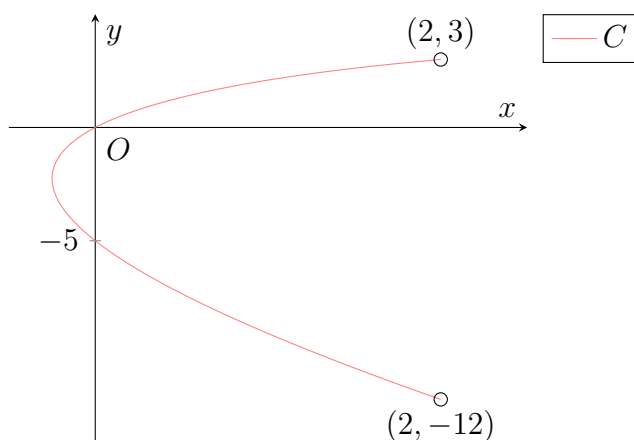


Problem 1.

The curve C has parametric equations

$$x = t^2 + t, y = 4t - t^2, -2 < t < 1$$

- (a) Sketch C , indicating the coordinates of the end-points and the axial intercepts (if any) of this curve.
- (b) Find the coordinates of the point(s) of intersection between C and the line $8y - 12x = 5$.

Solution**Part (a)****Part (b)**

$$\begin{aligned}
 &8y - 12x = 5 \\
 \Rightarrow &8(4t - t^2) - 12(t^2 + t) = 5 \\
 \Rightarrow &32t - 8t^2 - 12t^2 - 12t - 5 = 0 \\
 \Rightarrow &-20t^2 + 20t - 5 = 0 \\
 \Rightarrow &t^2 - t + \frac{1}{4} = 0 \\
 \Rightarrow &\left(t - \frac{1}{2}\right)^2 = 0 \\
 \Rightarrow &t = \frac{1}{2}
 \end{aligned}$$

When $t = \frac{1}{2}$, we have that $x = \frac{3}{4}$ and $y = \frac{7}{4}$. Thus, C and the line $8y - 12x = 5$ intersect at $\left(\frac{3}{4}, \frac{7}{4}\right)$.

$$\boxed{\left(\frac{3}{4}, \frac{7}{4}\right)}$$

Problem 2.

(a) Without using a calculator, solve $\frac{4}{3+2x-x^2} \leq 1$.

(b) Hence, solve $\frac{4}{3+2|x|-x^2} \leq 1$.

Solution**Part (a)**

$$\begin{aligned}
 & \frac{4}{3+2x-x^2} \leq 1 \\
 \Rightarrow & \frac{4}{x^2-2x-3} \geq -1 \\
 \Rightarrow & \frac{4}{(x-3)(x+1)} + 1 \geq 0 \\
 \Rightarrow & \frac{4+(x-3)(x+1)}{(x-3)(x+1)} \geq 0 \\
 \Rightarrow & \frac{4+(x^2-2x-3)}{(x-3)(x+1)} \geq 0 \\
 \Rightarrow & \frac{x^2-2x+1}{(x-3)(x+1)} \geq 0 \\
 \Rightarrow & \frac{(x-1)^2}{(x-3)(x+1)} \geq 0
 \end{aligned}$$

We thus have that $x = 1$ is a solution. In the case when $(x-1)^2 > 0$,

$$\begin{aligned}
 & \frac{1}{(x-3)(x+1)} \geq 0 \\
 \Rightarrow & (x-3)(x+1) \geq 0
 \end{aligned}$$

whence $x < -1$ or $x > 3$. Putting everything together, we have

$$\boxed{x < -1 \vee x = 1 \vee x > 3}$$

Part (b)

$$\begin{aligned}
 & \frac{4}{3+2|x|-x^2} \leq 1 \\
 \Rightarrow & \frac{4}{3+2|x|-|x|^2} \leq 1
 \end{aligned}$$

From part (a), we have that $|x| < -1$, $|x| = 1$ or $|x| > 3$.

Case 1: $|x| < -1$. Since $|x| \geq 0$ this case yields no solutions.

Case 2: $|x| = 1$. We have $x = 1$ or $x = -1$.

Case 3: $|x| > 3$. We have $x > 3$ or $x < -3$.

$$\boxed{x < -3 \vee x = -1 \vee x = 1 \vee x > 3}$$

Problem 3.

The curve C_1 has equation

$$y = \frac{2x^2 + 2x - 2}{x - 1}$$

- (a) Sketch the graph of C_1 , stating the equations of any asymptotes and the coordinates of any axial intercepts and/or turning points.

The curve C_2 has equation

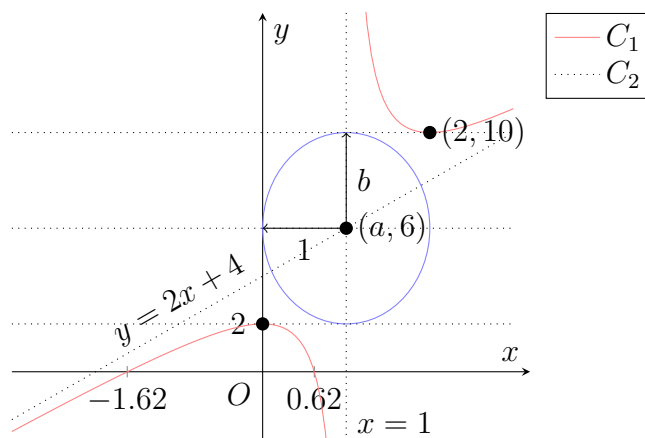
$$\frac{(x - a)^2}{1^2} + \frac{(y - 6)^2}{b^2} = 1$$

where $b > 0$. It is given that C_1 and C_2 have no points in common for all $a \in \mathbb{R}$.

- (b) By adding an appropriate curve in part (a), state the range of values of b , explaining your answer.
- (c) The function f is defined by

$$f(x) = \frac{2x^2 + 2x - 2}{x - 1}, \quad x < 1$$

- (i) By using the graph in part (a) or otherwise, explain why the inverse function f^{-1} does not exist.
- (ii) The domain of f is restricted to $[c, 1)$ such that c is the least value for which the inverse function f^{-1} exists. State the value of c and define f^{-1} clearly.

Solution**Part (a)****Part (b)**

Observe that C_2 describes an ellipse with vertical radius b and horizontal radius 1. Furthermore, the ellipse is centred at $(a, 6)$. Since C_1 and C_2 have no points in common for all $a \in \mathbb{R}$, the maximum y -value of the ellipse corresponds to the y -value of the minimum point $(2, 10)$ of C_1 . Similarly, the minimum y -value of the ellipse corresponds to the y -value of the maximum point $(0, 2)$ of C_1 . Thus, $2 < y < 10$, whence $b < \min \{|6 - 2|, |6 - 10|\} = 4$. Thus,

$$0 < b < 4$$

Part (c)

Subpart (i) Observe that $f(-1.62) = f(0.618) = 0$. Hence, there exist two different values of x in D_f that have the same image under f . Thus, f is not one-one. Hence, f^{-1} does not exist.

Subpart (ii)

$$c = 0$$

$$\begin{aligned}
 f(x) &= \frac{2x^2 + 2x - 2}{x - 1} \\
 \Rightarrow (x - 1)f(x) &= 2x^2 + 2x - 2 \\
 \Rightarrow xf(x) - x &= 2x^2 + 2x - 2 \\
 \Rightarrow 2x^2 + 2x - xf(x) - 2 + f(x) &= 0 \\
 \Rightarrow 2x^2 + (2 - f(x))x + (f(x) - 2) &= 0 \\
 \Rightarrow x &= \frac{-(2 - f(x)) \pm \sqrt{(2 - f(x))^2 - 4 \cdot 2 \cdot (f(x) - 2)}}{2 \cdot 2} \\
 &= \frac{f(x) - 2 \pm \sqrt{f(x)^2 - 12f(x) + 20}}{4} \\
 \Rightarrow f^{-1}(x) &= \frac{x - 2 \pm \sqrt{x^2 - 12x + 20}}{4}
 \end{aligned}$$

Note that $D_f = R_{f^{-1}} = [0, 1)$. We thus take the positive root. Also note that $R_f = D_{f^{-1}} = (-\infty, 2]$.

$$f^{-1} : x \mapsto \frac{x - 2 + \sqrt{x^2 - 12x + 20}}{4}, \quad x \in \mathbb{R}, x \leq 2$$