Problem 1.

Given that $y = 1$ when $x = 1$, find the particular solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2}{x}$.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x}$$

$$\implies \quad \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

$$\implies \int \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int \frac{1}{x}\,\mathrm{d}x$$

$$\implies \quad \int \frac{1}{y^2}\,\mathrm{d}y = \int \frac{1}{x}\,\mathrm{d}x$$

$$\implies \quad -\frac{1}{y} = \ln|x| + C_1$$

$$\implies \quad y = -\frac{1}{\ln|x| + C_1}$$

$$= \frac{1}{C - \ln|x|}$$

At $x = 1$, $y = 1$, $1 = \dfrac{1}{C - \ln|1|} \implies C = 1$.

$$\boxed{y = \frac{1}{1 - \ln|x|}}$$

Problem 2.

Two variables x and t are connected by the differential equation $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{kx}{10 - x}$, where $0 < x < 10$ and where k is a constant. It is given that $x = 1$ when $t = 0$ and that $x = 2$ when $t = 1$. Find the value of t when $x = 5$, given your answer to three s.f.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{kx}{10 - x}$$

$$\implies \quad \frac{10 - x}{x} \frac{\mathrm{d}x}{\mathrm{d}t} = k$$

$$\implies \int \frac{10 - x}{x} \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t = k \int \mathrm{d}t$$

$$\implies \quad \int \frac{10 - x}{x} \, \mathrm{d}x = k \int \mathrm{d}t$$

$$\implies \int \left(\frac{10}{x} - 1 \right) \mathrm{d}x = kt + C$$

$$\implies \quad 10 \ln x - x = kt + C$$

At $t = 0$, $-x = 1$, $C = -1$.

At $t = 1$, $x = 2$, $10 \ln 2 - 2 = k - 1 \implies k = \ln 2 - 1$.

When $x = 5$, $10 \ln 5 - 5 = (\ln 2 - 1)t - 1 \implies t = \dfrac{10 \ln 5 - 4}{10 \ln 2 - 1} = 2.04$ (3 s.f.).

$$\boxed{\text{When } x = 5, \ t = 2.04}$$

Problem 3.

Use the substitution $y = u - 2x$ to find the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{8x + 4y + 1}{4x + 2y + 1}$.

Solution

Note that $y = u - 2x \implies u = y + 2x$. Also, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x} - 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8x + 4y + 1}{4x + 2y + 1}$$

$$\implies \quad \frac{\mathrm{d}u}{\mathrm{d}x} - 2 = -\frac{8x + 4(u - 2x) + 1}{4x + 2(u - 2x) + 1}$$

$$= -\frac{4u + 1}{2u + 1}$$

$$\implies \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2 - \frac{4u + 1}{2u + 1}$$

$$= \frac{4u + 2}{2u + 1} - \frac{4u + 1}{2u + 1}$$

$$= \frac{1}{2u + 1}$$

$$\implies \quad (2u + 1)\frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\implies \quad \int (2u + 1)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int \mathrm{d}x$$

$$\implies \quad \int (2u + 1)\,\mathrm{d}u = \int \mathrm{d}x$$

$$\implies \quad u^2 + u = x + C$$

$$\implies (y + 2x)^2 + (y + 2x) = x + C$$

$$\implies \quad (y + 2x)^2 + y + x = C$$

$$\boxed{(y + 2x)^2 + y + x = C}$$

Problem 4.

By using the substitution $z = ye^{2x}$, find the general solution of the differential equation $\dfrac{dy}{dx} + 2y = xe^{-2x}$.

Find the particular solution of the differential equation given that $\dfrac{dy}{dx} = 1$ when $x = 0$.

Solution

Note that $z = ye^{2x} \implies \dfrac{dz}{dx} = \dfrac{dy}{dx}e^{2x} + 2ye^{2x} = \dfrac{dy}{dx}e^{2x} + 2z$. Hence, $\dfrac{dy}{dx} = \dfrac{dz}{dx}e^{-2x} - 2y$.

$$\frac{dy}{dx} + 2y = xe^{-2x}$$

$$\implies \frac{dz}{dx}e^{-2x} - 2y + 2y = xe^{-2x}$$

$$\implies \frac{dz}{dx} = x$$

$$\implies z = \frac{x^2}{2} + C$$

$$\implies ye^{2x} = \frac{x^2}{2} + C$$

$$\implies y = \frac{x^2}{2e^{2x}} + \frac{C}{e^{2x}}$$

$$\boxed{y = \frac{x^2}{2e^{2x}} + \frac{C}{e^{2x}}}$$

At $x = 0$, $\dfrac{dy}{dx} = 1$. Hence, $1 + 2y = 0 \implies y = -\dfrac{1}{2}$. Thus, $C = -\dfrac{1}{2}$.

$$\boxed{y = \frac{x^2 - 1}{2e^{2x}}}$$

Problem 5.

Find the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 6xy^3$.

Find its particular solution given that $y = 0.5$ when $x = 0$.

Determine the interval of validity for the particular solution.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6xy^3$$

$$\implies \frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} = 6x$$

$$\implies \int \frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = 6\int x\,\mathrm{d}x$$

$$\implies \int \frac{1}{y^3}\,\mathrm{d}y = 6\int x\,\mathrm{d}x$$

$$\implies -\frac{1}{2}\frac{1}{y^2} = 3x^2 + C_1$$

$$\implies \frac{1}{y^2} = C - 6x^2$$

$$\implies y^2 = \frac{1}{C - 6x^2}$$

$$\boxed{y^2 = \frac{1}{C - 6x^2}}$$

At $x = 0$, $y = 0.5$, $\dfrac{1}{4} = \dfrac{1}{C} \implies C = 4$.

$$\boxed{y^2 = \frac{1}{4 - 6x^2}}$$

Observe that we require $4 - 6x^2 > 0$, whence $-\sqrt{\dfrac{2}{3}} < x < \sqrt{\dfrac{2}{3}}$.

$$\boxed{\text{The interval of validity is } \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right).}$$

Problem 6.

(a) Find the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{3x}{x^2 + 1}$.

(b) What can you say about the gradient of every solution as $x \to \pm\infty$?

(c) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$. Hence sketch the graph of this solution.

Solution

Part (a)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x}{x^2 + 1} \\
&= \frac{3}{2}\frac{2x}{x^2 + 1} \\
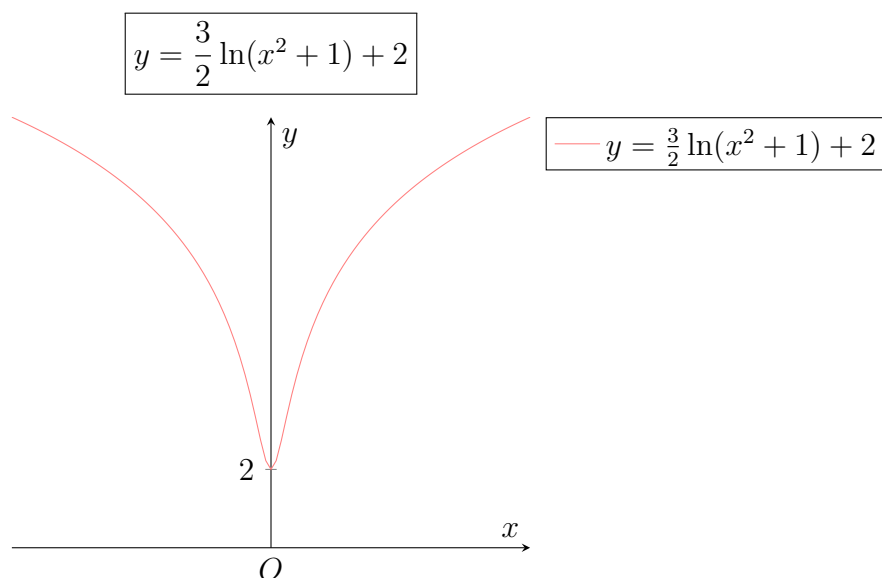\implies \quad y &= \frac{3}{2}\ln\left(x^2 + 1\right) + C
\end{aligned}$$

$$\boxed{y = \frac{3}{2}\ln(x^2 + 1) + C}$$

Part (b)

As $x \to \pm\infty$, $\dfrac{3x}{x^2 + 1} \to 0^+$. Hence, the gradient of every solution approaches 0 from above.

Part (c)

When $x = 0$ and $y = 2$, $C = 2$.

$$\boxed{y = \frac{3}{2}\ln(x^2 + 1) + 2}$$

Problem 7.

The variables x, y and z are connected by the following differential equations.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3 - 2z \tag{7.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z \tag{7.2}$$

(a) Given that $z < \dfrac{3}{2}$, solve equation 7.1 to find z in terms of x.

(b) Hence find y in terms of x.

(c) Use the result in part (b) to show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = a\frac{\mathrm{d}y}{\mathrm{d}x} + b$$

for constants a and b to be determined.

(d) The curve of the solution in part (b) passes through the points $(0, 1)$ and $(2, 3 + e^{-4})$. Sketch this curve, indicating its axial intercept and asymptote (if any).

Solution

Part (a)

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3 - 2z$$

$$\implies \frac{1}{3 - 2z}\frac{\mathrm{d}z}{\mathrm{d}x} = 1$$

$$\implies \int \frac{1}{3 - 2z}\frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}x = \int \mathrm{d}x$$

$$\implies \int \frac{1}{3 - 2z}\,\mathrm{d}z = \int \mathrm{d}x$$

$$\implies -\frac{1}{2}\ln(3 - 2z) = x + C_1$$

$$\implies \ln(3 - 2z) = C_2 - 2x$$

$$\implies 3 - 2z = C_3 e^{-2x}$$

$$\implies z = \frac{3}{2} - C_4 e^{-2x}$$

$$\boxed{z = \frac{3}{2} - Ae^{-2x},\ A > 0}$$

Part (b)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2} - C_4 e^{-2x}$$

$$\implies y = \int \left(\frac{3}{2} - C_4 e^{-2x}\right)\mathrm{d}x$$

$$= \frac{3}{2}x - \frac{C_4}{2}e^{-2x} + C_6$$

$$y = \frac{3}{2}x - \frac{A}{2}e^{-2x} + B$$

Part (c)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2} + C_4 e^{-2x}$$

$$\implies \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2C_4 e^{-2x}$$

$$= -2\left(\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{2}\right)$$
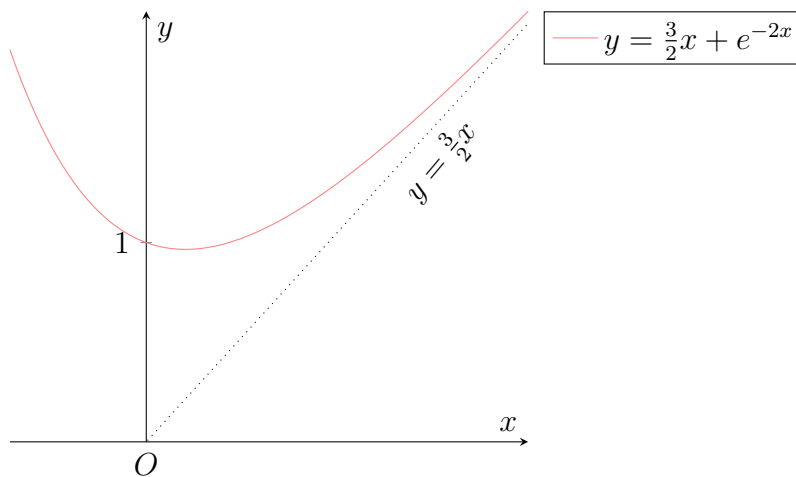
$$= -2\frac{\mathrm{d}y}{\mathrm{d}x} + 3$$

$$\boxed{a = -2,\ b = 3}$$

Part (d)

At $(0, 1)$, we obtain $1 = -\dfrac{A}{2} + B$.

At $(2, 3 + e^{-4})$, we obtain $3 + e^{-4} = 3 - \dfrac{A}{2}e^{-4} + B \implies 1 = -\dfrac{A}{2} + Be^4$.

Hence, $B = Be^4$, whence $B = 0$ and $A = -2$. The curve thus has equation $y = \dfrac{3}{2}x + e^{-2x}$.

Problem 8.

A bottle containing liquid is taken from a refrigerator and placed in a room where the termperature is a constant $20\,°\text{C}$. As the liquid warms up, the rate of increase of its temperature $\theta\,°\text{C}$ after time t minutes is proportional to the temperature difference $(20 - \theta)\,°\text{C}$. Initially the temperature of the liquid is $10\,°\text{C}$ and the rate of increase of the temperature is $1\,°\text{C}$ per minute. By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-t/10}$.

Find the time it takes the liquid to reach a temperature of $15\,°\text{C}$, and state what happens to θ for large values of t. Sketch a graph of θ against t.

Solution

Since $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} \propto (20 - \theta)$, we have $\dfrac{\mathrm{d}\theta}{\mathrm{d}t} = k(20 - \theta)$, where k is a constant. We now solve for θ.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = k(20 - \theta)$$
$$\implies \frac{1}{20 - \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} = k$$
$$\implies \int \frac{1}{20 - \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\mathrm{d}t = k\int \mathrm{d}t$$
$$\implies \int \frac{1}{20 - \theta}\,\mathrm{d}\theta = k\int \mathrm{d}t$$
$$\implies -\ln(20 - \theta) = kt + C_1$$
$$\implies \ln(20 - \theta) = C_2 - kt$$
$$\implies 20 - \theta = Ce^{-kt}$$
$$\implies \theta = 20 - Ce^{-kt}$$

At $t = 0$, $\theta = 10$. Hence, $10 = 20 - C \implies C = 10$. We also have $\left.\dfrac{\mathrm{d}\theta}{\mathrm{d}t}\right|_0 = 1$. Hence, $1 = k\left[20 - (20 - 10e^0)\right] = 10k \implies k = \dfrac{1}{10}$. Thus,
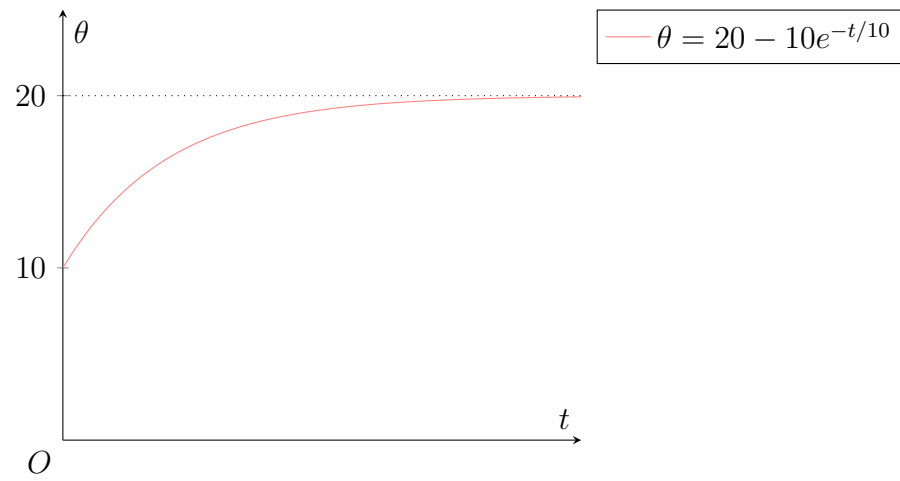
$$\theta = 20 - 10e^{-t/10}.$$

Consider $\theta = 15$.

$$15 = 20 - 10e^{-t/10}$$
$$\implies e^{-t/10} = \frac{1}{2}$$
$$\implies -\frac{t}{10} = -\ln 2$$
$$\implies t = 10\ln 2$$

It takes $10\ln 2$ minutes for the liquid to reach a temperature of $15\,°\text{C}$.

As $t \to \infty$, $\theta \to 20$.

Problem 9.

(a) Find $\int \dfrac{1}{100 - v^2}\, dx$.

(b) A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfies the differential equation

$$\frac{dv}{dt} = 10 - 0.1v^2.$$

(i) Find t in terms of v. Hence find the exact tiem the stone takes to reach a speed of 5 metres per second.

(ii) Find the speed of the stone after 1 second.

(iii) What happens to the speed of the stone for large values of t?

Solution

Part (a)

$$\int \frac{1}{100 - v^2}\, dv = \frac{1}{2(10)} \ln\left(\frac{10 + v}{10 - v}\right) + C$$
$$= \frac{1}{20} \ln\left(\frac{10 + v}{10 - v}\right) + C$$

$$\boxed{\int \frac{1}{100 - v^2}\, dv = \frac{1}{20} \ln\left(\frac{10 + v}{10 - v}\right) + C}$$

Part (b)

Subpart (i)

$$\frac{dv}{dt} = 10 - 0.1v^2$$
$$= \frac{1}{10}(100 - v^2)$$
$$\implies \frac{1}{100 - v^2}\frac{dv}{dt} = \frac{1}{10}$$
$$\implies \int \frac{1}{100 - v^2}\frac{dv}{dt}\, dt = \frac{1}{10}\int dt$$
$$\implies \int \frac{1}{100 - v^2}\, dv = \frac{1}{10}\int dt$$
$$\implies \frac{1}{20} \ln\left(\frac{10 + v}{10 - v}\right) + C = \frac{1}{10}t$$
$$\implies t = \frac{1}{2} \ln\left(\frac{10 + v}{10 - v}\right) + C$$

At $t = 0$, $v = 0$. Hence, $C = 0$.

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$$t = \frac{1}{2} \ln\left(\frac{10 + v}{10 - v}\right)$$

Subpart (ii) Consider $t = 1$.

$$\frac{1}{2} \ln\left(\frac{10 + v}{10 - v}\right) = 1$$

$$\implies \ln\left(\frac{10 + v}{10 - v}\right) = 2$$

$$\implies \frac{10 + v}{10 - v} = e^2$$

$$\implies 10 + v = e^2(10 - v)$$

$$\implies v(1 + e^2) = 10(e^2 - 1)$$

$$\implies v = \frac{10(e^2 - 1)}{e^2 + 1}$$

After 1 second, the speed of the stone is $\dfrac{10(e^2 - 1)}{e^2 + 1}$ m/s.

Subpart (iii) As $t \to \infty$, we have $\ln\left(\dfrac{10 + v}{10 - v}\right) \to \infty \implies \dfrac{10 + v}{10 - v} \to \infty$. Thus, $v \to 10^-$.

For large values of t, the speed of the stone approaches 10 m/s from below.

Problem 10.

Two scientists are investigating the change of a certain population of an animal species of size n thousand at time t years. It is known that due to its inability to reproduce effectively, the species is unable to replace itself in the long run.

(a) One scientist suggests that n and t are related by the differential equation $\dfrac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 10 - 6t$. Given that $n = 100$ when $t = 0$, show that the general solution of this differential equation is $n = 5t^2 - t^3 + Ct + 100$, where C is a constant. Sketch the solution curve of the particular solution when $C = 0$, stating the axial intercepts clearly.

(b) The other scientist suggests that n and t are related by the differential equation $\dfrac{\mathrm{d}n}{\mathrm{d}t} = 3 - 0.02n$. Find n in terms of t, given again that $n = 100$ when $t = 0$. Explain in simple terms what will eventually happen to the population using this model.

Which is a more appropriate model in modeling the population of the animal species?
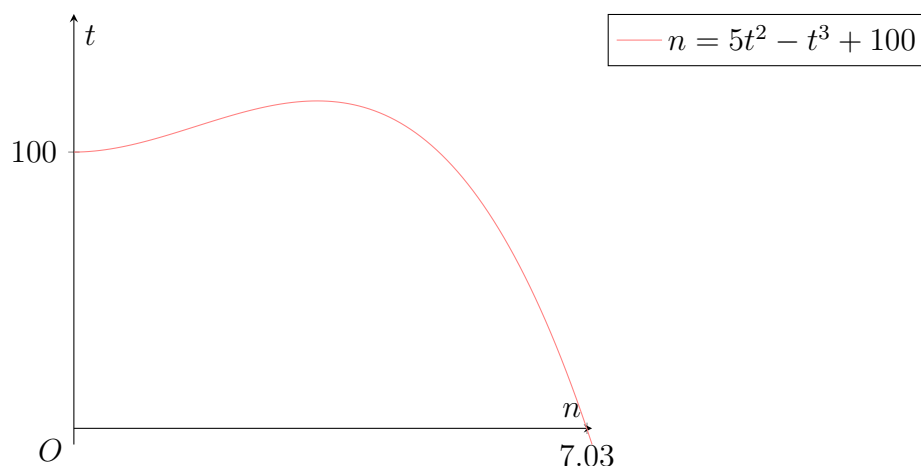
Solution

Part (a)

$$\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 10 - 6t$$
$$\implies \frac{\mathrm{d}n}{\mathrm{d}t} = \int (10 - 6t) \, \mathrm{d}\theta$$
$$= 10t - 3t^2 + C$$
$$\implies n = \int \left(10t - 3t^2 + C \right) \mathrm{d}t$$
$$= 5t^2 - t^3 + Ct + C'$$

When $t = 0$ and $n = 100$, we have $C' = 100$. Thus,

$$n = 5t^2 - t^3 + Ct + 100.$$

When $C = 0$, $n = 5t^2 - t^3 + 100$.

Part (b)

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 3 - 0.02n$$

$$= \frac{150 - n}{50}$$

$$\implies \frac{1}{150 - n}\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{1}{50}$$

$$\implies \int \frac{1}{150 - n}\frac{\mathrm{d}n}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{50}\int \mathrm{d}t$$

$$\implies \int \frac{1}{150 - n}\,\mathrm{d}n = \frac{1}{50}\int \mathrm{d}t$$

$$\implies -\ln(150 - n) = \frac{1}{50}t + C_1$$

$$\implies \ln(150 - n) = C_2 - \frac{1}{50}t$$

$$\implies 150 - n = Ce^{-t/50}$$

$$\implies n = 150 - Ce^{-t/50}$$

When $t = 0$ and $n = 100$, we have $C = 50$.

$$\boxed{n = 150 - 50e^{-t/50}}$$

As $t \to \infty$, $n \to 100$. Hence, the population will decrease before plateauing at 100 thousand.

The second model is more appropriate, as it account for the fact that the species will eventually go extinct ($n = 0$) due to the fact that they cannot replace itself in the long run.

Problem 11.

A rectangular tank has a horizontal base. Water is flowing into the tank at a constant rate, and flows out at a rate which is proportional to the depth of water in the tank. At time t seconds, the depth of water in the tank is x metres. If the depth is 0.5 m, it remains at this constant value. Show that $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -k(2x - 1)$, where k is a positive constant. When $t = 0$, the depth of water in the tank is 0.75 m and is decreasing at a rate of 0.01 m s^{-1}. Find the time at which the depth of water is 0.55 m.

Solution

Let k_i m/s be the rate at which water is flowing into the tank. Note that $k_i \geq 0$. Let the rate at which water is flowing out of the tank be $k_o x$ m/s. Then $\mathrm{d}x/\mathrm{d}t = k_i - k_o x$. At $x = 0.5$, the volume of water in the tank is constant, i.e. $\mathrm{d}x/\mathrm{d}t|_{0.5} = 0$. This gives $k_i - 0.5 k_o = 0$, whence $k_0 = 2k_i$. Thus, $\mathrm{d}x/\mathrm{d}t = k_i - 2k_i x = -k_i(2x - 1)$. Renaming k_i as k, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k(2x - 1)$$

as desired.

We now solve for t.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k(2x - 1)$$

$$\implies \quad \frac{1}{2x - 1}\frac{\mathrm{d}x}{\mathrm{d}t} = -k$$

$$\implies \quad \int \frac{1}{2x - 1}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = -k\int \mathrm{d}t$$

$$\implies \quad \int \frac{1}{2x - 1}\,\mathrm{d}x = -k\int \mathrm{d}t$$

$$\implies \frac{1}{2}\ln(2x - 1) + C_1 = -kt$$

$$\implies \quad \ln(2x - 1) + C_2 = -2kt$$

$$\implies \quad t = -\frac{1}{2k}\big(\ln(2x - 1) + C_2\big)$$

At $t = 0$, we have $x = 0.75$. This gives $0 = \ln(2 \cdot 0.75 - 1) + C_2$, whence $C_2 = \ln 2$. We also have $\mathrm{d}x/\mathrm{d}t|_0 = -0.01$. We thus obtain $-0.01 = -k(2 \cdot 0.75 - 1)$, whence $k = 0.02$. Thus,

$$t = -\frac{1}{0.04}\big(\ln(2x - 1) + \ln 2\big) = -25\ln(4x - 2).$$

Hence, when $x = 0.55$, we have $t = -25\ln 0.2 = 25\ln 5$.

The depth of the water is 0.55 m when $t = 25\ln 5$ s.

Problem 12.

In a model of mortgage repayment, the sum of money owed to the Building Society is denoted by x and the time is denoted by t. Both x and t are taken to be continuous variables. The sum of money owed to the Building Society increases, due to interest, at a rate proportional to the sum of money owed. Money is also repaid at a constant rate r.

When $x = a$, interest and repayment balance. Show that, for $x > 0$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{r}{a}(x - a)$.

Given that, when $t = 0$, $x = A$, find x in terms of t, r, a and A.

On a single, clearly labelled sketch, show the graph of x against t in the two cases:

  (a) $A > a$.

  (b) $A < a$.

State the circumstances under which the loan is repaid in a finite time T and show that, in this case, $T = \dfrac{a}{r} \ln \dfrac{a}{a - A}$.

Solution

Let the rate at which money is owned to the Building Society be kx. Then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = kx - r$. At $x = a$, interest and repayment balance, i.e. $\mathrm{d}x/\mathrm{d}t|_a = 0$. This gives $ka - r = 0 \implies k = r/a$. Thus,
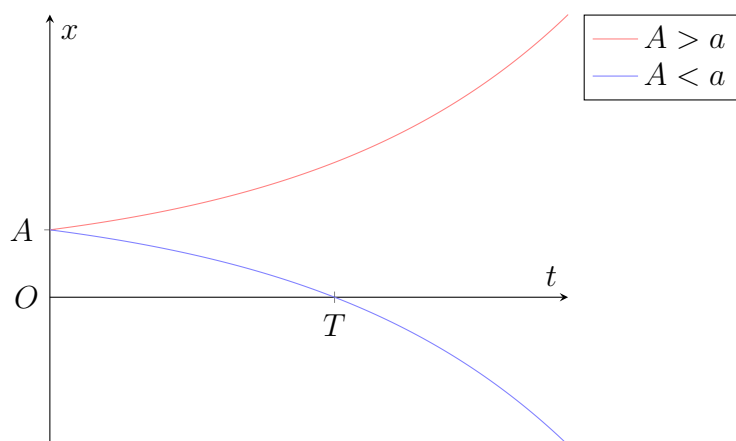
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r}{a}x - r = \frac{r}{a}(x - a).$$

We now solve for x.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r}{a}(x - a)$$
$$\implies \frac{1}{x - a}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r}{a}$$
$$\implies \int \frac{1}{x - a}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \frac{r}{a}\int \mathrm{d}t$$
$$\implies \int \frac{1}{x - a}\,\mathrm{d}x = \frac{r}{a}\int \mathrm{d}t$$
$$\implies \ln(x - a) = \frac{r}{a}t + C_1$$
$$\implies x - a = Ce^{rt/a}$$
$$\implies x = Ce^{rt/a} + a$$

When $t = 0$, we have $x = A$. This gives $A = C + a$, whence $C = A - a$. Thus,

$$\boxed{x = (A - a)e^{rt/a} + a}$$

For the loan to be repaid in finite time, $A < a$. At time T, the loan has been repaid, i.e. $x = 0$. Note that $C_1 = \ln C = \ln(A - a)$. Hence,

$$\frac{r}{a}T + \ln(A - a) = \ln(0 - a)$$

$$\implies \frac{r}{a}T = \ln a - \ln(A - a)$$

$$= \ln \frac{a}{A - a}$$

$$\implies T = \frac{a}{r} \ln \frac{a}{A - a}$$