Problem 1.

Without using a calculator, sketch the following graphs of conics.

(a) $y^2 - 4x = 12$

(b) $(x+1)^2 + y^2 = 4$

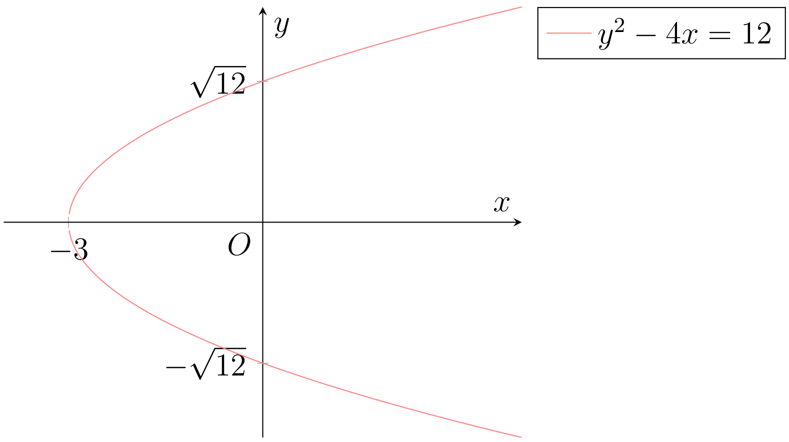
(c) $\dfrac{(x-3)^2}{9} + \dfrac{y^2}{2} = 1$

(d) $4x^2 + y^2 = 4$

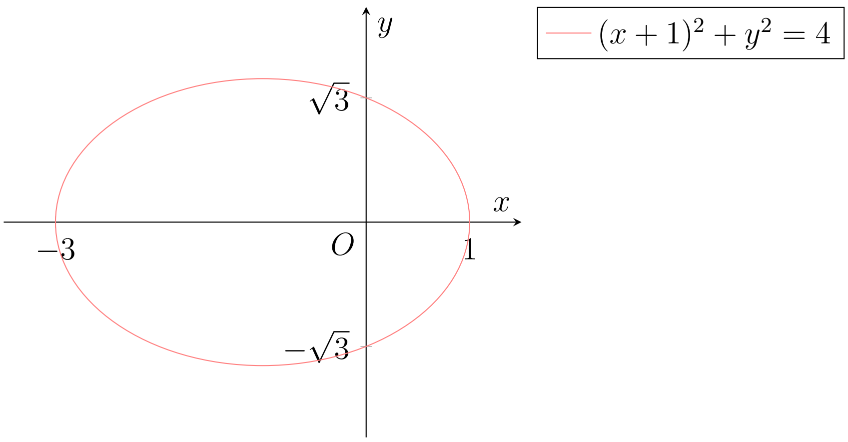
(e) $8y^2 - 2x^2 = 16$

Solution

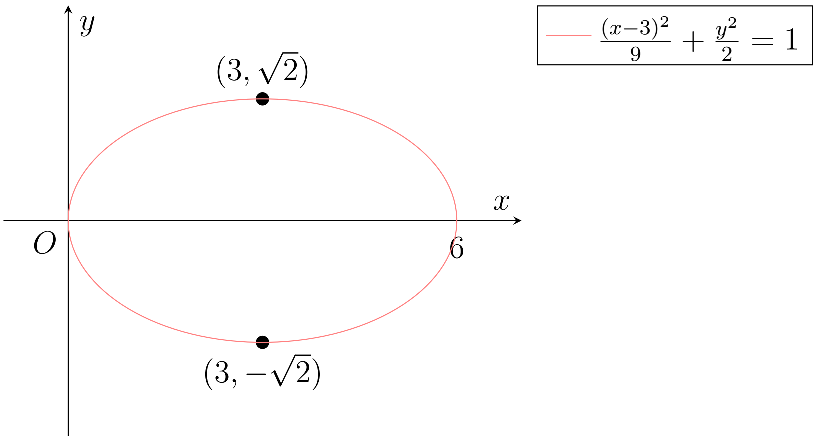
Part (a)



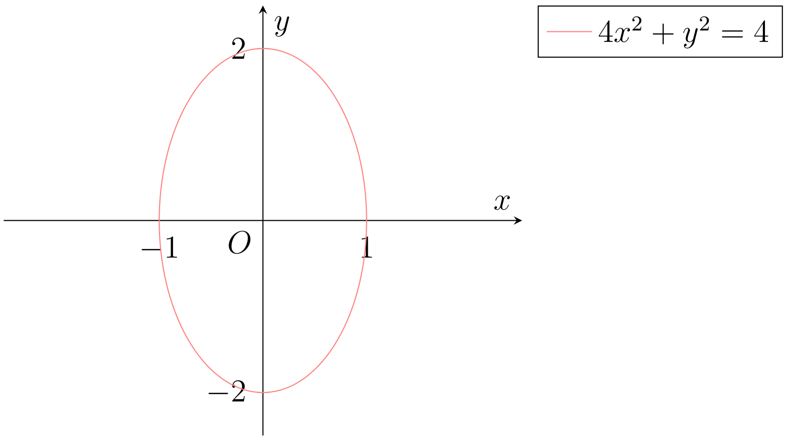
Part (b)



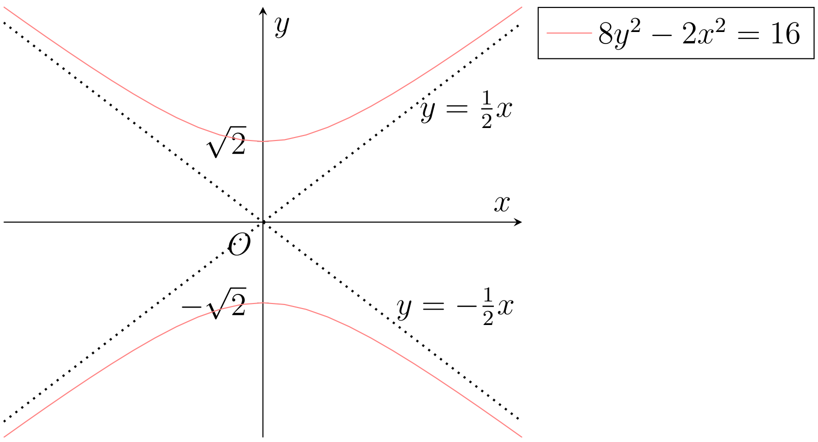
Part (c)



Part (d)



Part (e)



Problem 2.

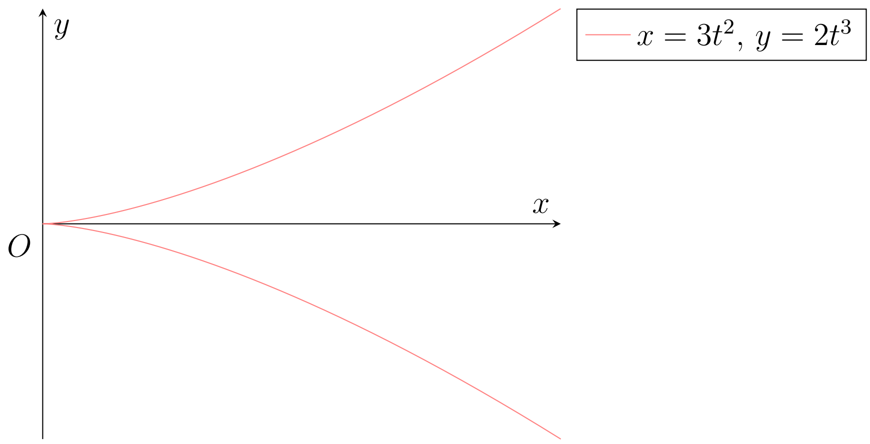
Sketch the curves defined by the following parametric equations, indicating the coordinates of any intersection with the axes.

(a) $x = 3t^2$, $y = 2t^3$

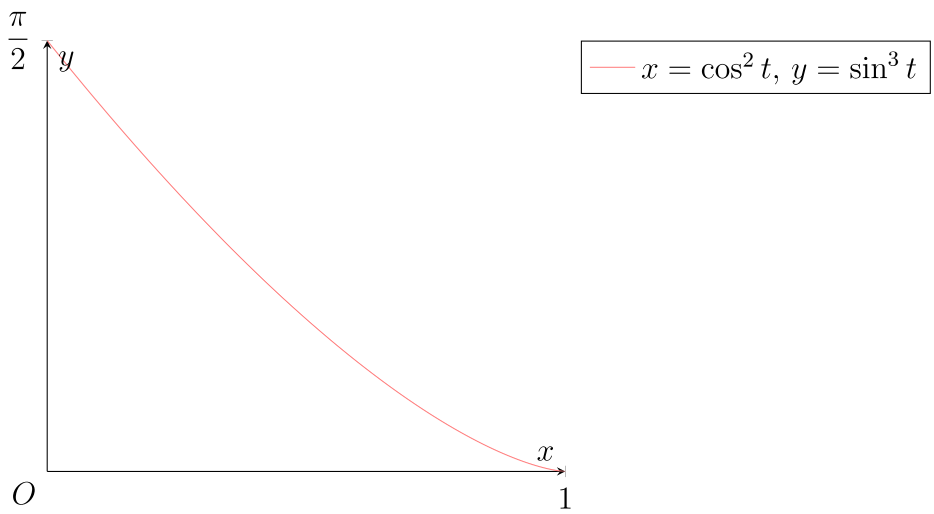
(b) $x = \cos^2 t$, $y = \sin^3 t$, $0 \le t \le \frac{\pi}{2}$

Solution

Part (a)



Part (b)



Problem 3.

Without using a calculator, sketch the following graphs of conics.

(a) $y^2 + 4y + x = 0$

(b) $x^2 + y^2 - 4x - 4y = 0$

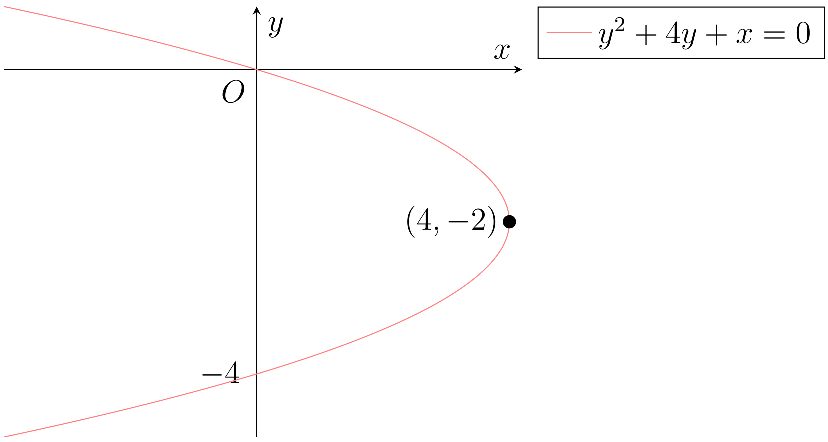
(c) $x^2 + 4y^2 - 2x - 24y + 33 = 0$

(d) $4x^2 - y^2 - 8x + 4y = 1$

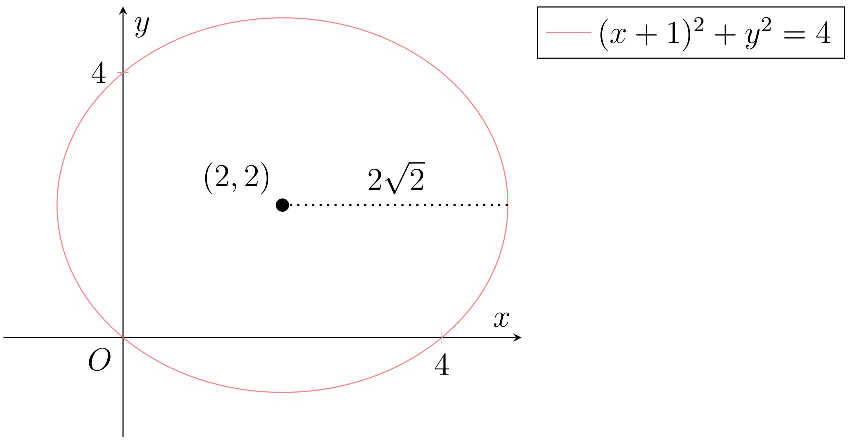
(e) $x = -\sqrt{17 - y^2}$

Solution

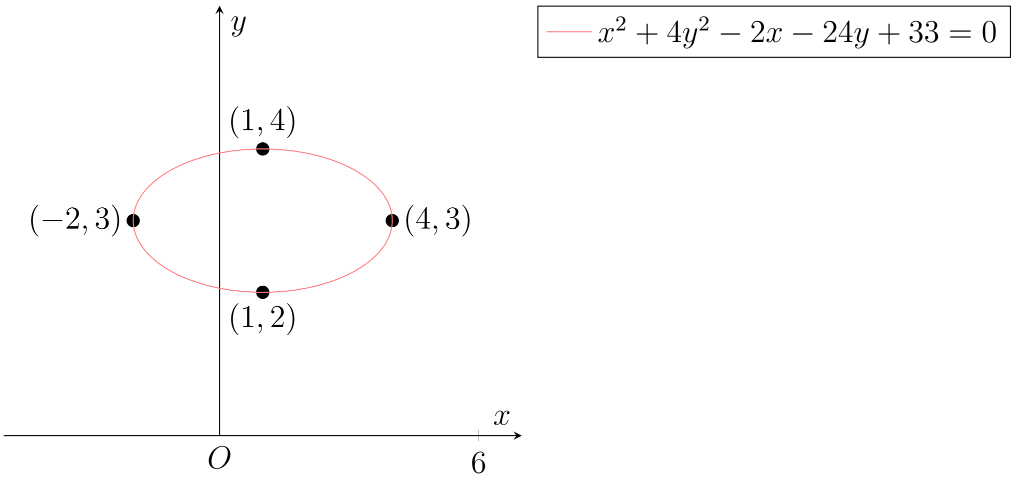
Part (a)



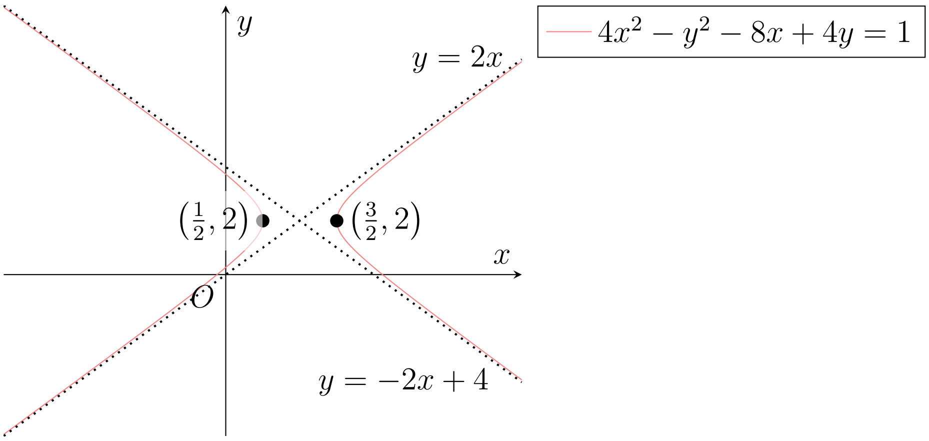
Part (b)



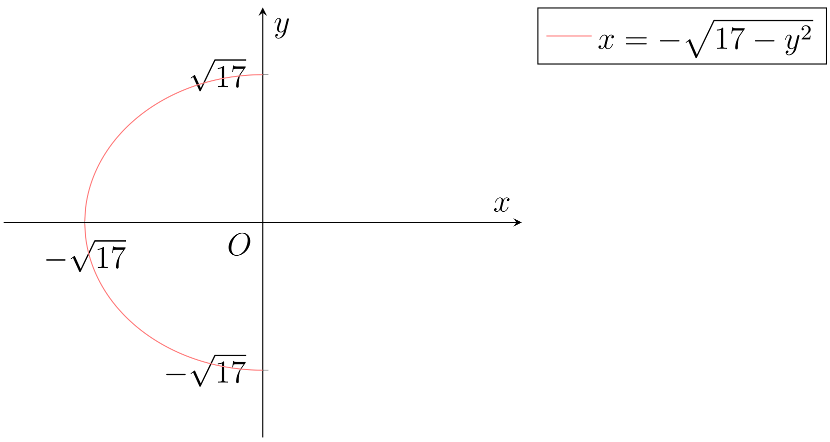
Part (c)



Part (d)



Part (e)



Problem 4.

Sketch the curves defined by the following parametric equations. Find also their respective Cartesian equations.

(a) $x = 4t + 3$, $y = 16t^2 - 9$, $t \in \mathbb{R}$

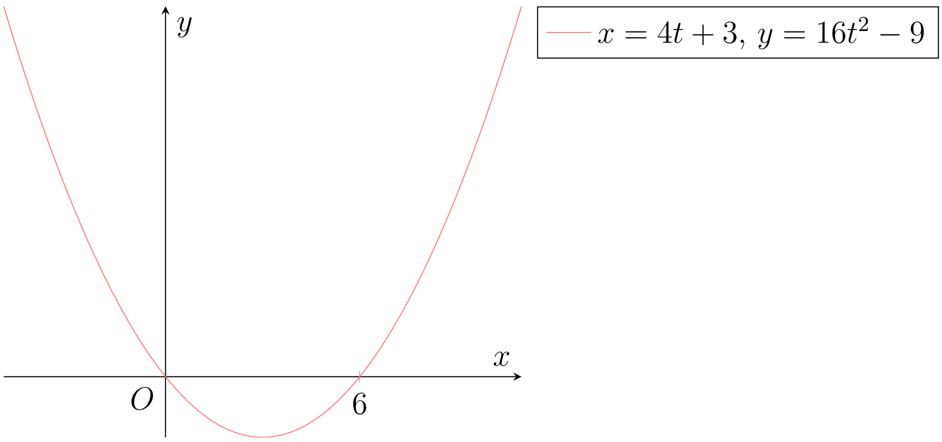
(b) $x = t^2$, $y = 2 \ln t$, $t \geq 1$

(c) $x = 1 + 2\cos\theta$, $y = 2\sin\theta - 1$, $0 \leq \theta \leq \dfrac{\pi}{2}$

(d) $x = t^2$, $y = \dfrac{2}{t}$, $t \neq 0$

Solution

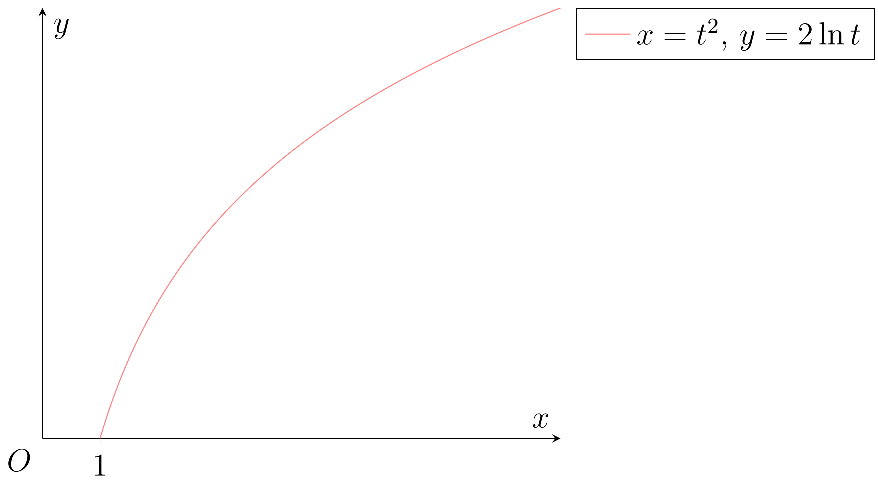
Part (a)



Since $x = 4t + 3$, we have $t = \dfrac{1}{4}(x - 3)$. Thus, $y = 16\left(\dfrac{1}{4}(x-3)\right)^2 - 9 = (x-3)^2 - 9$.

$$\boxed{y = (x-3)^2 - 9}$$

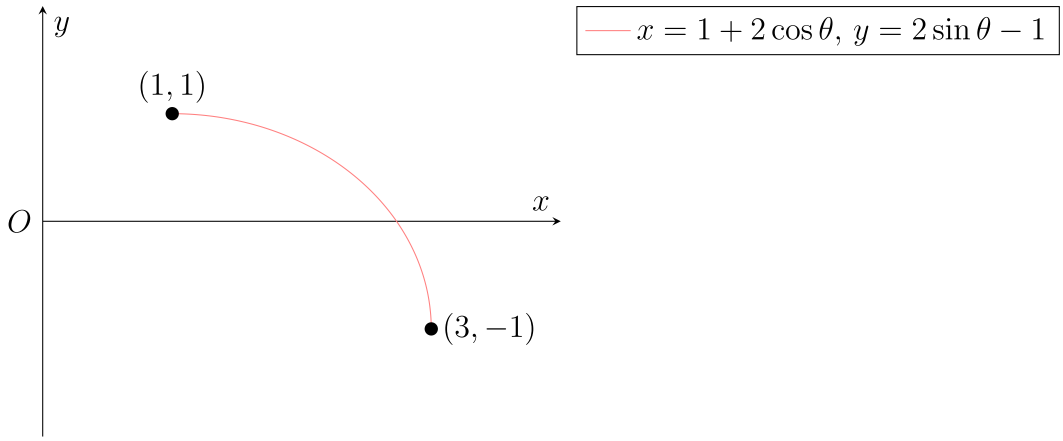
Part (b)



Since $x = t^2$ and $t \geq 1 > 0$, we have $t = \sqrt{x}$. Thus, $y = 2\ln(t) = 2\ln(\sqrt{x}) = \ln(x)$.

$$\boxed{y = \ln x}$$

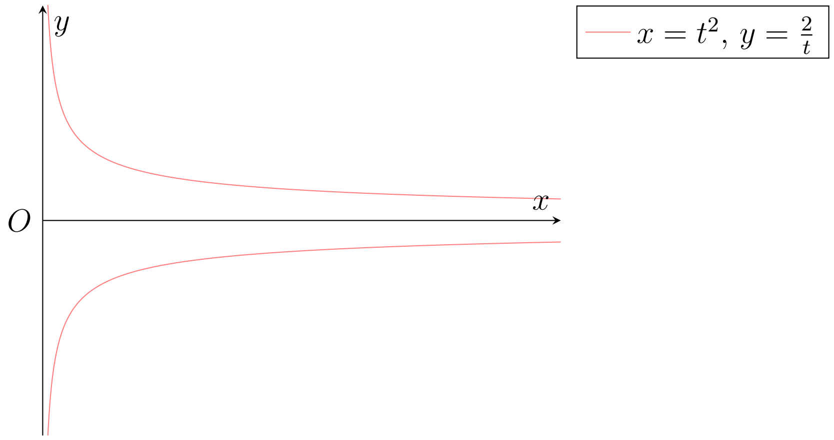
Part (c)



We have $2\cos\theta = x - 1$ and $2\sin\theta = y + 1$. Squaring both equations and adding them, we obtain $4\cos^2\theta + 4\sin^2\theta = (x-1)^2 + (y+1)^2$, which simplifies to $(x-1)^2 + (y+1)^2 = 4$.

$$\boxed{(x-1)^2 + (y+1)^2 = 4}$$

Part (d)



Since $x = t^2$, we have $t = \pm\sqrt{x}$. Hence, $y = \pm\dfrac{2}{\sqrt{x}}$.

$$\boxed{y = \pm\frac{2}{\sqrt{x}}}$$

Problem 5.

The curve C_1 has equation $y = \dfrac{x-2}{x+2}$. The curve C_2 has equation $\dfrac{x^2}{6} + \dfrac{y^2}{3} = 1$.

(a) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersections with the axes and the equations of any asymptotes.

(b) Show algebraically that the x-coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2 = (x+2)^2\,(6-x^2)$.

(c) Use your calculator to find these x-coordinates.

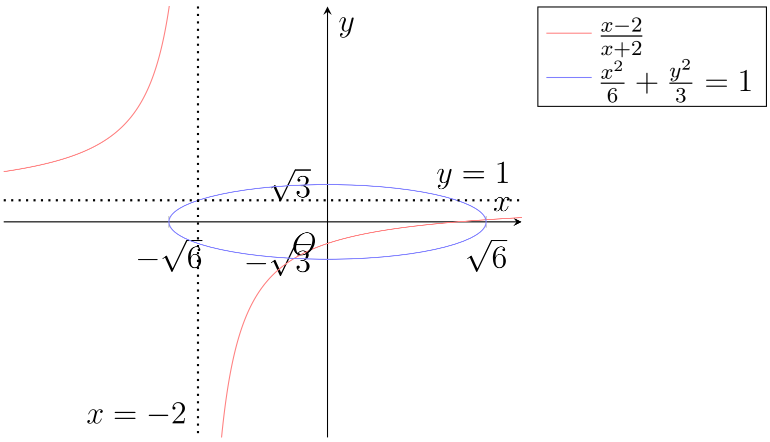
Another curve is defined parametrically by

$$x = \sqrt{6}\cos\theta,\ y = \sqrt{3}\sin\theta,\ -\pi \leq \theta \leq \pi$$

(d) Find the Cartesian equation of this curve and hence determine the number of roots to the equation $\sqrt{3}\sin\theta = \dfrac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}$ for $-\pi \leq \theta \leq \pi$.

Solution

Part (a)



Part (b)

From C_1, we have $y(x+2) = x-2$. Hence,

$$y^2(x+2)^2 = (x-2)^2$$

From C_2, we have $x^2 + 2y^2 = 6$. Hence,

$$y^2 = \frac{6-x^2}{2}$$

Putting both equations together, we have

$$\begin{aligned} (x-2)^2 &= \frac{(6-x^2)\,(x+2)^2}{2} \\ \implies 2(x-2)^2 &= \left(6-x^2\right)(x+2)^2 \end{aligned}$$

Part (c)

$$\boxed{x = -0.515 \vee x = 2.45}$$

Part (d)

Since $x = \sqrt{6}\cos\theta$ and $y = \sqrt{3}\sin\theta$, we have $x^2 = 6\cos^2\theta$ and $2y^2 = 6\sin^2\theta$. Adding both equations together, we have

$$\begin{aligned} x^2 + 2y^2 &= 6\cos^2\theta + 6\sin^2\theta \\ &= 6 \\ \implies \frac{x^2}{6} + \frac{y^2}{3} &= 1 \end{aligned}$$

$$\boxed{\frac{x^2}{6} + \frac{y^2}{3} = 1}$$

This is the equation that gives C_1. We further observe that the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}$ simplifies to $y = \frac{x-2}{x+2}$. This is the equation that gives C_2. Since there are two intersections between C_1 and C_2, there are thus two roots to the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}$.

$$\boxed{\text{There are 2 roots to } \sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}.}$$