Problem 1.

Given that $z = 3 - 2i$ and $w = 1 + 4i$, express in the form $a + bi$, where $a, b \in \mathbb{R}$:

(a) $z + 2w$

(b) zw

(c) z/w

(d) $(w - w^*)^3$

(e) z^4

Solution

Part (a)

$$\begin{aligned} z + 2w &= (3 - 2i) + 2(1 + 4i) \\ &= 3 - 2i + 2 + 8i \\ &= 5 + 6i \end{aligned}$$

$$\boxed{z + 2w = 5 + 6i}$$

Part (b)

$$\begin{aligned} zw &= (3 - 2i)(1 + 4i) \\ &= 3 + 12i - 2i + 8 \\ &= 11 + 10i \end{aligned}$$

$$\boxed{zw = 11 + 10i}$$

Part (c)

$$\begin{aligned} \frac{z}{w} &= \frac{3 - 2i}{1 + 4i} \\ &= \frac{(3 - 2i)(1 - 4i)}{(1 + 4i)(1 - 4i)} \\ &= \frac{3 - 12i - 2i - 8}{1^2 + 4^2} \\ &= \frac{-5 - 14i}{17} \\ &= -\frac{5}{17} - \frac{14}{17}i \end{aligned}$$

$$\boxed{\frac{z}{w} = -\frac{5}{17} - \frac{14}{17}i}$$

Part (d)

$$\begin{aligned}(w - w^*)^3 &= (2\,\mathrm{Im}\,(w)\,i)^3 \\ &= (8i)^3 \\ &= -512i\end{aligned}$$

$$\boxed{(w - w^*)^3 = -512i}$$

Part (e)

$$\begin{aligned}z^4 &= (3-2i)^4 \\ &= 3^4 + 4\cdot 3^3(-2i) + 6\cdot 3^2(-2i)^2 + 4\cdot 3(-2i)^3 + (-2i)^4 \\ &= 81 - 216i - 216 + 96i + 16 \\ &= -119 - 120i\end{aligned}$$

$$\boxed{z^4 = -119 - 120i}$$

Problem 2.

Is the following true or false in general?

(a) $\operatorname{Im}(zw) = \operatorname{Im}(z)\operatorname{Im}(w)$

(b) $\operatorname{Re}(zw) = \operatorname{Re}(z)\operatorname{Re}(w)$

Solution

Let $z = a + bi$ and $w = c + di$. Then $zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

Part (a)

$$\operatorname{Im}(zw) = ad + bc \neq bd = \operatorname{Im}(z)\operatorname{Im}(w)$$

$$\boxed{\text{The statement } \operatorname{Im}(zw) = \operatorname{Im}(z)\operatorname{Im}(w) \text{ is false in general.}}$$

Part (b)

$$\operatorname{Re}(zw) = ac - bd \neq ac = \operatorname{Re}(z)\operatorname{Re}(w)$$

$$\boxed{\text{The statement } \operatorname{Re}(zw) = \operatorname{Re}(z)\operatorname{Re}(w) \text{ is false in general.}}$$

Problem 3.

(a) Find the complex number z such that $\frac{z-2}{z} = 1 + i$.

(b) Given that $u = 2 + i$ and $v = -2 + 4i$, find in the form $a + bi$, where $a, b \in \mathbb{R}$, the complex number z such that $\frac{1}{z} = \frac{1}{u} + \frac{1}{v}$.

Solution

Part (a)

$$\begin{aligned} & \frac{z-2}{z} = 1 + i \\ \implies & z - 2 = z + iz \\ \implies & iz = -2 \\ \implies & z = -\frac{2}{i} \\ & \quad = 2i \end{aligned}$$

$$\boxed{z = 2i}$$

Part (b)

$$\begin{aligned} \frac{1}{z} &= \frac{1}{2+i} + \frac{1}{-2+4i} \\ &= \frac{2-i}{2^2+1^2} + \frac{-2-4i}{2^2+4^2} \\ &= \frac{8-4i}{20} + \frac{-2-4i}{20} \\ &= \frac{6-8i}{20} \\ &= \frac{3-4i}{10} \\ \implies z &= \frac{10}{3-4i} \\ &= 10 \cdot \frac{3+4i}{3^2+4^2} \\ &= \frac{6}{5} + \frac{8}{5}i \end{aligned}$$

$$\boxed{z = \frac{6}{5} + \frac{8}{5}i}$$

Problem 4.

The complex numbers z and w are $1 + ai$ and $b - 2i$ respectively, where a and b are real and a is negative. Given that $zw^* = 8i$, find the exact values of a and b.

Solution

$$\begin{aligned} && zw^* &= 8i \\ \implies && (1+ai)(b+2i) &= 8i \\ \implies && b + 2i + abi - 2a &= 8i \\ \implies && (b-2a) + (-6+ab)i &= 0 \end{aligned}$$

Comparing real parts, we have $b - 2a = 0$, whence $b = 2a$. Comparing imaginary parts, we have $-6 + ab = 0$, whence $2a^2 = 6 \implies a = -\sqrt{3} \implies b = -2\sqrt{3}$.

$$\boxed{a = -\sqrt{3},\ b = -2\sqrt{3}}$$

Problem 5.

Find, in the form $x + iy$, the two complex numbers z satisfying both of the equations

$$\frac{z}{z^*} = \frac{3}{5} + \frac{4}{5}i \quad \text{and} \quad zz^* = 5.$$

Solution

Multiplying both equations together, we have $z^2 = 3 + 4i$. Let $z = x + iy$, with $x, y \in \mathbb{R}$. We thus have $z^2 = x^2 - y^2 + 2ixy = 3 + 4i$. Comparing real and imaginary parts, we obtain the following system:

$$\begin{cases} x^2 - y^2 = 3 \\ 2xy = 4 \end{cases}$$

Squaring the second equation yields $x^2y^2 = 4$. From the first equation, we have $x^2 = 3+y^2$. Thus, $y^2(3 + y^2) = 4 \implies y^2 = 1 \implies y = \pm 1 \implies x = \pm 2$. Hence, $z = 2 + i$ or $z = -2 - i$.

$$\boxed{z = 2 + i \vee -2 - i}$$

Problem 6.

(a) Given that $iw + 3z = 2 + 4i$ and $w + (1 - i)z = 2 - i$, find z and w in the form of $x + iy$, where x and y are real numbers.

(b) Determine the value of k such that $z = \dfrac{1 - ki}{\sqrt{3} + i}$ is purely imaginary, where $k \in \mathbb{R}$.

Solution

Part (a)

Let $w = a + bi$ and $z = c + di$. From the first equation, we have

$$\begin{aligned} && iw + 3z &= 2 + 4i \\ \implies && i(a + bi) + 3(c + di) &= 2 + 4i \\ \implies && ai - b + 3c + 3di &= 2 + 4i \\ \implies && (-b + 3c) + (a + 3d)i &= 2 + 4i \end{aligned}$$

From the second equation, we have

$$\begin{aligned} && w + (1 - i)z &= 2 - i \\ \implies && a + bi + (1 - i)(c + di) &= 2 - i \\ \implies && a + bi + c + di - ci + d &= 2 - i \\ \implies && (a + c + d) + (b - c + d)i &= 2 - i \end{aligned}$$

Comparing real and imaginary parts from the two resultant equations, we have the following system:

$$\begin{cases} -b + 3c = 2 \\ a + 3d = 4 \\ a + c + d = 2 \\ b - c + d = -1 \end{cases}$$

which has the unique solution $a = 1$, $b = -2$, $c = 0$ and $d = 1$. Hence, $w = 1 - 2i$ and $z = i$.

$$\boxed{w = 1 - 2i,\ z = i}$$

Part (b)

$$\begin{aligned} z &= \frac{1 - ki}{\sqrt{3} + i} \\ &= \frac{(1 - ki)(\sqrt{3} - i)}{\sqrt{3}^2 + 1^2} \\ &= \frac{1}{4}(\sqrt{3} - i - k\sqrt{3}i - k) \\ &= \frac{1}{4}\left[(\sqrt{3} - k) - (1 + k\sqrt{3})i\right] \end{aligned}$$

Since z is purely imaginary, $\operatorname{Re}(z) = 0$. Hence, $\sqrt{3} - k = 0 \implies k = \sqrt{3}$.

$$\boxed{k = \sqrt{3}}$$

Problem 7.

(a) The complex number $x + iy$ is such that $(x + iy)^2 = i$. Find the possible values of the real numbers x and y, giving your answers in exact form.

(b) Hence, find the possible values of the complex number w such that $w^2 = -i$.

Solution

Part (a)

Note that $(x + iy)^2 = x^2 - y^2 + 2xyi = i$. Comparing real and imaginary parts, we have

$$\begin{cases} x^2 - y^2 = 0 \\ 2xy = 1 \end{cases}$$

Note that the second equation implies that both x and y have the same sign. Hence, from the first equation, we have $x = y$. Thus, $x^2 = 1 \implies x = y = \pm\frac{1}{\sqrt{2}}$.

$$\boxed{x = y = \pm\frac{1}{\sqrt{2}}}$$

Part (b)

$$\begin{aligned} & w^2 = -i \\ \implies & (w^*)^2 = i \\ \implies & w^* = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i \\ \implies & w = \pm\frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}i \end{aligned}$$

$$\boxed{w = \pm\frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}i}$$

Problem 8.

(a) The roots of the equation $z^2 = -8i$ are z_1 and z_2. Find z_1 and z_2 in Cartesian form $x + iy$, showing your working.

(b) Hence, or otherwise, find in Cartesian form the roots w_1 and w_2 of the equation $w^2 + 4w + (4 + 2i) = 0$.

Solution

Part (a)

Let $z = x + iy$ where $x, y \in \mathbb{R}$. Then $(x + iy)^2 = x^2 - y^2 + 2xyi = -8i$. Comparing real and imaginary parts, we have the following system:

$$\begin{cases} x^2 - y^2 = 0 \\ 2xy = 8 \end{cases}$$

From the second equation, we know that x and y have opposite signs. Hence, from the first equation, we have that $x = -y$. Thus, $x^2 = 4 \implies x = \pm 2 \implies y = \mp 2$. Thus, $z = \pm 2(1 - i)$, whence $z_1 = 2 - 2i$ and $z_2 = -2 + 2i$.

$$\boxed{z_1 = 2 - 2i,\ z_2 = -2 + 2i}$$

Part (b)

$$\begin{aligned} & w^2 + 4w + (4 + 2i) = 0 \\ \implies \quad & (w + 2)^2 = -2i \\ \implies \quad & (2w + 4)^2 = -8i \\ \implies \quad & 2w + 4 = \pm 2(1 - i) \\ \implies \quad & w + 2 = \pm(1 - i) \\ \implies \quad & w = 2 \pm (1 - i) \end{aligned}$$

$$\boxed{w_1 = 3 - i,\ w_2 = -1 - i}$$

Problem 9.

One of the roots of the equations $2x^3 - 9x^2 + 2x + 30 = 0$ is $3 + i$. Find the other roots of the equation.

Solution

Let $P(x) = 2x^3 - 9x^2 + 2x + 30$. Since $P(x)$ is a polynomial with real coefficients, and $P(3 + i) = 0$, we have that $(3 + i)^* = 3 - i$ is a root of $P(x)$. Let a be the real root of $P(x)$. We hence have

$$P(x) = 2x^3 - 9x^2 + 2x + 30 = 2(x - a)\left[x - (3 + i)\right]\left[x - (3 - i)\right]$$

Comparing constants,

$$\begin{aligned} & 2 \cdot -a \cdot -(3+i) \cdot -(3-i) = 30 \\ \implies & 2 \cdot -a \cdot (3^2 + 1^2) = 30 \\ \implies & a = -\frac{30}{2 \cdot 10} \\ & \quad = -\frac{3}{2} \end{aligned}$$

$$\boxed{\text{The other roots are } 3 - i \text{ and } -\frac{3}{2}.}$$

Problem 10.

Obtain a cubic equation having 2 and $\frac{5}{4} - \frac{\sqrt{7}}{4}i$ as two of its roots, in the form $az^3 + bz^2 + cz + d = 0$, where a, b, c and d are real integral coefficients to be determined.

Solution

Let $P(z) = az^3 + bz^2 + cz + d$. Since $P(z)$ is a polynomial with real coefficients, and $P\left(\frac{5}{4} - \frac{\sqrt{7}}{4}i\right) = 0$, we have that $\left(\frac{5}{4} - \frac{\sqrt{7}}{4}i\right)^* = \frac{5}{4} + \frac{\sqrt{7}}{4}i$ is also a root of $P(z)$. Hence,

$$\begin{aligned}
P(z) &= k(z-2)\left[z - \left(\frac{5}{4} - \frac{\sqrt{7}}{4}i\right)\right]\left[z - \left(\frac{5}{4} + \frac{\sqrt{7}}{4}i\right)\right] \\
&= k(z-2)\left[\left(z - \frac{5}{4}\right) + \frac{\sqrt{7}}{4}i\right]\left[\left(z - \frac{5}{4}\right) - \frac{\sqrt{7}}{4}i\right] \\
&= k(z-2)\left[\left(z - \frac{5}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right] \\
&= k(z-2)\left(z^2 - \frac{5}{2}z + \frac{25}{16} + \frac{7}{16}\right) \\
&= k(z-2)\left(z^2 - \frac{5}{2}z + 2\right) \\
&= 2k(z-2)(2z^2 - 5z + 4) \\
&= 2k(2z^3 - 5z^2 + 4z - 4z^2 + 10z - 8) \\
&= 2k(2z^3 - 9z^2 + 14z - 8)
\end{aligned}$$

Taking $k = \frac{1}{2}$, we have $P(z) = 2z^3 - 9z^2 + 14z - 8$, whence $a = 2$, $b = -9$, $c = 14$ and $d = -8$.

$$\boxed{2z^3 - 9z^2 + 14z - 8 = 0}$$

Problem 11.

(a) Verify that $-1+5i$ is a root of the equation $w^2+(-1-8i)w+(-17+7i)=0$. Hence, or otherwise, find the second root of the equation in Cartesian form, $p+iq$, showing your working.

(b) The equation $z^3-5z^2+16z+k=0$, where k is a real constant, has a root $z=1+ai$, where a is a positive real constant. Find the values of a and k, showing your working.

Solution

Part (a)

Let $P(w)=w^2+(-1-8i)w+(-17+7i)$. Consider $P(-1+5i)$.

$$\begin{aligned} P(-1+5i) &= (-1+5i)^2+(-1-8i)(-1+5i)+(-17+7i) \\ &= (1-10i-25)+(1-5i+8i+40)+(-17+7i) \\ &= (1-25+1+40-17)+(-10-5+8+7)i \\ &= 0 \end{aligned}$$

Hence, $-1+5i$ is a root of $w^2+(-1-8i)w+(-17+7i)=0$.

Let α be the other root of the equation. By Vieta's formula, we have

$$\begin{aligned} & & \alpha+(-1+5i) &= -\frac{-1-8i}{1} \\ & & &= 1+8i \\ \implies & & \alpha &= 2+3i \end{aligned}$$

$$\boxed{\text{The second root of the equation is } 2+3i.}$$

Part (b)

Let $P(z)=z^3-5z^2+16z+k$. Then $P(1+ai)=0$.

$$\begin{aligned} & & P(1+ai) &= 0 \\ \implies & & (1+ai)^3-5(1+ai)^2+16(1+ai)+k &= 0 \\ \implies & & \left[1+3ai+3(ai)^2+(ai)^3\right]-5(1+2ai-a^2)+(16+16ai)+k &= 0 \\ \implies & & 1+3ai-3a^2-a^3i-5-10ai+5a^2+16+16ai+k &= 0 \\ \implies & & (12+k+2a^2)+(9-a^2)ai &= 0 \end{aligned}$$

Comparing real and imaginary parts, we have $9-a^2=0 \implies a=3$ and $12+k+2a^2=0 \implies k=-30$.

$$\boxed{a=3,\ k=-30}$$