

Problem 1.

Omitted.

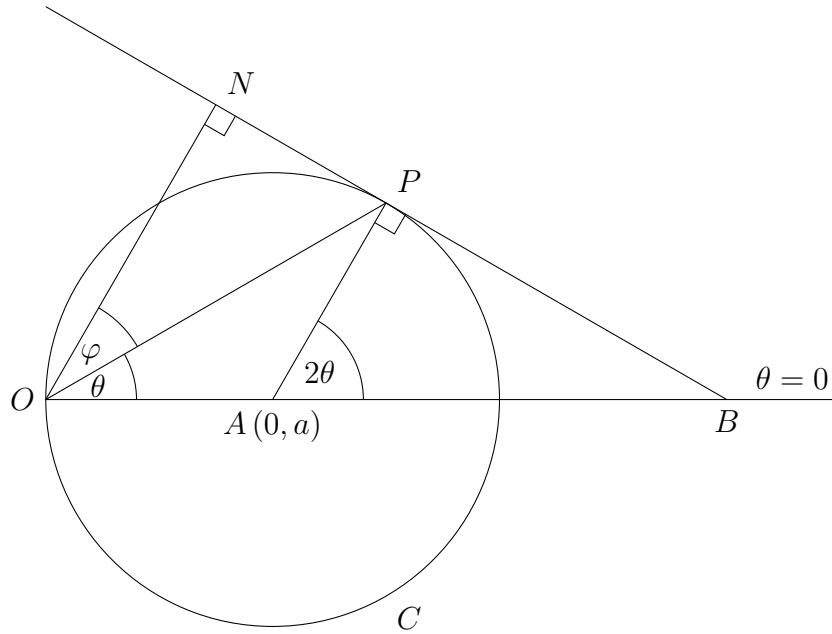
Problem 2.

A point P lies on the curve C with polar equation $r = 2a \cos \theta$, $-\frac{\pi}{2} < \theta \leq \frac{\pi}{2}$, where a is a positive constant. The point N is the foot of the perpendicular from the pole to the tangent of C at P .

- Sketch C , P and N on the same diagram.
- By considering the polar coordinates of N , show that, as P varies, the locus of N is given by the polar equation $r = a(1 + \cos \theta)$, $-\pi < \theta \leq \pi$.

Solution

Part (a)



Part (b)

Consider the diagram above. Let $N(\theta + \varphi, r)$. Let B be the intersection between the tangent at P and the half-line $\theta = 0$. Note that C is a circle centered at A with radius a .

Since angle at centre is twice angle at circumference, $\angle PAB = 2\angle POA = 2\theta$. Since tangent to circle is parallel to radius, $AP \perp PB$. Hence, $ON \parallel AP \implies \angle NOA = \angle PAB \implies \varphi = \theta$.

Since $\cos \angle NOP = \frac{ON}{OP}$, we have

$$\begin{aligned}
 ON &= OP \cos \theta \\
 &= (2a \cos \theta) \cos \theta \\
 &= 2a \cos^2 \theta \\
 &= a(2 \cos^2 \theta - 1 + 1) \\
 &= a(\cos 2\theta + 1)
 \end{aligned}$$

Hence, $N(\theta + \varphi, r) = (2\theta, a(\cos 2\theta + 1)) = (\theta, a(\cos \theta + 1))$. Thus, the locus of N is given by $r = a(1 + \cos \theta)$.

Problem 3.

The sequence $\{u_n\}$ is given by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n, \quad n \in \mathbb{Z}^+$$

together with terms $u_1 = a$ and $u_2 = b$.

- (a) Find the expression of u_n in terms of a and b .
- (b) Find algebraically the possible limits of $\frac{u_n}{u_{n-1}}$.

Solution

Part (a)

Consider the characteristic equation of the recurrence relation.

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ \implies (x - 2)(x - 3) &= 0 \end{aligned}$$

Hence, the roots of the characteristic equation are 2 and 3. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n$$

Substituting $n = 1$ and $n = 2$, we have

$$\begin{cases} 2A + 3B = a \\ 4A + 9B = b \end{cases}$$

which has solution $A = \frac{3a - b}{2}$ and $B = \frac{b - 2a}{3}$. Thus,

$$u_n = \frac{3a - b}{2} \cdot 2^n + \frac{b - 2a}{3} \cdot 3^n$$

Part (b)

Let $L = \lim_{n \rightarrow \infty} \frac{u_n}{u_{n-1}}$.

$$\begin{aligned} &u_{n+2} = 5u_{n+1} - 6u_n \\ \implies &\frac{u_{n+2}}{u_{n+1}} = 5 - 6 \cdot \frac{u_n}{u_{n+1}} \\ \implies &\lim_{n \rightarrow \infty} \frac{u_{n+2}}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(5 - 6 \cdot \frac{u_n}{u_{n+1}} \right) \\ \implies &L = 5 - \frac{6}{L} \\ \implies &L^2 - 5L + 6 = 0 \\ \implies &(L - 2)(L - 3) = 0 \\ \implies &L = 2 \vee 3 \end{aligned}$$

The possible limits are 2 and 3.

Problem 4.

Omitted.

Problem 5.

The points A and B have Cartesian coordinates $(a, 0)$ and $(-a, 0)$ respectively, where a is a positive constant. The point P is such that $AP \cdot BP = a^2$. The curve C describes the locus of P .

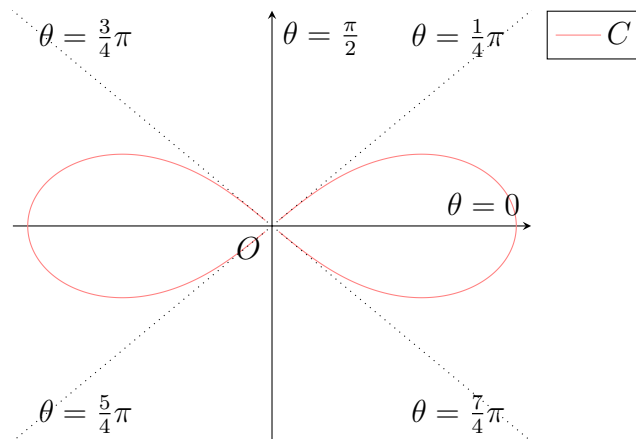
- Show that C has polar equation $r^2 = 2a^2 \cos 2\theta$, $0 \leq \theta \leq 2\pi$.
- Sketch the graph of C , indicating all key features and symmetries of the curve.
- Find the exact area of the region enclosed by the curve C .

Solution

Part (a)

Let $P(x, y)$.

$$\begin{aligned}
 & \Rightarrow AP \cdot BP = a^2 \\
 & \Rightarrow AP^2 \cdot BP^2 = a^4 \\
 & \Rightarrow ((x-a)^2 + y^2)((x+a)^2 + y^2) = a^4 \\
 & \Rightarrow (x-a)^2(x+a)^2 + y^2(x+a)^2 + y^2(x-a)^2 + y^4 = a^4 \\
 & \Rightarrow (x^2 - a^2)^2 + y^2((x+a)^2 + (x-a)^2) + y^4 = a^4 \\
 & \Rightarrow (x^4 - 2a^2x^2 + a^4) + y^2(x^2 + 2ax + a^2 + x^2 - 2ax + a^2) + y^4 = a^4 \\
 & \Rightarrow x^4 - 2a^2x^2 + 2y^2(x^2 + a^2) + y^4 = 0 \\
 & \Rightarrow x^4 - 2a^2x^2 + 2x^2y^2 + 2a^2y^2 + y^4 = 0 \\
 & \Rightarrow (x^4 + 2x^2y^2 + y^4) + (2a^2y^2 - 2a^2x^2) = 0 \\
 & \Rightarrow (x^2 + y^2)^2 + 2a^2(y^2 - x^2) = 0 \\
 & \Rightarrow (r^2)^2 + 2a^2((r \sin \theta)^2 - (r \cos \theta)^2) = 0 \\
 & \Rightarrow r^4 + 2a^2r^2(\sin^2 \theta - \cos^2 \theta) = 0 \\
 & \Rightarrow r^4 - 2a^2r^2 \cos 2\theta = 0 \\
 & \Rightarrow r^2 - 2a^2 \cos 2\theta = 0 \\
 & \Rightarrow r^2 = 2a^2 \cos 2\theta
 \end{aligned}$$

Part (b)**Part (c)**

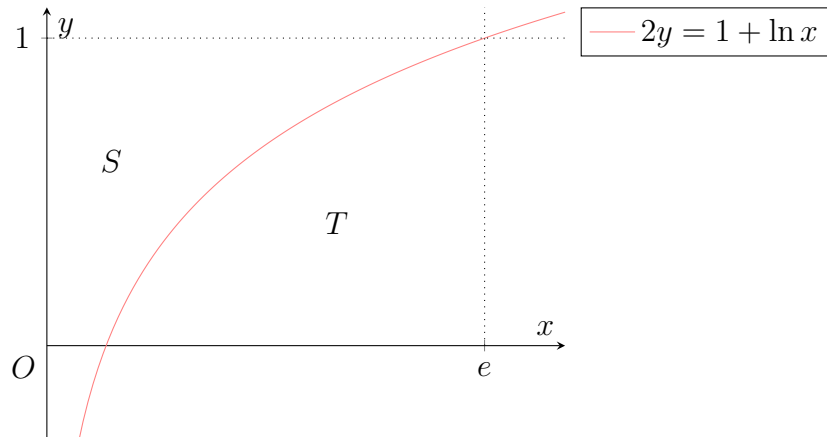
$$\begin{aligned}
 \text{Area} &= 2 \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, d\theta \\
 &= \int_{-\pi/4}^{\pi/4} 2a^2 \cos 2\theta \, d\theta \\
 &= a^2 \int_{-\pi/2}^{\pi/2} \cos u \, du \\
 &= a^2 [\sin u]_{-\pi/2}^{\pi/2} \\
 &= 2a^2
 \end{aligned}$$

$$\begin{aligned}
 u &= 2\theta \\
 du &= 2 \, d\theta
 \end{aligned}$$

The area of the region enclosed by C is $2a^2$ units².

Problem 6.

In the diagram, the curve with equation $2y = 1 + \ln x$ for $x > 0$ divides the rectangle bounded by the axes, the lines $y = 1$ and $x = e$ into two regions, S and T .



- (a) Show that the volume of the solid generated when S is rotated completely about the x -axis is given by $2\pi \int_0^1 F(y) \, dy$, where $F(y)$ is a function to be determined.
- (b) Find the exact value of $\int_0^1 F(y) \, dy$.
- (c) By using the result in part (b), find the exact value of $\int_{e^{-1}}^e (\ln x + 1)^2 \, dx$.
- (d) The arc of the curve between the x -intercept and $x = e$ is rotated through 2π radians about the x -axis. Find the area of the surface generated.

Without further calculation, deduce the area of the surface generated when the arc of a curve with equation $y = e^{2x-1}$ between the y -intercept and $y = e$ is rotated through 2π radians about the y -axis, justifying your answer.

Solution

Part (a)

Note that $2y = 1 + \ln x \implies x = e^{2y-1}$. Using the shell method,

$$\begin{aligned} \text{Volume}_S &= 2\pi \int_0^1 xy \, dy \\ &= 2\pi \int_0^1 ye^{2y-1} \, dy \end{aligned}$$

Part (b)

$$\begin{aligned} \int_0^1 ye^{2y-1} \, dy &= \frac{1}{e} \int_0^1 ye^{2y} \, dy \\ &= \frac{1}{4e} \int_0^2 ue^u \, du \end{aligned}$$

$$\begin{aligned} u &= 2y \\ du &= 2 \, dy \end{aligned}$$

	D	I
+	u	e^u
-	1	e^u
+	0	e^u

$$\begin{aligned}
 \int_0^1 ye^{2y-1} dy &= \frac{1}{4e} [ue^u - e^u]_0^2 \\
 &= \frac{1}{4e} (e^2 + 1) \\
 &= \frac{1}{4} (e + e^{-1})
 \end{aligned}$$

$$\boxed{\int_0^1 ye^{2y-1} dy = \frac{1}{4} (e + e^{-1})}$$

Part (c)

Let $I = \int_{e^{-1}}^e (\ln x + 1)^2 dx$. Observe that the volume of the solid generated when T is rotated completely about the x -axis is given by

$$\begin{aligned}
 \text{Volume}_T &= \int_{e^{-1}}^e \left(\frac{1 + \ln x}{2} \right)^2 dx \\
 &= \frac{\pi}{4} \int_{e^{-1}}^e (\ln x + 1)^2 dx \\
 &= \frac{\pi}{4} I
 \end{aligned}$$

Note that when the entire rectangle is rotated completely about the x -axis, its volume is given by πe . Hence,

$$\begin{aligned}
 \text{Volume}_S + \text{Volume}_T &= \pi e \\
 \implies 2\pi \cdot \frac{1}{4} (e - e^{-1}) + \frac{\pi}{4} I &= \pi e \\
 \implies 2(e - e^{-1}) + I &= 4e \\
 \implies I &= 4e - 2(e + e^{-1}) \\
 &= 2(e - e^{-1})
 \end{aligned}$$

$$\boxed{\int_{e^{-1}}^e (\ln x + 1)^2 dx = 2(e - e^{-1})}$$

Part (d)

Note that $2y = 1 + \ln x \implies \frac{dy}{dx} = \frac{1}{2x}$. Hence,

$$\text{Surface area} = 2\pi \int_{e^{-1}}^e y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx$$

$$\begin{aligned} &= 2\pi \int_{e^{-1}}^e \left(\frac{1 + \ln x}{2} \right) \sqrt{1 + \left(\frac{1}{2x} \right)^2} dx \\ &= 10.3 \text{ (3 s.f.)} \end{aligned}$$

The area of the surface generated is 10.3 units².

The curve described is the inverse of $2y = 1 + \ln x$. Hence, the surface area generated by the new curve is the same as that of $2y = 1 + \ln x$.

The surface area generated by the new curve is also 10.3 units².

Problem 7.

Omitted.

Problem 8.

The function g is given by $g(x) = x^{3/2} - 2\sqrt{x} - 1$ for $x \geq 0$. Show that the equation $g(x) = 0$ has only one real root, $x = \alpha$. State an integer n such that $n < \alpha < n + 1$.

- (a) To find an approximate value for α , the following rearrangements of $g(x) = 0$ are suggested as a basis for the iteration method of the form $x_{n+1} = f(x_n)$.

- $x = \frac{1}{4} (x^{3/2} - 1)^2$
- $x = 2 + \frac{1}{\sqrt{x}}$
- $x = x^{5/2} - x^{3/2}$

- (i) By considering $f'(x)$, identify the iteration method which converges to α . Use a graph to explain why the chosen iteration method converges to α .
- (ii) Using the appropriate iteration method found in part(a)(i), and $x_0 = n$, find the value of α , correct to 4 decimal places, and demonstrate how to verify its correctness.
- (b) (i) Show that a Newton-Raphson iteration method for the root α is given by

$$x_{n+1} = \frac{x_n^{3/2} + 2x_n^{1/2} + 2}{3x_n^{1/2} - 2x_n^{-1/2}}$$

- (ii) Explain why the Newton-Raphson iteration method fails when the initial value x_0 is less than or equal to $\frac{2}{3}$.

Solution

Let $y = \sqrt{x}$. Hence, $g(x)$ is equivalent to $y^3 - 2y - 1$. Let this function be $h(y)$. Observe that $h'(y) = 3y^2 - 2$. Thus, the sole stationary point of $h(y)$ occurs at $y = \sqrt{\frac{2}{3}}$. Note that we reject $y = -\sqrt{\frac{2}{3}}$ since $y = \sqrt{x} \geq 0$.

y	$\left(\sqrt{\frac{2}{3}}\right)^-$	$\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}\right)^+$
$h'(y)$	-ve	0	+ve

From the First Derivative Test, we see that $y = \sqrt{\frac{2}{3}}$ is a minimum point. Hence, $h(y)$ is strictly decreasing on the interval $\left[0, \sqrt{\frac{2}{3}}\right)$ and strictly increasing on the interval $\left(\sqrt{\frac{2}{3}}, \infty\right)$. Since $h(0) = -1 < 0$, we have that $h(y) < 0$ for all $y \leq \sqrt{\frac{2}{3}}$. Since $h(y)$

is strictly increasing for $y > \sqrt{\frac{2}{3}}$, and $h\left(\sqrt{\frac{2}{3}}\right) < 0$, $h(y)$ has only one real root. Thus, $g(x)$ has only one real root.

Observe that $g(2)g(3) = (-1)(0.732) < 0$. Hence, $\alpha \in (2, 3)$. Thus,

$$\boxed{n = 2}$$

Part (a)

Subpart (i)

Case 1: $f(x) = \frac{1}{4}(x^{3/2} - 1)^2$

$$\begin{aligned} f'(x) &= \frac{1}{4} \cdot 2(x^{3/2} - 1) \cdot \frac{3}{2}x^{1/2} \\ &= \frac{3}{4}x^{1/2}(x^{3/2} - 1) \end{aligned}$$

Note that for all $x \in (2, 3)$, $|f'(x)| > 1$. Hence, the iteration may not converge.

Case 2: $f(x) = 2 + \frac{1}{\sqrt{x}}$

$$f'(x) = -\frac{1}{2}x^{-3/2}$$

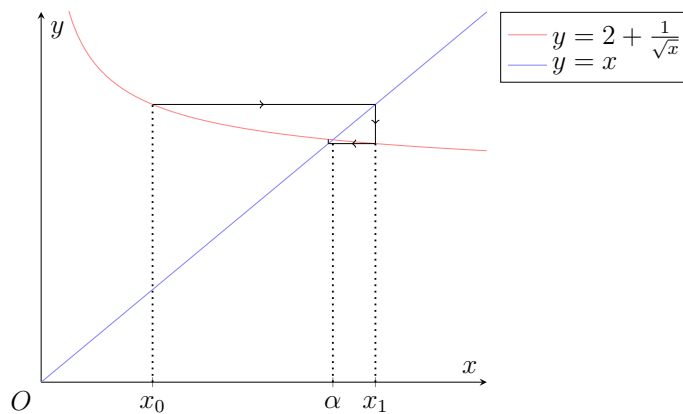
Note that for all $x \in (2, 3)$, $|f'(x)| < 1$. Hence, the iteration converges.

Case 3: $f(x) = x^{5/2} - x^{3/2}$

$$f'(x) = \frac{5}{2}x^{3/2} - \frac{3}{2}x^{1/2}$$

Note that for all $x \in (2, 3)$, $|f'(x)| > 1$. Hence, the iteration may not converge.

$$\boxed{x_{n+1} = 2 + \frac{1}{\sqrt{x_n}} \text{ converges to } \alpha.}$$



From the graph, the subsequent approximations get closer and closer to α . Hence, $x_{n+1} = 2 + \frac{1}{\sqrt{x_n}}$ converges to α .

Subpart (ii)

$$\begin{aligned}
 x_0 &= 2 \\
 \implies x_1 &= 2 + \frac{1}{\sqrt{x_0}} = 2.7171068 \\
 \implies x_2 &= 2 + \frac{1}{\sqrt{x_1}} = 2.6077813 \\
 \implies x_3 &= 2 + \frac{1}{\sqrt{x_2}} = 2.6192477 \\
 \implies x_4 &= 2 + \frac{1}{\sqrt{x_3}} = 2.6178908 \\
 \implies x_5 &= 2 + \frac{1}{\sqrt{x_4}} = 2.6180509 \\
 \implies x_6 &= 2 + \frac{1}{\sqrt{x_5}} = 2.6180320 \\
 \implies x_7 &= 2 + \frac{1}{\sqrt{x_6}} = 2.6180342
 \end{aligned}$$

Observe that $g(2.61795)g(2.61805) = (-1.5 \times 10^{-4})(2.9 \times 10^{-5}) < 0$. Hence, $\alpha \in (2.61795, 2.61805)$. Thus,

$$\boxed{\alpha = 2.6180 \text{ (4 d.p.)}}$$

Part (b)

Subpart (i)

Note that $g'(x) = \frac{3}{2}x^{1/2} - x^{-1/2}$. Thus,

$$\begin{aligned}
 x_{n+1} &= x_n - \frac{g(x_n)}{g'(x_n)} \\
 &= x_n - \frac{x_n^{3/2} - 2x_n^{1/2} - 1}{\frac{3}{2}x_n^{1/2} - x_n^{-1/2}} \\
 &= x_n - \frac{2x_n^{3/2} - 4x_n^{1/2} - 2}{3x_n^{1/2} - 2x_n^{-1/2}} \\
 &= \frac{x_n(3x_n^{1/2} - 2x_n^{-1/2}) - (2x_n^{3/2} - 4x_n^{1/2} - 2)}{3x_n^{1/2} - 2x_n^{-1/2}} \\
 &= \frac{x_n^{3/2} + 2x_n^{1/2} + 2}{3x_n^{1/2} - 2x_n^{-1/2}}
 \end{aligned}$$

Subpart (ii)

When $x_0 = \frac{2}{3}$, we have $3x_n^{1/2} - 2x_n^{-1/2} = 0$, whence x_1 is undefined. When $x_0 < \frac{2}{3}$, $x_1 < 0$. However, the iterative formula for x_n is valid only for $x_n > 0$. Thus, x_2 will be undefined. Hence, the method fails for $x_0 \leq \frac{2}{3}$.

Problem 9.

A point P resides on the circumference of a circular gear with centre C and radius a , which rolls in an anticlockwise direction externally without slipping on the circumference of a fixed circular axle with centre O and radius a . Figure 1 shows the initial position of P at $(a, 0)$ and Figure 2 shows its position $P(x, y)$ where OC makes an angle θ with the positive x -axis.

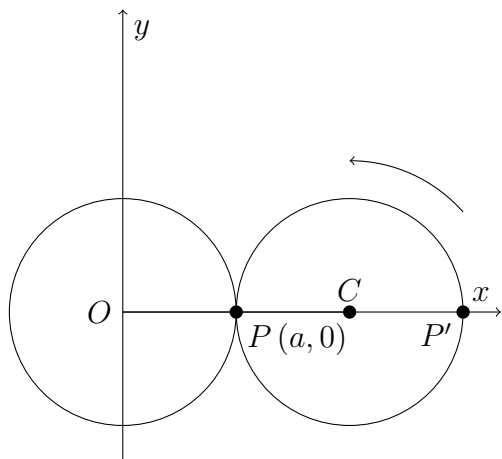


Figure 1

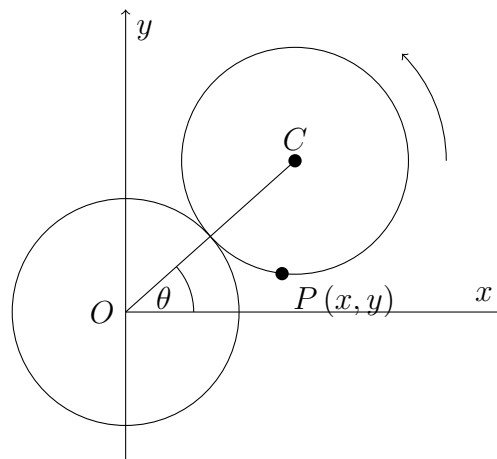


Figure 2

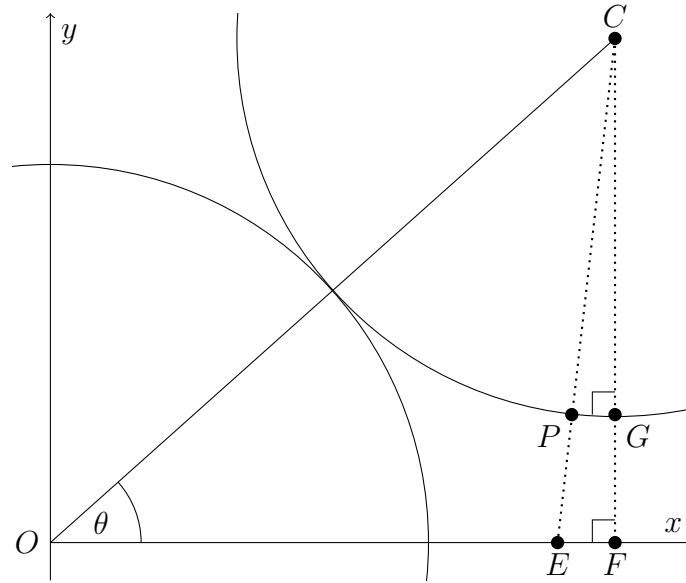
- (a) Show that the equation of one full revolution of the path of P can be represented by

$$\begin{aligned}x &= 2a \cos \theta - a \cos 2\theta \\y &= 2a \sin \theta - a \sin 2\theta\end{aligned}$$

- (b) Sketch the path of P for $0 \leq \theta \leq 2\pi$, indicating clearly the coordinates of the x -intercepts.
- (c) Without the use of a calculator, find the length of the path of P in terms of a .
- (d) A student commented that a point P' with the initial position $(3a, 0)$ (as shown in Figure 1) is further away from O than point P and therefore travels a longer path as the circular gear makes a full revolution around the fixed circular axle. Do you agree with the comment? Justify your answer.

Solution

Part (a)



Consider the above diagram. Let E be the intersection between the x -axis and CP extended. Let F be the point on the x -axis such that $CF \perp OF$. Let G be the point on the line CF such that $PG \perp CG \implies PG \parallel EF$.

By symmetry, $OE = EC$, whence $\triangle OEC$ is isosceles. Hence, $\angle COE = \angle OCE = \theta$. By the exterior angle theorem, $\angle CEF = 2\theta$. Since $PG \parallel EF$, we have $\angle CPG = \angle CEF = 2\theta$. Finally, from the angle sum of a triangle, we have $\angle ECF = \frac{\pi}{2} - \theta$.

$$\begin{aligned}\cos \angle COF &= \frac{OF}{OC} \implies \cos \theta = \frac{OF}{2a} \implies OF = 2a \cos \theta \\ \cos \angle CPG &= \frac{PG}{PC} \implies \cos 2\theta = \frac{PG}{a} \implies PG = a \cos 2\theta\end{aligned}$$

Since $x = OF - PG$, we have

$$x = 2a \cos \theta - a \cos 2\theta$$

$$\begin{aligned}\cos \angle OCF &= \frac{CF}{OC} \implies \cos \left(\frac{\pi}{2} - \theta \right) = \frac{CF}{2a} \implies CF = 2a \sin \theta \\ \cos \angle PCG &= \frac{CG}{PC} \implies \cos \left(\frac{\pi}{2} - 2\theta \right) = \frac{CG}{a} \implies CG = a \sin 2\theta\end{aligned}$$

Since $y = CF - CG$, we have

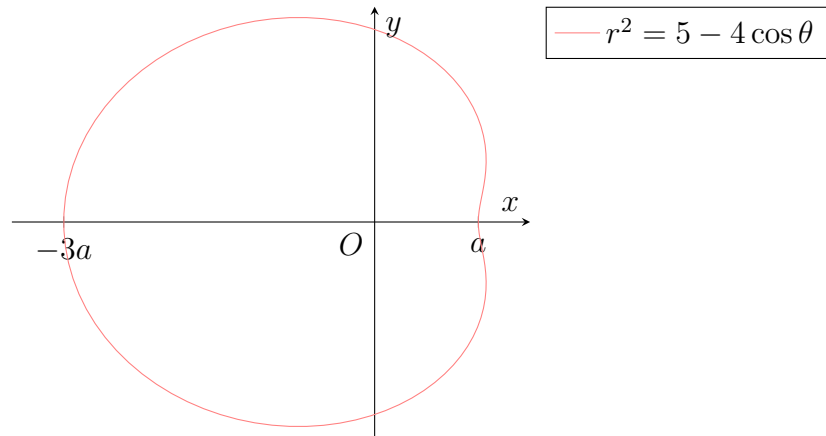
$$y = 2a \sin \theta - a \sin 2\theta$$

Hence, the path of P can be represented by

$$\begin{aligned}x &= 2a \cos \theta - a \cos 2\theta \\ y &= 2a \sin \theta - a \sin 2\theta\end{aligned}$$

Part (b)

$$\begin{aligned}
r^2 &= x^2 + y^2 \\
&= (2a \cos \theta - a \cos 2\theta)^2 + (2a \sin \theta - a \sin 2\theta)^2 \\
&= a^2 \left[(2 \cos \theta - \cos 2\theta)^2 + (2 \sin \theta - \sin 2\theta)^2 \right] \\
&= a^2 \left[4 \cos^2 \theta - 4 \cos \theta \cos 2\theta + \cos^2 2\theta + 4 \sin^2 \theta - 4 \sin \theta \sin 2\theta + \sin^2 2\theta \right] \\
&= a^2 \left[(4 \cos^2 \theta + 4 \sin^2 \theta) + (\cos^2 2\theta + \sin^2 2\theta) - 4(\cos \theta \cos 2\theta + \sin \theta \sin 2\theta) \right] \\
&= a^2(4 + 1 - 4 \cos \theta) \\
&= a^2(5 - 4 \cos \theta) \\
\Rightarrow r &= \pm a\sqrt{5 - 4 \cos \theta}
\end{aligned}$$

**Part (c)**

Note that we have $\frac{dx}{d\theta} = -2a \sin \theta + 2a \sin 2\theta$ and $\frac{dy}{d\theta} = 2a \cos \theta - 2a \cos 2\theta$. Thus,

$$\begin{aligned}
&\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 \\
&= (-2a \sin \theta + 2a \sin 2\theta)^2 + (2a \cos \theta - 2a \cos 2\theta)^2 \\
&= 4a^2 \left[(-\sin \theta + \sin 2\theta)^2 + (\cos \theta - \cos 2\theta)^2 \right] \\
&= 4a^2 \left[\sin^2 \theta - 2 \sin \theta \sin 2\theta + \sin^2 2\theta + \cos^2 \theta - 2 \cos \theta \cos 2\theta + \cos^2 2\theta \right] \\
&= 4a^2 \left[(\sin^2 \theta + \cos^2 \theta) + (\sin^2 2\theta + \cos^2 2\theta) - 2(\sin \theta \sin 2\theta + \cos \theta \cos 2\theta) \right] \\
&= 4a^2(1 + 1 - 2 \cos \theta) \\
&= 8a^2(1 - \cos \theta)
\end{aligned}$$

Hence, the length of the path of P is given by

$$\text{Length} = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi} \sqrt{8a^2(1 - \cos \theta)} \, d\theta \\
&= \sqrt{8}a \int_0^{2\pi} \sqrt{1 - \cos \theta} \, d\theta \\
&= \sqrt{8}a \int_0^{2\pi} \sqrt{2 \sin^2 \left(\frac{\theta}{2} \right)} \, d\theta \\
&= 4a \int_0^{2\pi} \sin \left(\frac{\theta}{2} \right) \, d\theta \\
&= 8a \int_0^{\pi} \sin u \, du \\
&= 8a [-\cos u]_0^{\pi} \\
&= 16a
\end{aligned}$$

$$\begin{aligned}
u &= \theta/2 \\
du &= d\theta/2
\end{aligned}$$

The length of the path of P is $16a$ units.

Part (d)

When the gear has rotated π radians, P ends up at $(-3a, 0)$. This is the reflection of P' in the y -axis. Indeed, P' also ends up at $(-a, 0)$, which is the reflection of P in the y -axis. Hence, the path taken by P' is exactly the path taken by P in the y -axis, but reflected in the y -axis. Hence, the total length travelled by P and P' are equal. Thus, the comment is incorrect.

Problem 10.

To promote Earth Day, the Earth Day Network organises an event that will be held for n consecutive days. Cash prizes will be awarded to participants throughout the event, with one lucky participant chosen each day. Let the budget for the cash prizes be $\$m$.

The amount of cash prize given out on the first day is a sum of $\$10$ and $\frac{1}{7}$ of the remaining budget, i.e. $\$ \left(10 + \frac{1}{7}(m - 10) \right)$. The amount of cash prize given out on the second day is a sum of $\$20$ and $\frac{1}{7}$ of the remaining budget. In general, the amount of cash prize given out on the k th day is a sum of $\$10k$ and $\frac{1}{7}$ of the remaining budget. Let u_k denote the amount of cash prize given out on the k th day, with $1 \leq k \leq n$, $k \in \mathbb{Z}^+$.

- (a) Write down the expression for u_k , in terms of u_1, u_2, \dots, u_{k-1} .
- (b) By considering $u_{k+1} - u_k$, show that $u_{k+1} = \frac{6}{7}u_k + \frac{60}{7}$.
- (c) Find u_k in the form $u_k = p \left(\frac{6}{7} \right)^k (m - 360) + q$, where p and q are constants to be determined.

It is given that $m = 4000$.

- (d) Find an expression of the total amount of cash prizes given out in n days. Hence or otherwise, explain if it is possible for the Earth Day Network to host this event for 2 weeks.
- (e) Find the set of values of k for which the daily cash prize will be less than 5% of the initial budget.
- (f) Determine the minimum budget that the Earth Day Network needs to have so that they can host this event for 2 weeks.

Solution

Part (a)

On the k th day, the total cash prizes that have already been given out is $u_1 + u_2 + \dots + u_{k-1}$. Hence,

$$u_k = 10k + \frac{1}{7} \left(m - 10k - (u_1 + u_2 + \dots + u_{k-1}) \right)$$

Part (b)

$$\begin{aligned} u_{k+1} - u_k &= 10(k+1) + \frac{1}{7} \left(m - 10(k+1) - (u_1 + u_2 + \dots + u_k) \right) \\ &\quad - \left(10k + \frac{1}{7} \left(m - 10k - (u_1 + u_2 + \dots + u_{k-1}) \right) \right) \end{aligned}$$

$$\begin{aligned}
&= 10 + \frac{1}{7}(-10 - u_k) \\
&= \frac{60}{7} - \frac{1}{7}u_k \\
\implies u_{k+1} &= \frac{60}{7} + \frac{6}{7}u_k
\end{aligned}$$

Part (c)

Let x be the constant such that $u_k + x = \frac{6}{7}(u_{k-1} + x)$. Then $-\frac{1}{7}x = \frac{60}{7} \implies x = -60$.

$$\begin{aligned}
u_k - 60 &= \frac{6}{7}(u_{k-1} - 60) \\
&= \left(\frac{6}{7}\right)^{k-1} (u_1 - 60) \\
\implies u_k &= \left(\frac{6}{7}\right)^{k-1} \left(10 + \frac{1}{7}(m - 10) - 60\right) + 60 \\
&= \left(\frac{6}{7}\right)^{k-1} \left(\frac{1}{7}m - \frac{360}{7}\right) + 60 \\
&= \left(\frac{6}{7}\right)^k \cdot \frac{7}{6} \left(\frac{1}{7}m - \frac{360}{7}\right) + 60 \\
&= \frac{1}{6} \left(\frac{6}{7}\right)^k (m - 360) + 60
\end{aligned}$$

$$u_k = \frac{1}{6} \left(\frac{6}{7}\right)^k (m - 360) + 60$$

Part (d)

Let S_n be the total amount of cash prizes given out in n days.

$$\begin{aligned}
S_n &= \sum_{k=1}^n u_k \\
&= \sum_{k=1}^n \left(\frac{1}{6} \left(\frac{6}{7}\right)^k (m - 360) + 60 \right) \\
&= \frac{m - 360}{6} \cdot \frac{6}{7} \cdot \frac{1 - (6/7)^n}{1 - 6/7} + 60n \\
&= (m - 360) \left(1 - \left(\frac{6}{7}\right)^n \right) + 60n
\end{aligned}$$

Substituting $m = 4000$ and $n = 14$, we have

$$\begin{aligned}
S_{14} &= (4000 - 360) \left(1 - \left(\frac{6}{7}\right)^{14} \right) + 60 \cdot 14 \\
&= 4059
\end{aligned}$$

Since $S_{14} > 4000 = m$, there is not enough money to pay out the cash prizes for 14 days, or 2 weeks.

It is not possible for the Earth Day Network to host this event for 2 weeks.

Part (e)

Consider $u_k < \frac{5}{100}m = 200$.

$$\begin{aligned}
 & u_k < 200 \\
 \Rightarrow & \frac{1}{6} \left(\frac{6}{7} \right)^k (4000 - 360) + 60 < 200 \\
 \Rightarrow & \left(\frac{6}{7} \right)^k < \frac{3}{13} \\
 \Rightarrow & k > \log_{6/7} \frac{3}{13} \\
 & = 9.5 \text{ (2 s.f.)}
 \end{aligned}$$

Note that $S_{13} = (4000 - 360) \left(1 - \left(\frac{6}{7} \right)^{13} \right) + 60 \cdot 13 = 3929 < 4000 = m$. Hence,

$$\{k \in \mathbb{N} : 10 \leq k \leq 13\}$$

Part (f)

Consider $S_{14} \leq m$.

$$\begin{aligned}
 & S_{14} \leq m \\
 \Rightarrow & (m - 360) \left(1 - \left(\frac{6}{7} \right)^{14} \right) + 60 \cdot 14 \leq m \\
 \Rightarrow & m \left(1 - \left(\frac{6}{7} \right)^{14} \right) - 360 \left(1 - \left(\frac{6}{7} \right)^{14} \right) + 840 \leq m \\
 \Rightarrow & m - m \left(\frac{6}{7} \right)^{14} - m \leq 360 \left(1 - \left(\frac{6}{7} \right)^{14} \right) - 840 \\
 \Rightarrow & -m \left(\frac{6}{7} \right)^{14} \leq 360 \left(1 - \left(\frac{6}{7} \right)^{14} \right) - 840 \\
 \Rightarrow & m \geq \frac{840 - 360 (1 - (6/7)^{14})}{(6/7)^{14}} \\
 & = 4514.285
 \end{aligned}$$

Earth Day Network needs to have a minimum budget of \$4514.29.