Problem 1.

- (a) Find $\int \frac{6x^3 + 2}{x^2 + 1} dx$.
- (b) Evaluate $\int_{2}^{4} x \ln x \, dx$ exactly.

Solution

Part (a)

$$\int \frac{6x^3 + 2}{x^2 + 1} dx = 6 \int \frac{x^3}{x^2 + 1} dx + 2 \int \frac{1}{x^2 + 1} dx$$

$$= 6 \int \frac{x^3}{x^2 + 1} dx + 2 \arctan x + C$$

$$= 6 \int \frac{x^2}{x^2 + 1} \cdot x dx + 2 \arctan x + C$$

$$= 6 \cdot \frac{1}{2} \int \frac{u - 1}{u} du + 2 \arctan x + C$$

$$= 3 \int \left(1 - \frac{1}{u}\right) du + 2 \arctan x + C$$

$$= 3 (u - \ln|u|) + 2 \arctan x + C$$

$$= 3 (x^2 + 1) - \ln(x^2 + 1) + 2 \arctan x + C$$

$$= 3x^2 - \ln(x^2 + 1) + 2 \arctan x + C$$

$$\int \frac{6x^3 + 2}{x^2 + 1} dx = 3x^2 - \ln(x^2 + 1) + 2 \arctan x + C$$

Part (b)

Note that $\frac{d}{dx}x \ln x = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$.

	D	I
+	$x \ln x$	1
_	$1 + \ln x$	x

Let $I = \int x \ln x \, \mathrm{d}x$.

$$I = \int x \ln x \, dx$$
$$= x^2 \ln x - \int (x + x \ln x) \, dx$$
$$= x^2 \ln x - \frac{1}{2}x^2 - I$$

$$\implies 2I = x^2 \ln x - \frac{1}{2}x^2 + C$$

$$\implies I = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Evaluating I from x = 2 to 4,

$$\int_{2}^{4} x \ln x \, dx = \left[\frac{1}{2} x^{2} \ln x - \frac{1}{4} x^{2} \right]_{2}^{4}$$
$$= 14 \ln 2 - 3$$

$$\int_2^4 x \ln x \, \mathrm{d}x = 14 \ln 2 - 3$$

Problem 2.

- (a) Use the derivative of $\cos \theta$ to show that $\frac{d}{d\theta} \sec \theta = \sec \theta \tan \theta$.
- (b) Use the substitution $x = \sec \theta 1$ to find the exact value of $\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}} dx$.

Solution

Part (a)

$$\frac{d}{d\theta} \sec \theta = \frac{d}{d\theta} \frac{1}{\cos \theta}$$

$$= \frac{1}{\cos^2 \theta} \cdot \frac{d}{d\theta} \cos \theta$$

$$= \frac{1}{\cos^2 \theta} \cdot \sin \theta$$

$$= \frac{1}{\cos \theta} \cdot \frac{\sin \theta}{\cos \theta}$$

$$= \sec \theta \tan \theta$$

Part (b)

Consider the substitution $x = \sec \theta - 1 \implies dx = \sec \theta \tan \theta d\theta$. When x = 1, we have $\theta = \frac{\pi}{3}$. When $x = \sqrt{2} - 1$, we have $\theta = \frac{\pi}{4}$. Also note that $x + 1 = \sec \theta$. Consider $x^2 + 2x$.

$$x^{2} + 2x = (\sec \theta - 1)^{2} + 2(\sec \theta - 1)$$
$$= \sec^{2} \theta - 2\sec \theta + 1 + 2\sec \theta - 2$$
$$= \sec^{2} \theta - 1$$
$$= \tan^{2} \theta$$

Hence, $\sqrt{x^2 + 2x} = \tan \theta$. Hence,

$$\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}} dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec \theta \tan \theta}{\sec \theta \tan \theta} d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\theta$$
$$= \frac{\pi}{3} - \frac{\pi}{4}$$
$$= \frac{\pi}{12}$$

$$\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}} \, \mathrm{d}x = \frac{\pi}{12}$$

Problem 3.

The expression $\frac{x^2}{9-x^2}$ can be written in the form $A + \frac{B}{3-x} + \frac{C}{3+x}$.

- (a) Find the values of constants A, B and C.
- (b) Show that $\int_0^2 \frac{x^2}{9-x^2} dx = \frac{3}{2} \ln 5 2$.
- (c) Hence, find the value of $\int_0^2 \ln(9-x^2) dx$, giving your answer in terms of $\ln 5$.

Solution

Part (a)

$$\frac{x^2}{9-x^2} = \frac{-(9-x^2)+9}{9-x^2}$$

$$= -1 + \frac{9}{9-x^2}$$

$$= -1 + \frac{9}{(3-x)(3+x)}$$

$$= -1 + \frac{9/6}{3-x} + \frac{9/6}{3+x}$$

$$= -1 + \frac{3/2}{3-x} + \frac{3/2}{3+x}$$

$$A = -1, B = \frac{3}{2}, C = \frac{3}{2}$$

Part (b)

$$\int_0^2 \frac{x^2}{9 - x^2} dx = \int_0^2 \left(-1 + \frac{3/2}{3 - x} + \frac{3/2}{3 + x} \right) dx$$
$$= \left[-x - \frac{3}{2} \ln(3 - x) + \frac{3}{2} \ln(3 + x) \right]_0^2$$
$$= \frac{3}{2} \ln 5 - 2$$

Part (c)

$$\begin{array}{c|cc}
D & I \\
+ & \ln(9 - x^2) & 1 \\
- & -\frac{2x}{9 - x^2} & x
\end{array}$$

$$\int_0^2 \ln(9 - x^2) dx = \left[x \ln(9 - x^2) \right]_0^2 + 2 \int_0^2 \frac{x^2}{9 - x^2} dx$$
$$= 2 \ln 5 + 2 \left(\frac{3}{2} \ln 5 - 2 \right)$$
$$= 5 \ln 5 - 4$$

$$\int_0^2 \ln(9 - x^2) \, \mathrm{d}x = 5 \ln 5 - 4$$