

Problem 1.

Show that

$$\sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) = \frac{a}{b} \left(729 \cdot 9^{(n-1)^2} - 1 \right) - c(n-1)^3 - d(n-1)$$

where a , b , c and d are constants to be determined.

Solution

$$\begin{aligned} \sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) &= \sum_{r=1}^{(n-1)^2+3} 9^r - \sum_{r=1}^{(n-1)^2+3} (n-1) \\ &= \frac{9 \left(9^{(n-1)^2+3} - 1 \right)}{9 - 1} - (n-1) \left((n-1)^2 + 3 \right) \\ &= \frac{9}{8} \left(729 \cdot 9^{(n-1)^2} - 1 \right) - (n-1)^3 - 3(n-1) \end{aligned}$$

Problem 2.**Do not use a calculator in answering this question.**The sequence of positive numbers, u_n , satisfies the recurrence relation:

$$u_{n+1} = \sqrt{2u_n + 3}, \quad n = 1, 2, 3, \dots$$

- (a) If the sequence converges to m , find the value of m .
- (b) By using a graphical approach, explain why $m < u_{n+1} < u_n$ when $u_n > u_m$. Hence, determine the behaviour of the sequence when $u_1 > m$.

Solution**Part (a)**

Observe that

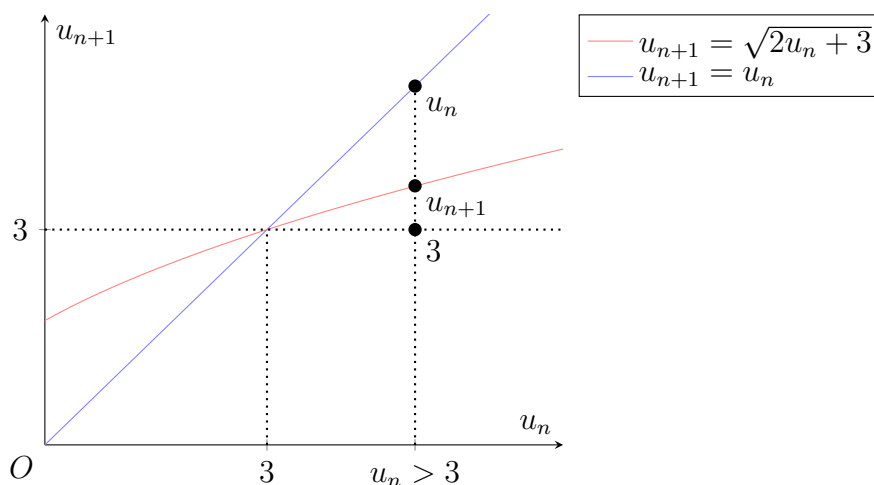
$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \sqrt{2u_{n-1} + 3} \\ &= \sqrt{2 \lim_{n \rightarrow \infty} u_{n-1} + 3} \\ &= \sqrt{2 \lim_{n \rightarrow \infty} u_n + 3} \end{aligned}$$

Since the sequence converges to m , we have $\lim_{n \rightarrow \infty} u_n = m$. Thus,

$$\begin{aligned} m &= \sqrt{2m + 3} \\ \implies m^2 &= 2m + 3 \\ \implies m^2 - 2m - 3 &= 0 \\ \implies (m - 3)(m + 1) &= 0 \end{aligned}$$

Thus, $m = 3$ or $m = -1$. Since u_n is always positive, we take $m = 3$.

$$\boxed{m = 3}$$

Part (b)

From the graph, if $u_n > 3$, then $3 < u_{n+1} < u_n$.

The sequence decreases and converges to 3.
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Problem 3.

Two expedition teams are to climb a vertical distance of 8100 m from the foot to the peak of a mountain. Team A plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Team B plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day.

- (a) Find the number of days required for Team A to reach the peak.
- (b) Explain why Team B will never be able to reach the peak.
- (c) At the end of the 15th day, Team B decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer.

Solution**Part (a)**

The vertical distance Team A plans to cover in a day can be described as a sequence in arithmetic progression with first term 400 and common difference -5 . In order to reach the peak, the total vertical distance covered by Team A has to be greater than 8100 m. Hence,

$$\frac{n}{2} (2(400) + (n-1)(-5)) \geq 8100$$

From the graphing calculator, $n \geq 24$. Hence, Team A requires 24 days to reach the peak.

24 days.

Part (b)

The vertical distance Team B plans to cover in the n th day can be described by the sequence U_n in geometric progression with first term 800 and common ratio $r = 0.9$. Let S_n^U be the n th partial sum of U_n . Since $|r| < 1$, the sum to infinity exists and is equal to

$$S_\infty^U = \frac{800}{1 - 0.9} = 8000$$

Hence, Team B will never be able to surpass 8 km in height. Thus, they will not reach the peak no matter how long they take.

Part (c)

The new vertical distance covered by Team B after Day 15 can be described by the sequence V_n in geometric progression with first term U_{15} and common ratio $r = 0.95$. Let S_n^V be the n th partial sum of V_n . Then,

$$S_n^V = \frac{U_{15} \cdot 0.95 (1 - (0.95)^n)}{1 - 0.95}$$

Note that

$$S_n^U = \frac{800(1 - (0.9)^n)}{1 - 0.9}$$

Hence, after Day 15, Team B has to climb another $8000 - S_{15}^U = 1747.13$ metres. Since $U_{15} = 183.01$, we have the inequality

$$\frac{183.01 \cdot 0.95(1 - (0.95)^n)}{1 - 0.95} \geq 1747.13$$

Using the graphing calculator, $n \geq 14$. Hence, Team B will need at least $15 + 14 = 29$ days to reach the peak.

Team A will reach the peak first.

Problem 4.

The function f is given by $f(x) = x^2 - 3x + 2 - e^{-x}$. It is known from graphical work that this equation has 2 roots $x = \alpha$ and $x = \beta$, where $\alpha < \beta$.

- (a) Show that $f(x) = 0$ has at least one root in the interval $[0, 1]$.

It is known that there is exactly one root in $[0, 1]$ where $x = \alpha$.

- (b) Starting with $x_0 = 0.5$, use an iterative method based on the form

$$x_{n+1} = p(x_n^2 + q - e^{-x_n})$$

where p and q are real numbers to be determined, to find the value of α correct to 3 decimal places. You should demonstrate that your answer has the required accuracy.

It is known that the other root $x = \beta$ lies in the interval $[2, 3]$.

- (c) With the aid of a clearly labelled diagram, explain why the method in (b) will fail to obtain any reasonable approximation to β , where x_0 is chosen such that $x_0 \in [2, 3]$, $x_0 \neq \beta$.

To obtain an approximation to β , another approach is used.

- (d) Use linear interpolation once in the interval $[2, 3]$ to find a first approximation to β , giving your answer to 2 decimal places. Explain whether this approximate is an overestimate or underestimate.
- (e) With your answer in (d) as the initial approximate, use the Newton-Raphson method to obtain β correct to 3 decimal places. Your process should terminate when you have two successive iterates that are equal when rounded to 3 decimal places.

Solution**Part (a)**

Observe that $f(0) = 1 > 0$ and $f(1) = -e^{-1} < 0$. Since f is continuous and $f(0)f(1) < 0$, there must be at least one root to $f(x) = 0$ in the interval $[0, 1]$.

Part (b)

Let $f(x) = 0$. Then,

$$\begin{aligned} x^2 - 3x + 2 - e^{-x} &= 0 \\ \implies x^2 + 2 - e^{-x} &= 3x \\ \implies x &= \frac{1}{3}(x^2 + 2 - e^{-x}) \end{aligned}$$

Hence, we should use an iterative method based on the form

$$x_{n+1} = \frac{1}{3}(x_n^2 + 2 - e^{-x_n})$$

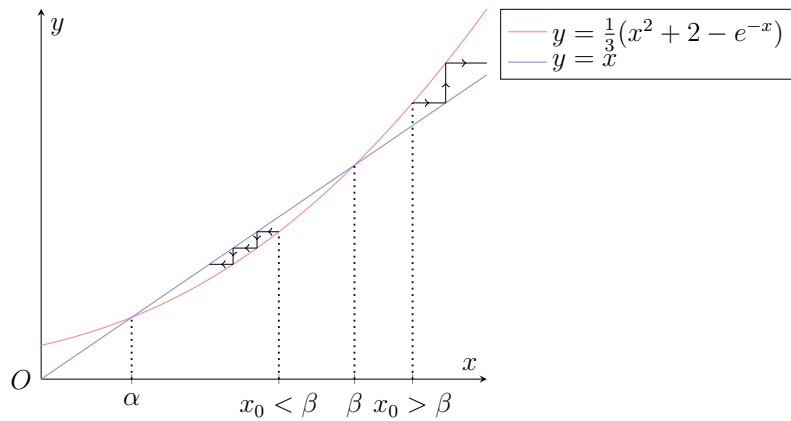
Starting with $x_0 = 0.5$,

$$\begin{aligned}
 x_1 &= \frac{1}{3} (x_0^2 + 2 - e^{-x_0}) = 0.54782 \\
 \Rightarrow x_2 &= \frac{1}{3} (x_1^2 + 2 - e^{-x_1}) = 0.57396 \\
 \Rightarrow x_3 &= \frac{1}{3} (x_2^2 + 2 - e^{-x_2}) = 0.58871 \\
 \Rightarrow x_4 &= \frac{1}{3} (x_3^2 + 2 - e^{-x_3}) = 0.59718 \\
 \Rightarrow x_5 &= \frac{1}{3} (x_4^2 + 2 - e^{-x_4}) = 0.60208 \\
 \Rightarrow x_6 &= \frac{1}{3} (x_5^2 + 2 - e^{-x_5}) = 0.60494 \\
 \Rightarrow x_7 &= \frac{1}{3} (x_6^2 + 2 - e^{-x_6}) = 0.60662 \\
 \Rightarrow x_8 &= \frac{1}{3} (x_7^2 + 2 - e^{-x_7}) = 0.60759 \\
 \Rightarrow x_9 &= \frac{1}{3} (x_8^2 + 2 - e^{-x_8}) = 0.60817 \\
 \Rightarrow x_{10} &= \frac{1}{3} (x_9^2 + 2 - e^{-x_9}) = 0.60851 \\
 \Rightarrow x_{11} &= \frac{1}{3} (x_{10}^2 + 2 - e^{-x_{10}}) = 0.60870
 \end{aligned}$$

Since $f(0.6085) = 0.000606 > 0$ and $f(0.6095) = -0.000632 < 0$, we have that $\alpha \in (0.6085, 0.6095)$. Hence,

$\alpha = 0.609 \text{ (3 d.p.)}$

Part (c)



From the diagram, we see that whether we chose $x_0 < \beta$ or $x_0 > \beta$, the approximates move away from the root β . In fact, if we choose $x_0 < \beta$, the approximates converge to the root α instead.

Part (d)

Using linear interpolation on the interval $[2, 3]$,

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = 2.06 \text{ (2 d.p.)}$$

$$\boxed{\beta = 2.06 \text{ (2 d.p.)}}$$

Observe that $f(2.06) = -0.039 < 0$ and $f(3) = 1.950 > 0$. Hence, $\beta \in (2.06, 3)$. Thus,

$$\boxed{\beta = 2.06 \text{ is an underestimate.}}$$

Part (e)

Observe that $f'(x) = 2xx - 3 + e^{-x}$. Using the Newton-Raphson method with $x_1 = 2.06$,

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.11118 = 2.111 \text{ (3 d.p.)} \\ \implies x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 2.10935 = 2.109 \text{ (3 d.p.)} \\ \implies x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 2.10935 = 2.109 \text{ (3 d.p.)} \end{aligned}$$

Hence,

$$\boxed{\beta = 2.109 \text{ (3 d.p.)}}$$