

Problem 1.

(a) Show, algebraically, that the derivative of the function

$$\ln(1+x) - \frac{2x}{x+2}$$

is never negative.

(b) Hence show that $\ln(1+x) \geq \frac{2x}{x+2}$ when $x \geq 0$.

Solution

$$\text{Let } f(x) = \ln(1+x) - \frac{2x}{x+2} = \ln(1+x) - 2 + \frac{4}{x+2}.$$

Part (a)

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - \frac{4}{(x+2)^2} \\ &= \frac{(x+2)^2 - 4(1+x)}{(1+x)(x+2)^2} \\ &= \frac{x^2}{(1+x)(x+2)^2} \end{aligned}$$

Given that $\ln(1+x)$ is defined, it must be that $1+x > 0$. We also know that $x^2 \geq 0$ and $(x+2)^2 \geq 0$. Hence, $f'(x) \geq 0$ for all x in the domain of f and is thus never negative.

Part (b)

Note that $f(0) = \ln(1+0) - 2 + \frac{4}{0+2} = 0$. Since $f'(x)$ is never negative, $f(x)$ is increasing. Hence, for all $x \geq 0$,

$$\begin{aligned} f(x) &\geq f(0) \\ \implies \ln(1+x) - \frac{2x}{x+2} &\geq 0 \\ \implies \ln(1+x) &\geq \frac{2x}{x+2} \end{aligned}$$

Problem 2.

The equation of a curve is $y = ax^2 - 2bx + c$, where a , b and c are constants, with $a > 0$.

- Using differentiation, find the coordinates of the turning point on the curve, in terms of a , b and c . State whether it is a maximum point or a minimum point.
- Given that the turning point of the curve lies on the line $y = x$, find an expression for c in terms of a and b . Show that in this case, whatever the value of b , $c \geq -\frac{1}{4a}$.
- Find the numerical values of a , b and c when the curve passes through the point $(0, 6)$ and has a turning point at $(2, 2)$.

Solution**Part (a)**

For stationary points, $\frac{dy}{dx} = 0$.

$$\begin{aligned}
 \frac{dy}{dx} &= 0 \\
 \implies 2ax - 2b &= 0 \\
 \implies ax - b &= 0 \\
 \implies x &= \frac{b}{a} \\
 \implies y &= a \cdot \frac{b^2}{a^2} - 2b \cdot \frac{b}{a} + c \\
 &= \frac{b^2}{a} - 2 \cdot \frac{b^2}{a} + c \\
 &= -\frac{b^2}{a} + c
 \end{aligned}$$

Since $a > 0$, the graph of y is concave upwards. Thus, there is a maximum point at $\left(\frac{b}{a}, -\frac{b^2}{a} + c\right)$.

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Part (b)

Since the turning point $\left(\frac{b}{a}, -\frac{b^2}{a} + c\right)$ lies on the line $y = x$,

$$\begin{aligned}
 \frac{b}{a} &= -\frac{b^2}{a} + c \\
 \implies c &= \frac{b}{a} + \frac{b^2}{a} \\
 &= \frac{1}{a}(b + b^2)
 \end{aligned}$$

Consider the stationary points of c with respect to b . For stationary points, $\frac{dc}{db} = 0$.

$$\begin{aligned}\frac{dc}{db} &= 0 \\ \implies \frac{1}{a}(1 + 2b) &= 0 \\ \implies 1 + 2b &= 0 \\ \implies b &= -\frac{1}{2}\end{aligned}$$

b	$\left(-\frac{1}{2}\right)^{-}$	$-\frac{1}{2}$	$\left(-\frac{1}{2}\right)^{+}$
$\frac{dc}{db}$	-ve	0	+ve

By the First Derivative Test, c achieves a minimum when $b = -\frac{1}{2}$. Observe that when $b = -\frac{1}{2}$, we have $c = \frac{1}{a} \left(-\frac{1}{2} + \left(-\frac{1}{2} \right)^2 \right) = -\frac{1}{4a}$. Thus, $c \geq -\frac{1}{4a}$ whatever the value of b .

Part (c)

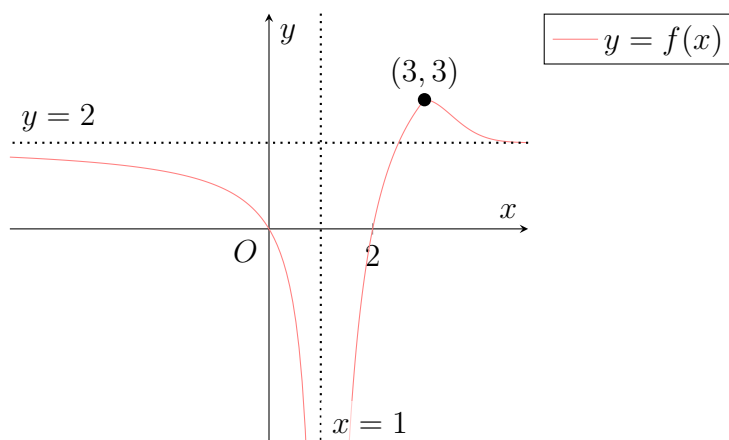
Since the curve passes through $(0, 6)$, it is obvious to see that $c = 6$. Furthermore, since the curve has a turning point at $(2, 2)$, we know that $\frac{b}{a} = 2$ and $-\frac{b^2}{a} + c = 2$.

$$\begin{aligned}-\frac{b^2}{a} + c &= 2 \\ \implies -\frac{b^2}{a} + 6 &= 2 \\ \implies -\frac{b^2}{a} &= -4 \\ \implies \frac{b^2}{a} &= 4 \\ \implies \frac{b}{a} \cdot b &= 4 \\ \implies 2 \cdot b &= 4 \\ \implies b &= 2 \\ \implies a &= 1\end{aligned}$$

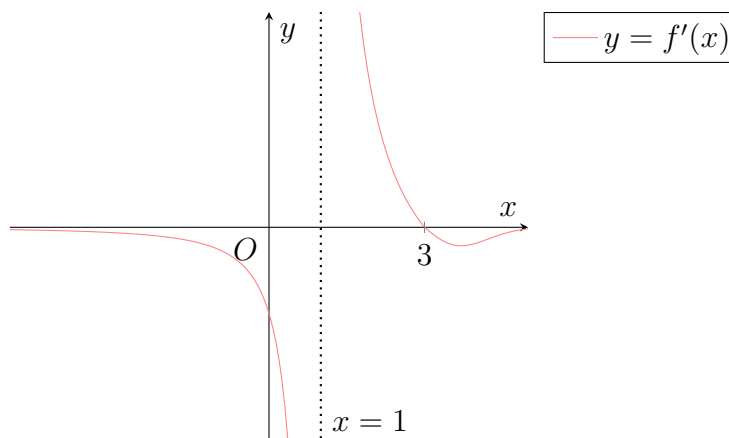
$a = 1, b = 2, c = 6$

Problem 3.

The diagram below shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.



Solution



Problem 4.

The curve C has equation

$$x - y = (x + y)^2$$

It is given that C has only one turning point.

- (a) Show that $1 + \frac{dy}{dx} = \frac{2}{2x + 2y + 1}$.
- (b) Hence, or otherwise, show that $\frac{d^2y}{dx^2} = -\left(1 + \frac{dy}{dx}\right)^2$.
- (c) Hence, state, with a reason, whether the turning point is a maximum or a minimum.

Solution**Part (a)**

Implicitly differentiating the given equation,

$$\begin{aligned}
 1 - \frac{dy}{dx} &= 2(x + y) \left(1 + \frac{dy}{dx}\right) \\
 \implies 1 - \frac{dy}{dx} &= (2x + 2y) + (2x + 2y) \frac{dy}{dx} \\
 \implies (2x + 2y + 1) \frac{dy}{dx} &= 1 - (2x + 2y) \\
 \implies \frac{dy}{dx} &= \frac{1 - (2x + 2y)}{2x + 2y + 1} \\
 &= \frac{1 - (2x + 2y + 1) + 1}{2x + 2y + 1} \\
 &= \frac{2}{2x + 2y + 1} - 1 \\
 \implies 1 + \frac{dy}{dx} &= \frac{2}{2x + 2y + 1}
 \end{aligned} \tag{4.1}$$

Part (b)

Implicitly differentiating Equation 4.1,

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= -\frac{2}{(2x + 2y + 1)^2} \left(2 + 2\frac{dy}{dx}\right) \\
 &= -\frac{2^2}{(2x + 2y + 1)^2} \left(1 + \frac{dy}{dx}\right) \\
 &= -\left(\frac{2}{2x + 2y + 1}\right)^2 \left(1 + \frac{dy}{dx}\right) \\
 &= -\left(1 + \frac{dy}{dx}\right)^2 \left(1 + \frac{dy}{dx}\right) \\
 &= -\left(1 + \frac{dy}{dx}\right)^3
 \end{aligned}$$

Part (c)

For turning points, $\frac{dy}{dx} = 0$. Hence $\frac{d^2y}{dx^2} = -1(1+0)^2 = -1 < 0$. Thus, the turning point is a maximum.

The turning point is a maximum.
