Problem 1.

Without using a calculator, sketch the following graphs and determine their symmetries.

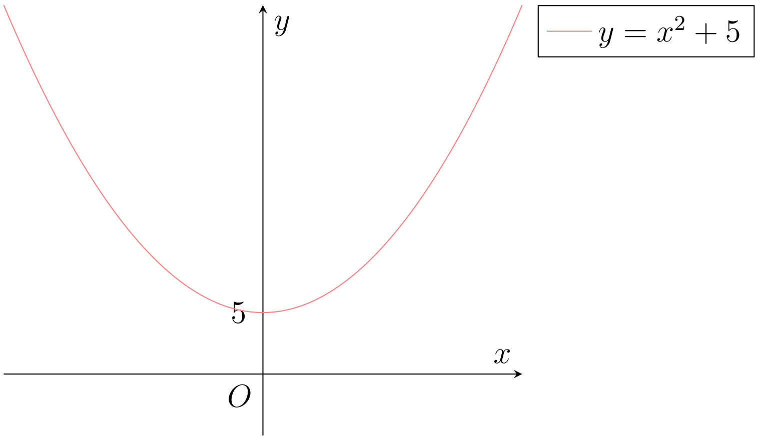
(a) $y = x^2 + 5$

(b) $y = 2x - x^3$

(c) $y = x^2 - 4x + 3$

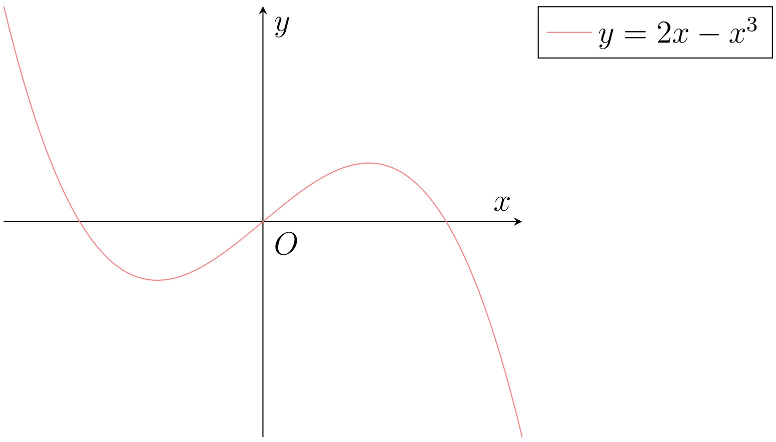
Solution

Part (a)



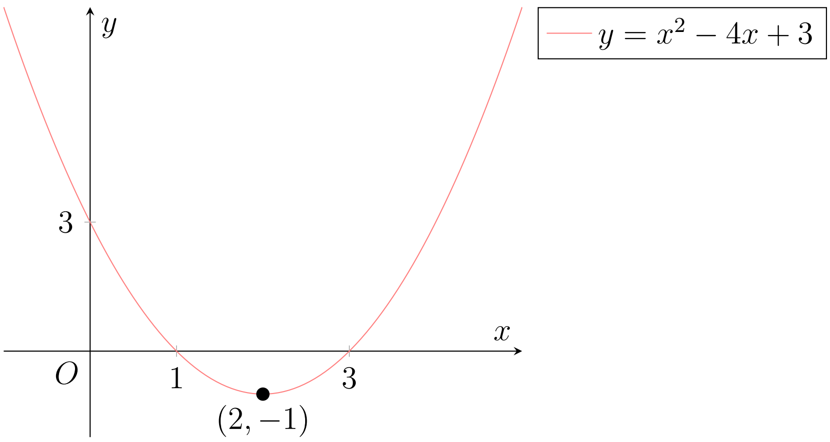
$\boxed{\text{Symmetry: } x = 0}$

Part (b)



$\boxed{\text{Symmetry: } (0, 0)}$

Part (c)



Symmetry: $x = 2$

Problem 2.

Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

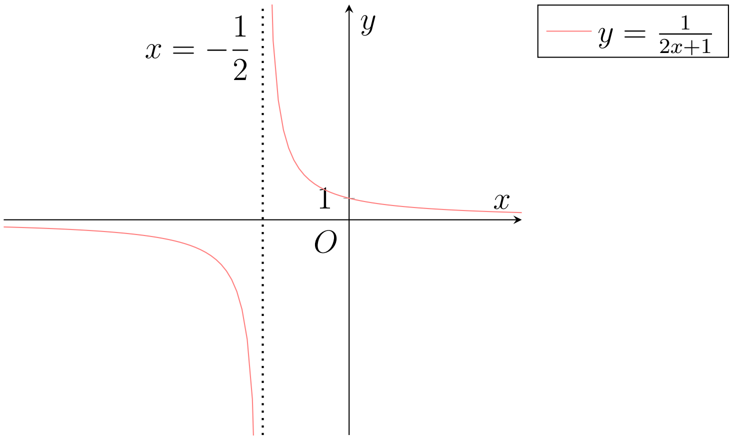
(a) $y = \dfrac{1}{2x+1}$

(b) $y = \dfrac{3x}{x-2}$

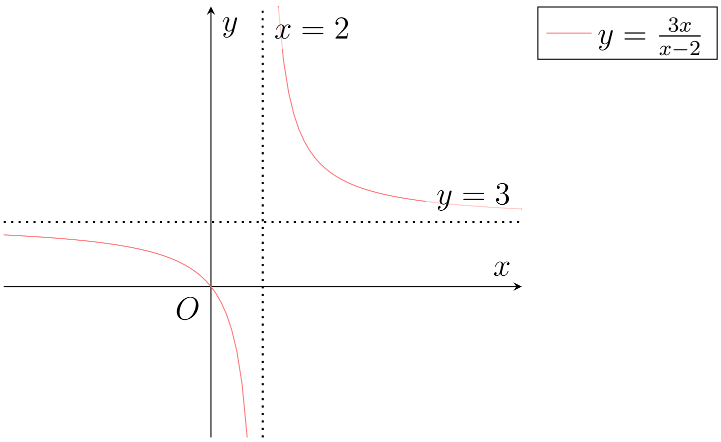
(c) $y = \dfrac{x^2+x-6}{2x-2}$

Solution

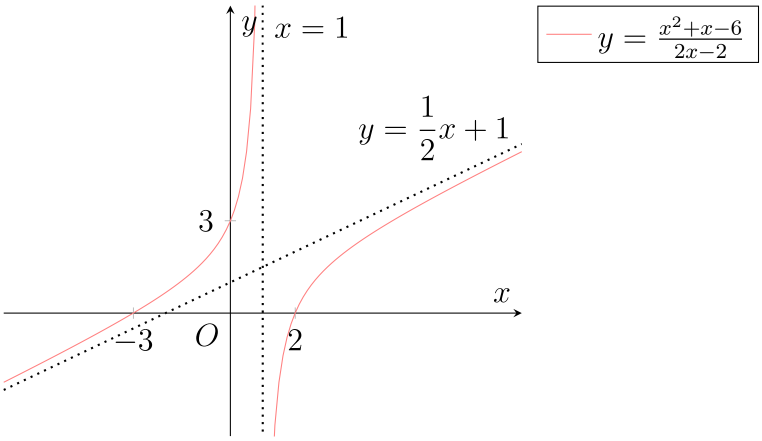
Part (a)



Part (b)



Part (c)



Problem 3.

Sketch the following graphs

(a) $x^2 + 2x + 2y + 4 = 0$

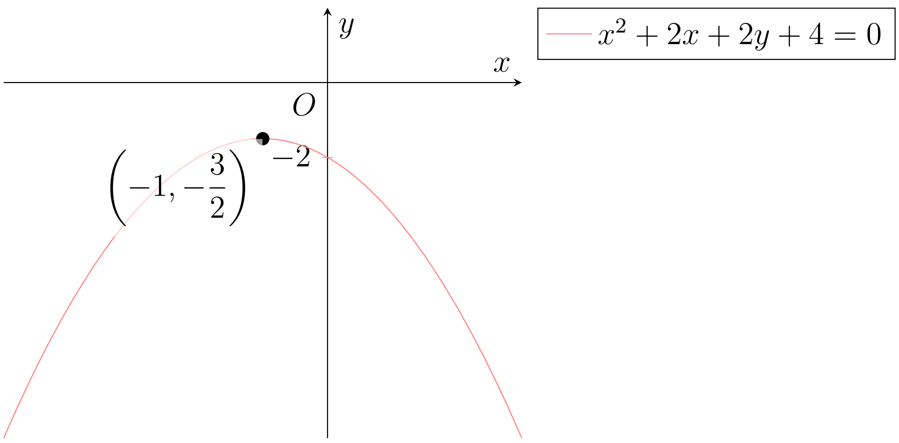
(b) $y^2 = x - 9$

(c) $y^2 = (x-2)^4 + 5$

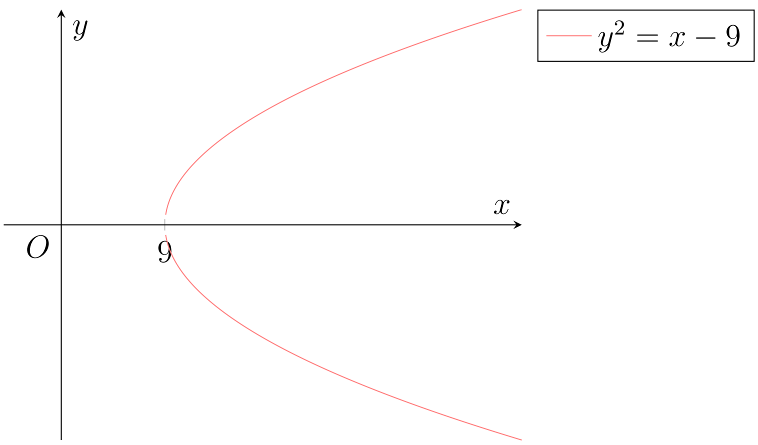
(d) $y = \tan \frac{1}{2}x$, $-2\pi \leq x \leq 2\pi$

Solution

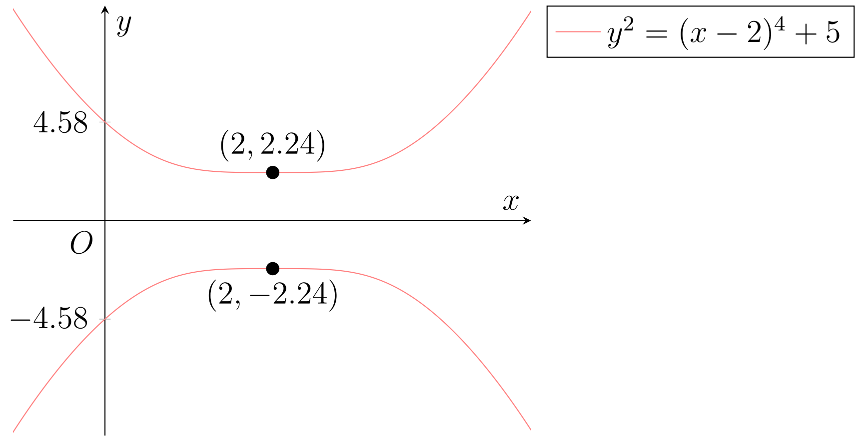
Part (a)



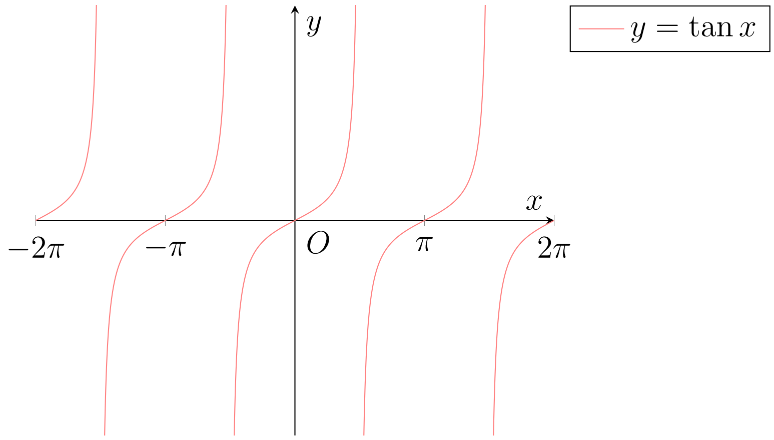
Part (b)



Part (c)



Part (d)



Problem 4.

Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

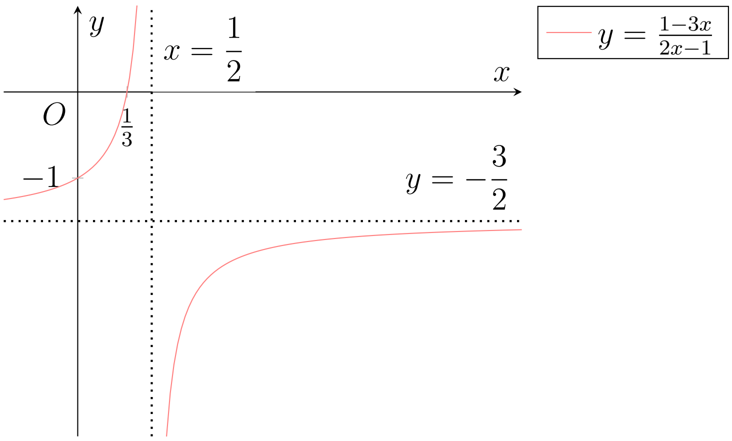
(a) $y = \dfrac{1-3x}{2x-1}$

(b) $y = \dfrac{ax}{x-a}$, $a < 0$

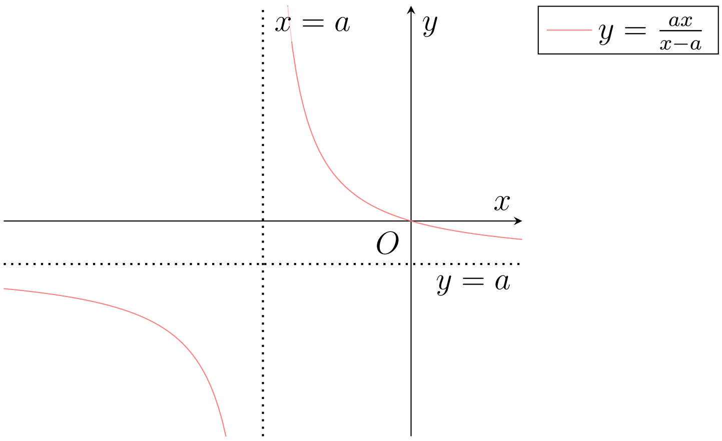
(c) $y = -\dfrac{b(x+3a)}{x+a}$, $a, b > 0$

Solution

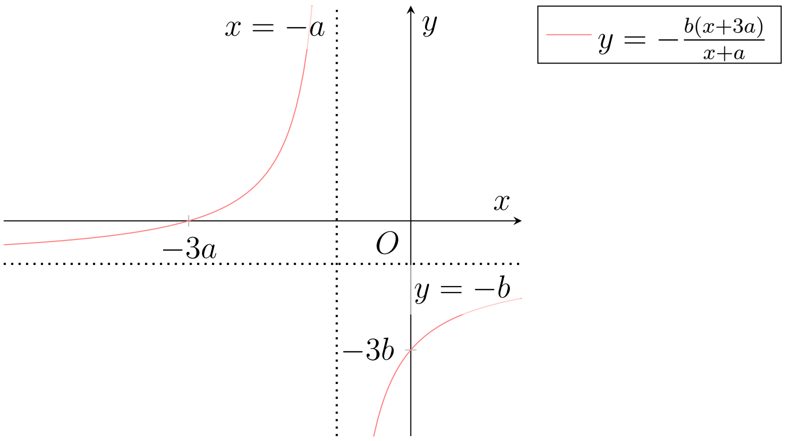
Part (a)



Part (b)



Part (c)



Problem 5.

Sketch the following curves and find the coordinates of any turning points on the curves.

(a) $y = x + 2\sin x$, $0 \leq x \leq 2\pi$

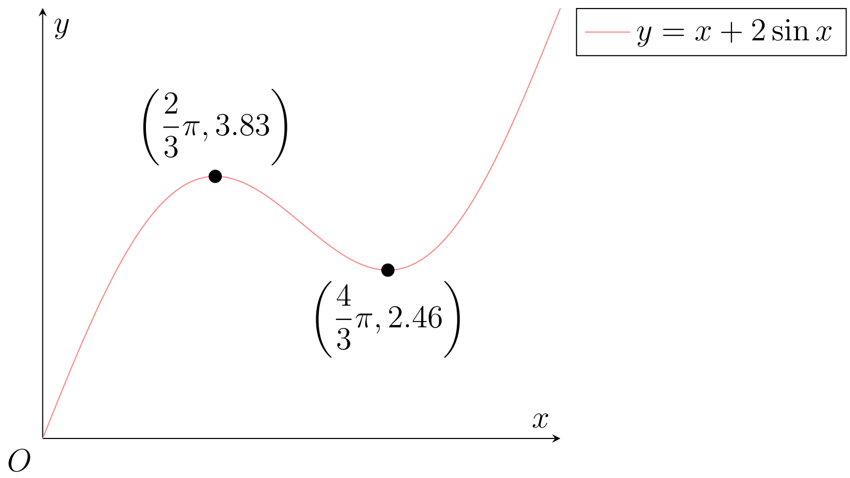
(b) $y = \dfrac{x}{\ln x}$, $x > 0$, $x \neq 1$

(c) $y = xe^{-x}$

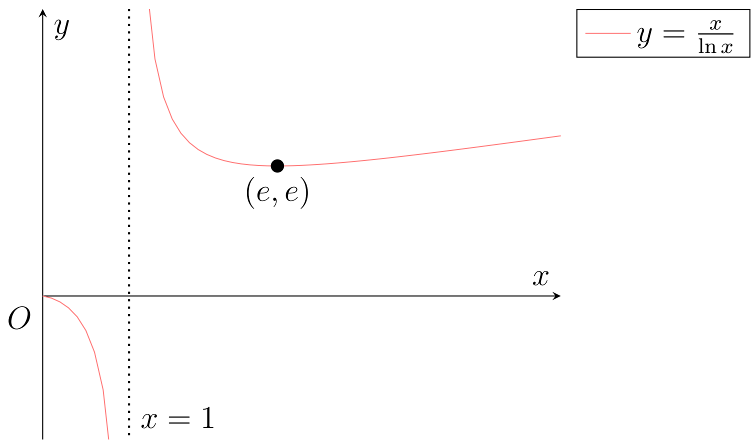
(d) $y = xe^{-x^2}$

Solution

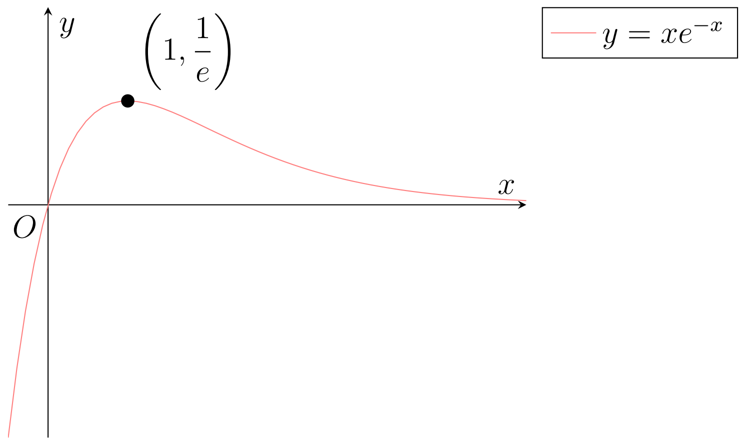
Part (a)



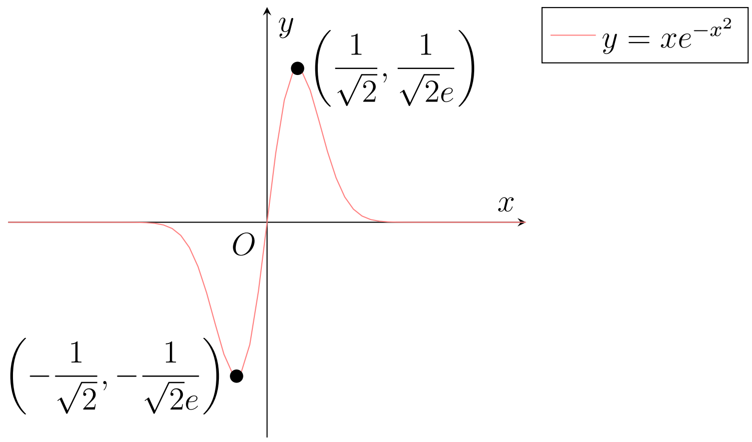
Part (b)



Part (c)


y
$\left(1, \frac{1}{e}\right)$
$y = xe^{-x}$
x
O

Part (d)


y
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}e}\right)$
$y = xe^{-x^2}$
x
O
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}e}\right)$

Problem 6.

The equation of a curve C is $y = 1 + \dfrac{6}{x-3} - \dfrac{24}{x+3}$.

(a) Explain why $y = 1$ and $x = 3$ are asymptotes to the curve.

(b) Find the coordinates of the points where C meets the axes.

(c) Sketch C.

Solution

Part (a)

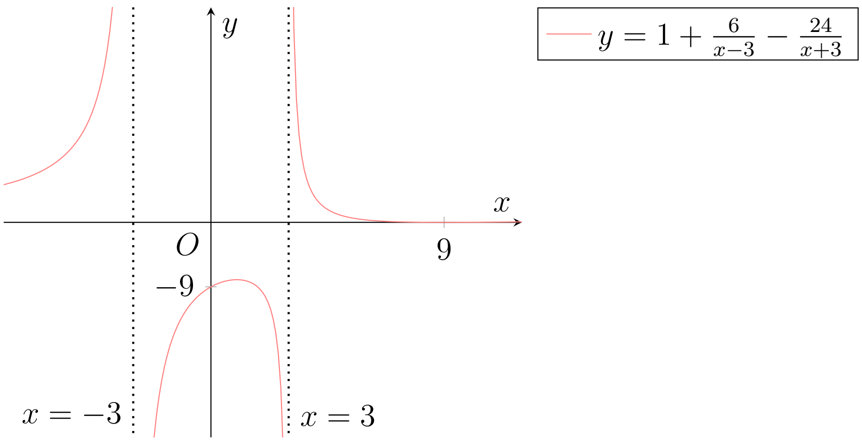
As $x \to \pm\infty$, $y \to 1$. Hence, $y = 1$ is an asymptote to C. As $x \to 3^{\pm}$, $y \to \pm\infty$. Hence, $x = 3$ is an asymptote to C.

Part (b)

When $x = 0$, $y = -9$. When $y = 0$, $x = 9$.

$$\boxed{C \text{ meets the axes at } (0, -9) \text{ and } (9, 0).}$$

Part (c)



Problem 7.

The curve C has equation $y = \dfrac{ax^2 + bx}{x+2}$, where $x \neq -2$. It is given that C has an asymptote $y = 1 - 2x$.

(a) Show (do not verify) that $a = -2$ and $b = -3$.

(b) Using an algebraic method, find the set of values that y can take.

(c) Sketch C, showing clearly the positions of any axial intercept(s), asymptote(s) and stationary point(s).

(d) Deduce that the equation $x^4 + 2x^3 + 2x^2 + 3x = 0$ has exactly one real non-zero root.

Solution

Part (a)

$$\begin{aligned} y &= \frac{ax^2 + bx}{x+2} \\ &= \frac{(ax + b - 2a)(x+2) - 2(b - 2a)}{x+2} \\ &= ax + b - 2a - \frac{2(b-2a)}{x+2} \end{aligned}$$

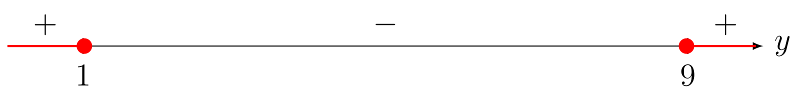
Since C has an asymptote $y = 1 - 2x$, we have $a = -2$ and $b - 2a = 1$, whence $b = -3$.

Part (b)

$$\begin{aligned} && y &= \frac{-2x^2 + -3x}{x+2} \\ \implies && y(x+2) &= -2x^2 - 3x \\ \implies && 2x^2 + (3+y)x + 2y &= 0 \end{aligned}$$

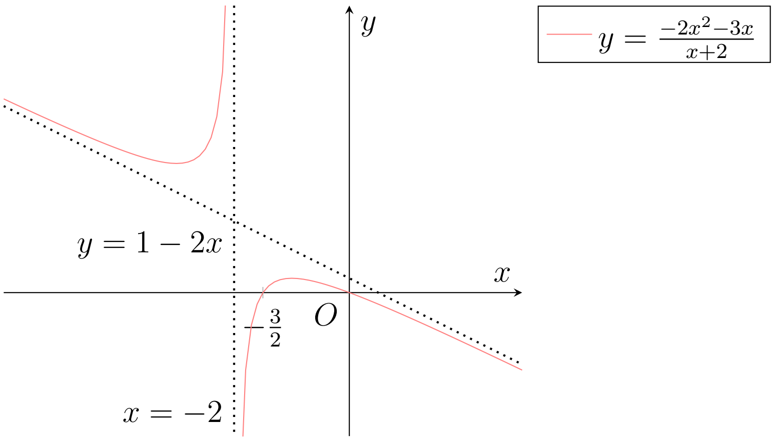
For all values that y can take on, there exists a solution to $2x^2 + (3+y)x + 2y = 0$. Hence, $\Delta \geq 0$.

$$\begin{aligned} & (3+y)^2 - 4(2)(2y) \geq 0 \\ \implies & 9 + 6y + y^2 - 16y \geq 0 \\ \implies & y^2 - 10y + 9 \geq 0 \\ \implies & (y-1)(y-9) \geq 0 \end{aligned}$$



$$\boxed{\{y \in \mathbb{R} \colon y \leq 1 \lor y \geq 9\}}$$

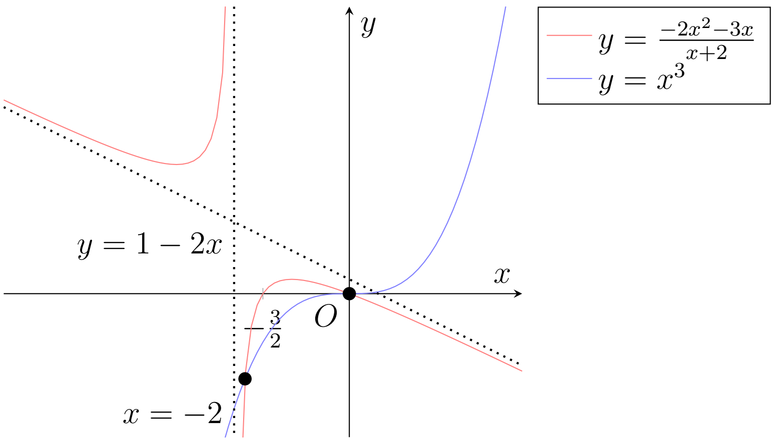
Part (c)



Part (d)

$$\begin{aligned} & x^4 + 2x^3 + 2x^2 + 3x = 0 \\ \implies \quad & x^4 + 2x^3 = -2x^2 - 3x \\ \implies \quad & x^3(x+2) = -2x^2 - 3x \\ \implies \quad & x^3 = \frac{-2x^2 - 3x}{x+2} \end{aligned}$$

This motivates us to plot $y = x^3$ and $y = \dfrac{-2x^2 - 3x}{x+2}$ on the same graph.



We thus see that $y = x^3$ intersects $y = \dfrac{-2x^2 - 3x}{x+2}$ twice, with one intersection point being the origin. Thus, there is only one real non-zero root to $x^4 + 2x^3 + 2x^2 + 3x = 0$.

Problem 8.

The curve C is defined by the equation $y = \dfrac{x}{x^2 - 5x + 4}$.

(a) Write down the equations of the asymptotes.

(b) Sketch C, indicating clearly the axial intercept(s), asymptote(s) and turning point(s).

(c) Find the positive value k such that the equation $\dfrac{x}{x^2 - 5x + 4} = kx$ has exactly 2 distinct real roots.

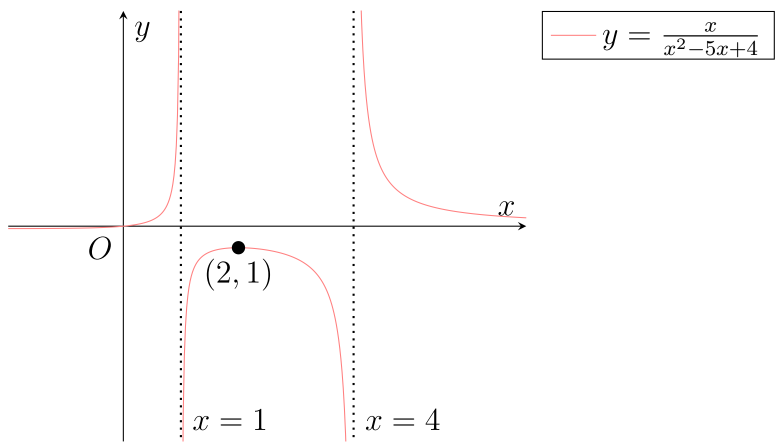
Solution

Part (a)

As $x \to \pm\infty$, $y \to 0$. Hence, $y = 0$ is an asymptote. Observe that $x^2 - 5x + 4 = (x-1)(x-4)$. Hence, $x = 1$ and $x = 4$ are asymptotes.

$$\boxed{\text{Asymptotes: } y = 0,\ x = 1,\ x = 4}$$

Part (b)



Part (c)

Note that $x = 0$ is always a root of $\dfrac{x}{x^2 - 5x + 4} = kx$. We thus aim to find the value of k such that $\dfrac{x}{x^2 - 5x + 4} = kx$ has only one non-zero root.

We observe that if $k > 0$, $y = kx$ will intersect with $y = \dfrac{x}{x^2 - 5x + 4}$ at least twice: before $x = 1$ and after $x = 4$. In order to have only one non-zero root, we must force the intersection point that comes before $x = 1$ to be at the origin $(0, 0)$. Hence, k is tangential to C at $(0, 0)$, thus giving $k = \left.\dfrac{dC}{dx}\right|_{x=0}$.

$$
\begin{aligned}
k &= \left.\frac{dC}{dx}\right|_{x=0} \\
&= \left.\frac{d}{dx}\frac{x}{x^2-5x+4}\right|_{x=0} \\
&= \left.\frac{3x^2-10x+4}{(x^2-5x+4)^2}\right|_{x=0} \\
&= \frac{1}{4}
\end{aligned}
$$

$$\boxed{k = \frac{1}{4}}$$