Problem 1.

Expand $(1+2x)^{-\frac{1}{3}}$, where $|x| < \dfrac{1}{2}$, as a series of ascending powers of x, up to an including the term in x^2, simplifying the coefficients.

By choosing $x = \dfrac{1}{14}$, find an approximate value of $\sqrt[3]{7}$ in the form $\dfrac{p}{q}$, where p and q are to be determined.

Using your calculator, calculate the numerical value of $\sqrt[3]{7}$. Compare this value to the approximate value found, and with reference to the value of x chosen, comment on the accuracy of your approximation.

Solution

$$\begin{aligned}(1+2x)^{-\frac{1}{3}} &= 1 + -\frac{1}{3} \cdot 2x + \frac{-\frac{1}{3}(-\frac{1}{3}-1)}{2} \cdot (2x)^2 + \dots \\ &= 1 - \frac{2}{3}x + \frac{8}{9}x^2 + \dots\end{aligned}$$

Substituting $x = \dfrac{1}{14}$,

$$\begin{aligned} & (1 + 2 \cdot \frac{1}{14})^{-\frac{1}{3}} = 1 - \frac{2}{3} \cdot \frac{1}{14} + \frac{8}{9}\left(\frac{1}{14}\right)^2 + \dots \\ \implies \quad & \left(\frac{8}{7}\right)^{-\frac{1}{3}} \approx \frac{422}{441} \\ \implies \quad & \left(\frac{7}{8}\right)^{\frac{1}{3}} \approx \frac{422}{441} \\ \implies \quad & (7)^{\frac{1}{3}} \cdot \frac{1}{2} \approx \frac{422}{441} \\ \implies \quad & \sqrt[3]{7} \approx \frac{844}{441} \\ & = 1.9138 \text{ (5 s.f.)}\end{aligned}$$

Since $\sqrt[3]{7} = 1.9129$ (5 s.f.), the approximation is accurate.

Problem 2.

In the triangle ABC, $AB = 1$, $BC = 3$ and angle $ABC = \theta$ radians. Given that θ is a sufficiently small angle, show that

$$AC \approx (4 + 3\theta^2)^{\frac{1}{2}} \approx a + b\theta^2$$

for constants a and b to be determined.

Solution

By cosine rule,

$$\begin{aligned}
& AC^2 = AB^2 + BC^2 - 2 \cdot AB \cdot BC \cdot \cos ABC \\
\implies & AC^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \cos\theta \\
& = 10 - 6\cos\theta
\end{aligned}$$

Since θ is sufficiently small, $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$. Hence,

$$\begin{aligned}
& AC^2 \approx 10 - 6\left(1 - \frac{\theta^2}{2}\right) \\
& = 4 + 3\theta^2 \\
\implies & AC \approx (4 + 3\theta^2)^{\frac{1}{2}} \\
& = 2\left(1 + \frac{3}{4}\theta^2\right)^{\frac{1}{2}} \\
& \approx 2\left(1 + \frac{1}{2} \cdot \frac{3}{4}\theta^2\right) \\
& = 2 + \frac{3}{4}\theta^2
\end{aligned}$$

Problem 3.

Given that $y = \ln \sec x$, show that

(a) $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 2\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}$

(b) the value of $\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4}$ when $x = 0$ is 2.

Write down the Maclaurin series for $\ln \sec x$ up to and including the term in x^4. By substituting $x = \dfrac{\pi}{4}$, show that $\ln 2 \approx \dfrac{\pi^2}{16} + \dfrac{\pi^4}{1536}$.

Solution

Part (a)

Note that $y = \ln \sec x = -\ln \cos x$. Hence,

$$e^{-y} = \cos x \tag{3.1}$$

Implicitly differentiating Equation 3.1,

$$\begin{aligned} & e^{-y} \cdot (-y') = -\sin x \\ \implies & y' e^{-y} = \sin x \end{aligned} \tag{3.2}$$

Implicitly differentiating Equation 3.2,

$$\begin{aligned} & y' e^{-y} \cdot (-y') + e^{-y} \cdot y'' = \cos x \\ \implies & y' e^{-y} \cdot (-y') + e^{-y} \cdot y'' = e^{-y} \\ \implies & y'' - (y')^2 = 0 \end{aligned} \tag{3.3}$$

Implicitly differentiating Equation 3.3,

$$\begin{aligned} & y^{(3)} - 2 \cdot y' \cdot y'' = 0 \\ \implies & y^{(3)} = 2y''y' \end{aligned} \tag{3.4}$$

Part (b)

Implicitly differentiating Equation 3.4,

$$y^{(4)} = 2\left(y^{(3)}y' + (y'')^2\right) \tag{3.5}$$

Substituting $x = 0$ into Equations 3.1, 3.2, 3.3, 3.4 and 3.5, we see that

$$\begin{aligned} y &= 0 \\ y' &= 0 \\ y'' &= 1 \\ y^{(3)} &= 0 \\ y^{(4)} &= 2 \end{aligned}$$

Thus, $\left.\dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4}\right|_{x=0} = 2$.

$$\begin{aligned}
\ln\sec x &= \sum_{n=0}^{\infty} \frac{y^{(n)}}{n!} x^n \\
&= \frac{1}{2}x^2 + \frac{1}{12}x^4 + \dots
\end{aligned}$$

Substituting $x = \dfrac{\pi}{4}$,

$$\begin{aligned}
\ln\sec\frac{\pi}{4} &= \frac{1}{2}\left(\frac{\pi}{4}\right)^2 + \frac{1}{12}\left(\frac{\pi}{4}\right)^4 + \dots \\
\implies \quad \ln\sqrt{2} &= \frac{\pi^2}{32} + \frac{\pi^2}{3072} + \dots \\
\implies \quad \frac{1}{2}\ln 2 &= \frac{\pi^2}{32} + \frac{\pi^2}{3072} + \dots \\
\implies \quad \ln 2 &= \frac{\pi^2}{16} + \frac{\pi^2}{1536} + \dots \\
\implies \quad \ln 2 &\approx \frac{\pi^2}{16} + \frac{\pi^2}{1536}
\end{aligned}$$