

Problem 1.

- (a) Solve $z^4 = -4 - 4\sqrt{3}i$, expressing your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.
- (b) Sketch the roots on an Argand diagram.
- (c) Hence, solve $w^4 = -1 + \sqrt{3}i$, expressing your answers in a similar form.

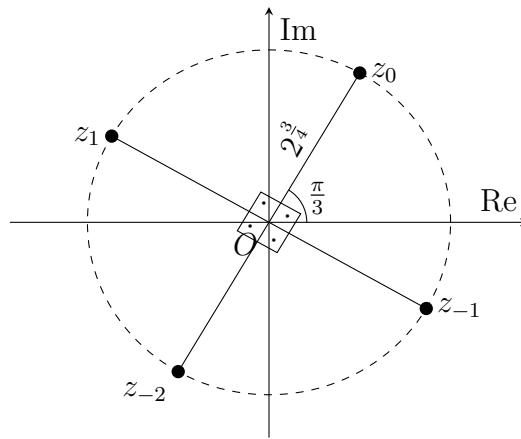
Solution**Part (a)**

Observe that $-4 - 4\sqrt{3}i = 8 \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right) = 8 \left(\cos \frac{4}{3}\pi + i \sin \frac{4}{3}\pi \right) = 8e^{i\frac{4}{3}\pi} = 8e^{i\frac{4}{3}\pi + 2k\pi i}$ for all $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} z^4 &= 8e^{i\frac{4}{3}\pi + 2k\pi i} \\ \implies z &= 8^{\frac{1}{4}}e^{i\frac{1}{3}\pi + \frac{1}{2}k\pi i} \\ &= 2^{\frac{3}{4}}e^{i\frac{2+3k}{6}\pi} \end{aligned}$$

Taking $k = -2, -1, 0, 1$:

$$z_{-2} = 2^{\frac{3}{4}}e^{-i\frac{2}{3}\pi}, z_{-1} = 2^{\frac{3}{4}}e^{-i\frac{1}{6}\pi}, z_0 = 2^{\frac{3}{4}}e^{i\frac{1}{3}\pi}, z_1 = 2^{\frac{3}{4}}e^{i\frac{5}{6}\pi}$$

Part (b)**Part (c)**

Observe that $w^4 = -1 + \sqrt{3}i = \frac{1}{4}(-4 + 4\sqrt{3}i) = \frac{1}{4}(z^4)^* = 2^{-2}(z^*)^4$. Hence, $w = 2^{-\frac{1}{2}}z^*$.

$$w_{-2} = 2^{\frac{1}{4}}e^{i\frac{2}{3}\pi}, w_{-1} = 2^{\frac{1}{4}}e^{i\frac{1}{6}\pi}, w_0 = 2^{\frac{1}{4}}e^{-i\frac{1}{3}\pi}, w_1 = 2^{\frac{1}{4}}e^{-i\frac{5}{6}\pi}$$

Problem 2.

Let

$$C = 1 - \binom{2n}{1} \cos \theta + \binom{2n}{2} \cos 2\theta - \binom{2n}{3} \cos 3\theta + \dots + \cos 2n\theta$$

$$S = -\binom{2n}{1} \sin \theta + \binom{2n}{2} \sin 2\theta - \binom{2n}{3} \sin 3\theta + \dots + \sin 2n\theta$$

where n is a positive integer.

Show that $C = (-4)^n \cos(n\theta) \sin^{2n}(\theta/2)$, and find the corresponding expression for S .

Solution

$$\begin{aligned} C + iS &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \cos k\theta + \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k i \sin k\theta \\ &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (\cos k\theta + i \sin k\theta) \\ &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k e^{ik\theta} \\ &= \sum_{k=0}^{2n} \binom{2n}{k} (-e^{i\theta})^k \\ &= (1 - e^{i\theta})^{2n} \\ &= (e^{i\theta/2})^{2n} (e^{-i\theta/2} - e^{i\theta/2})^{2n} \\ &= (e^{i\theta/2})^{2n} (e^{i\theta/2} - e^{-i\theta/2})^{2n} \\ &= (e^{i\theta/2})^{2n} (2i \sin(\theta/2))^{2n} \\ &= e^{in\theta} 2^{2n} i^{2n} \sin^{2n}(\theta/2) \\ &= e^{in\theta} 4^n (-1)^n \sin^{2n}(\theta/2) \\ &= (\cos n\theta + i \sin n\theta) (-4)^n \sin^{2n}(\theta/2) \end{aligned}$$

Comparing real and imaginary parts, we have $C = (-4)^n \cos(n\theta) \sin^{2n}(\theta/2)$ and

$$\boxed{S = (-4)^n \sin(n\theta) \sin^{2n}(\theta/2)}$$

Problem 3.

Given that $z = \cos \theta + i \sin \theta$, show that

$$(a) \quad z - \frac{1}{z} = 2i \sin \theta,$$

$$(b) \quad z^n + z^{-n} = 2 \cos n\theta.$$

Hence, show that

$$\sin^6 \theta = \frac{1}{32}(10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta)$$

Find a similar expression for $\cos^6 \theta$, and hence express $\cos^6 \theta - \sin^6 \theta$ in the form $a \cos 2\theta + b \cos 6\theta$.

Solution**Part (a)**

$$\begin{aligned} z - \frac{1}{z} &= z - z^{-1} \\ &= e^{i\theta} - e^{-i\theta} \\ &= (\cos \theta + i \sin \theta) - (\cos(-\theta) + i \sin(-\theta)) \\ &= (\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta) \\ &= 2i \sin \theta \end{aligned}$$

Part (b)

$$\begin{aligned} z^n + z^{-n} &= (e^{i\theta})^n + (e^{i\theta})^{-n} \\ &= e^{in\theta} + e^{-in\theta} \\ &= (\cos n\theta + i \sin n\theta) + (\cos(-n\theta) + i \sin(-n\theta)) \\ &= (\cos n\theta + i \sin n\theta) + (\cos n\theta - i \sin n\theta) \\ &= 2 \cos n\theta \end{aligned}$$

$$\begin{aligned} \sin^6 \theta &= \frac{1}{(2i)^6} (2i \sin \theta)^6 \\ &= -\frac{1}{64} (z - z^{-1})^6 \\ &= -\frac{1}{64} \left[\binom{6}{0} z^6 - \binom{6}{1} z^4 + \binom{6}{2} z^2 - \binom{6}{3} z^0 + \binom{6}{4} z^{-2} - \binom{6}{5} z^{-4} + \binom{6}{6} z^{-6} \right] \\ &= -\frac{1}{64} [-20 + 15(z^2 + z^{-2}) - 6(z^4 + z^{-4}) + (z^6 + z^{-6})] \\ &= -\frac{1}{64} [-20 + 15(2 \cos 2\theta) - 6(2 \cos 4\theta) + 2 \cos 6\theta] \\ &= \frac{1}{32} (10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta) \end{aligned}$$

$$\begin{aligned}\cos^6 \theta &= \frac{1}{2^6} (2 \cos \theta)^6 \\&= \frac{1}{64} (z + z^{-1})^6 \\&= \frac{1}{64} \left[\binom{6}{0} z^6 + \binom{6}{1} z^4 + \binom{6}{2} z^2 + \binom{6}{3} z^0 + \binom{6}{4} z^{-2} + \binom{6}{5} z^{-4} + \binom{6}{6} z^{-6} \right] \\&= \frac{1}{64} [20 + 15(z^2 + z^{-2}) + 6(z^4 + z^{-4}) + (z^6 + z^{-6})] \\&= \frac{1}{64} [20 + 15(2 \cos 2\theta) + 6(2 \cos 4\theta) + 2 \cos 6\theta] \\&= \frac{1}{32} (10 + 15 \cos 2\theta + 6 \cos 4\theta + \cos 6\theta)\end{aligned}$$

$$\begin{aligned}\cos^6 \theta - \sin^6 \theta &= \left[\frac{1}{32} (10 + 15 \cos 2\theta + 6 \cos 4\theta + \cos 6\theta) \right] \\&\quad - \left[\frac{1}{32} (10 - 15 \cos 2\theta + 6 \cos 4\theta - \cos 6\theta) \right] \\&= \frac{1}{32} (30 \cos 2\theta + 2 \cos 6\theta) \\&= \frac{15}{16} \cos 2\theta + \frac{1}{16} \cos 6\theta\end{aligned}$$

$$\boxed{a = \frac{15}{16}, b = \frac{1}{16}}$$