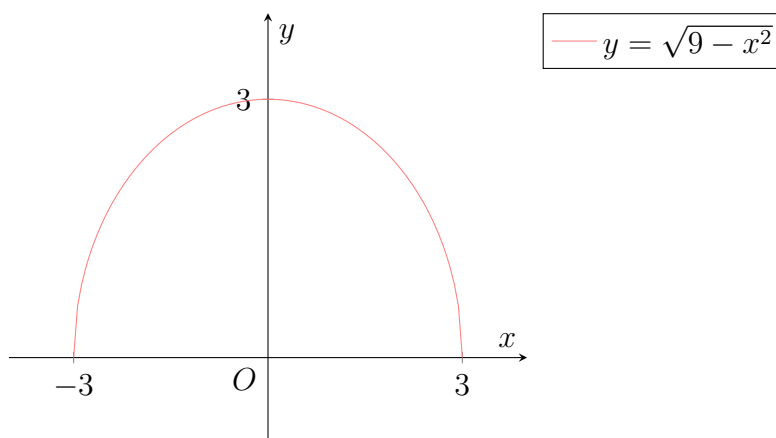


**Problem 1.**

Sketch the following graphs and determine whether each graph represents a function for the given domain.

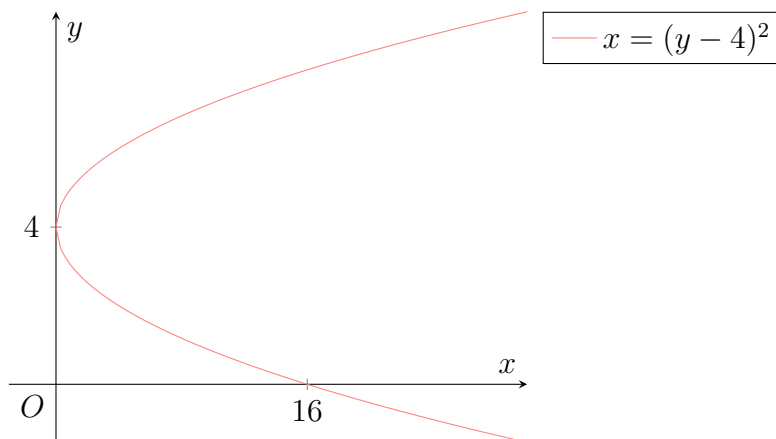
(a)  $y = \sqrt{9 - x^2}$ ,  $-3 \leq x \leq 3$

(b)  $x = (y - 4)^2$ ,  $y \in \mathbb{R}$

**Solution****Part (a)**

$y = \sqrt{9 - x^2}$  passes the vertical line test for  $-3 \leq x \leq 3$  and is hence a function.

$y = \sqrt{9 - x^2}$ ,  $-3 \leq x \leq 3$  is a function.

**Part (b)**

$x = (y - 4)^2$  does not pass the vertical line test for  $y \in \mathbb{R}$  and is hence not a function.

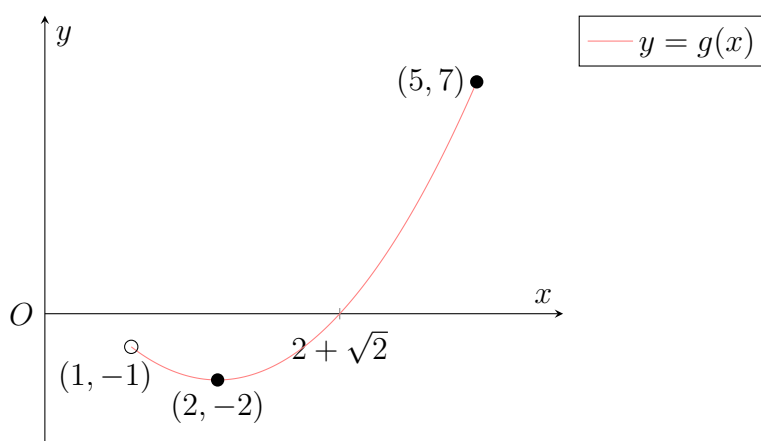
$x = (y - 4)^2$ ,  $y \in \mathbb{R}$  is not a function.

**Problem 2.**

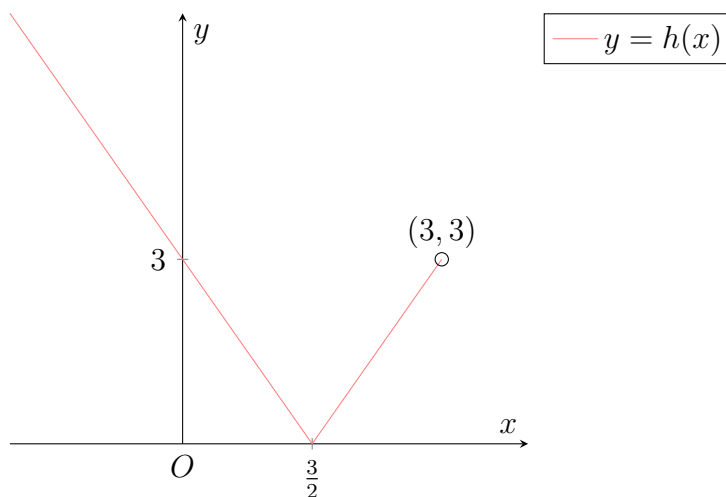
Sketch the graph and find the range for each the following functions.

(a)  $g: x \mapsto x^2 - 4x + 2, 1 < x \leq 5$

(b)  $h: x \mapsto |2x - 3|, x < 3$

**Solution****Part (a)**

$$R_g = [-2, 7)$$

**Part (b)**

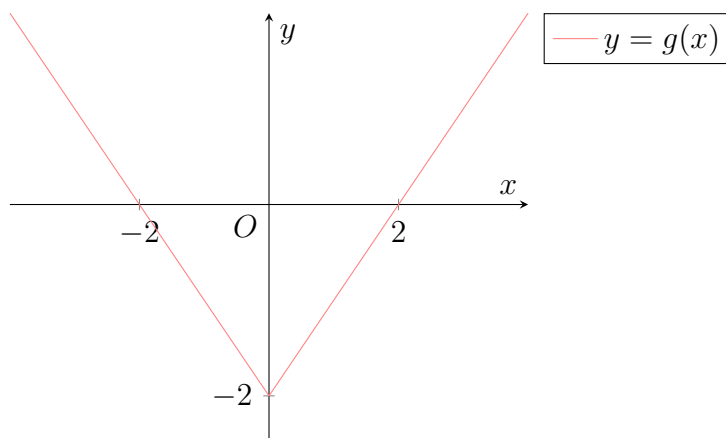
$$R_h = [0, \infty)$$

**Problem 3.**

For each of the following functions, sketch its graph and determine if the function is one-one. If it is, find its inverse in a similar form.

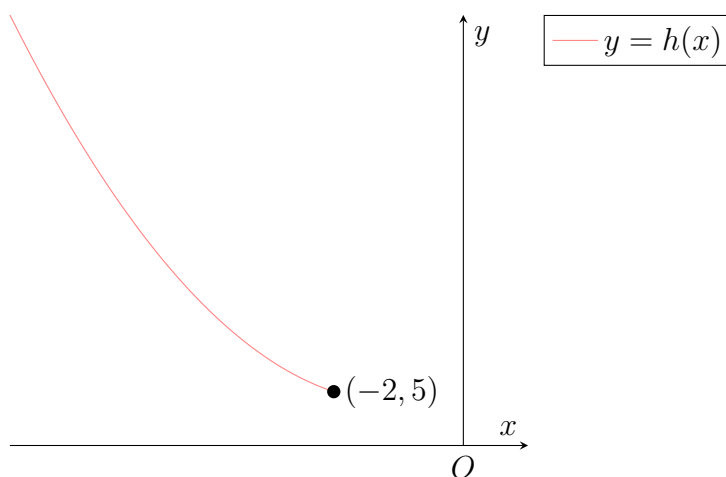
(a)  $g: x \mapsto |x| - 2, x \in \mathbb{R}$

(b)  $h: x \mapsto x^2 + 2x + 5, x \leq -2$

**Solution****Part (a)**

$y = g(x)$  does not pass the horizontal line test. Hence,  $g$  is not one-one.

$g$  is not one-one.

**Part (b)**

$y = h(x)$  passes the horizontal line test. Hence,  $h$  is one-one.

$h$  is not one-one.

Note that  $y = h(x) \implies x = h^{-1}(y)$ . Now consider  $y = h(x)$ .

$$\begin{aligned}
 & y = h(x) \\
 \implies & y = x^2 + 2x + 5 \\
 \implies & y = x^2 + 2x + 1 + 4 \\
 \implies & y = (x + 1)^2 + 4 \\
 \implies & (x + 1)^2 = y - 4 \\
 \implies & x + 1 = \pm \sqrt{y - 4}
 \end{aligned}$$

Now, since  $x \leq -2$ , we have  $x + 1 \leq -1$ . Hence, we reject  $x + 1 = \sqrt{y - 4}$  since  $\sqrt{y - 4} \geq 0$ .

$$\begin{aligned}
 \implies & x + 1 = -\sqrt{y - 4} \\
 \implies & x = -1 - \sqrt{y - 4}
 \end{aligned}$$

Hence,  $h^{-1}(x) = -1 - \sqrt{x - 4}$ . Note that  $D_{h^{-1}} = R_h = [5, \infty)$ . Hence,

$$\boxed{h^{-1}: x \mapsto -1 - \sqrt{x - 4}, x \geq 5}$$

**Problem 4.**

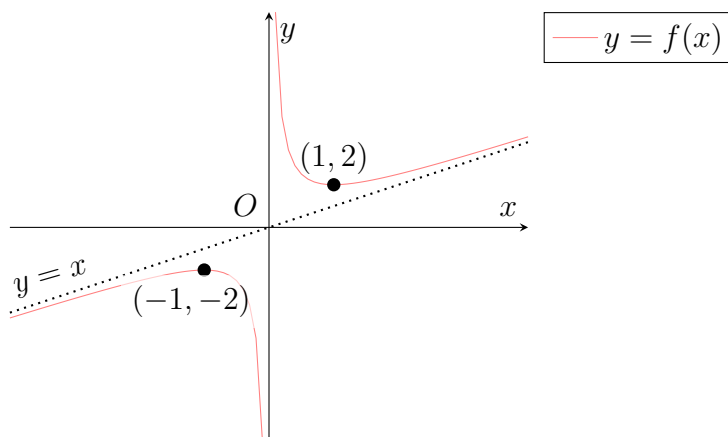
The function  $f$  is defined by

$$f: x \mapsto x + \frac{1}{x}, x \neq 0$$

- (a) Sketch the graph of  $f$  and explain why  $f^{-1}$  does not exist.
- (b) The function  $h$  is defined by  $h: x \mapsto f(x)$ ,  $x \in \mathbb{R}$ ,  $x \geq \alpha$ , where  $\alpha \in \mathbb{R}^+$ . Find the smallest value of  $\alpha$  such that the inverse function of  $h$  exists.

Using this value of  $\alpha$ ,

- (c) State the range of  $h$ .
- (d) Express  $h^{-1}$  in a similar form and sketch on a single diagram, the graphs of  $h$  and  $h^{-1}$ , showing clearly their geometrical relationship.

**Solution****Part (a)**

$y = f(x)$  does not pass the horizontal line test. Hence,  $f$  is not one-one. Hence,  $f^{-1}$  does not exist.

**Part (b)**

Consider  $f'(x) = 0$  for  $x > 0$ .

$$\begin{aligned} f'(x) &= 0 \\ \implies 1 - \frac{1}{x^2} &= 0 \\ \implies x^2 &= 1 \\ \implies x &= 1 \quad (\text{rej. } x = -1 \because x > 0) \end{aligned}$$

Looking at the graph of  $y = f(x)$ , we see that  $f(x)$  achieves a minimum at  $x = 1$ . Hence,  $f$  is increasing for all  $x \geq 1$ . Thus, the smallest value of  $\alpha$  is 1.

$$\boxed{\min \alpha = 1}$$

**Part (c)**

Note  $f(1) = 2$ . Hence, from the graph,

$$R_h = [2, \infty)$$

**Part (d)**

Note that  $y = h(x) \implies x = h^{-1}(y)$ . Now consider  $y = h(x)$ .

$$\begin{aligned} & y = h(x) \\ \implies & y = x + \frac{1}{x} \\ \implies & xy = x^2 + 1 \\ \implies & x^2 - yx + 1 = 0 \\ \implies & x = \frac{1}{2}(y \pm \sqrt{y^2 - 4}) \end{aligned}$$

Note that  $f(2) = \frac{5}{2}$ . Since  $2 = \frac{1}{2} \left( \frac{5}{2} + \sqrt{\left(\frac{5}{2}\right)^2 - 4} \right)$  and  $2 \neq \frac{1}{2} \left( \frac{5}{2} - \sqrt{\left(\frac{5}{2}\right)^2 - 4} \right)$ , we reject  $x = \frac{1}{2}(y - \sqrt{y^2 - 4})$ . Hence,  $h^{-1}(x) = \frac{1}{2}(x + \sqrt{x^2 - 4})$ . Note that  $D_{f^{-1}} = R_f = [2, \infty)$ . Thus,

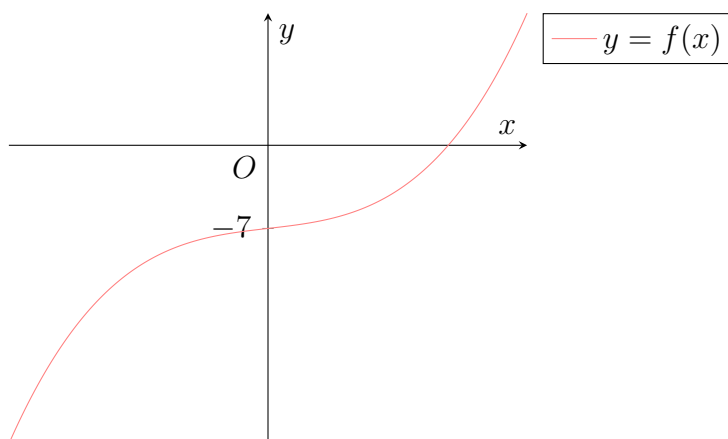
$$h^{-1}: x \mapsto \frac{1}{2} (x + \sqrt{x^2 - 4}), x \geq 2$$

**Problem 5.**

The function  $f$  is defined as follows:

$$f: x \mapsto x^3 + x - 7, x \in \mathbb{R}$$

- By using a graphical method or otherwise, show that the inverse of  $f$  exists.
- Solve exactly the equation  $f^{-1}(x) = 0$ . Sketch the graph of  $f^{-1}$  together with the graph of  $f$  on the same diagram.
- Find, in exact form, the coordinates of the intersection point(s) of the graphs of  $f$  and  $f^{-1}$ .
- Given that the gradient of the tangent to the curve with equation  $y = f^{-1}(x)$  is  $\frac{1}{4}$  at the point with  $x = p$ , find the possible values of  $p$ .

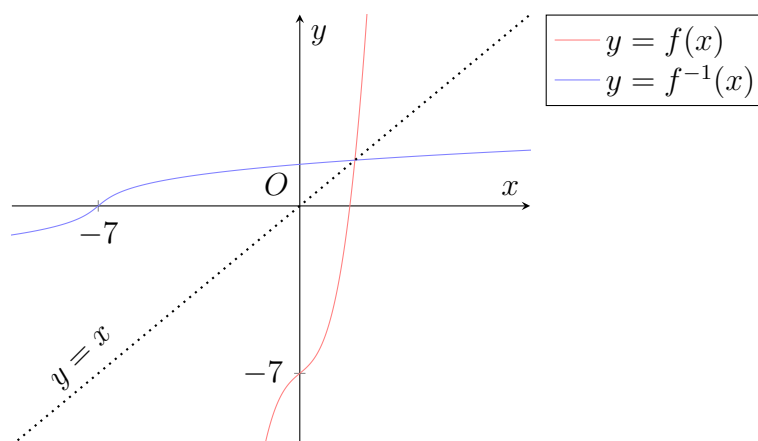
**Solution****Part (a)**

$y = f(x)$  passes the horizontal line test. Hence,  $f$  is one-one. Thus,  $f^{-1}$  exists.

**Part (b)**

$$\begin{aligned} f^{-1}(x) &= 0 \\ \implies x &= f(0) \\ \implies x &= 0^3 + 0 - 7 \\ &= -7 \end{aligned}$$

$$\boxed{x = -7}$$

**Part (c)**

Let  $(\alpha, \beta)$  be the coordinates of the intersection between  $f(x)$  and  $f^{-1}$ . From the graph, we see that  $\alpha = \beta$ , hence  $f(\alpha) = \alpha$ .

$$\begin{aligned}
 f(\alpha) &= \alpha \\
 \implies \alpha^3 + \alpha - 7 &= \alpha \\
 \implies \alpha^3 &= 7 \\
 \implies \alpha &= \sqrt[3]{7}
 \end{aligned}$$

$$\boxed{(\sqrt[3]{7}, \sqrt[3]{7})}$$

**Part (d)**

$$\begin{aligned}
 [f^{-1}(x)]' &= \frac{1}{f'(f^{-1}(x))} \\
 \implies [f^{-1}(x)]' \Big|_{x=p} &= \frac{1}{f'(f^{-1}(x))} \Big|_{x=p} \\
 \implies \frac{1}{4} &= \frac{1}{f'(f^{-1}(x))} \Big|_{x=p} \\
 \implies f'(f^{-1}(x)) \Big|_{x=p} &= 4
 \end{aligned}$$

Note that  $f'(x) = 3x^2 + 1$ .

$$\begin{aligned}
 \implies (3 \cdot f^{-1}(x)^2 + 1) \Big|_{x=p} &= 4 \\
 \implies 3 \cdot f^{-1}(p)^2 + 1 &= 4 \\
 \implies f^{-1}(p)^2 &= 1 \\
 \implies f^{-1}(p) &= \pm 1
 \end{aligned}$$



**Case 1:**  $f^{-1}(p) = 1$

$$\begin{aligned} f^{-1}(p) &= 1 \\ \implies p &= f(1) \\ \implies p &= 1^3 + 1 - 7 \\ &= -5 \end{aligned}$$

**Case 2:**  $f^{-1}(p) = -1$

$$\begin{aligned} f^{-1}(p) &= -1 \\ \implies p &= f(-1) \\ \implies p &= (-1)^3 - 1 - 7 \\ &= -9 \end{aligned}$$

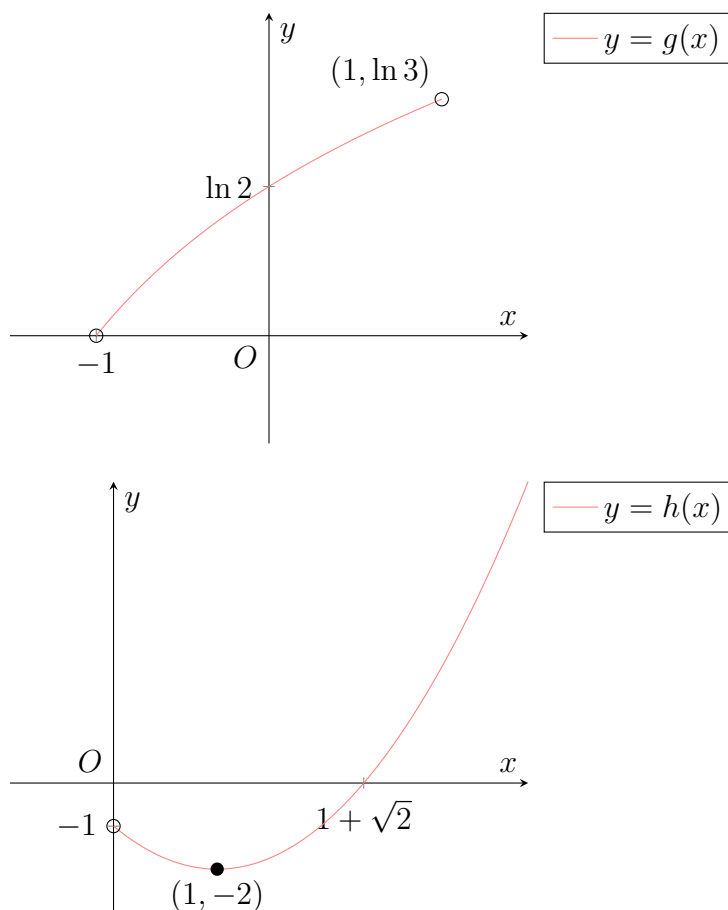
$$\boxed{p = -9 \vee -5}$$

**Problem 6.**

The functions  $g$  and  $h$  are defined as follows:

$$\begin{aligned} g: x &\mapsto \ln(x+2), & x &\in (-1, 1) \\ h: x &\mapsto x^2 - 2x - 1, & x &\in \mathbb{R}^+ \end{aligned}$$

- Sketch, on separate diagrams, the graphs of  $g$  and  $h$ .
- Determine whether the composite function  $gh$  exists.
- Give the rule and domain of the composite function  $hg$  and find its range.
- The image of  $a$  under the composite function  $hg$  is -1.5. Find the value of  $a$ .

**Solution****Part (a)****Part (b)**

Observe that  $R_h = [-2, \infty)$  and  $D_g = (-1, 1)$ . Hence,  $R_h \not\subseteq D_g$ . Thus,  $gh$  does not exist.

 $gh$  does not exist.

**Part (c)**

$$\begin{aligned} hg(x) &= h(\ln(x+2)) \\ &= \ln(x+2)^2 - 2\ln(x+2) - 1 \end{aligned}$$

Note that  $D_{hg} = D_g = (-1, 1)$ .

$$\boxed{hg: x \mapsto \ln(x+2)^2 - 2\ln(x+2) - 1, x \in (-1, 1)}$$

Observe that  $h$  is decreasing on the interval  $(0, 1]$  and increasing on the interval  $[1, \infty)$ . Note that  $R_g = (0, \ln 3)$ . Hence,

$$\begin{aligned} R_{hg} &= [-2, \max\{h(0), h(\ln 3)\}) \\ &= [-2, -1) \end{aligned}$$

**Part (d)**

Note that  $h(x) = (x-1)^2 - 2$ . Hence,  $h^{-1}(x) = 1 + \sqrt{x+2}$  (we reject  $h^{-1}(x) = 1 - \sqrt{x+2}$  since  $R_{h^{-1}} = D_h = \mathbb{R}^+$ ). Further note that  $g^{-1} = e^x - 2$ .

$$\begin{aligned} hg(a) &= -1.5 \\ \implies g(a) &= h^{-1}(-1.5) \\ &= 1 + \sqrt{-1.5 + 2} \\ &= 1 + \frac{1}{\sqrt{2}} \\ \implies a &= g^{-1}\left(1 + \frac{1}{\sqrt{2}}\right) \\ &= e^{1 + \frac{1}{\sqrt{2}}} - 2 \end{aligned}$$

$$\boxed{a = e^{1 + \frac{1}{\sqrt{2}}} - 2}$$

**Problem 7.**

The functions  $f$  and  $g$  are defined as follows:

$$\begin{aligned} f: x &\mapsto 3 - x, & x &\in \mathbb{R} \\ g: x &\mapsto \frac{4}{x}, & x &\in \mathbb{R}, x \neq 0 \end{aligned}$$

- (a) Show that the composite function  $fg$  exists and express the definition of  $fg$  in a similar form. Find the range of  $fg$ .
- (b) Find, in similar form,  $g^2$  and  $g^3$ , and deduce  $g^{2017}$ .
- (c) Find the set of values of  $x$  for which  $g(x) = g^{-1}(x)$ .

**Solution****Part (a)**

Note that  $R_g = \mathbb{R} \setminus \{0\}$  and  $D_g = \mathbb{R}$ . Hence,  $R_g \subseteq D_g$ . Thus,  $fg$  exists.

$$\begin{aligned} fg(x) &= f\left(\frac{4}{x}\right) \\ &= 3 - \frac{4}{x} \end{aligned}$$

Observe that  $D_{fg} = D_g = \mathbb{R} \setminus \{0\}$ .

$$\boxed{fg: x \mapsto 3 - \frac{4}{x}, x \in \mathbb{R} \setminus \{0\}}$$

Since  $\frac{4}{x}$  can take on any value except 0, then  $fg(x) = 3 - \frac{4}{x}$  can take on any value except 3.

$$\boxed{R_{fg} = \mathbb{R} \setminus \{3\}}$$

**Part (b)**

$$\begin{aligned} g^2(x) &= g\left(\frac{4}{x}\right) \\ &= \frac{4}{\frac{4}{x}} \\ &= x \end{aligned}$$

$$\boxed{g^2: x \mapsto x, x \in \mathbb{R} \setminus \{0\}}$$

$$\begin{aligned}
 g^3(x) &= g(g^2(x)) \\
 &= g(x) \\
 &= \frac{4}{x}
 \end{aligned}$$

$$g^3: x \mapsto \frac{4}{x}, x \in \mathbb{R} \setminus \{0\}$$

$$\begin{aligned}
 g^{2017} &= g^{2016}(g(x)) \\
 &= (g^2)^{1008}(g(x)) \\
 &= g(x) \\
 &= \frac{4}{x}
 \end{aligned}$$

$$g^{2017}: x \mapsto \frac{4}{x}, x \in \mathbb{R} \setminus \{0\}$$

**Part (c)**

$$\begin{aligned}
 g(x) &= g^{-1}(x) \\
 \implies g^2(x) &= x
 \end{aligned}$$

From the definition of  $g^2(x)$ , we know that  $g^2(x) = x$  for all  $x$  in  $D_{g^2}$ .

$$\mathbb{R} \setminus \{0\}$$

**Problem 8.**

The function  $f$  is defined by

$$f(x) = \begin{cases} 2x + 1, & 0 \leq x < 2 \\ (x - 4)^2 + 1, & 2 \leq x < 4 \end{cases}$$

It is further given that  $f(x) = f(x + 4)$  for all real values of  $x$ .

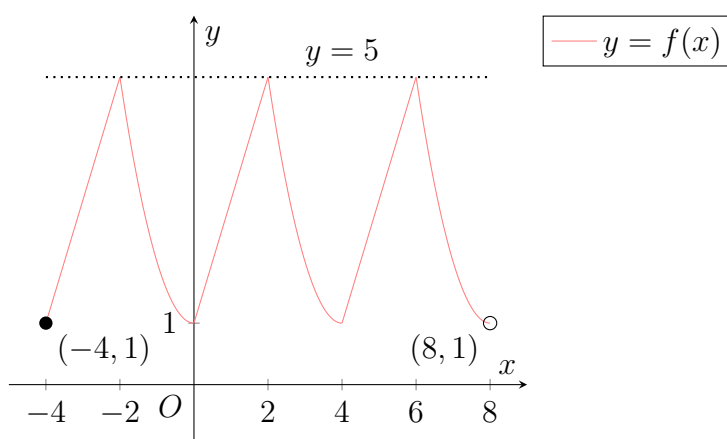
- (a) Find the values of  $f(1)$  and  $f(5)$  and hence explain why  $f$  is not one-one.
- (b) Sketch the graph of  $y = f(x)$  for  $-4 \leq x < 8$ .
- (c) Find the range of  $f$  for  $-4 \leq x < 8$ .

**Solution****Part (a)**

$$\begin{aligned} f(1) &= 2(1) + 1 \\ &= 3 \\ f(5) &= f(1 + 4) \\ &= f(1) \\ &= 3 \end{aligned}$$

$$\boxed{f(1) = 3, f(5) = 3}$$

Since  $f(1) = f(5)$ , but  $1 \neq 5$ ,  $f$  is not one-one.

**Part (b)****Part (c)**

$$\boxed{R_f = [1, 5]}$$

**Problem 9.**

- (a) The function  $f$  is given by  $f: x \mapsto 1 + \sqrt{x}$  for  $x \in \mathbb{R}^+$ .
- (i) Find  $f^{-1}(x)$  and state the domain of  $f^{-1}$ .
  - (ii) Find  $f^2(x)$  and the range of  $f^2$ .
  - (iii) Show that if  $f^2(x) = x$  then  $x^3 - 4x^2 + 4x - 1 = 0$ . Hence find the value of  $x$  for which  $f^2(x) = x$ . Explain why this value of  $x$  satisfies the equation  $f(x) = f^{-1}(x)$ .
- (b) The function  $g$ , with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1, & n = 0 \\ 2 + g\left(\frac{1}{2}n\right), & n \text{ even} \\ 1 + g(n-1), & n \text{ odd} \end{cases}$$

- (i) Find  $g(4)$ ,  $g(7)$  and  $g(12)$ .
- (ii) Does  $g$  have an inverse? Justify your answer.

**Solution****Part (a)****Subpart (i)**

Let  $y = f(x)$ . Then  $x = f^{-1}(y)$ .

$$\begin{aligned} y &= f(x) \\ \implies y &= 1 + \sqrt{x} \\ \implies \sqrt{x} &= y - 1 \\ \implies x &= (y - 1)^2 \end{aligned}$$

$$\boxed{f^{-1}(x) = (x - 1)^2}$$

Observe that  $D_{f^{-1}} = R_f = (1, \infty)$ .

$$\boxed{D_{f^{-1}} = (1, \infty)}$$

**Subpart (ii)**

$$\begin{aligned} f^2(x) &= f(1 + \sqrt{x}) \\ &= 1 + \sqrt{1 + \sqrt{x}} \end{aligned}$$

$$\boxed{f^2(x) = 1 + \sqrt{1 + \sqrt{x}}}$$

Observe that  $\sqrt{1 + \sqrt{x}} > 1$ . Hence,  $1 + \sqrt{1 + \sqrt{x}} > 1 + 1 = 2$ .

$$\boxed{R_{f^2} = (2, \infty)}$$

**Subpart (iii)**

$$\begin{aligned}
 & f^2(x) = x \\
 \implies & 1 + \sqrt{1 + \sqrt{x}} = x \\
 \implies & \sqrt{1 + \sqrt{x}} = x - 1 \\
 \implies & 1 + \sqrt{x} = (x - 1)^2 \\
 \implies & \sqrt{x} = (x - 1)^2 - 1 \\
 \implies & = x(x - 2) \\
 \implies & x = (x(x - 2))^2 \\
 \implies & x(x - 2)^2 = 1 \quad (\because x \neq 0) \\
 \implies & x(x^2 - 4x + 4) = 1 \\
 \implies & x^3 - 4x^2 + 4x = 1 \\
 \implies & x^3 - 4x^2 + 4x - 1 = 0
 \end{aligned}$$

Hence, if  $f^2(x) = x$ , then  $x^3 - 4x^2 + 4x - 1 = 0$ .

$$\begin{aligned}
 & f^2(x) = x \\
 \implies & x^3 - 4x^2 + 4x - 1 = 0 \\
 \implies & (x - 1)(x^2 - 3x + 1) = 0
 \end{aligned}$$

Hence,  $x = 1$  or  $(x^2 - 3x + 1) = 0$ . However, since  $x \geq 2$ ,  $x$  cannot be 1. We thus consider  $(x^2 - 3x + 1) = 0$ .

$$\begin{aligned}
 & (x^2 - 3x + 1) = 0 \\
 \implies & x = \frac{3 \pm \sqrt{5}}{2}
 \end{aligned}$$

Observe that  $\frac{3 - \sqrt{5}}{2} < 2$  and  $\frac{3 + \sqrt{5}}{2} > 2$ . Thus, we reject  $x = \frac{3 - \sqrt{5}}{2}$  and take  $x = \frac{3 + \sqrt{5}}{2}$ .

$$\boxed{x = \frac{3 + \sqrt{5}}{2}}$$

Consider  $f(x) = f^{-1}(x)$ . Applying  $f$  on both sides of the equation, we have  $f^2(x) = f(x)$ . Since  $x = \frac{3 + \sqrt{5}}{2}$  satisfies  $f^2(x) = f(x)$ , it also satisfies  $f(x) = f^{-1}(x)$ .



**Part (b)****Subpart (i)**

$$\begin{aligned}g(4) &= 2 + g(2) \\&= 2 + 2 + g(1) \\&= 2 + 2 + 1 + g(0) \\&= 2 + 2 + 1 + 1 \\&= 6\end{aligned}$$

$$\begin{aligned}g(7) &= 1 + g(6) \\&= 1 + 2 + g(3) \\&= 1 + 2 + 1 + g(2) \\&= 1 + 2 + 1 + (g(4) - 2) \\&= 1 + 2 + 1 + 6 - 2 \\&= 8\end{aligned}$$

$$\begin{aligned}g(12) &= 2 + g(6) \\&= 2 + (g(7) - 1) \\&= 2 + 8 - 1 \\&= 9\end{aligned}$$

$g(4) = 6, g(7) = 8, g(12) = 9$
---------------------------------

**Subpart (ii)**

Consider  $g(5)$  and  $g(6)$ .

$$\begin{aligned}g(5) &= 1 + g(4) \\&= 1 + 6 \\&= 7 \\g(6) &= g(7) - 1 \\&= 8 - 1 \\&= 7\end{aligned}$$

Since  $g(5) = g(6)$ , but  $5 \neq 6$ ,  $g$  is not one-one. Hence,  $g^{-1}$  does not exist.

$g$ does not have an inverse.
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