

Problem 1.

Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

$$(a) \begin{cases} a + 2b - 3c = -5 \\ -2a - 4b - 6c = 10 \\ 3a + 7b - 2c = -13 \end{cases}$$

$$(b) \begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$$

$$(c) \begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$$

$$(d) \begin{cases} \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 5 \\ \frac{2}{p} - \frac{3}{q} - \frac{4}{r} = -11 \\ \frac{3}{p} + \frac{2}{q} - \frac{1}{r} = -6 \end{cases}$$

$$(e) \begin{cases} 2 \sin \alpha - \cos \beta + 3 \tan \gamma = 3 \\ 4 \sin \alpha + 2 \cos \beta - 2 \tan \gamma = 2 \\ 6 \sin \alpha - 3 \cos \beta + \tan \gamma = 9 \end{cases}, \text{ where } 0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq 2\pi, \text{ and } 0 \leq \gamma < \pi.$$

Solution**Part (a)**

Unique solution: $a = -9, b = 2, c = 0$

Part (b)

No solution.

Part (c)

Infinitely many solutions: $x_1 = 8 - t, x_2 = t - 3, x_3 = t$

Part (d)

$$\frac{1}{p} = 2, \frac{1}{q} = -3, \frac{1}{r} = 6$$

Unique solution: $p = \frac{1}{2}, q = -\frac{1}{3}, r = \frac{1}{6}$

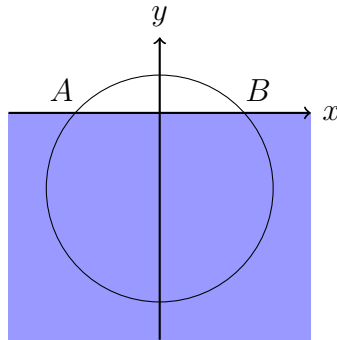
Part (e)

$$\sin \alpha = 1, \cos \beta = -1, \tan \gamma = 0$$

Unique solution: $\alpha = \frac{\pi}{2}, \beta = \pi, \gamma = 0$

Problem 2.

The following figure shows the circular cross section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0 \quad (2.1)$$

A and B are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when AB is 40 cm apart horizontally, determine the values of a , b and c by forming a system of linear equations.

Solution

Since $AB = 40$, we have $A(-20, 0)$ and $B(20, 0)$. We also know $(0, 10)$ lies on the circle. Substituting these points into Equation 2.1,

$$\begin{cases} -20a + c = -400 \\ 20a + c = -400 \\ 10b + c = -100 \end{cases}$$

Solving the system of equations,

$a = 0, b = 30, c = -400$

Problem 3.

Find the exact solution set of the following inequalities.

(a) $x^2 - 2 \geq 0$

(b) $4x^2 - 12x + 10 > 0$

(c) $x^2 + 4x + 13 < 0$

(d) $x^3 < 6x - x^2$

(e) $x^2(x - 1)(x + 3) \geq 0$

Solution**Part (a)**

$$\begin{aligned} x^2 - 2 &\geq 0 \\ \implies x^2 &\geq 2 \\ \implies x &\leq -\sqrt{2} \vee x \geq \sqrt{2} \end{aligned}$$

Solution set: $\{x \in \mathbb{R} : x \leq -\sqrt{2} \vee x \geq \sqrt{2}\}$

Part (b)

$$\begin{aligned} 4x^2 - 12x + 10 &> 0 \\ \implies x^2 - 3x + \frac{5}{2} &> 0 \\ \implies x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{2} &> 0 \\ \implies \left(x - \frac{3}{2}\right)^2 + \frac{19}{4} &> 0 \end{aligned}$$

Since $\left(x - \frac{3}{2}\right)^2 \geq 0$, all $x \in \mathbb{R}$ satisfy the inequality.

Solution set: \mathbb{R}

Part (c)

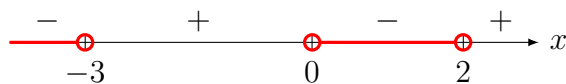
$$\begin{aligned} x^2 + 4x + 13 &< 0 \\ \implies x^2 + 4x + 4 + 9 &< 0 \\ \implies (x + 2)^2 + 9 &< 0 \end{aligned}$$

Since $(x + 2)^2 \geq 0$, there is no solution to the inequality.

Solution set: \emptyset

Part (d)

$$\begin{aligned}x^3 &< 6x - x^2 \\ \implies x(x+3)(x-2) &< 0\end{aligned}$$

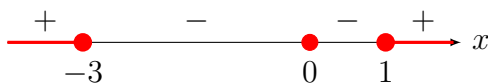


$$\implies x < -3 \vee 0 < x < 2$$

Solution set: $\{x \in \mathbb{R} : x < -3 \vee 0 < x < 2\}$

Part (e)

$$x^2(x-1)(x+3) \geq 0$$



$$\implies x \leq -3 \vee x = 0 \vee x \geq 1$$

Solution set: $\{x \in \mathbb{R} : x \leq -3 \vee x = 0 \vee x \geq 1\}$

Problem 4.

Find the exact solution set of the following inequalities.

(a) $|3x + 5| < 4$

(b) $|x - 2| < 2x$

Solution**Part (a)**

$$|3x + 5| < 4$$

Case 1: $3x + 5 < 4$

$$\begin{aligned} 3x + 5 &< 4 \\ \implies 3x &< -1 \\ \implies x &< -\frac{1}{3} \end{aligned}$$

Case 2: $-(3x + 5) < 4$

$$\begin{aligned} -(3x + 5) &< 4 \\ \implies -3x - 5 &< 4 \\ \implies -3x &< 9 \\ \implies x &> -3 \end{aligned}$$

Combining both inequalities, we have $-3 < x < -\frac{1}{3}$.

Solution set: $\left\{x \in \mathbb{R}: -3 < x < -\frac{1}{3}\right\}$

Part (b)

Case 1: $x - 2 < 2x$

$$\begin{aligned} x - 2 &< 2x \\ \implies x &> -2 \end{aligned}$$

Case 2: $-(x - 2) < 2x$

$$\begin{aligned} -(x - 2) &< 2x \\ \implies -x + 2 &< 2x \\ \implies 3x &> 2 \\ \implies x &> \frac{2}{3} \end{aligned}$$

Combining both inequalities, we have $x > \frac{2}{3}$.

Solution set: $\left\{x \in \mathbb{R} : x > \frac{2}{3}\right\}$

Problem 5.

It is given that $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a , b , c and d are constants. Given that the curve with equation $y = p(x)$ is symmetrical about the y -axis, and that it passes through the points with coordinates $(1, 2)$ and $(2, 11)$, find the values of a , b , c and d .

Solution

We know that $(1, 2)$ and $(2, 11)$ lie on the curve. Since $y = p(x)$ is symmetrical about the y -axis, we have that $(-1, 2)$ and $(-2, 11)$ also lie on the curve. Substituting these points into $y = p(x)$, we obtain the following system of equations.

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving the system of equations,

$a = 0, b = -2, c = 0, d = 3$

Problem 6.

Mr Mok invested \$50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of \$3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

Solution

Let a , b and c be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a + b + c = 50000 \\ \frac{6}{100}a + \frac{8}{100}b + \frac{10}{100}c = 3700 \\ a = 2c \end{cases}$$

Solving the system of equations, we have $a = 30000$, $b = 5000$ and $c = 15000$.

Mr Mok invested \$30,000, \$5,000 and \$15,000 in Funds A, B and C respectively.

Problem 7.

Solve the following inequalities with exact answers.

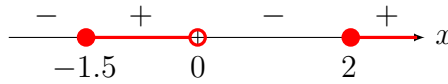
(a) $2x - 1 \geq \frac{6}{x}$

(b) $x - \frac{1}{x} < 1$

(c) $-1 < \frac{2x+3}{x-1} < 1$

Solution**Part (a)**

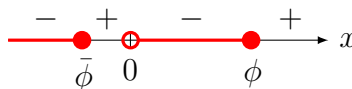
$$\begin{aligned}
& 2x - 1 \geq \frac{6}{x}, \quad x \neq 0 \\
\Rightarrow & x^2(2x - 1) \geq 6x \\
\Rightarrow & x^2(2x - 1) - 6x \geq 0 \\
\Rightarrow & x(x(2x - 1) - 6) \geq 0 \\
\Rightarrow & x(2x^2 - x - 6) \geq 0 \\
\Rightarrow & x(2x + 3)(x - 2) \geq 0
\end{aligned}$$



$$-\frac{3}{2} \leq x < 0 \vee x \geq 2$$

Part (b)

$$\begin{aligned}
& x - \frac{1}{x} < 1, \quad x \neq 0 \\
\Rightarrow & x^3 - x < x^2 \\
\Rightarrow & x^3 - x^2 - x < 0 \\
\Rightarrow & x(x^2 - x - 1) < 0 \\
\Rightarrow & x(x - \bar{\phi})(x - \phi) < 0
\end{aligned}$$



$$x \leq \bar{\phi} \vee 0 < x \leq \phi$$

Part (c)

$$\begin{aligned} & -1 < \frac{2x+3}{x-1} < 1 \\ \Rightarrow & -1 < \frac{(2x-2)+5}{x-1} < 1 \\ \Rightarrow & -1 < 2 + \frac{5}{x-1} < 1 \\ \Rightarrow & -3 < \frac{5}{x-1} < -1 \\ \Rightarrow & -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5} \\ \Rightarrow & -5 < x-1 < -\frac{5}{3} \\ \Rightarrow & -4 < x < -\frac{2}{3} \end{aligned}$$

$-4 < x < -\frac{2}{3}$

Problem 8.

Without using a calculator, solve the inequality $\frac{x^2 + x + 1}{x^2 + x - 2} < 0$.

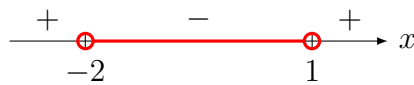
Solution

Observe that

$$\begin{aligned}x^2 + x + 1 &= x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 \\&= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \\&> 0\end{aligned}$$

Thus, the inequality reduces to $\frac{1}{x^2 + x - 2} < 0$

$$\begin{aligned}\frac{1}{x^2 + x - 2} &< 0 \\ \implies x^2 + x - 1 &< 0 \\ \implies (x - 1)(x + 2) &< 0\end{aligned}$$

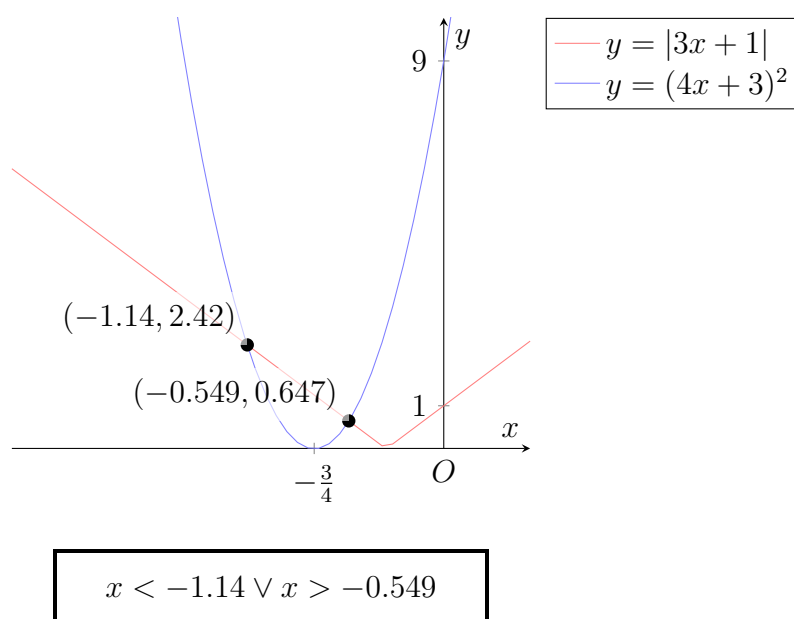
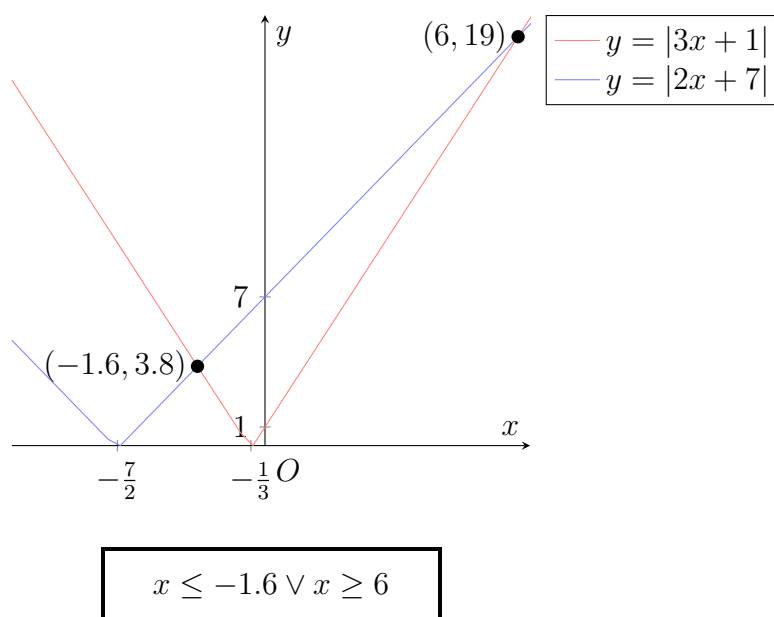


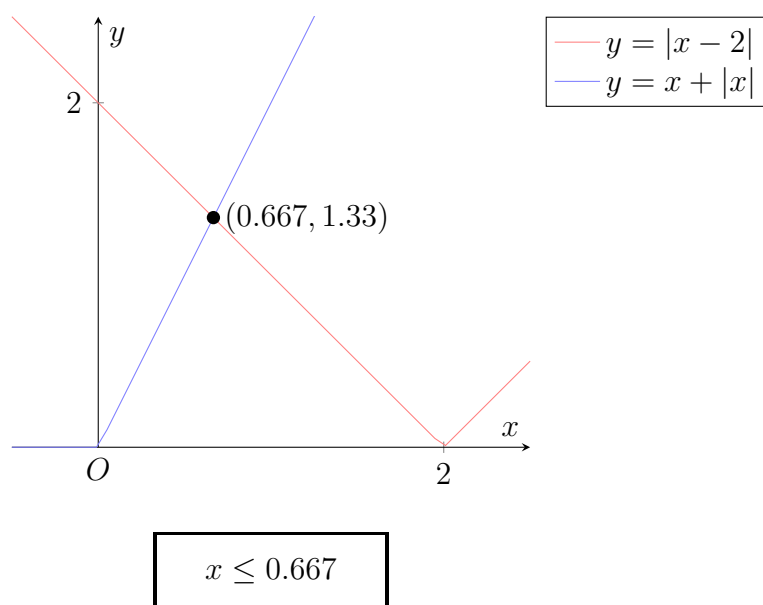
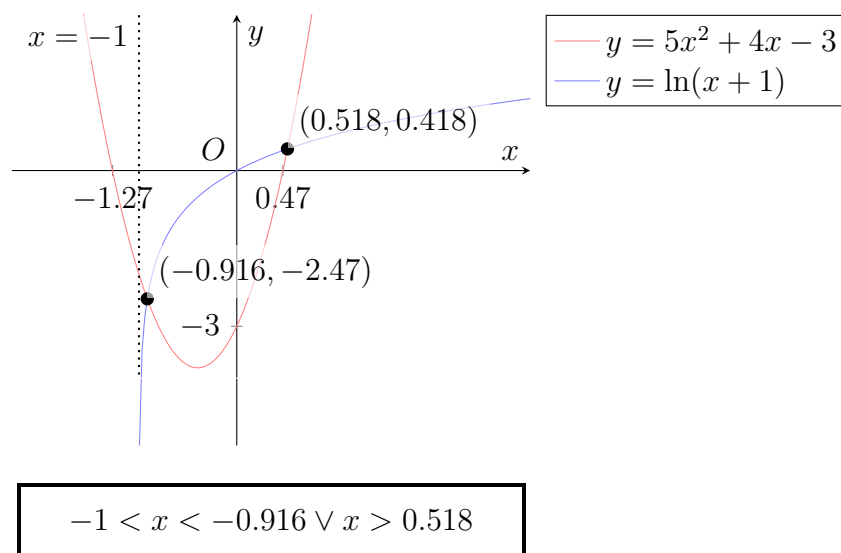
$$\boxed{-2 < x < 1}$$

Problem 9.

Solve the following inequalities using a graphical method.

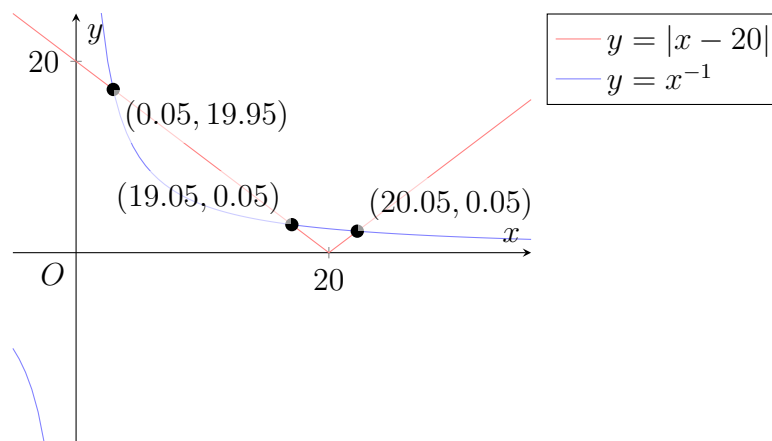
- (a) $|3x + 1| < (4x + 3)^2$
 (b) $|3x + 1| \geq |2x + 7|$
 (c) $|x - 2| \geq x + |x|$
 (d) $5x^2 + 4x - 3 > \ln(x + 1)$

Solution**Part (a)****Part (b)**

Part (c)**Part (d)**

Problem 10.

Sketch the graphs of $y = |x - 20|$ and $y = \frac{1}{x}$ on the same diagram. Hence or otherwise, solve the inequality $|x - 20| < \frac{1}{x}$, leaving your answers correct to 2 decimal places.

Solution

$$0 < x < 0.05 \vee 19.95 < x < 20.05$$

Problem 11.

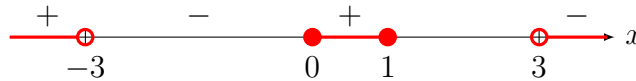
Solve the inequality $\frac{x-9}{x^2-9} \leq 1$. Hence, solve the inequalities

(a) $\frac{|x|-9}{x^2-9} \leq 1$

(b) $\frac{x+9}{x^2-9} \geq -1$

Solution

$$\begin{aligned}
 & \frac{x-9}{x^2-9} \leq 1 \\
 \implies & (x-9)(x^2-9) \leq (x^2-9)^2 \\
 \implies & x^3 - 9x^2 - 9x + 81 \leq x^4 - 18x^2 + 81 \\
 \implies & x^4 - x^3 - 9x^2 + 9x \geq 0 \\
 \implies & (x+3)x(x-1)(x-3) \geq 0
 \end{aligned}$$



$$x < -3 \vee 0 \leq x \leq 1 \vee x > 3$$

Part (a)

Consider the substitution $x \mapsto |x|$ on the inequality $\frac{x-9}{x^2-9} \leq 1$. This yields our desired inequality $\frac{|x|-9}{x^2-9} \leq 1$. Hence,

$$|x| < -3 \vee 0 \leq |x| \leq 1 \vee |x| > 3$$

Case 1: $|x| < -3$

No solution.

Case 2: $0 \leq |x| \leq 1$

$$-1 \leq x \leq 1$$

Case 3: $|x| > 3$

$$x < -3 \vee x > 3$$

Combining all three cases, we finally have

$$x < -3 \vee -1 \leq x \leq 1 \vee x > 3$$

Part (b)

Consider the substitution $x \mapsto -x$ on the inequality $\frac{x-9}{x^2-9} \leq 1$. This yields $\frac{-x-9}{x^2-9} \leq 1$, which is equivalent to our desired inequality $\frac{x+9}{x^2-9} \geq -1$. Hence,

$$-x < -3 \vee 0 \leq -x \leq 1 \vee -x > 3$$

Case 1: $-x < -3$

$$x > 3$$

Case 2: $0 \leq -x \leq 1$

$$-1 \leq x \leq 0$$

Case 3: $-x > 3$

$$x < -3$$

Combining all three cases, we finally have

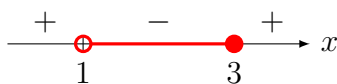
$$x < -3 \vee -1 \leq x \leq 0 \vee x > 3$$

Problem 12.

Solve the inequality $\frac{x-5}{1-x} \geq 1$. Hence, solve $0 < \frac{1-\ln x}{\ln x-5} \leq 1$.

Solution

$$\begin{aligned} \frac{x-5}{1-x} &\geq 1, & x &\neq 1 \\ \implies (x-5)(1-x) &\geq (1-x)^2 \\ \implies 2x^2 - 8x + 6 &\leq 0 \\ \implies 2(x-1)(x-3) &\leq 0 \end{aligned}$$



$$1 < x \leq 3$$

Consider the substitution $x \mapsto \ln x$ on the inequality $\frac{x-5}{1-x} \geq 1$. This yields our desired inequality $0 < \frac{1-\ln x}{\ln x-5} \leq 1$. Hence,

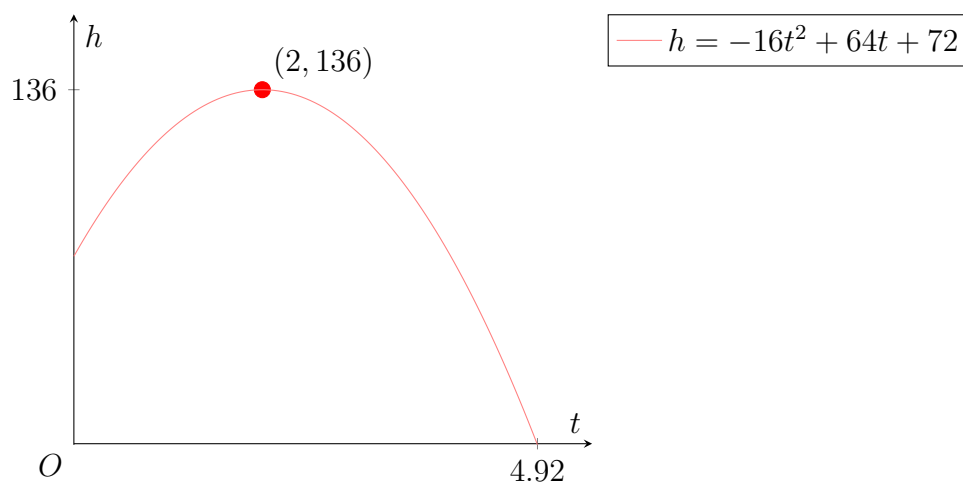
$$\begin{aligned} 1 &< \ln x \leq 3 \\ \implies e &< x \leq e^3 \end{aligned}$$

$$e < x \leq e^3$$

Problem 13.

A small rocket is launched from a height of 72 m from the ground. The height of the rocket in metres, h , is represented by the equation $h = -16t^2 + 64t + 72$, where t is the time in seconds after the launch.

- (a) Sketch the graph of h against t .
- (b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

Solution**Part (a)****Part (b)**

$$\begin{aligned}
 h &\geq 100 \\
 \implies -16t^2 + 64t + 72 &\geq 100 \\
 \implies -16t^2 + 64t - 28 &\geq 0 \\
 \implies 16t^2 - 64t + 28 &\geq 0 \\
 \implies 4(2t - 1)(2t - 7) &\leq 0
 \end{aligned}$$



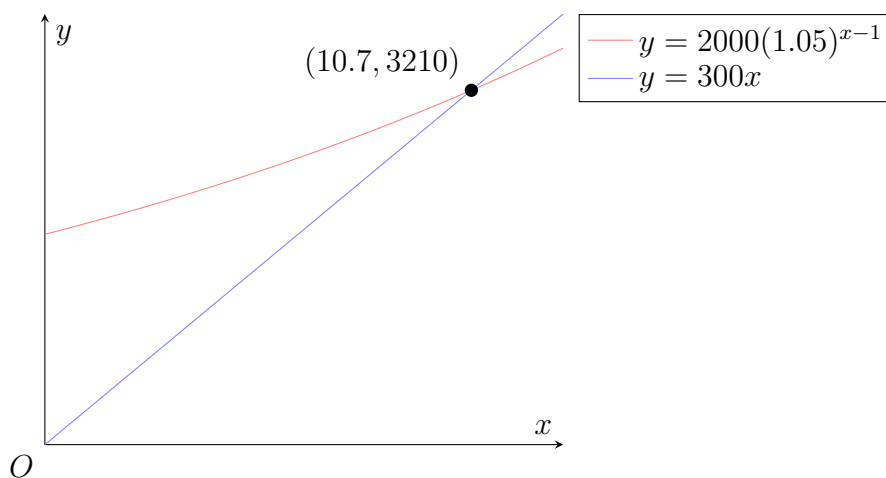
The rocket will remain at or above 100 m from the ground for $\frac{7}{2} - \frac{1}{2} = 3$ seconds.

Problem 14.

Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of $2000(1.05)^{n-1}$ dollars in the n th quarter, where n is a positive integer. She also gives a regular donation of $\$300n$ in the n th quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

Solution

Consider the curves $y = 2000(1.05)^{x-1}$ and $y = 300x$.



Xinxin will stop donating in the $\lceil 10.7 \rceil = 11$ th quarter.