Problem 1.

Without using a graphing calculator, show that the equation $x^3 + 2x^2 - 2 = 0$ has exactly one positive root.

This root is denoted by α and is to be found using two different iterative methods, starting with the same initial approximation in each case.

(a) Show that α is a root of the equation $x = \sqrt{\dfrac{2}{x+2}}$, and use the iterative formula $x_{n+1} = \sqrt{\dfrac{2}{x_n+2}}$, with $x_1 = 1$, to find α correct to 2 significant figures.

(b) Use the Newton-Raphson method, with $x_1 = 1$, to find α correct to 3 significant figures.

Solution

Let $f(x) = x^3 + 2x^2 - 2 = 0$. Consider $f'(x) = 3x^2 + 4x$. Observe that for all $x > 0$, we have $f'(x) > 0$. Hence, $f(x)$ is increasing for all positive x. Note that $f(0) = -2 < 0$ and $f(1) = 1 > 0$. Thus, $f(x)$ has exactly one positive root.

Part (a)

We know $f(\alpha) = 0$. Hence,

$$\begin{aligned}
& \alpha^3 + 2\alpha^2 - 2 = 0 \\
\implies & \alpha^2(\alpha+2) = 2 \\
\implies & \alpha^2 = \frac{2}{\alpha+2} \\
\implies & \alpha = \sqrt{\frac{2}{\alpha+2}} \qquad \left(\text{rej. } \alpha = -\sqrt{\frac{2}{\alpha+2}} \because \alpha > 0\right)
\end{aligned}$$

Thus, α is a root of the equation $x = \sqrt{\dfrac{2}{x+2}}$.

$$\begin{aligned}
& x_1 = 1 \\
\implies & x_2 = \sqrt{\frac{2}{x_1+2}} = 0.81650 \\
\implies & x_3 = \sqrt{\frac{2}{x_2+2}} = 0.84268 \\
\implies & x_4 = \sqrt{\frac{2}{x_3+2}} = 0.83879
\end{aligned}$$

$$\boxed{\alpha = 0.84 \text{ (2 s.f.)}}$$

Part (b)

$$\begin{aligned}
&x_1 = 1 \\
\implies &x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.857143 \\
\implies &x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.839545 \\
\implies &x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.839287 \\
\implies &x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 0.839287
\end{aligned}$$

$$\boxed{\alpha = 0.839 \text{ (3 s.f.)}}$$

Problem 2.

(a) Show that the tangent at the point $(e, 1)$ to the graph $y = \ln x$ passes through the origin, and deduce that the line $y = mx$ cuts the graph $y = \ln x$ in two points provided that $0 < m < \frac{1}{e}$.

(b) For each root of the equation $\ln x = \frac{1}{3}x$, find an integer n such that the interval $n < x < n + 1$ contains the root. Using linear interpolation, based on $x = n$ and $x = n + 1$, find a first approximation to the smaller root, giving your answer to 1 decimal place. Using your first approximation, obtain, by the Newton-Raphson method, a second approximation to the smaller root, giving your answer to 2 decimal plces.

Solution

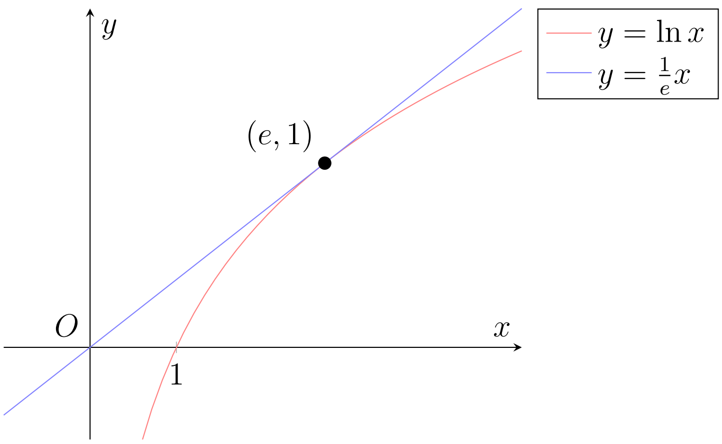
Part (a)

Using the point slope formula, we see that the equation of the tangent at the point $(e, 1)$ is given by

$$\begin{aligned} y - 1 &= \left.\frac{dy}{dx}\right|_{x=e} (x - e) \\ \implies \quad y &= \left.\frac{1}{x}\right|_{x=e} (x - e) + 1 \\ \implies \quad y &= \frac{1}{e}(x - e) + 1 \\ \implies \quad y &= \frac{1}{e}x \end{aligned}$$

Since $x = 0$, $y = 0$ is clearly a solution, the tangent at the point $(e, 1)$ passes through the origin.

From the graph below, it is clear that for $y = mx$ to intersect $y = \ln x$ twice, we must have $0 < m < \frac{1}{e}$.



Part (b)

Consider $f(x) = \frac{1}{3}x - \ln x$. Let α be the smaller root to $f(x) = 0$.

Observe that $f(1) = 1 > 0$ and $f(2) = -0.03 < 0$. Thus, for the smaller root α, $n = 1$.

$$\boxed{\text{Smaller root: } n = 1}$$

Observe that $f(4) = -0.05 < 0$ and $f(5) = 0.06 > 0$. Hence, for the larger root β, $n = 4$.

$$\boxed{\text{Larger root: } n = 4}$$

Using linear interpolation, we have that α is approximately equal to x_1, where

$$\begin{aligned} x_1 &= \frac{1f(2) - 2f(1)}{f(2) - f(1)} \\ &= 1.9 \text{ (1 d.p.)} \end{aligned}$$

$$\boxed{\text{First approximation} = 1.9}$$

Using the Newton-Raphson method,

$$\begin{aligned} x_1 &= 1.9 \\ \implies x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 1.85585 \\ \implies x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 1.85718 \end{aligned}$$

$$\boxed{\alpha = 1.86 \text{ (2 d.p.)}}$$

Problem 3.

Find the exact coordinates of the turning points on the graph of $y = f(x)$ where $f(x) = x^3 - x^2 - x - 1$. Deduce that the equation $f(x) = 0$ has only one real root α, and prove that α lies between 1 and 2. Use the Newton-Raphson method applied to the equation $f(x) = 0$ to find a second approximation x_2 to α, taking x_1, the first approximation, to be 2. With reference to a graph of $y = f(x)$, explain why all further approximations to α by this process are always larger than α.

Solution

For turning points, $f'(x) = 0$.

$$\begin{aligned} & f'(x) = 0 \\ \implies \quad & 3x^2 - 2x - 1 = 0 \\ \implies & (3x+1)(x-1) = 0 \end{aligned}$$

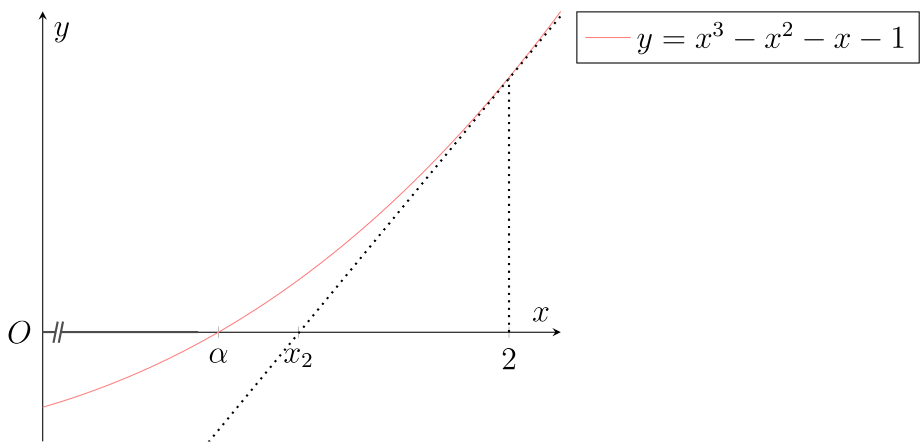
Hence, $x = -\frac{1}{3}$ or $x = 1$. When $x = -\frac{1}{3}$, we have $y = -0.815$, giving the coordinate $(-\frac{1}{3}, -0.815)$. When $x = 1$, we have $y = -2$, giving the coordinate $(1, -2)$.

$$\boxed{\text{The coordinates of the turning points are } (-\tfrac{1}{3}, -0.815) \text{ and } (1, -2).}$$

Observe that $f(x)$ is strictly increasing for all $x > 1$. Since $f(1) = -2 < 0$ and $f(2) = 1 > 0$, the equation $f(x) = 0$ has only one real root.

Using the Newton-Raphson method with $x_1 = 2$, we have $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{13}{7}$.

$$\boxed{x_2 = \frac{13}{7}}$$



Since x_2 lies on the right of α, the Newton-Raphson method gives an over-estimation given an initial approximation of 2. Thus, all further approximations to α will also be larger than α.

Problem 4.

A curve C has equation $y = x^5 + 50x$. Find the least value of $\frac{dy}{dx}$ and hence give a reason why the equation $x^5 + 50x = 10^5$ has exactly one real root. Use the Newton-Raphson method, with a suitable first approximation, to find, correct to 4 decimal places, the root of the equation $x^5+50x = 10^5$. You should demonstrate that your answer has the required accuracy.

Solution

Since $y = x^5+50x$, we know that $\frac{dy}{dx} = 5x^4+50$. Since $x^4 \geq 0$ for all real x, the minimum value of $\frac{dy}{dx}$ is 50.

$$\boxed{\min \frac{dy}{dx} = 50}$$

Let $f(x) = x^5+50x$. Since $\min \frac{df}{dx} = 50 > 0$, we have that $f(x)$ is a strictly increasing function. Thus, $f(x)$ will intersect only once with the line $y = 10^5$. Hence, the equation $x^5 + 50x = 10^5$ has exactly one real root.

Observe that $f(9) = -40901 < 0$ and $f(10) = 50 > 0$. Thus, there must be a root on the interval $(9, 10)$. We now use the Newton-Raphson method with $x_1 = 9$ as the first approximation.

$$\begin{aligned} & x_1 = 9 \\ \implies & x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 10.2178921 \\ \implies & x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 10.0017491 \\ \implies & x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 9.9901221 \\ \implies & x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 9.9899912 \\ \implies & x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 9.9899900 \end{aligned}$$

$$\boxed{\text{The root is 9.9900 (4 d.p.).}}$$

Observe that $f(9.98995) = -2.00 < 0$ and $f(9.99005) = 3.00 > 0$. Hence, the root lies in the interval $(9.98995, 9.99005)$. Thus, the calculated root has the required accuracy.

Problem 5.

(a) A function f is such that $f(4) = 1.158$ and $f(5) = -3.381$, correct to 3 decimal places in each case. Assuming that there is a value of x between 4 and 5 for which $f(x) = 0$, use linear interpolation to estimate this value.

For the case when $f(x) = \tan x$, and x is measured in radians, the value of $f(4)$ and $f(5)$ are as given above. Explain, with the aid of a sketch, why linear interpolation using these values does not give an approximation to a solution of the equation $\tan x = 0$.

(b) Show, by means of a graphical argument or otherwise, that the equation $\ln(x-1) = -2x$ has exactly one real root, and show that this root lies between 1 and 2.

The equation may be written in the form $\ln(x-1) + 2x = 0$. Show that neither $x = 1$ nor $x = 2$ is a suitable initial value for the Newton-Rapson method in this case.

The equation may also be written in the form $x - 1 - e^{-2x} = 0$. For this form, use two applications of the Newton-Raphson method, starting with $x = 1$, to obtain an approximation to the root, giving 3 decimal places in your answer.

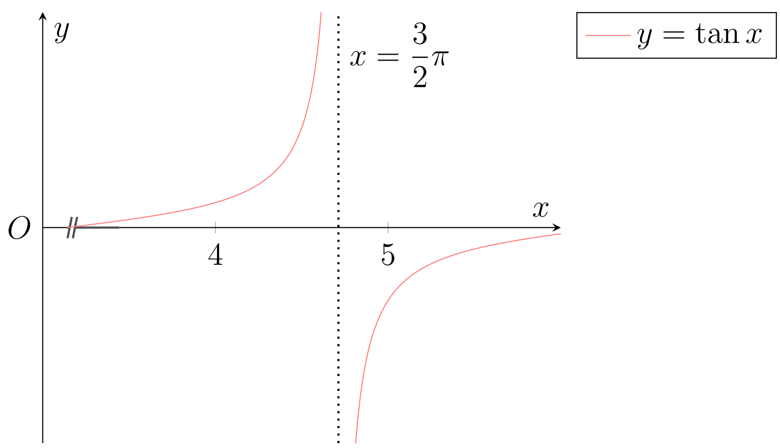
Solution

Part (a)

Let the root of $f(x) = 0$ be α. Using linear interpolation on the interval $[4, 5]$, we have

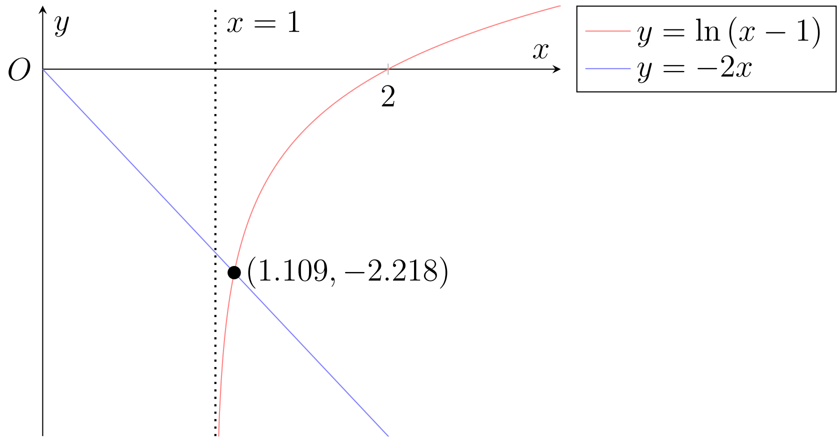
$$\begin{aligned}\alpha &= \frac{4f(5) - 5f(4)}{f(5) - f(4)} \\ &= 4.255 \text{ (3 d.p.)}\end{aligned}$$

$$\boxed{\alpha = 4.255 \text{ (3 d.p.)}}$$



$f(x) = \tan x$ is not continuous on the interval $[4, 5]$ as there is a vertical asymptote at $x = \frac{3}{2}\pi$. Since linear interpolation requires a function be continuous, linear interpolation does not give an approximation to a solution of the equation $\tan x = 0$.

Part (b)



Since there is only one intersection between the graphs $y = \ln(x-1)$ and $y = -2x$, there is only one real root to the equatoin $\ln(x-1) = -2x$. Furthermore, since $y = -2x$ is negative for all $x > 0$ and $y = \ln(x-1)$ is negative for all $1 < x < 2$, it follows that the root must lie between 1 and 2.

Let $f(x) = \ln(x-1) + 2x$. Then $f'(x) = \frac{1}{x-1} + 2$.

Using the Newton-Raphson method with the initial approximation $x_1 = 1$, we see that $x_2 = 1 - \frac{f(1)}{f'(1)}$. However, $f'(1)$ is undefined. Thus, $x_1 = 1$ is not a suitable initial value.

Using the Newton-Raphson method with the initial approximation $x_2 = 2$, we see that $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1$, whence $x_3 = 1 - \frac{f(1)}{f'(1)}$. Once again, $f'(1)$ is undefined. Thus, $x_1 = 2$ is also not a suitable initial value.

Let $g(x) = x - 1 - e^{-2x}$. Then $g'(x) = 1 + 2e^{-2x}$. Using the Newton-Raphson method with the initial approximation $x_1 = 1$, we have

$$\begin{aligned} & x_1 = 1 \\ \implies & x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.106507 \\ \implies & x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.108857 \end{aligned}$$

$$\boxed{x = 1.109 \text{ (3 d.p.)}}$$

Problem 6.

The equation $x = 3\ln x$ has two roots α and β, where $1 < \alpha < 2$ and $4 < \beta < 5$. Using the iterative formula $x_{n+1} = F(x_n)$, where $F(x) = 3\ln x$, and starting with $x_0 = 4.5$, find the value of β correct to 3 significant figures. Find a suitable $F(x)$ for computing α.

Solution

$$\begin{aligned}
&x_0 = 4.5 \\
\implies &x_1 = F(x_0) = 4.51223 \\
\implies &x_2 = F(x_1) = 4.52038 \\
\implies &x_3 = F(x_2) = 4.52579 \\
\implies &x_4 = F(x_3) = 4.52937 \\
\implies &x_5 = F(x_4) = 4.53175 \\
\implies &x_6 = F(x_5) = 4.53333 \\
\implies &x_7 = F(x_6) = 4.53437 \\
\implies &x_8 = F(x_7) = 4.53506
\end{aligned}$$

$$\boxed{\beta = 4.54 \text{ (3 s.f.)}}$$

$$\begin{aligned}
&x = 3\ln x \\
\implies &\frac{1}{3}x = \ln x \\
\implies &x = e^{\frac{1}{3}x}
\end{aligned}$$

Observe that $\frac{d}{dx}(e^{\frac{1}{3}x}) = \frac{1}{3}e^{\frac{1}{3}x} < 1$ for all $1 < x < 2$. Thus, $F(x) = e^{\frac{1}{3}x}$ is suitable for computing α as the iterative formula $x_{n+1} = F(x_n)$ will converge.

$$\boxed{F(x) = e^{\frac{1}{3}x}}$$

Problem 7.

Show that the cubic equation $x^3 + 3x - 15 = 0$ has only one real root. This root is near $x = 2$. The cubic equation can be written in any one of the forms below:

(a) $x = \frac{1}{3}(15 - x^3)$

(b) $x = \frac{15}{x^2 + 3}$

(c) $x = (15 - 3x)^{\frac{1}{3}}$

Determine which of these forms would be suitable for the use of the iterative formula $x_{r+1} = F(x_r)$, where $r = 1, 2, 3, \ldots$.

Hence, find the root correct to 3 decimal places.

Solution

Let $f(x) = x^3 + 3x - 15$. Then $f'(x) = 3x^2 + 3 > 0$ for all real x. Hence, f is strictly increasing. Since f is continuous, $f(x) = 0$ has only one real root.

Part (a)

Let $g_1(x) = \frac{1}{3}(15 - x^3)$. Then $g_1'(x) = -x$. For values of x near 2, $|g_1'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_1(x_n)$ will diverge. Thus, $g_1(x)$ is unsuitable.

Part (b)

Let $g_2(x) = \frac{15}{x^2+3}$. Then $g_2'(x) = \frac{-30x}{(x^2+3)^2}$. For values of x near 2, $|g_2'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_2(x_n)$ will diverge. Thus, $g_2(x)$ is unsuitable.

Part (c)

Let $g_3(x) = (15 - 3x)^{\frac{1}{3}}$. Then $g_3'(x) = -(15 - 3x)^{-\frac{2}{3}}$. For values of x near 2, $|g_3'(x)| < 1$. Hence, the iterative formula $x_{n+1} = g_3(x_n)$ will converge. Thus, $g_3(x)$ is suitable.

$$
\begin{aligned}
& x_1 = 2 \\
\implies & x_2 = g_3(x_1) = 2.080084 \\
\implies & x_3 = g_3(x_2) = 2.061408 \\
\implies & x_4 = g_3(x_3) = 2.065793 \\
\implies & x_5 = g_3(x_4) = 2.064765
\end{aligned}
$$

$$\boxed{x = 2.065 \text{ (3 d.p.)}}$$

Problem 8.

The equation of a curve is $y = f(x)$. The curve passes through the points $(a, f(a))$ and $(b, f(b))$, where $0 < a < b$, $f(a) > 0$ and $f(b) < 0$. The equation $f(x) = 0$ has precisely one root α such that $a < \alpha < b$. Derive an expression, in terms of a, b, $f(a)$ and $f(b)$, for the estimated value of α based on linear interpolation.

Let $f(x) = 3e^{-x} - x$. Show that $f(x) = 0$ has a root α such that $1 < \alpha < 2$, and that for all x, $f'(x) < 0$ and $f''(x) > 0$. Obtain an estimate of α using lienar interpolation to 2 decimal places, and explain by means of a sketch whether the value obtained is an over-estimate or an under-estimate.

Use one application of the Newton-Raphson method to obtain a better estimate of α, giving your answer to 2 decimal places.

Solution

We begin by finding the equation of the line that passes through both $(a, f(a))$ and $(b, f(b))$. Using the point-slope formula, we have

$$y - f(a) = \frac{f(a) - f(b)}{a - b}(x - a)$$

α is hence approximately the root of the above equation. Thus,

$$\begin{aligned} -f(a) &= \frac{f(a) - f(b)}{a - b}(\alpha - a) \\ \implies \qquad \alpha &= -f(a) \cdot \frac{a - b}{f(a) - f(b)} + a \\ &= \frac{bf(a) - af(a)}{f(a) - f(b)} + \frac{af(a) - af(b)}{f(a) - f(b)} \\ &= \frac{bf(a) - af(b)}{f(a) - f(b)} \end{aligned}$$

$$\boxed{\alpha = \frac{bf(a) - af(b)}{f(a) - f(b)}}$$

Observe that $f(1) = 0.10 > 0$ and $f(2) = -1.6 < 0$. Since f is continuous, there exists a root $\alpha \in (1, 2)$.

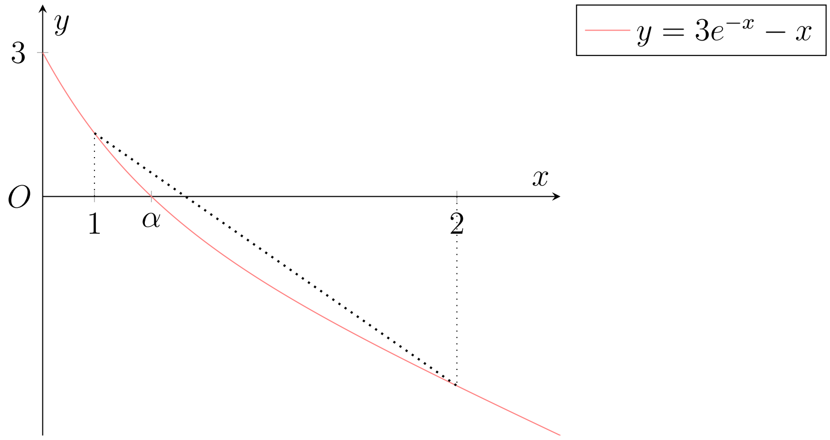
Note that $f'(x) = -3e^{-x} - 1$ and $f''(x) = 3e^{-x}$. Since $e^{-x} > 0$ for all x, we have that $f'(x) < 0$ and $f''(x) > 0$ for all x.

Using linear interpolation on the interval $(1, 2)$, we have

$$\begin{aligned} \alpha &= \frac{2 \cdot f(1) - 1 \cdot f(2)}{f(1) - f(2)} \\ &= 1.0610 \end{aligned}$$

$$\boxed{\alpha = 1.06 \text{ (2 d.p.)}}$$

Since $f'(x) < 0$ and $f''(x) > 0$, we know that f is strictly decreasing and has an upwards concave shape. This gives the following sketch of $f(x)$.

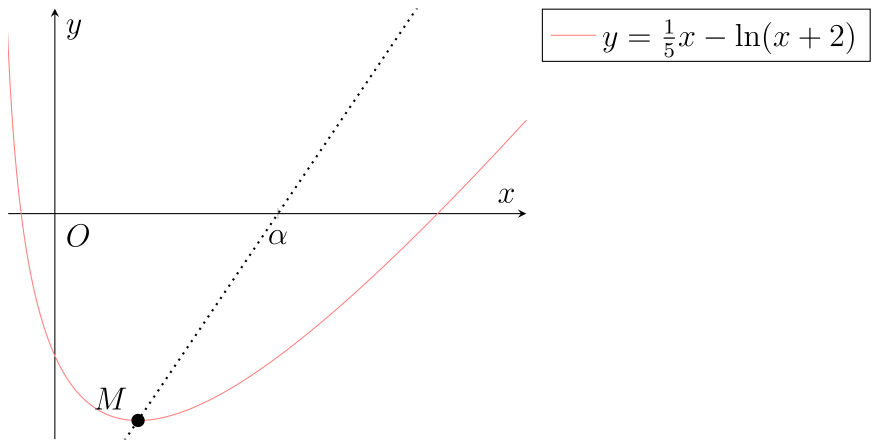


Hence, the value obtained is an over-estimate.

Using the Newton-Raphson method with the initial approximation $x_1 = 1.06$, we get $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.05$.

$$\boxed{\alpha = 1.05 \text{ (2 d.p.)}}$$

Problem 9.



The diagram shows a sketch of the graph $y = \dfrac{1}{3}x - \ln(x+2)$. Find the x-coordinate of the minimum point M on the graph, and verify that y is positive when $x = 20$.

Show that the gradient of the curve is always less than $\dfrac{1}{5}$. Hence, by considering the line through M having gradient $\dfrac{1}{5}$, show that the positive root of the equation $\dfrac{1}{3}x - \ln(x+2) = 0$ is greater than 8.

Use linear interpolation, once only, on the interval $[8, 20]$, to find an approximate value a for this positive root, giving your answer to 1 decimal place.

Using a as an initial value, carry out one application of the Newton-Raphson method to obtain another approximation to the positive root, giving your answer to 2 decimal places.

Solution

Let the x-coordinate of M be x_M. Since M is a minimum, we know that $\left.\dfrac{dy}{dx}\right|_{x=x_M} = 0$.

$$\begin{aligned}
& \left.\frac{dy}{dx}\right|_{x=x_M} = 0 \\
\implies & \left.\frac{1}{5} - \frac{1}{x+2}\right|_{x=x_M} = 0 \\
\implies & \frac{1}{5} - \frac{1}{x_M+2} = 0 \\
\implies & x_M = 3
\end{aligned}$$

$$\boxed{x_M = 3}$$

Subsituting $x = 20$ into the equation of the curve gives $y = 4 - \ln 22 = 0.909 > 0$.

We know that $\dfrac{dy}{dx} = \dfrac{1}{5} - \dfrac{1}{x+2}$, hence $\dfrac{dy}{dx} < \dfrac{1}{5}$ for all $x > -2$. Since the domain of the curve is $x > -2$, $\dfrac{dy}{dx}$ is always less than $\dfrac{1}{5}$.

Let $(\alpha, 0)$ be the coordinates of the root of the line through M having gradient $\dfrac{1}{5}$. We know that the coordinates of M are $(3, \dfrac{3}{5} - \ln 5)$. Hence,

$$
\begin{aligned}
\frac{\frac{3}{5} - \ln 5 - 0}{3 - \alpha} &= \frac{1}{5} \\
\implies \qquad 3 - \alpha &= 3 - 5\ln 5 \\
\implies \qquad \alpha &= 5\ln 5 \\
&= 8.05 \\
&> 8
\end{aligned}
$$

Since the gradient of the curve is always less than $\frac{1}{5}$, α represents the lowest possible value of the positive root of the curve. Hence, the positive root of the equation $\frac{1}{5}x - \ln(x+2) = 0$ is greater than 8.

Let $f(x) = \frac{1}{5}x - \ln(x+2)$. Using linear interpolation on the interval $[8, 20]$, we have

$$
\begin{aligned}
a &= \frac{8f(20) - 20f(8)}{f(20) - f(8)} \\
&= 13.2
\end{aligned}
$$

$$\boxed{a = 13.2 \text{ (1 d.p.)}}$$

Using the Newton-Raphson method with the initial approximation $x_1 = 13.2$, we have $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 13.81$.

$$\boxed{a = 13.81 \text{ (2 d.p.)}}$$

Problem 10.

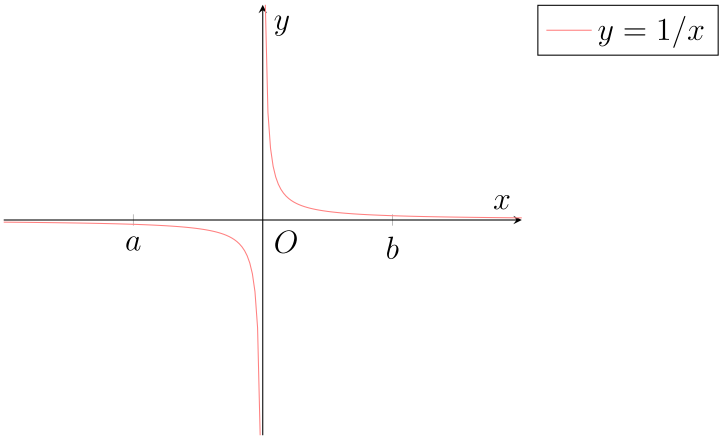
(a) The function f is such that $f(a)f(b) < 0$, where $a < b$. A student concludes that the equation $f(x) = 0$ has exactly one root in the interval (a, b). Draw sketches to illustrate two distinct ways in which the student could be wrong.

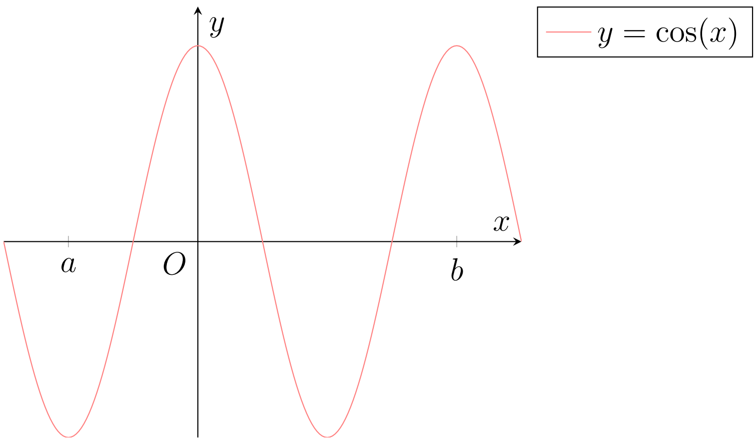
(b) The equation $\sec^2 x - e^2 = 0$ has a root α in the interval $[1.5, 2.5]$. A student uses linear interpolation once on this interval to find an approximation to α. Find the approximation to α given by this method and comment on the suitability of the method in this case.

(c) The equation $\sec^2 x - e^x = 0$ also has a root β in the interval $(0.1, 0.9)$. Use the Newton-Raphson method, with $f(x) = \sec^2 x - e^x$ and initial approximation 0.5, to find a sequence of approximations $\{x_1, x_2, x_3, \ldots\}$ to β. Describe what is happening to x_n for large n, and use a graph of the function to explain why the sequence is not converging to β.

Solution

Part (a)





Part (b)

Let $f(x) = \sec^2 x - e^x$. Using linear interpolation on the interval $[1.5, 2.5]$,

$$\begin{aligned} a &= \frac{1.5f(2.5) - 2.5f(1.5)}{f(2.5) - f(1.5)} \\ &= 1.06 \end{aligned}$$

$$\boxed{a = 1.06 \text{ (2 d.p.)}}$$

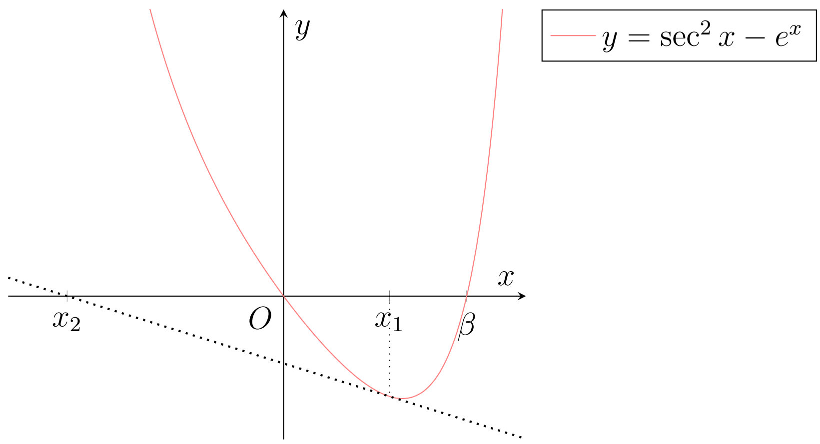
$\sec^2 x$ is not continuous on the interval $[1.5, 2.5]$ due to the presence of an asymptote at $x = \frac{\pi}{2}$. Hence, linear interpolation is not suitable in this case.

Part (c)

We know $f'(x) = 2\sec^2 x \tan x - e^x$. Using the Newton-Raphson method with the initial approximation $x_1 = 0.5$,

$$\begin{aligned} x_1 &= 0.5 \\ \implies x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = -1.02272 \\ \implies x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = -0.75526 \\ \implies x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = -0.40306 \\ \implies x_5 &= x_4 - \frac{f(x_4)}{f'(x_4)} = -0.09667 \\ \implies x_6 &= x_5 - \frac{f(x_5)}{f'(x_5)} = -0.00466 \\ \implies x_7 &= x_6 - \frac{f(x_6)}{f'(x_6)} = -0.00000 \end{aligned}$$

As $n \to \infty$, $x_n \to 0^-$.



The initial approximation of $x_1 = 0.5$ is past the turning point. Hence, all subsequent approximations will converge to the root at 0 instead of the root at β. Thus, the sequence does not converge to β.

Problem 11.

The function f is given by $f(x) = \sqrt{1-x^2} + \cos x - 1$ for $0 \leq x \leq 1$. It is known, from graphical work, that the equation $f(x) = 0$ has a single root $x = \alpha$.

(a) Express g(x) in terms of x, where $g(x) = x - \dfrac{f(x)}{f'(x)}$.

A student attempts to use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to calculate the value of α correct to 3 decimal places.

(b) (i) The student first uses an initial approximation to α of $x_1 = 0$. Explain why this will be unsuccessful in finding a value for α.

(ii) The student next uses an initial approximation to α of $x_1 = 1$. Explain why this will also be unsuccessful in finding a value for α.

(iii) The student then uses an initial approximate to α of $x_1 = 0.5$. Investigate what happens in this case.

(iv) By choosing a suitable value for x_1, use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to determine α correct to 3 decimal places.

Solution

Part (a)

We know $f'(x) = \dfrac{-x}{\sqrt{1-x^2}} - \sin x$. Hence,

$$\boxed{g(x) = x - \frac{\sqrt{1-x^2} + \cos x - 1}{\frac{-x}{\sqrt{1-x^2}} - \sin x}}$$

Part (b)

Subpart (i)

Observe that $f'(0) = 0$. Hence, $g(0)$ is undefined. Thus, starting with an initial approximation of $x_1 = 0$ will be unsuccessful in finding a value for α.

Subpart (ii)

Observe that $\sqrt{1-x^2}$ is 0 when $x = 1$. Hence $f'(0)$ is undefined. Thus, $g(0)$ is also undefined. Hence, starting with an initial approximation of $x_1 = 1$ will also be unsuccessful in finding a value for α.

Subpart (iii)

When $x_1 = 0.5$, we have $x_2 = g(x_1) = 1.20$. Since $g(x)$ is only defined for $0 \leq x \leq 1$, $x_3 = g(x_2)$ is undefined. Hence, an initial approximation of $x_1 = 0.5$ will also be unsuccessful in finding a value for α.

Subpart (iv)

Using the Newton-Raphson method with $x_1 = 0.9$, we have

$$\begin{aligned} & x_1 = 0.9 \\ \implies & x_2 = g(x_1) = 0.92019 \\ \implies & x_3 = g(x_2) = 0.91928 \\ \implies & x_4 = g(x_3) = 0.91928 \end{aligned}$$

$$\boxed{\alpha = 0.919 \text{ (3 d.p.)}}$$