Problem 1.

Sketch the curve with parametric equations

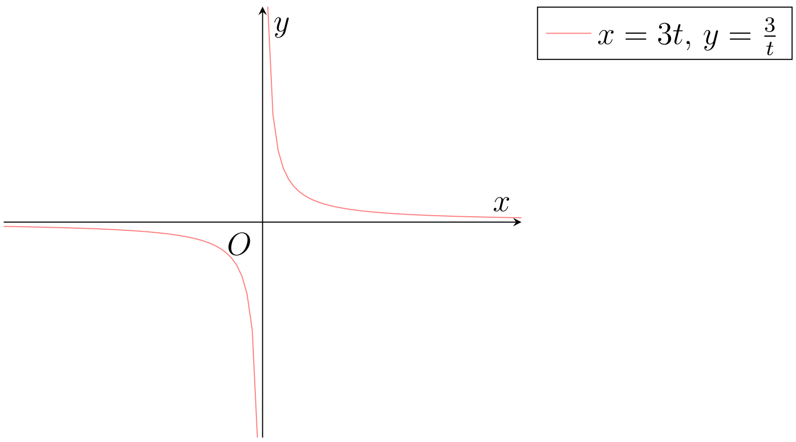
$$x = 3t,\ y = \frac{3}{t}$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q.

(a) Show that the normal at P has equation $2y = 8x - 45$.

(b) Find the value of t at Q.

Solution



Part (a)

Consider $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \\ &= \left(-\frac{3}{t^2}\right) \cdot \frac{1}{3} \\ &= -\frac{1}{t^2}\end{aligned}$$

Hence, the tangent to the curve has gradient $-\dfrac{1}{t^2}$, whence the normal to the curve has gradient $\dfrac{-1}{-\frac{1}{t^2}} = t^2$. Thus, the normal to the curve at P has gradient $2^2 = 4$. Note that P has coordinates $\left(3 \cdot 2, \dfrac{3}{2}\right) = \left(6, \dfrac{3}{2}\right)$. Using the point-slope formula,

$$
\begin{aligned}
y - \frac{3}{2} &= 4(x-6) \\
\implies y - \frac{3}{2} &= 4x - 24 \\
\implies \quad y &= 4x - 24 + \frac{3}{2} \\
\implies \quad 2y &= 8x - 48 + 3 \\
&= 8x - 45
\end{aligned}
$$

Thus, the normal at P has equation $2y = 8x - 45$.

Part (b)

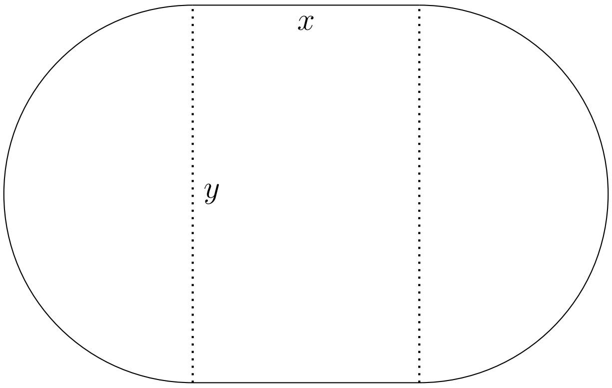
Observe that $x = 3t \implies t = \frac{x}{3} \implies y = \frac{3}{\frac{x}{3}} = \frac{9}{x}$. Substituting $y = \frac{9}{x}$ into the equation of the normal at P,

$$
\begin{aligned}
2 \cdot \frac{9}{x} &= 8x - 45 \\
\implies \quad \frac{18}{x} &= 8x - 45 \\
\implies \quad 18 &= 8x^2 - 45x \\
\implies 8x^2 - 45x - 18 &= 0 \\
\implies (x-6)(8x+3) &= 0
\end{aligned}
$$

Hence, $x = -\frac{3}{8}$ at Q. Note that we reject $x = 6$ since $x = 6$ at P. Thus, $t = \frac{-\frac{3}{8}}{3} = -\frac{1}{8}$ at Q.

$$\boxed{t = -\frac{1}{8}}$$

Problem 2.



A pond with a constant depth of 1 m is being designed for a park. The pond comprises a rectangle x m by y m and two semicircles of diameter y m, as shown in the diagram. The cost to build a boundary around the pond is \$30 per metre for straight parts and \$60 per metre for the semicircular parts. Given that the budget for the boundary is fixed at \$6000, using differentation or otherwise, find in terms of π, the exact values of x and y which give the pond a maximum volume.

Solution

Observe that the total length of the straight parts is $2x$ m and the total length of the semicircular parts is $2 \cdot \frac{1}{2}\pi y = \pi y$ m. Hence,

$$\begin{aligned} && 30 \cdot 2x + 60 \cdot \pi y &= 6000 \\ \implies && 60x + 60\pi y &= 6000 \\ \implies && x + \pi y &= 100 \\ \implies && x &= 100 - \pi y \end{aligned}$$

Let $V(y)$ m^3 be the volume of the pond.

$$\begin{aligned} V(y) &= 1 \cdot \left(\pi \left(\frac{y}{2}\right)^2 + xy\right) \\ &= \frac{\pi}{4}y^2 + xy \\ &= \frac{\pi}{4}y^2 + (100 - \pi y)y \\ &= \frac{\pi}{4}y^2 + 100y - \pi y^2 \\ &= -\frac{3}{4}\pi y^2 + 100y \end{aligned}$$

Consider the stationary points of $V(y)$. For stationary points, $V'(y) = 0$.

$$\begin{aligned} && V'(y) &= 0 \\ \implies && -\frac{3}{4}\pi \cdot 2y + 100 &= 0 \\ \implies && y &= \frac{200}{3}\pi \end{aligned}$$

y	$\left(\frac{200}{3}\pi\right)^-$	$\frac{200}{3}\pi$	$\left(\frac{200}{3}\pi\right)^+$
$V'(y)$	+ve	0	-ve

By the First Derivative Test, the maximum volume of the pond is achieved when $y = \frac{200}{3}\pi$. Thus, $x = 100 - \pi y = \frac{100}{3}$.

$$\boxed{x = \frac{100}{3},\ y = \frac{200}{3}\pi}$$

Problem 3.

A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm^2. Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm^{2}s^{-1}, find the rate of increase at this instant of

(a) the length of the cylinder,

(b) the volume of the cylinder,

(c) the radius of the cylinder.

Solution

Let $A = x$ cm^2 be the cross-sectional area of the cylinder. Then $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}x} \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\mathrm{d}x}{\mathrm{d}t}$, whence $\left.\dfrac{\mathrm{d}A}{\mathrm{d}t}\right|_{x=5} = \left.\dfrac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=5} = 0.025$.

Part (a)

Let $L = 20x$ cm be the length of the cylinder. Then $\dfrac{\mathrm{d}L}{\mathrm{d}t} = \dfrac{\mathrm{d}L}{\mathrm{d}x} \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t} = 20 \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t}$. Hence, $\left.\dfrac{\mathrm{d}L}{\mathrm{d}t}\right|_{x=5} = \left.\left(20 \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t}\right)\right|_{x=5} = 20 \cdot 0.025 = 0.5$.

$$\boxed{\text{The length of the cylinder is increasing at a rate of 0.5 cm/s.}}$$

Part (b)

Let $V = AL = 20x^2$ cm^3 be the volume of the cylinder. Then $\dfrac{\mathrm{d}V}{\mathrm{d}t} = \dfrac{\mathrm{d}V}{\mathrm{d}x} \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t} = 40x \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t}$. Hence, $\left.\dfrac{\mathrm{d}V}{\mathrm{d}t}\right|_{x=5} = \left.\left(40x \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t}\right)\right|_{x=5} = 40 \cdot 5 \cdot 0.025 = 5$.

$$\boxed{\text{The volume of the cylinder is increasing at a rate of 5 cm}^3\text{/s.}}$$

Part (c)

Let R cm be the radius of the cylinder. Since $\pi R^2 = A = x$, we have $R = \sqrt{\dfrac{x}{\pi}} = \dfrac{\sqrt{x}}{\sqrt{\pi}}$. Hence, $\dfrac{\mathrm{d}R}{\mathrm{d}t} = \dfrac{\mathrm{d}R}{\mathrm{d}x} \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{1}{2\sqrt{x}} \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t}$. Thus, $\left.\dfrac{\mathrm{d}R}{\mathrm{d}t}\right|_{x=5} = \left.\left(\dfrac{1}{\sqrt{\pi}} \cdot \dfrac{1}{2\sqrt{x}} \cdot \dfrac{\mathrm{d}x}{\mathrm{d}t}\right)\right|_{x=5} = \dfrac{1}{\sqrt{\pi}} \cdot \dfrac{1}{2\sqrt{5}} \cdot 0.025 = 0.00315$ (3 s.f.).

$$\boxed{\text{The radius of the cylinder is increasing at a rate of 0.00315 cm/s.}}$$

Problem 4.

The curve C has equation $2^{-y} = x$. The point A on C has x-coordinate a where $a > 0$. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{a \ln 2}$ at A and find the equation of the tangent to C at A. Hence find the equation of the tangent to C which passes through the origin.

The straight line $y = mx$ intersects C at 2 distinct points. Write down the range of values of m.

Solution

Implicitly differentiating the given equation,

$$\begin{aligned} 2^{-y} \cdot \ln 2 \cdot -y' &= 1 \\ \implies \quad x \cdot \ln 2 \cdot -y' &= 1 \\ \implies \quad y' &= -\frac{1}{x \ln 2} \end{aligned}$$

At A, $x = a$. Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{a \ln 2}$.

Note that $2^{-y} = x \implies y = -\log_2 x$. Hence, A has coordinates $(a, -\log_2 a)$. Using the point-slope formula, the tangent to C at A has equation

$$\begin{aligned} y - (-\log_2 a) &= -\frac{1}{a \ln 2}(x - a) \\ \implies \quad y &= -\frac{x}{a \ln 2} + \frac{1}{\ln 2} - \log_2 a \\ &= -\frac{x}{a \ln 2} + \frac{1}{\ln 2} - \frac{\ln a}{\ln 2} \\ &= -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \end{aligned}$$

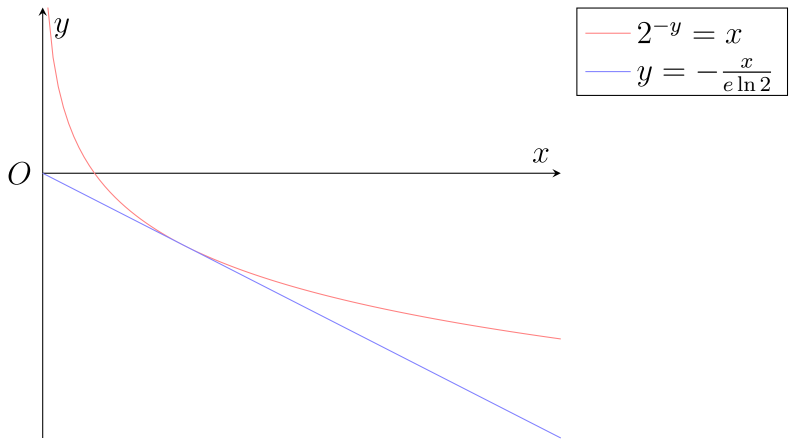
$$\boxed{\text{The tangent to } C \text{ at } A \text{ has equation } y = -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2}.}$$

Consider the straight line $y = mx$ that is tangent to C and passes through the origin.

$$\begin{aligned} 0 &= -\frac{0}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \\ \implies 1 - \ln a &= 0 \\ \implies \quad a &= e \end{aligned}$$

Hence, the equation of the tangent to C that passes through the origin is $y = -\frac{x}{e \ln 2}$.

Consider the graph of $2^{-y} = x$.



Hence, m must be strictly between $-\dfrac{1}{e \ln 2}$ and 0.

$$\boxed{m \in \left(-\frac{1}{e \ln 2}, 0\right)}$$