Problem 1.

The points A and B have position vectors relative to the origin O, denoted by $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. The point P lies on AB such that $AP : PB = \lambda : 1$. The point Q lies on OP extended such that $OP = 2PQ$ and $\overrightarrow{BQ} = \overrightarrow{OA} + \mu\overrightarrow{OB}$. Find the values of the real constants λ and μ.

Solution

By the Ratio Theorem,

$$\begin{aligned}
&& \overrightarrow{OP} &= \frac{\mathbf{a}+\lambda\mathbf{b}}{1+\lambda} \\
\implies && \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\
&& &= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{OP} \\
&& &= \frac{3}{2} \cdot \frac{\mathbf{a}+\lambda\mathbf{b}}{1+\lambda} \\
&& \overrightarrow{BQ} &= \overrightarrow{OA} + \mu\overrightarrow{OB} \\
\implies && \overrightarrow{OB} + \overrightarrow{BQ} &= \overrightarrow{OA} + (1+\mu)\overrightarrow{OB} \\
\implies && \overrightarrow{OQ} &= \overrightarrow{OA} + (1+\mu)\overrightarrow{OB} \\
\implies && \frac{3}{2} \cdot \frac{\mathbf{a}+\lambda\mathbf{b}}{1+\lambda} &= \mathbf{a} + (1+\mu)\mathbf{b}
\end{aligned}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel, we have the following system:

$$\begin{cases} \frac{3}{2} \cdot \frac{1}{1+\lambda} &= 1 \\ \frac{3}{2} \cdot \frac{\lambda}{1+\lambda} &= 1+\mu \end{cases}$$

which has the unique solution $\lambda = \frac{1}{2}$ and $\mu = -\frac{1}{2}$.

$$\boxed{\lambda = \frac{1}{2},\ \mu = -\frac{1}{2}}$$

Problem 2.

Given that $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ where $\lambda \in \mathbb{R}$, find the possible value(s) of λ for which the angle between $\mathbf{p}$ and $\mathbf{k}$ is $45°$.

Solution

$$\begin{aligned}
\mathbf{p} &= \lambda\mathbf{a} + (1-\lambda)\mathbf{b} \\
&= \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} \\
&= \begin{pmatrix} 4 - 3\lambda \\ -2 + 3\lambda \\ 6 - 6\lambda \end{pmatrix} \\
\implies |\mathbf{p}|^2 &= (4-3\lambda)^2 + (-2+3\lambda)^2 + (6-6\lambda)^2 \\
&= 54\lambda^2 - 108\lambda + 56
\end{aligned}$$

Since the angle between $\mathbf{p}$ and $\mathbf{k}$ is $45°$,

$$\begin{aligned}
&& \cos 45° &= \frac{\mathbf{p} \cdot \mathbf{k}}{|\mathbf{p}||\mathbf{k}|} \\
\implies && \frac{1}{\sqrt{2}} &= \frac{(4-3\lambda)\cdot 0 + (-2+3\lambda)\cdot 0 + (6-6\lambda)\cdot 1}{|\mathbf{p}|\cdot 1} \\
\implies && \frac{|\mathbf{p}|}{\sqrt{2}} &= 6 - 6\lambda \\
\implies && \frac{|\mathbf{p}|^2}{2} &= (6-6\lambda)^2 \\
\implies && \frac{54\lambda^2 - 108\lambda + 56}{2} &= 36\lambda^2 - 72\lambda + 36 \\
\implies && 9\lambda^2 - 18\lambda + 8 &= 0 \\
\implies && (3\lambda - 2)(3\lambda - 4) &= 0
\end{aligned}$$

Hence, $\lambda = \frac{2}{3}, \frac{4}{3}$.

$$\boxed{\lambda = \frac{2}{3}, \frac{4}{3}}$$

Problem 3.

(a) $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$. State the relation between the directions of $\mathbf{a}$ and $\mathbf{b}$, and find $|\mathbf{b}|$.

(b) $\mathbf{a}$ is a non-zero vector such that $\mathbf{a} = \sqrt{3}$ and $\mathbf{b}$ is a unit vector. Given that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and the angle between them is $\frac{5}{6}\pi$, find the exact value of the length of projection of $\mathbf{a}$ on $\mathbf{b}$. By considering $(2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b})$, or otherwise, find the exact value of $|2\mathbf{a} + \mathbf{b}|$.

Solution

Part (a)

$\mathbf{a}$ and $\mathbf{b}$ either have the same or opposite direction.

Let $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$.

$$\begin{aligned} & \mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} \\ \implies & \mathbf{a} = (\mathbf{a} \cdot \lambda\mathbf{a})\lambda\mathbf{a} \\ \implies & \mathbf{a} = \lambda^2\mathbf{a} \\ \implies & \lambda^2 = 1 \end{aligned}$$

Hence, $\lambda = \pm 1$, whence $\mathbf{b} = \pm\mathbf{a}$. Let $A = |\mathbf{a}| = |\mathbf{b}|$. Note that the cosine of the angle between $\mathbf{a}$ and $\mathbf{b}$ is ± 1.

$$\begin{aligned} A &= |a| \\ &= |(\mathbf{a} \cdot \mathbf{b})\mathbf{b}| \\ &= |\mathbf{a} \cdot \mathbf{b}||\mathbf{b}| \\ &= |\mathbf{a}||\mathbf{b}||\pm 1||\mathbf{b}| \\ &= A^3 \end{aligned}$$

Hence, $A = 1$. Note that we reject $A = 0$ since $\mathbf{a}$ is a non-zero vector. Thus, $|\mathbf{b}| = 1$.

$$\boxed{|\mathbf{b}| = 1}$$

Part (b)

$$\begin{aligned} |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| \cos \frac{5}{6}\pi \\ &= \sqrt{3} \cdot 1 \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= -\frac{3}{2} \\ \implies \text{length of projection of } \mathbf{a} \text{ on } \mathbf{b} &= \left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| \\ &= |\mathbf{a} \cdot \mathbf{b}| \\ &= \left|-\frac{3}{2}\right| \end{aligned}$$

$$= \frac{3}{2}$$

$$\boxed{\text{length of projection of } \mathbf{a} \text{ on } \mathbf{b} = \frac{3}{2}}$$

$$\begin{aligned}
|2\mathbf{a} + \mathbf{b}|^2 &= (2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) \\
&= 2\mathbf{a} \cdot 2\mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot 2\mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\
&= 4\mathbf{a} \cdot \mathbf{a} + 4\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\
&= 4 \cdot 3 + 4 \cdot \left(-\frac{3}{2}\right) + 1^2 \\
&= 7 \\
\implies \quad |2\mathbf{a} + \mathbf{b}| &= \sqrt{7}
\end{aligned}$$

$$\boxed{|2\mathbf{a} + \mathbf{b}| = \sqrt{7}}$$

Problem 4.

The points A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ given by $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$, respectively. The point P lies on AB produced such that $AP = 2AB$, and the point Q is the mid-point of AC.

(a) Show that PQ is perpendicular to AQ.

(b) Find the area of the triangle APQ.

(c) Find a vector perpendicular to the plane ABC.

(d) Find the cosine of the angle between $\overrightarrow{AD}$ and $\overrightarrow{BD}$.

Solution

We recenter the vectors such that $\mathbf{a}$ is the origin. This gives $\mathbf{a}' = \begin{pmatrix}0\\0\\0\end{pmatrix}$, $\mathbf{b}' = \begin{pmatrix}0\\0\\-1\end{pmatrix}$, $\mathbf{c}' = \begin{pmatrix}2\\0\\-2\end{pmatrix}$, $\mathbf{d} = \begin{pmatrix}3\\-3\\-4\end{pmatrix}$. Hence, $\overrightarrow{OP}'$ is clearly $\begin{pmatrix}0\\0\\-2\end{pmatrix}$, while $\overrightarrow{OQ}' = \frac{1}{2}\mathbf{c}' = \begin{pmatrix}1\\0\\-1\end{pmatrix}$

Part (a)

$$\begin{aligned}
\overrightarrow{PQ} \cdot \overrightarrow{AQ} &= \overrightarrow{PQ}' \cdot \overrightarrow{OQ}' \\
&= \left(\overrightarrow{OQ}' - \overrightarrow{OP}'\right) \cdot (\overrightarrow{OQ}') \\
&= \begin{pmatrix}1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\-1\end{pmatrix} \\
&= 1 + 0 - 1 \\
&= 0
\end{aligned}$$

Since $\overrightarrow{PQ} \cdot \overrightarrow{AQ} = 0$, the lines PQ and AQ must be perpendicular.

Part (b)

$$\begin{aligned}
\text{Area}\,\triangle APQ &= \frac{1}{2}\left|\overrightarrow{AP} \times \overrightarrow{AQ}'\right| \\
&= \frac{1}{2}\left|\overrightarrow{OP}' \times \overrightarrow{OQ}'\right| \\
&= \frac{1}{2}\left|\begin{pmatrix}0\\0\\-2\end{pmatrix} \times \begin{pmatrix}1\\0\\-1\end{pmatrix}\right| \\
&= \frac{1}{2}\left|\begin{pmatrix}0\\-2\\0\end{pmatrix}\right| \\
&= 1
\end{aligned}$$

$$\boxed{\text{Area}\,\triangle APQ = 1}$$

Part (c)

$$\begin{aligned}
\mathbf{b}' \times \mathbf{c}' &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\boxed{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \text{ is perpendicular to the plane } ABC.}$$

Part (d)

Let the angle between $\overrightarrow{AD}$ and $\overrightarrow{BD}$ be θ. Note that $\overrightarrow{BD} = \mathbf{d}' - \mathbf{b}' = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.

$$\begin{aligned}
\cos\theta &= \frac{\overrightarrow{AD} \cdot \overrightarrow{BD}}{\left|\overrightarrow{AD}\right|\left|\overrightarrow{BD}\right|} \\
&= \frac{1}{\sqrt{3^2 + (-3)^2 + (-3)^2} \cdot 3\sqrt{1^2 + (-1)^2 + (-1)^2}} \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} \cdot 3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\
&= \frac{1}{3\sqrt{102}} \cdot 3\Big(3 \cdot 1 + (-3) \cdot (-1) + (-4) \cdot (-1)\Big) \\
&= \frac{10}{\sqrt{102}}
\end{aligned}$$

$$\boxed{\cos\theta = \frac{10}{\sqrt{102}}}$$