

**Problem 1.**

Find

(a)  $\int \frac{1}{\sqrt{3-2x}} dx$

(b)  $\int \frac{1}{3-2x} dx$

(c)  $\int \frac{1}{3-2x^2} dx$

(d)  $\int \frac{1}{\sqrt{3-2x^2}} dx$

(e)  $\int \frac{x}{\sqrt{3-2x^2}} dx$

(f)  $\int \frac{1}{3+4x+2x^2} dx$

**Solution****Part (a)**

$$\begin{aligned}
 \int \frac{1}{\sqrt{3-2x}} dx &= -\frac{1}{2} \int \frac{1}{\sqrt{u}} du \\
 &= -\frac{1}{2} \cdot 2u^{\frac{1}{2}} + C \\
 &= -u^{\frac{1}{2}} + C \\
 &= -(3-2x)^{\frac{1}{2}} + C
 \end{aligned}$$

$$\begin{aligned}
 u &= 3-2x \\
 du &= -2x dx \\
 dx &= -\frac{1}{2} du
 \end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{3-2x}} dx = -(3-2x)^{\frac{1}{2}} + C}$$

**Part (b)**

$$\begin{aligned}
 \int \frac{1}{3-2x} dx &= -\frac{1}{2} \int \frac{1}{u} du \\
 &= -\frac{1}{2} \ln |u| + C \\
 &= -\frac{1}{2} \ln |3-2x| + C
 \end{aligned}$$

$$\begin{aligned}
 u &= 3-2x \\
 du &= -2x dx \\
 dx &= -\frac{1}{2} du
 \end{aligned}$$

$$\boxed{\int \frac{1}{3-2x} dx = -\frac{1}{2} \ln |3-2x| + C}$$

**Part (c)**

$$\begin{aligned}
 \int \frac{1}{3-2x^2} dx &= \frac{1}{2} \int \frac{1}{\frac{3}{2}-x^2} dx \\
 &= \frac{1}{2} \cdot \left(1/2\sqrt{\frac{3}{2}}\right) \ln \left( \frac{\sqrt{\frac{3}{2}}+x}{\sqrt{\frac{3}{2}}-x} \right) + C \\
 &= \frac{1}{2\sqrt{6}} \ln \left( \frac{\sqrt{3}+\sqrt{2}x}{\sqrt{3}-\sqrt{2}x} \right) + C
 \end{aligned}$$

$$\boxed{\int \frac{1}{3-2x^2} dx = \frac{1}{2\sqrt{6}} \ln \left( \frac{\sqrt{3}+\sqrt{2}x}{\sqrt{3}-\sqrt{2}x} \right) + C}$$

**Part (d)**

$$\begin{aligned}
 \int \frac{1}{\sqrt{3-2x^2}} dx &= \frac{1}{\sqrt{2}} \int \frac{1}{\sqrt{\frac{3}{2}-x^2}} dx \\
 &= \frac{1}{\sqrt{2}} \arcsin \left( x / \sqrt{\frac{3}{2}} \right) + C \\
 &= \frac{\sqrt{2}}{2} \arcsin \left( \frac{\sqrt{6}x}{3} \right) + C
 \end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{3-2x^2}} dx = \frac{\sqrt{2}}{2} \arcsin \left( \frac{\sqrt{6}x}{3} \right) + C}$$

**Part (e)**

$$\begin{aligned}
 \int \frac{x}{\sqrt{3-2x^2}} dx &= -\frac{1}{4} \int \frac{1}{\sqrt{u}} du \\
 &= -\frac{1}{4} \cdot 2u^{\frac{1}{2}} + C \\
 &= -\frac{1}{2} \sqrt{3-2x^2} + C
 \end{aligned}$$

$$\boxed{\int \frac{x}{\sqrt{3-2x^2}} dx = -\frac{1}{2} \sqrt{3-2x^2} + C}$$

$$\begin{aligned}
 u &= 3-2x^2 \\
 du &= -4x dx \\
 x dx &= -\frac{1}{4} du
 \end{aligned}$$

**Part (f)**

$$\begin{aligned}\int \frac{1}{3+4x+2x^2} dx &= \int \frac{1}{2(x+1)^2+1} dx \\&= \frac{1}{2} \int \frac{1}{(x+1)^2+\frac{1}{2}} dx \\&= \frac{1}{2} \cdot \left(1/\sqrt{\frac{1}{2}}\right) \arctan\left(\frac{x+1}{1/\sqrt{\frac{1}{2}}}\right) + C \\&= \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2}(x+1)\right) + C\end{aligned}$$

$$\boxed{\int \frac{1}{3+4x+2x^2} dx = \frac{\sqrt{2}}{2} \arctan\left(\sqrt{2}(x+1)\right) + C}$$

**Problem 2.**

Find

(a)  $\int \frac{\sec^2 3x}{\tan 3x} dx$

(b)  $\int \cos(3x + \alpha) dx$ , where  $\alpha$  is a constant

(c)  $\int \cos^2 3x dx$

(d)  $\int e^{1-2x} dx$

**Solution****Part (a)**

$$\begin{aligned}\int \frac{\sec^2 3x}{\tan 3x} dx &= \int \frac{1 + \tan^2 3x}{\tan 3x} dx \\&= \int (\cot 3x + \tan 3x) dx \\&= \frac{1}{3} \ln \sin 3x + \frac{1}{3} \ln \sec 3x + C \\&= \frac{1}{3} (\ln \sin 3x + \ln \sec 3x) + C \\&= \frac{1}{3} \ln(\sin 3x \sec 3x) + C \\&= \frac{1}{3} \ln \tan 3x + C\end{aligned}$$

$$\boxed{\int \frac{\sec^2 3x}{\tan 3x} dx = \frac{1}{3} \ln \tan 3x + C}$$

**Part (b)**

$$\begin{aligned}\int \cos(3x + \alpha) dx &= \frac{1}{3} \int \cos u du \\&= \frac{1}{3} \sin u + C \\&= \frac{1}{3} \sin(3x + \alpha) + C\end{aligned}$$

$$\boxed{\int \cos(3x + \alpha) dx = \frac{1}{3} \sin(3x + \alpha) + C}$$

$$\begin{aligned}u &= 3x + \alpha \\du &= 3 dx \\dx &= \frac{1}{3} du\end{aligned}$$

**Part (c)**

Recall that  $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ . Hence,  $\cos^2 3x = \frac{1 + \cos 6x}{2}$ .

$$\begin{aligned}\int \cos^2 3x \, dx &= \int \frac{1 + \cos 6x}{2} \, dx \\&= \frac{1}{2} \int (1 + \cos 6x) \, dx \\&= \frac{1}{2} \left( x + \frac{1}{6} \sin 6x \right) + C \\&= \frac{1}{2}x + \frac{1}{12} \sin 6x + C\end{aligned}$$

$$\boxed{\int \cos^2 3x \, dx = \frac{1}{2}x + \frac{1}{12} \sin 6x + C}$$

**Part (d)**

$$\begin{aligned}\int e^{1-2x} \, dx &= -\frac{1}{2} \int e^u \, du \\&= -\frac{1}{2} e^u + C \\&= -\frac{1}{2} e^{1-2x} + C\end{aligned}$$

$$\boxed{\int e^{1-2x} \, dx = -\frac{1}{2} e^{1-2x} + C}$$

$$\begin{aligned}u &= 1 - 2x \\du &= -2 \, dx \\dx &= -\frac{1}{2} \, du\end{aligned}$$

**Problem 3.**

Find

(a)  $\int 2x\sqrt{3x^2 - 5} \, dx$

(b)  $\int \frac{x^2 - 1}{\sqrt{x^3 - 3x}} \, dx$

(c)  $\int \sin x \sqrt{\cos x} \, dx$

(d)  $\int e^{2x}(1 - e^{2x})^4 \, dx$

**Solution****Part (a)**

$$\int 2x\sqrt{3x^2 - 5} \, dx = \frac{1}{6} \int 2\sqrt{u} \, du$$

$$= \frac{1}{3} \int \sqrt{u} \, du$$

$$= \frac{1}{3} \cdot \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{2}{9} u^{\frac{3}{2}} + C$$

$$= \frac{2}{9} (3x^2 - 5)^{\frac{3}{2}} + C$$

$$\boxed{\int 2x\sqrt{3x^2 - 5} \, dx = \frac{2}{9} (3x^2 - 5)^{\frac{3}{2}} + C}$$

$$\begin{aligned} u &= 3x^2 - 5 \\ du &= 6x \, dx \\ x \, dx &= \frac{1}{6} du \end{aligned}$$

**Part (b)**

$$\int \frac{x^2 - 1}{\sqrt{x^3 - 3x}} \, dx = \frac{1}{3} \int \frac{du}{\sqrt{u}}$$

$$= \frac{1}{3} \left( 2u^{\frac{1}{2}} \right) + C$$

$$= \frac{2}{3} \sqrt{x^3 - 3x} + C$$

$$\boxed{\int \frac{x^2 - 1}{\sqrt{x^3 - 3x}} \, dx = \frac{2}{3} \sqrt{x^3 - 3x} + C}$$

$$\begin{aligned} u &= x^3 - 3x \\ du &= 3x^2 - 3 \, dx \\ x^2 - 1 \, dx &= \frac{1}{3} du \end{aligned}$$

**Part (c)**

$$\begin{aligned}\int \sin x \sqrt{\cos x} \, dx &= - \int \sqrt{u} \, du \\ &= -\frac{2}{3} u^{\frac{3}{2}} + C \\ &= -\frac{2}{3} (\cos x)^{\frac{3}{2}} + C\end{aligned}$$

$$\boxed{\int \sin x \sqrt{\cos x} \, dx = -\frac{2}{3} (\cos x)^{\frac{3}{2}} + C}$$

$$\begin{aligned}u &= \cos x \\ du &= -\sin x \, dx \\ \sin x \, dx &= -du\end{aligned}$$

**Part (d)**

$$\begin{aligned}\int e^{2x} (1 - e^{2x})^4 \, dx &= -\frac{1}{2} \int u^4 \, du \\ &= -\frac{1}{2} \cdot \frac{u^5}{5} + C \\ &= -\frac{1}{10} (1 - e^{2x})^5 + C\end{aligned}$$

$$\boxed{\int e^{2x} (1 - e^{2x})^4 \, dx = -\frac{1}{10} (1 - e^{2x})^5 + C}$$

$$\begin{aligned}u &= 1 - e^{2x} \\ du &= -2e^{2x} \, dx \\ e^{2x} \, dx &= -\frac{1}{2} du\end{aligned}$$

**Problem 4.**

Find

(a)  $\int \frac{1}{\sqrt{x}(1-\sqrt{x})} dx$

(b)  $\int \frac{3x}{x+3} dx$

(c)  $\int \frac{\sin x + \cos x}{\sin x - \cos x} dx$

**Solution****Part (a)**

$$\begin{aligned}
 \int \frac{1}{\sqrt{x}(1-\sqrt{x})} dx &= -2 \int \frac{1}{u} du \\
 &= -2 \ln |u| + C \\
 &= -2 \ln |1 - \sqrt{x}| + C
 \end{aligned}$$

$$\begin{aligned}
 u &= 1 - \sqrt{x} \\
 du &= -\frac{1}{2\sqrt{x}} dx \\
 \frac{1}{\sqrt{x}} dx &= -2 du
 \end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{x}(1-\sqrt{x})} dx = -2 \ln |1 - \sqrt{x}| + C}$$

**Part (b)**

$$\begin{aligned}
 \int \frac{3x}{x+3} dx &= \int \left( 3 - \frac{9}{x+3} \right) dx \\
 &= 3x - 9 \ln |x+3| + C
 \end{aligned}$$

$$\boxed{\int \frac{3x}{x+3} dx = 3x - 9 \ln |x+3| + C}$$

**Part (c)**

$$\begin{aligned}
 \int \frac{\sin x + \cos x}{\sin x - \cos x} dx &= \int \frac{1}{u} du \\
 &= \ln |u| + C \\
 &= \ln |\sin x - \cos x| + C
 \end{aligned}$$

$$\begin{aligned}
 u &= \sin x - \cos x \\
 du &= \cos x + \sin x dx
 \end{aligned}$$

$$\boxed{\int \frac{\sin x + \cos x}{\sin x - \cos x} dx = \ln |\sin x - \cos x| + C}$$



**Problem 5.**

Find

(a)  $\int \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx$

(b)  $\int (\sin x)(\cos x)(e^{\cos 2x}) dx$

**Solution****Part (a)**

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = -2 \int e^u du$$

$$= -2e^u + C$$

$$= -2e^{-\sqrt{x}} + C$$

$$\boxed{\int \frac{e^{-\sqrt{x}}}{\sqrt{x}} dx = -2e^{-\sqrt{x}} + C}$$

$$\begin{aligned} u &= -\sqrt{x} \\ du &= -\frac{1}{2\sqrt{x}} dx \\ \frac{1}{\sqrt{x}} dx &= -2 du \end{aligned}$$

**Part (b)**

$$\int (\sin x)(\cos x)(e^{\cos 2x}) dx = \frac{1}{2} \int \sin 2x \cdot e^{\cos 2x} dx$$

$$= -\frac{1}{4} \int e^u du$$

$$= -\frac{1}{4} e^u + C$$

$$= -\frac{1}{4} e^{\cos 2x} + C$$

$$\boxed{\int (\sin x)(\cos x)(e^{\cos 2x}) dx = -\frac{1}{4} e^{\cos 2x} + C}$$

$$\begin{aligned} u &= \cos 2x \\ du &= -2 \sin 2x dx \\ \sin 2x dx &= -\frac{1}{2} du \end{aligned}$$

**Problem 6.**

Find

(a)  $\int \tan^2 2x \, dx$

(b)  $\int \frac{1}{1 + \cos 2t} \, dt$

(c)  $\int \sin\left(\frac{5}{2}\theta\right) \cos\left(\frac{1}{2}\theta\right) \, d\theta$

(d)  $\int \tan^4 x \, dx$

**Solution****Part (a)**

$$\begin{aligned} \int \tan^2 2x \, dx &= \int (\sec^2 2x - 1) \, dx \\ &= \frac{1}{2} \tan 2x - x + C \end{aligned}$$

$$\boxed{\int \tan^2 2x \, dx = \frac{1}{2} \tan 2x - x + C}$$

**Part (b)**

$$\begin{aligned} \int \frac{1}{1 + \cos 2t} \, dt &= \int \frac{1}{1 + (2 \cos^2 t - 1)} \, dt \\ &= \frac{1}{2} \int \frac{1}{\cos^2 t} \, dt \\ &= \frac{1}{2} \int \sec^2 t \, dt \\ &= \frac{1}{2} \tan t + C \end{aligned}$$

$$\boxed{\int \frac{1}{1 + \cos 2t} \, dt = \frac{1}{2} \tan t + C}$$

**Part (c)**

$$\begin{aligned} \int \sin\left(\frac{5}{2}\theta\right) \cos\left(\frac{1}{2}\theta\right) \, d\theta &= \frac{1}{2} \int \left( \sin\left(\frac{5}{2}\theta + \frac{1}{2}\theta\right) + \sin\left(\frac{5}{2}\theta - \frac{1}{2}\theta\right) \right) \, d\theta \\ &= \frac{1}{2} \int (\sin 3\theta + \sin 2\theta) \, d\theta \\ &= \frac{1}{2} \left( -\frac{1}{3} \cos 3\theta - \frac{1}{2} \cos 2\theta \right) + C \\ &= -\frac{1}{6} \cos 3\theta - \frac{1}{4} \cos 2\theta + C \end{aligned}$$

$$\int \sin\left(\frac{5}{2}\theta\right) \cos\left(\frac{1}{2}\theta\right) d\theta = -\frac{1}{6} \cos 3\theta - \frac{1}{4} \cos 2\theta + C$$

**Part (d)**

Note that

$$\int \tan^2 x \, dx = \tan x - x + C$$

and

$$\begin{aligned} \int \tan^2 x \sec^2 x \, dx &= \int u^2 \, du \\ &= \frac{1}{3} u^3 + C \\ &= \frac{1}{3} \tan^3 x + C \end{aligned}$$

$$\begin{aligned} u &= \tan x \\ du &= \sec^2 x \, dx \end{aligned}$$

Hence,

$$\begin{aligned} \int \tan^4 x \, dx &= \int \tan^2 x \cdot \tan^2 x \, dx \\ &= \int \tan^2 x (\sec^2 x - 1) \, dx \\ &= \int (\tan^2 x \sec^2 x - \tan^2 x) \, dx \\ &= \frac{1}{3} \tan^3 x - (\tan x - x) + C \\ &= \frac{1}{3} \tan^3 x - \tan x + x + C \end{aligned}$$

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C$$

**Problem 7.**

Find

(a)  $\int \frac{1}{4x^2 + 2x + 10} dx$

(b)  $\int \frac{x^2}{1 - x^2} dx$

(c)  $\int \frac{1}{\sqrt{3} + 2x - x^2} dx$

**Solution****Part (a)**

$$\begin{aligned}
\int \frac{1}{4x^2 + 2x + 10} dx &= \int \frac{1}{\left(2x + \frac{1}{2}\right)^2 + \frac{39}{4}} dx \\
&= 4 \int \frac{1}{(4x + 1)^2 + 39} dx \\
&= 4 \cdot \frac{1}{4} \cdot \frac{1}{\sqrt{39}} \arctan\left(\frac{4x + 1}{\sqrt{39}}\right) + C \\
&= \frac{1}{\sqrt{39}} \arctan\left(\frac{4x + 1}{\sqrt{39}}\right) + C
\end{aligned}$$

$$\boxed{\int \frac{1}{4x^2 + 2x + 10} dx = \frac{1}{\sqrt{39}} \arctan\left(\frac{4x + 1}{\sqrt{39}}\right) + C}$$

**Part (b)**

$$\begin{aligned}
\int \frac{x^2}{1 - x^2} dx &= \int \left( \frac{1}{1 - x^2} - 1 \right) dx \\
&= \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) - x + C
\end{aligned}$$

$$\boxed{\int \frac{x^2}{1 - x^2} dx = \frac{1}{2} \ln\left(\frac{1 + x}{1 - x}\right) - x + C}$$

**Part (c)**

$$\begin{aligned}
\int \frac{1}{\sqrt{3} + 2x - x^2} dx &= \int \frac{1}{\sqrt{2^2 - (x - 1)^2}} dx \\
&= \arcsin\left(\frac{x - 1}{2}\right) + C
\end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{3} + 2x - x^2} dx = \arcsin\left(\frac{x - 1}{2}\right) + C}$$

**Problem 8.**

Evaluate the following without the use of graphic calculator:

(a)  $\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4 \cot \frac{x}{2} \csc^2 \frac{x}{2} dx$

(b)  $\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx$

(c)  $\int_0^1 \frac{2}{(1+x)(1+x^2)} dx$

(d)  $\int_{-4}^{-2} \frac{x^3+2}{x^2-1} dx$

**Solution****Part (a)**

$$\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4 \cot \frac{x}{2} \csc^2 \frac{x}{2} dx = -4 \int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} -\cot \frac{x}{2} \csc \frac{x}{2} \cdot \csc \left( \frac{1}{2}x \right) dx$$

$$= -8 \int_2^{\frac{2}{\sqrt{3}}} u du$$

$$= -8 \left[ \frac{1}{2} u^2 \right]_2^{\frac{2}{\sqrt{3}}}$$

$$= \frac{32}{3}$$

$$\boxed{\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4 \cot \frac{x}{2} \csc^2 \frac{x}{2} dx = \frac{32}{3}}$$

$$\begin{aligned} u &= \csc \frac{x}{2} \\ du &= -\frac{1}{2} \cot \frac{x}{2} \csc \frac{x}{2} \\ 2 du &= -\cot \frac{x}{2} \csc \frac{x}{2} \\ x = \frac{1}{3}\pi &\implies u = 2 \\ x = \frac{2}{3}\pi &\implies u = \frac{2}{\sqrt{3}} \end{aligned}$$

**Part (b)**

$$\begin{aligned}\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx &= \int_0^4 \frac{\frac{1}{2}(2x+1) + \frac{3}{2}}{\sqrt{2x+1}} dx \\ &= \frac{1}{2} \int_0^4 \left( \frac{2x+1}{\sqrt{2x+1}} + \frac{3}{\sqrt{2x+1}} \right) dx\end{aligned}$$

$$= \frac{1}{2} \int_0^4 \left( \sqrt{2x+1} + \frac{3}{\sqrt{2x+1}} \right) dx$$

$$= \frac{1}{4} \int_1^9 \left( \sqrt{u} + \frac{3}{\sqrt{u}} \right) du$$

$$= \frac{1}{4} \left[ \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{3u^{\frac{1}{2}}}{\frac{1}{2}} \right]_1^9$$

$$= \frac{22}{3}$$

$$\boxed{\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx = \frac{22}{3}}$$

$$\begin{aligned}u &= 2x+1 \\ du &= 2 dx \\ dx &= \frac{1}{2} du \\ x=0 &\implies u=1 \\ x=4 &\implies u=9\end{aligned}$$

**Part (c)**

$$\int_0^1 \frac{2}{(1+x)(1+x^2)} dx = \int_0^1 \left( \frac{1}{1+x} + \frac{1}{1+x^2} - \frac{x}{1+x^2} \right) dx$$

$$= [\ln|1+x|]_0^1 + [\arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2} dx$$

$$= \ln 2 + \frac{\pi}{4} - \frac{1}{2} \int_1^2 \frac{1}{u} du$$

$$= \ln 2 + \frac{\pi}{4} - \frac{1}{2} [\ln|u|]_1^2$$

$$= \ln 2 + \frac{\pi}{4} - \frac{1}{2} \ln 2$$

$$= \frac{1}{2} \ln 2 + \frac{\pi}{4}$$

$$\boxed{\int_0^1 \frac{2}{(1+x)(1+x^2)} dx = \frac{1}{2} \ln 2 + \frac{\pi}{4}}$$

$$\begin{aligned}u &= 1+x^2 \\ du &= 2x dx \\ x dx &= \frac{1}{2} du \\ x=0 &\implies u=1 \\ x=1 &\implies u=2\end{aligned}$$

**Part (d)**

$$\begin{aligned}\int_{-4}^{-2} \frac{x^3 + 2}{x^2 - 1} dx &= \int_{-4}^{-2} \left( x + \frac{x + 2}{x^2 - 1} \right) dx \\&= \left[ \frac{x^2}{2} \right]_{-4}^{-2} + \int_{-4}^{-2} \frac{x + 2}{(x - 1)(x + 1)} dx \\&= -6 + \int_{-4}^{-2} \left( \frac{3/2}{x - 1} - \frac{1/2}{x + 1} \right) dx \\&= -6 + \left[ \frac{3}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| \right]_{-4}^{-2} \\&= -6 + 2 \ln 3 - \frac{3}{2} \ln 5\end{aligned}$$

$$\boxed{\int_{-4}^{-2} \frac{x^3 + 2}{x^2 - 1} dx = -6 + 2 \ln 3 - \frac{3}{2} \ln 5}$$

**Problem 9.**

Using the given substitution, find

$$(a) \int \frac{x}{(2x+3)^3} dx \quad [u = 2x + 3]$$

$$(b) \int \frac{1}{e^x + 4e^{-x}} dx \quad [u = e^x]$$

$$(c) \int_0^{\sqrt{2}} \sqrt{4-y^2} dy \quad [y = 2 \sin \theta]$$

$$(d) \int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin \theta} d\theta \quad \left[ t = \tan \frac{\theta}{2} \right]$$

**Solution****Part (a)**

$$\begin{aligned} \int \frac{x}{(2x+3)^3} dx &= \frac{1}{4} \int \frac{u-3}{u^3} du \\ &= \frac{1}{4} \int \left( \frac{1}{u^2} - \frac{3}{u^3} \right) du \\ &= \frac{1}{4} \left( \frac{u^{-1}}{-1} - \frac{3u^{-2}}{-2} \right) + C \\ &= \frac{3}{8}(2x+3)^{-2} - \frac{1}{4}(2x+3)^{-1} + C \end{aligned}$$

$$\begin{aligned} u &= 2x + 3 \\ x &= (u - 3)/2 \\ dx &= \frac{1}{2} du \end{aligned}$$

$$\boxed{\int \frac{x}{(2x+3)^3} dx = \frac{3}{8}(2x+3)^{-2} - \frac{1}{4}(2x+3)^{-1} + C}$$

**Part (b)**

$$\begin{aligned} \int \frac{1}{e^x + 4e^{-x}} dx &= \int \frac{e^x}{e^{2x} + 4} dx \\ &= \int \frac{u}{u^2 + 4} \cdot \frac{1}{u} du \\ &= \int \frac{1}{u^2 + 2^2} du \\ &= \frac{1}{2} \arctan \left( \frac{u}{2} \right) + C \\ &= \frac{1}{2} \arctan \left( \frac{e^x}{2} \right) + C \end{aligned}$$

$$\begin{aligned} u &= e^x \\ \ln u &= x \\ dx &= \frac{1}{u} du \end{aligned}$$



$$\int \frac{1}{e^x + 4e^{-x}} dx = \frac{1}{2} \arctan\left(\frac{e^x}{2}\right) + C$$

**Part (c)**

$$\int_0^{\sqrt{2}} \sqrt{4-y^2} dy = 2 \int_0^{\frac{\pi}{4}} \cos \theta \sqrt{4 - (2 \sin \theta)^2} d\theta$$

$$= 2 \int_0^{\frac{\pi}{4}} \cos \theta \sqrt{4 - 4 \sin^2 \theta} d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} \cos \theta \sqrt{1 - \sin^2 \theta} d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} \cos^2 \theta d\theta$$

$$= 4 \int_0^{\frac{\pi}{4}} \frac{1 + \cos 2\theta}{2} d\theta$$

$$= 2 \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{4}}$$

$$= 1 + \frac{\pi}{2}$$

$$\int_0^{\sqrt{2}} \sqrt{4-y^2} dy = 1 + \frac{\pi}{2}$$

$$\begin{aligned} y &= 2 \sin \theta \\ dy &= 2 \cos \theta d\theta \\ y = 0 &\implies \theta = 0 \\ y = \sqrt{2} &\implies \theta = \pi/4 \end{aligned}$$

**Part (d)**

Consider the substitution  $t = \tan \frac{\theta}{2} \implies \theta = 2 \arctan t$ . Hence,  $d\theta = \frac{2}{1+t^2} dt$ . Also note that  $\sin \theta = \sin(2 \arctan t) = 2 \sin(\arctan t) \cos(\arctan t) = 2 \cdot \frac{t}{\sqrt{1+t^2}} \cdot \frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$ .

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin \theta} d\theta = \int_0^1 \frac{2/(1+t^2)}{1 + 2t/(1+t^2)} du$$

$$= 2 \int_0^1 \frac{1}{t^2 + 2t + 1} dt$$

$$= 2 \int_0^1 \frac{1}{(t+1)^2} dt$$

$$= 2 \left[ -\frac{1}{t+1} \right]_0^1$$

$$= 1$$

$$\begin{aligned} t &= \tan \frac{\theta}{2} \\ \theta = 0 &\implies t = 0 \\ \theta = \frac{\pi}{2} &\implies t = 1 \end{aligned}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{1 + \sin \theta} \, d\theta = 1$$

**Problem 10.**

Find

(a)  $\int \ln(2x + 1) \, dx$

(b)  $\int x \arctan(x^2) \, dx$

(c)  $\int e^{-2x} \cos 2x \, dx$

(d)  $\int_0^2 x^2 e^{-x} \, dx$

**Solution****Part (a)**

$$\int \ln(2x + 1) \, dx = \frac{1}{2} \int \ln u \, du$$

$$\begin{aligned} u &= 2x + 1 \\ du &= 2 \, dx \\ dx &= \frac{1}{2} \, du \end{aligned}$$

$D$	$I$
$+$ $\ln u$	$u$
$-$ $\frac{1}{u}$	$\ln u$

$$\begin{aligned} \int \ln(2x + 1) \, dx &= \frac{1}{2} \left( u \ln u - \int u \cdot \frac{1}{u} \, du \right) \\ &= \frac{1}{2} (u \ln u - u) + C \\ &= \frac{1}{2} ((2x + 1) \ln(2x + 1) - (2x + 1)) + C \\ &= x \ln(2x + 1) + \frac{1}{2} \ln(2x + 1) - x + C \end{aligned}$$

$$\boxed{\int \ln(2x + 1) \, dx = x \ln(2x + 1) + \frac{1}{2} \ln(2x + 1) - x + C}$$

**Part (b)**

$$\int x \arctan(x^2) \, dx = \frac{1}{2} \int \arctan u \, du$$

$$\begin{aligned} u &= x^2 \\ du &= 2x \, dx \\ x \, dx &= \frac{1}{2} \, du \end{aligned}$$

	$D$	$I$
+	$\arctan u$	1
-	$\frac{1}{1+u^2}$	$u$

$$\int x \arctan(x^2) dx = \frac{1}{2} \left( u \arctan u - \int \frac{u}{1+u^2} du \right)$$

$$\begin{aligned} v &= 1 + u^2 \\ dv &= 2u du \\ u du &= \frac{1}{2} dv \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left( u \arctan u - \frac{1}{2} \int \frac{1}{v} dv \right) \\ &= \frac{1}{2} \left( u \arctan u - \frac{1}{2} \ln |v| \right) + C \\ &= \frac{1}{2} \left( u \arctan u - \frac{1}{2} \ln(1 + u^2) \right) + C \\ &= \frac{1}{2} \left( x^2 \arctan x^2 - \frac{1}{2} \ln(1 + x^4) \right) + C \\ &= \frac{1}{2} x^2 \arctan x^2 - \frac{1}{4} \ln(1 + x^4) + C \end{aligned}$$

$$\boxed{\int x \arctan(x^2) dx = \frac{1}{2} x^2 \arctan x^2 - \frac{1}{4} \ln(1 + x^4) + C}$$

**Part (c)**

	$D$	$I$
+	$e^{-2x}$	$\cos 2x$
-	$-2e^{-2x}$	$\frac{1}{2} \sin 2x$
+	$4e^{-2x}$	$-\frac{1}{4} \cos 2x$

$$\begin{aligned} \int e^{-2x} \cos 2x dx &= \frac{1}{2} e^{-2x} \sin 2x - \frac{1}{2} e^{-2x} \cos 2x - \int e^{-2x} \cos 2x dx \\ \Rightarrow 2 \int e^{-2x} \cos 2x dx &= \frac{1}{2} e^{-2x} \sin 2x - \frac{1}{2} e^{-2x} \cos 2x + C \\ \Rightarrow \int e^{-2x} \cos 2x dx &= \frac{1}{2} \left( \frac{1}{2} e^{-2x} \sin 2x - \frac{1}{2} e^{-2x} \cos 2x \right) + C \\ &= \frac{1}{4} e^{-2x} \sin 2x - \frac{1}{4} e^{-2x} \cos 2x + C \end{aligned}$$

$$\boxed{\int e^{-2x} \cos 2x dx = \frac{1}{4} e^{-2x} \sin 2x - \frac{1}{4} e^{-2x} \cos 2x + C}$$

**Part (d)**

	$D$	$I$
+	$x^2$	$e^{-x}$
-	$2x$	$-e^{-x}$
+	$2$	$e^{-x}$
-	$0$	$-e^{-x}$

$$\begin{aligned}\int_0^2 x^2 e^{-x} \, dx &= \left[ -x^2 e^{-x} - 2x e^{-x} - 2e^{-x} \right]_0^2 \\ &= 2 - 10e^{-2}\end{aligned}$$

$$\boxed{\int_0^2 x^2 e^{-x} \, dx = 2 - 10e^{-2}}$$

**Problem 11.**

- (a) Show that  $\frac{d}{dx} \ln(\sec x + \tan x) = \sec x$ .
- (b) Find  $\int x \sin x \, dx$ .
- (c) Find the exact value of  $\int_0^{\frac{1}{4}\pi} (x \sin x) \ln(\sec x + \tan x) \, dx$ .

**Solution****Part (a)**

$$\begin{aligned}
 \frac{d}{dx} \ln(\sec x + \tan x) &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\
 &= \sec x \cdot \frac{\tan x + \sec x}{\sec x + \tan x} \\
 &= \sec x
 \end{aligned}$$

**Part (b)**

	$D$	$I$
+	$x$	$\sin x$
−	$1$	$-\cos x$
+	$0$	$-\sin x$

$$\int x \sin x \, dx = -x \cos x + \sin x + C$$

**Part (c)**

$D$	$I$
$+$ $\ln(\sec x + \tan x)$	$x \sin x$
$-$ $\sec x$	$-x \cos x + \sin x$

$$\begin{aligned}
& \int_0^{\frac{\pi}{4}} (x \sin x) \ln(\sec x + \tan x) \, dx \\
&= [\ln(\sec x + \tan x) (-x \cos x + \sin x)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec x (-x \cos x + \sin x) \, dx \\
&= [\ln(\sec x + \tan x) (-x \cos x + \sin x)]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (-x + \tan x) \, dx \\
&= \left[ \ln(\sec x + \tan x) (-x \cos x + \sin x) - \frac{x^2}{2} - \ln |\cos x| \right]_0^{\frac{\pi}{4}} \\
&= \frac{\sqrt{2}}{2} \left(1 - \frac{\pi}{4}\right) \ln(\sqrt{2} + 1) + \frac{\pi^2}{32} - \frac{1}{2} \ln 2
\end{aligned}$$

$$\boxed{\int_0^{\frac{\pi}{4}} (x \sin x) \ln(\sec x + \tan x) \, dx = \frac{\sqrt{2}}{2} \left(1 - \frac{\pi}{4}\right) \ln(\sqrt{2} + 1) + \frac{\pi^2}{32} - \frac{1}{2} \ln 2}$$

**Problem 12.**

- (a) Use the fact that  $7 \cos x - 4 \sin x = \frac{3}{2}(\cos x + \sin x) + \frac{11}{2}(\cos x - \sin x)$  to find the exact value of  $\int_0^{\frac{\pi}{2}} \frac{7 \cos x - 4 \sin x}{\cos x + \sin x} dx$ .
- (b) Use integration by parts to find the exact value of  $\int_1^e (\ln x)^2 dx$ .

**Solution****Part (a)**

$$\begin{aligned}
 \int_0^{\frac{\pi}{2}} \frac{7 \cos x - 4 \sin x}{\cos x + \sin x} dx &= \int_0^{\frac{\pi}{2}} \frac{\frac{3}{2}(\cos x + \sin x) + \frac{11}{2}(\cos x - \sin x)}{\cos x + \sin x} dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{3(\cos x + \sin x) + 11(\cos x - \sin x)}{\cos x + \sin x} dx \\
 &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left( 3 + 11 \cdot \frac{\cos x - \sin x}{\cos x + \sin x} \right) dx \\
 &= \frac{1}{2} [3x + 11 \ln |\cos x + \sin x|]_0^{\frac{\pi}{2}} \\
 &= \frac{3}{4} \pi
 \end{aligned}$$

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{7 \cos x - 4 \sin x}{\cos x + \sin x} dx = \frac{3}{4} \pi}$$

**Part (b)**

	$D$	$I$
+	$\ln x$	$\ln x$
-	$\frac{1}{x}$	$x \ln x - x$

$$\begin{aligned}
 \int_1^e (\ln x)^2 dx &= [\ln x (x \ln x - x)]_1^e - \int_1^e \frac{1}{x} (x \ln x - x) dx \\
 &= [\ln x (x \ln x - x)]_1^e - \int_1^e (\ln x - 1) dx \\
 &= [\ln x (x \ln x - x) - (x \ln x - x - x)]_1^e \\
 &= e - 2
 \end{aligned}$$

$$\boxed{\int_1^e (\ln x)^2 dx = e - 2}$$



**Problem 13.**

- (a) Solve the inequality  $x^2 + 2x - 3 < 0$ .  
 (b) Without using the graphing calculator, evaluate

(i)  $\int_{-4}^4 |x^2 + 2x - 3| \, dx$

(ii)  $\int_0^2 x |x^2 + 2x - 3| \, dx$

**Solution****Part (a)**

$$\begin{aligned}
 & x^2 + 2x - 3 < 0 \\
 \implies & (x + 1)^2 - 4 < 0 \\
 \implies & (x + 1)^2 < 4 \\
 \implies & -2 < x + 1 < 2 \\
 \implies & -3 < x < 1
 \end{aligned}$$

$$\boxed{-3 < x < 1}$$

**Part (b)****Subpart (i)**

Let  $F(x) = \int (x^2 + 2x - 3) \, dx = \frac{1}{3}x^3 + x^2 - 3x + C$ . Then,

$$\begin{aligned}
 & \int_{-4}^4 |x^2 + 2x - 3| \, dx \\
 &= \int_{-4}^{-3} |x^2 + 2x - 3| \, dx + \int_{-3}^1 |x^2 + 2x - 3| \, dx + \int_1^4 |x^2 + 2x - 3| \, dx \\
 &= \int_{-4}^{-3} x^2 + 2x - 3 \, dx - \int_{-3}^1 x^2 + 2x - 3 \, dx + \int_1^4 x^2 + 2x - 3 \, dx \\
 &= \left( F(-3) - F(-4) \right) - \left( F(1) - F(-3) \right) + \left( F(4) - F(1) \right) \\
 &= 40
 \end{aligned}$$

$$\boxed{\int_{-4}^4 |x^2 + 2x - 3| \, dx = 40}$$

**Subpart (ii)**

Let  $F(x) = \int x(x^2 + 2x - 3) \, dx = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + C$ . Then,

$$\begin{aligned}\int_0^2 x |x^2 + 2x - 3| \, dx &= \int_0^1 x |x^2 + 2x - 3| \, dx + \int_1^2 x |x^2 + 2x - 3| \, dx \\ &= -\int_0^1 x(x^2 + 2x - 3) \, dx + \int_1^2 x(x^2 + 2x - 3) \, dx \\ &= -\left(F(1) - F(0)\right) + \left(F(2) - F(1)\right) \\ &= \frac{9}{2}\end{aligned}$$

$$\boxed{\int_0^2 x |x^2 + 2x - 3| \, dx = \frac{9}{2}}$$

**Problem 14.**

The indefinite integral  $\int \frac{P(x)}{x^3 + 1} dx$ , where  $P(x)$  is a polynomial in  $x$ , is denoted by  $I$ .

- (a) Find  $I$  when  $P(x) = x^2$ .
- (b) By writing  $x^3 + 1 = (x + 1)(x^2 + Ax + B)$ , where  $A$  and  $B$  are constants, find  $I$  when
- (i)  $P(x) = x^2 - x + 1$
- (ii)  $P(x) = x + 1$
- (c) Using the results of parts (a) and (b), or otherwise, find  $I$  when  $P(x) = 1$ .

**Solution****Part (a)**

$$\begin{aligned}\int \frac{x^2}{x^3 + 1} dx &= \frac{1}{3} \int \frac{1}{u} du \\ &= \frac{1}{3} \ln |u| + C \\ &= \frac{1}{3} \ln |x^3 + 1| + C\end{aligned}$$

$$\begin{aligned}u &= x^3 + 1 \\ du &= 3x^2 dx \\ x^2 dx &= \frac{1}{3} du\end{aligned}$$

$$\boxed{I = \frac{1}{3} \ln |x^3 + 1| + C}$$

**Part (b)**

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

**Subpart (i)**

$$\begin{aligned}\int \frac{x^2 - x + 1}{x^3 + 1} dx &= \int \frac{x^2 - x + 1}{(x + 1)(x^2 - x + 1)} dx \\ &= \int \frac{1}{x + 1} dx \\ &= \ln |x + 1| + C\end{aligned}$$

$$\boxed{I = \ln |x + 1| + C}$$

**Subpart (ii)**

$$\begin{aligned}
\int \frac{x+1}{x^3+1} dx &= \int \frac{x+1}{(x+1)(x^2-x+1)} dx \\
&= \int \frac{1}{x^2-x+1} dx \\
&= \int \frac{1}{(x-\frac{1}{2})^2 + \frac{3}{4}} dx \\
&= \frac{1}{\sqrt{3/4}} \arctan\left(\frac{x-\frac{1}{2}}{\sqrt{3/4}}\right) + C \\
&= \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C
\end{aligned}$$

$$I = \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$

**Part (c)**

Observe that  $1 = \frac{1}{2}((x^2 - x + 1) - x^2 + (x + 1))$ . Hence,

$$\begin{aligned}
\int \frac{1}{x^3+1} dx &= \frac{1}{2} \left( \int \frac{x^2-x+1}{x^3+1} dx - \int \frac{x^2}{x^3+1} dx + \int \frac{x+1}{x^3+1} dx \right) \\
&= \frac{1}{2} \left( \ln|x+1| - \frac{1}{3} \ln|x^3+1| + \frac{2}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) \right) + C \\
&= \frac{1}{2} \ln|x+1| - \frac{1}{6} \ln|x^3+1| + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C
\end{aligned}$$

$$I = \frac{1}{2} \ln|x+1| - \frac{1}{6} \ln|x^3+1| + \frac{1}{\sqrt{3}} \arctan\left(\frac{2x-1}{\sqrt{3}}\right) + C$$