Problem 1.

Differentiate $\arccos\left(\sqrt{1-4x}\right)$ with respect to x, simplifying your answer.

Solution

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\arccos\left(\sqrt{1-4x}\right) &= -\frac{1}{\sqrt{1-(1-4x)}}\cdot\frac{1}{2\sqrt{1-4x}}\cdot -4 \\ &= \frac{2}{\sqrt{4x}\sqrt{1-4x}} \\ &= \frac{1}{\sqrt{x-4x^2}}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\arccos\left(\sqrt{1-4x}\right) = \frac{1}{\sqrt{x-4x^2}}}$$

Problem 2.

It is given that x and y satisfy the equation $xy^2 = \ln(x^2 e^y) - \frac{2e}{x}$.

(a) Verify that $(e, 0)$ satisfies the equation.

(b) Hence, show that at $y = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{k}{e}$, where k is a constant to be determined.

Solution

Part (a)

Substituting $x = e$ and $y = 0$ into the given equation,

$$\begin{aligned} \text{LHS} &= e \cdot 0^2 = 0 \\ \text{RHS} &= \ln\left(e^2 \cdot e^0\right) - \frac{2e}{e} = 2 - 2 = 0 \end{aligned}$$

Hence, LHS = RHS. Thus, $(e, 0)$ satisfies the equation.

Part (b)

From the given equation, we have

$$xy^2 = 2\ln x + y - \frac{2e}{x}.$$

Implicitly differentiating yields

$$x \cdot 2y \frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = \frac{2}{x} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2e}{x^2}.$$

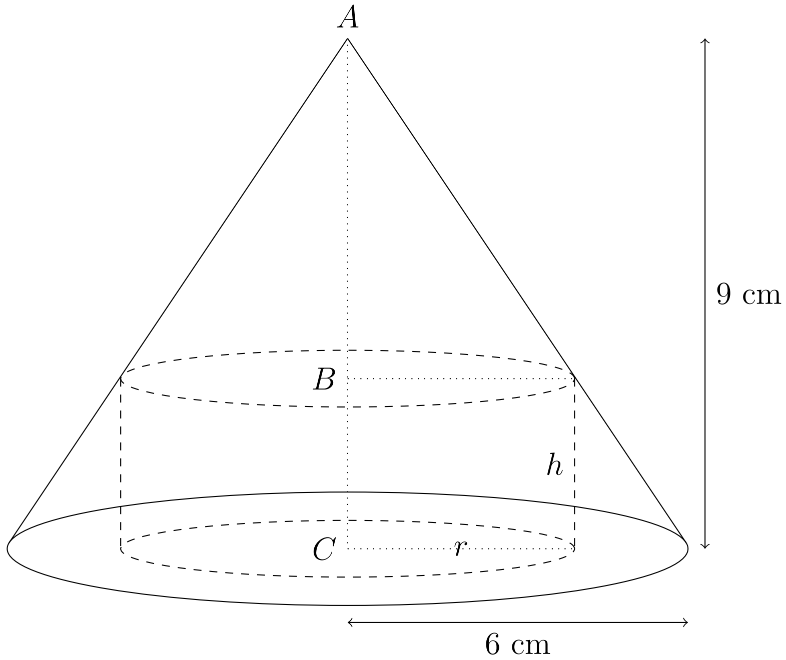
Substituting $x = e$ and $y = 0$ into the above equation gives

$$\begin{aligned} & e \cdot 2 \cdot 0 \frac{\mathrm{d}y}{\mathrm{d}x} + 0^2 = \frac{2}{e} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2e}{e^2} \\ \implies & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4}{e} \end{aligned}$$

$$\boxed{k = -4}$$

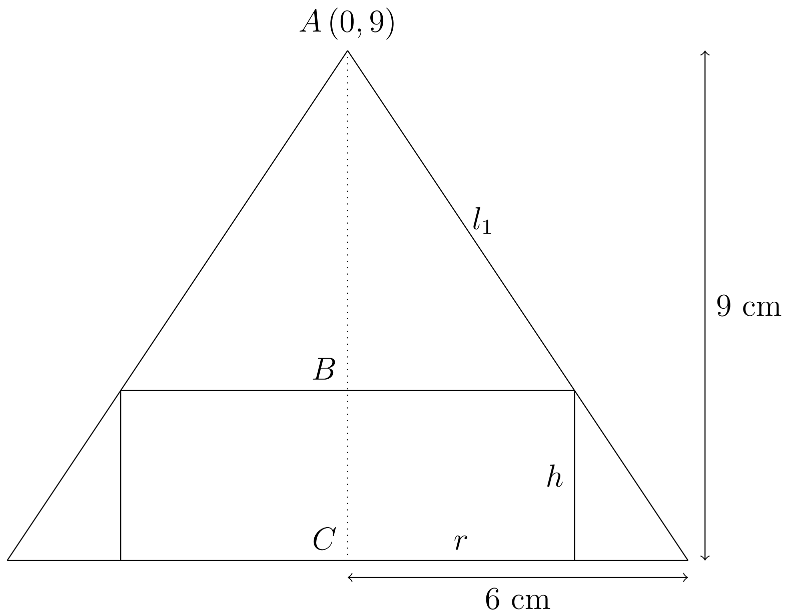
Problem 3.

A toy manufacturer wants to make a toy in the shape of a right circular cone with a cylinder drilled out, as shown in the diagram below. The cylinder is inscribed in the cone. The circumference of the top of the cylinder is in contact with the inner surface of the cone and the base of the cylinder is level with the base of the cone. The base radius of the cylinder is r cm and the base radius of the cone is 6 cm. The height of the cylinder, BC, is h cm and the height of the cone, AC is 9 cm.



Using differentiation, find the minimum volume of the toy, V cm^3, in terms of π.

Solution



Consider the diagram above. Let C be the origin. Note that l_1 has gradient $-\frac{9}{6} = -\frac{3}{2}$.

Hence, l_1 has equation

$$l_1 : y = 9 - \frac{3}{2}x.$$

When $x = r$, we have $y = 9 - \frac{3}{2}r$. Thus, the height of the cylinder is $\left(9 - \frac{3}{2}r\right)$ cm. Let the volume of the cylinder be V_1 cm^3.

$$\begin{aligned} V_1 &= \pi r^2 h \\ &= \pi r^2 \left(9 - \frac{3}{2}r\right) \\ &= 9\pi r^2 - \frac{3}{2}\pi r^3 \end{aligned}$$

For stationary points, $\frac{\mathrm{d}V_1}{\mathrm{d}r} = 0$.

$$\begin{aligned} & \frac{\mathrm{d}V_1}{\mathrm{d}r} = 0 \\ \implies & 18\pi r - \frac{9}{2}\pi r^2 = 0 \\ \implies & \frac{9}{2}\pi r(4 - r) = 0 \end{aligned}$$

Hence, V_1 has a stationary point when $r = 4$. Note that we reject $r = 0$ since $r > 0$.

r	4^-	4	4^+
$\frac{\mathrm{d}V_1}{\mathrm{d}r}$	+ve	0	−ve

By the first derivative test, V_1 attains a maximum when $r = 4$. Hence,

$$\begin{aligned} \min V &= \text{Volume of cone} - \max V_1 \\ &= \left(\frac{1}{3}\pi \cdot 6^2 \cdot 9\right) - \left(9\pi \cdot 4^2 - \frac{3}{2}\pi \cdot 4^3\right) \\ &= 60\pi \end{aligned}$$

$\boxed{\text{The minimum volume of the toy is } 60\pi \text{ cm}^3.}$

Problem 4.

A curve C has parametric equations

$$x = 2\theta + \sin 2\theta,\ y = \cos 2\theta,\ 0 \leq \theta \leq \pi.$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$, expressing your answer in terms of only a single trigonometric function.

(b) Hence, find the coordinates of point Q, on C, whose tangent is parallel to the y-axis.

Solution

Part (a)

Note that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2 + 2\cos 2\theta$ while $\frac{\mathrm{d}y}{\mathrm{d}\theta} = -2\sin 2\theta$. Hence,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} \\
&= \frac{-2\sin 2\theta}{2 + 2\cos 2\theta} \\
&= -\frac{\sin 2\theta}{1 + \cos 2\theta} \\
&= -\frac{2\sin\theta\cos\theta}{1 + (2\cos^2\theta - 1)} \\
&= -\frac{2\sin\theta\cos\theta}{2\cos^2\theta} \\
&= -\frac{\sin\theta}{\cos\theta} \\
&= -\tan\theta
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\tan\theta}$$

Part (b)

Since the tangent at Q is parallel to the y-axis, the derivative $\mathrm{d}y/\mathrm{d}x = -\tan\theta$ must be undefined there. Hence, $\cos\theta = 0 \implies \theta = \pi/2$. Substituting $\theta = \pi/2$ into the given parametric equations, we obtain $x = \pi$ and $y = -1$, whence $Q(\pi, -1)$.

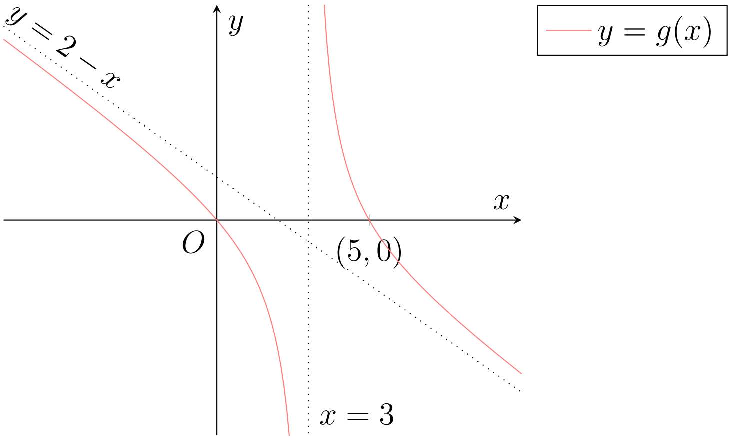
$$\boxed{Q(\pi, -1)}$$

Problem 5.

(a) A function is defined as $f(x) = a(2-x)^2 - b$, where a and b are positive constants such that $a < 1$ and $b > 4$.

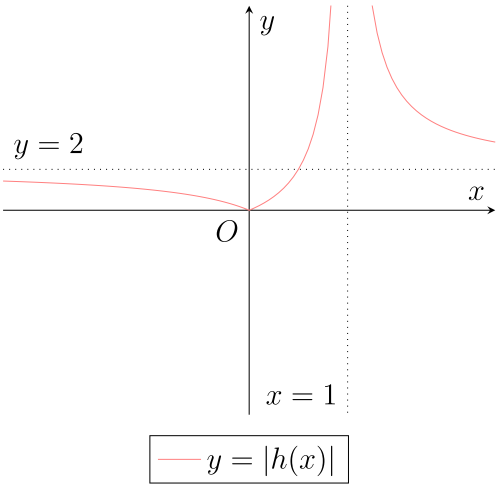
State a sequence of transformations that will transform the curve with equation $y = x^2$ on to the curve with equation $y = f(x)$.

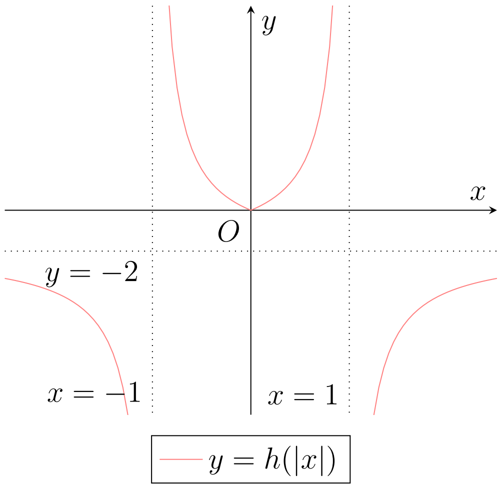
(b) The diagram shows the graph of $y = g(x)$. The lines $x = 3$ and $y = 2 - x$ are asymptotes to the curve, and the graph passes through the points $(0, 0)$ and $(5, 0)$.



Sketch the graph of $y = \dfrac{1}{g(x)}$, indicating clearly the coordinates of any axial intercepts (where applicable) and the equations of any asymptotes.

(c) Given the graphs of $y = |h(x)|$ and $y = h(|x|)$ below, sketch the two possible graphs of $y = h(x)$.



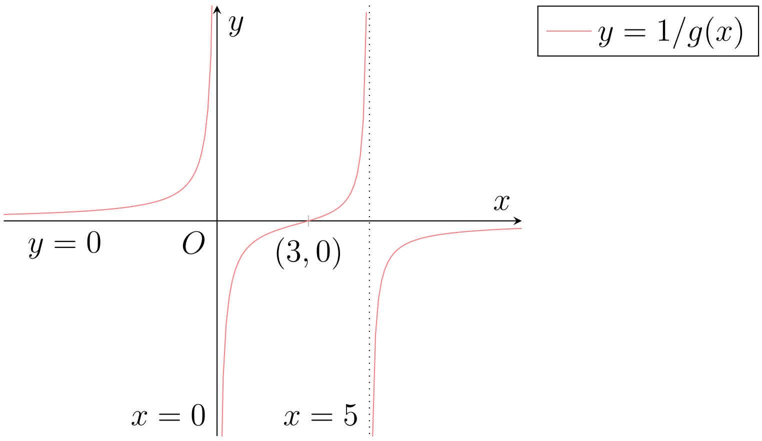


Solution

Part (a)

1. Translate the graph 2 units in the positive x-direction.
2. Scale the graph by a factor of a parallel to the y-axis.
3. Translate the graph b units in the negative y-direction.

Part (b)



Part (c)