Problem 1.

Two biological cultures, X and Y, react with each other, and their volumes at time t are x and y respectively, in appropriate units. Their rates of growth are modelled by the simultaneous equations

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= (2-x)y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{y^2}{x}\end{aligned}$$

When $t=0$, $x=y=1$.

(a) Show that $x = \dfrac{2y^2}{1+y^2}$.

(b) Find and simplify expressions for y and x in terms of t.

(c) Sketch the graph of y against x for $0 < t < \dfrac{\pi}{2}$.

Solution

Part (a)

Note that $x, y > 0$ since they represent volume. Also, for $x \in (0,2)$, we have $\dfrac{\mathrm{d}x}{\mathrm{d}t} = (2-x)y > 0$. When $x=2$, we have $\dfrac{\mathrm{d}x}{\mathrm{d}t} = 0$. Hence, $0 < x \leq 2$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}\\
&= \frac{y^2/x}{(2-x)y}\\
&= \frac{y}{x(2-x)}\\
\implies \qquad \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x(2-x)}\\
\implies \int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int \frac{1}{x(2-x)}\,\mathrm{d}x\\
\implies \qquad \int \frac{1}{y}\,\mathrm{d}y &= \int \frac{1}{x(2-x)}\,\mathrm{d}x\\
&= \frac{1}{2}\int\left(\frac{1}{x} + \frac{1}{2-x}\right)\mathrm{d}x\\
\implies \qquad \ln y &= \frac{1}{2}\left[\ln x - \ln(2-x)\right] + C_1\\
&= \ln\sqrt{\frac{x}{2-x}} + C_1\\
\implies \qquad y &= C_2\sqrt{\frac{x}{2-x}}
\end{aligned}$$

At $t = 0$, $x = y = 1$. Hence, $1 = C_2\sqrt{\dfrac{1}{2-1}} \implies C_2 = 1$.

$$\begin{aligned}
&& y &= \sqrt{\frac{x}{2-x}} \\
&\implies & y^2 &= \frac{x}{2-x} \\
&\implies & (2-x)y^2 &= x \\
&\implies & 2y^2 - xy^2 &= x \\
&\implies & x + xy^2 &= 2y^2 \\
&\implies & x(1+y^2) &= 2y^2 \\
&\implies & x &= \frac{2y^2}{1+y^2}
\end{aligned}$$

Part (b)

$$\begin{aligned}
&& \frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{y^2}{x} \\
&& &= \frac{y^2}{2y^2/(1+y^2)} \\
&& &= \frac{1}{2}(1+y^2) \\
&\implies & \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{1}{2} \\
&\implies & \int \frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t &= \int \frac{1}{2}\,\mathrm{d}t \\
&\implies & \int \frac{1}{1+y^2}\,\mathrm{d}y &= \int \frac{1}{2}\,\mathrm{d}t \\
&\implies & \arctan y &= \frac{1}{2}t + C \\
&\implies & y &= \tan\left(\frac{1}{2}t + C\right)
\end{aligned}$$

At $t = 0$, $y = 1$. Hence, $1 = \tan C \implies C = \dfrac{\pi}{4}$.

$$\begin{aligned}
y &= \tan\left(\frac{1}{2}t + \frac{\pi}{4}\right) \\
&= \frac{1 - \cos(t + \pi/2)}{\sin(t + \pi/2)} \\
&= \frac{1 + \sin t}{\cos t} \\
&= \sec t + \tan t
\end{aligned}$$

$$\boxed{y = \sec t + \tan t}$$

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= (2-x)y \\
&= (2-x)\sqrt{\frac{x}{2-x}} \\
&= \sqrt{x(2-x)} \\
\implies \quad \frac{1}{\sqrt{x(2-x)}}\frac{\mathrm{d}x}{\mathrm{d}t} &= 1 \\
\implies \int \frac{1}{\sqrt{x(2-x)}}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= \int \mathrm{d}t \\
\implies \quad \int \frac{1}{\sqrt{x(2-x)}}\,\mathrm{d}x &= \int \mathrm{d}t \\
\implies \quad \int \frac{1}{\sqrt{x}\sqrt{2-x}}\,\mathrm{d}x &= t + C_1 \\
\implies \quad \int \frac{2u}{u\sqrt{2-u^2}}\,\mathrm{d}u &= t + C_1 \\
\implies \quad 2\int \frac{1}{\sqrt{2-u^2}}\,\mathrm{d}u &= t + C_1 \\
\implies \quad 2\arcsin\left(\frac{u}{\sqrt{2}}\right) &= t + C_1 \\
\implies \quad 2\arcsin\left(\sqrt{\frac{x}{2}}\right) &= t + C_1 \\
\implies \quad \arcsin\left(\sqrt{\frac{x}{2}}\right) &= \frac{1}{2}t + C_2 \\
\implies \quad \sqrt{\frac{x}{2}} &= \sin\left(\frac{1}{2}t + C_2\right) \\
\implies \quad \frac{x}{2} &= \sin^2\left(\frac{1}{2}t + C_2\right) \\
\implies \quad x &= 2\sin^2\left(\frac{1}{2}t + C_2\right)
\end{aligned}
$$

$$
\begin{aligned}
u &= \sqrt{x} \\
\mathrm{d}x &= 2u\,\mathrm{d}u
\end{aligned}
$$

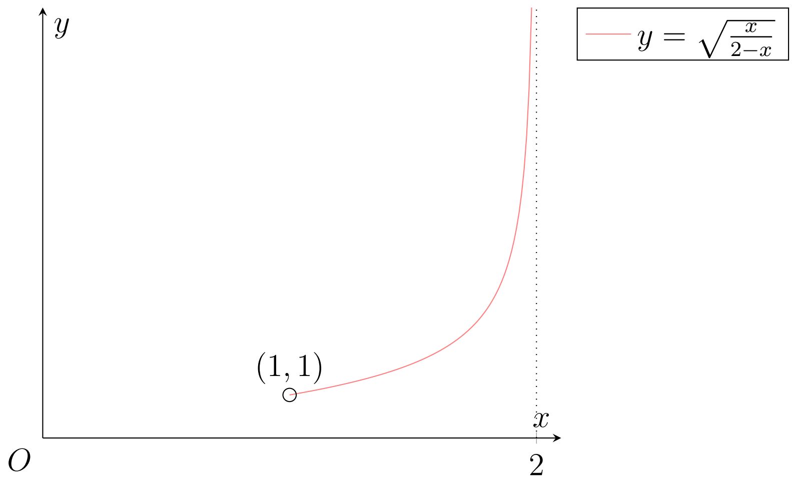
At $t = 0$, $x = 1$. Hence, $1 = 2\sin C_2$, whence $C_2 = \frac{\pi}{4}$.

$$
\begin{aligned}
x &= 2\sin^2\left(\frac{1}{2}t + \frac{\pi}{4}\right) \\
&= 1 - \cos\left(t + \frac{\pi}{2}\right) \\
&= 1 + \sin t
\end{aligned}
$$

$$\boxed{x = 1 + \sin t}$$

Part (c)

Note that $0 < t < \frac{\pi}{2} \implies 1 < x < 2$.


y
$y = \sqrt{\frac{x}{2-x}}$
$(1, 1)$
x
O
2

Problem 2.

Find the general solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 10x = 0,$$

Find the particular solution such that $y \to 0$ as $x \to 0$.

Show, on a single diagram, a sketch of this particular solution and one typical member of the family, F of solution curves for which $\frac{\mathrm{d}y}{\mathrm{d}x}$ is positive whenever x is positive.

Show that there is a straight line which passes through the maximum point of every member of F and find its equation.

Solution

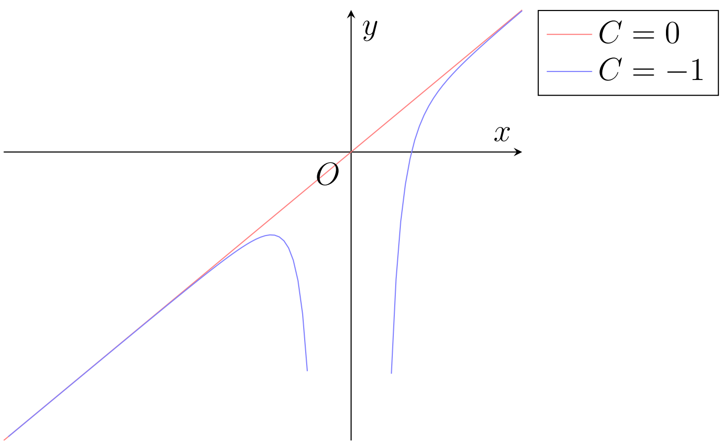
$$\begin{aligned}
& x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 10x = 0 \\
\implies \quad & x^4\frac{\mathrm{d}y}{\mathrm{d}x} + 4x^3y = 10x^4 \\
\implies \quad & \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4y\right) = 10x^4 \\
\implies \quad & x^4y = \int 10x^4 \,\mathrm{d}x \\
\implies \quad & x^4y = 2x^5 + C \\
\implies \quad & y = 2x + Cx^{-4}
\end{aligned}$$

$$\boxed{y = 2x + Cx^{-4}}$$

As $x \to 0$, $x^{-4} \to \infty$. Hence, C must be 0.

$$\boxed{y = 2x}$$

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 4Cx^{-5} > 0 \implies C < \frac{1}{2}x^5$. Since $x > 0$, we hence have the constraint $C \leq 0$ for members of F.



Consider the stationary points of members of F. For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Hence,

$$\begin{aligned} & x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 10x = 0 \\ \implies \quad & 4y - 10x = 0 \\ \implies \quad & y = \frac{5}{2}x \end{aligned}$$

Differentiating the original differential equation with respect to x, we obtain

$$\begin{aligned} & x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 10x = 0 \\ \implies \quad & x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}y}{\mathrm{d}x} - 10 = 0 \\ \implies \quad & x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 10 \\ \implies \quad & \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{10}{x} \end{aligned}$$

Note that for members of F, we have that $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for $x > 0$. Hence, there are no stationary points when $x > 0$. That is, any stationary point must occur when $x < 0$. (Indeed, there is a stationary point when $x = \sqrt[5]{2C} < 0$.) Furthermore, when $x < 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$. Hence, all stationary points must be a maximum point. Thus, $\boxed{y = \frac{5}{2}x}$ passes through the maximum point of every member of F.

Problem 3.

(a) The variables x and y are related by the differential equation

$$x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy + y = 0.$$

(i) Find the general solution of this differential equation, expressing y in terms of x.

(ii) Find the particular solution for which $y = -e$ when $x = 1$. Obtain the coordinates of the turning point of the solution curve of this particular solution and sketch the curve for $x > 0$.

(b) Find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + xy = e^x x^2,$$

expressing y in terms of x.

Solution

Part (a)

Subpart (i)

$$\begin{aligned}
& & x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy + y &= 0 \\
&\implies & x^2\frac{\mathrm{d}y}{\mathrm{d}x} &= y(2x-1) \\
&\implies & \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2x-1}{x^2} \\
& & &= \frac{2}{x} - \frac{1}{x^2} \\
&\implies & \int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int \left(\frac{2}{x} - \frac{1}{x^2}\right)\mathrm{d}x \\
&\implies & \int \frac{1}{y}\,\mathrm{d}y &= \int \left(\frac{2}{x} - \frac{1}{x^2}\right)\mathrm{d}x \\
&\implies & \ln|y| &= 2\ln(x) + \frac{1}{x} + C_1 \\
&\implies & y &= C_2 x^2 e^{1/x}
\end{aligned}$$

$$\boxed{y = C_2 x^2 e^{1/x}}$$

Subpart (ii)

When $x = 1$, $y = -e$. Hence, $-e = C_2 \cdot 1^2 \cdot e^1 \implies C_2 = -1$.

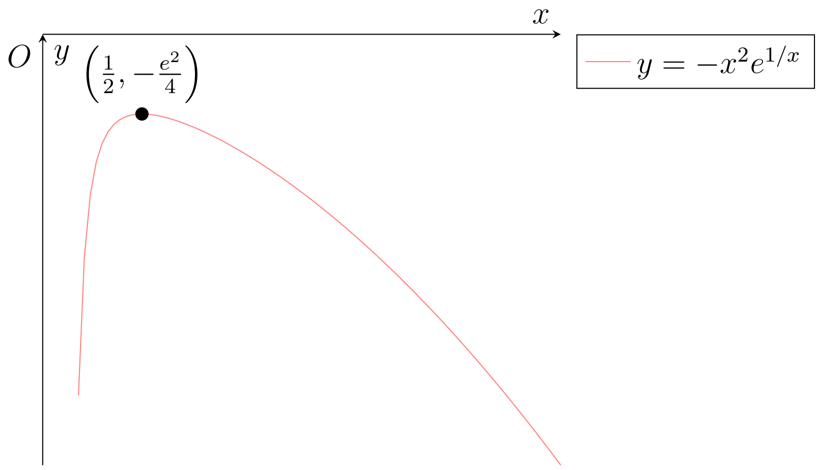
$$\boxed{y = -x^2 e^{1/x}}$$

For stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Hence, $y(2x-1) = 0$, whence $x = \frac{1}{2}$. Note that we reject $y = 0$ since $e^{1/x} \neq 0$ and $x \neq 0$ due to the presence of a $\frac{1}{x}$ term. Hence, y has a stationary point at $\left(\frac{1}{2}, -\frac{e^2}{4}\right)$.

Differentiating the original differential equation with respect to x, we obtain

$$\begin{aligned} x^2\frac{\mathrm{d}y}{\mathrm{d}x} - 2xy + y &= 0 \\ \implies x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies \qquad x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2y &= 0 \\ \implies \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{2y}{x^2} \end{aligned}$$

Hence, at $\left(\frac{1}{2}, -\frac{e^2}{4}\right)$, we have $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{-e^2/2}{1/4} < 0$. Thus, $\boxed{\left(\frac{1}{2}, -\frac{e^2}{4}\right)}$ is a maximum point and is thus a turning point.



Part (b)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + xy &= e^x x^2 \\ \implies e^{\frac{1}{2}x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + xe^{\frac{1}{2}x^2}y &= e^{\frac{1}{2}x^2+x}x^2 \\ \implies \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{\frac{1}{2}x^2}y\right) &= e^{\frac{1}{2}x^2+x}x^2 \\ \implies \qquad e^{\frac{1}{2}x^2}y &= \int e^{\frac{1}{2}x^2+x}x^2 \,\mathrm{d}x \end{aligned}$$

Suppose $\int e^{\frac{1}{2}x^2+x}x^2 \,\mathrm{d}x = P(x)e^{\frac{1}{2}x^2+x} + C$ for some function $P(x)$. Differentiating both sides with respect to x, we obtain

$$x^2e^{\frac{1}{2}x^2+x} = e^{\frac{1}{2}x^2+x}\left[(x+1)P(x) + P'(x)\right],$$

whence

$$x^2 = (x+1)P(x) + P'(x).$$

Thus, $P(x)$ is a polynomial of degree 1. Let $P(x) = ax + b$. For some constants a and b. Then

$$x^2 = ax^2 + (a+b)x + (a+b).$$

Comparing coefficients of x^2, x and constant terms, we have $a = 1$ and $a + b = 0 \implies b = -1$. Thus,

$$\int x^2 e^{\frac{1}{2}x^2+x} \, \mathrm{d}x = (x-1)e^{\frac{1}{2}x^2+x} + C.$$

Thus,

$$\boxed{y = (x-1)e^x + Ce^{-\frac{1}{2}x^2}}$$