

Problem 1.

Find $\sum_{r=0}^n (n^2 + 1 - 3r)$ in terms of n , giving your answer in factorised form.

Solution

$$\begin{aligned}\sum_{r=0}^n (n^2 + 1 - 3r) &= (n+1)(n^2 + 1) + 3 \cdot \frac{n(n+1)}{2} \\ &= (n+1) \left(n^2 + \frac{3}{2}n + 1 \right) \\ &= \frac{1}{2}(n+1)(2n^2 + 3n + 2)\end{aligned}$$

$$\boxed{\sum_{r=0}^n (n^2 + 1 - 3r) = \frac{1}{2}(n+1)(2n^2 + 3n + 2)}$$

Problem 2.

Given that $\sum_{k=1}^n k! (k^2 + 1) = (n + 1)! n$, find $\sum_{k=1}^{n-1} (k + 1)! (k^2 + 2k + 2)$.

Solution

$$\begin{aligned}
 \sum_{k=1}^{n-1} (k + 1)! (k^2 + 2k + 2) &= \sum_{k-1=1}^{k-1=n-1} ((k - 1) + 1)! ((k - 1)^2 + 2(k - 1) + 2) \\
 &= \sum_{k=2}^n k! (k^2 + 1) \\
 &= \sum_{k=1}^n k! (k^2 + 1) - 1! (1^2 + 1) \\
 &= (n + 1)! n - 2
 \end{aligned}$$

$$\sum_{k=1}^{n-1} (k + 1)! (k^2 + 2k + 2) = (n + 1)! n - 2$$

Problem 3.

Given that $\sum_{r=1}^n = \frac{1}{6}n(n+1)(2n+1)$, find $\sum_{r=N+1}^{2N} (7^{r+1} + 3r^2)$ in terms of N , simplifying your answer.

Solution

$$\begin{aligned}
 \sum_{r=N+1}^{2N} (7^{r+1} + 3r^2) &= \sum_{r=N+1}^{2N} 7^{r+1} + 3 \sum_{r=N+1}^{2N} r^2 \\
 &= \frac{7^{(N+1)+1}(7^N - 1)}{7 - 1} + 3 \left(\sum_{r=1}^{2N} r^2 - \sum_{r=1}^N r^2 \right) \\
 &= \frac{7^{N+2}(7^N - 1)}{6} + 3 \left(\frac{1}{6}(2N)(2N+1)(2 \cdot 2N+1) - \frac{1}{6}N(N+1)(2N+1) \right) \\
 &= \frac{7^{N+2}(7^N - 1)}{6} + \frac{1}{2} (2N(2N+1)(4N+1) - N(N+1)(2N+1)) \\
 &= \frac{7^{N+2}(7^N - 1)}{6} + \frac{1}{2} N(2N+1)(2 \cdot (4N+1) - (N+1)) \\
 &= \frac{7^{N+2}(7^N - 1)}{6} + \frac{1}{2} N(2N+1)(7N+1)
 \end{aligned}$$

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|---|
| $\sum_{r=N+1}^{2N} (7^{r+1} + 3r^2) = \frac{7^{N+2}(7^N - 1)}{6} + \frac{1}{2} N(2N+1)(7N+1)$ |
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Problem 4.

Let $f(r) = \frac{3}{r-1}$.

(a) Show that $f(r+1) - f(r) = -\frac{3}{r(r-1)}$.

(b) Hence find in terms of N , the sum of the series $S_N = \sum_{r=2}^N \frac{1}{r(r-1)}$.

(c) Explain why $\sum_{r=2}^{\infty} \frac{1}{r(r-1)}$ is a convergent series, and find the value of the sum to infinity.

(d) Using the result from part (b), find $\sum_{r=2}^N \frac{1}{r(r+1)}$.

Solution**Part (a)**

$$\begin{aligned} f(r+1) - f(r) &= \frac{3}{(r+1)-1} - \frac{3}{r-1} \\ &= \frac{3(r-1) - 3r}{r(r-1)} \\ &= -\frac{3}{r(r-1)} \end{aligned}$$

Part (b)

$$\begin{aligned} S_N &= \sum_{r=2}^N \frac{1}{r(r-1)} = -\frac{1}{3} \sum_{r=2}^N -\frac{3}{r(r-1)} \\ &= -\frac{1}{3} \left(\sum_{r=2}^N f(r+1) - \sum_{r=2}^N f(r) \right) \\ &= -\frac{1}{3} \left(\sum_{r=3}^{N+1} f(r) - \sum_{r=2}^N f(r) \right) \\ &= -\frac{1}{3} \left(\left(\sum_{r=3}^N f(r) + f(N+1) \right) - \left(f(2) + \sum_{r=3}^N f(r) \right) \right) \\ &= -\frac{1}{3} (f(N+1) - f(2)) \\ &= -\frac{1}{3} \left(\frac{3}{N+1-1} - \frac{3}{2-1} \right) \\ &= 1 - \frac{1}{N} \end{aligned}$$

$$S_N = 1 - \frac{1}{N}$$

Part (c)

Consider $\lim_{n \rightarrow \infty} S_n$.

$$\begin{aligned}\lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} \left(1 - \frac{1}{N}\right) \\ &= 1 - 0 \\ &= 1\end{aligned}$$

Since 1 is a constant, S_N is a convergent series.

The value of the sum to infinity is 1.

Part (d)

$$\begin{aligned}\sum_{r=2}^N \frac{1}{r(r+1)} &= \sum_{r=3}^{N+1} \frac{1}{(r-1)r} \\ &= \sum_{r=2}^N \frac{1}{r(r-1)} - \frac{1}{2(2-1)} + \frac{1}{(N+1)N} \\ &= 1 - \frac{1}{N} - \frac{1}{2} + \frac{1}{N(N+1)} \\ &= \frac{1}{2} + \frac{1 - (N+1)}{N(N+1)} \\ &= \frac{1}{2} - \frac{N}{N(N+1)} \\ &= \frac{1}{2} - \frac{1}{N+1}\end{aligned}$$

$$\sum_{r=2}^N \frac{1}{r(r+1)} = \frac{1}{2} - \frac{1}{N+1}$$

Problem 5.

Sketch the graph of $y = \frac{4\lambda - x^2}{x^2 + \lambda}$ in each of the following two cases:

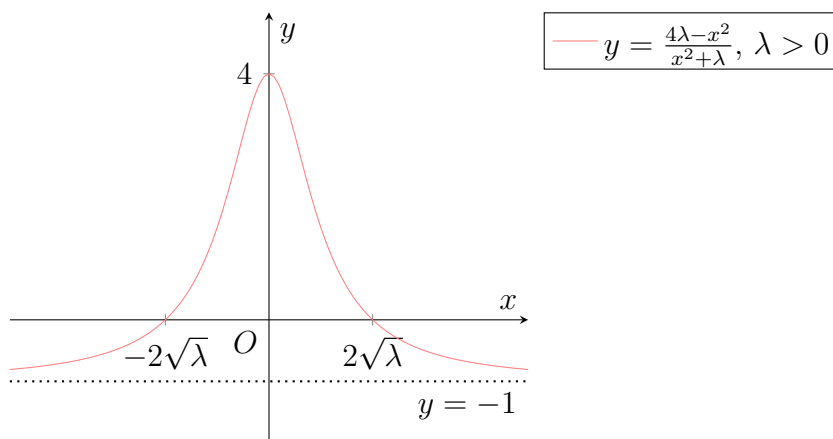
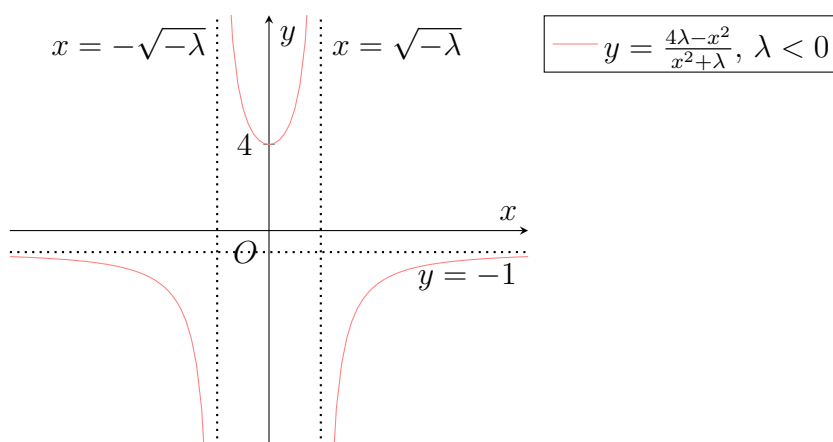
(a) $\lambda > 0$,

(b) $\lambda < 0$.

By using your graph in part (b) and considering a suitable graph whose Cartesian equation is to be stated, find the positive value of h such that the equation

$$\left(\frac{4\lambda - x^2}{hx^2 + h\lambda} \right)^2 = 1 + \frac{x^2}{\lambda}$$

where $\lambda < 0$, has only one real root. State the value of the real root for this value of h .

Solution**Part (a)****Part (b)**

Consider $\left(\frac{4\lambda - x^2}{hx^2 + h\lambda}\right)^2 = 1 + \frac{x^2}{\lambda}$.

$$\begin{aligned} & \left(\frac{4\lambda - x^2}{hx^2 + h\lambda}\right)^2 = 1 + \frac{x^2}{\lambda} \\ \Rightarrow & \left(\frac{1}{h} \cdot \frac{4\lambda - x^2}{x^2 + \lambda}\right)^2 = 1 + \frac{x^2}{\lambda} \\ \Rightarrow & \left(\frac{1}{h} \cdot y\right)^2 = 1 + \frac{x^2}{\lambda} \\ \Rightarrow & \frac{y^2}{h^2} = 1 + \frac{x^2}{\lambda} \\ \Rightarrow & -\frac{x^2}{\lambda} + \frac{y^2}{h^2} = 1 \\ \Rightarrow & -\frac{x^2}{\sqrt{\lambda}^2} + \frac{y^2}{h^2} = 1 \end{aligned}$$

We hence plot an ellipse centered at the origin with horizontal radius $\sqrt{\lambda}$ and vertical radius h .

For only one real root, the ellipse must meet the above graph at exactly one point. Hence, $h = 4$ and the corresponding root is $x = 0$.

