Problem 1.

Omitted.

Problem 2.

A curve E has polar equation $r = \dfrac{3}{\sqrt{\cos^2\theta + (\sin\theta - \cos\theta)^2}}$ for $0 \le \theta < 2\pi$.

(a) Taking the polar axis as the positive x-axis, find the Cartesian equation of E, leaving your answer in the form $ax^2 + bxy + cy^2 = 9$, where a, b and c are constants to be determined.

(b) Hence or otherwise, find the exact Cartesian coordinates of the point(s) of intersection between E and the graph with polar equation $r = \dfrac{1}{\sin\theta - \cos\theta}$ for $\dfrac{\pi}{4} < \theta < \dfrac{5\pi}{4}$.

Solution

Part (a)

$$\begin{aligned}
&& r &= \frac{3}{\sqrt{\cos^2\theta + (\sin\theta - \cos\theta)^2}} \\
\implies && r\sqrt{\cos^2\theta + (\sin\theta - \cos\theta)^2} &= 3 \\
\implies && r^2\left(\cos^2\theta + (\sin\theta - \cos\theta)^2\right) &= 9 \\
\implies && r^2\left(2\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta\right) &= 9 \\
\implies && 2(r\cos\theta)^2 - 2(r\cos\theta)(r\sin\theta) + (r\sin\theta)^2 &= 9 \\
\implies && 2x^2 - 2xy + y^2 &= 9
\end{aligned}$$

$$\boxed{E : 2x^2 - 2xy + y^2 = 9}$$

Part (b)

$$\begin{aligned}
&& r &= \frac{1}{\sin\theta - \cos\theta} \\
\implies && r(\sin\theta - \cos\theta) &= 1 \\
\implies && r\sin\theta - r\cos\theta &= 1 \\
\implies && y - x &= 1 \\
\implies && y &= x + 1
\end{aligned}$$

Substituting $y = x + 1$ into the Cartesian equation of E,

$$\begin{aligned}
&& 2x^2 - 2xy + y^2 &= 9 \\
\implies && 2x^2 - 2x(x+1) + (x+1)^2 &= 9 \\
\implies && 2x^2 - 2x^2 - 2x + x^2 + 2x + 1 &= 9 \\
\implies && x^2 &= 8 \\
\implies && x &= \pm\sqrt{8} \\
&& &= \pm 2\sqrt{2} \\
\implies && y &= 1 \pm 2\sqrt{2}
\end{aligned}$$

$$\boxed{\left(2\sqrt{2}, 1 + 2\sqrt{2}\right),\ \left(-2\sqrt{2}, 1 - 2\sqrt{2}\right)}$$

Problem 3.

The function f is defined by

$$f(x) = 2 - \frac{1}{x},\ x \in \mathbb{R}, x \geq 1$$

(a) Write down expressions for $f^2(x)$, $f^3(x)$ and $f^4(x)$ in the form $\dfrac{ax+b}{cx+d}$, where a, b, c and d are integers. Hence, make a conjecture for $f^n(x)$ in terms of n.

(b) Prove your conjecture for all positive integers n.

(c) Let A be the largest subset of the real numbers such that when the domain of f is replaced with A, $f^n(x)$ is defined for all positive integers n. State A.

Solution

Part (a)

$$\begin{aligned}
f^2(x) &= 2 - \frac{1}{f(x)} \\
&= 2 - \left(2 - \frac{1}{x}\right)^2 \\
&= \frac{3x-2}{2x-1} \\
\implies f^3(x) &= 2 - \frac{1}{f^2(x)} \\
&= 2 - \left(\frac{3x-2}{2x-1}\right)^{-1} \\
&= \frac{4x-3}{3x-2} \\
\implies f^4(x) &= 2 - \frac{1}{f^3(x)} \\
&= 2 - \left(\frac{4x-3}{3x-2}\right)^{-1} \\
&= \frac{5x-4}{4x-3}
\end{aligned}$$

$$\boxed{f^2(x) = \frac{3x-2}{2x-1},\ f^3(x) = \frac{4x-3}{3x-2},\ f^4(x) = \frac{5x-4}{4x-3}}$$

Conjecture. For all $n \in \mathbb{N}$, $f^n(x) = \dfrac{(n+1)x-n}{nx-(n-1)}$.

Part (b)

Let P_n be the statement that $f^n(x) = \dfrac{(n+1)x-n}{nx-(n-1)}$ for $n \in \mathbb{N}$.

Base Case. $n = 1 : f^1(x) = 2 - \frac{1}{x} = \frac{2x-1}{x-0}$. Hence, P_1 is true.

Inductive Hypothesis. Assume P_k is true for some $k \in \mathbb{N}$.

Inductive Step. Consider $f^{k+1}(x)$.

$$\begin{aligned} f^{k+1}(x) &= 2 - \frac{1}{f^k(x)} \\ &= 2 - \left(\frac{(k+1)x-k}{kx-(k-1)}\right)^{-1} \\ &= 2 - \frac{kx-(k-1)}{(k+1)x-k} \\ &= \frac{2\left[(k+1)x-k\right]-\left[kx-(k-1)\right]}{(k+1)x-k} \\ &= \frac{(k+2)x-(k+1)}{(k+1)-k} \end{aligned}$$

Hence, $P_k \implies P_{k+1}$. Since P_1 is true, by induction, P_n is true for all $n \in \mathbb{N}$.

Part (c)

Since $f^n(x) = \frac{(n+1)x-n}{nx-(n-1)}$ for all $n \in \mathbb{N}$, we have the restriction $nx-(n-1) \neq 0 \implies x \neq \frac{n-1}{n}$. Hence,

$$\boxed{A = \mathbb{R} \setminus \left\{n \in \mathbb{N} : \frac{n-1}{n}\right\}}$$

Problem 4.

A curve T has polar equation $r = \sqrt{\dfrac{2}{\sin 3\theta}}$ where $-\pi < \theta \leq \pi$.

(a) Determine the range of values of θ for which the value of r is undefined. Hence, state the equations of the asymptotes of T in polar form.

(b) Hence, sketch T, indicating clearly, in polar form, the equations of the asymptotes and any lines of symmetry, and the polar coordinates of any points where r attains stationary values.

Solution

Part (a)

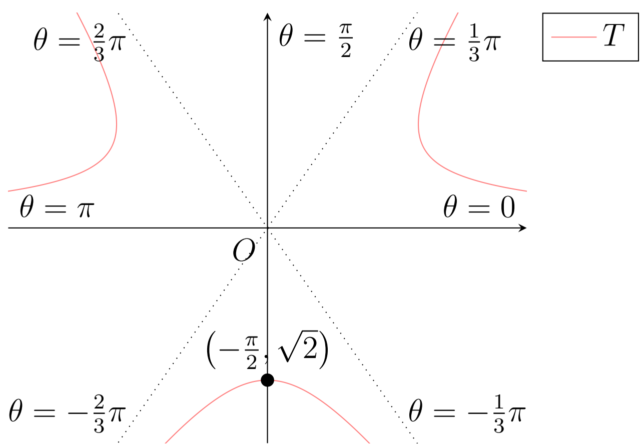
For r to be undefined, either $\sin 3\theta = 0$ or $\dfrac{2}{\sin 3\theta} < 0$. In the latter case, $\sin 3\theta < 0$. Hence, it suffices to determine the range of values of θ such that $\sin 3\theta \leq 0$. Note that $-\pi < \theta \leq \pi \implies -3\pi < 3\theta \leq 3\pi$. Hence,

$$3\theta \in (-3\pi, -2\pi] \cup [-\pi, 0] \cup [\pi, 2\pi] \cup \{3\pi\}$$

$$\boxed{r \text{ is undefined for } \theta \in \left(-\pi, -\frac{2}{3}\pi\right] \cup \left[-\frac{1}{3}\pi, 0\right] \cup \left[\frac{1}{3}\pi, \frac{2}{3}\pi\right] \cup \{\pi\}.}$$

$$\boxed{\text{Asymptotes: } \theta = -\frac{2}{3}\pi, -\frac{1}{3}\pi, 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi.}$$

Part (b)



From the graph, we see that the only stationary point of T occurs when $\theta = -\dfrac{\pi}{2}$, where $r = \sqrt{2}$.

Problem 5.

Omitted.

Problem 6.

(a) The sequence $\{X_n\}$ is given by $X_0 = 6$ and

$$X_n = \frac{1}{4}X_{n-1} + 2^{1-n},\ n \geq 1$$

By multiplying the recurrence relation throughout by 2^n, use a suitable substitution to determine X_n as a function of n.

(b) The sequence of real numbers $\{u_n\}$ is defined by $u_1 = a$, and the recurrence relation

$$u_{n+1} = \frac{u_n^2 + 5}{2u_n + 4},\ n \geq 1$$

(i) Given that the sequence converges to a limit l, find all possible values of l.

(ii) With the aid of a graphing calculator, determine the long-term behaviour of the sequence when $a = -2.01$ and when $a = -1.99$.

(iii) Show that $u_{n+1} > -2$ if $u_n > -2$ and $u_{n+1} < -2$ if $u_n < -2$. Hence, explain the difference in the behaviour of the sequence when $a = -2.01$ and $a = -1.99$.

Solution

Part (a)

$$\begin{aligned} X_n &= \frac{1}{4}X_{n-1} + 2^{1-n} \\ \implies 2^n X_n &= \frac{1}{4} \cdot 2^n X_{n-1} + 2 \\ \implies 2^n X_n &= \frac{1}{2} \cdot 2^{n-1} X_{n-1} + 2 \end{aligned}$$

Let $Y_n = 2^n X_n$. We have

$$Y_n = \frac{1}{2}Y_{n-1} + 2$$

Let k be a constant such that $Y_n + k = \frac{1}{2}(Y_{n-1} + k) \implies \frac{1}{2}k - k = 2 \implies k = -4$.

$$\begin{aligned} Y_n - 4 &= \frac{1}{2}(Y_{n-1} - 4) \\ \implies Y_n - 4 &= \frac{1}{2^n}(Y_0 - 4) \\ \implies \quad Y_n &= \frac{1}{2^n}(2^0 X_0 - 4) + 4 \\ &= \frac{1}{2^n} \cdot 2 + 4 \\ &= 2^{1-n} + 2^2 \\ \implies \quad X_n &= 2^{-n} Y_n \\ &= 2^{1-2n} + 2^{2-n} \end{aligned}$$

$$\boxed{X_n = 2^{1-2n} + 2^{2-n}}$$

Part (b)

Subpart (i)
Let $l = \lim_{n\to\infty} u_n$.

$$\begin{aligned} & l = \frac{l^2+5}{2l+4} \\ \implies & l(2l+4) = l^2+5 \\ \implies & l^2+4l-5=0 \\ \implies & (l+5)(l-1)=0 \end{aligned}$$

Hence, $l=-5$ or $l=1$.

$$\boxed{l = -5 \vee 1}$$

Subpart (ii)

When $a=-2.01$, the sequence is increase and converges to -5.

When $a=-1.99$, the sequence is decreasing and converges to 1.

Subpart (iii)

$$\begin{aligned} u_{n+1} &= \frac{u_n^2+5}{2u_n+4} \\ &= \frac{1}{2}u_n - 1 + \frac{9}{2u_n+4} \end{aligned}$$

Suppose $u_n > -2$. Then $2u_n+4>0$. Hence,

$$\begin{aligned} u_{n+1} &= \frac{1}{2}u_n - 1 + \frac{9}{2u_n+4} \\ &> \frac{1}{2}\cdot -2 - 1 + 0 \\ &> -2 \end{aligned}$$

Thus, $u_{n+1} > -2$.

Suppose $u_n < -2$. Then $2u_n+4<0$. Hence,

$$\begin{aligned} u_{n+1} &= \frac{1}{2}u_n - 1 + \frac{9}{2u_n+4} \\ &< \frac{1}{2}\cdot -2 - 1 + 0 \\ &< -2 \end{aligned}$$

Thus, $u_{n+1} < -2$.

When $a=-2.01<-2$, all terms in the sequence are less than -2. Hence, the sequence converges to -5 since the other limiting value (1) is greater than -2.

When $a=-1.99>-2$, all terms in the sequence are greater than -2. Hence, the sequence converges to 1 since the other limiting value (-5) is less than -2.

Problem 7.

After the Omega variant of a new virus emerged in 2023, a group of epidemiologists sought to model the spread of the virus in Singapore. In their model, the increase in cases from week $n-1$ to n is modelled as k times the increase in cases from week $n-2$ to $n-1$, where k is known as the weekly infection growth rate. Let x_n be the total number of Omega variant cases in Singapore within the first n weeks of the outbreak.

(a) Show that x_n is defined by the recurrence relation $x_n = ax_{n-1} + bx_{n_2}$, where a and b are constants to be determined in terms of k.

8 Omega variant cases were reported in the first week of the outbreak, while 15 new cases were reported the week after.

(b) Given that $k \neq 0$ and $k \neq 1$, solve the recurrence relation and obtain an expression for x_n of the form $\alpha + \beta\left(k^{n-1} - 1\right)$, where α and β are constants to be determined in terms of k.

(c) Determine the long-term behaviour of x_n for the cases where $k > 1$ and $0 < k < 1$, and hence explain in context what the long-term spread of the Omega variant in Singapore will be for each case.

(d) State a limitation of using this model to predict the spread of the Omega variant in Singapore.

(e) Show that x_n is in arithmetic progression if the weekly infection growth rate is 1, and state the value of its common difference.

In one simulation, the weekly infection growth rate is estimated to be 4. After the implementation of a mass vaccination programme in the fifth week of the outbreak, the weekly infection growth rate is estimated to decrease to 1, such that the increase in cases from week 5 to 6 is equal to the increase from week 4 to 5. The weekly infection growth rate then remains unchanged for the rest of the outbreak. Alert Orange is triggered when the 10000th case is reported.

(f) Under this simulation, determine the week in which Alert Orange will be triggered.

Solution

Part (a)

Let Δx_n be the increase in cases from week n to $n+1$. We have the relations

$$x_n = x_{n-1} + \Delta x_{n-1} \text{ and } \Delta x_n = k\Delta x_{n-1}$$

Note that $x_{n-1} = x_{n-2} + \Delta x_{n-2} \implies \Delta x_{n-2} = x_{n-1} - x_{n-2}$. Then

$$\begin{aligned} x_n &= x_{n-1} + \Delta x_{n-1} \\ &= x_{n-1} + k\Delta x_{n-2} \\ &= x_{n-1} + k\left(x_{n-1} - x_{n-2}\right) \\ &= (1+k)x_{n-1} - kx_{n-2} \end{aligned}$$

Part (b)

Consider the characteristic equation of the recurrence relation.

$$\begin{aligned} & x^2 - (1+k)x + k = 0 \\ \implies \quad & (x-1)(x-k) = 0 \end{aligned}$$

Hence, the roots of the characteristic equation are 1 and k. Since $k \neq 1$, we have

$$\begin{aligned} x_n &= A \cdot 1^n + B \cdot k^n \\ &= A + Bk^n \end{aligned}$$

Consider $n = 1, 2$. Since $x_1 = 8$ and $x_2 = 23$, we have the system

$$\begin{cases} A + \ \ kB = 8 \\ A + k^2 B = 23 \end{cases}$$

We hence have $A = 8 - kB = 23 - k^2 B$. Thus,

$$\begin{aligned} & 8 - kB = 23 - k^2 B \\ \implies & Bk^2 - Bk = 15 \\ \implies & \qquad B = \frac{15}{k(k-1)} \\ \implies & \qquad A = 8 - \frac{15}{k-1} \end{aligned}$$

Putting the values of A and B back into our recurrence relation, we obtain

$$\begin{aligned} x_n &= \left(8 - \frac{15}{k-1}\right) + \left(\frac{15}{k(k-1)}\right) k^n \\ &= 8 - \frac{15}{k-1} + \frac{15}{k-1} k^{n-1} \\ &= 8 + \frac{15}{k-1}\left(k^{n-1} - 1\right) \end{aligned}$$

$$\boxed{x_n = 8 + \frac{15}{k-1}\left(k^{n-1} - 1\right)}$$

Part (c)

Suppose $k > 1$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left[8 + \frac{15}{k-1}\left(k^{n-1} - 1\right)\right] \to \infty$$

Hence, everyone in Singapore will get the Omega variant, i.e. the Omega variant will not stop spreading.

Suppose $0 < k < 1$. Then

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \left[8 + \frac{15}{k-1}\left(k^{n-1} - 1\right)\right] = 8 - \frac{15}{k-1} = 8 + \frac{15}{1-k}$$

Hence, a total of $8 + \dfrac{15}{1-k}$ people in Singapore will get the Omega variant, i.e. the Omega variant will eventually stop spreading.

Part (d)

The weekly infection growth rate k is unlikely to be a constant. For instance, as most of the population gets the Omega variant, there are fewer people available for the Omega variant to infect, hence k will start to decrease.

Part (e)

Observe that

$$\begin{aligned} x_n - x_{n-1} &= (x_{n-1} + \Delta x_{n-1}) - x_{n-1} \\ &= \Delta x_{n-1} \end{aligned}$$

If the weekly infection growth rate is 1, Δx_n is constant, whence x_n is in arithmetic progression.

$$\boxed{\text{The common difference is 15.}}$$

Part (f)

When $k = 4$,

$$\begin{aligned} x_4 &= 8 + \frac{15}{3}\left(4^3 - 1\right) = 323 \\ x_5 &= 8 + \frac{15}{3}\left(4^4 - 1\right) = 1283 \end{aligned}$$

Hence, $\Delta x_4 = 1283 - 323 = 960$. Let x'_n denote the total number of Omega variant cases in Singapore within the first n weeks of the outbreak in the given simulation. Since k becomes 1 starting from week 5, we have

$$\begin{aligned} x'_n &= x_5 + \Delta x_5(n-5),\ n \geq 5 \\ &= 1283 + 960(n-5) \\ &= 960n - 3517 \end{aligned}$$

Consider $x'_n \geq 10000$.

$$\begin{aligned} x'_n &\geq 10000 \\ \implies 960n - 3517 &\geq 10000 \\ \implies 960n &\geq 13517 \\ \implies n &\geq \frac{13517}{960} \\ &= 14.1 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{Alert Orange will be triggered in week 15.}}$$