

**Problem 1.**

Estimate, using the trapezium rule, the values of the following definite integrals, taking the number ordinates given in each case.

(a)  $\int_{-\pi/2}^0 \frac{1}{1 + \cos \theta} d\theta$ , 3 ordinates

(b)  $\int_{-0.4}^{0.2} \frac{x^2 - 4x + 1}{4x - 4} dx$ , 4 ordinates

**Solution****Part (a)**

Let  $f(\theta) = \frac{1}{1 + \cos \theta}$ .

$$\int_{-\pi/2}^0 \frac{1}{1 + \cos \theta} d\theta \approx \frac{1}{2} \cdot \frac{0 - (-\pi/2)}{3 - 1} \cdot \left[ f\left(-\frac{\pi}{2}\right) + 2f\left(-\frac{\pi}{4}\right) + f(0) \right] = 1.05$$

$$\boxed{\int_{-\pi/2}^0 \frac{1}{1 + \cos \theta} d\theta \approx 1.05}$$

**Part (b)**

Let  $f(x) = \frac{x^2 - 4x + 1}{4x - 4}$ .

$$\int_{-0.4}^{0.2} \frac{x^2 - 4x + 1}{4x - 4} dx \approx \frac{1}{2} \cdot \frac{0.2 - (-0.4)}{4 - 1} \cdot \left[ f(-0.4) + 2(f(-0.2) + f(0)) + f(0.2) \right] = -0.183$$

$$\boxed{\int_{-0.4}^{0.2} \frac{x^2 - 4x + 1}{4x - 4} dx \approx -0.183}$$

**Problem 2.**

Use the trapezium rule with intervals of width 0.5 to obtain an approximation to  $\int_2^{3.5} \ln \frac{1}{x} \, dx$ , giving your answer to 2 decimal places.

**Solution**

$$\begin{aligned}\int_2^{3.5} \ln \frac{1}{x} \, dx &= - \int_2^{3.5} \ln x \, dx \\ &\approx - \left( \frac{1}{2} \cdot \frac{3.5 - 2}{4 - 1} \cdot [\ln 2 + 2(\ln 2.5 + \ln 3) + \ln 3.5] \right) \\ &= -1.49 \text{ (2 d.p.)}\end{aligned}$$

$$\boxed{\int_2^{3.5} \ln \frac{1}{x} \, dx = -1.49}$$

**Problem 3.**

Estimate, using Simpson's rule, the values of the following definite integrals, taking the number of ordinates given in each case.

(a)  $\int_{-\pi/2}^0 \frac{1}{1 + \cos \theta} d\theta$ , 3 ordinates

(b)  $\int_0^{0.4} \sqrt{1 - x^2} dx$ , 5 ordinates

**Solution****Part (a)**

Let  $f(\theta) = \frac{1}{1 + \cos \theta}$ .

$$\int_{-\pi/2}^0 \frac{1}{1 + \cos \theta} d\theta \approx \frac{1}{3} \cdot \frac{0 - (-\pi/2)}{3 - 1} \cdot \left[ f\left(-\frac{\pi}{2}\right) + 4f\left(-\frac{\pi}{4}\right) + f(0) \right] = 1.01$$

$$\boxed{\int_{-\pi/2}^0 \frac{1}{1 + \cos \theta} d\theta \approx 1.01}$$

**Part (b)**

Let  $f(x) = \sqrt{1 - x^2}$ .

$$\int_0^{0.4} \sqrt{1 - x^2} dx \approx \frac{1}{3} \cdot \frac{0.4 - 0}{5 - 1} \cdot \left[ f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + f(0.4) \right] = 0.389$$

$$\boxed{\int_0^{0.4} \sqrt{1 - x^2} dx \approx 0.389}$$

**Problem 4.**

Show, by means of substitution  $u = \sqrt{x}$ , that

$$\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} dx = \int_0^{0.5} 2e^{-u^2} du$$

Use the trapezium rule, with ordinates at  $u = 0$ ,  $u = 0.1$ ,  $u = 0.2$ ,  $u = 0.3$ ,  $u = 0.4$  and  $u = 0.5$ , to estimate the value of  $I = \int_0^{0.5} 2e^{-u^2} du$ , giving three decimal places in your answer.

Explain briefly why the trapezium rule cannot be used directly to estimate the value of  $\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} dx$ .

By using the first four terms of the expansion of  $e^{-x}$ , obtain an estimate for the integral  $\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} dx$ , giving three decimal places in your answer.

**Solution**

Note that  $u = \sqrt{x} \implies u^2 = x \implies 2u du = dx$ . Furthermore,  $x = 0 \implies u = 0$  and  $x = 0.25 \implies u = 0.5$ .

$$\begin{aligned} \int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} dx &= \int_0^{0.5} \frac{1}{u} e^{-u^2} \cdot 2u du \\ &= \int_0^{0.5} 2e^{-u^2} du \end{aligned}$$

Let  $f(u) = 2e^{-u^2}$ . Using the trapezium rule,

$$\begin{aligned} \int_0^{0.5} 2e^{-u^2} du &\approx \frac{1}{2} \cdot \frac{0.5 - 0}{5} \cdot [f(0) + 2(f(0.1) + f(0.2) + f(0.3) + f(0.4)) + f(0.5)] \\ &= 0.921 \text{ (3 d.p.)} \end{aligned}$$

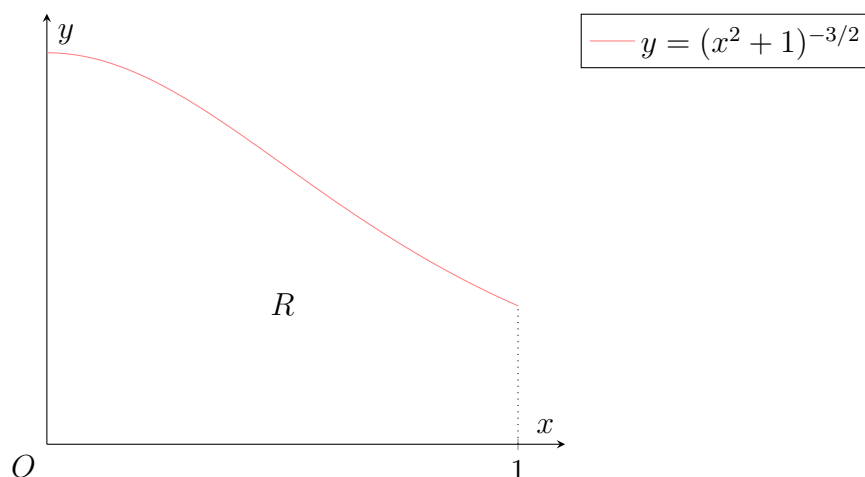
$$\boxed{I \approx 0.921}$$

At  $x = 0$ ,  $\frac{1}{\sqrt{x}} e^{-x}$  is undefined. Hence, the trapezium rule cannot be used.

Recall that  $e^{-x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \dots$

$$\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} dx \approx \int_0^{0.25} \frac{1}{\sqrt{x}} \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right) dx = 0.923 \text{ (3 d.p.)}$$

$$\boxed{\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} dx \approx 0.923}$$

**Problem 5.**

The diagram (not to scale) show the region  $R$  bounded by the axes, the curve  $y = (x^2 + 1)^{-3/2}$  and the line  $x = 1$ . The integral  $\int_0^1 (x^2 + 1)^{-3/2}$  is denoted by  $I$ .

- Use the trapezium rule and Simpson's rule, with ordinates at  $x = 0$ ,  $x = 0.5$  and  $x = 1$ , to estimate the value of  $I$  correct to 4 significant figures.
- Use the substitution  $x = \tan \theta$  to show that  $I = \frac{1}{2}\sqrt{2}$ . Comment on the approximations using the 2 rules and give a reason why one gives a better approximation than the other.
- By using the trapezium rule, with the same ordinates as in part (a), or otherwise, estimate the volume of the solid formed when  $R$  is rotated completely about the  $x$ -axis, giving your answer to 2 significant figures.

**Solution****Part (a)**

Let  $f(x) = (x^2 + 1)^{-3/2}$ . Using the trapezium rule,

$$I \approx \frac{1}{2} \cdot \frac{1 - 0}{3 - 1} \cdot [f(0) + 2f(0.5) + f(1)] = 0.6962 \text{ (4 s.f.)}$$

$$I \approx 0.6962$$

Using Simpson's rule,

$$I \approx \frac{1}{3} \cdot \frac{1 - 0}{3 - 1} \cdot [f(0) + 4f(0.5) + f(1)] = 0.7026 \text{ (4 s.f.)}$$

$$I \approx 0.7026$$

**Part (b)**

$$\begin{aligned}
\int_0^1 (x^2 + 1)^{-3/2} dx &= \int_0^{\pi/4} (\tan^2 \theta + 1)^{-3/2} \sec^2 \theta d\theta \\
&= \int_0^{\pi/4} (\sec^2 \theta)^{-3/2} \sec^2 \theta d\theta \\
&= \int_0^{\pi/4} \frac{1}{\sec \theta} d\theta \\
&= \int_0^{\pi/4} \cos \theta d\theta \\
&= [\sin \theta]_0^{\pi/4} \\
&= \frac{1}{2} \sqrt{2}
\end{aligned}$$

$$\begin{aligned}
x &= \tan \theta \\
dx &= \sec^2 \theta d\theta
\end{aligned}$$

The curve  $y = (x^2 + 1)^{-3/2}$  resembles a quadratic more than a line on the interval  $[0, 1]$ . Hence, Simpson's rule, which uses quadratic curves to approximate  $y$ , gives a better approximation for  $I$  than the trapezium rule, which uses lines to approximate  $y$ .

**Part (c)**

Let  $g(x) = (x^2 + 1)^{-3}$ .

$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 y^2 dx \\
&= \pi \int_0^1 (x^2 + 1)^{-3} dx \\
&\approx \pi \left( \frac{1}{2} \cdot \frac{1 - 0}{3 - 1} \cdot [g(0) + 2g(0.5) + g(1)] \right) \\
&= 1.7 \text{ (2 s.f.)}
\end{aligned}$$

The volume of the solid formed is 1.7 units <sup>3</sup> .
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**Problem 6.**

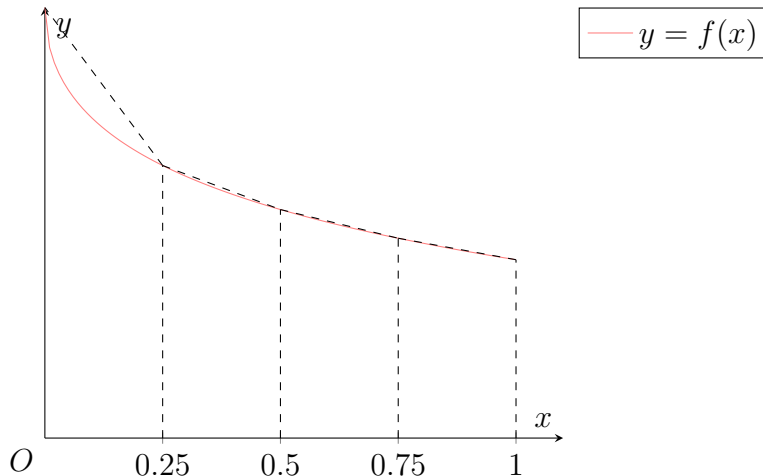
It is given that  $f(x) = \frac{1}{\sqrt{1+\sqrt{x}}}$ , and the integral  $\int_0^1 f(x) dx$  is denoted by  $I$ .

- Using the trapezium rule, with four trapezia of equal width, obtain an approximation  $I_1$  to the value of  $I$ , giving 3 decimal places in your answer.
- Explain, with the aid of a sketch, why  $I < I_1$ .
- Evaluate  $I_2$ , where  $I_2 = \frac{1}{3} \sum_{r=1}^3 f\left(\frac{1}{3}r\right)$ , giving 3 decimal places in your answer, and use the sketch in (b) to justify the inequality  $I > I_2$ .
- By means of a substitution  $\sqrt{x} = u - 1$ , show that the value of  $I$  is  $\frac{4}{3}(2 - \sqrt{2})$ .

**Solution****Part (a)**

$$I_1 = \frac{1}{2} \cdot \frac{1-0}{4} \cdot [f(0) + 2(f(0.25) + f(0.5) + f(0.75)) + f(1)] = 0.792 \text{ (3 d.p.)}$$

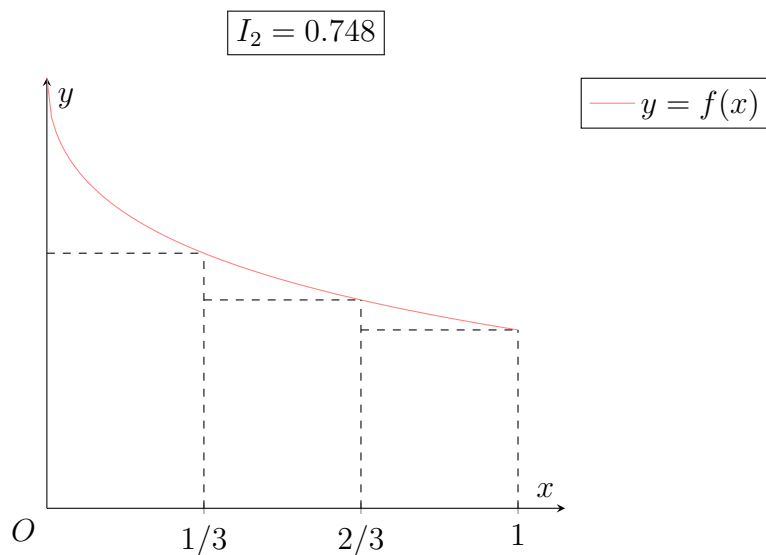
$$I_1 = 0.792$$

**Part (b)**

$I$  is the area under the curve  $y = f(x)$ , while  $I_1$  is the sum of the areas of the trapeziums. Hence, from the sketch,  $I_1 > I$ .

**Part (c)**

$$\begin{aligned} I_2 &= \frac{1}{3} \sum_{r=1}^3 f\left(\frac{1}{3}r\right) \\ &= \frac{1}{3} \left[ f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1) \right] \\ &= 0.748 \text{ (3 d.p.)} \end{aligned}$$



$I$  is the area under the curve  $y = f(x)$ , while  $I_2$  is the sum of the areas of the rectangles. Hence, from the sketch,  $I_2 < I$ .

### Part (d)

Note  $\sqrt{x} = u - 1 \implies x = u^2 - 2u + 1 \implies dx = (2u - 2) du$ . Furthermore,  $x = 0 \implies u = 1$  and  $x = 1 \implies u = 2$ .

$$\begin{aligned}
 \int_0^1 \frac{1}{\sqrt{1 + \sqrt{x}}} dx &= \int_1^2 \frac{1}{\sqrt{1 + (u - 1)}} \cdot (2u - 2) du \\
 &= 2 \int_1^2 \frac{u - 1}{\sqrt{u}} du \\
 &= 2 \left[ \frac{2}{3} u^{3/2} - 2u^{1/2} \right]_1^2 \\
 &= \frac{4}{3} (2 - \sqrt{2})
 \end{aligned}$$



**Problem 7.**

For  $0 < x < \pi$ , the curve  $C$  has the equation  $y = \ln \sin x$ . The region of the plane bounded by  $C$ , the  $x$ -axis and the lines  $x = \frac{\pi}{4}$  and  $x = \frac{\pi}{2}$  is rotated through  $2\pi$  radians about the  $x$ -axis.

Show that the surface area of the solid generated in this way is given by  $S$ , where

$$S = 2\pi \int_{\pi/4}^{\pi/2} \left| \frac{\ln \sin x}{\sin x} \right| dx$$

Use Simpson's rule with 5 ordinates to find an approximate value of  $S$ , giving your answer to 3 decimal places.

**Solution**

Note that  $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x = \cot x \implies 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cot^2 x = \csc^2 x$ .

$$\begin{aligned} S &= 2\pi \int_{\pi/4}^{\pi/2} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 2\pi \int_{\pi/4}^{\pi/2} \ln \sin x \sqrt{\csc^2 x} dx \\ &= 2\pi \int_{\pi/4}^{\pi/2} \ln \sin x |\csc x| dx \\ &= 2\pi \int_{\pi/4}^{\pi/2} |\ln \sin x| \left| \frac{1}{\sin x} \right| dx \\ &= 2\pi \int_{\pi/4}^{\pi/2} \left| \frac{\ln \sin x}{\sin x} \right| dx \end{aligned}$$

$$\text{Let } f(x) = \left| \frac{\ln \sin x}{\sin x} \right|.$$

$$\begin{aligned} S &\approx 2\pi \left( \frac{1}{3} \cdot \frac{\pi/2 - \pi/4}{5 - 1} \cdot \left[ f\left(\frac{4}{16}\pi\right) + 4f\left(\frac{5}{16}\pi\right) + 2f\left(\frac{6}{16}\pi\right) + 4f\left(\frac{7}{16}\pi\right) + f\left(\frac{8}{16}\pi\right) \right] \right) \\ &= 0.670 \text{ (3 d.p.)} \end{aligned}$$

$$\boxed{S \approx 0.670}$$

**Problem 8.**

The value of the integral  $\int_{0.2}^{0.4} f(x) dx$  is to be estimated from information in the table below.

$x$	0.2	0.3	0.4
$f(x)$	1.2030	1.2441	1.2777

- Find the best possible estimate for the integral using the trapezium rule.
- Using the table of values above, find an approximate value for  $f''(0.3)$  and use your answer to explain why the estimate found in part (a) is likely to be smaller than the actual value.
- Estimate the integral using Simpson's rule and determine the equation of the curve used in this method.

**Solution****Part (a)**

$$\int_{0.2}^{0.4} f(x) dx \approx \frac{1}{2} \cdot \frac{0.4 - 0.2}{3 - 1} \cdot [f(0.2) + 2f(0.3) + f(0.4)] = 0.248$$

$$\boxed{\int_{0.2}^{0.4} f(x) dx \approx 0.248}$$

**Part (b)**

Note that  $f'(0.25) \approx \frac{f(0.3) - f(0.2)}{0.3 - 0.2} = 0.411$  and  $f'(0.35) \approx \frac{f(0.4) - f(0.3)}{0.4 - 0.3} = 0.336$ . Hence,

$$f''(0.30) \approx \frac{f'(0.35) - f'(0.25)}{0.35 - 0.25} = -0.75$$

$$\boxed{f''(0.30) \approx -0.75}$$

Since  $f''(0.3) < 0$ ,  $f(x)$  is concave downwards around  $x = 0.3$ . Hence, the estimate is likely to be smaller than the actual value.

**Part (c)**

$$\int_{0.2}^{0.4} f(x) dx \approx \frac{1}{3} \cdot \frac{0.4 - 0.2}{3 - 1} \cdot [f(0.2) + 4f(0.3) + f(0.4)] = 0.249$$

$$\boxed{\int_{0.2}^{0.4} f(x) dx \approx 0.249}$$

Let the equation of the quadratic used be  $P(x) = ax^2 + bx + c$ , where  $a, b, c \in \mathbb{R}$ . Since  $P(0.2) = f(0.2)$ ,  $P(0.3) = f(0.3)$  and  $P(0.4) = f(0.4)$ , we obtain the system

$$\begin{cases} (0.2)^2a + 0.2b + c = 1.2030 \\ (0.3)^2a + 0.3b + c = 1.2441 \\ (0.4)^2a + 0.4b + c = 1.2777 \end{cases}$$

which has the unique solution  $a = -0.375$ ,  $b = 0.5985$ ,  $c = 1.0983$ . Thus, the required equation is  $y = -0.375x^2 + 0.5985x + 1.0983$ .

$$\boxed{y = -0.375x^2 + 0.5985x + 1.0983}$$

**Problem 9.**

The curve  $C$  is given by  $y = \frac{1}{x}$ , where  $x > 0$ .

- (a) Apply the trapezium rule with ordinates at unit intervals to the function  $f : x \mapsto \frac{1}{x}$ ,

$x \in \mathbb{R}^+$ , to show that  $\ln n < \frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r}$  where  $n \geq 3$ .

- (b) Obtain the area of the trapezium bounded by the axis, the lines  $x = r \pm \frac{1}{2}$ , and the tangent to the curve  $y = \frac{1}{x}$  at the point  $\left(r, \frac{1}{r}\right)$ .

Hence, show that  $\sum_{r=2}^{n-1} \frac{1}{r} < \ln \left(\frac{2n-1}{3}\right)$ , where  $n \geq 3$ .

- (c) From these results, obtain numerical values between which the value of  $\sum_{r=2}^{99} \frac{1}{r}$  lies,

and show that  $4.110 < \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{100} < 4.205$ .

**Solution****Part (a)**

$$\begin{aligned} \int_1^n \frac{1}{x} dx &\approx \frac{1}{2} \cdot 1 \cdot \left[ \frac{1}{1} + 2 \left( \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} \right) + \frac{1}{n} \right] \\ &= \frac{1}{2} \left( 1 + \frac{1}{n} + 2 \sum_{r=2}^{n-1} \frac{1}{r} \right) \\ &= \frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r} \end{aligned}$$

Note that  $\frac{d^2y}{dx^2} \frac{1}{x} = \frac{2}{x^3} > 0$  for  $x > 0$ . Hence,  $y = \frac{1}{x}$  is concave upwards. Thus,

$$\frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r} > \int_1^n \frac{1}{x} dx = \ln n$$

**Part (b)**

Since  $\frac{dy}{dx} = -\frac{1}{x^2}$ , the equation of the tangent at  $x = r$  is given by

$$y - \frac{1}{r} = -\frac{1}{r^2}(x - r) \implies y = -\frac{1}{r^2}x + \frac{2}{r}$$

Area of trapezium centred at  $r$

$$\begin{aligned}
 &= \int_{r-1/2}^{r+1/2} \left( -\frac{1}{r^2}x + \frac{2}{r} \right) dx \\
 &= \left[ -\frac{1}{r^2} \cdot \frac{1}{2}x^2 + \frac{2}{r}x \right]_{r-1/2}^{r+1/2} \\
 &= \left[ -\frac{1}{2r^2} \left( r + \frac{1}{2} \right)^2 + \frac{2}{r} \left( r + \frac{1}{2} \right) \right] - \left[ -\frac{1}{2r^2} \left( r - \frac{1}{2} \right)^2 + \frac{2}{r} \left( r - \frac{1}{2} \right) \right] \\
 &= \frac{1}{2r^2} \left[ \left( r - \frac{1}{2} \right)^2 - \left( r + \frac{1}{2} \right)^2 \right] + \frac{2}{r} \left[ \left( r + \frac{1}{2} \right) - \left( r - \frac{1}{2} \right) \right] \\
 &= \frac{1}{2r^2} \cdot 2r \cdot -1 + \frac{2}{r} \cdot 1 \\
 &= \frac{-1}{r} + \frac{2}{r} \\
 &= \frac{1}{r}
 \end{aligned}$$

The area of the trapezium is  $\frac{1}{r}$  units<sup>2</sup>.

Observe that the area of the trapezium centred at  $r$  is less than the area under the curve  $y = \frac{1}{x}$  from  $r - \frac{1}{2}$  to  $r + \frac{1}{2}$ . That is,

$$\frac{1}{r} < \int_{r-1/2}^{r+1/2} \frac{1}{x} dx = \ln \left( r + \frac{1}{2} \right) - \ln \left( r - \frac{1}{2} \right)$$

Summing from  $r = 2$  to  $n - 1$ ,

$$\begin{aligned}
 \sum_{r=2}^{n-1} \frac{1}{r} &< \sum_{r=2}^{n-1} \left[ \ln \left( r + \frac{1}{2} \right) - \ln \left( r - \frac{1}{2} \right) \right] \\
 &= \sum_{r=2}^{n-1} \ln \left( r + \frac{1}{2} \right) - \sum_{r=2}^{n-1} \ln \left( r - \frac{1}{2} \right) \\
 &= \sum_{r=3}^n \ln \left( r - \frac{1}{2} \right) - \sum_{r=2}^{n-1} \ln \left( r - \frac{1}{2} \right) \\
 &= \ln \left( n - \frac{1}{2} \right) - \ln \left( 2 - \frac{1}{2} \right) \\
 &= \ln \left( \frac{n - 1/2}{3/2} \right) \\
 &= \ln \left( \frac{2n - 1}{3} \right)
 \end{aligned}$$

### Part (c)

Taking  $n = 100$ , we have

$$\frac{1}{2} + \frac{1}{2 \cdot 100} + \sum_{r=2}^{100-1} \frac{1}{r} > \ln 100 \implies \sum_{r=2}^{99} \frac{1}{r} > \ln 100 - \frac{1}{2} - \frac{1}{200} = 4.100$$

We also have

$$\sum_{r=2}^{100-1} \frac{1}{r} < \ln\left(\frac{2 \cdot 100 - 1}{3}\right) \implies \sum_{r=2}^{100-1} \frac{1}{r} < \ln\left(\frac{199}{3}\right) = 4.195$$

Putting both inequalities together, we obtain

$$4.100 < \sum_{r=2}^{99} \frac{1}{r} < 4.195$$

Adding  $\frac{1}{100} = 0.01$  to all sides of the inequality, we see that

$$4.110 < \sum_{r=2}^{100} \frac{1}{r} < 4.205$$