

Problem 1.

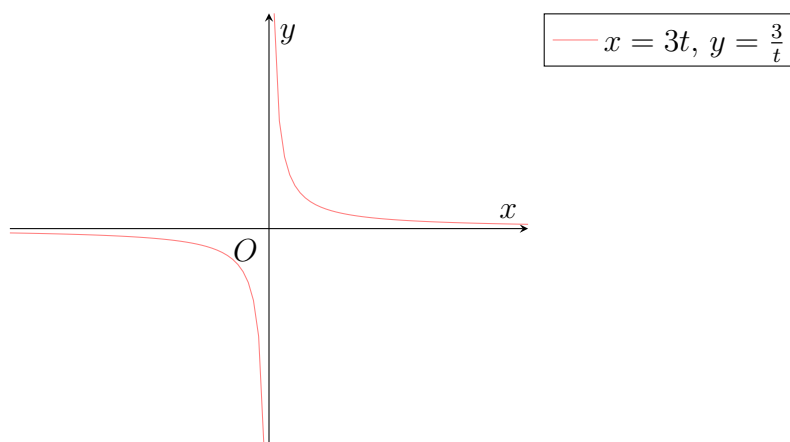
Sketch the curve with parametric equations

$$x = 3t, y = \frac{3}{t}$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q .

(a) Show that the normal at P has equation $2y = 8x - 45$.

(b) Find the value of t at Q .

Solution**Part (a)**

Consider $\frac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt} \right)^{-1} \\ &= \left(-\frac{3}{t^2} \right) \cdot \frac{1}{3} \\ &= -\frac{1}{t^2} \end{aligned}$$

Hence, the tangent to the curve has gradient $-\frac{1}{t^2}$, whence the normal to the curve has gradient $\frac{-1}{-\frac{1}{t^2}} = t^2$. Thus, the normal to the curve at P has gradient $2^2 = 4$. Note that P has coordinates $\left(3 \cdot 2, \frac{3}{2} \right) = \left(6, \frac{3}{2} \right)$. Using the point-slope formula,

$$\begin{aligned}
y - \frac{3}{2} &= 4(x - 6) \\
\implies y - \frac{3}{2} &= 4x - 24 \\
\implies y &= 4x - 24 + \frac{3}{2} \\
\implies 2y &= 8x - 48 + 3 \\
&= 8x - 45
\end{aligned}$$

Thus, the normal at P has equation $2y = 8x - 45$.

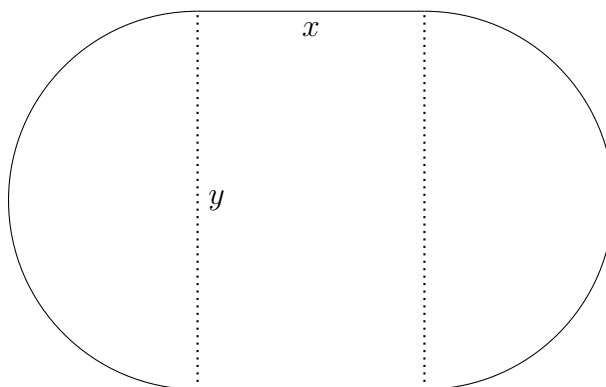
Part (b)

Observe that $x = 3t \implies t = \frac{x}{3} \implies y = \frac{3}{\frac{x}{3}} = \frac{9}{x}$. Substituting $y = \frac{9}{x}$ into the equation of the normal at P ,

$$\begin{aligned}
2 \cdot \frac{9}{x} &= 8x - 45 \\
\implies \frac{18}{x} &= 8x - 45 \\
\implies 18 &= 8x^2 - 45x \\
\implies 8x^2 - 45x - 18 &= 0 \\
\implies (x - 6)(8x + 3) &= 0
\end{aligned}$$

Hence $x = -\frac{3}{8}$ at Q . Note that we reject $x = 6$ since $x = 6$ at P . Thus, $t = \frac{-\frac{3}{8}}{3} = -\frac{1}{8}$ at Q .

$$t = -\frac{1}{8}$$

Problem 2.

A pond with a constant depth of 1 m is being designed for a park. The pond comprises a rectangle x m by y m and two semicircles of diameter y m, as shown in the diagram. The cost to build a boundary around the pond is \$30 per metre for straight parts and \$60 per metre for the semicircular parts. Given that the budget for the boundary is fixed at \$6000, using differentiation or otherwise, find in terms of π , the exact values of x and y which give the pond a maximum volume.

Solution

Observe that the total length of the straight parts is $2x$ m and the total length of the semicircular parts is $2 \cdot \frac{1}{2}\pi y = \pi y$ m. Hence,

$$\begin{aligned}
 30 \cdot 2x + 60 \cdot \pi y &= 6000 \\
 \implies 60x + 60\pi y &= 6000 \\
 \implies x + \pi y &= 100 \\
 \implies x &= 100 - \pi y
 \end{aligned}$$

Let $V(y)$ m³ be the volume of the pond.

$$\begin{aligned}
 V(y) &= 1 \cdot \left(\pi \left(\frac{y}{2} \right)^2 + xy \right) \\
 &= \frac{\pi}{4}y^2 + xy \\
 &= \frac{\pi}{4}y^2 + (100 - \pi y)y \\
 &= \frac{\pi}{4}y^2 + 100y - \pi y^2 \\
 &= -\frac{3}{4}\pi y^2 + 100y
 \end{aligned}$$

Consider the stationary points of $V(y)$. For stationary points, $V'(y) = 0$.

$$\begin{aligned}
 V'(y) &= 0 \\
 \implies -\frac{3}{4}\pi \cdot 2y + 100 &= 0 \\
 \implies y &= \frac{200}{3}\pi
 \end{aligned}$$

y	$\left(\frac{200}{3}\pi\right)^{-}$	$\frac{200}{3}\pi$	$\left(\frac{200}{3}\pi\right)^{+}$
$V'(y)$	+ve	0	-ve

By the First Derivative Test, the maximum volume of the pond is achieved when $y = \frac{200}{3}\pi$. Thus, $x = 100 - \pi y = \frac{100}{3}$.

$$x = \frac{100}{3}, y = \frac{200}{3}\pi$$

Problem 3.

A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm². Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm²s⁻¹, find the rate of increase at this instant of

- (a) the length of the cylinder,
- (b) the volume of the cylinder,
- (c) the radius of the cylinder.

Solution

Let $A = x$ cm² be the cross-sectional area of the cylinder. Then $\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = \frac{dx}{dt}$,
whence $\left. \frac{dA}{dt} \right|_{x=5} = \left. \frac{dx}{dt} \right|_{x=5} = 0.025$.

Part (a)

Let $L = 20x$ cm be the length of the cylinder. Then $\frac{dL}{dt} = \frac{dL}{dx} \cdot \frac{dx}{dt} = 20 \cdot \frac{dx}{dt}$. Hence,
 $\left. \frac{dL}{dt} \right|_{x=5} = \left(20 \cdot \frac{dx}{dt} \right) \Big|_{x=5} = 20 \cdot 0.025 = 0.5$.

The length of the cylinder is increasing at a rate of 0.5 cm/s.

Part (b)

Let $V = AL = 20x^2$ cm³ be the volume of the cylinder. Then $\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = 40x \cdot \frac{dx}{dt}$.
Hence, $\left. \frac{dV}{dt} \right|_{x=5} = \left(40x \cdot \frac{dx}{dt} \right) \Big|_{x=5} = 40 \cdot 5 \cdot 0.025 = 5$.

The volume of the cylinder is increasing at a rate of 5 cm³/s.

Part (c)

Let R cm be the radius of the cylinder. Since $\pi R^2 = A = x$, we have $R = \sqrt{\frac{x}{\pi}} = \frac{\sqrt{x}}{\sqrt{\pi}}$.
Hence, $\frac{dR}{dt} = \frac{dR}{dx} \cdot \frac{dx}{dt} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt}$. Thus, $\left. \frac{dR}{dt} \right|_{x=5} = \left(\frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{dx}{dt} \right) \Big|_{x=5} =$
 $\frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{5}} \cdot 0.025 = 0.00315$ (3 s.f.).

The radius of the cylinder is increasing at a rate of 0.00315 cm/s.

Problem 4.

The curve C has equation $2^{-y} = x$. The point A on C has x -coordinate a where $a > 0$. Show that $\frac{dy}{dx} = -\frac{1}{a \ln 2}$ at A and find the equation of the tangent to C at A . Hence find the equation of the tangent to C which passes through the origin.

The straight line $y = mx$ intersects C at 2 distinct points. Write down the range of values of m .

Solution

Implicitly differentiating the given equation,

$$\begin{aligned} 2^{-y} \cdot \ln 2 \cdot -y' &= 1 \\ \implies x \cdot \ln 2 \cdot -y' &= 1 \\ \implies y' &= -\frac{1}{x \ln 2} \end{aligned}$$

At A , $x = a$. Hence, $\frac{dy}{dx} = -\frac{1}{a \ln 2}$.

Note that $2^{-y} = x \implies y = -\log_2 x$. Hence, A has coordinates $(a, -\log_2 a)$. Using the point-slope formula, the tangent to C at A has equation

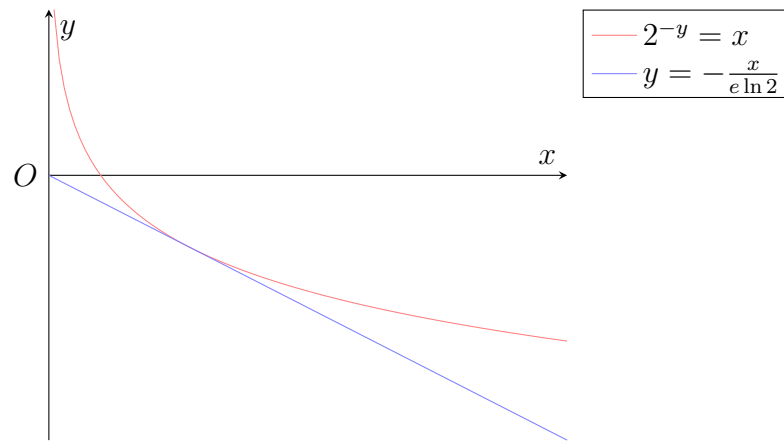
$$\begin{aligned} y - (-\log_2 a) &= -\frac{1}{a \ln 2}(x - a) \\ \implies y &= -\frac{x}{a \ln 2} + \frac{1}{\ln 2} - \log_2 a \\ &= -\frac{x}{a \ln 2} + \frac{1}{\ln 2} - \frac{\ln a}{\ln 2} \\ &= -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \end{aligned}$$

The tangent to C at A has equation $y = -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2}$.

Consider the straight line $y = mx$ that is tangent to C and passes through the origin.

$$\begin{aligned} 0 &= -\frac{0}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \\ \implies 1 - \ln a &= 0 \\ \implies a &= e \end{aligned}$$

Hence, the equation of the tangent to C that passes through the origin is $y = -\frac{x}{e \ln 2}$.
Consider the graph of $2^{-y} = x$.



Hence, m must be strictly between $-\frac{1}{e \ln 2}$ and 0.

$$m \in \left(-\frac{1}{e \ln 2}, 0\right)$$