Problem 1.

(a) Find the rectangular coordinates of the following points.

(i) $\left(3, -\frac{\pi}{4}\right)$

(ii) $(1, \pi)$

(iii) $\left(\frac{1}{2}, \frac{3}{2}\pi\right)$

(b) Find the polar coordinates of the following points.

(i) $(3, 3)$

(ii) $(-1, -\sqrt{3})$

(iii) $(2, 0)$

(iv) $(4, 2)$

Solution

Part (a)

Subpart (i)

Note that $r = 3$ and $\theta = -\frac{\pi}{4}$. This gives

$$\begin{aligned} x = r\cos\theta &= 3\cos\left(-\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2} \\ y = r\sin\theta &= 3\sin\left(-\frac{\pi}{4}\right) = -\frac{3\sqrt{2}}{2} \end{aligned}$$

$$\boxed{\left(\frac{3\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)}$$

Subpart (ii)

Note that $r = 1$ and $\theta = \pi$. This gives

$$\begin{aligned} x = r\cos\theta &= 1\cos\pi = -1 \\ y = r\sin\theta &= 1\sin\pi = 0 \end{aligned}$$

$$\boxed{(-1, 0)}$$

Subpart (iii)

Note that $r = \frac{1}{2}$ and $\theta = \frac{3}{2}\pi$. This gives

$$\begin{aligned} x = r\cos\theta &= \frac{1}{2}\cos\frac{3}{2}\pi = 0 \\ y = r\sin\theta &= \frac{1}{2}\sin\frac{3}{2}\pi = -\frac{1}{2} \end{aligned}$$

$$\boxed{\left(0, -\frac{1}{2}\right)}$$

Part (b)

Subpart (i)
Note that $x = 3$ and $y = -3$. This gives

$$\begin{aligned} r^2 = x^2 + y^2 &= 3^2 + (-3)^2 &&\implies r = 3\sqrt{2} \\ \tan\theta = \frac{y}{x} &= \frac{-3}{3} &&\implies \theta = -\frac{\pi}{4} \end{aligned}$$

$$\boxed{\left(3\sqrt{2}, -\frac{\pi}{4}\right)}$$

Subpart (ii)
Note that $x = -1$ and $y = -\sqrt{3}$. This gives

$$\begin{aligned} r^2 = x^2 + y^2 &= (-1)^2 + (-\sqrt{3})^2 &&\implies r = 2 \\ \tan\theta = \frac{y}{x} &= \frac{-1}{-\sqrt{3}} &&\implies \theta = \frac{\pi}{3} \end{aligned}$$

$$\boxed{\left(2, \frac{\pi}{3}\right)}$$

Subpart (iii)
Note that $x = 2$ and $y = 0$. This gives

$$\begin{aligned} r^2 = x^2 + y^2 &= 2^2 + 0^2 &&\implies r = 2 \\ \tan\theta = \frac{y}{x} &= \frac{0}{2} &&\implies \theta = 0 \end{aligned}$$

$$\boxed{(2, 0)}$$

Subpart (iv)
Note that $x = 4$ and $y = 2$. This gives

$$\begin{aligned} r^2 = x^2 + y^2 &= 4^2 + 2^2 &&\implies r = 2\sqrt{5} \\ \tan\theta = \frac{y}{x} &= \frac{2}{4} &&\implies \theta = \arctan\frac{1}{2} \end{aligned}$$

$$\boxed{\left(2\sqrt{5}, \arctan\frac{1}{2}\right)}$$

Problem 2.

Rewrite the following equations in polar form.

(a) $2x^2 + 3y^2 = 4$

(b) $y = 2x^2$

Solution

Part (a)

$$\begin{aligned}
& 2x^2 + 3y^2 = 4 \\
\implies & 2(r\cos\theta)^2 + 3(r\sin\theta)^2 = 4 \\
\implies & 2r^2\cos^2\theta + 3r^2\sin^2\theta = 4 \\
\implies & r^2(2\cos^2\theta + 3\sin^2\theta) = 4 \\
\implies & r^2(2 + \sin^2\theta) = 4 \\
\implies & r^2 = \frac{4}{2 + \sin^2\theta}
\end{aligned}$$

$$\boxed{r^2 = \frac{4}{2 + \sin^2\theta}}$$

Part (b)

$$\begin{aligned}
& y = 2x^2 \\
\implies & \frac{y}{x} = 2x \\
\implies & \tan\theta = 2r\cos\theta \\
\implies & r = \frac{1}{2}\tan\theta\sec\theta
\end{aligned}$$

$$\boxed{r = \frac{1}{2}\tan\theta\sec\theta}$$

Problem 3.

Rewrite the following equations in rectangular form.

(a) $r = \dfrac{1}{1 - 2\cos\theta}$

(b) $r = \sin\theta$

Solution

Part (a)

$$
\begin{aligned}
r &= \frac{1}{1 - 2\cos\theta} \\
\implies r(1 - 2\cos\theta) &= 1 \\
\implies r - 2r\cos\theta &= 1 \\
\implies r - 2x &= 1 \\
\implies r &= 2x + 1 \\
\implies r^2 &= (2x+1)^2 \\
\implies &= 4x^2 + 4x + 1 \\
\implies x^2 + y^2 &= 4x^2 + 4x + 1 \\
\implies y^2 &= 3x^2 + 4x + 1
\end{aligned}
$$

$$\boxed{y^2 = 3x^2 + 4x + 1}$$

Part (b)

$$
\begin{aligned}
r &= \sin\theta \\
\implies r^2 &= r\sin\theta \\
\implies x^2 + y^2 &= y
\end{aligned}
$$

$$\boxed{x^2 + y^2 = y}$$

Problem 4.

(a) Show that the curve with polar equation $r = 3a\cos\theta$, where a is a positive constant, is a circle. Write down its centre and radius.

(b) By finding the Cartesian equation, sketch the curve whose polar equation is $r = a\sec\left(\theta - \frac{\pi}{4}\right)$, where a is a positive constant.

Solution

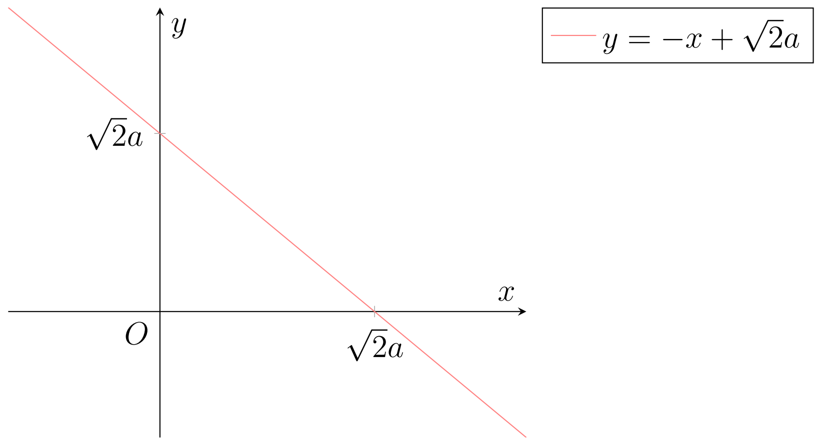
Part (a)

$$\begin{aligned}
&& r &= 3a\cos\theta \\
\implies && r^2 &= 3ar\cos\theta \\
\implies && x^2 + y^2 &= 3ax \\
\implies && x^2 - 3ax + y^2 &= 0 \\
\implies && x^2 - 3ax + \left(-\frac{3}{2}a\right)^2 + y^2 &= \left(-\frac{3}{2}a\right)^2 \\
\implies && \left(x - \frac{3}{2}a\right)^2 + y^2 &= \left(\frac{3}{2}a\right)^2
\end{aligned}$$

$$\boxed{\text{Center: } \left(\frac{3}{2}a, 0\right), \text{ Radius: } \frac{3}{2}a}$$

Part (b)

$$\begin{aligned}
&& r &= a\sec\left(\theta - \frac{\pi}{4}\right) \\
\implies && r &= \frac{a}{\cos\left(\theta - \frac{\pi}{4}\right)} \\
\implies && r\cos\left(\theta - \frac{\pi}{4}\right) &= a \\
\implies && r\left(\cos\theta\cos\frac{\pi}{4} + \sin\theta\sin\frac{\pi}{4}\right) &= a \\
\implies && r\left(\cos\theta \cdot \frac{1}{\sqrt{2}} + \sin\theta \cdot \frac{1}{\sqrt{2}}\right) &= a \\
\implies && r\left(\cos\theta + \sin\theta\right) &= \sqrt{2}a \\
\implies && r\cos\theta + r\sin\theta &= \sqrt{2}a \\
\implies && x + y &= \sqrt{2}a \\
\implies && y &= -x + \sqrt{2}a
\end{aligned}$$


$y = -x + \sqrt{2}a$
y
x
O
$\sqrt{2}a$
$\sqrt{2}a$

Problem 5.

Sketch the following polar curves, where r is non-negative and $0 \leq \theta \leq 2\pi$.

(a) $r = 2$

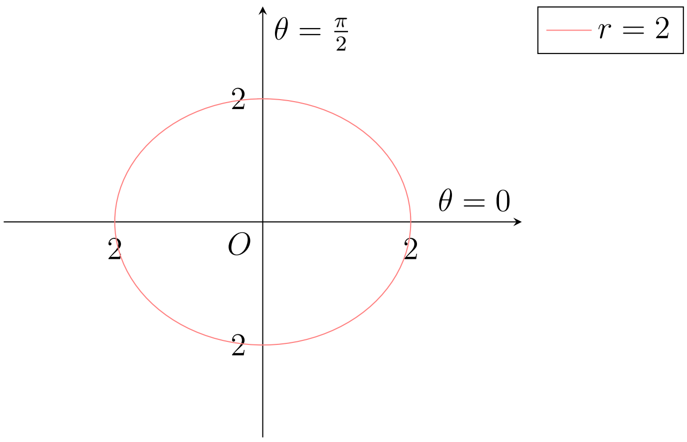
(b) $\theta = \dfrac{\pi}{4}$

(c) $r = \dfrac{1}{2}\theta$

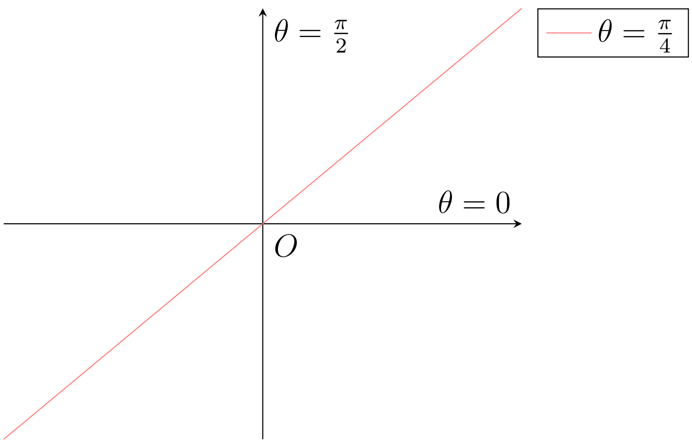
(d) $r = 2\csc\theta$

Solution

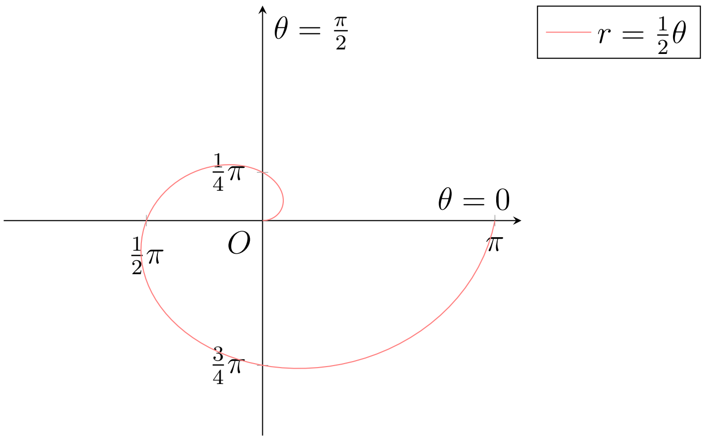
Part (a)



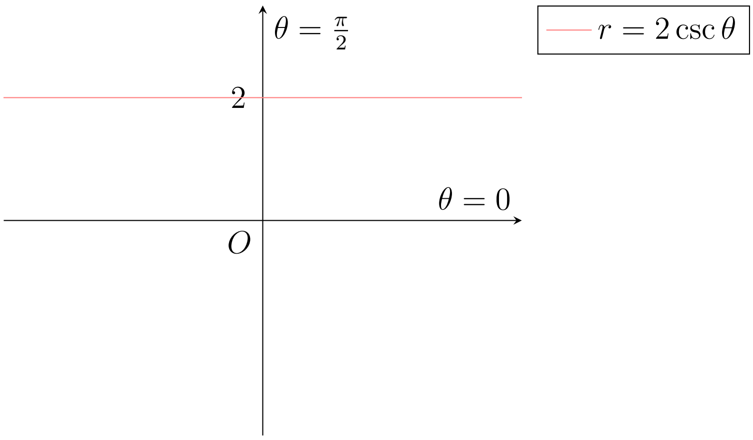
Part (b)



Part (c)

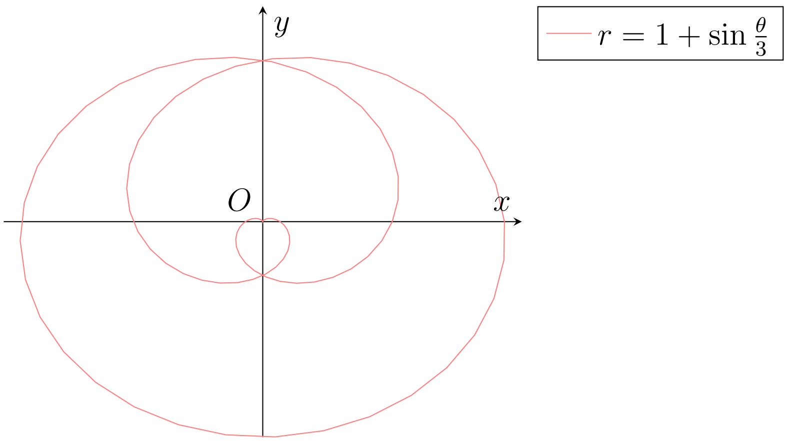

$\theta = \frac{\pi}{2}$
$r = \frac{1}{2}\theta$
$\frac{1}{4}\pi$
$\theta = 0$
$\frac{1}{2}\pi$
O
π
$\frac{3}{4}\pi$

Part (d)


$\theta = \frac{\pi}{2}$
$r = 2\csc\theta$
2
$\theta = 0$
O

Problem 6.

A sketch of the curve $r = 1 + \sin\dfrac{\theta}{3}$ is shown. Copy the diagram and indicate the x- and y-intercepts.



Solution

Observe that the curve is symmetric about the y-axis. Also observe that $\dfrac{\theta}{3} \in [0, 2\pi)$, hence we take $\theta \in [0, 6\pi)$.

For x-intercepts, $y = 0 \implies r\sin\theta = 0 \implies \theta = n\pi$, where $n \in \mathbb{Z}$. Due to the symmetry of the curve, we consider only $n = 0, 2, 4$.

Case 1: $n = 0 \implies r = 1 + \sin\dfrac{0}{3}\pi = 1$

Case 2: $n = 2 \implies r = 1 + \sin\dfrac{2}{3}\pi = 1 + \dfrac{\sqrt{3}}{2}$

Case 3: $n = 4 \implies r = 1 + \sin\dfrac{4}{3}\pi = 1 - \dfrac{\sqrt{3}}{2}$

Hence, the curve intersects the x-axis at $x = 1, 1 + \dfrac{\sqrt{3}}{2}, 1 - \dfrac{\sqrt{3}}{2}$. Correspondingly, the curve also intersects the x-axis at $x = -1, -1 - \dfrac{\sqrt{3}}{2}, -1 + \dfrac{\sqrt{3}}{2}$.

For y-intercepts, $x = 0 \implies r\cos\theta = 0 \implies \theta = \left(n + \dfrac{1}{2}\right)\pi$, where $n \in \mathbb{Z}$. Due to the restriction on θ, we consider $n \in [0, 5)$.

Case 1: $n = 0,\ \theta = \left(0 + \dfrac{1}{2}\right)\pi \implies r = 1 + \sin\dfrac{0 + \frac{1}{2}}{3}\pi = \dfrac{3}{2}$

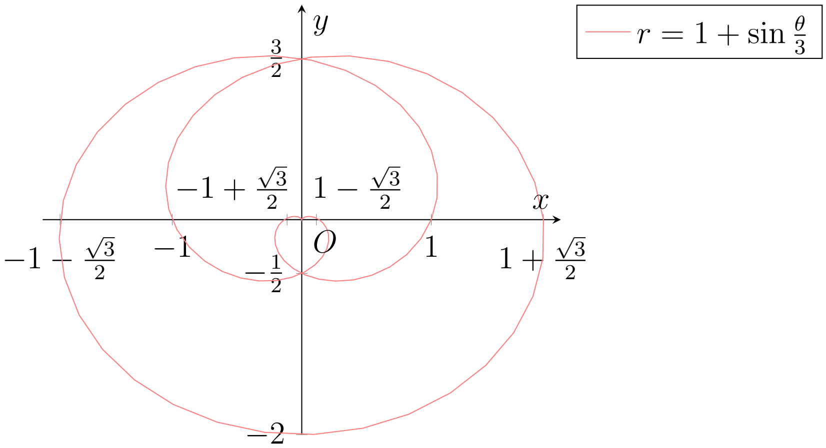
Case 2: $n = 1,\ \theta = \left(1 + \dfrac{1}{2}\right) \implies r = 1 + \sin\dfrac{1 + \frac{1}{2}}{3}\pi = 2$

Case 3: $n = 2,\ \theta = \left(2 + \dfrac{1}{2}\right) \implies r = 1 + \sin\dfrac{2 + \frac{1}{2}}{3}\pi = \dfrac{3}{2}$

Case 4: $n = 3,\ \theta = \left(3 + \frac{1}{2}\right) \implies r = 1 + \sin\frac{3+\frac{1}{2}}{3}\pi = \frac{1}{2}$

Case 5: $n = 4,\ \theta = \left(4 + \frac{1}{2}\right) \implies r = 1 + \sin\frac{4+\frac{1}{2}}{3}\pi = 0$

Hence, the curve intersects the y-axis at $y = -2, -\frac{1}{2}, \frac{3}{2}$.



Problem 7.

(a) A graph has polar equation $r = \dfrac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha}$, where α is a constant. Express the equation in Cartesian form. Hence, sketch the graph in the case $\alpha = \dfrac{\pi}{4}$, giving the Cartesian coordinates of the intersection with the axes.

(b) A graph has Cartesian equation $(x^2 + y^2)^2 = 4x^2$. Express the equation in polar form. Hence, or otherwise, sketch the graph.

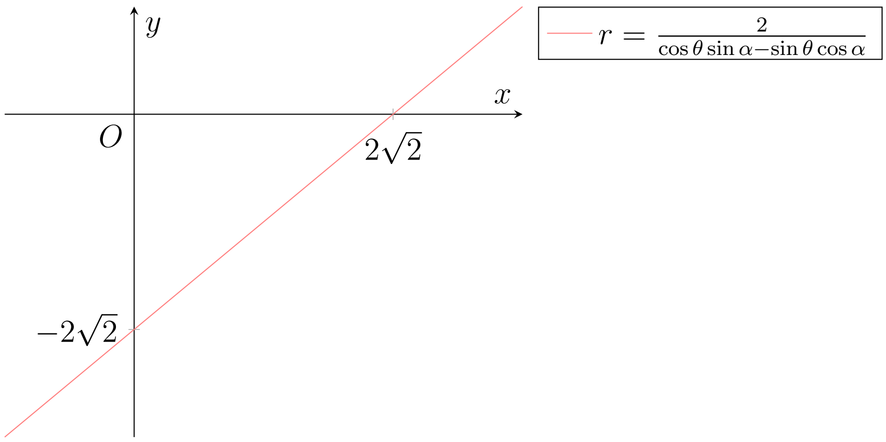
Solution

Part (a)

$$\begin{aligned} r &= \frac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha} \\ \implies r\cos\theta\sin\alpha - r\sin\theta\cos\alpha &= 2 \\ \implies x\sin\alpha - y\cos\alpha &= 2 \\ \implies y\cos\alpha &= x\sin\alpha - 2 \\ \implies y &= x\tan\alpha - 2\sec\alpha \end{aligned}$$

$$\boxed{y = x\tan\alpha - 2\sec\alpha}$$

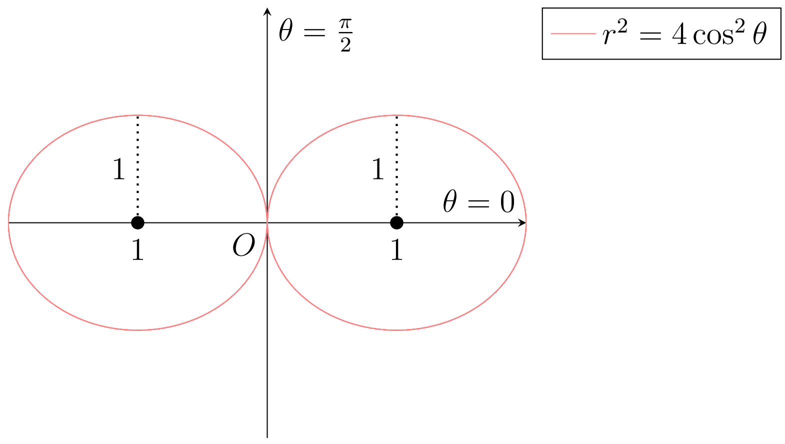
When $\alpha = \dfrac{\pi}{4}$, we have $y = x\tan\dfrac{\pi}{4} - 2\sec\dfrac{\pi}{4} = x - 2\sqrt{2}$.



Part (b)

$$\begin{aligned} (x^2 + y^2)^2 &= 4x^2 \\ \implies \left(r^2\right)^2 &= 4(r\cos\theta)^2 \\ \implies r^4 &= 4r^2\cos^2\theta \\ \implies r^2 &= 4\cos^2\theta \end{aligned}$$

$$\boxed{r^2 = 4\cos^2\theta}$$


$\theta = \frac{\pi}{2}$
$r^2 = 4\cos^2\theta$
1
1
$\theta = 0$
1
O
1

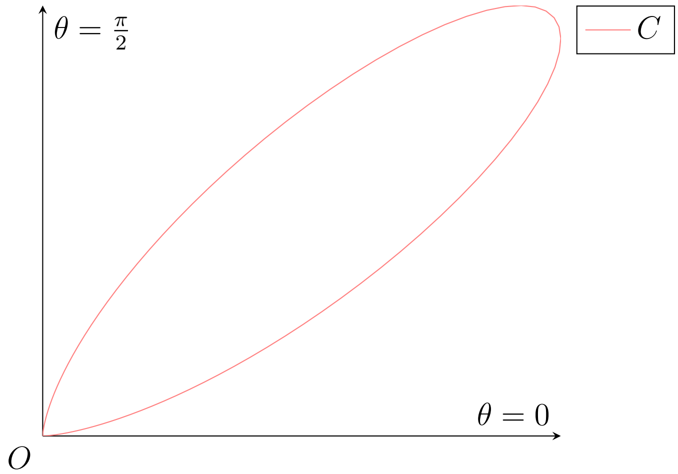
Problem 8.

Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \le \theta \le \frac{\pi}{2}$.

Solution

$$\begin{aligned}
& x^5 + y^5 = 5bx^2y^2 \\
\implies & (r\cos\theta)^5 + (r\sin\theta)^5 = 5b(r\cos\theta)^2(r\sin\theta)^2 \\
\implies & r^5\cos^5\theta + r^5\sin^5\theta = 5b\cdot r^2\cos^2\theta\cdot r^2\sin^2\theta \\
\implies & r^5\left(\cos^5\theta + \sin^5\theta\right) = r^4\cdot 5b\cos^2\theta\sin^2\theta \\
\implies & r\left(\cos^5\theta + \sin^5\theta\right) = 5b\cos^2\theta\sin^2\theta \\
\implies & r = \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}
\end{aligned}$$

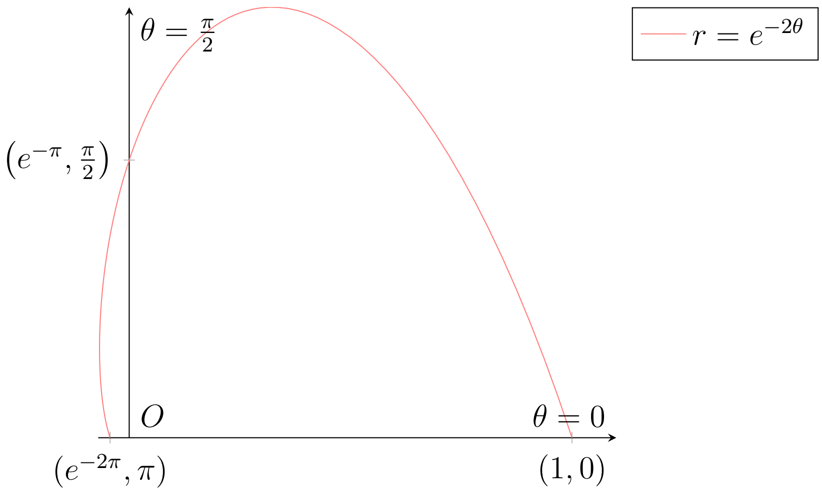
$$\boxed{r = \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}}$$



Problem 9.

The equation of a curve, in polar coordinates, is $r = e^{-2\theta}$, for $0 \le \theta \le \pi$. Sketch the curve, indicating clearly the polar coordinates of any axial intercepts.

Solution



Problem 10.

Suppose that a long thin rod with one end fixed at the pole of a polar coordinate system rotates counter-clockwise at the constant rate of 0.5 rad/sec. At time $t = 0$, a bug on the rod is 10 mm from the pole and is moving outward along the rod at a constant speed of 2 mm/sec. Find an equation of the form $r = f(\theta)$ for the part of motion of the bug, assuming that $\theta = 0$ when $t = 0$. Sketch the path of the bug on the polar coordinate system for $0 \leq t \leq 4\pi$.

Solution

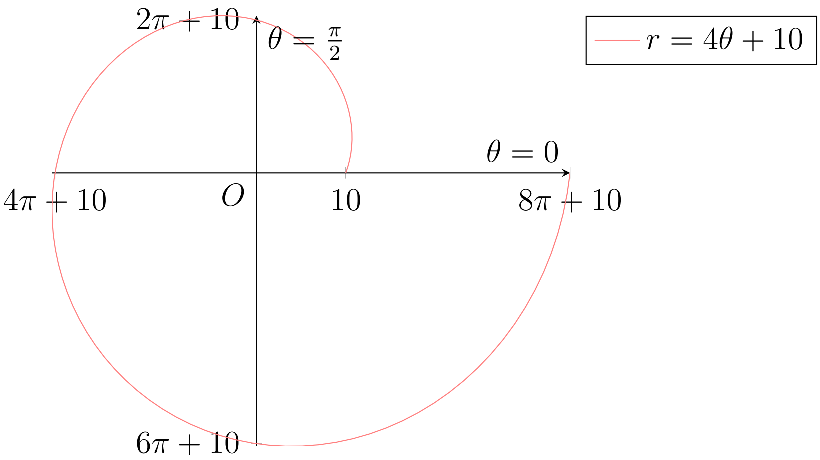
Let $\theta(t)$ and $r(t)$ be functions of time, with $\theta(0) = 0$ and $r(0) = 10$. We know that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = 2$. Hence,

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\theta} \\ &= \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^{-1} \\ &= 2 \cdot (0.5)^{-1} \\ &= 4\end{aligned}$$

Integrating with respect to θ, we have $r = 4\theta + C$. Evaluating the equation at $t = 0$, we see that $10 = 4 \cdot 0 + C \implies C = 10$. Thus,

$$\boxed{r = 4\theta + 10}$$

Since $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5$ and $\theta(0) = 0$, we have $\theta = 0.5t$. Hence, $0 \leq t \leq 4\pi \implies 0 \leq \theta \leq 2\pi$.



Problem 11.

The equation, in polar coordinates, of a curve C is $r = ae^{\frac{1}{2}\theta}$, $0 \leq \theta \leq 2\pi$, where a is a positive constant. Write down, in terms of θ, the Cartesian coordinates, x and y, of a general point P on the curve. Show that the gradient at P is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\tan\theta + 2}{1 - 2\tan\theta}$.

Hence, show that the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$. Sketch the curve C.

Solution

Note that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = ae^{\frac{1}{2}\theta}\cos\theta$ and $y = ae^{\frac{1}{2}\theta}\sin\theta$. Hence,

$$\boxed{P\left(ae^{\frac{1}{2}\theta}\cos\theta, ae^{\frac{1}{2}\theta}\sin\theta\right)}$$

Observe that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = ae^{\frac{1}{2}\theta} \cdot \frac{1}{2} = \frac{1}{2}r$. Hence,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} \\
&= \frac{\frac{1}{2}r\sin\theta + r\cos\theta}{\frac{1}{2}r\cos\theta - r\sin\theta} \\
&= \frac{\frac{1}{2}\sin\theta + \cos\theta}{\frac{1}{2}\cos\theta - \sin\theta} \\
&= \frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} \\
&= \frac{\tan\theta + 2}{1 - 2\tan\theta}
\end{aligned}$$

Let $\overrightarrow{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$ represent the tangent line and let $\overrightarrow{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ represent $\overrightarrow{OP}$. We hence have

$$\begin{aligned}
\overrightarrow{T} &= \begin{bmatrix} 1 \\ \frac{dy}{dx} \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ \frac{\tan\theta + 2}{1 - 2\tan\theta} \end{bmatrix} \\
&= \frac{1}{1 - 2\tan\theta}\begin{bmatrix} 1 - 2\tan\theta \\ \tan\theta + 2 \end{bmatrix} \\
\overrightarrow{P} &= \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} ae^{\frac{1}{2}\theta}\cos\theta \\ ae^{\frac{1}{2}\theta}\sin\theta \end{bmatrix} \\
&= ae^{\frac{1}{2}\theta}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}
\end{aligned}$$

Let $\overrightarrow{T'} = \begin{bmatrix} 1 - 2\tan\theta \\ \tan\theta + 2 \end{bmatrix}$ and $\overrightarrow{P'} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$. Then the angle α between $\overrightarrow{T}$ and $\overrightarrow{P}$ is equal to the angle between $\overrightarrow{T'}$ and $\overrightarrow{P'}$. By the definition of the dot product, we have

$$\overrightarrow{T'} \cdot \overrightarrow{P'} = T_1'P_1' + T_2'P_2' = \|T'\| \, \|P'\| \cos\alpha$$

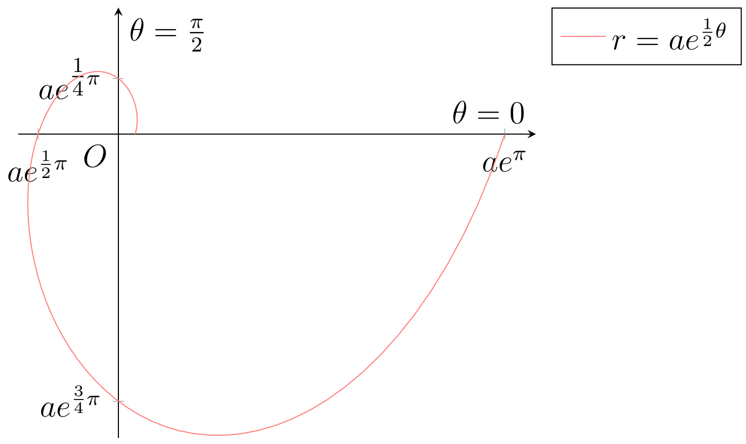
whence

$$\begin{aligned}
\cos\alpha &= \frac{T_1'P_1' + T_2'P_2'}{\|T'\| \, \|P'\|} \\
&= \frac{(1 - 2\tan\theta)\cos\theta + (\tan\theta + 2)\sin\theta}{\sqrt{(1 - 2\tan\theta)^2 + (\tan\theta + 2)^2} \cdot \sqrt{\cos^2\theta + \sin^2\theta}} \\
&= \frac{(1 - 2\tan\theta)\cos\theta + (\tan\theta + 2)\sin\theta}{\sqrt{(1 - 2\tan\theta)^2 + (\tan\theta + 2)^2}} \\
&= \frac{\cos\theta - 2\sin\theta + \tan\theta\sin\theta + 2\sin\theta}{\sqrt{1 - 4\tan\theta + 4\tan^2\theta + \tan^2\theta + 4\tan\theta + 4}} \\
&= \frac{\cos\theta + \tan\theta\sin\theta}{\sqrt{5\tan^2\theta + 5}} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\cos\theta\sqrt{5\tan^2\theta + 5}} \\
&= \frac{1}{\sqrt{\cos^2\theta\,(5\tan^2\theta + 5)}} \\
&= \frac{1}{\sqrt{5\sin^2 + 5\cos^2\theta}} \\
&= \frac{1}{\sqrt{5}}
\end{aligned}$$

Thus, $\alpha = \arccos\frac{1}{\sqrt{5}}$. Since $\tan(\arccos x) = \frac{\sqrt{1 - x^2}}{x}$,

$$\begin{aligned}
\tan\alpha &= \tan\left(\arccos\frac{1}{\sqrt{5}}\right) \\
&= \frac{\sqrt{1 - \left(\frac{1}{\sqrt{5}}\right)^2}}{\frac{1}{\sqrt{5}}} \\
&= \sqrt{5} \cdot \sqrt{\frac{4}{5}} \\
&= \sqrt{4} \\
&= 2
\end{aligned}$$

Hence, the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$.


$\theta = \frac{\pi}{2}$
$r = ae^{\frac{1}{2}\theta}$
$ae^{\frac{1}{4}\pi}$
$\theta = 0$
$ae^{\frac{1}{2}\pi}$
O
ae^{π}
$ae^{\frac{3}{4}\pi}$

Problem 12.

The polar equation of a curve is given by $r = e^\theta$ where $0 \le \theta \le \frac{\pi}{2}$. Cartesian axes are taken at the pole O. Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$.

Solution

Recall that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = e^\theta \cos\theta$ and $y = e^\theta \sin\theta$.

$$\boxed{x = e^\theta \cos\theta,\ y = e^\theta \sin\theta}$$

Note that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = e^\theta \cdot -\sin\theta + e^\theta\cos\theta = e^\theta(\cos\theta - \sin\theta)$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = e^\theta\cos\theta + e^\theta\sin\theta = e^\theta(\cos\theta + \sin\theta)$. Hence,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \\ &= \frac{\mathrm{d}y}{\mathrm{d}\theta} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^{-1} \\ &= \frac{e^\theta(\cos\theta + \sin\theta)}{e^\theta(\cos\theta - \sin\theta)} \\ &= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}\end{aligned}$$

At $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$,

$$\begin{aligned}x &= e^{\frac{\pi}{2}} \cos\frac{\pi}{2} = 0 \\ y &= e^{\frac{\pi}{2}} \sin\frac{\pi}{2} = e^{\frac{\pi}{2}} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cos\frac{\pi}{2} + \sin\frac{\pi}{2}}{\cos\frac{\pi}{2} - \sin\frac{\pi}{2}} = -1\end{aligned}$$

By the point-slope formula, the equation of the tangent line at $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$ is given by

$$\begin{aligned} & y - e^{\frac{\pi}{2}} = -1(x - 0) \\ \implies & \qquad y = -x + e^{\frac{\pi}{2}}\end{aligned}$$

$$\boxed{y = -x + e^{\frac{\pi}{2}}}$$

Problem 13.

A curve C has polar equation $r = a\cot\theta$, $0 < \theta \leq \pi$, where a is a positive constant.

(a) Show that $y = a$ is an asymptote of C.

(b) Find the tangent at the pole.

Hence, sketch C and find the Cartesian equation of C in the form $y^2(x^2 + y^2) = bx^2$, where b is a constant to be determined.

Solution

Part (a)

$$\begin{aligned} r &= a\cot\theta \\ &= a\frac{\cos\theta}{\sin\theta} \\ \implies r\sin\theta &= a\cos\theta \\ \implies \qquad y &= a\cos\theta \end{aligned}$$

Observe that as $\theta \to 0$, $r \to \infty$. Hence, there is an asymptote at $\theta = 0$. Since $\cos\theta = 1$ when $\theta = 0$, the line $y = a\cos\theta = a$ is an asymptote of C.

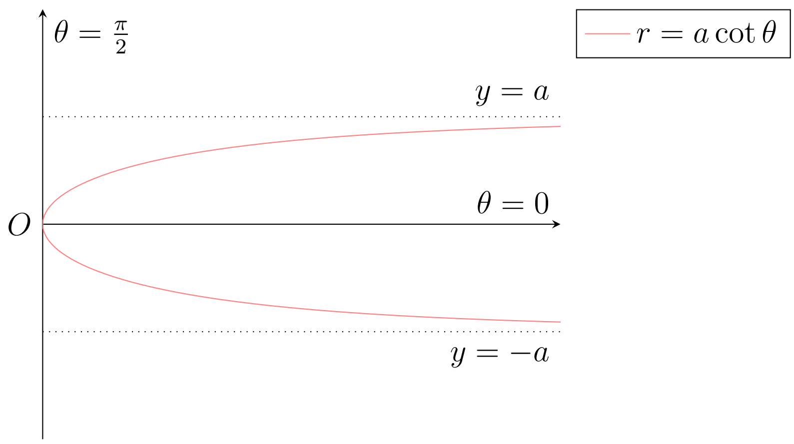
Part (b)

For tangents at the pole, $r = 0$.

$$\begin{aligned} r &= 0 \\ \implies a\cot\theta &= 0 \\ \implies \qquad \theta &= \frac{\pi}{2} \end{aligned}$$

Hence, the tangent at the pole is $\theta = \frac{\pi}{2}$.

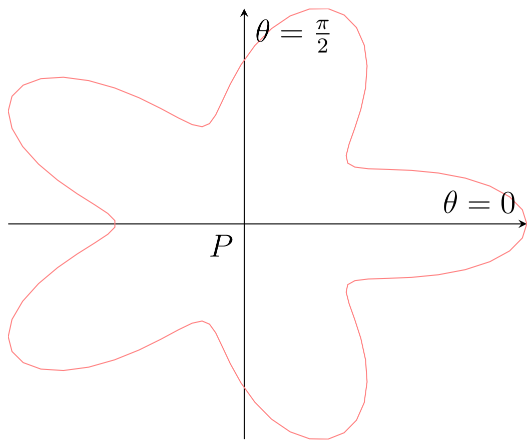
$$\boxed{\theta = \frac{\pi}{2}}$$



$$
\begin{aligned}
r &= a \cot\theta \\
&= a\frac{\cos\theta}{\sin\theta} \\
&= a\frac{r\cos\theta}{r\sin\theta} \\
\implies \qquad r^2 &= a^2\left(\frac{r\cos\theta}{r\sin\theta}\right)^2 \\
\implies \qquad x^2+y^2 &= a^2\left(\frac{x}{y}\right)^2 \\
\implies y^2(x^2+y^2) &= a^2x^2
\end{aligned}
$$

$$\boxed{b = a^2}$$

Problem 14.



Relative to the pole P and the initial line $\theta = 0$, the polar equation of the curve shown is either

i. $r = a + b \sin n\theta$, or

ii. $r = a + b \cos n\theta$

where a, b and n are positive constants. State, with a reason, whether the equation is (i) or (ii) and state the value of n.

The maximum value of r is $\dfrac{11}{2}$ and the minimum value of r is $\dfrac{5}{2}$. Find the values of a and b.

Solution

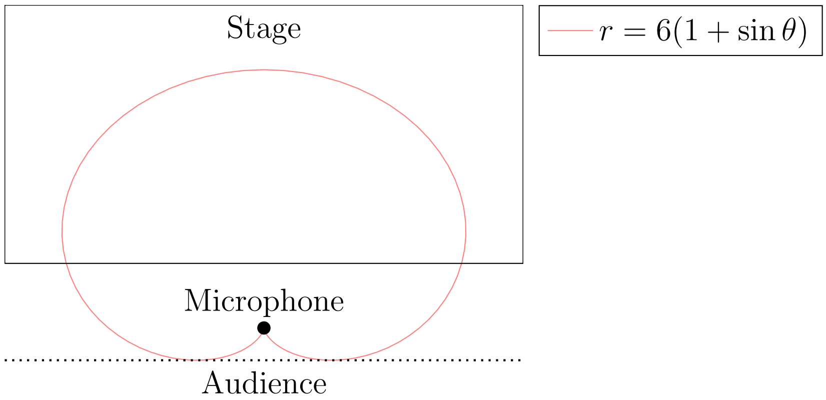
Since the curve is symmetrical about the horizontal half-line $\theta = 0$, the polar equation of the curve is a function of $\cos n\theta$ only. Hence, the polar equation of the curve is $r = a + b \cos n\theta$.

$$\boxed{\text{The equation is } r = a + b \cos n\theta, \text{ with } n = 5.}$$

Observe that the maximum value of r is achieved when $\cos 5\theta = 1$, whence $r = a + b$. Thus, $a + b = \dfrac{11}{2}$. Also observe that the minimum value of r is achieved when $\cos 5\theta = -1$, whence $r = a - b$. Thus, $a - b = \dfrac{5}{2}$. Solving, we get $a = 4$ and $b = \dfrac{3}{2}$.

$$\boxed{a = 4,\ b = \frac{3}{2}}$$

Problem 15.



Sound engineers often use a microphone with a cardioid acoustic pickup pattern to record live performances because it reduces pickup from the audience. Suppose a cardioid microphone is placed 3 metres from the front of the stage, and the boundary of the optimal pickup region is given by the cardioid with polar equation

$$r = 6(1 + \sin\theta)$$

where r is measured in metres and the microphone is at the pole.

Find the minimum distance from the front of the stage the first row of the audience can be seated such that the microphone does not pick up noise from the audience.

Solution

$$\begin{aligned}
r &= 6(1 + \sin\theta) \\
&= 6\left(1 + \frac{y}{r}\right) \\
\implies \quad r^2 &= 6(r + y) \\
&= 6r + 6y \\
\implies r^2 - 6r - 6y &= 0 \\
\implies \quad r &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(-6y)}}{2(1)} \\
&= 3 \pm \sqrt{9 + 6y} \\
\implies \quad 9 + 6y &= (r - 3)^2
\end{aligned}$$

Observe that $(r-3)^2 \geq 0$. Hence, $9 + 6y \geq 0 \implies y \geq -1.5$. Thus, the furthest distance the audience has to be from the stage is $|-1.5| + 3 = 4.5$ m.

$$\boxed{4.5 \text{ m}}$$

Problem 16.

To design a flower pendant, a designer starts off with a curve C_1, given by the Cartesian equation

$$\left(x^2+y^2\right)^2 = a^2\left(3x^2-y^2\right)$$

where a is a positive constant.

(a) Show that a corresponding polar equation of C_1 is $r^2 = a^2(1+2\cos 2\theta)$.

(b) Find the equations of the tangents to C_1 at the pole.

Another curve C_2 is obtained by rotating C_1 anti-clockwise about the origin by $\frac{\pi}{3}$ radians.

(c) State a polar equation of C_2.

(d) Sketch C_1 and C_2 on the same diagram, stating clearly the exact polar coordinates of the points of intersection of the curves with the axes. Find also the exact polar coordinates of the points of intersection with C_1 and C_2.

The curve C_3 is obtained by reflecting C_2 in the line $\theta = \frac{\pi}{2}$.

(e) State a polar equation of C_3.

(f) The designer wishes to enclose the 3 curves inside a circle given by the polar equation $r = r_1$. State the minimum value of r_1 in terms of a.

Solution

Part (a)

$$\begin{aligned}
& & \left(x^2+y^2\right)^2 &= a^2\left(3x^2-y^2\right) \\
&\implies & (r^2)^2 &= a^2\left(3(r\cos\theta)^2-(r\sin\theta)^2\right) \\
&\implies & r^4 &= a^2r^2\left(3\cos^2\theta-\sin^2\theta\right) \\
&\implies & r^2 &= a^2\left(3\cos^2\theta-\sin^2\theta\right) \\
& & &= a^2\left(2\cos^2\theta+(1-\sin^2\theta)-\sin^2\theta\right) \\
& & &= a^2\left(1+2\cos^2\theta-2\sin^2\theta\right) \\
& & &= a^2\left(1+2\cos 2\theta\right)
\end{aligned}$$

Part (b)

For tangents at the pole, $r = 0$.

$$\begin{aligned}
& & r &= 0 \\
&\implies & a^2\left(1+2\cos 2\theta\right) &= 0 \\
&\implies & 1+2\cos 2\theta &= 0 \\
&\implies & \cos 2\theta &= -\frac{1}{2}
\end{aligned}$$

Note that $0 \leq 2\theta \leq 2\pi$. Hence, $2\theta = \frac{2}{3}\pi, \frac{4}{3}\pi$, whence $\theta = \frac{1}{3}\pi, \frac{2}{3}\pi$. For full lines, we also have $\theta = \frac{2}{3}\pi$ and $\theta = \frac{5}{3}\pi$.

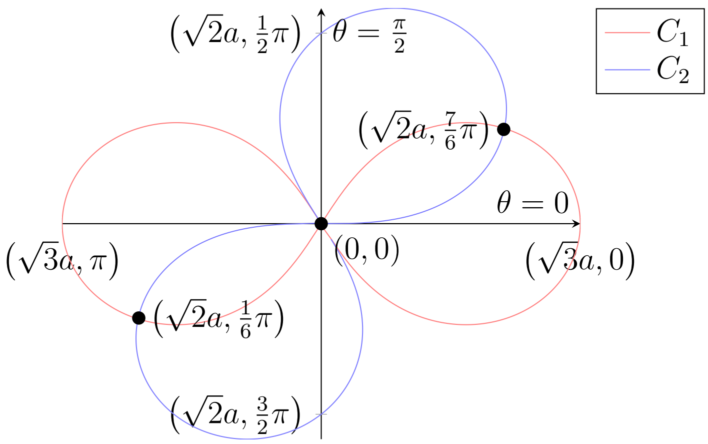
$$\boxed{\theta = \frac{1}{3}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi, \frac{5}{3}\pi}$$

Part (c)

$$\begin{aligned} r^2 &= a^2\left(1 + 2\cos\left(2\left(\theta - \frac{\pi}{3}\right)\right)\right) \\ &= a^2\left(1 + 2\cos\left(2\theta - \frac{2}{3}\pi\right)\right) \end{aligned}$$

$$\boxed{r^2 = a^2\left(1 + 2\cos\left(2\theta - \frac{2}{3}\pi\right)\right)}$$

Part (d)



Consider the horizontal intercepts of C_1. When $\theta = 0$,

$$\begin{aligned} r^2 &= a^2(1 + 2\cos 0) \\ &= 3a^2 \\ \implies r &= \sqrt{3}a \end{aligned}$$

Hence, C_1 intercepts the horizontal axis at $\left(\sqrt{3}a, 0\right)$. Due to symmetry, there is another intercept at $\left(\sqrt{3}a, \pi\right)$.

Consider the vertical intercepts of C_2. When $\theta = \frac{\pi}{2}$,

$$\begin{aligned} r^2 &= a^2\left(1 + 2\cos\left(2 \cdot \frac{\pi}{2} - \frac{2}{3}\pi\right)\right) \\ &= 2a^2 \\ \implies r &= \sqrt{2}a \end{aligned}$$

Hence, C_2 intercepts the horizontal axis at $\left(\sqrt{2}a, \frac{\pi}{2}\right)$. Due to symmetry, there is another intercept at $\left(\sqrt{2}a, \frac{3}{2}\pi\right)$.

Now consider the intersections between C_1 and C_2.

$$\begin{aligned}
& C_1 = C_2 \\
\implies a^2 (1 + 2\cos 2\theta) &= a^2 \left(1 + 2\cos\left(2\theta - \frac{2}{3}\pi\right)\right) \\
\implies \cos 2\theta &= \cos\left(2\theta - \frac{2}{3}\pi\right) \\
\implies \cos 2\theta &= \cos 2\theta \cos \frac{2}{3}\pi + \sin 2\theta \sin \frac{2}{3}\pi \\
\implies \cos 2\theta &= \cos 2\theta \cdot -\frac{1}{2} + \sin 2\theta \cdot \frac{\sqrt{3}}{2} \\
\implies 1 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}\tan 2\theta \\
\implies \tan 2\theta &= \sqrt{3}
\end{aligned}$$

Hence, $2\theta = \frac{1}{3}\pi, \frac{4}{3}\pi, \frac{7}{3}\pi, \frac{10}{3}\pi$, whence $\theta = \frac{1}{6}\pi, \frac{2}{3}\pi, \frac{1}{6}\pi + \pi, \frac{2}{3}\pi + \pi$. Due to symmetry, we only consider the cases when $\theta = \frac{1}{6}\pi$ and $\theta = \frac{2}{3}\pi$.

Case 1: $\theta = \frac{1}{6}\pi \implies r^2 = a^2\left(1 + 2\cos 2 \cdot \frac{1}{6}\pi\right) = 2a^2 \implies r = \sqrt{2}a$. Hence, C_1 and C_2 intersect at $\left(\sqrt{2}a, \frac{1}{6}\pi\right)$. Due to symmetry, there is another intersection point at $\left(\sqrt{2}a, \frac{7}{6}\pi\right)$.

Case 2: $\theta = \frac{2}{3}\pi \implies r^2 = a^2\left(1 + 2\cos 2 \cdot \frac{2}{3}\pi\right) = 0 \implies r = 0$. This is a trivial case.

Part (e)

Consider the map $\theta \mapsto \theta + \frac{2}{3}\pi$.

$$\begin{aligned}
r^2 &= a^2\left(1 + 2\cos\left(2\left(\theta + \frac{2}{3}\pi\right) - \frac{2}{3}\pi\right)\right) \\
&= a^2\left(1 + 2\cos\left(2\theta + \frac{2}{3}\pi\right)\right)
\end{aligned}$$

$$\boxed{r^2 = a^2\left(1 + 2\cos\left(2\theta + \frac{2}{3}\pi\right)\right)}$$

Part (f)

$$\boxed{r_1 = \sqrt{3}a}$$