

Problem 1.

Calculate the exact length of each of the arcs of the following curves.

- (a) $y^3 = x^2$ for $-1 \leq x \leq 1$.
- (b) $x = t^2 - 1$, $y = t^3 + 1$ from $t = 0$ to $t = 1$.
- (c) $r = a \cos \theta$ from $\theta = 0$ to $\theta = \pi/2$.

Solution**Part (a)**

Note that $y^3 = x^2 \implies y = x^{2/3} \implies \frac{dy}{dx} = \frac{2}{3}x^{-1/3}$.

$$\begin{aligned}
 \text{Length} &= \int_{-1}^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\
 &= \int_{-1}^1 \sqrt{1 + \left(\frac{2}{3}x^{-1/3}\right)^2} dx \\
 &= \int_{-1}^1 \sqrt{1 + \frac{4}{9}x^{-2/3}} dx \\
 &= 2 \int_0^1 \sqrt{1 + \frac{4}{9}x^{-2/3}} dx \\
 &= 2 \int_0^1 \sqrt{x^{-2/3} \left(x^{2/3} + \frac{4}{9}\right)} dx \\
 &= 2 \int_0^1 x^{-1/3} \sqrt{x^{2/3} + \frac{4}{9}} dx \\
 &= 2 \int_{4/9}^{13/9} \sqrt{u} \cdot \frac{3}{2} du \\
 &= 3 \left[\frac{2}{3} u^{3/2} \right]_{4/9}^{13/9} \\
 &= \frac{2}{27} (13\sqrt{13} - 8)
 \end{aligned}$$

$$\begin{aligned}
 u &= x^{2/3} + \frac{4}{9} \\
 du &= \frac{2}{3}x^{-1/3} dx
 \end{aligned}$$

The arc length of the curve is $\frac{2}{27} (13\sqrt{13} - 8)$ units.

Part (b)

Since the arc length of a curve is invariant under translation, it suffices to find the arc length of the curve with parametric equations $x = t^2, y = t^3$, $0 \leq t \leq 1$. The Cartesian equation of this curve is $y = x^{3/2}$, $0 \leq x \leq 1$, which is the inverse of $y = x^{2/3}$, $0 \leq x \leq 1$. From part (a), the required arc length is $\frac{1}{2} \cdot \frac{2}{27} (13\sqrt{13} - 8) = \frac{1}{27} (13\sqrt{13} - 8)$.

The arc length of the curve is $\frac{1}{27} (13\sqrt{13} - 8)$ units.

Part (c)

Since $r = a \cos \theta$, $0 \leq \theta \leq \frac{\pi}{2}$ describes the top half of a circle with centre $\left(\frac{a}{2}, 0\right)$ and diameter a , the arc length of the curve is $\frac{1}{2} \cdot \pi a = \frac{\pi}{2}a$.

The arc length of the curve is $\frac{\pi}{2}a$ units.

Problem 2.

Find the exact areas of the surfaces generated by completely rotating the following arcs about the (i) x -axis and (ii) y -axis.

- (a) The line $2y = x$ between the origin and the point $(4, 2)$.
 (b) The curve $x = t^3 - 3t + 2$, $y = 3(t^2 - 1)$, $t \in \mathbb{R}$ from $t = 1$ to $t = 2$.

Solution**Part (a)****Subpart (i)**

When rotated about the x -axis, the curve forms a cone with slant height $\sqrt{4^2 + 2^2} = 2\sqrt{5}$ and radius 2. Hence, the required surface area is $\pi \cdot 2 \cdot 2\sqrt{5} = 4\sqrt{5}\pi$.

The surface area is $4\sqrt{5}\pi$ units².

Subpart (ii)

When rotated about the y -axis, the curve forms a cone with slant height $\sqrt{4^2 + 2^2} = 2\sqrt{5}$ and radius 4. Hence, the required surface area is $\pi \cdot 4 \cdot 2\sqrt{5} = 8\sqrt{5}\pi$.

The surface area is $8\sqrt{5}\pi$ units².

Part (b)

Note that $x = t^3 - 3t + 2 \implies \frac{dx}{dt} = 3t^2 - 3$ and $y = 3(t^2 - 1) \implies \frac{dy}{dt} = 6t$.

Subpart (i)

$$\begin{aligned}
 \text{Area} &= 2\pi \int_1^2 y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= 2\pi \int_1^2 3(t^2 - 1) \sqrt{(3t^2 - 3)^2 + (6t)^2} dt \\
 &= 6\pi \int_1^2 (t^2 - 1) \sqrt{9t^4 + 18t^2 + 9} dt \\
 &= 6\pi \int_1^2 (t^2 - 1) \sqrt{(3t^2 + 3)^2} dt \\
 &= 18\pi \int_1^2 (t^2 - 1)(t^2 + 1) dt \\
 &= 18\pi \int_1^2 (t^4 - 1) dt \\
 &= 18\pi \left[\frac{1}{5}t^5 - t \right]_1^2 \\
 &= \frac{468}{5}\pi
 \end{aligned}$$

The surface area is $\frac{468}{5}\pi$ units².

Subpart (ii)

$$\begin{aligned}\text{Area} &= 2\pi \int_1^2 x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\&= 2\pi \int_1^2 (t^3 - 3t + 2) \sqrt{(3t^2 - 3)^2 + (6t)^2} dt \\&= 6\pi \int_1^2 (t^3 - 3t + 2) (t^2 + 1) dt \\&= 6\pi \int_1^2 (t^5 - 2t^3 + 2t^2 - 3t + 2) dt \\&= 6\pi \left[\frac{1}{6}t^6 - \frac{2}{4}t^4 - \frac{2}{3}t^3 - \frac{3}{2}t^2 + 2t \right]_1^2 \\&= 31\pi\end{aligned}$$

The surface area is 31π units².

Problem 3.

The section of the curve $y = e^x$ between $x = 0$ and $x = 1$ is rotated through one revolution about

- (a) the x -axis.
- (b) the y -axis.

Find the numerical values of the areas of the surfaces obtained.

Solution**Part (a)**

$$\begin{aligned}\text{Area} &= 2\pi \int_0^1 y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} dx \\ &= 22.9 \text{ (3 s.f.)}\end{aligned}$$

The surface area is 22.9 units.

Part (b)

Note that $y = e^x \implies x = \ln y$ and $\frac{dy}{dx} = e^x \implies \frac{dx}{dy} = e^{-x}$.

$$\begin{aligned}\text{Area} &= 2\pi \int_1^e x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= 2\pi \int_0^1 \ln y \sqrt{1 + e^{-2x}} dx \\ &= 7.05 \text{ (3 s.f.)}\end{aligned}$$

The surface area is 7.05 units.

Problem 4.

The curve $y^2 = \frac{1}{3}x(1-x)^2$ has a loop between $x = 0$ and $x = 1$. Prove that the total length of the loop is $\frac{4\sqrt{3}}{3}$.

Solution

Since the curve is even in y , it is symmetric about the x -axis. We thus only consider the part of the curve above the x -axis, i.e. $y \geq 0$, where $y = \sqrt{\frac{1}{3}x(1-x)^2}$. Differentiating,

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2} \left[\frac{1}{3}x(1-x)^2 \right]^{-1/2} \cdot \frac{1}{3} [(1-x)^2 + x \cdot 2(1-x) \cdot -1] \\ &= \frac{1}{6} \left[\frac{1}{3}x(1-x)^2 \right]^{-1/2} (3x^2 - 4x + 1) \\ &= \frac{1}{6} \left[\frac{1}{3}x(1-x)^2 \right]^{-1/2} (3x-1)(x-1) \\ &= \frac{\sqrt{3}}{6} x^{-1/2} (x-1)^{-1} (3x-1)(x-1) \\ &= \frac{3x-1}{2\sqrt{3x}}\end{aligned}$$

Hence,

$$\begin{aligned}\text{Length} &= 2 \int_0^1 \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx \\ &= 2 \int_0^1 \sqrt{1 + \left(\frac{3x-1}{2\sqrt{3x}} \right)^2} dx \\ &= 2 \int_0^1 \sqrt{1 + \frac{(3x-1)^2}{12x}} dx \\ &= \frac{2}{\sqrt{12}} \int_0^1 \sqrt{\frac{12x + 9x^2 - 6x + 1}{x}} dx \\ &= \frac{\sqrt{3}}{3} \int_0^1 \sqrt{\frac{9x^2 + 6x + 1}{x}} dx \\ &= \frac{\sqrt{3}}{3} \int_0^1 \sqrt{\frac{(3x+1)^2}{x}} dx \\ &= \frac{\sqrt{3}}{3} \int_0^1 \frac{3x+1}{\sqrt{x}} dx \\ &= \frac{\sqrt{3}}{3} \left[\frac{3}{3/2} x^{3/2} + \frac{1}{1/2} x^{1/2} \right]_0^1 \\ &= \frac{4\sqrt{3}}{3}\end{aligned}$$

Problem 5.

The tangent at a point P on the curve $x = a \left(t - \frac{1}{3}t^3 \right)$, $y = at^2$ cuts the x -axis at T . Prove that the distance of the point T from the origin O is half the length of the arc OP .

Solution

Let P be the point on the curve where $t = t_P$. Note that $x = a \left(t - \frac{1}{3}t^3 \right) \Rightarrow \frac{dx}{dt} = a(1 - t^2)$ and $y = at^2 \Rightarrow \frac{dy}{dt} = 2at$.

$$\begin{aligned}
 \text{Length of arc } OP &= \int_0^{t_P} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^{t_P} \sqrt{[a(1 - t^2)]^2 + (2at)^2} dt \\
 &= a \int_0^{t_P} \sqrt{t^4 + 2t^2 + 1} dt \\
 &= a \int_0^{t_P} \sqrt{(t^2 + 1)^2} dt \\
 &= a \int_0^{t_P} (t^2 + 1) dt \\
 &= a \left[\frac{1}{3}t^3 + t \right]_0^{t_P} \\
 &= a \left(\frac{1}{3}t_P^3 + t_P \right)
 \end{aligned}$$

Note that $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{2at}{a(1 - t^2)} = \frac{2t}{1 - t^2}$. Hence, the equation of the tangent at P is given by

$$y - at_P^2 = \frac{2t_P}{1 - t_P^2} \left[x - a \left(t_P - \frac{1}{3}t_P^3 \right) \right]$$

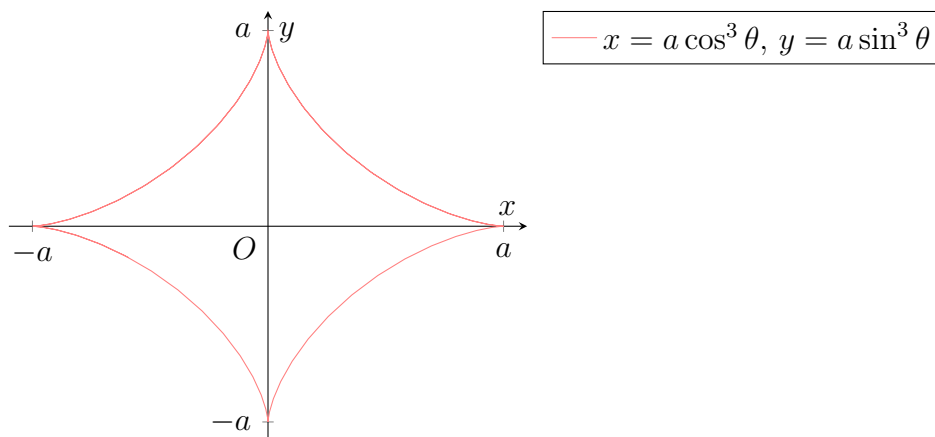
At T , $x = OT$ and $y = 0$. Hence,

$$\begin{aligned}
 -at_P^2 &= \frac{2t_P}{1 - t_P^2} \left[OT - a \left(t_P - \frac{1}{3}t_P^3 \right) \right] \\
 \Rightarrow -at_P &= \frac{2}{1 - t_P^2} \left[OT - a \left(t_P - \frac{1}{3}t_P^3 \right) \right] \\
 \Rightarrow OT &= \frac{-at_P(1 - t_P^2)}{2} + a \left(t_P - \frac{1}{3}t_P^3 \right) \\
 &= \frac{1}{2}a \left(t_P^3 - t_P + 2t_P - \frac{2}{3}t_P^3 \right) \\
 &= \frac{1}{2}a \left(\frac{1}{3}t_P^3 + t_P \right) \\
 &= \frac{1}{2} \cdot \text{length of arc } OP
 \end{aligned}$$

Problem 6.

Sketch the curve whose parametric equations are $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, $a > 0$.

- (a) Find the total length of the curve.
- (b) The portion of the curve in the first quadrant is revolved through four right angles about the x -axis. Prove that the area of the surface thus formed is $\frac{6}{5}\pi a^2$.

Solution**Part (a)**

By symmetry, we only consider the length of the curve in the first quadrant. Note that $x = 0 \implies \theta = \frac{\pi}{2}$ and $x = a \implies \theta = 0$. Also, $x = a \cos^3 \theta \implies \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta$ and $y = a \sin^3 \theta \implies \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$.

$$\begin{aligned}
 \text{Length} &= 4 \int_0^{\pi/2} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\
 &= 4 \int_0^{\pi/2} \sqrt{(-3a \cos^2 \theta \sin \theta)^2 + (3a \sin^2 \theta \cos \theta)^2} d\theta \\
 &= 12a \int_0^{\pi/2} \sqrt{\cos^4 \theta \sin^2 \theta + \sin^4 \theta \cos^2 \theta} d\theta \\
 &= 12a \int_0^{\pi/2} \sqrt{\cos^2 \theta \sin^2 \theta (\cos^2 \theta + \sin^2 \theta)} d\theta \\
 &= 12a \int_0^{\pi/2} \cos \theta \sin \theta d\theta \\
 &= 12a \left[\frac{\sin^2 \theta}{2} \right]_0^{\pi/2} \\
 &= 6a
 \end{aligned}$$

The total length of the curve is $6a$ units.

Part (b)

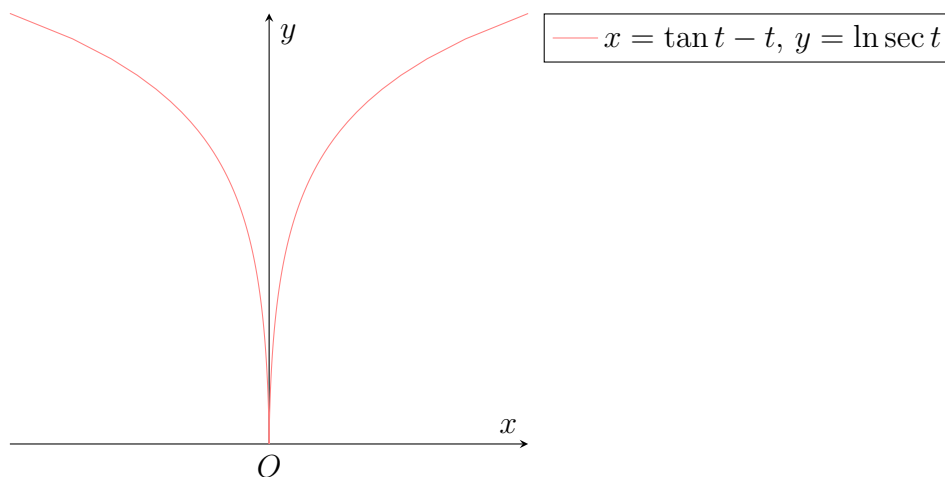
$$\begin{aligned}\text{Area} &= 2\pi \int_0^{\pi/2} x \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta \\&= 2\pi \int_0^{\pi/2} a \cos^3 \theta \cdot 3a \cos \theta \sin \theta d\theta \\&= 3\pi a^2 \cdot 2 \int_0^{\pi/2} \sin \theta \cos^4 \theta d\theta \\&= 3\pi a^2 B(1, 5/2) \\&= 3\pi a^2 \cdot \frac{\Gamma(1)\Gamma(5/2)}{\Gamma(1 + 5/2)} \\&= 3\pi a^2 \cdot \frac{\Gamma(5/2)}{\Gamma(5/2) \cdot 5/2} \\&= 3\pi a^2 \cdot \frac{2}{5} \\&= \frac{6}{5}\pi a^2\end{aligned}$$

Problem 7.

The parametric equations of a curve are given by

$$x = \tan t - t, \quad y = \ln \sec t, \quad t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

- Sketch the curve.
- Prove that the arc length of the curve measured from the origin to the point $\left(1 - \frac{\pi}{4}, \frac{1}{2} \ln 2\right)$ is $\sqrt{2} - 1$.
- The arc in (b) is rotated about the x -axis through an angle of 360° . Find the exact surface area formed.

Solution**Part (a)****Part (b)**

Note that $x = 0 \implies t = 0$ and $x = 1 - \frac{\pi}{4} \implies t = \frac{\pi}{4}$. Further, $x = \tan t - t \implies \frac{dx}{dt} = \sec^2 t - 1 = \tan^2 t$ and $y = \ln \sec t \implies \frac{dy}{dt} = \tan t$.

$$\begin{aligned}
 \text{Length} &= \int_0^{\pi/4} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^{\pi/4} \sqrt{(\tan^2 t)^2 + (\tan t)^2} dt \\
 &= \int_0^{\pi/4} \tan t \sqrt{\tan^2 t + 1} dt \\
 &= \int_0^{\pi/4} \tan t \sec t dt \\
 &= [\sec t]_0^{\pi/4} \\
 &= \sqrt{2} - 1
 \end{aligned}$$

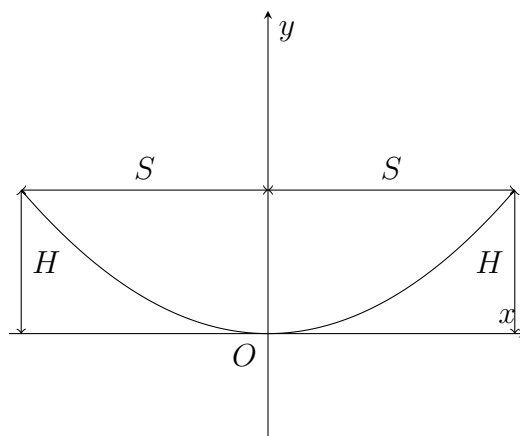
Part (c)

$$\begin{aligned}
 \text{Area} &= 2\pi \int_0^{\pi/4} y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= 2\pi \int_0^{\pi/4} \ln \sec t \cdot \tan t \sec t dt
 \end{aligned}$$

D	I
$+$ $\ln \sec t$	$\tan t \sec t$
$-$ $\tan t$	$\sec t$

$$\begin{aligned}
 &= 2\pi \left[[\sec t \ln \sec t]_0^{\pi/4} - \int_0^{\pi/4} \tan t \sec t dt \right] \\
 &= 2\pi \left[\sqrt{2} \cdot \frac{1}{2} \ln 2 - (\sqrt{2} - 1) \right] \\
 &= \sqrt{2}\pi (\ln 2 - 2 + \sqrt{2})
 \end{aligned}$$

The surface area is $\sqrt{2}\pi (\ln 2 - 2 + \sqrt{2})$ units².

Problem 8.

The diagram shows a cable for a suspension bridge, which has the shape of a parabola with equation $y = kx^2$. The suspension bridge has a total span $2S$ and the height of the cable relative to the lowest point is H at each end. Show that the total length of the cable is $L = 2 \int_0^S \sqrt{1 + \frac{4H^2}{S^4} x^2} dx$.

- (a) Engineers from country A proposed a suspension bridge across a strait of 8 km wide to country B . The plan included suspension towers 380 m high at each end. Find the length of the parabolic cable for this proposed bridge to the nearest metre.
- (b) By using the result $\frac{d}{dx} \ln(x + \sqrt{a^2 + x^2}) = \frac{1}{\sqrt{a^2 + x^2}}$ or otherwise, find L in terms of S and H .

Solution

By symmetry, we only need to consider the length of the curve where $x \geq 0$. Since (S, H) is on the curve, $H = kS^2 \implies k = \frac{H}{S^2}$. Note that $y = kx^2 \implies \frac{dy}{dx} = 2kx = \frac{2H}{S^2}x$. Hence,

$$\begin{aligned} L &= 2 \int_0^S \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \\ &= 2 \int_0^S \sqrt{1 + \frac{4H^2}{S^4} x^2} dx \end{aligned}$$

Part (a)

Note that $2S = 8000 \implies S = 4000$ and $H = 380$. Hence,

$$\begin{aligned} L &= 2 \int_0^{4000} \sqrt{1 + \frac{4(380)^2}{(4000)^4} x^2} dx \\ &= 8048 \text{ (to nearest integer)} \end{aligned}$$

The cable is 8048 m long.

Part (b)

Consider the integral $I = \int \sqrt{1 + (kx)^2} dx$.

$$\begin{aligned} I &= \int \sqrt{1 + (kx)^2} dx \\ &= \frac{1}{k} \int \sqrt{1 + \tan^2 \theta} \sec^2 \theta d\theta \\ &= \frac{1}{k} \int \sec^3 \theta d\theta \end{aligned}$$

$$\begin{aligned} \tan \theta &= kx \\ \sec^2 \theta d\theta &= k dx \end{aligned}$$

D	I
$+$ $\sec \theta$	$\sec^2 \theta$
$-$ $\sec \theta \tan \theta$	$\tan t$

$$\begin{aligned} &= \frac{1}{k} \left(\sec \theta \tan \theta - \int \sec \theta \tan^2 \theta d\theta \right) \\ &= \frac{1}{k} \left(\sec \theta \tan \theta - \int \sec \theta (\sec^2 \theta - 1) d\theta \right) \\ &= \frac{1}{k} \left(\sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta \right) \\ &= \frac{1}{k} (\sec \theta \tan \theta - I + \ln |\sec \theta + \tan \theta|) \\ \implies 2I &= \frac{1}{k} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C \end{aligned}$$

$$\implies I = \frac{1}{2k} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) + C$$

$$\begin{aligned} \sec \theta &= \sqrt{\tan^2 \theta + 1} \\ &= \sqrt{(kx)^2 + 1} \end{aligned}$$

$$= \frac{1}{2k} \left[\sqrt{(kx)^2 + 1} \cdot kx + \ln \left| \sqrt{(kx)^2 + 1} + kx \right| \right] + C$$

In our case, $k = \frac{2H}{S^2} > 0$.

$$\begin{aligned} L &= 2 \left[\frac{1}{2} \cdot \frac{S^2}{2H} \left(\sqrt{\left(\frac{2H}{S^2} x \right)^2 + 1} \cdot \frac{2H}{S^2} x + \ln \left(\sqrt{\left(\frac{2H}{S^2} x \right)^2 + 1} + \frac{2H}{S^2} x \right) \right) \right]_0^S \\ &= \frac{S^2}{2H} \left(\sqrt{\left(\frac{2H}{S^2} S \right)^2 + 1} \cdot \frac{2H}{S^2} S + \ln \left(\sqrt{\left(\frac{2H}{S^2} S \right)^2 + 1} + \frac{2H}{S^2} S \right) \right) \\ &= \sqrt{\left(\frac{2H}{S^2} S \right)^2 + 1} \cdot S + \frac{S^2}{2H} \ln \left(\sqrt{\left(\frac{2H}{S^2} S \right)^2 + 1} + \frac{2H}{S^2} S \right) \\ &= \sqrt{\left(\frac{2H}{S} \right)^2 + 1} \cdot S + \frac{S^2}{2H} \ln \left(\sqrt{\left(\frac{2H}{S} \right)^2 + 1} + \frac{2H}{S} \right) \end{aligned}$$

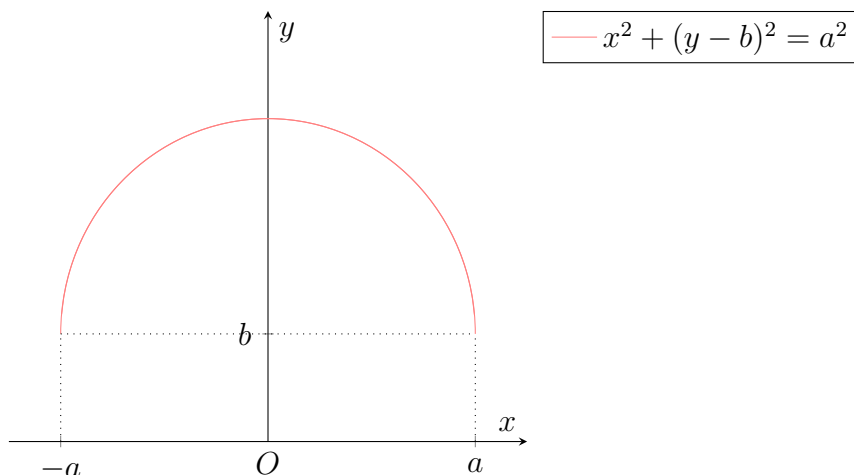
$$\begin{aligned} &= \sqrt{\frac{4H^2}{S^2} + 1} \cdot S + \frac{S^2}{2H} \ln \left(\sqrt{\frac{4H^2}{S^2} + 1} + \frac{2H}{S} \right) \\ &= \sqrt{4H^2 + S^2} + \frac{S^2}{2H} \ln \left(\frac{\sqrt{4H^2 + S^2} + 2H}{S} \right) \end{aligned}$$

$$L = \sqrt{4H^2 + S^2} + \frac{S^2}{2H} \ln \left(\frac{\sqrt{4H^2 + S^2} + 2H}{S} \right)$$

Problem 9.

Sketch the semicircle with equation $x^2 + (y - b)^2 = a^2$, $y \geq b$ where a and b are positive constants.

A solid is formed by rotating the region bounded by the semicircle and its diameter on the line $y = b$ about the x -axis through 4 right angles. Find the total surface area of the solid.

Solution

Note that $x^2 + (y - b)^2 = a^2 \implies y = b + \sqrt{a^2 - x^2}$ since $y \geq b \implies y - b \geq 0$. Hence,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{a^2 - x^2}} \cdot -2x = -\frac{x}{\sqrt{a^2 - x^2}}.$$

$$\begin{aligned}
 \text{Area} &= 2\pi \int_{-a}^a y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx + 2\pi \cdot b \cdot 2a \\
 &= 2\pi \int_{-a}^a (b + \sqrt{a^2 - x^2}) \sqrt{1 + \left(-\frac{x}{\sqrt{a^2 - x^2}}\right)^2} dx + 4\pi ab \\
 &= 4\pi \int_0^a (b + \sqrt{a^2 - x^2}) \sqrt{\frac{(a^2 - x^2) + x^2}{a^2 - x^2}} dx + 4\pi ab \\
 &= 4\pi \int_0^a (b + \sqrt{a^2 - x^2}) \sqrt{\frac{a^2}{a^2 - x^2}} dx + 4\pi ab \\
 &= 4\pi a \int_0^a (b + \sqrt{a^2 - x^2}) \frac{1}{\sqrt{a^2 - x^2}} dx + 4\pi ab \\
 &= 4\pi a \left(b \int_0^a \frac{1}{\sqrt{a^2 - x^2}} dx + \int_0^a dx \right) + 4\pi ab \\
 &= 4\pi a \left(b \left[\arcsin \frac{x}{a} \right]_0^a + [x]_0^a \right) + 4\pi ab \\
 &= 4\pi a \left(b \cdot \frac{\pi}{2} + a \right) + 4\pi ab \\
 &= 2\pi^2 ab + 4\pi a^2 + 4\pi ab
 \end{aligned}$$

The total surface area is $2\pi^2 ab + 4\pi a^2 + 4\pi ab$ units².

Problem 10.

Using polar coordinates with pole O , the curve C has the equation $r = ae^{\theta/k}$, where a and k are positive constants and $0 \leq \theta \leq 2\pi$. The points A and B on the curve corresponds to $\theta = 0$ and $\theta = \beta$ respectively where $0 < \beta < \pi$. The length of the arc AB is denoted by q and the area of the sector OAB is denoted by Q .

- (a) Show that $Q = \frac{1}{4}ka^2(e^{2\beta/k} - 1)$.
- (b) Show that $q = a(1 + k^2)^{1/2}(e^{\beta/k} - 1)$.
- (c) Deduce from the results of parts (a) and (b) that, for large values of k , $\frac{Q}{q} \approx \frac{1}{2}a$.
- (d) Draw a sketch of C for the case where k is large and explain how the result in part (c) can be deduced from the sketch.

Solution**Part (a)**

$$\begin{aligned}
 Q &= \frac{1}{2} \int_0^\beta r^2 d\theta \\
 &= \frac{1}{2} \int_0^\beta (ae^{\theta/k})^2 d\theta \\
 &= \frac{1}{2}a^2 \int_0^\beta e^{2\theta/k} d\theta \\
 &= \frac{1}{2}a^2 \left[\frac{e^{2\theta/k}}{2/k} \right]_0^\beta \\
 &= \frac{1}{4}a^2k(e^{2\beta/k} - 1)
 \end{aligned}$$

Part (b)

Note that $r = ae^{\theta/k} \implies \frac{dr}{d\theta} = \frac{ae^{\theta/k}}{k} = \frac{r}{k}$.

$$\begin{aligned}
 q &= \int_0^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta \\
 &= \int_0^\beta \sqrt{r^2 + \frac{r^2}{k^2}} d\theta \\
 &= \sqrt{1 + k^{-2}} \int_0^\beta r d\theta \\
 &= \sqrt{1 + k^{-2}} \int_0^\beta ae^{\theta/k} d\theta \\
 &= a\sqrt{1 + k^{-2}} \left[\frac{e^{\theta/k}}{1/k} \right]_0^\beta
 \end{aligned}$$

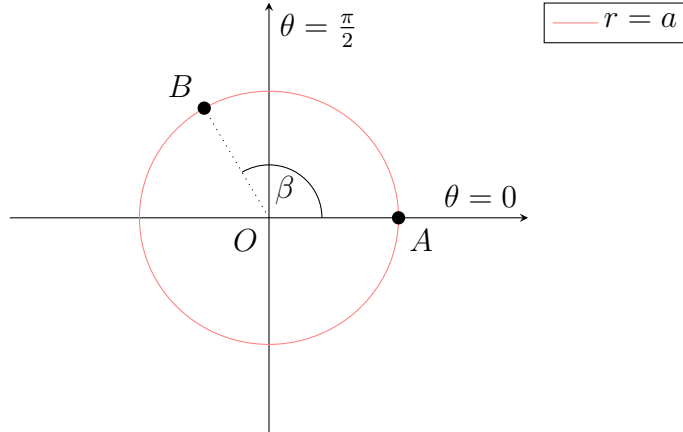
$$\begin{aligned}
&= ak\sqrt{1+k^{-2}}(e^{\beta/k}-1) \\
&= a\sqrt{k^2+1}(e^{\beta/k}-1)
\end{aligned}$$

Part (c)

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{Q}{q} &= \lim_{k \rightarrow \infty} \frac{\frac{1}{4}a^2k(e^{2\beta/k}-1)}{a\sqrt{k^2+1}(e^{\beta/k}-1)} \\
&= \frac{1}{4}a \lim_{k \rightarrow \infty} \left(\frac{k}{\sqrt{k^2+1}} \cdot \frac{e^{2\beta/k}-1}{e^{\beta/k}-1} \right) \\
&= \frac{1}{4}a \lim_{k \rightarrow \infty} \left(\frac{k}{\sqrt{k^2+1}} \right) \lim_{k \rightarrow \infty} \left(\frac{e^{2\beta/k}-1}{e^{\beta/k}-1} \right) \\
&= \frac{1}{4}a \cdot 1 \cdot \lim_{k \rightarrow \infty} \left(\frac{2 \cdot e^{\beta/k} \cdot d/dk(e^{\beta/k})}{d/dk(e^{\beta/k})} \right) \\
&= \frac{1}{2}a \cdot \lim_{k \rightarrow \infty} e^{\beta/k} \\
&= \frac{1}{2}a
\end{aligned}$$

Part (d)

Note that $\lim_{k \rightarrow \infty} r = \lim_{k \rightarrow \infty} ae^{\theta/k} = a$.



As $k \rightarrow \infty$, the curve becomes a circle. Hence, Q is the area of a sector with angle β , and q is the arc length of a sector with angle β . Thus, $\frac{Q}{q} = \left(\frac{\beta}{2\pi} \cdot \pi a^2 \right) / \left(\frac{\beta}{2\pi} \right) 2\pi a = \frac{1}{2}a$.