Problem 1.

Show that

$$\sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) = \frac{a}{b}\left(729 \cdot 9^{(n-1)^2} - 1\right) - c(n-1)^3 - d(n-1)$$

where a, b, c and d are constants to be determined.

Solution

$$\begin{aligned}\sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) &= \sum_{r=1}^{(n-1)^2+3} 9^r - \sum_{r=1}^{(n-1)^2+3} (n-1) \\ &= \frac{9\left(9^{(n-1)^2+3} - 1\right)}{9-1} - (n-1)\left((n-1)^2 + 3\right) \\ &= \frac{9}{8}\left(729 \cdot 9^{(n-1)^2} - 1\right) - (n-1)^3 - 3(n-1)\end{aligned}$$

Problem 2.

Do not use a calculator in answering this question.

The sequence of positive numbers, u_n, satisfies the recurrence relation:

$$u_{n+1} = \sqrt{2u_n + 3}, \qquad n = 1, 2, 3, \ldots$$

(a) If the sequence converges to m, find the value of m.

(b) By using a graphical approach, explain why $m < u_{n_1} < u_n$ when $u_n > u_m$. Hence, determine the behaviour of the sequence when $u_1 > m$.

Solution

Part (a)

Observe that

$$\begin{aligned}\lim_{n\to\infty} u_n &= \lim_{n\to\infty} \sqrt{2u_{n-1} + 3}\\ &= \sqrt{2\lim_{n\to\infty} u_{n-1} + 3}\\ &= \sqrt{2\lim_{n\to\infty} u_n + 3}\end{aligned}$$

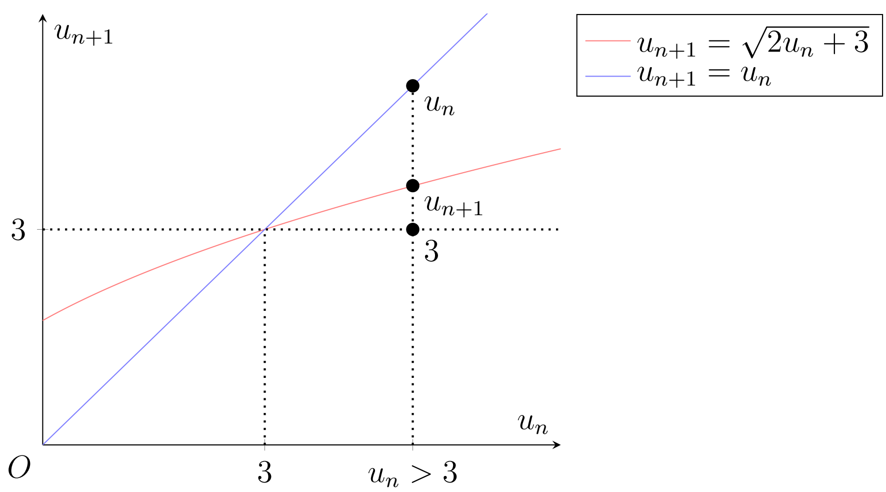
Since the sequence converges to m, we have $\lim_{n\to\infty} u_n = m$. Thus,

$$\begin{aligned} && m &= \sqrt{2m+3}\\ \implies && m^2 &= 2m+3\\ \implies && m^2 - 2m - 3 &= 0\\ \implies && (m-3)(m+1) &= 0\end{aligned}$$

Thus, $m = 3$ or $m = -1$. Since u_n is always positive, we take $m = 3$.

$$\boxed{m = 3}$$

Part (b)



From the graph, if $u_n > 3$, then $3 < u_{n+1} < u_n$.

$$\boxed{\text{The sequence decreases and converges to 3.}}$$

Problem 3.

Two expedition teams are to climb a vertical distance of 8100 m from the foot to the peak of a mountain. Team A plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Team B plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day.

(a) Find the number of days required for Team A to reach the peak.

(b) Explain why Team B will never be able to reach the peak.

(c) At the end of the 15th day, Team B decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer.

Solution

Part (a)

The vertical distance Team A plans to cover in a day can be described as a sequence in arithmetic progression with first term 400 and common difference -5. In order to reach the peak, the total vertical distance covered by Team A has to be greater than 8100 m. Hence,

$$\frac{n}{2}\left(2(400)+(n-1)(-5)\right) \geq 8100$$

From the graphing calculator, $n \geq 24$. Hence, Team A requires 24 days to reach the peak.

$$\boxed{\text{24 days.}}$$

Part (b)

The vertical distance Team B plans to cover in the nth day can be described by the sequence U_n in geometric progression with first term 800 and common ratio $r = 0.9$. Let S_n^U be the nth partial sum of U_n. Since $|r| < 1$, the sum to infinity of exists and is equal to

$$S_\infty^U = \frac{800}{1-0.9} = 8000$$

Hence, Team B will never be able to surpass 8 km in height. Thus, they will not reach the peak no matter how long they take.

Part (c)

The new vertical distance covered by Team B after Day 15 can be described by the sequence V_n in geometric progression with first term U_{15} and common ratio $r = 0.95$. Let S_n^V be the nth partial sum of V_n. Then,

$$S_n^V = \frac{U_{15} \cdot 0.95\left(1-(0.95)^n\right)}{1-0.95}$$

Note that

$$S_n^U = \frac{800\left(1-(0.9)^n\right)}{1-0.9}$$

Hence, after Day 15, Team B has to climb another $8000 - S_{15}^U = 1747.13$ metres. Since $U_{15} = 183.01$, we have the inequality

$$\frac{183.01 \cdot 0.95\left(1-(0.95)^n\right)}{1-0.95} \geq 1747.13$$

Using the graphing calculator, $n \geq 14$. Hence, Team B will need at least $15 + 14 = 29$ days to reach the peak.

$$\boxed{\text{Team } A \text{ will reach the peak first.}}$$

Problem 4.

The function f is given by $f(x) = x^2 - 3x + 2 - e^{-x}$. It is known from graphical work that this equation has 2 roots $x = \alpha$ and $x = \beta$, where $\alpha < \beta$.

(a) Show that $f(x) = 0$ has at least one root in the interval $[0, 1]$.

It is known that there is exactly one root in $[0, 1]$ where $x = \alpha$.

(b) Starting with $x_0 = 0.5$, use an iterative method based on the form

$$x_{n+1} = p\left(x_n^2 + q - e^{-x_n}\right)$$

where p and q are real numbers to be determined, to find the value of α correct to 3 decimal places. You should demonstrate that your answer has the required accuracy.

It is known that the other root $x = \beta$ lies in the interval $[2, 3]$.

(c) With the aid of a clearly labelled diagram, explain why the method in (b) will fail to obtain any reasonable approximation to β, where x_0 is chosen such that $x_0 \in [2, 3]$, $x_0 \neq \beta$.

To obtain an approximation to β, another approach is used.

(d) Use linear interpolation once in the interval $[2, 3]$ to find a first approximation to β, giving your answer to 2 decimal places. Explain whether this approximate is an overestimate or underestimate.

(e) With your answer in (d) as the initial approximate, use the Newton-Raphson method to obtain β correct to 3 decimal places. Your process should terminate when you have two successive iterates that are equal when rounded to 3 decimal places.

Solution

Part (a)

Observe that $f(0) = 1 > 0$ and $f(1) = -e^{-1} < 0$. Since f is continuous and $f(0)f(1) < 0$, there must be at least one root to $f(x) = 0$ in the interval $[0, 1]$.

Part (b)

Let $f(x) = 0$. Then,

$$\begin{aligned} & x^2 - 3x + 2 - e^{-x} = 0 \\ \implies \quad & x^2 + 2 - e^{-x} = 3x \\ \implies \quad & x = \frac{1}{3}\left(x^2 + 2 - e^{-x}\right) \end{aligned}$$

Hence, we should use an iterative method based on the form

$$x_{n+1} = \frac{1}{3}\left(x_n^2 + 2 - e^{-x_n}\right)$$

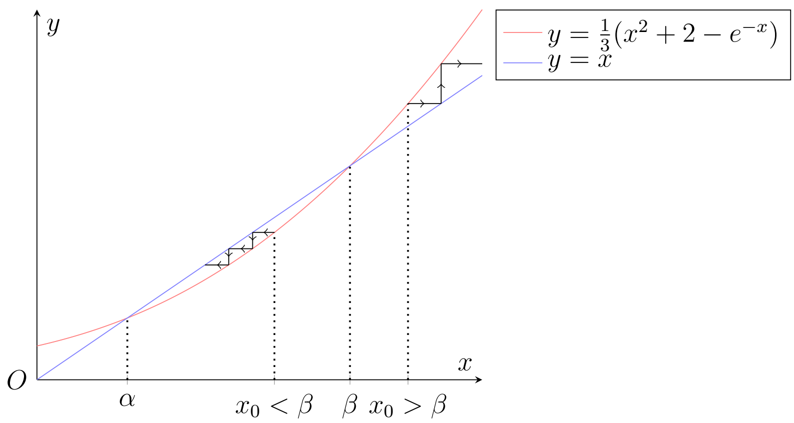
Starting with $x_0 = 0.5$,

$$\begin{aligned}
x_1 &= \frac{1}{3}\left(x_0^2 + 2 - e^{-x_0}\right) &= 0.54782 \\
\implies x_2 &= \frac{1}{3}\left(x_1^2 + 2 - e^{-x_1}\right) &= 0.57396 \\
\implies x_3 &= \frac{1}{3}\left(x_2^2 + 2 - e^{-x_2}\right) &= 0.58871 \\
\implies x_4 &= \frac{1}{3}\left(x_3^2 + 2 - e^{-x_3}\right) &= 0.59718 \\
\implies x_5 &= \frac{1}{3}\left(x_4^2 + 2 - e^{-x_4}\right) &= 0.60208 \\
\implies x_6 &= \frac{1}{3}\left(x_5^2 + 2 - e^{-x_5}\right) &= 0.60494 \\
\implies x_7 &= \frac{1}{3}\left(x_6^2 + 2 - e^{-x_6}\right) &= 0.60662 \\
\implies x_8 &= \frac{1}{3}\left(x_7^2 + 2 - e^{-x_7}\right) &= 0.60759 \\
\implies x_9 &= \frac{1}{3}\left(x_8^2 + 2 - e^{-x_8}\right) &= 0.60817 \\
\implies x_{10} &= \frac{1}{3}\left(x_9^2 + 2 - e^{-x_9}\right) &= 0.60851 \\
\implies x_{11} &= \frac{1}{3}\left(x_{10}^2 + 2 - e^{-x_{10}}\right) &= 0.60870
\end{aligned}$$

Since $f(0.6085) = 0.000606 > 0$ and $f(0.6095) = -0.000632 < 0$, we have that $\alpha \in (0.6085, 0.6095)$. Hence,

$$\boxed{\alpha = 0.609 \text{ (3 d.p.)}}$$

Part (c)



From the diagram, we see that whether we chose $x_0 < \beta$ or $x_0 > \beta$, the approximates move away from the root β. In fact, if we choose $x_0 < \beta$, the approximates converge to the root α instead.

Part (d)

Using linear interpolation on the interval $[2, 3]$,

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = 2.06 \text{ (2 d.p.)}$$

$$\boxed{\beta = 2.06 \text{ (2 d.p.)}}$$

Observe that $f(2.06) = -0.039 < 0$ and $f(3) = 1.950 > 0$. Hence, $\beta \in (2.06, 3)$. Thus,

$$\boxed{\beta = 2.06 \text{ is an underestimate.}}$$

Part (e)

Observe that $f'(x) = 2xx - 3 + e^{-x}$. Using the Newton-Raphson method with $x_1 = 2.06$,

$$\begin{aligned} x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} &= 2.11118 = 2.111 \text{ (3 d.p.)} \\ \implies x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} &= 2.10935 = 2.109 \text{ (3 d.p.)} \\ \implies x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} &= 2.10935 = 2.109 \text{ (3 d.p.)} \end{aligned}$$

Hence,

$$\boxed{\beta = 2.109 \text{ (3 d.p.)}}$$