n \\
&= \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} \\
&= \frac{1}{(1-x)^2}
\end{aligned}$$

Multiplying by x on both sides gives

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Hence, the Abel regularization of $\sum_{n=0}^{\infty} n(-1)^n$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} n(-1)^n &= \lim_{x\to 1^-} \sum_{n=0}^{\infty} n(-1)^n x^n \\ &= \lim_{x\to 1^-} \sum_{n=0}^{\infty} n(-x)^n \\ &= \lim_{x\to 1^-} \frac{-x}{(1-(-x))^2} \\ &= -\frac{1}{4}\end{aligned}$$

Finally,

$$\begin{aligned}\ln(1+\cos x) &= \ln 2 - \frac{1}{2}x^2 \cdot \frac{1}{2} + \frac{3}{24}x^4 \cdot -\frac{1}{4} + \frac{1}{24}x^4 \cdot \frac{1}{2} + \dots \\ &= \ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \dots\end{aligned}$$

$$\boxed{\ln(1+\cos x) = \ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \dots}$$

Problem 4.

(a) Find the first three terms of the MacLaurin series for $e^x(1+\sin 2x)$.

(b) It is given that the first two terms of this series are equal to the first two terms in the series expansion, in ascending powers of x, of $\left(1+\frac{4}{3}x\right)^n$. Find n and show that the third terms in each of these series are equal.

Solution

Part (a)

$$\begin{aligned}
f(x) &= e^x(1+\sin 2x) \\
&= e^x + e^x \sin 2x \\
&= e^x + e^x \operatorname{Im}\left(e^{i2x}\right) \\
&= e^x + \operatorname{Im}\left(e^x e^{i2x}\right) \\
&= e^x + \operatorname{Im}\left(e^{x(1+2i)}\right) \\
\implies f^{(n)}(x) &= e^x + \operatorname{Im}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{x(1+2i)}\right) \\
&= e^x + \operatorname{Im}\left((1+2i)^n e^{x(1+2i)}\right) \\
&= e^x + \operatorname{Im}\left(\left(\sqrt{5}e^{i\arctan 2}\right)^n e^{x(1+2i)}\right) \\
&= e^x + \operatorname{Im}\left(5^{\frac{n}{2}} e^{in\arctan 2} e^{x(1+2i)}\right) \\
&= e^x + 5^{\frac{n}{2}} e^x \operatorname{Im}\left(e^{i(n\arctan 2+2x)}\right) \\
&= e^x + 5^{\frac{n}{2}} e^x \sin(n\arctan 2 + 2x) \\
\implies f^{(n)}(0) &= 1 + 5^{\frac{n}{2}} e^x \sin(n\arctan 2)
\end{aligned}$$

Hence,

$$\begin{aligned}
f^{(0)}(0) &= 1 + 5^{\frac{0}{2}} e^x \sin(0\arctan 2) \\
&= 1 \\
f^{(1)}(0) &= 1 + 5^{\frac{1}{2}} e^x \sin(1\arctan 2) \\
&= 1 + \sqrt{5} \cdot \frac{2}{\sqrt{5}} \\
&= 3 \\
f^{(2)}(0) &= 1 + 5^{\frac{2}{2}} e^x \sin(2\arctan 2) \\
&= 1 + 5 \cdot 2\sin(\arctan 2)\cos(\arctan 2) \\
&= 1 + 5 \cdot 2 \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \\
&= 5
\end{aligned}$$

Thus,

$$\begin{aligned}
e^x(1+\sin 2x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n \\
&= \frac{1}{0!}x^0 + \frac{3}{1!}x^1 + \frac{5}{2!}x^2 + \dots \\
&= 1 + 3x + \frac{5}{2}x^2 + \dots
\end{aligned}$$

$$\boxed{e^x(1+\sin 2x) = 1 + 3x + \frac{5}{2}x^2 + \dots}$$

Part (b)

By the Binomial Theorem,

$$\begin{aligned}
\left(1+\frac{4}{3}x\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{4}{3}x\right)^k 1^{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{4}{3}\right)^k x^k \\
&= \binom{n}{0} \left(\frac{4}{3}\right)^0 x^0 + \binom{n}{1} \left(\frac{4}{3}\right)^1 x^1 + \dots \\
&= 1 + \frac{4}{3}nx + \dots
\end{aligned}$$

Comparing the coefficient of x terms, we have $3 = \frac{4}{3}n$, whence $n = \frac{9}{4}$. Hence, the third term is in the expansion of $\left(1+\frac{4}{3}x\right)^n$ is given by

$$\begin{aligned}
\binom{9/4}{2} \left(\frac{4}{3}\right)^2 x^2 &= \frac{9/4 \cdot (9/4-1)}{2} \left(\frac{4}{3}\right)^2 x^2 \\
&= \frac{5}{2}x^2
\end{aligned}$$

Hence, the third terms in each of these series are equal.

Problem 5.

(a) Show that the first three non-zero terms in the expansion of $\left(\frac{8}{x^3}-1\right)^{1/3}$ in ascending powers of x are in the form $\frac{a}{x}+bx^2+cx^5$, where a, b and c are constants to be determined.

(b) By putting $x=\frac{2}{3}$ in your result, obtain an approximation for $\sqrt[3]{26}$ in the form of a fraction in its lowest terms.

A student put $x=6$ into the expansion to obtain an approximation of $\sqrt[3]{26}$. Comment on the suitability of this choice of x for the approximation of $\sqrt[3]{26}$.

Solution

Part (a)

$$\begin{aligned}
\left(\frac{8}{x^3}-1\right)^{\frac{1}{3}} &= \frac{2}{x}\left(1-\frac{x^3}{8}\right)^{\frac{1}{3}} \\
&= \frac{2}{x}\sum_{k=0}^{\infty}\binom{1/3}{k}\left(-\frac{x^3}{8}\right)^k \\
&= \frac{2}{x}\left[\binom{1/3}{0}\left(-\frac{x^3}{8}\right)^0+\binom{1/3}{1}\left(-\frac{x^3}{8}\right)^1+\binom{1/3}{2}\left(-\frac{x^3}{8}\right)^2+\dots\right] \\
&= \frac{2}{x}\left(1+\frac{1}{3}\cdot-\frac{x^3}{8}+\frac{1/3\cdot(1/3-1)}{2}\cdot\frac{x^6}{64}+\dots\right) \\
&= \frac{2}{x}-\frac{x^2}{12}-\frac{x^5}{288}+\dots
\end{aligned}$$

Part (b)

Evaluating the above equation at $x=\frac{2}{3}$,

$$\begin{aligned}
\left(\frac{8}{(2/3)^3}-1\right)^{1/3} &= \frac{2}{2/3}-\frac{(2/3)^2}{12}-\frac{(2/3)^5}{288}+\dots \\
\implies \qquad \sqrt[3]{26} &= 3-\frac{1}{27}-\frac{1}{2187} \\
&= \frac{6479}{2187}
\end{aligned}$$

$$\boxed{\sqrt[3]{26}=\frac{6479}{2187}}$$

Since $|6|>1$, the binomial expansion of $\left(\frac{8}{x^3}-1\right)^{1/3}$ does not hold. Hence, $x=6$ is not an appropriate choice.

Problem 6.

Let $f(x) = e^x \sin x$.

(a) Sketch the graph of $y = f(x)$ for $-3 \le x \le 3$.

(b) Find the series expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3.

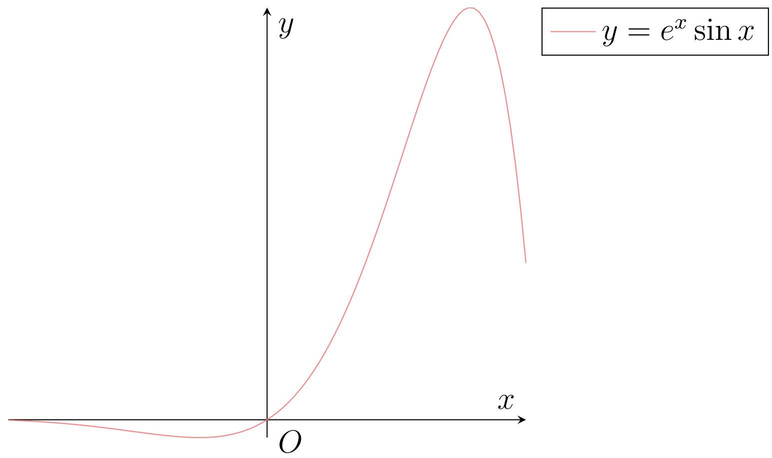
Denote the answer to part (b) by $g(x)$.

(c) On the same diagram, sketch the graph of $y = f(x)$ and $y = g(x)$. Label the two graphs clearly.

(d) Find, for $-3 \le x \le 3$, the set of values of x for which the value of $g(x)$ is within ± 0.5 of the value of $f(x)$.

Solution

Part (a)



Part (b)

$$\begin{aligned}
f(x) &= e^x \sin x \\
&= e^x \operatorname{Im}\left(e^{ix}\right) \\
&= \operatorname{Im}\left(e^x e^{ix}\right) \\
&= \operatorname{Im}\left(e^{x(1+i)}\right) \\
\implies f^{(n)}(x) &= \operatorname{Im}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{x(1+i)}\right) \\
&= \operatorname{Im}\left((1+i)^n e^{x(1+i)}\right) \\
&= \operatorname{Im}\left(\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^n e^{x(1+i)}\right) \\
&= \operatorname{Im}\left(2^{\frac{n}{2}} e^x e^{i\frac{\pi}{4}n} e^{ix}\right) \\
&= 2^{\frac{n}{2}} e^x \operatorname{Im}\left(e^{i\left(\frac{\pi}{4}n + x\right)}\right) \\
&= 2^{\frac{n}{2}} e^x \sin\left(\tfrac{\pi}{4}n + x\right)
\end{aligned}$$

Evaluating $f^{(n)}(x)$ at $x = 0$,

$$f^{(n)}(x) = 2^{\frac{n}{2}} \sin\left(\left(\tfrac{\pi}{4}n\right)\right)$$

Hence,

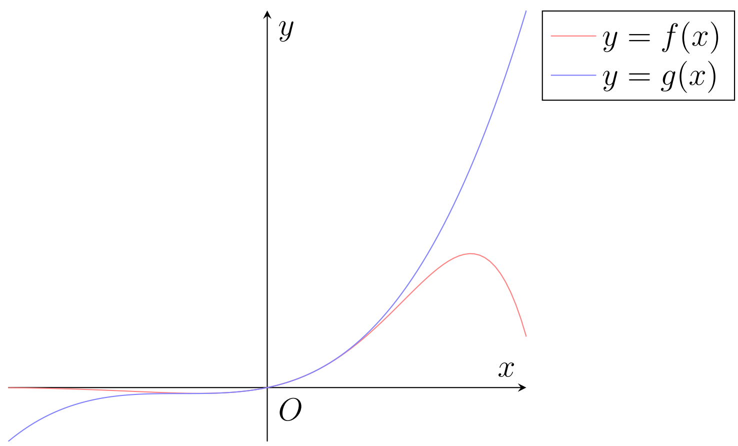
$$\begin{aligned}
f(0) &= 2^{\frac{0}{2}} \sin\left(\tfrac{\pi}{4} \cdot 0\right) = 0 \\
f'(0) &= 2^{\frac{1}{2}} \sin\left(\tfrac{\pi}{4} \cdot 1\right) = 1 \\
f''(0) &= 2^{\frac{2}{2}} \sin\left(\tfrac{\pi}{4} \cdot 2\right) = 2 \\
f^{(3)}(0) &= 2^{\frac{3}{2}} \sin\left(\tfrac{\pi}{4} \cdot 2\right) = 2
\end{aligned}$$

Thus,

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
&= \frac{f^{(0)}(0)}{0!} x^0 + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots \\
&= x + x^2 + \frac{1}{3} x^3 + \dots
\end{aligned}$$

$$\boxed{f(x) = x + x^2 + \frac{1}{3}x^3 + \dots}$$

Part (c)



Part (d)

Consider $|f(x) - g(x)| \leq 0.5$ for $-3 \leq x \leq 3$, where $g(x) = x + x^2 + \dfrac{1}{3}x^3$.

Case 1: $f(x) - g(x) \leq 0.5$

$$\begin{aligned}
& f(x) - g(x) \leq 0.5 \\
\implies & e^x \sin x - \left(x + x^2 + \frac{1}{3}x^3\right) \leq 0.5 \\
\implies & \qquad\qquad x \geq -1.96
\end{aligned}$$

Case 2: $-\left[f(x)-g(x)\right] \leq 0.5$

$$\begin{aligned}
& -\left[f(x)-g(x)\right] \leq 0.5 \\
\implies & g(x)-f(x) \leq 0.5 \\
\implies & x+x^2+\frac{1}{3}x^3-e^x\sin x \leq 0.5 \\
\implies & x \leq 1.56
\end{aligned}$$

Putting both inequalities together, we have

$$\boxed{-1.96 \leq x \leq 1.56}$$

Problem 7.

It is given that $y = \dfrac{1}{1+\sin 2x}$. Show that, when $x = 0$, $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 8$. Find the first three terms of the MacLaurin series for y.

(a) Use the series to obtain an approximate value for $\displaystyle\int_{-0.1}^{0.1} y \,\mathrm{d}x$, leaving your answer as a fraction in its lowest terms.

(b) Find the first two terms of the MacLaurin series for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(c) Write down the equation of the tangent at the point where $x = 0$ on the curve $y = \dfrac{1}{1+\sin 2x}$.

Solution

$$
\begin{aligned}
y &= \frac{1}{1+\sin 2x} && (7.1)\\
\implies y' &= -\frac{1}{(1+\sin 2x)^2} \cdot (\cos 2x \cdot 2)\\
&= -2y^2 \cos 2x && (7.2)\\
\implies y'' &= -2\left(\cos 2x \cdot 2y \cdot y' + y^2 \cdot -\sin 2x \cdot 2\right)\\
&= -4\left(y \cdot y' \cos 2x - y^2 \sin 2x\right) && (7.3)
\end{aligned}
$$

From Equations 7.1, 7.2 and 7.3,

$$
\begin{aligned}
y(0) &= 1\\
y'(0) &= -2\\
y''(0) &= 8
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{1}{1+\sin 2x} &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n\\
&= \frac{y(0)}{0!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \dots\\
&= 1 - 2x + 4x^2 + \dots
\end{aligned}
$$

Part (a)

$$
\begin{aligned}
\int_{-0.1}^{0.1} y \,\mathrm{d}x &\approx \int_{-0.1}^{0.1} \left(1 - 2x + 4x^2\right) \mathrm{d}x\\
&= \left[x - 2 \cdot \frac{1}{2}x^2 + 4 \cdot \frac{1}{3}x^3\right]_{-0.1}^{0.1}\\
&= \frac{76}{275}
\end{aligned}
$$

$$\boxed{\int_{-0.1}^{0.1} y \,\mathrm{d}x \approx \frac{76}{275}}$$

Part (b)

$$\begin{aligned} y' &= \frac{\mathrm{d}}{\mathrm{d}x} y \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(1 - 2x + 4x^2 + \ldots\right) \\ &= -2 + 8x + \ldots \end{aligned}$$

$$\boxed{y' = -2 + 8x + \ldots}$$

Part (c)

Using the point-slope formula,

$$\begin{aligned} y - 1 &= -2(x - 0) \\ \implies \quad y &= -2x + 1 \end{aligned}$$

$$\boxed{y = -2x + 1}$$

Problem 8.

It is given that $y = e^{\arcsin 2x}$.

(a) Show that $(1-4x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4y$.

(b) By further differentiating this result, find the MacLaurin series for y in ascending powers of x, up to an including the term in x^3.

(c) Hence, find an approximation value of $e^{\frac{\pi}{2}}$, by substituting a suitable value of x in the MacLaurin series for y.

(d) Suggest one way to improve the accuracy of the approximated value obtained.

Solution

Part (a)

$$y = e^{\arcsin 2x} \tag{8.1}$$

$$\implies \ln y = \arcsin 2x \tag{8.2}$$

Implicitly differentiating Equation 8.2,

$$\begin{aligned}\frac{y'}{y} &= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1-4x^2}}\end{aligned}$$

$$\implies y'\sqrt{1-4x^2} = 2y \tag{8.3}$$

Implicitly differentiating Equation 8.3,

$$\begin{aligned} y''\sqrt{1-4x^2} + y'\frac{1}{2\sqrt{1-4x^2}} \cdot -8x &= 2y' \\ \implies \left(1-4x^2\right) y'' - 4xy' &= 2y'\sqrt{1-4x^2} \\ &= 2\left(\frac{2y}{\sqrt{1-4x^2}}\right)\sqrt{1-4x^2} \\ &= 4y \end{aligned} \tag{8.4}$$

Part (b)

Implicitly differentiating Equation 8.4,

$$y^{(3)}(1-4x^2) + y'' \cdot -8x - 4(xy'' + y') = 4y'$$

$$\implies y^{(3)}(1-4x^2) - 12xy'' - 8y' = 0 \tag{8.5}$$

From Equations 8.1, 8.3, 8.4 and 8.5,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 2 \\ y''(0) &= 4 \\ y^{(3)}(0) &= 16 \end{aligned}$$

Hence,

$$
\begin{aligned}
y &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\
&= \frac{y(0)}{1!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \dots \\
&= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \dots
\end{aligned}
$$

$$
\boxed{y = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \dots}
$$

Part (c)

Consider $y = e^{\frac{\pi}{2}} \implies \arcsin 2x = \frac{\pi}{2} \implies x = \frac{1}{2} \cdot \sin\frac{\pi}{2} = \frac{1}{2}$. Hence, substituting $x = \frac{1}{2}$ in the MacLaurin series for y,

$$
\begin{aligned}
e^{\frac{\pi}{2}} &\approx 1 + 2 \cdot \frac{1}{2} + 2\left(\frac{1}{2}\right)^2 + \frac{8}{3}\left(\frac{1}{2}\right)^3 \\
&= \frac{17}{6}
\end{aligned}
$$

$$
\boxed{e^{\frac{\pi}{2}} \approx \frac{17}{6}}
$$

Part (d)

More terms of the MacLaurin series of y could be considered.

Problem 9.

The curve $y = f(x)$ passes through the point $(0, 1)$ and satisfies the equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{6-2y}{\cos 2x}$.

(a) Find the MacLaurin series of $f(x)$, up to and including the term in x^3.

(b) Using standard results given in the List of Formulae (MF27), express $\dfrac{1-\sin x}{\cos x}$ as a power series of x, up to and including the term in x^3.

(c) Using the two power series you have found, show to this degree of approximation, that $f(x)$ can be expressed as $a(\tan 2x - \sec 2x) + b$, where a and b are constants to be determined.

Solution

Part (a)

$$\begin{aligned} y' &= \frac{6-2y}{\cos 2x} \\ \implies y'\cos 2x &= 6 - 2y \end{aligned} \tag{9.1}$$

Implicitly differentiating Equation 9.1,

$$\begin{aligned} -\sin 2x \cdot 2 \cdot y' + y''\cos 2x &= -2y' \\ \implies \quad -2y'\sin 2x + y''\cos 2x &= -2y' \end{aligned} \tag{9.2}$$

Implicitly differentiating Equation 9.2,

$$\begin{aligned} -2\left(y''\sin 2x + y'\cos 2x \cdot 2\right) + \left(y'' \cdot -\sin 2x \cdot 2 + y^{(3)}\cos 2x\right) &= -2y'' \\ \implies \quad -4y'\cos 2x - 3y''\sin 2x + y^{(3)}\cos 2x &= -2y'' \end{aligned} \tag{9.3}$$

Given that y passes through the point $(0, 1)$, and from Equations 9.1, 9.2 and 9.3,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 4 \\ y''(0) &= -8 \\ y^{(3)}(0) &= 32 \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= \frac{y(0)}{0!}x^0 + \frac{y'(0)}{1!}x^1 + \frac{y''(0)}{2!}x^2 + \frac{y^{(3)}(0)}{3!}x^3 + \dots \\ &= 1 + 4x - 4x^2 + \frac{16}{3}x^3 + \dots \end{aligned}$$

$$\boxed{f(x) = 1 + 4x - 4x^2 + \frac{16}{3}x^3 + \dots}$$

Part (b)

Observe that
$$\frac{1-\sin x}{\cos x} = \sec x - \tan x$$
Since $\sec x$ is even, $\sec x$ only contributes even powers of x to the power series expansion of $\frac{1-\sin x}{\cos x}$. Likewise, since $\tan x$ is odd, $\tan x$ only contributes odd powers of x to the power series expansion of $\frac{1-\sin x}{\cos x}$.

Let $f(x) = \sec x$ and $g(x) = \tan x$.

$$\begin{aligned}
f(x) &= \sec x \\
\implies f'(x) &= \ln(\sec x + \tan x) \\
&= \ln(f(x) + g(x)) \\
\implies f''(x) &= \frac{f'(x) + g'(x)}{f(x) + g(x)}
\end{aligned}$$

$$\begin{aligned}
g(x) &= \tan x \\
\implies \quad g'(x) &= \sec^2(x) \\
&= f^2(x) \\
\implies \quad g''(x) &= 2f(x)f'(x) \\
\implies g^{(3)}(x) &= 2f(x)f''(x) + 2\left(f'(x)\right)^2
\end{aligned}$$

Evaluating the above derivatives at $x = 0$, we have

$$\begin{aligned}
f(0) &= 1, & g(0) &= 0 \\
f'(0) &= 0, & g'(0) &= 1 \\
f''(0) &= 1, & g''(0) &= 0 \\
& & g^{(3)}(0) &= 2
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{1-\sin x}{\cos x} &= \sec x - \tan x \\
&= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n - \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n \\
&= \left(1 + \frac{1}{2}x^2 + \dots\right) - \left(x + \frac{1}{3}x^3 + \dots\right) \\
&= 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots
\end{aligned}$$

$$\boxed{\frac{1-\sin x}{\cos x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots}$$

Part (c)

$$\begin{aligned}
a(\tan 2x - \sec 2x) + b &= -a(\sec 2x - \tan 2x) + b \\
&= -a\left(1 - 2x + \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3 + \dots\right) + b \\
&\approx -a\left(1 - 2x + \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3\right) + b \\
&= -a\left(1 - 2x + 2x^2 - \frac{8}{3}x^3\right) + b \\
&= a\left(-1 + 2x - 2x^2 + \frac{8}{3}x^3\right) + b \\
&= a\left(-1 + \frac{1}{2}\left(f(x) - 1\right)\right) + b \\
&= -\frac{3}{2}a + b + \frac{a}{2}f(x)
\end{aligned}$$

Hence,

$$\frac{a}{2}f(x) - \frac{3}{2}a + b \approx a(\tan 2x - \sec 2x) + b$$

In order to obtain an approximation for $f(x)$, we need $\frac{a}{2} = 1$ and $-\frac{3}{2}a + b = 0$, whence $a = 2$ and $b = 3$.

$$\boxed{a = 2,\ b = 3}$$

Problem 10.

Given that x is sufficiently small for x^3 and higher powers of x to be neglected, and that $13 - 59\sin x = 10(2 - \cos 2x)$, find a quadratic equation for x and hence solve for x.

Solution

$$\begin{aligned}
13 - 59\sin x &= 10\left(2 - \cos 2x\right) \\
&= 10\left(2 - \left(1 - 2\sin^2 x\right)\right) \\
&= 10\left(1 + 2\sin^2 x\right) \\
&= 10 + 20\sin^2 x \\
\implies \quad 20\sin^2 x + 59\sin x - 3 &= 0 \\
\implies (20\sin x - 1)(\sin x + 3) &= 0
\end{aligned}$$

Hence, $\sin x = \frac{1}{20}$. Note that we reject $\sin x = -3$ since $|\sin x| \leq 1$. Since x is sufficiently small for x^3 and higher powers of x to be neglected, $\sin x \approx x$. Thus, $x \approx \frac{1}{20}$.

$$\boxed{x \approx \frac{1}{20}}$$

Problem 11.

In triangle ABC, angle $A = \frac{\pi}{3}$ radians, angle $B = \left(\frac{\pi}{3} + x\right)$ radians and angle $C = \left(\frac{\pi}{3} - x\right)$ radians, where x is small. The lengths of the sides BC, CA and AB are denoted by a, b and c respectively. Show that $b - c \approx \frac{2ax}{\sqrt{3}}$.

Solution

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Hence,

$$\begin{aligned} b &= a \cdot \frac{\sin B}{\sin A} = a \cdot \frac{\sin B}{\sqrt{3}/2} = \frac{2a}{\sqrt{3}} \sin B \\ c &= a \cdot \frac{\sin C}{\sin A} = a \cdot \frac{\sin C}{\sqrt{3}/2} = \frac{2a}{\sqrt{3}} \sin C \end{aligned}$$

This gives

$$\begin{aligned} b - c &= \frac{2a}{\sqrt{3}} (\sin B - \sin C) \\ &= \frac{2a}{\sqrt{3}} \left(\sin\left(\frac{\pi}{3} + x\right) - \sin\left(\frac{\pi}{3} - x\right)\right) \\ &= \frac{2a}{\sqrt{3}} \left(\left(\sin \frac{\pi}{3} \cos x + \cos \frac{\pi}{3} \sin x\right) - \left(\sin \frac{\pi}{3} \cos x - \cos \frac{\pi}{3} \sin x\right)\right) \\ &= \frac{2a}{\sqrt{3}} \cdot 2 \cos \frac{\pi}{3} \sin x \\ &= \frac{2a}{\sqrt{3}} \cdot 2 \cdot \frac{1}{2} \sin x \\ &= \frac{2a}{\sqrt{3}} \sin x \end{aligned}$$

Since x is small, $\sin x \approx x$. Hence, $b - c \approx \frac{2ax}{\sqrt{3}}$.

Problem 12.

D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, and diverges when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$. When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty}\frac{x^r}{r!}$ converges for all real values of x and state the sum to infinity of this series, in terms of x.

Solution

Let $a_n = \frac{x^n}{n!}$ and consider $\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right|$.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right| &= \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\bigg/\frac{x^n}{n!}\right| \\
&= \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\cdot\frac{n!}{(n+1)!}\right| \\
&= \lim_{n\to\infty}\left|\frac{x}{n+1}\right| \\
&= 0
\end{aligned}$$

Since $\lim_{n\to\infty}\left|\frac{a_{n+1}}{n}\right| < 1$ for all $x \in \mathbb{R}$, it follows by D'Alembert's ratio test that $\sum_{r=0}^{\infty}\frac{x^r}{r!}$ converges for all real values of x. The sum to infinity of the series in question is e^x.