Problem 1.

It is given that $f(r) = \dfrac{1}{(r+1)(r+2)}$.

(a) Show that $f(r-1) - f(r) = \dfrac{2}{r(r+1)(r+2)}$ and find $\displaystyle\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$ in terms of n.

(b) (i) Deduce the exact value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$.

(ii) For $n > 3$, deduce an expression for $\displaystyle\sum_{r=3}^{N-1} \frac{1}{r(r-1)(r-2)}$ in terms of N.

Solution

Part (a)

$$\begin{aligned}
f(r-1) - f(r) &= \frac{1}{(r-1+1)(r-1+2)} - \frac{1}{(r+1)(r+2)} \\
&= \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \\
&= \frac{(r+2)-r}{r(r+1)(r+2)} \\
&= \frac{2}{r(r+1)(r+2)}
\end{aligned}$$

$$\begin{aligned}
\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} &= \frac{1}{2}\sum_{r=1}^{n} \frac{2}{r(r+1)(r+2)} \\
&= \frac{1}{2}\sum_{r=1}^{n} \Big(f(r-1) - f(r)\Big) \\
&= \frac{1}{2}\left(\sum_{r=1}^{n} f(r-1) - \sum_{r=1}^{n} f(r)\right) \\
&= \frac{1}{2}\left(\sum_{r=0}^{n-1} f(r) - \sum_{r=1}^{n} f(r)\right) \\
&= \frac{1}{2}\left(\left[f(0) + \sum_{r=1}^{n-1} f(r)\right] - \left[\sum_{r=1}^{n-1} f(r) + f(n)\right]\right) \\
&= \frac{1}{2}\Big(f(0) - f(n)\Big) \\
&= \frac{1}{2}\left(\frac{1}{1\cdot 2} - \frac{1}{(n+1)(n+2)}\right) \\
&= \frac{1}{4} - \frac{1}{2(n+1)(n+2)}
\end{aligned}$$

$$\boxed{\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}}$$

Part (b)

Subpart (i)

$$\begin{aligned}
\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} &= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} \\
&= \lim_{n\to\infty} \left(\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right) \\
&= \frac{1}{4}
\end{aligned}$$

$$\boxed{\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}}$$

Subpart (ii)

$$\begin{aligned}
\sum_{r=3}^{N-1} \frac{1}{r(r-1)(r-2)} &= \sum_{r=1}^{N-3} \frac{1}{(r+2)(r+1)r} \\
&= \frac{1}{4} - \frac{1}{2(N-3+1)(N-3+2)} \\
&= \frac{1}{4} - \frac{1}{2(N-2)(N-1)}
\end{aligned}$$

$$\boxed{\sum_{r=3}^{N-1} \frac{1}{r(r-1)(r-2)} = \frac{1}{4} - \frac{1}{2(N-2)(N-1)}}$$

Problem 2.

(a) A geometric progression G has positive first term a, a common ratio r and sum to infinity S. The sum to infinity of the even-numbered terms of G, i.e. the second, fourth, sixth, ... terms, is $-\frac{1}{2}S$.

(i) Find the value of r.

(ii) In another geometric progression H, each term is the modulus of the corresponding term of G. Given that the third term of G is 2, show that the sum to infinity of H is 27.

(b) An arithmetic progression has first term 1000 and common difference -1.4. Determine, with clear workings, the value of the first negative term of the sequence and the sum of all the positive terms.

Solution

Part (a)

Subpart (i)

Let the nth term of G be $a_n = ar^{n-1}$. Since the sum to infinity of S exists, $|r| < 1$.

$$\begin{aligned}\sum_{n=1}^{\infty} a_{2n} &= \sum_{n=1}^{\infty} ar^{2n-1} \\ &= \frac{a}{r}\sum_{n=1}^{\infty}\left(r^2\right)^n \\ &= \frac{a}{r}\cdot\frac{r^2}{1-r^2} \\ &= \frac{ar}{1-r^2}\end{aligned}$$

Note that $S = \frac{a}{1-r}$. Thus, we have

$$\begin{aligned} & -\frac{1}{2}\cdot\frac{a}{1-r} = \frac{ar}{1-r^2} \\ \implies & -\frac{1}{2}\cdot\frac{1}{1-r} = \frac{r}{1-r^2} \\ \implies & 1-r^2 = -2r(1-r) \\ \implies & 3r^2-2r-1=0 \\ \implies & (3r+1)(r-1)=0 \end{aligned}$$

Hence, $r = -\frac{1}{3}$. Note that we reject $r = 1$ since $|r| < 1$.

$$\boxed{r = -\frac{1}{3}}$$

Subpart (ii)

Since $a_3 = 2 = a\left(-\frac{1}{3}\right)^2$, we have $a = 18$. Let the nth term of H be $b_n = |a_n| = \left|18\left(-\frac{1}{3}\right)^{n-1}\right| = 18\left(\frac{1}{3}\right)^{n-1}$. Hence, the sum to infinity of H is given by

$$\sum_{n=1}^{\infty} b_n = \frac{18}{1-1/3} = 27$$

Part (b)

Let the nth term of the arithmetic progression be $a_n = 1000 - 1.4(n-1) = 1001.4 - 1.4n$. Consider $a_n < 0$.

$$\begin{aligned} & a_n < 0 \\ \implies & 1001.4 - 1.4n < 0 \\ \implies & 1.4n > 1001.4 \\ \implies & n > \frac{1001.4}{1.4} \\ & = 715.3 \end{aligned}$$

Hence, the first negative term of the sequence is achieved when $n = 716$. Thus, the value of the first negative term is $a_{716} = 1001.4 - 1.4 \cdot 716 = -1$.

$$\boxed{\text{The value of the first negative term is } -1.}$$

$$\begin{aligned} \sum_{n=1}^{715} a_n &= \sum_{n=1}^{715} \Big(1001.4 - 1.4n\Big) \\ &= 1001.4 \cdot 715 - 1.4 \cdot \frac{715 \cdot 716}{2} \\ &= 357643 \end{aligned}$$

$$\boxed{\text{The sum of all the positive terms is } 357643.}$$

Problem 3.

Omitted.

Problem 4.

Referring to the pole O, the curve C has polar equation $r = \cot\theta$, where $\frac{\pi}{6} < \theta < \frac{\pi}{2}$.

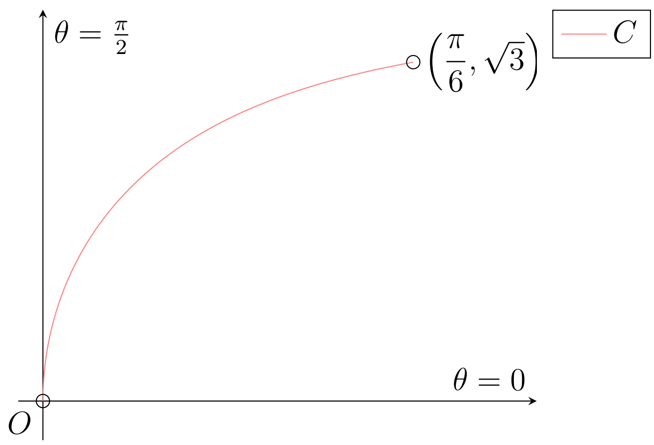
(a) Sketch the curve C.

(b) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{r(r^2+2)}$. Determine the exact range of values of the gradient of C.

(c) Obtain a Cartesian equation of C in the form $y = f(x)$.

Solution

Part (a)



Part (b)

Note that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\csc^2\theta = -(1+r^2)$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \\
&= \frac{-(1+r^2)\sin\theta + r\cos\theta}{-(1+r^2)\cos\theta - r\sin\theta} \\
&= \frac{-(1+r^2) + r\cot\theta}{-(1+r^2)\cot\theta - r} \\
&= \frac{-(1+r^2) + r^2}{-(1+r^2)r - r} \\
&= \frac{(1+r^2) - r^2}{(1+r^2)r + r} \\
&= \frac{1}{r(2+r^2)}
\end{aligned}$$

Observe that $r \in (0, \sqrt{3})$. Since $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{r(r^2+2)}$ is continuous and decreasing on the interval $(0, \sqrt{3})$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} \in \left(\frac{1}{\sqrt{3}(\sqrt{3}^2+2)}, \infty\right) = \left(\frac{1}{5\sqrt{3}}, \infty\right)$.

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} \in \left(\frac{1}{5\sqrt{3}}, \infty\right)}$$

Part (c)

$$\begin{aligned}
& & r &= \cot\theta \\
\implies & & r\sin\theta &= \cos\theta \\
\implies & & y &= \cos\arctan\frac{y}{x} \\
& & &= \frac{x}{\sqrt{x^2+y^2}} \\
\implies & & y^2 &= \frac{x^2}{x^2+y^2} \\
\implies & & y^2\left(x^2+y^2\right) &= x^2 \\
\implies & y^4 + x^2y^2 - x^2 = 0 & & \\
\implies & & y^2 &= \frac{-x^2+\sqrt{x^4+4x^2}}{2} & (*) \\
\implies & & y &= \sqrt{\frac{-x^2+\sqrt{x^4+4x^2}}{2}} & (*)
\end{aligned}$$

Note that in the steps marked $(*)$, we reject the negative branch since $y^2 \geq 0$ and $y > 0$ in the given domain.

$$\boxed{y = \sqrt{\frac{-x^2+\sqrt{x^4+4x^2}}{2}}}$$

Problem 5.

Relative to an origin O, an object is placed at point P with coordinates $(-4, c, c)$, where c is a positive real constant, and there is a mirror plane with equation $x + y + z = 1$. It is known that the shortest distance between P and the mirror is $3\sqrt{3}$.

(a) Show that $c = 7$.

A point A has coordinates $(-15, 17, 5)$.

(b) Find the coordinates of A', the point of reflection of A in the mirror.

A laser beam is directed from A towards a point on the mirror and is reflected to reach the object at P.

(c) Find the acute angle that the laser beam makes with the mirror.

Solution

Part (a)

We have that the mirror is defined by the vector equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1$. Note that the point with position vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is on the mirror. Thus,

$$\begin{aligned} \text{Shortest distance between } P \text{ and mirror} &= \left| \left[\begin{pmatrix} -4 \\ c \\ c \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| \Bigg/ \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{3}} \left| \begin{pmatrix} -5 \\ c \\ c \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| \\ &= \frac{\sqrt{3}}{3} \left| -5 + 2c \right| \end{aligned}$$

We are given that the shortest distance between P and the mirror is $3\sqrt{3}$ units. Hence,

$$\begin{aligned} \frac{\sqrt{3}}{3} \left| -5 + 2c \right| &= 3\sqrt{3} \\ \implies \quad \left| -5 + 2c \right| &= 9 \end{aligned}$$

Case 1: $-5 + 2c > 0 \implies -5 + 2c = 9 \implies c = 7$.

Case 2: $-5 + 2c < 0 \implies -5 + 2c = -9 \implies c = -2$ which cannot be since c is positive. Hence, $c = 7$ as required.

Part (b)

Let F be a point on the mirror (i.e. $\overrightarrow{OF} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 1$) such that $\overrightarrow{AF} = \lambda \begin{pmatrix}1\\1\\1\end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

$$\begin{aligned}
& \overrightarrow{AF} = \lambda \begin{pmatrix}1\\1\\1\end{pmatrix} \\
\implies & \overrightarrow{OF} - \begin{pmatrix}-15\\17\\5\end{pmatrix} = \lambda \begin{pmatrix}1\\1\\1\end{pmatrix} \\
\implies & \left[\overrightarrow{OF} - \begin{pmatrix}-15\\17\\5\end{pmatrix}\right] \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} \\
\implies & \overrightarrow{OF} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}-15\\17\\5\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \lambda \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} \\
\implies & 1 - 7 = 3\lambda \\
\implies & \lambda = -2 \\
\implies & \overrightarrow{AF} = -2 \begin{pmatrix}1\\1\\1\end{pmatrix}
\end{aligned}$$

Note that since A' is the reflection of A in the mirror, $\overrightarrow{AF} = \overrightarrow{FA'}$.

$$\begin{aligned}
& \overrightarrow{AF} = \overrightarrow{FA'} \\
\implies & \overrightarrow{AA'} = 2\overrightarrow{AF} \\
\implies & \overrightarrow{OA'} - \overrightarrow{OA} = 2\overrightarrow{AF} \\
\implies & \overrightarrow{OA'} - \begin{pmatrix}-15\\17\\5\end{pmatrix} = 2 \cdot -2 \begin{pmatrix}1\\1\\1\end{pmatrix} \\
\implies & \overrightarrow{OA'} = \begin{pmatrix}-4\\-4\\-4\end{pmatrix} + \begin{pmatrix}-15\\17\\5\end{pmatrix} \\
& = \begin{pmatrix}-19\\13\\1\end{pmatrix}
\end{aligned}$$

$$\boxed{A'(-19, 13, 1)}$$

Part (c)

Let θ be the acute angle the laser beam makes with the mirror. Note that $\overrightarrow{A'P} = \begin{pmatrix} 15 \\ -6 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$. Hence, the line $A'P$ has direction vector $\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}$.

$$\begin{aligned} \sin\theta &= \frac{\left|\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|}{\left|\begin{pmatrix} 5 \\ -2 \\ 2 \end{pmatrix}\right|\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|} \\ &= \frac{5}{\sqrt{99}} \\ \implies \quad \theta &= 0.527 \text{ (3 s.f.)} \end{aligned}$$

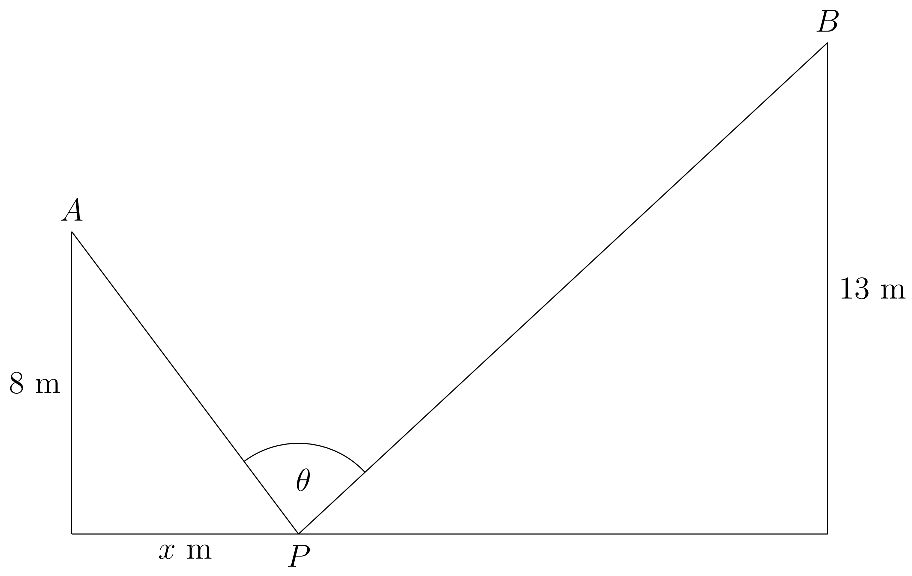
$$\boxed{\text{The laser beams makes an acute angle of 0.527 with the mirror.}}$$

Problem 6.

Omitted.

Problem 7.

A straight street of width 20 metres is bounded on its parallel sides by two vertical walls, one of height 13 metres, the other of height 8 metres. The intensity of light at point P at ground level on the street is proportional to the angle θ radians, where $\theta = \angle APB$ as shown in the cross-sectional diagram below.



(a) Given that the distance of P from the base of the wall of height 8 metres is x metres ($0 \leq x \leq 20$), show that

$$\theta = \arctan \frac{x}{8} + \arctan \frac{20-x}{13}$$

(b) Find an expression for $\dfrac{\mathrm{d}\theta}{\mathrm{d}x}$.

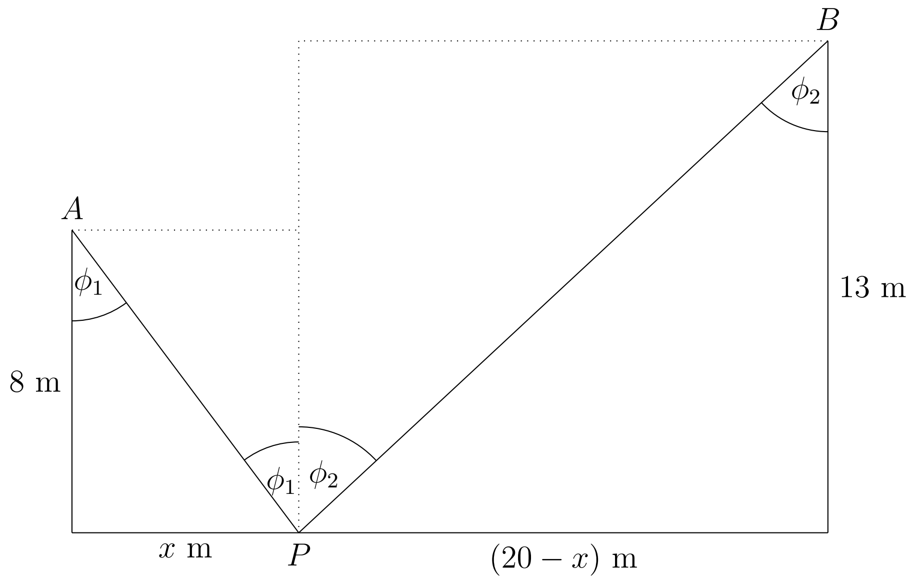
(c) Hence, determine the value of x corresponding to the maximum light intensity at P. Give your answer to four significant figures. You need not justify that the value of x obtained gives the maximum light intensity at P.

(d) Find the minimum value of θ as x varies.

(e) The point P moves across the street from the base of A to the base of B with speed 0.5 ms^{-1}. Determine the rate of change of θ with respect to time when P is at the midpoint of the street.

Solution

Part (a)



Consider the diagram above. It is clear that $\theta = \phi_1 + \phi_2$. Observe that $\tan\phi_1 = \dfrac{x}{8}$ and $\tan\phi_2 = \dfrac{20-x}{13}$. Thus, $\theta = \arctan\dfrac{x}{8} + \arctan\dfrac{20-x}{13}$.

Part (b)

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}x} &= \frac{1}{1+\left(\frac{x}{8}\right)^2}\cdot\frac{1}{8} + \frac{1}{1+\left(\frac{20-x}{13}\right)^2}\cdot\left(-\frac{1}{13}\right)\\ &= \frac{8^2}{8^2+x^2}\cdot\frac{1}{8} + \frac{13^2}{13^2+(20-x)^2}\cdot\left(-\frac{1}{13}\right)\\ &= \frac{8}{x^2+64} + \frac{13}{(20-x)^2+169}\end{aligned}$$

Part (c)

At stationary points, $\dfrac{\mathrm{d}\theta}{\mathrm{d}x} = 0$. Hence,

$$\frac{8}{x^2+64} + \frac{13}{(20-x)^2+169} = 0$$

From G.C., we have $x = 10.05$ and $x = -74.05$ (4 s.f.). Since $0 \le x \le 20$, we take $x = 10.05$.

$$\boxed{x = 10.05 \text{ (4 s.f.)}}$$

Part (d)

Since there is only one stationary point in the interval $[0, 20]$, and it is a maximum, the minimum value of θ occurs either at $x = 0$ or $x = 20$, i.e. the extreme ends of the interval.

$$x = 0 : \theta = \arctan \frac{20}{13} = 0.994 \text{ (3 s.f.)}$$
$$x = 20 : \theta = \arctan \frac{20}{8} = 1.19 \text{ (3 s.f.)}$$

$$\boxed{\text{The minimum value of } \theta \text{ is } 0.994.}$$

Part (e)

Let the time elapsed be t s. We have $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5$. Also note that when P is at the midpoint of the street, $x = 10$.

$$\begin{aligned} \left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{x=10} &= \left.\frac{\mathrm{d}\theta}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=10} \\ &= \left(\frac{8}{10^2 + 64} + \frac{13}{(20-10)^2 + 169}\right) \cdot 0.5 \\ &= 0.000227 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{The rate of change of } \theta \text{ is } 0.000227 \text{ rad per second.}}$$