Problem 1.

In response to a massive ecosystem-wide destruction by goats on the island of Isabela in Ecuador, Project Isabela was started on the first day of 1997 to eliminate all goats on the island. Goat elimination was done by hunting at a constant rate. Suppose that the goat population, P (in thousands), can be modelled by the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P}{4}\left(1 - \frac{P}{150}\right) - H,$$

where t is measured in months and H is measured in thousands.

(a) State, in context, the significance of the term H.

(b) Find the greatest integer value of H for which it is still possible for some goats to survive in the long run.

(c) Based on the answer from part (b), discuss the long-term behaviour of the goat population for different initial populations.

The hunters involved in Project Isabela finally managed to eliminate all the goats on the island of Isabela on the first day of 2006.

(d) State an inequality that must be satisfied by H.

(e) Given that the initial goat population was 100 thousand, find the value of H, correc to 3 decimal places.

Solution

Part (a)

H represents the number of goats killed (in thousands) per month.

Part (b)

Consider the equilibrium points of the differential equation.

$$\begin{aligned}
&& \frac{\mathrm{d}P}{\mathrm{d}t} &= 0 \\
\implies && \frac{P}{4}\left(1 - \frac{P}{150}\right) - H &= 0 \\
\implies && -\frac{1}{600}\left(P^2 - 150P + 600H\right) &= 0 \\
\implies && P^2 - 150P + 600H &= 0
\end{aligned}$$

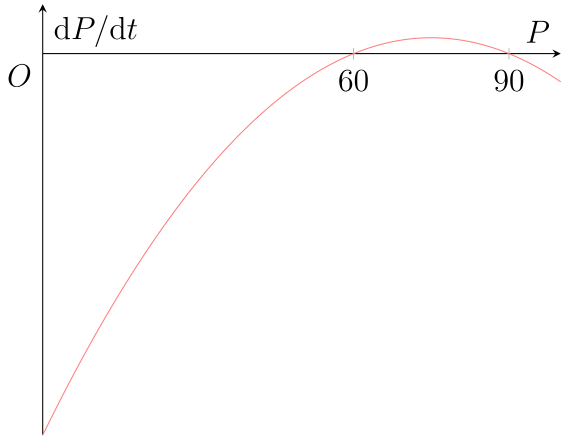
Hence, by the quadratic formula,

$$P = \frac{150 \pm \sqrt{150^2 - 4(600H)}}{2} = 75 \pm 5\sqrt{225 - 24H}.$$

For it to be possible for goats to survive in the long term, there must be at least one equilibrium point. That is, $\sqrt{225 - 24H} \geq 0 \implies H \leq 9.375$. Thus, the maximum integer value of H is $\boxed{9}$.

Part (c)

When $H = 9$, the equilibrium points are $P = 75 \pm 5\sqrt{225 - 24 \cdot 9} = 60, 90$. Let the initial population be P_0.



When $P_0 = 0$, there are no goats initially. Hence, the population will remain at 0.

When $0 < P_0 < 60$, $\mathrm{d}P/\mathrm{d}t < 0$. Hence, the population of goats will decrease towards 0.

When $P_0 = 60$, $\mathrm{d}P/\mathrm{d}t = 0$. Hence, the population of goats will remain at 60 thousand.

When $60 < P_0 < 90$, $\mathrm{d}P/\mathrm{d}t > 0$. Hence, the population of goats will increase towards 90.

When $P_0 = 90$, $\mathrm{d}P/\mathrm{d}t = 0$. Hence, the population of goats will remain at 90 thousand.

When $P_0 > 90$, $\mathrm{d}P/\mathrm{d}t < 0$. Hence, the population of goats will decrease towards 90.

Part (d)

$$\boxed{H > 9.375}$$

Part (e)

Note that $t = 10 \cdot 12 = 120$, $P(0) = 100$ and $P(120) = 0$.

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{P}{4}\left(1 - \frac{P}{150}\right) - H \\
&= -\frac{1}{600}\left(P^2 - 150P + 600H\right) \\
&= -\frac{1}{600}\left[(P-75)^2 + (600H - 75^2)\right] \\
\implies \quad \frac{1}{(P-75)^2 + (600H-75^2)}\frac{\mathrm{d}P}{\mathrm{d}t} &= -\frac{1}{600} \\
\implies \quad \int \frac{1}{(P-75)^2 + (600H-75^2)}\frac{\mathrm{d}P}{\mathrm{d}t}\,\mathrm{d}t &= \int -\frac{1}{600}\,\mathrm{d}t \\
\implies \quad \int \frac{1}{(P-75)^2 + (600H-75^2)}\,\mathrm{d}P &= -\frac{1}{600}t + C \\
\implies \frac{1}{\sqrt{600H - 75^2}}\arctan\left(\frac{P-75}{\sqrt{600H-75^2}}\right) &= -\frac{1}{600}t + C
\end{aligned}$$

Note that $600H - 75^2 > 0$ since $H > 9.375$.

When $t = 0$, $P = 100$. Hence,

$$C = \frac{1}{\sqrt{600H - 75^2}} \arctan\left(\frac{25}{\sqrt{600H - 75^2}}\right)$$

When $t = 120$, $P = 0$. Hence,

$$\frac{1}{\sqrt{600H - 75^2}} \arctan\left(\frac{-75}{\sqrt{600H - 75^2}}\right) = -\frac{120}{600} + \frac{1}{\sqrt{600H - 75^2}} \arctan\left(\frac{25}{\sqrt{600H - 75^2}}\right)$$

Let $X = \dfrac{1}{\sqrt{600H - 75^2}}$. The above equation simplifies to

$$X \arctan(-75X) = -\frac{1}{5} + X \arctan(25X)$$

which has the solution $X = 0.079667$. Note that we reject $X = -0.079667$ since $X \geq 0$. We thus have

$$H = \frac{1}{600}\left(\frac{1}{0.079667^2} + 75^2\right) = \boxed{9.638} \text{ (3 d.p.)}$$