Problem 1.

(a) Show, algebraically, that the derivative of the function

$$\ln(1+x) - \frac{2x}{x+2}$$

is never negative.

(b) Hence, show that $\ln(1+x) \geq \dfrac{2x}{x+2}$ when $x \geq 0$.

Solution

Let $f(x) = \ln(1+x) - \dfrac{2x}{x+2} = \ln(1+x) - 2 + \dfrac{4}{x+2}$.

Part (a)

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - \frac{4}{(x+2)^2} \\ &= \frac{(x+2)^2 - 4(1+x)}{(1+x)(x+2)^2} \\ &= \frac{x^2}{(1+x)(x+2)^2} \end{aligned}$$

Given that $\ln(1+x)$ is defined, it must be that $1+x > 0$. We also know that $x^2 \geq 0$ and $(x+2)^2 \geq 0$. Hence, $f'(x) \geq 0$ for all x in the domain of f and is thus never negative.

Part (b)

Note that $f(0) = \ln(1+0) - 2 + \dfrac{4}{0+2} = 0$. Since $f'(x)$ is never negative, $f(x)$ is increasing. Hence, for all $x \geq 0$,

$$\begin{aligned} & f(x) \geq f(0) \\ \implies & \ln(1+x) - \frac{2x}{x+2} \geq 0 \\ \implies & \ln(1+x) \geq \frac{2x}{x+2} \end{aligned}$$

Problem 2.

The equation of a curve is $y = ax^2 - 2bx + c$, where a, b and c are constants, with $a > 0$.

(a) Using differentiation, find the coordinates of the turning point on the curve, in terms of a, b and c. State whether it is a maximum point or a minimum point.

(b) Given that the turning point of the curve lies on the line $y = x$, find an expression for c in terms of a and b. Show that in this case, whatever the value of b, $c \geq -\dfrac{1}{4a}$.

(c) Find the numerical values of a, b and c when the curve passes through the point $(0, 6)$ and has a turning point at $(2, 2)$.

Solution

Part (a)

For stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\implies 2ax - 2b &= 0 \\
\implies ax - b &= 0 \\
\implies x &= \frac{b}{a} \\
\implies y &= a \cdot \frac{b}{a}^2 - 2b \cdot \frac{b}{a} + c \\
&= \frac{b^2}{a} - 2 \cdot \frac{b^2}{a} + c \\
&= -\frac{b^2}{a} + c
\end{aligned}$$

Since $a > 0$, the graph of y is concave upwards. Thus, there is a maximum point at $\left(\dfrac{b}{a}, -\dfrac{b^2}{a} + c\right)$.

$$\boxed{\text{There is maximum point at } \left(\frac{b}{a}, -\frac{b^2}{a} + c\right).}$$

Part (b)

Since the turning point $\left(\dfrac{b}{a}, -\dfrac{b^2}{a} + c\right)$ lies on the line $y = x$,

$$\begin{aligned}
\frac{b}{a} &= -\frac{b^2}{a} + c \\
\implies c &= \frac{b}{a} + \frac{b^2}{a} \\
&= \frac{1}{a}(b + b^2)
\end{aligned}$$

Consider the stationary points of c with respect to b. For stationary points, $\frac{\mathrm{d}c}{\mathrm{d}b} = 0$.

$$\begin{aligned} \frac{\mathrm{d}c}{\mathrm{d}b} &= 0 \\ \implies \frac{1}{a}(1+2b) &= 0 \\ \implies \quad 1+2b &= 0 \\ \implies \quad b &= -\frac{1}{2} \end{aligned}$$

b	$\left(-\frac{1}{2}\right)^-$	$-\frac{1}{2}$	$\left(-\frac{1}{2}\right)^+$
$\frac{\mathrm{d}c}{\mathrm{d}b}$	-ve	0	+ve

By the First Derivative Test, c achieves a minimum when $b = -\frac{1}{2}$. Observe that when $b = -\frac{1}{2}$, we have $c = \frac{1}{a}\left(-\frac{1}{2} + \left(-\frac{1}{2}\right)^2\right) = -\frac{1}{4a}$. Thus, $c \geq -\frac{1}{4a}$ whatever the value of b.

Part (c)

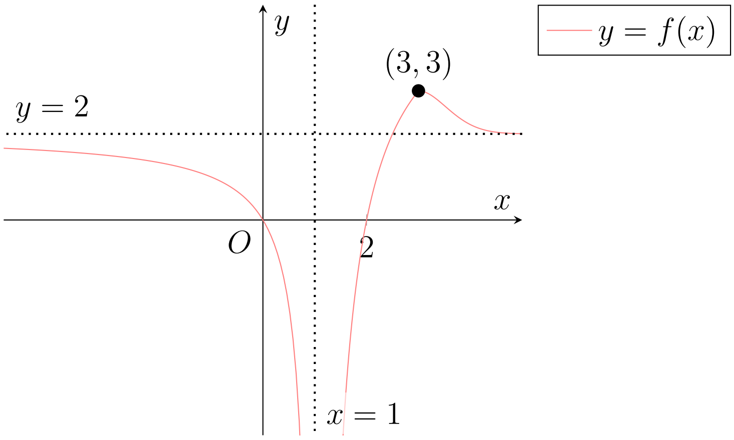
Since the curve passes through $(0, 6)$, it is obvious to see that $c = 6$. Furthermore, since the curve has a turning point at $(2, 2)$, we know that $\frac{b}{a} = 2$ and $-\frac{b^2}{a} + c = 2$.

$$\begin{aligned} -\frac{b^2}{a} + c &= 2 \\ \implies -\frac{b^2}{a} + 6 &= 2 \\ \implies \quad -\frac{b^2}{a} &= -4 \\ \implies \quad \frac{b^2}{a} &= 4 \\ \implies \quad \frac{b}{a} \cdot b &= 4 \\ \implies \quad 2 \cdot b &= 4 \\ \implies \quad b &= 2 \\ \implies \quad a &= 1 \end{aligned}$$

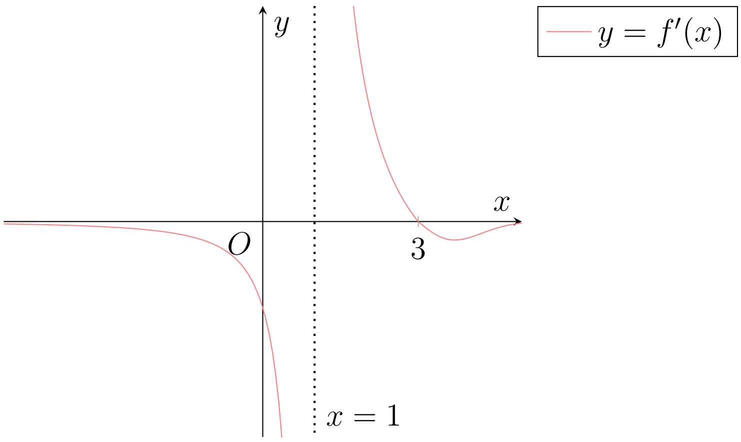
$$\boxed{a = 1,\ b = 2,\ c = 6}$$

Problem 3.

The diagram below shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.



Solution



Problem 4.

The curve C has equation

$$x - y = (x + y)^2$$

It is given that C has only one turning point.

(a) Show that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x + 2y + 1}$.

(b) Hence, or otherwise, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

(c) Hence, state, with a reason, whether the turning point is a maximum or a minimum.

Solution

Part (a)

Implicitly differentiating the given equation,

$$\begin{aligned}
& & 1 - \frac{\mathrm{d}y}{\mathrm{d}x} &= 2(x + y)\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&\implies & 1 - \frac{\mathrm{d}y}{\mathrm{d}x} &= (2x + 2y) + (2x + 2y)\frac{\mathrm{d}y}{\mathrm{d}x} \\
&\implies & (2x + 2y + 1)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 - (2x + 2y) \\
&\implies & \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1 - (2x + 2y)}{2x + 2y + 1} \\
& & &= \frac{1 - (2x + 2y + 1) + 1}{2x + 2y + 1} \\
& & &= \frac{2}{2x + 2y + 1} - 1 \\
&\implies & 1 + \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2}{2x + 2y + 1} \qquad (4.1)
\end{aligned}$$

Part (b)

Implicitly differentiating Equation 4.1,

$$\begin{aligned}
\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\frac{2}{(2x + 2y + 1)^2}\left(2 + 2\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= -\frac{2^2}{(2x + 2y + 1)^2}\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= -(\frac{2}{2x + 2y + 1})^2\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
&= -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^3
\end{aligned}$$

Part (c)

For turning points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Hence, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -1(1+0)^2 = -1 < 0$. Thus, the turning point is a maximum.

$$\boxed{\text{The turning point is a maximum.}}$$