Problem 1.

Solve these recurrence relations together with the initial conditions.

(a) $u_n = 2u_{n-1}$, for $n \geq 1$, $u_0 = 3$

(b) $u_n = 3u_{n-1} + 7$, for $n \geq 1$, $u_0 = 5$

Solution

Part (a)

$$\begin{aligned} u_n &= 2^n \cdot u_0 \\ &= 3 \cdot 2^n \end{aligned}$$

$$\boxed{u_n = 3 \cdot 2^n}$$

Part (b)

Let k be a constant such that $u_n + k = 3(u_{n-1} + k)$. Then $2k = 7 \implies k = \frac{7}{2}$. Hence,

$$\begin{aligned} u_n + \frac{7}{2} &= 3\left(u_{n-1} + \frac{7}{2}\right) \\ \implies u_n + \frac{7}{2} &= 3^n\left(u_0 + \frac{7}{2}\right) \\ \implies \qquad u_n &= 3^n\left(5 + \frac{7}{2}\right) - \frac{7}{2} \\ &= \frac{17}{2} \cdot 3^n - \frac{7}{2} \end{aligned}$$

$$\boxed{u_n = \frac{17}{2} \cdot 3^n - \frac{7}{2}}$$

Problem 2.

Solve these recurrence relations together with the initial conditions.

(a) $u_n = 5u_{n-1} - 6u_{n-2}$, for $n \geq 2$, $u_0 = 1$, $u_1 = 0$

(b) $u_n = 4u_{n-2}$, for $n \geq 2$, $u_0 = 0$, $u_1 = 4$

(c) $u_n = 4u_{n-1} - 4u_{n-2}$, for $n \geq 2$, $u_0 = 6$, $u_1 = 8$

(d) $u_n = -6u_{n-1} - 9u_{n-2}$, for $n \geq 2$, $u_0 = 3$, $u_1 = -3$

(e) $u_n = 2u_{n-1} - 2u_{n-2}$, for $n \geq 2$, $u_0 = 2$, $u_1 = 6$

Solution

Part (a)

Consider the characteristic equation of u_n.

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ \implies (x-2)(x-3) &= 0 \end{aligned}$$

Hence, 2 and 3 are the roots of the characteristic equation. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n$$

Since $u_0 = 1$,

$$\begin{aligned} A \cdot 2^0 + B \cdot 3^0 &= 1 \\ \implies \qquad A + B &= 1 \end{aligned} \tag{2.1}$$

Since $u_1 = 0$,

$$\begin{aligned} A \cdot 2^1 + B \cdot 3^1 &= 0 \\ \implies \qquad 2A + 3B &= 0 \end{aligned} \tag{2.2}$$

Solving Equations 2.1 and 2.2 simultaneously, we have $A = 3$ and $B = -2$. Thus,

$$\boxed{u_n = 3 \cdot 2^n + 2 \cdot 3^n}$$

Part (b)

Consider the characteristic equation of u_n.

$$\begin{aligned} x^2 - 4 &= 0 \\ \implies (x+2)(x-2) &= 0 \end{aligned}$$

Hence, -2 and 2 are the roots of the characteristic equation. Thus,

$$u_n = A \cdot (-2)^n + B \cdot 2^n$$

Since $u_0 = 0$,

$$\begin{aligned} A \cdot (-2)^0 + B \cdot 2^0 &= 0 \\ \implies \qquad A + B &= 0 \end{aligned} \tag{2.3}$$

Since $u_1 = 4$,

$$\begin{aligned} A \cdot (-2)^1 + B \cdot 2^1 &= 4 \\ \implies \qquad -2A + 2B &= 4 \end{aligned} \tag{2.4}$$

Solving Equations 2.3 and 2.4 simultaneously, we have $A = -1$ and $B = 1$. Thus,

$$\boxed{u_n = -(-2)^n + 2^n}$$

Part (c)

Consider the characteristic equation of u_n.

$$\begin{aligned} x^2 - 4x + 4 &= 0 \\ \implies \qquad (x-2)^2 &= 0 \end{aligned}$$

Hence, 2 is the only root of the characteristic equation. Thus,

$$u_n = (A + Bn)2^n$$

Since $u_0 = 6$,

$$\begin{aligned} (A + B \cdot 0)2^0 &= 6 \\ \implies \qquad A &= 6 \end{aligned}$$

Since $u_1 = 8$,

$$\begin{aligned} (A + B \cdot 1)2^1 &= 8 \\ \implies \qquad A + B &= 4 \\ \implies \qquad B &= -2 \end{aligned}$$

Since $A = 6$ and $B = -2$, we have

$$\boxed{u_n = (6 - 2n)2^n}$$

Part (d)

Consider the characteristic equation of u_n.

$$\begin{aligned} x^2 + 6x + 9 &= 0 \\ \implies \qquad (x+3)^2 &= 0 \end{aligned}$$

Hence, -3 is the only root of the characteristic equation. Thus,

$$u_n = (A + Bn)(-3)^n$$

Since $u_0 = 3$,

$$\begin{aligned} && (A + B \cdot 0)2^0 &= 3 \\ \implies && A &= 3 \end{aligned}$$

Since $u_1 = -3$,

$$\begin{aligned} && (A + B \cdot 1)2^1 &= -3 \\ \implies && A + B &= 1 \\ \implies && B &= -2 \end{aligned}$$

Since $A = 3$ and $B = -2$, we have

$$\boxed{u_n = (3 - 2n)2^n}$$

Part (e)

Consider the characteristic equation of u_n, $x^2 - 2x + 2 = 0$. Solving the characteristic equation using the quadratic formula,

$$\begin{aligned} x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1} \\ &= \frac{2 \pm \sqrt{-4}}{2} \\ &= \frac{2 \pm 4i}{2} \\ &= 1 \pm 2i \\ &= \sqrt{2} \exp\left(\pm i \frac{\pi}{4}\right) \end{aligned}$$

Hence,

$$u_n = A \cdot \sqrt{2}^n \cos\left(\frac{\pi}{4} n\right) + B \cdot \sqrt{2}^n \sin\left(\frac{\pi}{4} n\right)$$

Since $u_0 = 2$,

$$\begin{aligned} && A \cdot \sqrt{2}^0 \cos\left(\frac{\pi}{4} \cdot 0\right) + B \cdot \sqrt{2}^0 \sin\left(\frac{\pi}{4} \cdot 0\right) &= 2 \\ \implies && A \cdot 1 \cdot 1 + B \cdot 1 \cdot 0 &= 2 \\ \implies && A &= 2 \end{aligned}$$

Since $u_1 = 6$,

$$\begin{aligned} && A \cdot \sqrt{2}^1 \cos\left(\frac{\pi}{4} \cdot 1\right) + B \cdot \sqrt{2}^1 \sin\left(\frac{\pi}{4} \cdot 1\right) &= 6 \\ \implies && A \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} + B \cdot \sqrt{2} \cdot \frac{1}{\sqrt{2}} &= 6 \\ \implies && A + B &= 6 \\ \implies && B &= 4 \end{aligned}$$

Since $A = 2$ and $B = 4$,

$$\begin{aligned} u_n &= 2 \cdot \sqrt{2}^n \cos\left(\frac{\pi}{4}n\right) + 4 \cdot \sqrt{2}^n \sin\left(\frac{\pi}{4}n\right) \\ &= \sqrt{2}^{n+2} \cos\left(\frac{\pi}{4}n\right) + \sqrt{2}^{n+4} \sin\left(\frac{\pi}{4}n\right) \end{aligned}$$

Problem 3.

(a) A sequence is defined by the formula $b_n = \dfrac{n!n!}{(2n)!} \cdot 2^n$, where $n \in \mathbb{Z}^+$. Show that the sequence satisfies the recurrence relation $b_{n+1} = \dfrac{n+1}{2n+1} b_n$.

(b) A sequence is defined recursively by the formula

$$u_{n+1} = 2u_n + 3, \qquad n \in \mathbb{Z}_0^+,\ u_0 = a$$

Show that $u_n = 2^n a + 3\left(2^n - 1\right)$.

Solution

Part (a)

$$\begin{aligned} b_{n+1} &= \frac{(n+1)!(n+1)!}{(2(n+1))!} \cdot 2^{n+1} \\ &= \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot 2^{n+1} \\ &= \frac{(n+1)n!(n+1)n!}{(2n+2)(2n+1)(2n)!} \cdot 2 \cdot 2^n \\ &= \frac{(n+1)n!(n+1)n!}{(n+1)(2n+1)(2n)!} \cdot 2^n \\ &= \frac{(n+1)n!n!}{(2n+1)(2n)!} \cdot 2^n \\ &= \frac{n+1}{2n+1} \cdot \frac{n!n!}{(2n)!} \cdot 2^n \\ &= \frac{n+1}{2n+1} b_n \end{aligned}$$

Part (b)

Let k be a constant such that $u_{n+1} + k = 2(u_n + k)$. Then $k = 3$. Hence,

$$\begin{aligned} u_{n+1} + 3 &= 2(u_n + 3) \\ \implies \quad u_n + 3 &= 2^n(u_0 + 3) \\ &= 2^n(a + 3) \\ \implies \quad u_n &= 2^n(a + 3) - 3 \\ &= a \cdot 2^n + 3 \cdot 2^n - 3 \\ &= a \cdot 2^n + 3\left(2^n - 1\right) \end{aligned}$$

Problem 4.

The volume of water, in litres, in a storage tank decreases by 10% by the end of each day. However, 90 litres of water is also pumped into the tank at the end of each day. The volume of water in the tank at the end of n days is denoted by x_n and x_0 is the initial volume of water in the tank.

(a) Write down a recurrence relation to represent the above situation.

(b) Show that $x_n = 0.9^n(x_0 - 900) + 900$.

(c) Deduce the amount of water in the tank when n becomes very large.

Solution

Part (a)

$$\boxed{x_{n+1} = 0.9x_n + 90,\ n \in \mathbb{N}}$$

Part (b)

Let k be a constant such that $x_{n+1} + k = 0.9(x_n + k)$. Then $-\frac{1}{10}k = 90 \implies k = -900$. Hence,

$$\begin{aligned} & x_{n+1} - 900 = 0.9(x_n - 900) \\ \implies & x_n - 900 = 0.9^n(x_0 - 900) \\ \implies & x_n = 0.9^n(x_0 - 900) + 900 \end{aligned}$$

Part (c)

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \left(0.9^n(x_0 - 900) + 900\right) \\ &= \lim_{n\to\infty} (0(x_0 - 900) + 900) \\ &= 900 \end{aligned}$$

$$\boxed{\text{When } n \text{ becomes very large, the amount of water in the tank converges to 900 litres.}}$$

Problem 5.

A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year, two dividends are awarded and reinvested into the fund. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

(a) Find a recurrence relation $\{P_n\}$ where P_n is the amount at the start of the nth year if no money is ever withdrawn.

(b) How much is in the account after n years if no money is ever withdrawn?

Solution

Part (a)

$$\begin{aligned} P_{n+2} &= P_{n+1} + 0.2P_{n+1} + 0.45P_n \\ &= 1.2P_{n+1} + 0.45P_n \end{aligned}$$

$$\boxed{P_{n+2} = 1.2P_{n+1} + 0.45P_n}$$

Part (b)

Consider the characteristic equation of P_n.

$$\begin{aligned} & x^2 - 1.2x - 0.45 = 0 \\ \implies & \quad 20x^2 - 24x - 9 = 0 \\ \implies & (10x+3)(2x-3) = 0 \end{aligned}$$

Hence, $-\dfrac{3}{10}$ and $\dfrac{3}{2}$ are the roots of the characteristic equation. Thus,

$$P_n = A\left(-\frac{3}{10}\right)^n + B\left(\frac{3}{2}\right)^n$$

Since $P_0 = 0$,

$$A\left(-\frac{3}{10}\right)^0 + B\left(\frac{3}{2}\right)^0 = 0 \implies A + B = 0$$

Since $P_1 = 100000$,

$$A\left(-\frac{3}{10}\right)^1 + B\left(\frac{3}{2}\right)^1 = 100000 \implies -3A + 15B = 1000000$$

Solving both equations simultaneously, we have $A = -\dfrac{500000}{9}$ and $B = \dfrac{500000}{9}$. Thus,

$$P_n = \frac{500000}{9}\left(\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right)$$

$$\boxed{\text{There will be } \$\left(\frac{500000}{9}\left(\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right)\right) \text{ in the account after } n \text{ years.}}$$

Problem 6.

A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbit produces another pair each month.

(a) Find a recurrence relation $\{f_n\}$ where f_n is the total number of pairs of rabbits, assuming that no rabbits ever die.

(b) What is the number of pairs of rabbits at the end of the nth month, assuming that no rabbits ever die?

Solution

Part (a)

$$\boxed{f_{n+2} = f_{n+1} + f_n,\ n \geq 2,\ f_1 = 1, f_2 = 1}$$

Part (b)

Consider the characteristic equation of f_n, $x^2 - x - 1 = 0$. Using the quadratic formula, the roots of the characteristic equation are $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$.

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$$

Since $f_1 = 1$,

$$\begin{aligned} & A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1 \\ \implies \quad & A(1+\sqrt{5}) + B(1-\sqrt{5}) = 2 \\ \implies \quad & (A+B) + \sqrt{5}(A-B) = 2 \end{aligned} \tag{6.1}$$

Since $f_2 = 1$,

$$A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1$$

Since $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$ are both roots of $x^2 - x - 1 = 0$, they both satisfy the equation $x^2 = x + 1$. Hence,

$$\begin{aligned} & A\left(\frac{1+\sqrt{5}}{2} + 1\right) + B\left(\frac{1-\sqrt{5}}{2} + 1\right) = 1 \\ \implies \quad & A(1+\sqrt{5}+2) + B(1-\sqrt{5}+2) = 2 \\ \implies \quad & 3(A+B) + \sqrt{5}(A-B) = 2 \end{aligned} \tag{6.2}$$

From Equations 6.1 and 6.2, we see that $A + B = 0 \implies A = -B$, and $\sqrt{5}(A - B) = 2 \implies A = \dfrac{2}{\sqrt{5}} + B$. Thus, $-B = \dfrac{2}{\sqrt{5}} + B \implies B = -\dfrac{1}{\sqrt{5}} \implies A = \dfrac{1}{\sqrt{5}}$. Hence,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\boxed{\text{After } n \text{ months, there will be } \left(\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right) \text{ pairs of rabbits.}}$$

Problem 7.

For $n \in \{2^j : j \in \mathbb{Z}, j \geq 1\}$, it is given that $T_n = 3T_{\frac{n}{2}} + 17$, where $T_1 = 4$. By considering the substitution $n = 2^i$ and another suitable substitution, show that the recurrence relation can be expressed in the form

$$t_i = 3t_{i-1} + 17, \qquad i \in \mathbb{Z}^+$$

Hence, find an expression for T_n in terms of n.

Solution

$$T_n = 3T_{\frac{n}{2}} + 17, \qquad T_1 = 4$$

Let $n = 2^i \iff i = \log_2 n$.

$$\begin{aligned} T_{2^i} &= 3T_{\frac{2^i}{2}} + 17, \qquad T_{2^0} = 4 \\ \implies T_{2^i} &= 3T_{2^{i-1}} + 17, \qquad T_{2^0} = 4 \end{aligned}$$

Let $t_i = T_{2^i}$.

$$t_i = 3t_{i-1} + 17, \qquad t_0 = 4$$

Let k be a constant such that $t_i + k = 3(t_{i-1} + k)$. Then $2k = 17 \implies k = \frac{17}{2}$. Hence,

$$\begin{aligned} t_i + \frac{17}{2} &= 3\left(t_{i-1} + \frac{17}{2}\right) \\ \implies t_i + \frac{17}{2} &= 3^i\left(t_0 + \frac{17}{2}\right) \\ &= 3^i\left(4 + \frac{17}{2}\right) \\ &= 3^i \cdot \frac{25}{2} \\ \implies \qquad t_i &= \frac{25}{2} \cdot 3^i - \frac{17}{2} \\ \implies \qquad T_{2^i} &= \frac{25}{2} \cdot 3^i - \frac{17}{2} \\ \implies \qquad T_n &= \frac{25}{2} \cdot 3^{\log_2 n} - \frac{17}{2} \end{aligned}$$

$$\boxed{T_n = \frac{25}{2} \cdot 3^{\log_2 n} - \frac{17}{2}}$$

Problem 8.

Consider the sequence $\{a_n\}$ given by the recurrence relation

$$a_{n+1} = 2a_n + 5^n, \qquad n \geq 1$$

(a) Given that $a_n = k\left(5^n\right)$ satisfies the recurrent relation, find the value of the constant k.

(b) Hence, by considering the sequence $\{b_n\}$ where $b_n = a_n - k(5^n)$, find the particular solution to the recurrence relation for which $a_1 = 2$.

Solution

Part (a)

$$\begin{aligned}
a_{n+1} &= 2a_n + 5^n \\
\implies k\left(5^{n+1}\right) &= 2 \cdot k\left(5^n\right) + 5^n \\
\implies 5k \cdot 5^n &= 2k \cdot 5^n + 5^n \\
\implies 5k &= 2k + 1 \\
\implies 3k &= 1 \\
\implies k &= \frac{1}{3}
\end{aligned}$$

$$\boxed{k = \frac{1}{3}}$$

Part (b)

$$\begin{aligned}
b_n &= a_n - k\left(5^n\right) \\
&= a_n - \frac{1}{3} \cdot 5^n \\
&= 2a_{n-1} + 5^{n-1} - \frac{1}{3} \cdot 5^n \\
&= 2\left(a_{n-1} - \frac{1}{3} \cdot 5^{n-1}\right) + \frac{2}{3} \cdot 5^{n-1} + 5^{n-1} - \frac{1}{3} \cdot 5^n \\
&= 2b_{n-1} + \frac{5}{3} \cdot 5^{n-1} - \frac{1}{3} \cdot 5^n \\
&= 2b_{n-1} + \frac{5}{3} \cdot 5^{n-1} - \frac{5}{3} \cdot 5^{n-1} \\
&= 2b_{n-1}
\end{aligned}$$

Hence, $b_n = b_1 \cdot 2^{n-1}$. Note that $b_1 = a_1 - \frac{1}{3} \cdot 5^1 = 2 - \frac{1}{3} \cdot 5^1 = \frac{1}{3}$. Thus, $b_n = \frac{1}{3} \cdot 2^{n-1}$.

$$
\begin{aligned}
b_n &= \frac{1}{3} \cdot 2^{n-1} \\
\implies a_n - \frac{1}{3} \cdot 5^n &= \frac{1}{3} \cdot 2^{n-1} \\
\implies \qquad\qquad a_n &= \frac{1}{3} \cdot 2^{n-1} + \frac{1}{3} \cdot 5^n \\
&= \frac{1}{3} \left(\frac{1}{2} 2^n + 5^n \right) \\
&= \frac{1}{6} \left(2^n + 2 \cdot 5^n \right)
\end{aligned}
$$

$$
\boxed{a_n = \frac{1}{6} \left(2^n + 2 \cdot 5^n \right)}
$$

Problem 9.

The sequence $\{X_n\}$ is given by

$$\sqrt{X_{n+2}} = \frac{X_{n+1}}{X_n^2}, \qquad n \geq 1$$

By applying the natural logarithm to the recurrence relation, use a suitable substitution to find the general solution of the sequence, expressing your answer in trigonometric form.

Solution

$$\begin{aligned}
\sqrt{X_{n+2}} &= \frac{X_{n+1}}{X_n^2} \\
\implies \ln \sqrt{X_{n+2}} &= \ln \frac{X_{n+1}}{X_n^2} \\
\implies \frac{1}{2} \ln X_{n+2} &= \ln X_{n+1} - 2 \ln X_n \\
\implies \quad \ln X_{n+2} &= 2 \ln X_{n+1} - 4 \ln X_n
\end{aligned}$$

Let $L_n = \ln X_n \iff X_n = \exp(L_n)$. Then,

$$L_{n+2} = 2L_{n+1} - 4L_n$$

Consider the characteristic equation of L_n, $x^2 - 2x + 4 = 0$. Using the quadratic formula,

$$\begin{aligned}
x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} \\
&= \frac{2 \pm \sqrt{4 \cdot -3}}{2} \\
&= 1 \pm \sqrt{3} i \\
&= \sqrt{1 + \sqrt{3}^2} \exp\left(\left(i \arctan\left(\frac{\pm\sqrt{3}}{1}\right)\right)\right) \\
&= 2 \exp\left(\pm i \frac{\pi}{3}\right)
\end{aligned}$$

Thus, we can express L_n as

$$\begin{aligned}
L_n &= A \cdot 2^n \cos\left(\frac{\pi}{3} n\right) + B \cdot 2^n \sin\left(\frac{\pi}{3} n\right) \\
&= 2^n \left(A \cos\left(\frac{\pi}{3} n\right) + B \sin\left(\frac{\pi}{3} n\right)\right)
\end{aligned}$$

Thus,

$$\boxed{X_n = \exp\left(2^n \left(A \cos\left(\frac{\pi}{3} n\right) + B \sin\left(\frac{\pi}{3} n\right)\right)\right)}$$

Problem 10.

The sequence $\{X_n\}$ is given by $X_1 = 2$, $X_2 = 15$ and

$$X_{n+2} = 5\left(1 + \frac{1}{n+2}\right)X_{n+1} - 6\left(1 + \frac{2}{n+1}\right)X_n, \qquad n \geq 1$$

By dividing the recurrence relation throughout by $n+3$, use a suitable substitution to determine X_n as a function of n.

Solution

$$\begin{aligned}
X_{n+2} &= 5\left(1 + \frac{1}{n+2}\right)X_{n+1} - 6\left(1 + \frac{2}{n+1}\right)X_n \\
\implies \frac{1}{n+3}X_{n+2} &= \frac{1}{n+3}\cdot 5\left(1 + \frac{1}{n+2}\right)X_{n+1} - \frac{1}{n+3}\cdot 6\left(1 + \frac{2}{n+1}\right)X_n \\
&= 5\left(\frac{1}{n+3} + \frac{1}{(n+2)(n+3)}\right)X_{n+1} - 6\left(\frac{1}{n+3} + \frac{2}{(n+1)(n+3)}\right)X_n
\end{aligned}$$

Note that $\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$ and $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$. Thus,

$$\begin{aligned}
\frac{1}{n+3}X_{n+2} &= 5\left(\frac{1}{n+3} + \frac{1}{n+2} - \frac{1}{n+3}\right)X_{n+1} - 6\left(\frac{1}{n+3} + \frac{1}{n+1} - \frac{1}{n+3}\right)X_n \\
\implies \qquad &= 5\cdot\frac{1}{n+2}\cdot X_{n+1} - 6\cdot\frac{1}{n+1}\cdot X_n
\end{aligned}$$

Let $Y_n = \frac{n+1}{X}{}_n \iff X_n = (n+1)Y_n$. Then,

$$Y_{n+2} = 5Y_{n+1} - 6Y_n$$

Consider the characteristic equation of Y_n.

$$\begin{aligned}
x^2 - 5x + 6 &= 0 \\
\implies (x-2)(x-3) &= 0
\end{aligned}$$

Thus, 2 and 3 are the roots of the characteristic equation. Hence,

$$\begin{aligned}
Y_n &= A\cdot 2^n + B\cdot 3^n \\
\implies X_n &= (n+1)\left(A\cdot 2^n + B\cdot 3^n\right)
\end{aligned}$$

Since $X_1 = 2$,

$$\begin{aligned}
(1+1)\left(A\cdot 2^1 + B\cdot 3^1\right) &= 2 \\
\implies \qquad 2A + 3B &= 1 \qquad (10.1)
\end{aligned}$$

Since $X_2 = 15$,

$$\begin{aligned}
(2+1)\left(A\cdot 2^2 + B\cdot 3^2\right) & \\
\implies \qquad 4A + 9B &= 5 \qquad (10.2)
\end{aligned}$$

Solving Equations 10.1 and 10.2 simultaneously, we have $A = -1$ and $B = 1$. Thus,

$$\boxed{X_n = (n+1)\left(3^n - 2^n\right)}$$

Problem 11.

A logistics company set up an online platform providing delivery services to users on a monthly paid subscription basis. The company's sales manager models the number of subscribers that the company has at the end of each month. She notes that approximately 10% of the existing subscribers leave each month, and that there will be a constant number k of new subscribers in each subsequent month after the first.

Let T_n, $n \geq 1$, denote the number of subscribers the company has at the end of the nth month after the online platform was set up.

(a) Express T_{n+1} in terms of T_n.

The company has 250 subscribers at the end of the first month.

(b) Find T_n in terms of n and k.

(c) Find the least number of subscribers the company needs to attract in each subsequent month after the first if it aims to have at least 350 subscribers by the end of the 12th month.

Let $k = 50$ for the rest of the question.

The monthly running cost of the company is assumed to be fixed at \$4,000. The monthly subscription fee is \$10 per user which is charged at the end of each month.

(d) Given that the mth month is the first month in which the company's revenue up to and including that month is able to cover its cost up to and including that month, find the value of m.

(e) Using your answer to part (b), determine the long-term behaviour of the number of subscribers that the company has. Hence, explain whether this behaviour is appropriate in terms of long-term prospects for the company's success.

Solution

Part (a)

$$\boxed{T_{n+1} = 0.9T_n + k}$$

Part (b)

Let m be a constant such that $T_{n+1} + m = 0.9\left(T_n + m\right)$. Then $-0.1m = k \implies m = -10k$. Hence,

$$\begin{aligned}
T_{n+1} - 10k &= 0.9\left(T_n - 10k\right) \\
\implies \quad T_n - 10k &= 0.9^{n-1}\left(T_0 - 10k\right) \\
&= 0.9^{n-1}\left(250 - 10k\right) \\
\implies \quad T_n &= 0.9^{n-1}\left(250 - 10k\right) + 10k
\end{aligned}$$

$$\boxed{T_n = 0.9^{n-1}\left(250 - 10k\right) + 10k}$$

Part (c)

$$\begin{aligned}
& & T_{12} &\geq 350 \\
&\implies & 0.9^{12-1}\left(250-10k\right)+10k &\geq 350 \\
&\implies & 0.9^{11}\cdot 250-0.9^{11}\cdot 10k+10k &\geq 350 \\
&\implies & -0.9^{11}\cdot 10k+10k &\geq 350-0.9^{11}\cdot 250 \\
&\implies & \left(1-0.9^{11}\right)10k &\geq 350-0.9^{11}\cdot 250 \\
&\implies & k &\geq \frac{350-0.9^{11}\cdot 250}{10\left(1-0.9^{11}\right)} \\
& & &= 39.6 \text{ (3 s.f.)}
\end{aligned}$$

Since $k \in \mathbb{N}$, the least value of k is 40.

$$\boxed{\text{The company needs to attract at least 40 subscribers in each subsequent month.}}$$

Part (d)

Since $k = 50$,

$$\begin{aligned}
T_n &= 0.9^{n-1}\left(250-10\cdot 50\right)+10\cdot 50 \\
&= -250\cdot 0.9^{n-1}+500 \\
&= -250\cdot\frac{1}{0.9}\cdot 0.9^n+500 \\
&= -\frac{2500}{9}\cdot 0.9^n+500
\end{aligned}$$

Let $\$S_m$ be the total revenue for the first m months.

$$\begin{aligned}
S_m &= 10\sum_{n=1}^{m} T_n \\
&= 10\sum_{n=1}^{m}\left(-\frac{2500}{9}\cdot 0.9^n+500\right) \\
&= 10\left(-\frac{2500}{9}\cdot\frac{0.9(0.9^m-1)}{0.9-1}+500m\right) \\
&= 10\left(2500\left(0.9^m-1\right)+500m\right) \\
&= 25000\left(0.9^m-1\right)+5000m
\end{aligned}$$

Note that the total cost for the first m months is $\$4000m$. Hence, the total profit for the first m months is given by $\$(S_m-4000m)$.

$$\begin{aligned}
& & S_m-4000m &\geq 0 \\
&\implies & 25000\left(0.9^m-1\right)+5000m-4000m &\geq 0 \\
&\implies & 25000\left(0.9^m-1\right)+1000m &\geq 0 \\
&\implies & 25\left(0.9^m-1\right)+m &\geq 0 \\
&\implies & m &\geq 22.7 \text{ (3 s.f.)}
\end{aligned}$$

Since $m \in \mathbb{N}$, the least value of m is 23.

$$\boxed{m = 23}$$

Part (e)

$$\begin{aligned}\lim_{n\to\infty} T_n &= \lim_{n\to\infty} \big(0.9^{n-1}(250 - 10 \cdot 50) + 10 \cdot 50\big)\\ &= \lim_{n\to\infty} \big(-250 \cdot 0.9^{n-1} + 500\big)\\ &= -250 \cdot 0 + 500\\ &= 500\end{aligned}$$

As n becomes very large, the profit per month is $500 \cdot 10 - 4000 = 1000$ dollars.

$$\boxed{\text{This behaviour is appropriate.}}$$