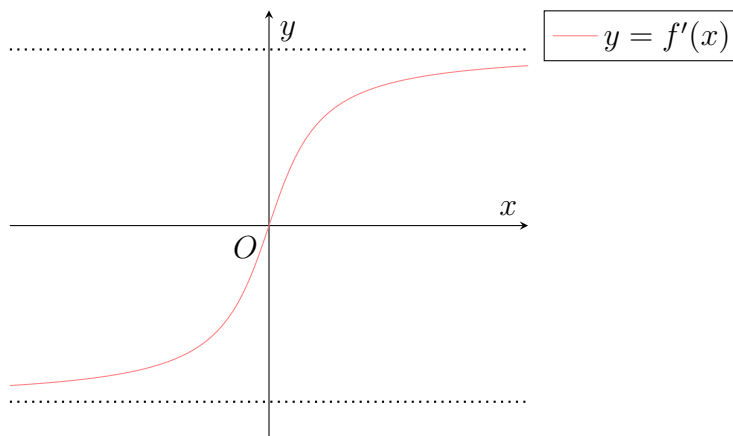


## Problem 1.

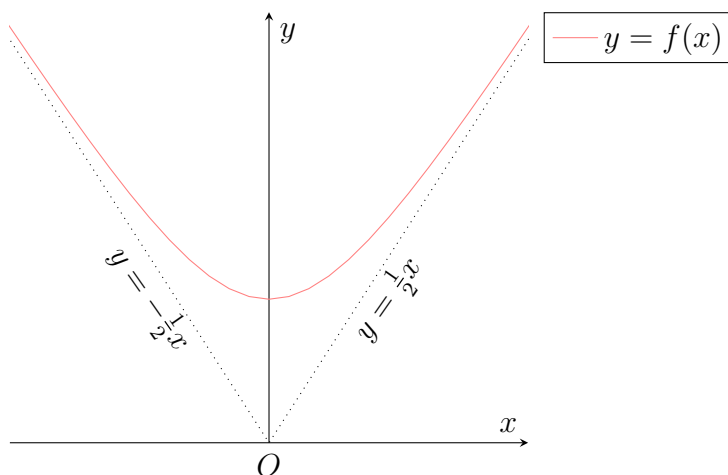
The graph of the first derivative of a function  $f$  is shown in the diagram below. It is symmetrical about the origin  $O$  and approaches the lines  $y = 0.5$  and  $y = -0.5$  for large values of  $x$ . Sketch the graph of  $y = f(x)$  given that it has a pair of asymptotes that intersect at the origin.



## Solution

$$\begin{aligned} \lim_{x \rightarrow \pm\infty} f'(x) &= \pm \frac{1}{2} \\ \Rightarrow \lim_{x \rightarrow \pm\infty} \int f'(x) \, dx &= \int \pm \frac{1}{2} x \, dx \\ \Rightarrow \lim_{x \rightarrow \pm\infty} f(x) &= \pm \frac{1}{2} x + C \end{aligned}$$

Hence,  $f(x)$  has asymptotes  $y = \pm \frac{1}{2}x + C$ , for some arbitrary constant  $C$ . Since both asymptotes meet at the origin,  $C = 0$ , whence  $f(x)$  has asymptotes  $y = \pm \frac{1}{2}x$ .



## Problem 2.

The terms in the sequence  $u_0, u_1, u_2, \dots$  satisfy the recurrence relation

$$u_{n+2} - u_{n+1} = r(u_{n+1} - u_n)$$

where  $r$  is a non-zero constant.

- (a) Find the general solution of this recurrence relation.
- (b) Given that  $u_0 = 0$  and the sequence converges to a finite value  $L$ , find an expression for  $u_n$  in terms of  $L, n$  and  $r$ . State a necessary condition on  $r$ .

## Solution

### Part (a)

$$\begin{aligned} u_{n+2} - u_{n+1} &= r(u_{n+1} - u_n) \\ \implies u_{n+2} &= (1+r)u_{n+1} - ru_n \end{aligned}$$

Consider the characteristic equation of the above recurrence relation.

$$\begin{aligned} x^2 - (1+r)x + r &= 0 \\ \implies (x-1)(x-r) &= 0 \end{aligned}$$

Hence, the roots of the characteristic equation are 1 and  $r$ . Thus, the general solution of the recurrence relation is given by

$$\begin{aligned} u_n &= A \cdot 1^n + B \cdot r^n \\ &= A + B \cdot r^n \end{aligned}$$

$$\boxed{u_n = A + B \cdot r^n}$$

### Part (b)

When  $n = 0$ , we have

$$\begin{aligned} A + B &= 0 \\ \implies B &= -A \end{aligned}$$

Since the sequence converges to a finite value, we know  $|r| < 1$ . Hence, considering  $n \rightarrow \infty$ , we have

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} (A + Br^n) \\ &= A \end{aligned}$$

whence  $B = -A = -L$ . Putting everything together, we have

$$\boxed{u_n = L - Lr^n, |r| < 1}$$

### Problem 3.

A curve is defined parametrically by  $x = \frac{t^2}{1+t^2}$ ,  $y = t^3 - \lambda t$ , where  $\lambda$  is a positive constant.

- Sketch the curve, stating the equation of its asymptote.
- Find in terms of  $\lambda$ , the  $x$ -coordinate of the point  $P$  where the curve intersects itself.
- Show that the area of the region bounded by the curve between  $P$  and the origin is given by an integral of the form

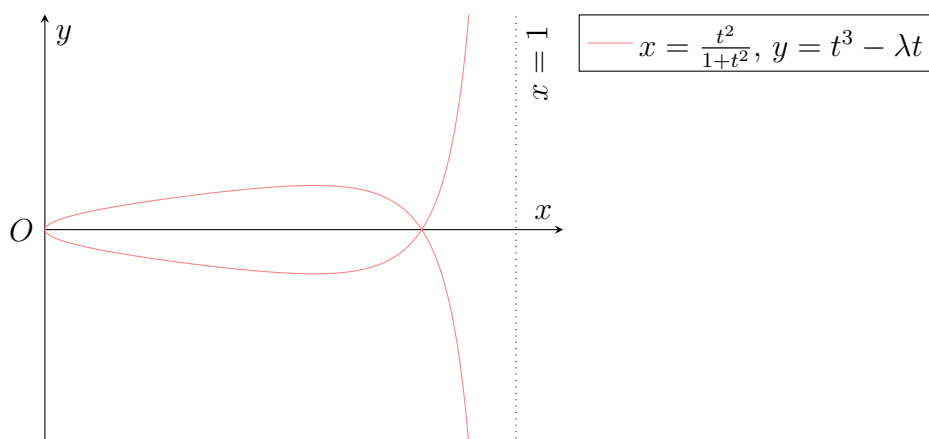
$$4 \int_0^{f(\lambda)} g(t^2) dt$$

where  $f(\lambda)$  is a function of  $\lambda$  and  $g(t^2)$  is a function of  $t^2$  to be determined.

### Solution

#### Part (a)

Note that  $\lim_{t \rightarrow \pm\infty} x = \lim_{t \rightarrow \pm\infty} \frac{t^2}{1+t^2} = 1$ . Hence, the curve has a vertical asymptote with equation  $x = 1$ .



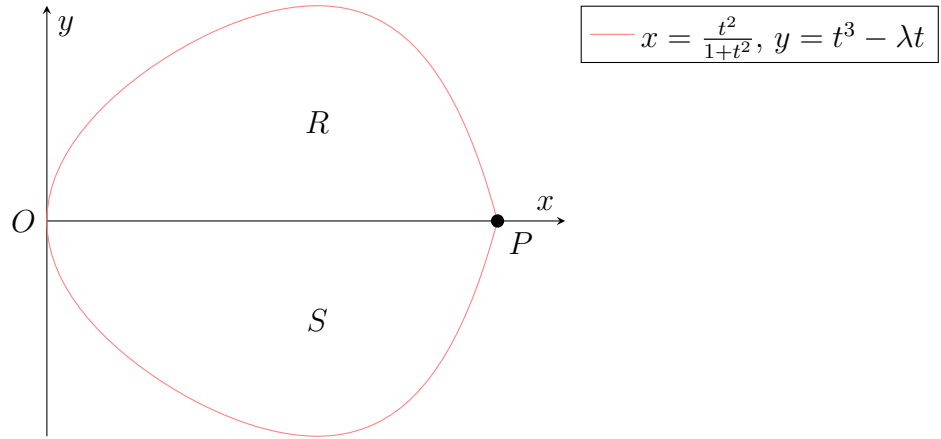
#### Part (b)

From the graph, the curve intersects itself when  $y = 0$  and  $x \neq 0 \implies t \neq 0$ .

$$\begin{aligned} y &= 0 \\ \implies t^3 - \lambda t &= 0 \\ \implies t^2 - \lambda &= 0 \end{aligned}$$

Hence,  $\lambda = t^2$ , whence  $x = \frac{t^2}{1+t^2} = \frac{\lambda}{1+\lambda}$ .

$$x = \frac{\lambda}{1+\lambda}$$

**Part (c)**

Let the region bounded by the curve between  $P$  and the origin be  $A$ . Let  $R$  be the region of  $A$  where  $y \geq 0$ . Let  $S$  be the region of  $A$  where  $y \leq 0$ . By symmetry,  $\text{Area } R = \text{Area } S$ . Hence,

$$\text{Area } A = 2 \text{Area } R$$

We will consider only region  $R$  for the rest of the solution. Note that  $R$  is bounded by the part of the curve where  $-\sqrt{\lambda} \leq t \leq 0$ . Also note that

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \frac{t^2}{1+t^2} \\ &= \frac{(1+t^2) \cdot 2t - t^2 \cdot 2t}{(1+t^2)^2} \\ &= \frac{2t}{(1+t^2)^2} \\ \implies dx &= \frac{2t}{(1+t^2)^2} dt \end{aligned}$$

Hence,

$$\begin{aligned} \text{Area } A &= 2 \text{Area } R \\ &= 2 \int_0^{-\sqrt{\lambda}} y \, dx \\ &= 2 \int_0^{-\sqrt{\lambda}} (t^3 - \lambda t) \cdot \frac{2t}{(1+t^2)^2} dt \\ &= 4 \int_0^{-\sqrt{\lambda}} \frac{t^4 - \lambda t^2}{(1+t^2)^2} dt \\ &= 4 \int_0^{-\sqrt{\lambda}} \frac{t^2(t^2 - \lambda)}{(1+t^2)^2} dt \end{aligned}$$

Hence,

$$f(\lambda) = -\sqrt{\lambda}, \quad g(t^2) = \frac{t^2(t^2 - \lambda)}{(1+t^2)^2}$$

## Problem 4.

It is given that the equation  $1 + \cos(\pi x) - 2\sqrt{x} = 0$  has a root  $\alpha$  in the interval  $[0, 1]$ .

- Use linear interpolation once on the interval  $[0, 1]$  to obtain an approximation  $x_1$  to  $\alpha$ .
- Using  $x_1$  as an initial estimate, apply the Newton-Raphson method to find  $\alpha$ , correct to 2 decimal places.
- With the help of an appropriate graph, explain how Newton-Raphson method using another initial estimate  $x_1^*$  in the interval  $[0, 1]$  fails to give an approximation to  $\alpha$ .

## Solution

Let  $f(x) = 1 + \cos(\pi x) - 2\sqrt{x}$ .

### Part (a)

Using linear interpolation on the interval  $[0, 1]$ ,

$$x_1 = \frac{1 \cdot f(0) - 0 \cdot f(1)}{f(0) - f(1)} = \frac{1}{2}$$

$$\boxed{x_1 = \frac{1}{2}}$$

### Part (b)

Note that  $f'(x) = -\sin(\pi x) \cdot \pi - \frac{2}{2\sqrt{x}} = -\pi \sin(\pi x) - \frac{1}{\sqrt{x}}$ .

$$x_1 = \frac{1}{2}$$

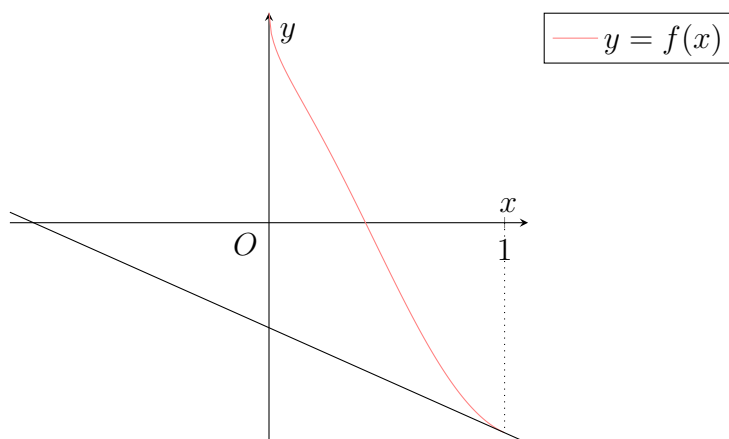
$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.40908$$

$$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.40964$$

$$\Rightarrow x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.40964$$

Since  $f(0.405) = 0.02 > 0$  and  $f(0.415) = -0.02 < 0$ ,  $\alpha \in (0.405, 0.415)$ . Thus,

$$\boxed{\alpha = 0.41 \text{ (2 d.p.)}}$$

**Part (c)**

When  $x_1^* = 1$ , the tangent to the curve intersects the  $x$ -axis at a negative  $x$ -value, thus giving a negative  $x_2$ . Since  $f(x)$  is only defined for  $x \geq 0$  due to the presence of  $\sqrt{x}$ ,  $f(x_2)$  and thus  $x_3$  will be undefined. Hence, the Newton-Raphson method will fail to give an approximation to  $\alpha$ .

## Problem 5.

- (a) For a positive constant  $a$ , there is an angle  $\phi$  such that  $\sin \phi = a$  and  $\frac{\pi}{2} < \phi < \pi$ .

Evaluate  $\int_{-1}^0 \frac{1}{\sqrt{1-a^2x^2}} dx$ , leaving your answer in terms of  $a$ ,  $\phi$  and  $\pi$ .

- (b) Using the substitution  $t = \tan \frac{x}{2}$ , show that

$$\int \frac{\cos x}{1 + \cos x - \sin x} dx = \int \frac{1+t}{1+t^2} dt$$

Hence determine  $\int \frac{\cos x}{1 + \cos x - \sin x} dx$ .

## Solution

### Part (a)

$$\begin{aligned} \int_{-1}^0 \frac{1}{\sqrt{1-a^2x^2}} dx &= \int_{-1}^0 \frac{1}{\sqrt{1-(ax)^2}} dx \\ &= \frac{1}{a} \int_{-a}^0 \frac{1}{\sqrt{1-u^2}} du \\ &= \frac{1}{a} [\arcsin u]_{-a}^0 \\ &= \frac{1}{a} \cdot -\arcsin(-a) \\ &= \frac{\arcsin a}{a} \\ &= \frac{\pi - \phi}{a} \end{aligned}$$

$$\begin{aligned} u &= ax \\ du &= a dx \end{aligned}$$

$$\boxed{\int_{-1}^0 \frac{1}{\sqrt{1-a^2x^2}} dx = \frac{\pi - \phi}{a}}$$

### Part (b)

Consider the substitution  $t = \tan \frac{x}{2}$ .

$$\begin{aligned} \sin x &= \frac{2 \sin(x/2) \cos(x/2)}{\cos^2(x/2) + \sin^2(x/2)} \\ &= \frac{2 \tan(x/2)}{1 + \tan^2(x/2)} \\ &= \frac{2t}{1+t^2} \\ \cos x &= \frac{\cos^2(x/2) - \sin^2(x/2)}{\cos^2(x/2) + \sin^2(x/2)} \end{aligned}$$

$$\begin{aligned}
&= \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} \\
&= \frac{1 - t^2}{1 + t^2}
\end{aligned}$$

Also note that

$$\begin{aligned}
t &= \tan \frac{x}{2} \\
\Rightarrow dt &= \frac{1}{2} \sec^2 \frac{x}{2} dx \\
\Rightarrow dt &= \frac{1}{2} \left( 1 + \tan^2 \frac{x}{2} \right) dx \\
&= \frac{1}{2} (1 + t^2) dx \\
\Rightarrow dx &= \frac{2}{1 + t^2} dt
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{\cos x}{1 + \cos x - \sin x} dx &= \int \frac{\frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\
&= \int \frac{2 \cdot \frac{1-t^2}{1+t^2}}{(1+t^2) + (1-t^2) - 2t} dt \\
&= \int \frac{2 \cdot \frac{1-t^2}{1+t^2}}{2-2t} dt \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{1-t} dt \\
&= \int \frac{\frac{(1-t)(1+t)}{1+t^2}}{1-t} dt \\
&= \int \frac{1+t}{1+t^2} dt \quad \square \\
&= \int \left( \frac{1}{1+t^2} + \frac{t}{1+t^2} \right) dt \\
&= \int \left( \frac{1}{1+t^2} + \frac{1}{2} \cdot \frac{2t}{1+t^2} \right) dt \\
&= \arctan t + \frac{1}{2} \ln |1+t^2| + C \\
&= \arctan \left( \tan \frac{x}{2} \right) + \frac{1}{2} \ln \left| 1 + \tan^2 \frac{x}{2} \right| + C \\
&= \frac{x}{2} + \frac{1}{2} \ln \left| \sec^2 \frac{x}{2} \right| + C \\
&= \frac{x}{2} + \ln \left| \sec \frac{x}{2} \right| + C
\end{aligned}$$

$$\boxed{\int \frac{\cos x}{1 + \cos x - \sin x} dx = \frac{x}{2} + \ln \left| \sec \frac{x}{2} \right| + C}$$



## Problem 6.

The curve  $G$  has equation  $y = \frac{x^2 - 2kx + k}{x - k}$ , where  $k$  is a non-zero constant and  $k \neq 1$ .

- State, in terms of  $k$ , the equations of the asymptotes of  $G$ .
- Determine the set of values for which  $G$  has two stationary points.
- Give a sketch of  $G$  for  $k > 1$ , stating in terms of  $k$ , the coordinates of the point of intersection of its asymptotes.
- With the help of your sketch in part (c), determine, in exact form, the value of  $m$  ( $m < 0$ ) such that the line  $y = m(x - k)$  is a line of symmetry of  $G$ .

## Solution

### Part (a)

$$\begin{aligned}
 y &= \frac{x^2 - 2kx + k}{x - k} \\
 &= \frac{x^2 - 2kx + k^2 + k - k^2}{x - k} \\
 &= \frac{(x - k)^2 + k - k^2}{x - k} \\
 &= x - k + \frac{k - k^2}{x - k}
 \end{aligned}$$

Hence,  $G$  has oblique asymptote  $y = x - k$  and vertical asymptote  $x = k$ .

$$\boxed{y = x - k, x = k}$$

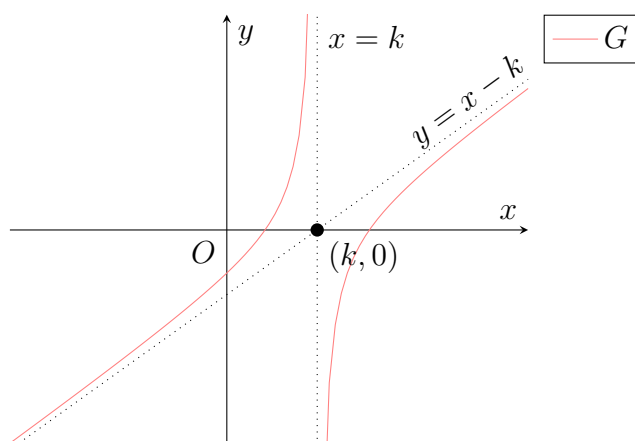
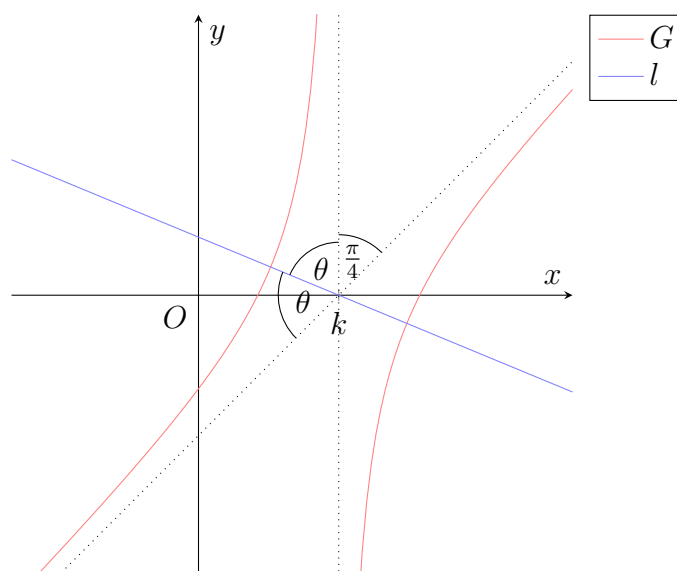
### Part (b)

For stationary points,  $\frac{dy}{dx} = 0$ .

$$\begin{aligned}
 \frac{dy}{dx} &= 0 \\
 \implies 1 - \frac{k - k^2}{(x - k)^2} &= 0 \\
 \implies (x - k)^2 &= k - k^2 \\
 \implies x - k &= \pm\sqrt{k - k^2}
 \end{aligned}$$

For  $G$  to have two stationary points,  $k - k^2 > 0$ , whence  $0 < k < 1$ .

$$\boxed{\{k \in \mathbb{R} : 0 < k < 1\}}$$

**Part (c)****Part (d)**

Let  $l$  be the line with equation  $y = m(x - k)$ . Since  $l$  is a line of symmetry of  $G$ ,  $l$  bisects the angle between the asymptotes. Since the asymptote  $y = x - k$  makes an angle  $\frac{\pi}{4}$  with the point  $(k, 0)$ , we have

$$\theta + \theta + \frac{\pi}{4} = \pi$$

whence  $\theta = \frac{3}{8}\pi$ . Thus,  $l$  makes an angle  $\theta + \frac{\pi}{4} = \frac{7}{8}\pi$  with the point  $(k, 0)$ , giving it a gradient of  $\tan \frac{7}{8}\pi$ . Hence,

$$m = \tan \frac{7}{8}\pi$$

## **Problem 7.**

**Omitted.**

## **Problem 8.**

**Omitted.**

## Problem 9.

It is given that  $I_n = \int_0^\pi \cos^n(2\theta) d\theta$ , where  $n$  is a positive integer.

- (a) Without using the calculator, evaluate  $I_2$ .
- (b) For  $n > 3$ , show that  $I_n = \frac{n-1}{n} I_{n-2}$ .
- (c) Deduce that for all odd values of  $n$ ,  $I_n$  is independent of  $n$ .
- (d) For even values of  $n$ , show that

$$I_n = \frac{n! \pi}{2^n \left[ \left( \frac{n}{2} \right)! \right]^2}$$

## Solution

### Part (a)

$$\begin{aligned} I_2 &= \int_0^\pi \cos^2(2\theta) d\theta \\ &= \int_0^\pi \frac{\cos 4\theta + 1}{2} d\theta \\ &= \int_0^{4\pi} \frac{\cos u + 1}{8} du \\ &= \frac{1}{8} [\sin u + u]_0^{4\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} u &= 4\theta \\ du &= 4 d\theta \end{aligned}$$

$$\boxed{I_2 = \frac{\pi}{2}}$$

### Part (b)

$$\begin{aligned} I_n &= \int_0^\pi \cos^n 2\theta d\theta \\ &= \int_0^\pi \cos 2\theta \cos^{n-1} 2\theta d\theta \end{aligned}$$

Note that  $\frac{d}{d\theta} \cos^{n-1} 2\theta = -2(n-1) \sin 2\theta \cos^{n-2} 2\theta$ . Integrating by parts, we have

	$D$	$I$
+	$\cos^{n-1} 2\theta$	$\cos 2\theta$
−	$-2(n-1) \sin 2\theta \cos^{n-2} 2\theta$	$\frac{1}{2} \sin 2\theta$

$$\begin{aligned}
I_n &= \left[ \frac{1}{2} \sin 2\theta \cos^{n-1} 2\theta \right]_0^\pi + (n-1) \int_0^\pi \sin^2 2\theta \cos^{n-2} 2\theta \, d\theta \\
&= (n-1) \int_0^\pi \sin^2 2\theta \cos^{n-2} 2\theta \, d\theta \\
&= (n-1) \int_0^\pi (1 - \cos^2 2\theta) \cos^{n-2} 2\theta \, d\theta \\
&= (n-1) \int_0^\pi (\cos^{n-2} 2\theta - \cos^n 2\theta) \, d\theta \\
&= (n-1) (I_{n-2} - I_n) \\
\implies nI_n &= (n-1)I_{n-2} \\
\implies I_n &= \frac{n-1}{n} I_{n-2}
\end{aligned}$$

**Part (c)**

Note that  $I_1 = \int_0^\pi \cos 2\theta \, d\theta = \frac{1}{2} [\sin 2\theta]_0^\pi = 0$ . For all odd  $n$ ,  $I_n$  will eventually reduce to  $I_1$  with the recurrence relation derived above. Hence,  $I_n = 0$  for odd  $n$ , which is independent of  $n$ .

**Part (d)**

$$\begin{aligned}
I_n &= \frac{n-1}{n} I_{n-2} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6} \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{3}{4} I_2 \\
&= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \dots \cdot \frac{3}{4} \cdot \frac{\pi}{2} \\
&= \frac{(n-1)(n-3)(n-5) \cdot \dots \cdot 3 \cdot 1}{n(n-2)(n-4) \cdot \dots \cdot 4 \cdot 2} \cdot \pi \\
&= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5) \cdot \dots \cdot 4 \cdot 3 \cdot 2 \cdot 1}{\left[ n(n-2)(n-4) \cdot \dots \cdot 4 \cdot 2 \right]^2} \cdot \pi \\
&= \frac{n! \pi}{\left[ n(n-2)(n-4) \cdot \dots \cdot 4 \cdot 2 \right]^2}
\end{aligned}$$

However, we have

$$\begin{aligned}
n(n-2)(n-4) \cdot \dots \cdot 2 &= \left( 2 \cdot \frac{n}{2} \right) \left( 2 \cdot \frac{n-2}{2} \right) \left( 2 \cdot \frac{n-4}{2} \right) \cdot \dots \cdot (2 \cdot 1) \\
&= 2^{n/2} \left[ \left( \frac{n}{2} \right) \left( \frac{n-2}{2} \right) \left( \frac{n-4}{2} \right) \cdot \dots \cdot 1 \right] \\
&= 2^{n/2} \left[ \left( \frac{n}{2} \right) \left( \frac{n}{2} - 1 \right) \left( \frac{n}{2} - 2 \right) \cdot \dots \cdot 1 \right]
\end{aligned}$$

$$= 2^{n/2} \left(\frac{n}{2}\right)!$$

Hence,

$$\begin{aligned} I_n &= \frac{n! \pi}{\left[2^{n/2} \left(\frac{n}{2}\right)!\right]^2} \\ &= \frac{n! \pi}{2^n \left[\left(\frac{n}{2}\right)!\right]^2} \end{aligned}$$

## Problem 10.

In a membership drive, a fitness club is trying to recruit new members. The sales manager models the number of members that the club has at the end of each month assuming that a certain portion  $p$  ( $0 < p < 1$ ) of its members in the previous month will be lost to competitors, and that it will recruit a constant number,  $k$ , of new members in each month.

Let  $M_n$  ( $n \geq 1$ ) be the number of members that the club has  $n$  months after the start of the membership drive.

- (a) Write down an expression for  $M_{n+1}$  in terms of  $M_n$ .
- (b) Given that the club has 500 members at the end of the first month, determine  $M_n$  in terms of  $n$ ,  $p$  and  $k$ .

The sales manager sets a target for the club membership to reach 750 by the end of 6 months.

- (c) Given that  $k = 80$ , show that to meet its target, the club needs to retain approximately 95% of its members, month-by-month.
- (d) Given that the club can only retain 90% of its members, month-by-month, find the least number of members it must recruit each month to meet or exceed its target.

## Solution

### Part (a)

$$M_{n+1} = (1 - p)M_n + k$$

### Part (b)

Let  $q$  be the constant such that  $M_{n+1} + q = (1 - p)(M_n + q)$ . Then  $(1 - p)q - q = k \implies q = -\frac{k}{p}$ .

$$\begin{aligned} M_{n+1} - \frac{k}{p} &= (1 - p) \left( M_n - \frac{k}{p} \right) \\ \implies M_n - \frac{k}{p} &= (1 - p)^{n-1} \left( M_1 - \frac{k}{p} \right) \\ &= (1 - p)^{n-1} \left( 500 - \frac{k}{p} \right) \\ \implies M_n &= (1 - p)^{n-1} \left( 500 - \frac{k}{p} \right) + \frac{k}{p} \end{aligned}$$

$$M_n = (1 - p)^{n-1} \left( 500 - \frac{k}{p} \right) + \frac{k}{p}$$



**Part (c)**

Consider  $M_6 \geq 750$  with  $k = 80$ .

$$\begin{array}{r} M_6 \geq 750 \\ \implies (1-p)^5 \left( 500 - \frac{80}{p} \right) + \frac{80}{p} \geq 750 \end{array}$$

From G.C.,  $p = 0.0495$  (3 s.f.). Hence, the club needs to retain  $(1-p) = 95.05\%$  of its members, month-by-month.

**Part (d)**

Consider  $M_6 \geq 750$  with  $p = 0.10$ .

$$\begin{array}{r} M_6 \geq 750 \\ \implies (1-p)^5 \left( 500 - \frac{k}{0.10} \right) + \frac{k}{0.10} \geq 750 \end{array}$$

From G.C.,  $k > 111.05$  (2 d.p.). Since  $k \in \mathbb{N}$ , the least  $k$  is 112.

The club must recruit at least 112 members each month.

## **Problem 11.**

**Omitted.**