Problem 1.

Find

(a) $\displaystyle\int \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{3-2x}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{1}{3-2x^2}\,\mathrm{d}x$

(d) $\displaystyle\int \frac{1}{\sqrt{3-2x^2}}\,\mathrm{d}x$

(e) $\displaystyle\int \frac{x}{\sqrt{3-2x^2}}\,\mathrm{d}x$

(f) $\displaystyle\int \frac{1}{3+4x+2x^2}\,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned} u &= 3-2x \\ \mathrm{d}u &= -2x\,\mathrm{d}x \\ \mathrm{d}x &= -\tfrac{1}{2}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= -\frac{1}{2}\cdot 2u^{\frac{1}{2}} + C \\ &= -u^{\frac{1}{2}} + C \\ &= -(3-2x)^{\frac{1}{2}} + C \end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{3-2x}}\,\mathrm{d}x = -(3-2x)^{\frac{1}{2}} + C}$$

Part (b)

$$\begin{aligned} u &= 3-2x \\ \mathrm{d}u &= -2x\,\mathrm{d}x \\ \mathrm{d}x &= -\tfrac{1}{2}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int \frac{1}{3-2x}\,\mathrm{d}x &= -\frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u \\ &= -\frac{1}{2}\ln|u| + C \\ &= -\frac{1}{2}\ln|3-2x| + C \end{aligned}$$

$$\boxed{\int \frac{1}{3-2x}\,\mathrm{d}x = -\frac{1}{2}\ln|3-2x| + C}$$

Part (c)

$$\begin{aligned}\int \frac{1}{3-2x^2}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{\frac{3}{2}-x^2}\,\mathrm{d}x \\ &= \frac{1}{2}\cdot\left(1\Big/2\sqrt{\frac{3}{2}}\right)\ln\left(\frac{\sqrt{\frac{3}{2}}+x}{\sqrt{\frac{3}{2}}-x}\right)+C \\ &= \frac{1}{2\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{2}x}{\sqrt{3}-\sqrt{2}x}\right)+C\end{aligned}$$

$$\boxed{\int \frac{1}{3-2x^2}\,\mathrm{d}x = \frac{1}{2\sqrt{6}}\ln\left(\frac{\sqrt{3}+\sqrt{2}x}{\sqrt{3}-\sqrt{2}x}\right)+C}$$

Part (d)

$$\begin{aligned}\int \frac{1}{\sqrt{3-2x^2}}\,\mathrm{d}x &= \frac{1}{\sqrt{2}}\int \frac{1}{\sqrt{\frac{3}{2}-x^2}}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2}}\arcsin\left(x\Big/\sqrt{\frac{3}{2}}\right)+C \\ &= \frac{\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{6}x}{3}\right)+C\end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{3-2x^2}}\,\mathrm{d}x = \frac{\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{6}x}{3}\right)+C}$$

Part (e)

$$\begin{aligned}u &= 3-2x^2 \\ \mathrm{d}u &= -4x\,\mathrm{d}x \\ x\,\mathrm{d}x &= -\tfrac{1}{4}\,\mathrm{d}u\end{aligned}$$

$$\begin{aligned}\int \frac{x}{\sqrt{3-2x^2}}\,\mathrm{d}x &= -\frac{1}{4}\int \frac{1}{\sqrt{u}}\,\mathrm{d}u \\ &= -\frac{1}{4}\cdot 2u^{\frac{1}{2}}+C \\ &= -\frac{1}{2}\sqrt{3-2x^2}+C\end{aligned}$$

$$\boxed{\int \frac{x}{\sqrt{3-2x^2}}\,\mathrm{d}x = -\frac{1}{2}\sqrt{3-2x^2}+C}$$

Part (f)

$$\begin{aligned}\int \frac{1}{3+4x+2x^2}\,\mathrm{d}x &= \int \frac{1}{2(x+1)^2+1}\,\mathrm{d}x \\ &= \frac{1}{2}\int \frac{1}{(x+1)^2+\frac{1}{2}}\,\mathrm{d}x \\ &= \frac{1}{2}\cdot\left(1\Big/\sqrt{\frac{1}{2}}\right)\arctan\left(\frac{x+1}{1\Big/\sqrt{\frac{1}{2}}}\right)+C \\ &= \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}(x+1)\right)+C\end{aligned}$$

$$\boxed{\int \frac{1}{3+4x+2x^2}\,\mathrm{d}x = \frac{\sqrt{2}}{2}\arctan\left(\sqrt{2}(x+1)\right)+C}$$

Problem 2.

Find

(a) $\displaystyle\int \frac{\sec^2 3x}{\tan 3x}\,\mathrm{d}x$

(b) $\displaystyle\int \cos(3x+\alpha)\,\mathrm{d}x$, where α is a constant

(c) $\displaystyle\int \cos^2 3x\,\mathrm{d}x$

(d) $\displaystyle\int e^{1-2x}\,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned}
\int \frac{\sec^2 3x}{\tan 3x}\,\mathrm{d}x &= \int \frac{1+\tan^2 3x}{\tan 3x}\,\mathrm{d}x \\
&= \int (\cot 3x + \tan 3x)\,\mathrm{d}x \\
&= \frac{1}{3}\ln\sin 3x + \frac{1}{3}\ln\sec 3x + C \\
&= \frac{1}{3}\left(\ln\sin 3x + \ln\sec 3x\right) + C \\
&= \frac{1}{3}\ln\left(\sin 3x\sec 3x\right) + C \\
&= \frac{1}{3}\ln\tan 3x + C
\end{aligned}$$

$$\boxed{\int \frac{\sec^2 3x}{\tan 3x}\,\mathrm{d}x = \frac{1}{3}\ln\tan 3x + C}$$

Part (b)

$$\begin{aligned}
u &= 3x+\alpha \\
\mathrm{d}u &= 3\,\mathrm{d}x \\
\mathrm{d}x &= \tfrac{1}{3}\,\mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
\int \cos(3x+\alpha)\,\mathrm{d}x &= \frac{1}{3}\int \cos u\,\mathrm{d}u \\
&= \frac{1}{3}\sin u + C \\
&= \frac{1}{3}\sin(3x+\alpha) + C
\end{aligned}$$

$$\boxed{\int \cos(3x+\alpha)\,\mathrm{d}x = \frac{1}{3}\sin(3x+\alpha) + C}$$

Part (c)

Recall that $cos^2\theta = \dfrac{1+\cos 2\theta}{2}$. Hence, $\cos^2 3x = \dfrac{1+\cos 6x}{2}$.

$$\begin{aligned}
\int \cos^2 3x \, \mathrm{d}x &= \int \frac{1+\cos 6x}{2} \, \mathrm{d}x \\
&= \frac{1}{2} \int (1+\cos 6x) \, \mathrm{d}x \\
&= \frac{1}{2} \left(x + \frac{1}{6} \sin 6x \right) + C \\
&= \frac{1}{2} x + \frac{1}{12} \sin 6x + C
\end{aligned}$$

$$\boxed{\int \cos^2 3x \, \mathrm{d}x = \frac{1}{2} x + \frac{1}{12} \sin 6x + C}$$

Part (d)

$$\begin{aligned}
u &= 1 - 2x \\
\mathrm{d}u &= -2 \, \mathrm{d}x \\
\mathrm{d}x &= -\tfrac{1}{2} \, \mathrm{d}u
\end{aligned}$$

$$\begin{aligned}
\int e^{1-2x} \, \mathrm{d}x &= -\frac{1}{2} \int e^u \, \mathrm{d}u \\
&= -\frac{1}{2} e^u + C \\
&= -\frac{1}{2} e^{1-2x} + C
\end{aligned}$$

$$\boxed{\int e^{1-2x} \, \mathrm{d}x = -\frac{1}{2} e^{1-2x} + C}$$

Problem 3.

Find

(a) $\displaystyle\int 2x\sqrt{3x^2-5}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{x^2-1}{\sqrt{x^3-3x}}\,\mathrm{d}x$

(c) $\displaystyle\int \sin x\sqrt{\cos x}\,\mathrm{d}x$

(d) $\displaystyle\int e^{2x}(1-e^{2x})^4\,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned} u &= 3x^2-5 \\ \mathrm{d}u &= 6x\,\mathrm{d}x \\ x\,\mathrm{d}x &= \tfrac{1}{6}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int 2x\sqrt{3x^2-5}\,\mathrm{d}x &= \frac{1}{6}\int 2\sqrt{u}\,\mathrm{d}u \\ &= \frac{1}{3}\int \sqrt{u}\,\mathrm{d}u \\ &= \frac{1}{3}\cdot\frac{2}{3}u^{\frac{3}{2}}+C \\ &= \frac{2}{9}u^{\frac{3}{2}}+C \\ &= \frac{2}{9}\left(3x^2-5\right)^{\frac{3}{2}}+C \end{aligned}$$

$$\boxed{\int 2x\sqrt{3x^2-5}\,\mathrm{d}x = \frac{2}{9}\left(3x^2-5\right)^{\frac{3}{2}}+C}$$

Part (b)

$$\begin{aligned} u &= x^3-3x \\ \mathrm{d}u &= 3x^2-3\,\mathrm{d}x \\ x^2-1\,\mathrm{d}x &= \tfrac{1}{3}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int \frac{x^2-1}{\sqrt{x^3-3x}}\,\mathrm{d}x &= \frac{1}{3}\int \frac{\mathrm{d}u}{\sqrt{u}} \\ &= \frac{1}{3}\left(2u^{\frac{1}{2}}\right)+C \\ &= \frac{2}{3}\sqrt{x^3-3x}+C \end{aligned}$$

$$\boxed{\int \frac{x^2-1}{\sqrt{x^3-3x}}\,\mathrm{d}x = \frac{2}{3}\sqrt{x^3-3x}+C}$$

Part (c)

$$\begin{aligned} u &= \cos x \\ \mathrm{d}u &= -\sin x \,\mathrm{d}x \\ \sin x \,\mathrm{d}x &= -\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int \sin x \sqrt{\cos x} \,\mathrm{d}x &= -\int \sqrt{u} \,\mathrm{d}u \\ &= -\frac{2}{3} u^{\frac{3}{2}} + C \\ &= -\frac{2}{3} (\cos x)^{\frac{3}{2}} + C \end{aligned}$$

$$\boxed{\int \sin x \sqrt{\cos x} \,\mathrm{d}x = -\frac{2}{3} (\cos x)^{\frac{3}{2}} + C}$$

Part (d)

$$\begin{aligned} u &= 1 - e^{2x} \\ \mathrm{d}u &= -2e^{2x} \,\mathrm{d}x \\ e^{2x} \,\mathrm{d}x &= -\tfrac{1}{2} \,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int e^{2x} (1 - e^{2x})^4 \,\mathrm{d}x &= -\frac{1}{2} \int u^4 \,\mathrm{d}u \\ &= -\frac{1}{2} \cdot \frac{u^5}{5} + C \\ &= -\frac{1}{10} (1 - e^{2x})^5 + C \end{aligned}$$

$$\boxed{\int e^{2x} (1 - e^{2x})^4 \,\mathrm{d}x = -\frac{1}{10} (1 - e^{2x})^5 + C}$$

Problem 4.

Find

(a) $\displaystyle\int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{3x}{x+3}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{\sin x+\cos x}{\sin x-\cos x}\,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned} u &= 1-\sqrt{x} \\ \mathrm{d}u &= -\tfrac{1}{2\sqrt{x}}\,\mathrm{d}x \\ \tfrac{1}{\sqrt{x}}\,\mathrm{d}x &= -2\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,\mathrm{d}x &= -2\int \frac{1}{u}\,\mathrm{d}u \\ &= -2\ln|u| + C \\ &= -2\ln\left|1-\sqrt{x}\right| + C \end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{x}(1-\sqrt{x})}\,\mathrm{d}x = -2\ln\left|1-\sqrt{x}\right| + C}$$

Part (b)

$$\begin{aligned} \int \frac{3x}{x+3}\,\mathrm{d}x &= \int \left(3 - \frac{9}{x+3}\right)\mathrm{d}x \\ &= 3x - 9\ln|x+3| + C \end{aligned}$$

$$\boxed{\int \frac{3x}{x+3}\,\mathrm{d}x = 3x - 9\ln|x+3| + C}$$

Part (c)

$$\begin{aligned} u &= \sin x - \cos x \\ \mathrm{d}u &= \cos x + \sin x\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \int \frac{\sin x+\cos x}{\sin x-\cos x}\,\mathrm{d}x &= \int \frac{1}{u}\,\mathrm{d}u \\ &= \ln|u| + C \\ &= \ln|\sin x - \cos x| + C \end{aligned}$$

$$\boxed{\int \frac{\sin x+\cos x}{\sin x-\cos x}\,\mathrm{d}x = \ln|\sin x - \cos x| + C}$$

Problem 5.

Find

(a) $\displaystyle\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$

(b) $\displaystyle\int (\sin x)(\cos x)(e^{\cos 2x})\,\mathrm{d}x$

Solution

Part (a)

$$\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = -2\int e^u\,\mathrm{d}u$$

$$\begin{aligned} u &= -\sqrt{x} \\ \mathrm{d}u &= -\tfrac{1}{2\sqrt{x}}\,\mathrm{d}x \\ \tfrac{1}{\sqrt{x}}\,\mathrm{d}x &= -2\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} &= -2e^u + C \\ &= -2e^{-\sqrt{x}} + C \end{aligned}$$

$$\boxed{\int \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x = -2e^{-\sqrt{x}} + C}$$

Part (b)

$$\int (\sin x)(\cos x)(e^{\cos 2x})\,\mathrm{d}x = \frac{1}{2}\int \sin 2x \cdot e^{\cos 2x}\,\mathrm{d}x$$

$$\begin{aligned} u &= \cos 2x \\ \mathrm{d}u &= -2\sin 2x\,\mathrm{d}x \\ \sin 2x\,\mathrm{d}x &= -\tfrac{1}{2}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{4}\int e^u\,\mathrm{d}u \\ &= -\frac{1}{4}e^u + C \\ &= -\frac{1}{4}e^{\cos 2x} + C \end{aligned}$$

$$\boxed{\int (\sin x)(\cos x)(e^{\cos 2x})\,\mathrm{d}x = -\frac{1}{4}e^{\cos 2x} + C}$$

Problem 6.

Find

(a) $\displaystyle\int \tan^2 2x \,\mathrm{d}x$

(b) $\displaystyle\int \frac{1}{1+\cos 2t} \,\mathrm{d}t$

(c) $\displaystyle\int \sin\left(\frac{5}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right) d\theta$

(d) $\displaystyle\int \tan^4 x \,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned}\int \tan^2 2x \,\mathrm{d}x &= \int \left(\sec^2 2x - 1\right) \mathrm{d}x \\ &= \frac{1}{2}\tan 2x - x + C\end{aligned}$$

$$\boxed{\int \tan^2 2x \,\mathrm{d}x = \frac{1}{2}\tan 2x - x + C}$$

Part (b)

$$\begin{aligned}\int \frac{1}{1+\cos 2t}\,\mathrm{d}t &= \int \frac{1}{1+(2\cos^2 t - 1)}\,\mathrm{d}t \\ &= \frac{1}{2}\int \frac{1}{\cos^2 t}\,\mathrm{d}t \\ &= \frac{1}{2}\int \sec^2 t\,\mathrm{d}t \\ &= \frac{1}{2}\tan t + C\end{aligned}$$

$$\boxed{\int \frac{1}{1+\cos 2t}\,\mathrm{d}t = \frac{1}{2}\tan t + C}$$

Part (c)

$$\begin{aligned}\int \sin\left(\frac{5}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right) d\theta &= \frac{1}{2}\int \left(\sin\left(\frac{5}{2}\theta + \frac{1}{2}\theta\right) + \sin\left(\frac{5}{2}\theta - \frac{1}{2}\theta\right)\right) d\theta \\ &= \frac{1}{2}\int (\sin 3\theta + \sin 2\theta)\, d\theta \\ &= \frac{1}{2}\left(-\frac{1}{3}\cos 3\theta - \frac{1}{2}\cos 2\theta\right) + C \\ &= -\frac{1}{6}\cos 3\theta - \frac{1}{4}\cos 2\theta + C\end{aligned}$$

$$\boxed{\int \sin\left(\frac{5}{2}\theta\right)\cos\left(\frac{1}{2}\theta\right)d\theta = -\frac{1}{6}\cos 3\theta - \frac{1}{4}\cos 2\theta + C}$$

Part (d)

Note that

$$\int \tan^2 x \, \mathrm{d}x = \tan x - x + C$$

and

$$\begin{aligned}\int \tan^2 x \sec^2 x \, \mathrm{d}x &= \int u^2 \, \mathrm{d}u \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}\tan^3 x + C\end{aligned}$$

$$\begin{aligned} u &= \tan x \\ \mathrm{d}u &= \sec^2 x \, \mathrm{d}x \end{aligned}$$

Hence,

$$\begin{aligned}\int \tan^4 x \, \mathrm{d}x &= \int \tan^2 x \cdot \tan^2 x \, \mathrm{d}x \\ &= \int \tan^2 x \left(\sec^2 x - 1\right) \mathrm{d}x \\ &= \int \left(\tan^2 x \sec^2 x - \tan^2\right) \mathrm{d}x \\ &= \frac{1}{3}\tan^3 x - (\tan x - x) + C \\ &= \frac{1}{3}\tan^3 x - \tan x + x + C\end{aligned}$$

$$\boxed{\int \tan^4 x \, \mathrm{d}x = \frac{1}{3}\tan^3 x - \tan x + x + C}$$

Problem 7.

Find

(a) $\displaystyle\int \frac{1}{4x^2+2x+10}\,\mathrm{d}x$

(b) $\displaystyle\int \frac{x^2}{1-x^2}\,\mathrm{d}x$

(c) $\displaystyle\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned}
\int \frac{1}{4x^2+2x+10}\,\mathrm{d}x &= \int \frac{1}{\left(2x+\frac{1}{2}\right)^2+\frac{39}{4}}\,\mathrm{d}x \\
&= 4\int \frac{1}{(4x+1)^2+39}\,\mathrm{d}x \\
&= 4\cdot\frac{1}{4}\cdot\frac{1}{\sqrt{39}}\arctan\left(\frac{4x+1}{\sqrt{39}}\right)+C \\
&= \frac{1}{\sqrt{39}}\arctan\left(\frac{4x+1}{\sqrt{39}}\right)+C
\end{aligned}$$

$$\boxed{\int \frac{1}{4x^2+2x+10}\,\mathrm{d}x = \frac{1}{\sqrt{39}}\arctan\left(\frac{4x+1}{\sqrt{39}}\right)+C}$$

Part (b)

$$\begin{aligned}
\int \frac{x^2}{1-x^2}\,\mathrm{d}x &= \int \left(\frac{1}{1-x^2}-1\right)\mathrm{d}x \\
&= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)-x+C
\end{aligned}$$

$$\boxed{\int \frac{x^2}{1-x^2}\,\mathrm{d}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)-x+C}$$

Part (c)

$$\begin{aligned}
\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x &= \int \frac{1}{\sqrt{2^2-(x-1)^2}}\,\mathrm{d}x \\
&= \arcsin\left(\frac{x-1}{2}\right)+C
\end{aligned}$$

$$\boxed{\int \frac{1}{\sqrt{3+2x-x^2}}\,\mathrm{d}x = \arcsin\left(\frac{x-1}{2}\right)+C}$$

Problem 8.

Evaluate the following without the use of graphic calculator:

(a) $\displaystyle\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4\cot\frac{x}{2}\csc^2\frac{x}{2}\,\mathrm{d}x$

(b) $\displaystyle\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x$

(c) $\displaystyle\int_0^1 \frac{2}{(1+x)(1+x^2)}\,\mathrm{d}x$

(d) $\displaystyle\int_{-4}^{-2} \frac{x^3+2}{x^2-1}\,\mathrm{d}x$

Solution

Part (a)

$$\begin{aligned} u &= \csc\tfrac{x}{2} \\ \mathrm{d}u &= -\tfrac{1}{2}\cot\tfrac{x}{2}\csc\tfrac{x}{2} \\ 2\,\mathrm{d}u &= -\cot\tfrac{x}{2}\csc\tfrac{x}{2} \\ x = \tfrac{1}{3}\pi &\implies u = 2 \\ x = \tfrac{2}{3}\pi &\implies u = \tfrac{2}{\sqrt{3}} \end{aligned}$$

$$\begin{aligned} \int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4\cot\frac{x}{2}\csc^2\frac{x}{2}\,\mathrm{d}x &= -4\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} -\cot\frac{x}{2}\csc\frac{x}{2}\cdot\csc\left(\frac{1}{2}x\right)\mathrm{d}x \\ &= -8\int_2^{\frac{2}{\sqrt{3}}} u\,\mathrm{d}u \\ &= -8\left[\frac{1}{2}u^2\right]_2^{\frac{2}{\sqrt{3}}} \\ &= \frac{32}{3} \end{aligned}$$

$$\boxed{\int_{\frac{1}{3}\pi}^{\frac{2}{3}\pi} 4\cot\frac{x}{2}\csc^2\frac{x}{2}\,\mathrm{d}x = \frac{32}{3}}$$

Part (b)

$$\begin{aligned}
\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x &= \int_0^4 \frac{\frac{1}{2}(2x+1)+\frac{3}{2}}{\sqrt{2x+1}}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^4 \left(\frac{2x+1}{\sqrt{2x+1}} + \frac{3}{\sqrt{2x+1}}\right)\mathrm{d}x \\
&= \frac{1}{2}\int_0^4 \left(\sqrt{2x+1} + \frac{3}{\sqrt{2x+1}}\right)\mathrm{d}x \\
&= \frac{1}{4}\int_1^9 \left(\sqrt{u} + \frac{3}{\sqrt{u}}\right)\mathrm{d}u \\
&= \frac{1}{4}\left[\frac{u^{\frac{3}{2}}}{3/2} + \frac{3u^{\frac{1}{2}}}{1/2}\right]_1^9 \\
&= \frac{22}{3}
\end{aligned}$$

$$\begin{aligned}
u &= 2x+1 \\
\mathrm{d}u &= 2\,\mathrm{d}x \\
\mathrm{d}x &= \tfrac{1}{2}\,\mathrm{d}u \\
x = 0 &\implies u = 1 \\
x = 4 &\implies u = 9
\end{aligned}$$

$$\boxed{\int_0^4 \frac{x+2}{\sqrt{2x+1}}\,\mathrm{d}x = \frac{22}{3}}$$

Part (c)

$$\begin{aligned}
\int_0^1 \frac{2}{(1+x)(1+x^2)}\,\mathrm{d}x &= \int_0^1 \left(\frac{1}{1+x} + \frac{1}{1+x^2} - \frac{x}{1+x^2}\right)\mathrm{d}x \\
&= [\ln|1+x|]_0^1 + [\arctan x]_0^1 - \int_0^1 \frac{x}{1+x^2}\,\mathrm{d}x \\
&= \ln 2 + \frac{\pi}{4} - \frac{1}{2}\int_1^2 \frac{1}{u}\,\mathrm{d}u \\
&= \ln 2 + \frac{\pi}{4} - \frac{1}{2}[\ln|u|]_1^2 \\
&= \ln 2 + \frac{\pi}{4} - \frac{1}{2}\ln 2 \\
&= \frac{1}{2}\ln 2 + \frac{\pi}{4}
\end{aligned}$$

$$\begin{aligned}
u &= 1+x^2 \\
\mathrm{d}u &= 2x\,\mathrm{d}x \\
x\,\mathrm{d}x &= \tfrac{1}{2}\,\mathrm{d}u \\
x = 0 &\implies u = 1 \\
x = 1 &\implies u = 2
\end{aligned}$$

$$\boxed{\int_0^1 \frac{2}{(1+x)(1+x^2)}\,\mathrm{d}x = \frac{1}{2}\ln 2 + \frac{\pi}{4}}$$

Part (d)

$$\begin{aligned}
\int_{-4}^{-2} \frac{x^3+2}{x^2-1}\,\mathrm{d}x &= \int_{-4}^{-2} \left(x + \frac{x+2}{x^2-1} \right) \mathrm{d}x \\
&= \left[\frac{x^2}{2} \right]_{-4}^{-2} + \int_{-4}^{-2} \frac{x+2}{(x-1)(x+1)}\,\mathrm{d}x \\
&= -6 + \int_{-4}^{-2} \left(\frac{3/2}{x-1} - \frac{1/2}{x+1} \right) \mathrm{d}x \\
&= -6 + \left[\frac{3}{2} \ln|x-1| - \frac{1}{2} \ln|x+1| \right]_{-4}^{-2} \\
&= -6 + 2\ln 3 - \frac{3}{2} \ln 5
\end{aligned}$$

$$\boxed{\int_{-4}^{-2} \frac{x^3+2}{x^2-1}\,\mathrm{d}x = -6 + 2\ln 3 - \frac{3}{2}\ln 5}$$

Problem 9.

Using the given substitution, find

(a) $\displaystyle\int \frac{x}{(2x+3)^3}\,\mathrm{d}x$ $[u = 2x+3]$

(b) $\displaystyle\int \frac{1}{e^x + 4e^{-x}}\,\mathrm{d}x$ $[u = e^x]$

(c) $\displaystyle\int_0^{\sqrt{2}} \sqrt{4-y^2}\,\mathrm{d}y$ $[y = 2\sin\theta]$

(d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,\mathrm{d}\theta$ $\left[t = \tan\dfrac{\theta}{2}\right]$

Solution

Part (a)

$$\begin{aligned} u &= 2x+3 \\ x &= (u-3)/2 \\ dx &= \tfrac{1}{2}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned}
\int \frac{x}{(2x+3)^3}\,\mathrm{d}x &= \frac{1}{4}\int \frac{u-3}{u^3}\,\mathrm{d}x \\
&= \frac{1}{4}\int \left(\frac{1}{u^2} - \frac{3}{u^3}\right)\mathrm{d}u \\
&= \frac{1}{4}\left(\frac{u^{-1}}{-1} - \frac{3u^{-2}}{-2}\right) + C \\
&= \frac{3}{8}(2x+3)^{-2} - \frac{1}{4}(2x+3)^{-1} + C
\end{aligned}$$

$$\boxed{\int \frac{x}{(2x+3)^3}\,\mathrm{d}x = \frac{3}{8}(2x+3)^{-2} - \frac{1}{4}(2x+3)^{-1} + C}$$

Part (b)

$$\begin{aligned} u &= e^x \\ \ln u &= x \\ dx &= \tfrac{1}{u}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned}
\int \frac{1}{e^x + 4e^{-x}}\,\mathrm{d}x &= \int \frac{e^x}{e^{2x}+4}\,\mathrm{d}x \\
&= \int \frac{u}{u^2+4}\cdot\frac{1}{u}\,\mathrm{d}u \\
&= \int \frac{1}{u^2+2^2}\,\mathrm{d}u \\
&= \frac{1}{2}\arctan\left(\frac{u}{2}\right) + C \\
&= \frac{1}{2}\arctan\left(\frac{e^x}{2}\right) + C
\end{aligned}$$

$$\boxed{\int \frac{1}{e^x + 4e^{-x}}\,\mathrm{d}x = \frac{1}{2}\arctan\left(\frac{e^x}{2}\right) + C}$$

Part (c)

$$\begin{aligned} y &= 2\sin\theta \\ \mathrm{d}y &= 2\cos\theta \,\mathrm{d}\theta \\ y = 0 &\implies \theta = 0 \\ y = \sqrt{2} &\implies \theta = \pi/4 \end{aligned}$$

$$\begin{aligned} \int_0^{\sqrt{2}} \sqrt{4-y^2}\,\mathrm{d}y &= 2\int_0^{\frac{\pi}{4}} \cos\theta\sqrt{4-(2\sin\theta)^2}\,\mathrm{d}\theta \\ &= 2\int_0^{\frac{\pi}{4}} \cos\theta\sqrt{4-4\sin^2\theta}\,\mathrm{d}\theta \\ &= 4\int_0^{\frac{\pi}{4}} \cos\theta\sqrt{1-\sin^2\theta}\,\mathrm{d}\theta \\ &= 4\int_0^{\frac{\pi}{4}} \cos^2\theta\,\mathrm{d}\theta \\ &= 4\int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta \\ &= 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}} \\ &= 1 + \frac{\pi}{2} \end{aligned}$$

$$\boxed{\int_0^{\sqrt{2}} \sqrt{4-y^2}\,\mathrm{d}y = 1 + \frac{\pi}{2}}$$

Part (d)

Consider the substitution $t = \tan\dfrac{\theta}{2} \implies \theta = 2\arctan t$. Hence, $\mathrm{d}\theta = \dfrac{2}{1+t^2}\,\mathrm{d}t$. Also note that $\sin\theta = \sin(2\arctan t) = 2\sin(\arctan t)\cos(\arctan t) = 2\cdot\dfrac{t}{\sqrt{1+t^2}}\cdot\dfrac{1}{\sqrt{1+t^2}} = \dfrac{2t}{1+t^2}$.

$$\begin{aligned} t &= \tan\tfrac{\theta}{2} \\ \theta = 0 &\implies t = 0 \\ \theta = \tfrac{\pi}{2} &\implies t = 1 \end{aligned}$$

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,\mathrm{d}\theta &= \int_0^1 \frac{2/(1+t^2)}{1+2t/(1+t^2)}\,\mathrm{d}u \\ &= 2\int_0^1 \frac{1}{t^2+2t+1}\,\mathrm{d}t \\ &= 2\int_0^1 \frac{1}{(t+1)^2}\,\mathrm{d}t \\ &= 2\left[-\frac{1}{t+1}\right]_0^1 \\ &= 1 \end{aligned}$$

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\,\mathrm{d}\theta = 1}$$

Problem 10.

Find

(a) $\displaystyle\int \ln(2x+1)\,\mathrm{d}x$

(b) $\displaystyle\int x\arctan(x^2)\,\mathrm{d}x$

(c) $\displaystyle\int e^{-2x}\cos 2x\,\mathrm{d}x$

(d) $\displaystyle\int_0^2 x^2 e^{-x}\,\mathrm{d}x$

Solution

Part (a)

$$\int \ln(2x+1)\,\mathrm{d}x = \frac{1}{2}\int \ln u\,\mathrm{d}u$$

$$\begin{aligned} u &= 2x+1 \\ \mathrm{d}u &= 2\,\mathrm{d}x \\ \mathrm{d}x &= \tfrac{1}{2}\,\mathrm{d}u \end{aligned}$$

	D	I
$+$	$\ln u$	1
$-$	$\frac{1}{u}$	u

$$\begin{aligned} \int \ln(2x+1)\,\mathrm{d}x &= \frac{1}{2}\left(u\ln u - \int u\cdot\frac{1}{u}\,\mathrm{d}u\right) \\ &= \frac{1}{2}\left(u\ln u - u\right) + C \\ &= \frac{1}{2}\left((2x+1)\ln(2x+1) - (2x+1)\right) + C \\ &= x\ln(2x+1) + \frac{1}{2}\ln(2x+1) - x + C \end{aligned}$$

$$\boxed{\int \ln(2x+1)\,\mathrm{d}x = x\ln(2x+1) + \frac{1}{2}\ln(2x+1) - x + C}$$

Part (b)

$$\int x\arctan(x^2)\,\mathrm{d}x = \frac{1}{2}\int \arctan u\,\mathrm{d}u$$

$$\begin{aligned} u &= x^2 \\ du &= 2x\,\mathrm{d}x \\ x\,\mathrm{d}x &= \tfrac{1}{2}\,\mathrm{d}u \end{aligned}$$

	D	I
$+$	$\arctan u$	1
$-$	$\frac{1}{1+u^2}$	u

$$\begin{aligned} v &= 1 + u^2 \\ \mathrm{d}v &= 2u \, \mathrm{d}u \\ u \, \mathrm{d}u &= \tfrac{1}{2} \, \mathrm{d}v \end{aligned}$$

$$\begin{aligned}
\int x \arctan(x^2) \, \mathrm{d}x &= \frac{1}{2} \left(u \arctan u - \int \frac{u}{1+u^2} \, \mathrm{d}u \right) \\
&= \frac{1}{2} \left(u \arctan u - \frac{1}{2} \int \frac{1}{v} \, \mathrm{d}v \right) \\
&= \frac{1}{2} \left(u \arctan u - \frac{1}{2} \ln|v| \right) + C \\
&= \frac{1}{2} \left(u \arctan u - \frac{1}{2} \ln(1+u^2) \right) + C \\
&= \frac{1}{2} \left(x^2 \arctan x^2 - \frac{1}{2} \ln(1+x^4) \right) + C \\
&= \frac{1}{2} x^2 \arctan x^2 - \frac{1}{4} \ln(1+x^4) + C
\end{aligned}$$

$$\boxed{\int x \arctan(x^2) \, \mathrm{d}x = \frac{1}{2} x^2 \arctan x^2 - \frac{1}{4} \ln(1+x^4) + C}$$

Part (c)

	D	I
$+$	e^{-2x}	$\cos 2x$
$-$	$-2e^{-2x}$	$\frac{1}{2} \sin 2x$
$+$	$4e^{-2x}$	$-\frac{1}{4} \cos 2x$

$$\begin{aligned}
\int e^{-2x} \cos 2x \, \mathrm{d}x &= \frac{1}{2} e^{-2x} \sin 2x - \frac{1}{2} e^{-2x} \cos 2x - \int e^{-2x} \cos 2x \, \mathrm{d}x \\
\implies 2 \int e^{-2x} \cos 2x \, \mathrm{d}x &= \frac{1}{2} e^{-2x} \sin 2x - \frac{1}{2} e^{-2x} \cos 2x + C \\
\implies \int e^{-2x} \cos 2x \, \mathrm{d}x &= \frac{1}{2} \left(\frac{1}{2} e^{-2x} \sin 2x - \frac{1}{2} e^{-2x} \cos 2x \right) + C \\
&= \frac{1}{4} e^{-2x} \sin 2x - \frac{1}{4} e^{-2x} \cos 2x + C
\end{aligned}$$

$$\boxed{\int e^{-2x} \cos 2x \, \mathrm{d}x = \frac{1}{4} e^{-2x} \sin 2x - \frac{1}{4} e^{-2x} \cos 2x + C}$$

Part (d)

	D	I
$+$	x^2	e^{-x}
$-$	$2x$	$-e^{-x}$
$+$	2	e^{-x}
$-$	0	$-e^{-x}$

$$\begin{aligned}\int_0^2 x^2e^{-x}\,\mathrm{d}x &= \left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^2 \\ &= 2 - 10e^{-2}\end{aligned}$$

$$\boxed{\int_0^2 x^2e^{-x}\,\mathrm{d}x = 2 - 10e^{-2}}$$

Problem 11.

(a) Show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) = \sec x$.

(b) Find $\displaystyle\int x \sin x \, \mathrm{d}x$.

(c) Find the exact value of $\displaystyle\int_0^{\frac{1}{4}\pi} (x\sin x)\ln(\sec x + \tan x)\, \mathrm{d}x$.

Solution

Part (a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\ln(\sec x + \tan x) &= \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} \\ &= \sec x \cdot \frac{\tan x + \sec x}{\sec x + \tan x} \\ &= \sec x\end{aligned}$$

Part (b)

	D	I
$+$	x	$\sin x$
$-$	1	$-\cos x$
$+$	0	$-\sin x$

$$\boxed{\int x \sin x \, \mathrm{d}x = -x\cos x + \sin x + C}$$

Part (c)

	D	I
$+$	$\ln(\sec x + \tan x)$	$x \sin x$
$-$	$\sec x$	$-x\cos x + \sin x$

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}} (x\sin x)\ln(\sec x + \tan x)\,\mathrm{d}x \\
&= \left[\ln(\sec x + \tan x)\,(-x\cos x + \sin x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec x\,(-x\cos x + \sin x)\,\mathrm{d}x \\
&= \left[\ln(\sec x + \tan x)\,(-x\cos x + \sin x)\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} (-x + \tan x)\,\mathrm{d}x \\
&= \left[\ln(\sec x + \tan x)\,(-x\cos x + \sin x) - \frac{x^2}{2} - \ln|\cos x|\right]_0^{\frac{\pi}{4}} \\
&= \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)\ln(\sqrt{2}+1) + \frac{\pi^2}{32} - \frac{1}{2}\ln 2
\end{aligned}$$

$$\boxed{\int_0^{\frac{\pi}{4}} (x\sin x)\ln(\sec x + \tan x)\,\mathrm{d}x = \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right)\ln(\sqrt{2}+1) + \frac{\pi^2}{32} - \frac{1}{2}\ln 2}$$

Problem 12.

(a) Use the fact that $7\cos x - 4\sin x = \dfrac{3}{2}(\cos x + \sin x) + \dfrac{11}{2}(\cos x - \sin x)$ to find the exact value of $\displaystyle\int_0^{\frac{\pi}{2}} \frac{7\cos x - 4\sin x}{\cos x + \sin x}\,\mathrm{d}x$.

(b) Use integration by parts to find the exact value of $\displaystyle\int_1^e (\ln x)^2\,\mathrm{d}x$.

Solution

Part (a)

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \frac{7\cos x - 4\sin x}{\cos x + \sin x}\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \frac{\frac{3}{2}(\cos x + \sin x) + \frac{11}{2}(\cos x - \sin x)}{\cos x + \sin x}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{3(\cos x + \sin x) + 11(\cos x - \sin x)}{\cos x + \sin x}\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(3 + 11 \cdot \frac{\cos x - \sin x}{\cos x + \sin x}\right)\mathrm{d}x \\
&= \frac{1}{2}\left[3x + 11\ln\lvert\cos x + \sin x\rvert\right]_0^{\frac{\pi}{2}} \\
&= \frac{3}{4}\pi
\end{aligned}$$

$$\boxed{\int_0^{\frac{\pi}{2}} \frac{7\cos x - 4\sin x}{\cos x + \sin x}\,\mathrm{d}x = \frac{3}{4}\pi}$$

Part (b)

	D	I
$+$	$\ln x$	$\ln x$
$-$	$\frac{1}{x}$	$x\ln x - x$

$$\begin{aligned}
\int_1^e (\ln x)^2\,\mathrm{d}x &= \left[\ln x\,(x\ln x - x)\right]_1^e - \int_1^e \frac{1}{x}\,(x\ln x - x)\,\mathrm{d}x \\
&= \left[\ln x\,(x\ln x - x)\right]_1^e - \int_1^e (\ln x - 1)\,\mathrm{d}x \\
&= \left[\ln x\,(x\ln x - x) - (x\ln x - x - x)\right]_1^e \\
&= e - 2
\end{aligned}$$

$$\boxed{\int_1^e (\ln x)^2\,\mathrm{d}x = e - 2}$$

Problem 13.

(a) Solve the inequality $x^2 + 2x - 3 < 0$.

(b) Without using the graphing calculator, evaluate

(i) $\displaystyle\int_{-4}^{4} \left|x^2 + 2x - 3\right| \mathrm{d}x$

(ii) $\displaystyle\int_{0}^{2} x\left|x^2 + 2x - 3\right| \mathrm{d}x$

Solution

Part (a)

$$\begin{aligned} && x^2 + 2x - 3 &< 0 \\ \implies && (x+1)^2 - 4 &< 0 \\ \implies && (x+1)^2 &< 4 \\ \implies && -2 < x + 1 &< 2 \\ \implies && -3 < x &< 1 \end{aligned}$$

$$\boxed{-3 < x < 1}$$

Part (b)

Subpart (i)

Let $F(x) = \int \left(x^2 + 2x - 3\right) \mathrm{d}x = \frac{1}{3}x^3 + x^2 - 3x + C$. Then,

$$\begin{aligned} &\int_{-4}^{4} \left|x^2 + 2x - 3\right| \mathrm{d}x \\ &= \int_{-4}^{-3} \left|x^2 + 2x - 3\right| \mathrm{d}x + \int_{-3}^{1} \left|x^2 + 2x - 3\right| \mathrm{d}x + \int_{1}^{4} \left|x^2 + 2x - 3\right| \mathrm{d}x \\ &= \int_{-4}^{-3} x^2 + 2x - 3 \,\mathrm{d}x - \int_{-3}^{1} x^2 + 2x - 3 \,\mathrm{d}x + \int_{1}^{4} x^2 + 2x - 3 \,\mathrm{d}x \\ &= \Big(F(-3) - F(-4)\Big) - \Big(F(1) - F(-3)\Big) + \Big(F(4) - F(1)\Big) \\ &= 40 \end{aligned}$$

$$\boxed{\int_{-4}^{4} \left|x^2 + 2x - 3\right| \mathrm{d}x = 40}$$

Subpart (ii)

Let $F(x) = \int x\,(x^2+2x-3)\,\mathrm{d}x = \frac{1}{4}x^4 + \frac{2}{3}x^3 - \frac{3}{2}x^2 + C$. Then,

$$\begin{aligned}
\int_0^2 x\left|x^2+2x-3\right|\,\mathrm{d}x &= \int_0^1 x\left|x^2+2x-3\right|\,\mathrm{d}x + \int_1^2 x\left|x^2+2x-3\right|\,\mathrm{d}x \\
&= -\int_0^1 x(x^2+2x-3)\,\mathrm{d}x + \int_1^2 x(x^2+2x-3)\,\mathrm{d}x \\
&= -\Big(F(1)-F(0)\Big) + \Big(F(2)-F(1)\Big) \\
&= \frac{9}{2}
\end{aligned}$$

$$\boxed{\int_0^2 x\left|x^2+2x-3\right|\,\mathrm{d}x = \frac{9}{2}}$$

Problem 14.

The indefinite integral $\int \frac{P(x)}{x^3+1}\,\mathrm{d}x$, where $P(x)$ is a polynomial in x, is denoted by I.

(a) Find I when $P(x) = x^2$.

(b) By writing $x^3 + 1 = (x+1)(x^2 + Ax + B)$, where A and B are constants, find I when

  (i) $P(x) = x^2 - x + 1$

  (ii) $P(x) = x + 1$

(c) Using the results of parts (a) and (b), or otherwise, find I when $P(x) = 1$.

Solution

Part (a)

$$\begin{aligned} u &= x^3 + 1 \\ \mathrm{d}u &= 3x^2\,\mathrm{d}x \\ x^2\,\mathrm{d}x &= \tfrac{1}{3}\,\mathrm{d}u \end{aligned}$$

$$\begin{aligned} \int \frac{x^2}{x^3+1}\,\mathrm{d}x &= \frac{1}{3}\int \frac{1}{u}\,\mathrm{d}u \\ &= \frac{1}{3}\ln|u| + C \\ &= \frac{1}{3}\ln\left|x^3+1\right| + C \end{aligned}$$

$$\boxed{I = \frac{1}{3}\ln|x^3+1| + C}$$

Part (b)

$$x^3 + 1 = (x+1)\left(x^2 - x + 1\right)$$

Subpart (i)

$$\begin{aligned} \int \frac{x^2 - x + 1}{x^3+1}\,\mathrm{d}x &= \int \frac{x^2 - x + 1}{(x+1)(x^2 - x + 1)}\,\mathrm{d}x \\ &= \int \frac{1}{x+1}\,\mathrm{d}x \\ &= \ln|x+1| + C \end{aligned}$$

$$\boxed{I = \ln|x+1| + C}$$

Subpart (ii)

$$\begin{aligned}
\int \frac{x+1}{x^3+1}\,\mathrm{d}x &= \int \frac{x+1}{(x+1)\left(x^2-x+1\right)}\,\mathrm{d}x \\
&= \int \frac{1}{x^2-x+1}\,\mathrm{d}x \\
&= \int \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}}\,\mathrm{d}x \\
&= \frac{1}{\sqrt{3/4}}\arctan\left(\frac{x-\frac{1}{2}}{\sqrt{3/4}}\right)+C \\
&= \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C
\end{aligned}$$

$$\boxed{I = \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C}$$

Part (c)

Observe that $1 = \dfrac{1}{2}\left(\left(x^2-x+1\right)-x^2+(x+1)\right)$. Hence,

$$\begin{aligned}
\int \frac{1}{x^3+1}\,\mathrm{d}x &= \frac{1}{2}\left(\int \frac{x^2-x+1}{x^3+1}\,\mathrm{d}x - \int \frac{x^2}{x^3+1}\,\mathrm{d}x + \int \frac{x+1}{x^3+1}\,\mathrm{d}x\right) \\
&= \frac{1}{2}\left(\ln|x+1| - \frac{1}{3}\ln\left|x^3+1\right| + \frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right)+C \\
&= \frac{1}{2}\ln|x+1| - \frac{1}{6}\ln\left|x^3+1\right| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C
\end{aligned}$$

$$\boxed{I = \frac{1}{2}\ln|x+1| - \frac{1}{6}\ln|x^3+1| + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C}$$