Problem 1.

The complex number w is such that $ww^* + 2w = 3 + 4i$, where w^* is the complex conjugate of w. Find w in the form $a + ib$, where a and b are real.

Solution

Note $ww^* = (\operatorname{Re} w)^2 + (\operatorname{Im} w)^2 \in \mathbb{R}$.

Taking the imaginary part of the given equation,

$$
\begin{aligned}
& \operatorname{Im}(ww^* + 2w) = \operatorname{Im}(3 + 4i) \\
\implies & \quad 2 \operatorname{Im}(w) = 4 \\
\implies & \quad \operatorname{Im}(w) = 2
\end{aligned}
$$

Taking the real part of the given equation,

$$
\begin{aligned}
& \operatorname{Re}(ww^* + 2w) = \operatorname{Re}(3 + 4i) \\
\implies & \left[(\operatorname{Re} w)^2 + 2^2\right] + 2 \operatorname{Re} w = 3 \\
\implies & (\operatorname{Re} w)^2 + 2 \operatorname{Re}(w) + 1 = 0 \\
\implies & (\operatorname{Re} w + 1)^2 = 0 \\
\implies & \operatorname{Re}(w) = -1
\end{aligned}
$$

Hence, $w = -1 + 2i$.

$$\boxed{w = -1 + 2i}$$

Problem 2.

Express $(3-i)^2$ in the form $a+ib$.

Hence, or otherwise, find the roots of the equation $(z+i)^2 = -8+6i$.

Solution

$$\begin{aligned}(3-i)^2 &= 3^2 - 2\cdot 3\cdot i + (i)^2 \\ &= 9 - 6i - 1 \\ &= 8 - 6i\end{aligned}$$

$$\boxed{(3-i)^2 = 8-6i}$$

$$\begin{aligned} & (z+i)^2 = -8+6i \\ \implies & -(z+i)^2 = 8-6i \\ \implies & i^2(z+i)^2 = 8-6i \\ \implies & (iz-1)^2 = 8-6i \\ \implies & iz-1 = \pm(3-i) \\ \implies & z = \frac{1}{i}(1\pm(3-i)) \\ & = -i(1\pm(3-i)) \\ & = -i \mp (3i+1) \\ & = -1-4i \vee 1+2i \end{aligned}$$

$$\boxed{z = -1-4i \vee 1+2i}$$

Problem 3.

(a) It is given that $z_1 = 1 + \sqrt{3}i$. Find the value of z_1^3, showing clearly how you obtain your answer.

(b) Given that $1 + \sqrt{3}i$ is a root of the equation

$$2z^3 + az^2 + bz + 4 = 0$$

find the values of the real numbers a and b. Hence, solve the above equation.

Solution

Part (a)

$$\begin{aligned} z_1 &= 1 + \sqrt{3}i \\ &= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \\ &= 2e^{i\pi/3} \\ \implies z_1^3 &= \left(2e^{i\pi/3}\right)^3 \\ &= 8e^{i\pi} \\ &= 8\left(\cos\pi + i\sin\pi\right) \\ &= -8 \end{aligned}$$

$$\boxed{z_1^3 = -8}$$

Part (b)

Since $1 + \sqrt{3}i$ is a root of the given equation, we have

$$\begin{aligned} & 2(1+\sqrt{3}i)^3 + a(1+\sqrt{3}i)^2 + b(1+\sqrt{3}i) + 4 = 0 \\ \implies & \quad 2 \cdot -8 + a(-2 + 2\sqrt{3}i) + b(1+\sqrt{3}i) + 400 \\ \implies & \quad (-2a + b) + \sqrt{3}(2a + b)i = 12 \end{aligned}$$

Comparing real and imaginary parts, we obtain $-2a + b = 12$ and $2a + b = 0$, whence $a = -3$ and $b = 6$.

$$\boxed{a = -3,\ b = 6}$$

Since the coefficients of $2z^3 + az^2 + bz + 4$ are all real, the second root is $(1+\sqrt{3}i)^* = 1 - \sqrt{3}i$. Let the third root be k. By Vieta's formula,

$$\begin{aligned} & (1+\sqrt{3}i)(1-\sqrt{3}i)k = -\frac{4}{2} \\ \implies & \quad 4k = -2 \\ \implies & \quad k = -\frac{1}{2} \end{aligned}$$

$$\boxed{\text{The roots of the equation are } 1+\sqrt{3}i,\ 1-\sqrt{3}i \text{ and } -\frac{1}{2}.}$$

Problem 4.

The complex number z is such that $az^2 + bz + a = 0$ where a and b are real constants. It is given that $z = z_0$ is a solution to this equation where $\operatorname{Im}(z_0) \neq 0$.

(a) Verify that $z = \dfrac{1}{z_0}$ is the other solution. Hence, show that $|z_0| = 1$.

Take $\operatorname{Im}(z_0) = \dfrac{1}{2}$ for the rest of the question.

(b) Find the possible complex numbers for z_0.

(c) If $\operatorname{Re}(z_0) > 0$, find b in terms of a.

Solution

Part (a)

$$a\left(\frac{1}{z_0}^2\right) + b\left(\frac{1}{z_0}\right) + a = \frac{1}{z_0}^2\left(a + bz_0 + az_0^2\right) = 0$$

Hence, $z = \dfrac{1}{z_0}$ is a root of the given equation.

Since $a, b \in \mathbb{R}$, by the conjugate root theorem, $z_0^* = \dfrac{1}{z_0}$.

$$\begin{aligned}
z_0^* &= \frac{1}{z_0} \\
\implies \quad z_0 z_0^* &= 1 \\
\implies \quad |z_0 z_0^*| &= 1 \\
\implies |z_0|\,|z_0^*| &= 1 \\
\implies |z_0|\,|z_0| &= 1 \\
\implies \quad |z_0| &= 1
\end{aligned}$$

Note that we reject $|z_0| = -1$ since $|z_0| > 0$.

Part (b)

Let $z_0 = x + \dfrac{1}{2}i$.

$$\begin{aligned}
\left|x + \frac{1}{2}i\right| &= 1 \\
\implies x^2 + \left(\frac{1}{2}\right)^2 &= 1^2 \\
\implies \quad x^2 &= \frac{3}{4} \\
\implies \quad x &= \pm\frac{\sqrt{3}}{2}
\end{aligned}$$

$$\boxed{z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}i \vee -\frac{\sqrt{3}}{2} + \frac{1}{2}i}$$

Part (c)

Since $\operatorname{Re}(z_0) > 0$, we have $z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. By Vieta's formula,

$$\begin{aligned}
z_0 + \frac{1}{z_0} &= -\frac{b}{a} \\
\implies z_0 + z_0^* &= -\frac{b}{a} \\
\implies 2\operatorname{Re}(z_0) &= -\frac{b}{a} \\
\implies \sqrt{3} &= -\frac{b}{a} \\
\implies b &= -\sqrt{3}a
\end{aligned}$$

$$\boxed{b = -\sqrt{3}a}$$