

Problem 1.

Find the position vector of the foot of the perpendicular from the point with position vector \mathbf{c} to the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. Leave your answers in terms of \mathbf{a} , \mathbf{b} and \mathbf{c} .

Solution

Let the foot of the perpendicular be F . We have that $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b}$ for some real λ , and $\overrightarrow{CF} \cdot \mathbf{b} = 0$.

$$\begin{aligned}\overrightarrow{CF} \cdot \mathbf{b} &= 0 \\ \Rightarrow (\overrightarrow{OF} - \overrightarrow{OC}) \cdot \mathbf{b} &= 0 \\ \Rightarrow (\mathbf{a} + \lambda\mathbf{b} - \mathbf{c}) \cdot \mathbf{b} &= 0 \\ \Rightarrow \lambda\mathbf{b} \cdot \mathbf{b} + (\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} &= 0 \\ \Rightarrow \lambda |\mathbf{b}|^2 &= (\mathbf{c} - \mathbf{a}) \cdot \mathbf{b} \\ \Rightarrow \lambda &= \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \\ \Rightarrow \overrightarrow{OF} &= \mathbf{a} + \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{OF} = \mathbf{a} + \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b}}$$

Problem 2.

The point O is the origin, and points A, B, C have position vectors given by $\overrightarrow{OA} = 6\mathbf{i}$, $\overrightarrow{OB} = 3\mathbf{j}$, $\overrightarrow{OC} = 4\mathbf{k}$. The point P is on the line AB between A and B , and is such that $AP = 2PB$. The point Q has position vector given by $\overrightarrow{OQ} = q\mathbf{i}$, where q is a scalar.

- Express, in terms of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, the vector \overrightarrow{CP} .
- Show that the line BQ has equation $\mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j})$, where t is a parameter. Give an equation of the line CP in a similar form.
- Find the value of q for which the lines CP and BQ are perpendicular.
- Find the sine of the acute angle between the lines CP and BQ in terms of q .

Solution

We have that $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

Part (a)

By the Ratio Theorem,

$$\begin{aligned}
 \overrightarrow{OP} &= \frac{2\overrightarrow{OB} + \overrightarrow{OA}}{1+2} \\
 &= \frac{1}{3} \left[2 \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \right] \\
 &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \\
 \Rightarrow \overrightarrow{CP} &= \overrightarrow{OP} - \overrightarrow{OC} \\
 &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \\
 &= \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}
 \end{aligned}$$

$$\boxed{\overrightarrow{CP} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}}$$

Part (b)

Note that $\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB} = \begin{pmatrix} q \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix}$. Thus, BQ is given by

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix}, t \in \mathbb{R} \\ \implies \mathbf{r} &= 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j}), t \in \mathbb{R} \end{aligned}$$

Note that $\overrightarrow{CP} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. Hence, CP is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, u \in \mathbb{R}$$

$$\boxed{CP : \mathbf{r} = 4\mathbf{k} + u(\mathbf{i} + \mathbf{j} - 2\mathbf{k}), u \in \mathbb{R}}$$

Part (c)

Since CP is perpendicular to BQ , we have $\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0$.

$$\begin{aligned} \overrightarrow{CP} \cdot \overrightarrow{BQ} &= 0 \\ \implies 2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} &= 0 \\ \implies q - 3 + 0 &= 0 \\ \implies q &= 3 \end{aligned}$$

$$\boxed{q = 3}$$

Part (d)

Let θ be the acute angle between CP and BQ .

$$\begin{aligned} \left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} \right| &= \left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} \right| \sin \theta \\ \implies \left| \begin{pmatrix} -6 \\ 2q \\ 3-q \end{pmatrix} \right| &= \left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} \right| \sin \theta \\ \implies \sqrt{(-6)^2 + (2q)^2 + (3-q)^2} &= \sqrt{1^2 + 1^2 + (-2)^2} \cdot \sqrt{q^2 + (-3)^2 + 0^2} \cdot \sin \theta \\ \implies \sqrt{36 + 4q^2 + 9 - 6q + q^2} &= \sqrt{6} \cdot \sqrt{q^2 + 9} \cdot \sin \theta \\ \implies \sqrt{5q^2 - 6q + 45} &= \sqrt{6(q^2 + 9)} \sin \theta \\ \implies \sqrt{5q^2 - 6q + 45} &= \sqrt{6q^2 + 54} \sin \theta \end{aligned}$$

$$\begin{aligned}\implies \sin \theta &= \frac{\sqrt{5q^2 - 6q + 45}}{\sqrt{6q^2 + 54}} \\ &= \sqrt{\frac{5q^2 - 6q + 45}{6q^2 + 54}}\end{aligned}$$

$$\boxed{\sin \theta = \sqrt{\frac{5q^2 - 6q + 45}{6q^2 + 54}}}$$

Problem 3.

Line l_1 passes through the point A with position vector $3\mathbf{i} - 2\mathbf{k}$ and is parallel to $-2\mathbf{i} + 4\mathbf{j} - \mathbf{j}$.
Line l_2 has Cartesian equation given by $\frac{x-1}{2} = y = z + 3$.

- Show that the two lines intersect and find the coordinates of their point of intersection.
- Find the acute angle between the two lines l_1 and l_2 . Hence, or otherwise, find the shortest distance from point A to line l_2 .
- Find the position vector of the foot N of the perpendicular from A to the line l_2 . The point B lies on the line AN produced and is such that N is the mid-point of AB . Find the position vector of B .

Solution

We have that

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$$

and

$$l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

Part (a)

Consider $l_1 = l_2$.

$$\begin{aligned} l_1 &= l_2 \\ \Rightarrow \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\ \Rightarrow \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 2\lambda + 2\mu = 2 \\ -4\lambda + \mu = 0 \\ \lambda + \mu = 1 \end{cases}$$

which has the unique solution $\mu = \frac{4}{5}$ and $\lambda = \frac{1}{5}$. Thus, the intersection point P has

position vector $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 13 \\ 4 \\ -11 \end{pmatrix}$ and thus has coordinates $\left(\frac{13}{5}, \frac{4}{5}, -\frac{11}{5}\right)$.

$$\boxed{\left(\frac{13}{5}, \frac{4}{5}, -\frac{11}{5}\right)}$$

Part (b)

Let θ be the acute angle between l_1 and l_2 .

$$\begin{aligned}
 \cos \theta &= \frac{\left| \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right|} \\
 &= \frac{|-4 + 4 - 1|}{\sqrt{21} \cdot \sqrt{6}} \\
 &= \frac{1}{\sqrt{126}} \\
 \Rightarrow \quad \theta &= \arccos \frac{1}{\sqrt{126}} \\
 &= 84.9^\circ \text{ (1 d.p.)} \\
 \boxed{\theta = 84.9^\circ}
 \end{aligned}$$

Note that

$$\begin{aligned}
 AP &= \sqrt{\left(\frac{17}{5} - 3\right)^2 + \left(-\frac{4}{5} - 0\right)^2 + \left(-\frac{9}{5} - (-2)\right)^2} \\
 &= \sqrt{\frac{21}{25}} \\
 &= \frac{\sqrt{21}}{5}
 \end{aligned}$$

Since $\sin \theta = \frac{AN}{AP}$, we have that $AN = AP \sin \theta$.

$$\begin{aligned}
 AN &= \frac{\sqrt{21}}{5} \sin \arccos \frac{1}{\sqrt{126}} \\
 &= \frac{\sqrt{21}}{5} \cdot \frac{\sqrt{(\sqrt{126})^2 - 1}}{\sqrt{126}} \\
 &= \frac{\sqrt{21}}{5} \cdot \frac{\sqrt{125}}{\sqrt{126}} \\
 &= \frac{\sqrt{21}}{5} \cdot \frac{5\sqrt{5}}{\sqrt{6} \cdot \sqrt{21}} \\
 &= \frac{\sqrt{5}}{\sqrt{6}} \\
 &= \sqrt{\frac{5}{6}}
 \end{aligned}$$

The shortest distance between A and l_2 is $\sqrt{\frac{5}{6}}$ units.

Part (c)

Since N is on l_2 , we have that $\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 03 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ for some real μ .

$$\begin{aligned}
 & \overrightarrow{AN} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \\
 \Rightarrow & (\overrightarrow{ON} - \overrightarrow{OA}) \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \\
 \Rightarrow & \left[\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \\
 \Rightarrow & \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \\
 \Rightarrow & 2(-2 + 2\mu) + \mu + (-1 + \mu) = 0 \\
 \Rightarrow & \mu = \frac{5}{6} \\
 \Rightarrow & \overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\
 & = \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}
 \end{aligned}$$

$$\boxed{\overrightarrow{ON} = \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}}$$

By the Ratio Theorem,

$$\begin{aligned}
 \overrightarrow{ON} &= \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} \\
 \Rightarrow \overrightarrow{OB} &= 2\overrightarrow{ON} - \overrightarrow{OA} \\
 &= \frac{2}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \\
 &= \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}
 \end{aligned}$$

$$\boxed{\overrightarrow{OB} = \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}}$$