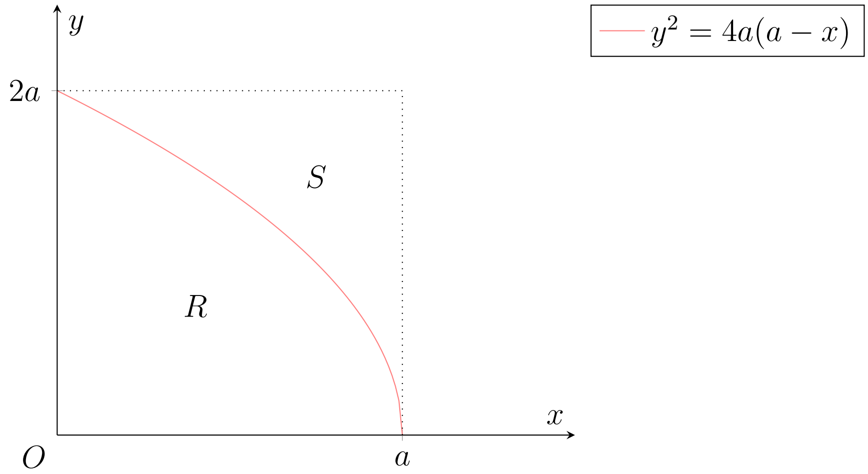
Problem 1.

The diagram shows the region R, which is bounded by the axes and the part of the curve $y^2 = 4a(a-x)$ lying in the first quadrant.

Find, in terms of a, the volume, V_x, of the solid formed when R is rotated completely about the x-axis.

The volume of the solid formed when R is rotated completely about the y-axis is V_y. Show that $V_y = \dfrac{8}{15}V_x$.

The region S, lying in the first quadrant, is bounded by the curve $y^2 = 4a(a-x)$ and the lines $x = a$ and $y = 2a$. Find, in terms of a, the volume of the solid formed when S is rotated completely about the y-axis.



Solution

$$\begin{aligned}
V_x &= \pi \int_0^a y^2 \,\mathrm{d}x \\
&= \pi \int_0^a 4a(a-x) \,\mathrm{d}x \\
&= 4\pi a \left[ax - \frac{1}{2}x^2\right]_0^a \\
&= 2\pi a^3
\end{aligned}$$

$$\boxed{V_x = 2\pi a^3 \text{ units}^3}$$

Note that $x = a - \dfrac{y^2}{4a} \implies x^2 = \left(a - \dfrac{y^2}{4a}\right)^2 = a^2 - \dfrac{1}{2}y^2 + \dfrac{1}{16a^2}y^4$. Hence,

$$\begin{aligned}
V_y &= \pi \int_0^{2a} x^2 \,\mathrm{d}y \\
&= \pi \int_0^{2a} a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4 \,\mathrm{d}y \\
&= \pi \left[a^2 y - \frac{1}{2}\cdot\frac{1}{3}y^3 + \frac{1}{16a^2}\cdot\frac{1}{5}y^5\right]_0^{2a}
\end{aligned}$$

$$
\begin{aligned}
&= \pi \left(2a^3 - \frac{8a^3}{6} + \frac{32a^5}{90a^2} \right) \\
&= \frac{16}{15}\pi a^3 \\
&= \frac{8}{15}(2\pi a^3) \\
&= \frac{8}{15}V_x
\end{aligned}
$$

$$
\begin{aligned}
\text{Volume } S &= \text{Volume of Cylinder} - V_y \\
&= \pi \cdot a^2 \cdot 2a - \frac{16}{15}\pi a^3 \\
&= \frac{14}{15}\pi a^3
\end{aligned}
$$

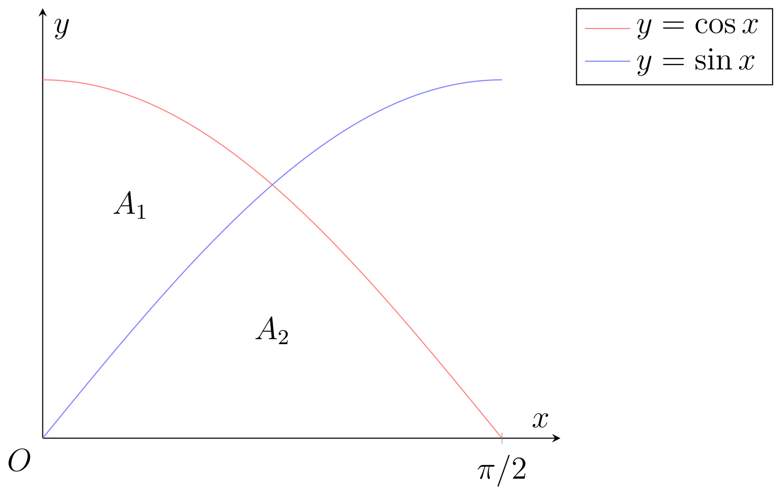
$$\boxed{\text{The volume required is } \frac{14}{15}\pi a^3 \text{ units}^3.}$$

Problem 2.

The region bounded by the axes and the curve $y = \cos x$ from $x = 0$ to $x = \dfrac{1}{2}\pi$ is divided into two parts, of areas A_1 and A_2, by the curve $y = \sin x$.

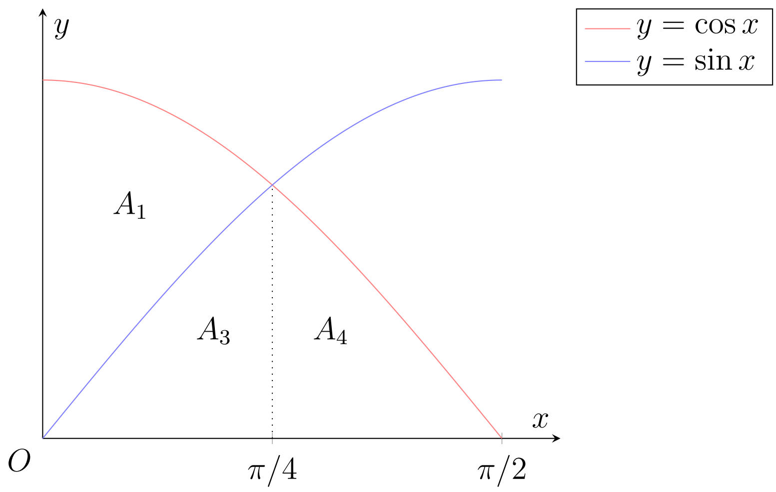
(a) Prove that $A_2 = \sqrt{2}A_1$.

(b) Prove Find the volume of the solid obtained when the region with area A_2 is rotated about the y-axis through 2π radians. Give your answer in exact form.



Solution

Part (a)



Let A_3 and A_4 be the areas as defined on the diagram above. By the symmetry of $y = \sin x$ and $y = \cos x$ about $x = \pi/4$, we have $A_3 = A_4$.

$$\begin{aligned} A_3 &= \int_0^{\pi/4} \sin x \,\mathrm{d}x \\ &= [-\cos x]_0^{\pi/4} \\ &= 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

$$\begin{aligned}
A_1 &= \int_0^{\pi/4} \cos x \, \mathrm{d}x - A_3 \\
&= [\sin x]_0^{\pi/4} - \left(1 - \frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2} \\
&= \sqrt{2} - 1 \\
\implies A_2 &= 2A_3 \\
&= 2\left(1 - \frac{\sqrt{2}}{2}\right) \\
&= 2 - \sqrt{2} \\
&= \sqrt{2}\left(\sqrt{2} - 1\right) \\
&= \sqrt{2}A_1
\end{aligned}$$

Part (b)

Let V_3 and V_4 be the volumes of the solids obtained when A_3 and A_4 are rotated about the y-axis through 2π radians, respectively.

$$\begin{aligned}
V_3 &= 2\pi \int_0^{\pi/4} xy \, \mathrm{d}x \\
&= 2\pi \int_0^{\pi/4} x \sin x \, \mathrm{d}x
\end{aligned}$$

	D	I
$+$	x	$\sin x$
$-$	1	$-\cos x$
$+$	0	$-\sin x$

$$\begin{aligned}
\implies V_3 &= 2\pi \left[-x \cos x + \sin x\right]_0^{\pi/4} \\
&= 2\pi \left[\left(-\frac{\pi}{4}\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) - (0 + \sin 0)\right] \\
&= 2\pi \left(-\frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \\
&= 2\pi \cdot \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right) \\
&= \sqrt{2}\pi\left(1 - \frac{\pi}{4}\right)
\end{aligned}$$

$$\begin{aligned}
V_4 &= 2\pi \int_{\pi/4}^{\pi/2} xy \, \mathrm{d}x \\
&= 2\pi \int_{\pi/4}^{\pi/2} x \cos x \, \mathrm{d}x
\end{aligned}$$

	D	I
$+$	x	$\cos x$
$-$	1	$\sin x$
$+$	0	$-\cos x$

$$
\begin{aligned}
\implies \qquad V_4 &= 2\pi \left[x \sin x + \cos x\right]_{\pi/4}^{\pi/2} \\
&= 2\pi \left(\left[\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right] - \left[\frac{\pi}{4} \sin \frac{\pi}{4} + \cos \frac{\pi}{4}\right] \right) \\
&= 2\pi \left[\frac{\pi}{2} - \left(\frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \right] \\
&= 2\pi \left[\frac{\pi}{2} - \frac{\sqrt{2}}{2} \left(1 + \frac{\pi}{4}\right) \right] \\
&= \pi^2 - \sqrt{2}\pi \left(1 + \frac{\pi}{4}\right) \\
\implies \text{Required volume} &= V_3 + V_4 \\
&= \sqrt{2}\pi \left(1 - \frac{\pi}{4}\right) + \pi^2 - \sqrt{2}\pi \left(1 + \frac{\pi}{4}\right) \\
&= \pi^2 - 2 \cdot \sqrt{2}\pi \cdot \frac{\pi}{4} \\
&= \pi^2 - \frac{\sqrt{2}}{2}\pi^2
\end{aligned}
$$

$$\boxed{\text{The required volume is } \left(\pi^2 - \frac{\sqrt{2}}{2}\pi^2\right) \text{ units}^3.}$$

Problem 3.

A curve has parametric equations

$$x = \cos^2 t,\ y = \sin^3 t,\ 0 \le t \le \frac{1}{2}\pi$$

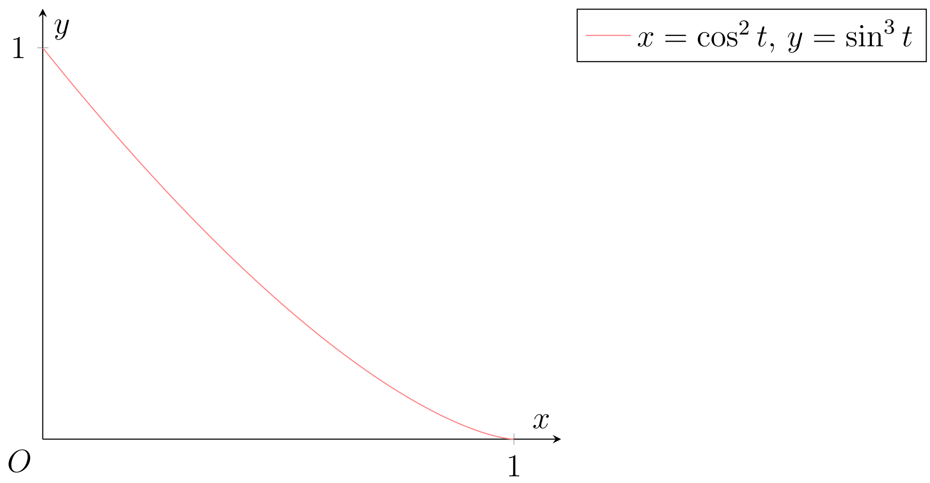
(a) Sketch the curve.

(b) Show that the area under the curve for $0 \le t \le \frac{1}{2}\pi$ is $2\int_0^{\pi/2} \cos t \sin^4 t \,\mathrm{d}t$, and find the exact value of the area.

(c) Find the volume of the solid obtained when the region in (b) is rotated about the y-axis through 2π radians.

Solution

Part (a)



Part (b)

Note that $x = 0 \implies t = \frac{\pi}{2}$ and $x = 1 \implies t = 0$. Hence,

$$\begin{aligned}
\text{Area} &= \int_0^1 y \,\mathrm{d}x \\
&= \int_{\pi/2}^0 y \frac{\mathrm{d}x}{\mathrm{d}t} \,\mathrm{d}t \\
&= \int_{\pi/2}^0 \sin^3 t \cdot (-2\cos t \sin t) \,\mathrm{d}t \\
&= 2\int_0^{\pi/2} \cos t \sin^4 t \,\mathrm{d}t \\
&= B\left(\frac{5}{2}, 1\right) \\
&= \frac{\Gamma(5/2)\,\Gamma(1)}{\Gamma(5/2+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(5/2) \cdot 1}{5/2 \cdot \Gamma(5/2)} \\
&= \frac{2}{5}
\end{aligned}
$$

$$\boxed{\text{The area under the curve is } \frac{2}{5} \text{ units}^2.}$$

Part (c)

$$
\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xy \, \mathrm{d}x \\
&= 2\pi \int_{\pi/2}^0 \cos^2 t \sin^3 t \cdot (-2 \cos t \sin t) \, \mathrm{d}t \\
&= 2\pi \cdot 2 \int_0^{\pi/2} \cos^3 t \sin^4 t \, \mathrm{d}t \\
&= 2\pi \cdot B(5/2, 2) \\
&= 2\pi \cdot \frac{\Gamma(5/2)\,\Gamma(2)}{\Gamma(5/2 + 2)} \\
&= 2\pi \cdot \frac{\Gamma(5/2) \cdot 1}{7/2 \cdot 5/2 \cdot \Gamma(5/2)} \\
&= 2\pi \cdot \frac{2}{7} \cdot \frac{2}{5} \\
&= \frac{8}{35}\pi
\end{aligned}
$$

$$\boxed{\text{The required volume is } \frac{8}{35}\pi \text{ units}^3.}$$

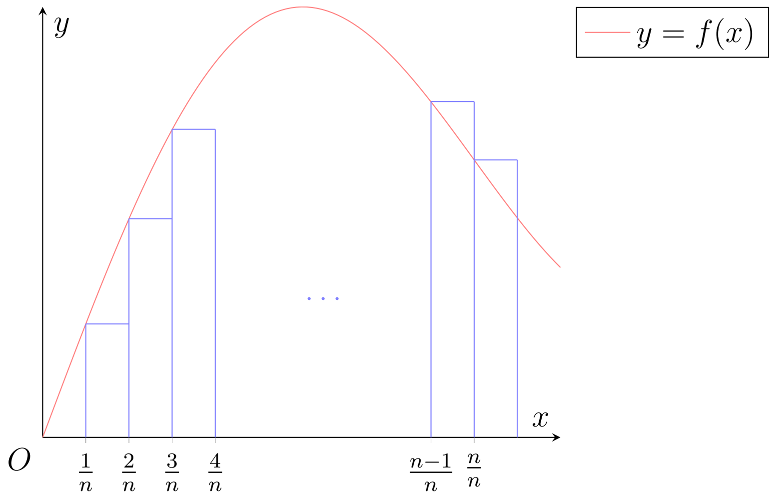
Problem 4.

(a) Given that f is a continuous function, explain, with the aid of a sketch, why the value of
$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right]$$
is $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$.

(b) Hence, evaluate $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left(\frac{\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}}{\sqrt[3]{n}}\right)$.

Solution

Part (a)



The area of the rectangles in the above figure is given by
$$\frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right]$$

This gives an approximation of the signed area under the curve from $x = \frac{1}{n}$ to $x = \frac{n}{n} = 1$. As $n \to \infty$, the widths of the rectangles become smaller and the approximation becomes exact. Hence,
$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \ldots + f\left(\frac{n}{n}\right)\right] = \int_0^1 f(x)\,\mathrm{d}x$$

Part (b)

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\left(\frac{\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}}{\sqrt[3]{n}}\right) &= \lim_{n\to\infty} \frac{1}{n}\left[\sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \ldots + \sqrt[3]{\frac{n}{n}}\right] \\ &= \int_0^1 \sqrt[3]{x}\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{1}{1/3+1}x^{1/3+1}\right]_0^1 \\
&= \frac{3}{4}
\end{aligned}
$$

$$\boxed{\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)=\frac{3}{4}}$$

Problem 5.

The function f satisfies $f'(x) > 0$ for $a \leq x \leq b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b f(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y$$

where $\alpha = f(a)$ and $\beta = f(b)$. Interpret this formula geometrically by means of a sketch where α and a are positive. Verify this result for the case where $f(x) = e^{2x}$, $a = 0$, $b = 1$.

Prove similarly and interpret geometrically the formula

$$2\pi \int_a^b x f(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2\,\mathrm{d}y$$

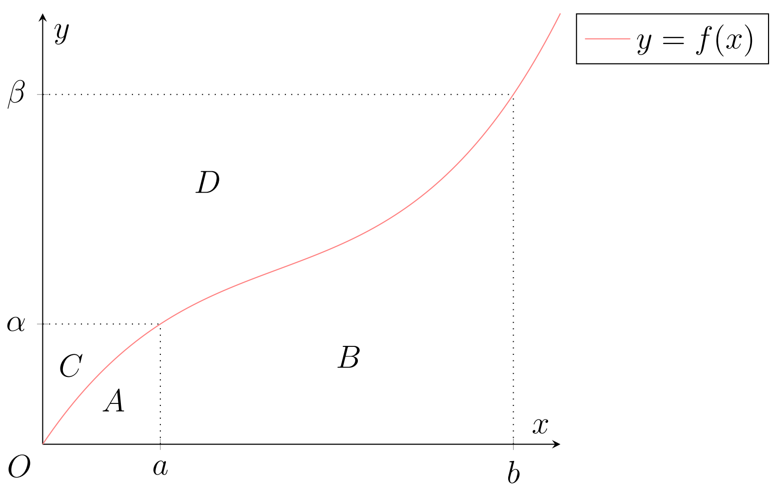
Solution

$$\begin{aligned} y &= f(x) \\ \implies \mathrm{d}y &= f'(x)\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \int_\alpha^\beta g(y)\,\mathrm{d}y &= \int_a^b f^{-1}(f(x)) f'(x)\,\mathrm{d}x \\ &= \int_a^b x f'(x)\,\mathrm{d}x \end{aligned}$$

	D	I
$+$	x	$f'(x)$
$-$	1	$f(x)$

$$\begin{aligned} &= [x f(x)]_a^b - \int_a^b f(x)\,\mathrm{d}x \\ &= b\beta - a\alpha - \int_a^b f(x)\,\mathrm{d}x \\ \implies \int_a^b f(x)\,\mathrm{d}x &= b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y \end{aligned}$$



Consider the above diagram. We clearly have $\text{Area}(A+C) = a\alpha$, $\text{Area}(A+B+C+D) = b\beta$, $\text{Area}\,B = \int_a^b f(x)\,\mathrm{d}x$ and $\text{Area}\,D = \int_\alpha^\beta g(y)\,\mathrm{d}y$. Thus,

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &= \text{Area}\,B \\ &= \text{Area}(A+B+C+D) - \text{Area}(A+C) - \text{Area}\,D \\ &= b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y\end{aligned}$$

Standard Way.

$$\int_0^1 e^{2x}\,\mathrm{d}x = \left[\frac{1}{2}e^{2x}\right]_0^1 = \frac{1}{2}e^2 - \frac{1}{2}$$

Via Formula. Let $f(x) = e^{2x}$. Then $g(x) = \frac{1}{2}\ln x$. Hence, $\alpha = g(0) = 1$ and $\beta = g(1) = e^2$. Invoking the above formula,

$$\begin{aligned}\int_0^1 e^{2x}\,\mathrm{d}x &= 1\cdot e^2 - 0\cdot 1 - \int_1^{e^2} \frac{1}{2}\ln x\,\mathrm{d}x \\ &= e^2 - \frac{1}{2}\left[x\ln x - x\right]_1^{e^2} \\ &= e^2 - \frac{1}{2}\left[\left(e^2\ln e^2 - e^2\right) - (\ln 1 - 1)\right] \\ &= e^2 - \frac{1}{2}\left(e^2+1\right) \\ &= \frac{1}{2}e^2 - \frac{1}{2}\end{aligned}$$

Hence, the formula holds for the above case.

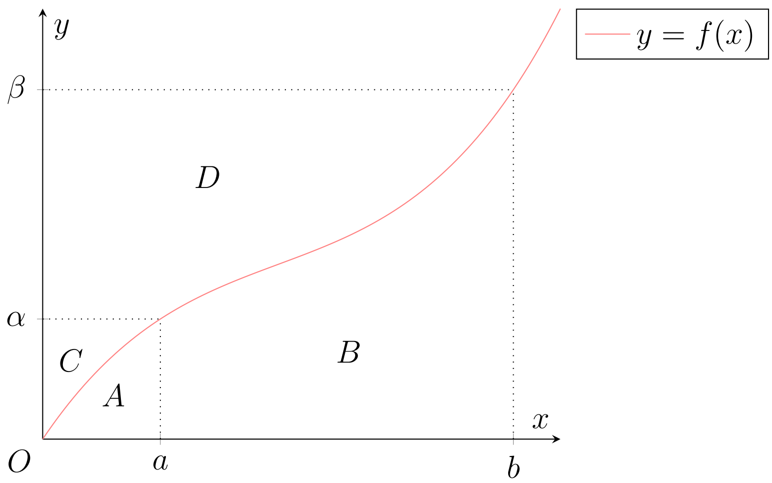
$$\begin{aligned}\int_\alpha^\beta [g(y)]^2\,\mathrm{d}y &= \int_\alpha^\beta \left[f^{-1}(f(x))\right]^2 f'(x)\,\mathrm{d}x \\ &= \int_a^b x^2 f'(x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned} y &= f(x) \\ \implies \mathrm{d}y &= f'(x)\,\mathrm{d}x\end{aligned}$$

	D	I
$+$	x^2	$f'(x)$
$-$	$2x$	$f(x)$

$$= \left[x^2 f(x)\right]_a^b - 2\int_a^b x f(x)\,\mathrm{d}x$$

$$
\begin{aligned}
&= b^2\beta - a^2\alpha - 2\int_a^b xf(x)\,\mathrm{d}x \\
\implies \quad 2\int_a^b xf(x)\,\mathrm{d}x &= b^2\beta - a^2\alpha - \int_\alpha^\beta [g(y)]^2\,\mathrm{d}y \\
\implies 2\pi\int_a^b xf(x)\,\mathrm{d}x &= \pi\left(b^2\beta - a^2\alpha\right) - \pi\int_\alpha^\beta [g(y)]^2\,\mathrm{d}y
\end{aligned}
$$



Let Volume R represent the volume of the solid obtained when the region R is rotated completely about the y-axis.

We clearly have Volume$(A+B+C+D) = \pi b^2\beta$, Volume$(A+C) = \pi a^2\alpha$, Volume $B = 2\pi\int_a^b xf(x)\,\mathrm{d}x$ (using the shell method), and Volume $D = \pi\int_\alpha^\beta [g(y)]^2\,\mathrm{d}y$ (using the disc method). Thus,

$$
\begin{aligned}
2\pi\int_a^b xf(x)\,\mathrm{d}x &= \text{Volume } B \\
&= \text{Volume}(A+B+C+D) - \text{Volume}(A+C) - \text{Volume } D \\
&= \pi b^2\beta - \pi a^2\alpha - \pi\int_\alpha^\beta [g(y)]^2\,\mathrm{d}y \\
&= \pi\left(b^2\beta - a^2\alpha\right) - \pi\int_\alpha^\beta [g(y)]^2\,\mathrm{d}y
\end{aligned}
$$