# Problem 1.

Without using a calculator, sketch the following graphs and determine their symmetries.

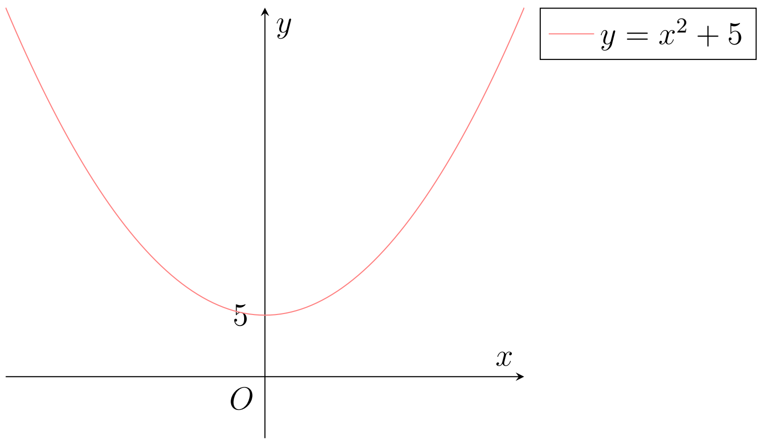
(a) $y = x^2 + 5$

(b) $y = 2x - x^3$

(c) $y = x^2 - 4x + 3$

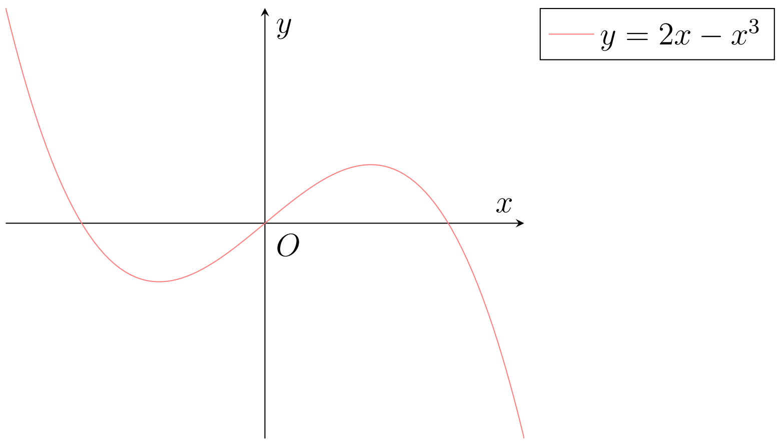
## Solution

### Part (a)



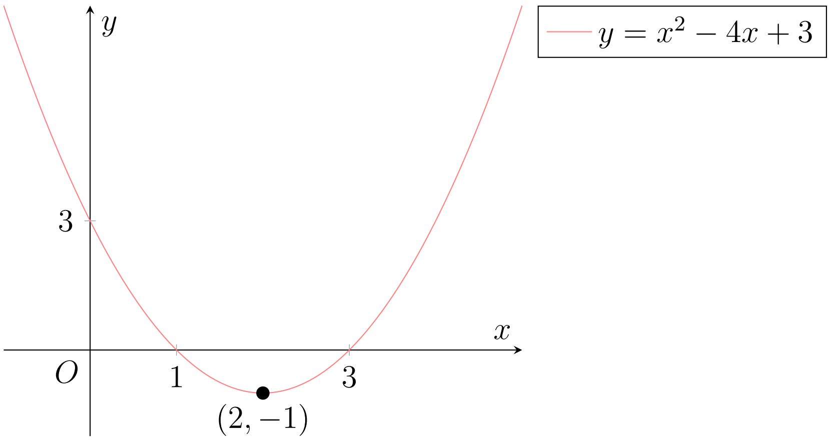

$\boxed{\text{Symmetry: } x = 0}$

### Part (b)




$\boxed{\text{Symmetry: } (0, 0)}$

**Part (c)**




$\boxed{\text{Symmetry: } x = 2}$

# Problem 2.

Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

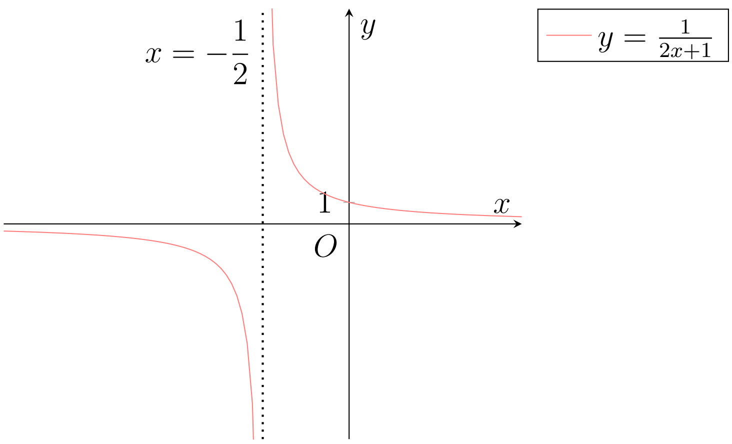
(a) $y = \dfrac{1}{2x+1}$

(b) $y = \dfrac{3x}{x-2}$

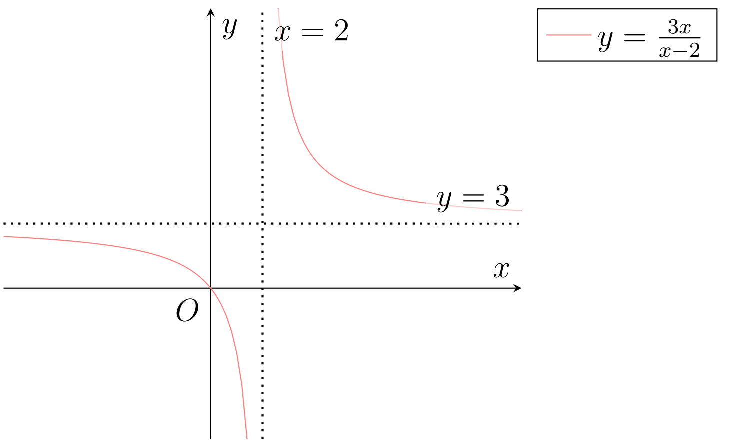
(c) $y = \dfrac{x^2+x-6}{2x-2}$

## Solution

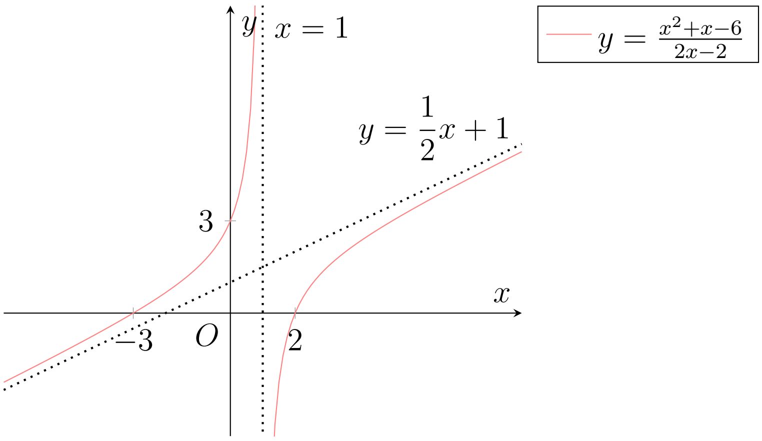
### Part (a)




### Part (b)



**Part (c)**



# Problem 3.

Sketch the following graphs

(a) $x^2 + 2x + 2y + 4 = 0$

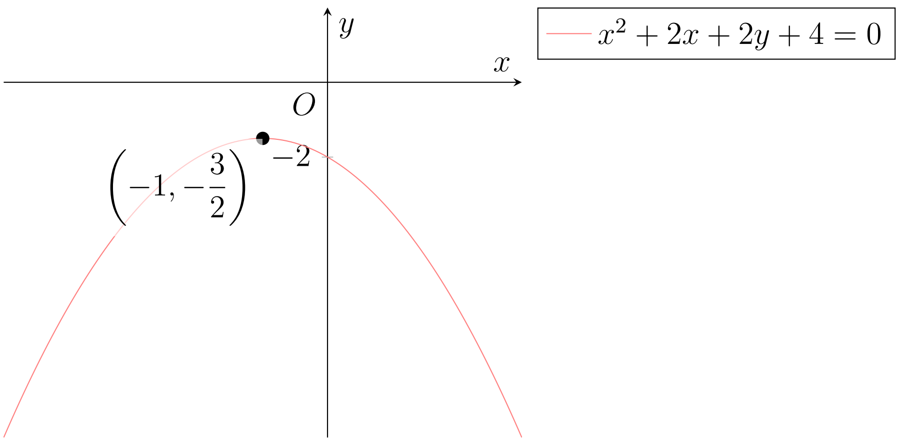
(b) $y^2 = x - 9$

(c) $y^2 = (x-2)^4 + 5$

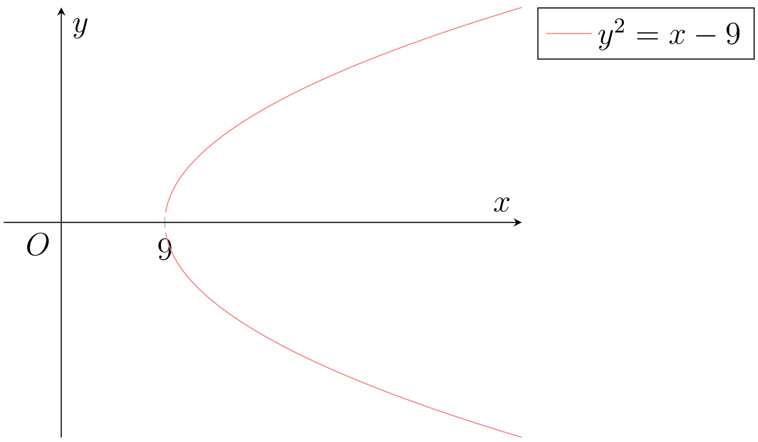
(d) $y = \tan\left(\dfrac{1}{2}x\right)$, $-2\pi \leq x \leq 2\pi$

## Solution

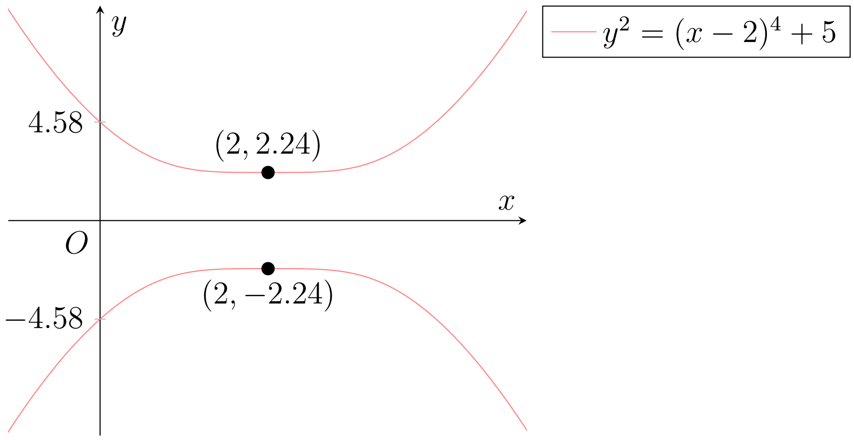
### Part (a)



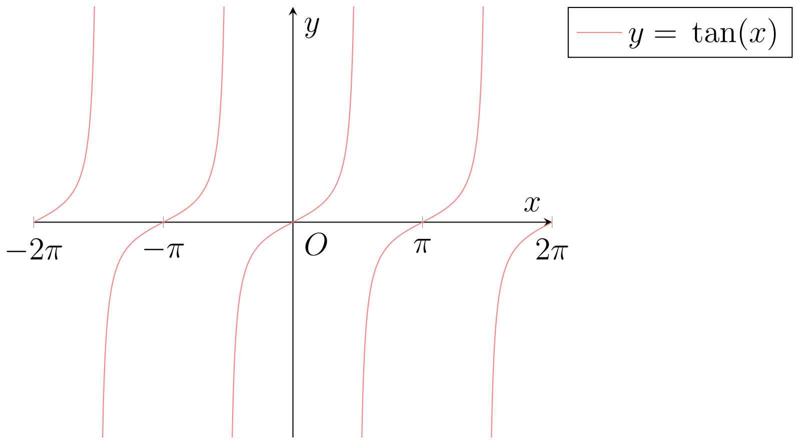

### Part (b)



**Part (c)**




**Part (d)**



# Problem 4.

Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

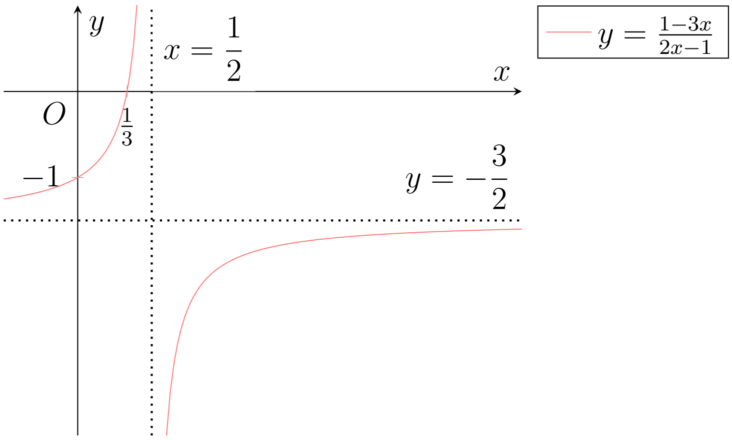
(a) $y = \dfrac{1-3x}{2x-1}$

(b) $y = \dfrac{ax}{x-a}$, $a < 0$

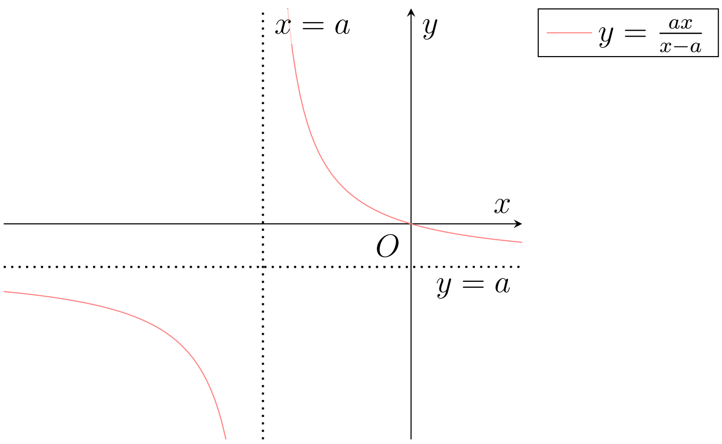
(c) $y = -\dfrac{b(x+3a)}{x+a}$, $a, b > 0$

## Solution

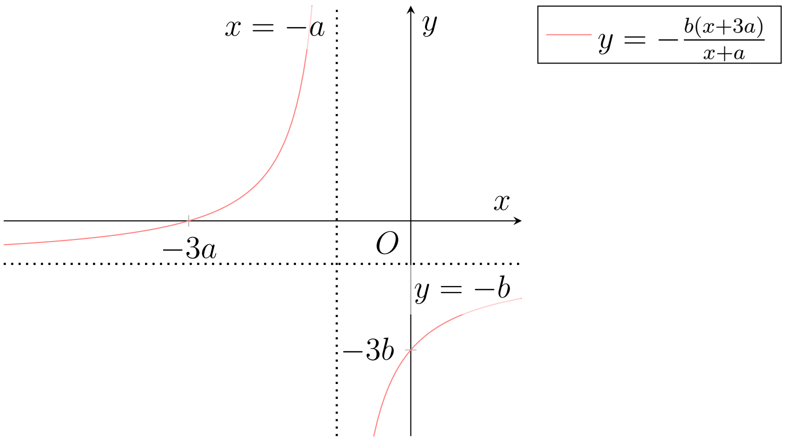
### Part (a)




### Part (b)



**Part (c)**



# Problem 5.

Sketch the following curves and find the coordinates of any turning points on the curves.

(a) $y = x + 2\sin x$, $0 \leq x \leq 2\pi$

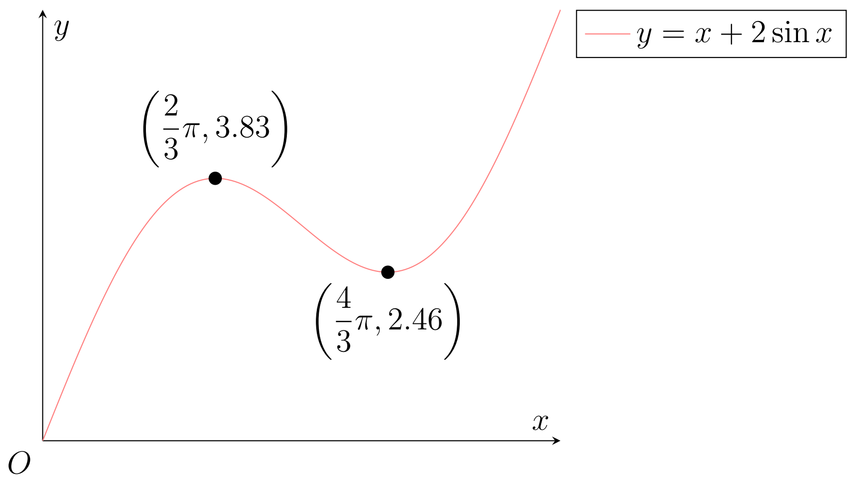
(b) $y = \dfrac{x}{\ln x}$, $x > 0$, $x \neq 1$

(c) $y = xe^{-x}$

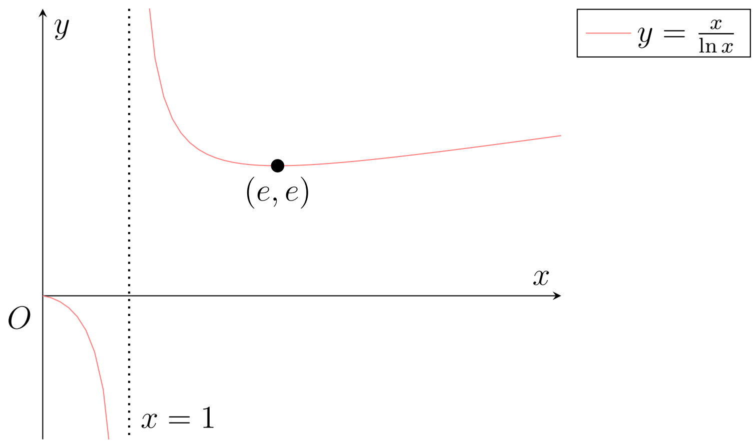
(d) $y = xe^{-x^2}$

## Solution

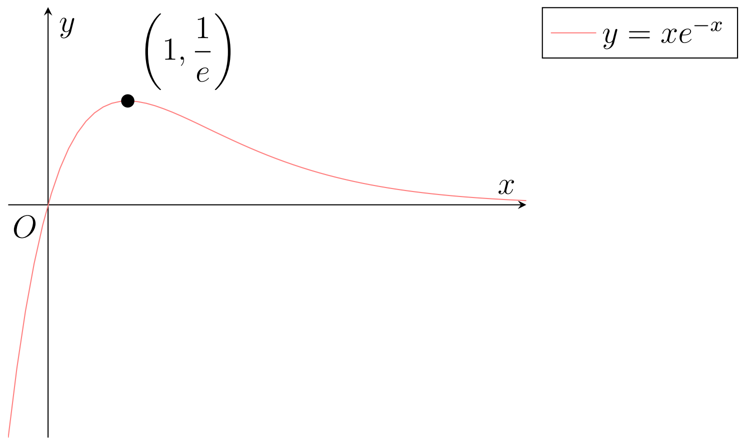
### Part (a)



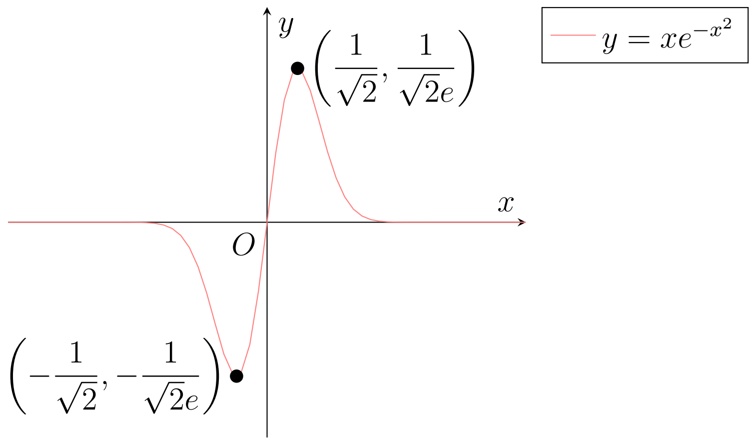

### Part (b)



**Part (c)**


$y$
$\left(1, \frac{1}{e}\right)$
$y = xe^{-x}$
$x$
$O$

**Part (d)**


$y$
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}e}\right)$
$y = xe^{-x^2}$
$x$
$O$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}e}\right)$

# Problem 6.

The equation of a curve $C$ is $y = 1 + \dfrac{6}{x-3} - \dfrac{24}{x+3}$.

(a) Explain why $y = 1$ and $x = 3$ are asymptotes to the curve.

(b) Find the coordinates of the points where $C$ meets the axes.

(c) Sketch $C$.

## Solution

### Part (a)

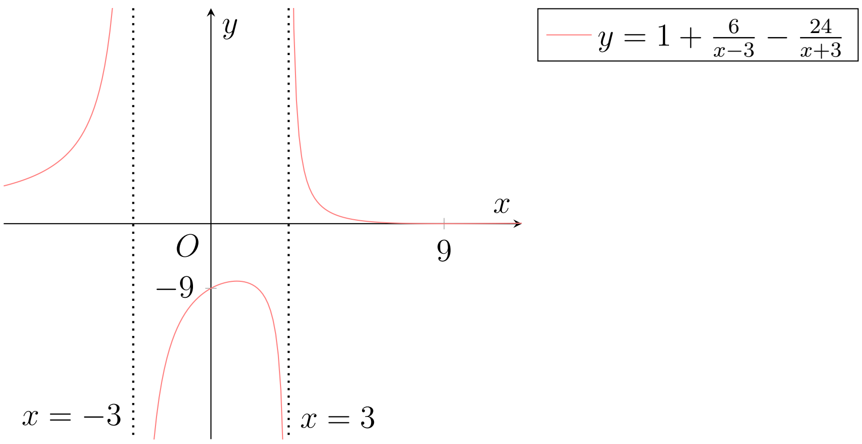
As $x \to \pm\infty$, $y \to 1$. Hence, $y = 1$ is an asymptote to $C$. As $x \to 3^{\pm}$, $y \to \pm\infty$. Hence, $x = 3$ is an asymptote to $C$.

### Part (b)

When $x = 0$, $y = -9$. When $y = 0$, $x = 9$.

$$\boxed{C \text{ meets the axes at } (0, -9) \text{ and } (9, 0).}$$

### Part (c)



# Problem 7.

The curve $C$ has equation $y = \dfrac{ax^2 + bx}{x + 2}$, where $x \neq -2$. It is given that $C$ has an asymptote $y = 1 - 2x$.

(a) Show (do not verify) that $a = -2$ and $b = -3$.

(b) Using an algebraic method, find the set of values that $y$ can take.

(c) Sketch $C$, showing clearly the positions of any axial intercept(s), asymptote(s) and stationary point(s).

(d) Deduce that the equation $x^4 + 2x^3 + 2x^2 + 3x = 0$ has exactly one real non-zero root.

## Solution

### Part (a)

$$\begin{aligned} y &= \frac{ax^2 + bx}{x + 2} \\ &= \frac{(ax + b - 2a)(x + 2) - 2(b - 2a)}{x + 2} \\ &= ax + b - 2a - \frac{2(b - 2a)}{x + 2} \end{aligned}$$

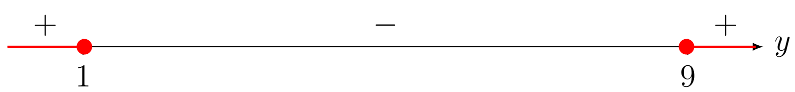
Since $C$ has an asymptote $y = 1 - 2x$, we have $a = -2$ and $b - 2a = 1$, whence $b = -3$.

### Part (b)

$$\begin{aligned} && y &= \frac{-2x^2 + -3x}{x + 2} \\ \implies && y(x + 2) &= -2x^2 - 3x \\ \implies && 2x^2 + (3 + y)x + 2y &= 0 \end{aligned}$$

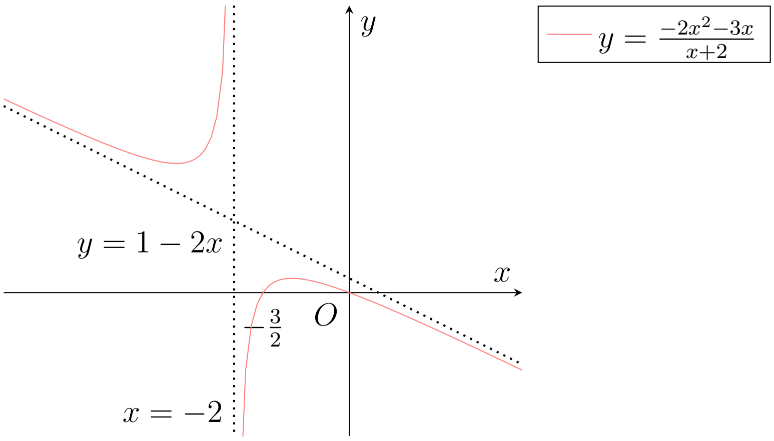
For all values that $y$ can take on, there exists a solution to $2x^2 + (3 + y)x + 2y = 0$. Hence, $\Delta \geq 0$.

$$\begin{aligned} && (3 + y)^2 - 4(2)(2y) &\geq 0 \\ \implies && 9 + 6y + y^2 - 16y &\geq 0 \\ \implies && y^2 - 10y + 9 &\geq 0 \\ \implies && (y - 1)(y - 9) &\geq 0 \end{aligned}$$




$$\boxed{\{y \in \mathbb{R} \colon y \leq 1 \lor y \geq 9\}}$$

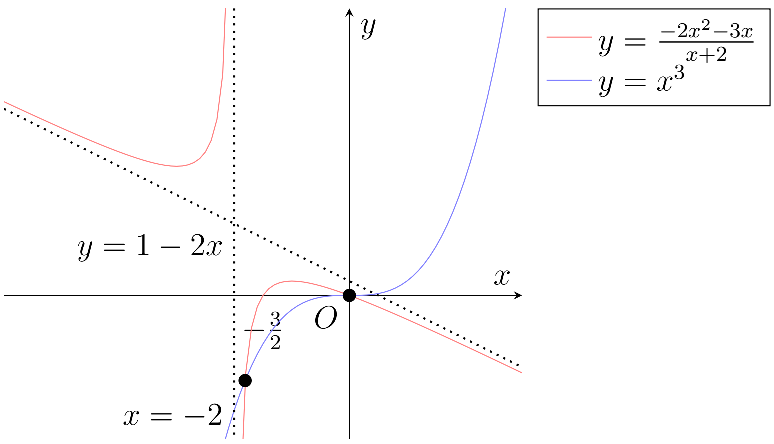
**Part (c)**




**Part (d)**

$$\begin{aligned} & x^4 + 2x^3 + 2x^2 + 3x = 0 \\ \implies \quad & x^4 + 2x^3 = -2x^2 - 3x \\ \implies \quad & x^3(x+2) = -2x^2 - 3x \\ \implies \quad & x^3 = \frac{-2x^2 - 3x}{x+2} \end{aligned}$$

This motivates us to plot $y = x^3$ and $y = \dfrac{-2x^2 - 3x}{x+2}$ on the same graph.




We thus see that $y = x^3$ intersects $y = \dfrac{-2x^2 - 3x}{x+2}$ twice, with one intersection point being the origin. Thus, there is only one real non-zero root to $x^4 + 2x^3 + 2x^2 + 3x = 0$.

# Problem 8.

The curve $C$ is defined by the equation $y = \dfrac{x}{x^2 - 5x + 4}$.

(a) Write down the equations of the asymptotes.

(b) Sketch $C$, indicating clearly the axial intercept(s), asymptote(s) and turning point(s).

(c) Find the positive value $k$ such that the equation $\dfrac{x}{x^2 - 5x + 4} = kx$ has exactly 2 distinct real roots.

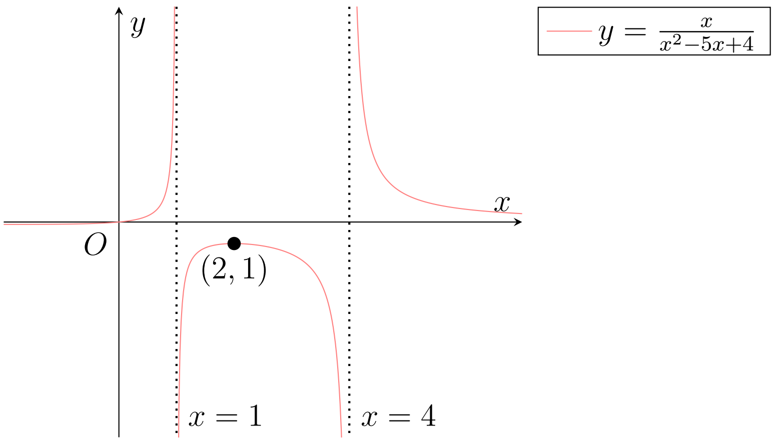
## Solution

### Part (a)

As $x \to \pm\infty$, $y \to 0$. Hence, $y = 0$ is an asymptote. Observe that $x^2 - 5x + 4 = (x-1)(x-4)$. Hence, $x = 1$ and $x = 4$ are asymptotes.

$$\boxed{\text{Asymptotes: } y = 0,\ x = 1,\ x = 4}$$

### Part (b)




### Part (c)

Note that $x = 0$ is always a root of $\dfrac{x}{x^2 - 5x + 4} = kx$. We thus aim to find the value of $k$ such that $\dfrac{x}{x^2 - 5x + 4} = kx$ has only one non-zero root.

We observe that if $k > 0$, $y = kx$ will intersect with $y = \dfrac{x}{x^2 - 5x + 4}$ at least twice: before $x = 1$ and after $x = 4$. In order to have only one non-zero root, we must force the intersection point that comes before $x = 1$ to be at the origin $(0, 0)$. Hence, $k$ is tangential to $C$ at $(0, 0)$, thus giving $k = \left.\dfrac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=0}$.

$$
\begin{aligned}
k &= \left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=0} \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{x^2-5x+4}\right|_{x=0} \\
&= \left.\frac{3x^2-10x+4}{(x^2-5x+4)^2}\right|_{x=0} \\
&= \frac{1}{4}
\end{aligned}
$$

$$\boxed{k = \frac{1}{4}}$$