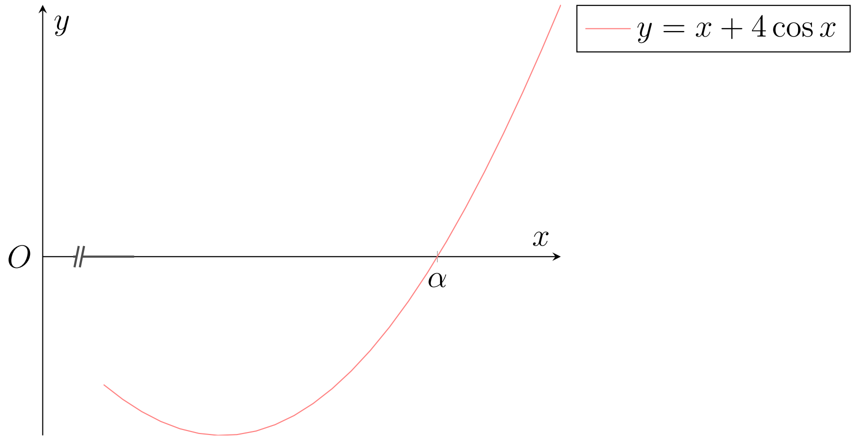
Problem 1.

By considering the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, or otherwise, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

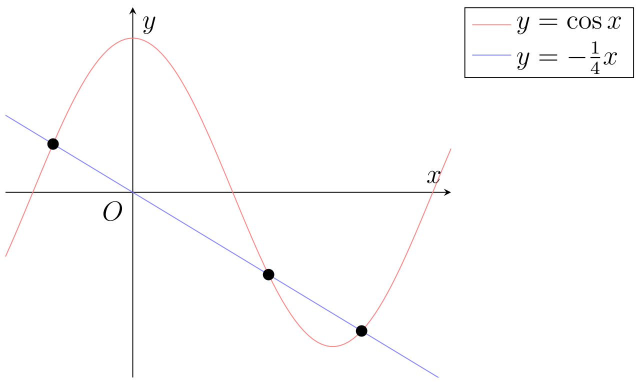
Use linear interpolation, once only, on the interval $[-1.5, 1]$ to find an approximation to the negative root of the equation $x + 4\cos x = 0$ correct to 2 decimal places.



The diagram shows part of the graph of $y = x + 4\cos x$ near the larger positive root, α, of the equation $x + 4\cos x = 0$. Explain why, when using the Newton-Raphson method to find α, an initial approximation which is smaller than α may not be satisfactory.

Use the Newton-Raphson method to find α correct to 2 significant figures. You should demonstrate that your answer has the required accuracy.

Solution



Note that $x + 4\cos x = 0 \implies \cos x = -\frac{1}{4}x$. Plotting the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, we see that there is one negative root and two positive roots. Hence, the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

Let $f(x) = x + 4\cos x$. Let β be the negative root of the equation $f(x) = 0$. Using linear interpolation on the interval $[-1.5. -1]$,

$$\begin{aligned}\beta &= \frac{-1.5f(-1) - (-1)f(-1.5)}{f(-1) - f(1.5)} \\ &= -1.24 \text{ (2 d.p.)}\end{aligned}$$

$$\boxed{\beta = -1.24 \text{ (2 d.p.)}}$$

There is a minimum at $x = m$ such that m is between the two positive roots. Hence, when using the Newton-Raphson method, an initial approximation which is smaller than m would result in subsequent approximations being further away from the desired root α. Hence, an initial approximation that is smaller than α may not be satisfactory.

We know from the above graph that $\alpha \in \left(\pi, \frac{3}{2}\pi\right)$. Following the above discussion, we pick $\frac{3}{2}\pi$ as our initial approximation.

$$\begin{aligned}
x_1 &= \frac{3}{2}\pi \\
\implies x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 3.7699 \\
\implies x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 3.6106 \\
\implies x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 3.5955 \\
\implies x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} = 3.5953
\end{aligned}$$

Since $f(3.55) = -0.1 < 0$ and $f(3.65) = 0.2 > 0$, $\alpha \in (3.55, 3.65)$. Hence, $\alpha = 3.6$ (2 s.f.).

$$\boxed{\alpha = 3.6 \text{ (2 s.f.)}}$$

Problem 2.

Find the coordinates of the stationary points on the graph $y = x^3 + x^2$. Sketch the graph and hence write down the set of values of the constant k for which the equation $x^3 + x^2 = k$ has three distinct real roots.

The positive root of the equation $x^3 + x^2 = 0.1$ is denoted by α.

(a) Find a first approximation to α by linear interpolation on the interval $0 \leq x \leq 1$.

(b) With the aid of a suitable figure, indicate why, in this case, linear interpolation does not give a good approximation to α.

(c) Find an alternative first approximation to α by using the fact that if x is small then x^3 is negligible when compared to x^2.

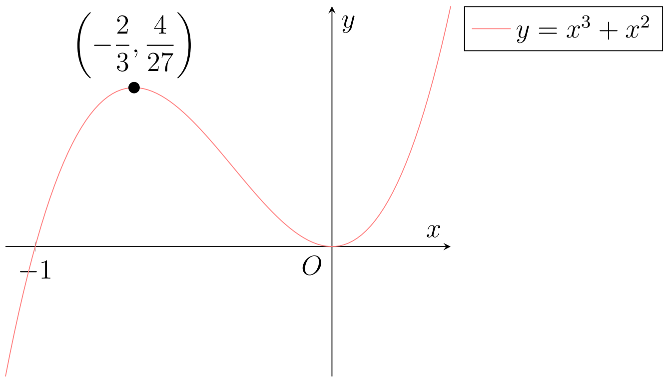
Solution

For stationary points, $y' = 0$.

$$\begin{aligned} & y' = 0 \\ \implies & 3x^2 + 2x = 0 \\ \implies & x(3x+2) = 0 \end{aligned}$$

Hence, $x = 0$ or $x = -\dfrac{2}{3}$. When $x = 0$, $y = 0$. When $x = -\dfrac{2}{3}$, $y = \dfrac{4}{27}$. Thus, the coordinates of the stationary points of $y = x^3 + x^2$ are $(0, 0)$ and $\left(-\dfrac{2}{3}, \dfrac{4}{27}\right)$.

$$\boxed{(0,0),\ \left(-\frac{2}{3}, \frac{4}{27}\right)}$$



Therefore, $k \in \left(0, \dfrac{4}{27}\right)$.

$$\boxed{\left\{k \in \mathbb{R} \colon 0 < k < \frac{4}{27}\right\}}$$

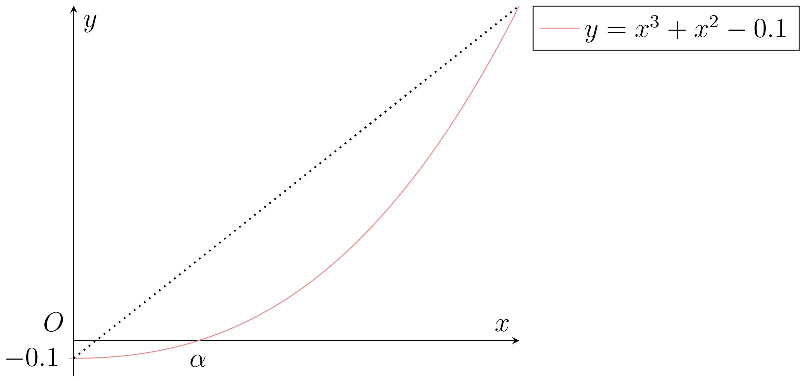
Part (a)

Let $f(x) = x^2 + x^2 - 0.1$. Using linear interpolation on the interval $[0, 1]$,

$$\begin{aligned}\alpha &= \frac{0f(1) - 1f(0)}{f(1) - f(0)} \\ &= \frac{1}{20}\end{aligned}$$

$$\boxed{\alpha = \frac{1}{20}}$$

Part (b)



On the interval $[0, 1]$, the gradient of $y = x^3 + x^2 - 0.1$ changes considerably. Hence, linear interpolation gives an approximation much less than the actual value.

Part (c)

For small x, x^3 is negligible when compared to x^2. Consider $g(x) = x^2 - 0.1$. Then the positive root of $g(x) = 0$ is approximately α. Hence, an alternative approximation to α is $\sqrt{0.1} = 0.316$ (3 s.f.).

$$\boxed{\alpha = 0.316 \text{ (3 s.f.)}}$$

Problem 3.

The equation $2\cos x - x = 0$ has a root α in the interval $[1, 1.2]$. Iterations of the form $x_{n+1} = F(x_n)$ are based on each of the following rearrangements of the equation:

(a) $x = 2\cos x$

(b) $x = \cos x + \frac{1}{2}x$

(c) $x = \frac{2}{3}(\cos x + x)$

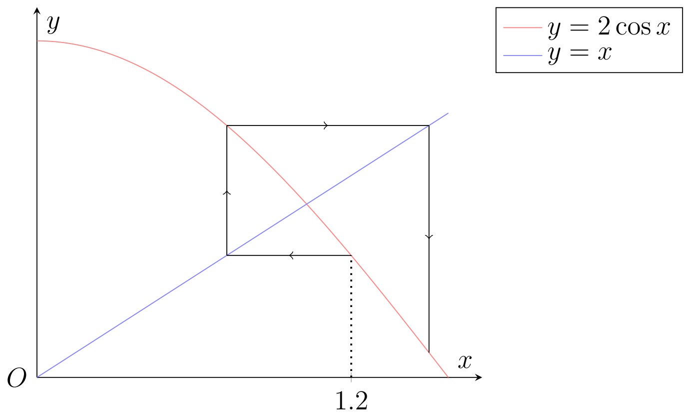
Determine which iteration will converge to α and illustrate your answer by a 'staircase' or 'cobweb' diagram. Use the most appropriate iteration with $x_1 = 1$, to find α to 4 significant figures. You should demonstrate that your answer has the required accuracy.

Solution

Part (a)

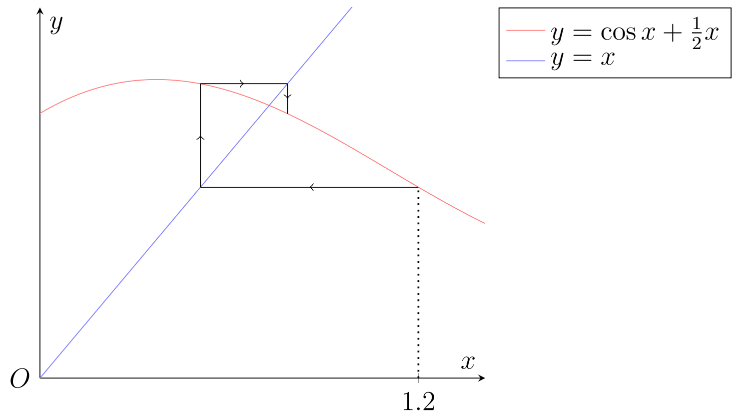
Consider $f(x) = 2\cos x$. Then $f'(x) = -2\sin x$.

Observe that $\sin x$ is increasing on $[1, 1.2]$. Since $\sin 1 > \frac{1}{2}$, $|f'(x)| > 1$ for all $x \in [1, 1.2]$. Thus, fixed-point iteration fails and will not converge to α.



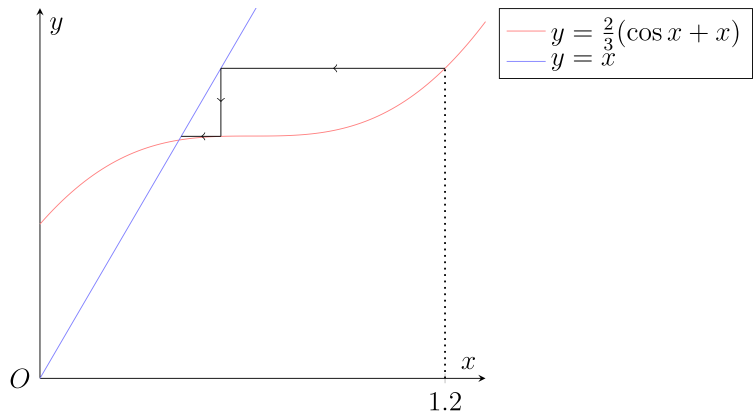
Part (b)

Consider $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} - \left(\sin x - \frac{1}{2}\right)$. Since $0 \le \sin x \le 1$ for $x \in \left[0, \frac{\pi}{2}\right]$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we know $-\frac{1}{2} \le \sin x - \frac{1}{2} \le \frac{1}{2}$ for $x \in [1, 1.2]$. Thus, $0 \le \left|\sin x - \frac{1}{2}\right| \le \frac{1}{2}$ for $x \in [1, 1.2]$. Hence, fixed-point iteration will work and converge to α.



Part (c)

Consider $f(x) = \dfrac{2}{3}(\cos x + x)$. Then $f'(x) = \dfrac{2}{3}(-\sin x + 1)$. For fixed-point iteration to converge to α, we need $|f'(x)| < 1$ for x near α. It thus suffices to show that $|-\sin x + 1| < \dfrac{3}{2}$ for all $x \in [1, 1.2]$. Observe that $1 - \sin x$ is strictly decreasing and positive for $x \in \left[0, \dfrac{\pi}{2}\right]$. Since $1 - \sin 1 < \dfrac{3}{2}$, and $[1, 1.2] \subset \left[0, \dfrac{\pi}{2}\right]$, we have that $|-\sin x + 1| < \dfrac{3}{2}$ for all $x \in [1, 1.2]$. Thus, $|f'(x)| < 1$ for x near α. Hence, fixed-point iteration will work and converge to α.



For $x \in [1, 1.2]$, $\left|\dfrac{2}{3}(-\sin x + 1)\right| < \left|-\sin x + \dfrac{1}{2}\right| < 1$. Thus, $x_{n+1} = \dfrac{2}{3}(\cos x_n + x_n)$ is the most suitable iteration as it will converge to α the quickest. Using $F(x) = \dfrac{2}{3}(\cos x + x)$ with $x_1 = 1$,

$$\begin{aligned} x_1 &= 1 \\ \implies x_2 &= F(x_1) = 1.02687 \\ \implies x_3 &= F(x_2) = 1.02958 \\ \implies x_4 &= F(x_3) = 1.02984 \\ \implies x_5 &= F(x_4) = 1.02986 \end{aligned}$$

Since $F(1.0295) > 1.0295$ and $F(1.0305) < 1.0305$, $\alpha \in (1.0295, 1.0305)$. Hence, $\alpha = 1.030$ (4 s.f.).

$$\boxed{\alpha = 1.030 \text{ (4 s.f.)}}$$