Problem 1.

Sketch the curve with parametric equations

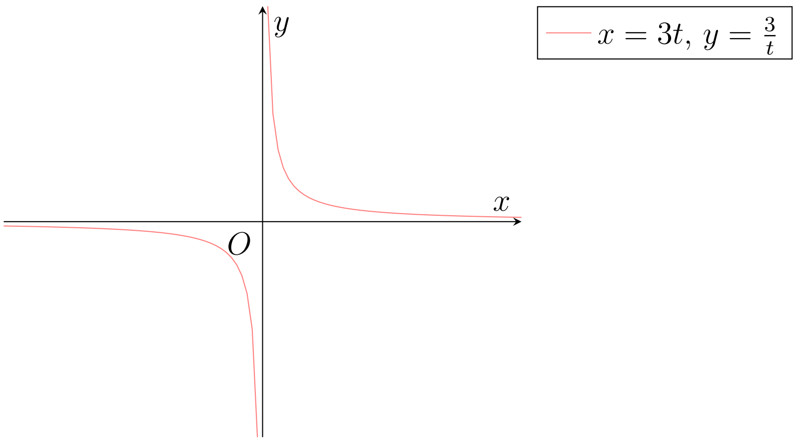
$$x = 3t,\ y = \frac{3}{t}$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q.

(a) Show that the normal at P has equation $2y = 8x - 45$.

(b) Find the value of t at Q.

Solution



Part (a)

Consider $\dfrac{dy}{dx}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \\ &= \left(-\frac{3}{t^2}\right) \cdot \frac{1}{3} \\ &= -\frac{1}{t^2}\end{aligned}$$

Hence, the tangent to the curve has gradient $-\dfrac{1}{t^2}$, whence the normal to the curve has gradient $\dfrac{-1}{-\frac{1}{t^2}} = t^2$. Thus, the normal to the curve at P has gradient $2^2 = 4$. Note that P has coordinates $\left(3 \cdot 2, \dfrac{3}{2}\right) = \left(6, \dfrac{3}{2}\right)$. Using the point-slope formula,

$$\begin{aligned}
& y-\frac{3}{2}=4(x-6) \\
\implies & y-\frac{3}{2}=4x-24 \\
\implies & y=4x-24+\frac{3}{2} \\
\implies & 2y=8x-48+3 \\
& \quad =8x-45
\end{aligned}$$

Thus, the normal at P has equation $2y=8x-45$.

Part (b)

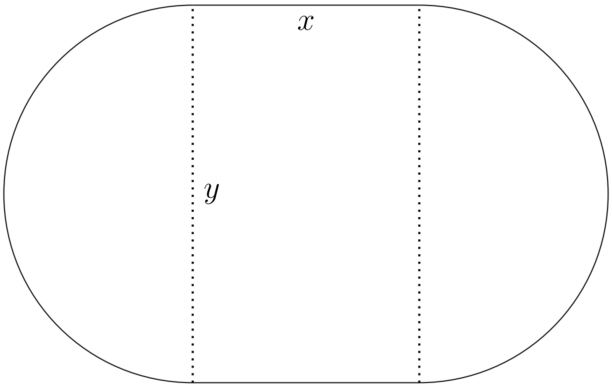
Observe that $x=3t \implies t=\frac{x}{3} \implies y=\frac{3}{\frac{x}{3}}=\frac{9}{x}$. Substituting $y=\frac{9}{x}$ into the equation of the normal at P,

$$\begin{aligned}
& 2\cdot\frac{9}{x}=8x-45 \\
\implies & \frac{18}{x}=8x-45 \\
\implies & 18=8x^2-45x \\
\implies & 8x^2-45x-18=0 \\
\implies & (x-6)(8x+3)=0
\end{aligned}$$

Hence $x=-\frac{3}{8}$ at Q. Note that we reject $x=6$ since $x=6$ at P. Thus, $t=\frac{-\frac{3}{8}}{3}=-\frac{1}{8}$ at Q.

$$\boxed{t=-\frac{1}{8}}$$

Problem 2.



A pond with a constant depth of 1 m is being designed for a park. The pond comprises a rectangle x m by y m and two semicircles of diameter y m, as shown in the diagram. The cost to build a boundary around the pond is $30 per metre for straight parts and $60 per metre for the semicircular parts. Given that the budget for the boundary is fixed at $6000, using differentation or otherwise, find in terms of π, the exact values of x and y which give the pond a maximum volume.

Solution

Observe that the total length of the straight parts is $2x$ m and the total length of the semicircular parts is $2 \cdot \frac{1}{2}\pi y = \pi y$ m. Hence,

$$
\begin{aligned}
& 30 \cdot 2x + 60 \cdot \pi y = 6000 \\
\implies \quad & 60x + 60\pi y = 6000 \\
\implies \quad & x + \pi y = 100 \\
\implies \quad & x = 100 - \pi y
\end{aligned}
$$

Let $V(y)$ m^3 be the volume of the pond.

$$
\begin{aligned}
V(y) &= 1 \cdot \left(\pi \left(\frac{y}{2}\right)^2 + xy\right) \\
&= \frac{\pi}{4}y^2 + xy \\
&= \frac{\pi}{4}y^2 + (100 - \pi y)y \\
&= \frac{\pi}{4}y^2 + 100y - \pi y^2 \\
&= -\frac{3}{4}\pi y^2 + 100y
\end{aligned}
$$

Consider the stationary points of $V(y)$. For stationary points, $V'(y) = 0$.

$$
\begin{aligned}
& V'(y) = 0 \\
\implies & -\frac{3}{4}\pi \cdot 2y + 100 = 0 \\
\implies & y = \frac{200}{3}\pi
\end{aligned}
$$

y	$\left(\frac{200}{3}\pi\right)^-$	$\frac{200}{3}\pi$	$\left(\frac{200}{3}\pi\right)^+$
$V'(y)$	+ve	0	-ve

By the First Derivative Test, the maximum volume of the pond is achieved when $y = \frac{200}{3}\pi$. Thus, $x = 100 - \pi y = \frac{100}{3}$.

$$\boxed{x = \frac{100}{3},\, y = \frac{200}{3}\pi}$$

Problem 3.

A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm^2. Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm^{2}s^{-1}, find the rate of increase at this instant of

(a) the length of the cylinder,

(b) the volume of the cylinder,

(c) the radius of the cylinder.

Solution

Let $A = x$ cm^2 be the cross-sectional area of the cylinder. Then $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = \frac{dx}{dt}$, whence $\left.\frac{dA}{dt}\right|_{x=5} = \left.\frac{dx}{dt}\right|_{x=5} = 0.025$.

Part (a)

Let $L = 20x$ cm be the length of the cylinder. Then $\frac{dL}{dt} = \frac{dL}{dx} \cdot \frac{dx}{dt} = 20 \cdot \frac{dx}{dt}$. Hence, $\left.\frac{dL}{dt}\right|_{x=5} = \left.\left(20 \cdot \frac{dx}{dt}\right)\right|_{x=5} = 20 \cdot 0.025 = 0.5$.

$$\boxed{\text{The length of the cylinder is increasing at a rate of 0.5 cm/s.}}$$

Part (b)

Let $V = AL = 20x^2$ cm^3 be the volume of the cylinder. Then $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = 40x \cdot \frac{dx}{dt}$. Hence, $\left.\frac{dV}{dt}\right|_{x=5} = \left.\left(40x \cdot \frac{dx}{dt}\right)\right|_{x=5} = 40 \cdot 5 \cdot 0.025 = 5$.

$$\boxed{\text{The volume of the cylinder is increasing at a rate of 5 cm/s.}}$$

Part (c)

Let R cm be the radius of the cylinder. Since $\pi R^2 = A = x$, we have $R = \sqrt{\frac{x}{\pi}} = \frac{\sqrt{x}}{\sqrt{\pi}}$. Hence, $\frac{dR}{dt} = \frac{dR}{dx} \cdot \frac{dx}{dt} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt}$. Thus, $\left.\frac{dR}{dt}\right|_{x=5} = \left.\left(\frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt}\right)\right|_{x=5} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{5}} \cdot 0.025 = 0.00315$ (3 s.f.).

$$\boxed{\text{The radius of the cylinder is increasing at a rate of 0.00315 cm/s.}}$$

Problem 4.

The curve C has equation $2^{-y} = x$. The point A on C has x-coordinate a where $a > 0$. Show that $\frac{dy}{dx} = -\frac{1}{a \ln 2}$ at A and find the equation of the tangent to C at A. Hence find the equation of the tangent to C which passes through the origin.

The straight line $y = mx$ intersects C at 2 distinct points. Write down the range of values of m.

Solution

Implicitly differentiating the given equation,

$$\begin{aligned} & 2^{-y} \cdot \ln 2 \cdot -y' = 1 \\ \implies & x \cdot \ln 2 \cdot -y' = 1 \\ \implies & y' = -\frac{1}{x \ln 2} \end{aligned}$$

At A, $x = a$. Hence, $\frac{dy}{dx} = -\frac{1}{a \ln 2}$.

Note that $2^{-y} = x \implies y = -\log_2 x$. Hence, A has coordinates $(a, -\log_2 a)$. Using the point-slope formula, the tangent to C at A has equation

$$\begin{aligned} & y - (-\log_2 a) = -\frac{1}{a \ln 2}(x - a) \\ \implies & y = -\frac{x}{a \ln 2} + \frac{1}{\ln 2} - \log_2 a \\ & = -\frac{x}{a \ln 2} + \frac{1}{\ln 2} - \frac{\ln a}{\ln 2} \\ & = -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \end{aligned}$$

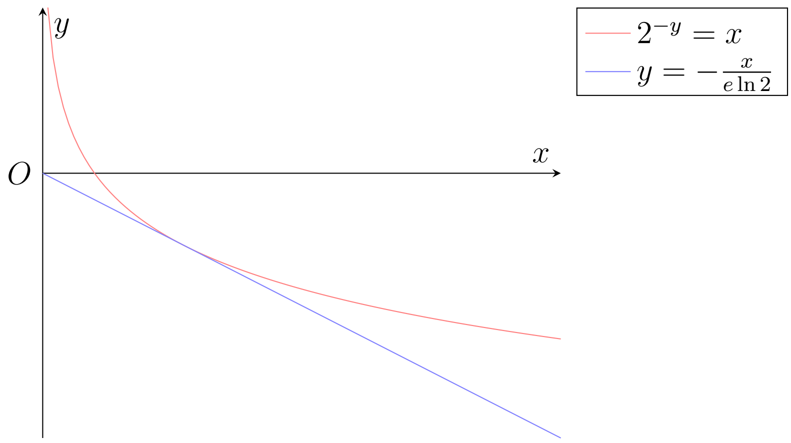
$$\boxed{\text{The tangent to } C \text{ at } A \text{ has equation } y = -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2}.}$$

Consider the straight line $y = mx$ that is tangent to C and passes through the origin.

$$\begin{aligned} & 0 = -\frac{0}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \\ \implies & 1 - \ln a = 0 \\ \implies & a = e \end{aligned}$$

Hence, the equation of the tangent to C that passes through the origin is $y = -\frac{x}{e \ln 2}$.

Consider the graph of $2^{-y} = x$.



Hence, m must be strictly between $-\dfrac{1}{e \ln 2}$ and 0.

$$\boxed{m \in \left(-\frac{1}{e \ln 2}, 0\right)}$$