Problem 1.

(a) Solve $z^4 = -4 - 4\sqrt{3}i$, expressing your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Sketch the roots on an Argand diagram.

(c) Hence, solve $w^4 = -1 + \sqrt{3}i$, expressing your answers in a similar form.

Solution

Part (a)

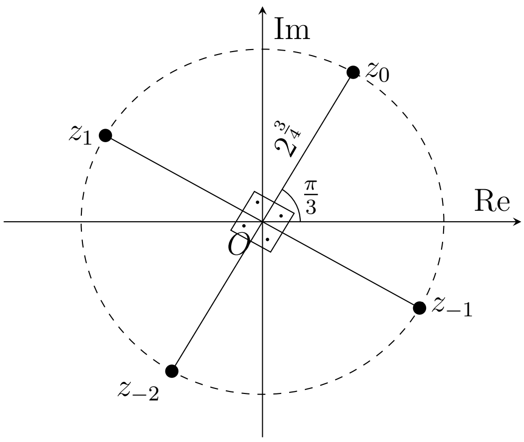
Observe that $-4 - 4\sqrt{3}i = 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 8\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) = 8e^{i\frac{4}{3}\pi} = 8e^{i\frac{4}{3}\pi + 2k\pi i}$ for all $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} z^4 &= 8e^{i\frac{4}{3}\pi + 2k\pi i} \\ \implies \quad z &= 8^{\frac{1}{4}} e^{i\frac{1}{3}\pi + \frac{1}{2}k\pi i} \\ &= 2^{\frac{3}{4}} e^{i\frac{2+3k}{6}\pi} \end{aligned}$$

Taking $k = -2, -1, 0, 1$:

$$\boxed{z_{-2} = 2^{\frac{3}{4}} e^{-i\frac{2}{3}\pi},\ z_{-1} = 2^{\frac{3}{4}} e^{-i\frac{1}{6}\pi},\ z_0 = 2^{\frac{3}{4}} e^{i\frac{1}{3}\pi},\ z_1 = 2^{\frac{3}{4}} e^{i\frac{5}{6}\pi}}$$

Part (b)



Part (c)

Observe that $w^4 = -1 + \sqrt{3}i = \frac{1}{4}(-4 + 4\sqrt{3}i) = \frac{1}{4}(z^4)^* = 2^{-2}(z^*)^4$. Hence, $w = 2^{-\frac{1}{2}}z^*$.

$$\boxed{w_{-2} = 2^{\frac{1}{4}} e^{i\frac{2}{3}\pi},\ w_{-1} = 2^{\frac{1}{4}} e^{i\frac{1}{6}\pi},\ w_0 = 2^{\frac{1}{4}} e^{-i\frac{1}{3}\pi},\ w_1 = 2^{\frac{1}{4}} e^{-i\frac{5}{6}\pi}}$$

Problem 2.

Let

$$C = 1 - \binom{2n}{1}\cos\theta + \binom{2n}{2}\cos 2\theta - \binom{2n}{3}\cos 3\theta + \ldots + \cos 2n\theta$$
$$S = -\binom{2n}{1}\sin\theta + \binom{2n}{2}\sin 2\theta - \binom{2n}{3}\sin 3\theta + \ldots + \sin 2n\theta$$

where n is a positive integer.

Show that $C = (-4)^n \cos(n\theta)\sin^{2n}(\theta/2)$, and find the corresponding expression for S.

Solution

$$\begin{aligned}
C + iS &= \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k\cos k\theta + \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k i\sin k\theta \\
&= \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k(\cos k\theta + i\sin k\theta) \\
&= \sum_{k=0}^{2n}\binom{2n}{k}(-1)^k e^{ik\theta} \\
&= \sum_{k=0}^{2n}\binom{2n}{k}(-e^{i\theta})^k \\
&= (1 - e^{i\theta})^{2n} \\
&= \left(e^{i\theta/2}\right)^{2n}\left(e^{-i\theta/2} - e^{i\theta/2}\right)^{2n} \\
&= \left(e^{i\theta/2}\right)^{2n}\left(e^{i\theta/2} - e^{-i\theta/2}\right)^{2n} \\
&= \left(e^{i\theta/2}\right)^{2n}(2i\sin(\theta/2))^{2n} \\
&= e^{in\theta}2^{2n}i^{2n}\sin^{2n}(\theta/2) \\
&= e^{in\theta}4^n(-1)^n\sin^{2n}(\theta/2) \\
&= (\cos n\theta + i\sin n\theta)(-4)^n\sin^{2n}(\theta/2)
\end{aligned}$$

Comparing real and imaginary parts, we have $C = (-4)^n\cos(n\theta)\sin^{2n}(\theta/2)$ and

$$\boxed{S = (-4)^n\sin(n\theta)\sin^{2n}(\theta/2)}$$

Problem 3.

Given that $z = \cos\theta + \imath \sin\theta$, show that

(a) $z - \dfrac{1}{z} = 2i\sin\theta$,

(b) $z^n + z^{-n} = 2\cos n\theta$.

Hence, show that

$$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$$

Find a similar expression for $\cos^6\theta$, and hence express $\cos^6\theta - \sin^6\theta$ in the form $a\cos 2\theta + b\cos 6\theta$.

Solution

Part (a)

$$\begin{aligned}
z - \frac{1}{z} &= z - z^{-1} \\
&= e^{i\theta} - e^{-i\theta} \\
&= (\cos\theta + i\sin\theta) - (\cos(-\theta) + i\sin(-\theta)) \\
&= (\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta) \\
&= 2i\sin\theta
\end{aligned}$$

Part (b)

$$\begin{aligned}
z^n + z^{-n} &= \left(e^{i\theta}\right)^n + \left(e^{i\theta}\right)^{-n} \\
&= e^{in\theta} + e^{-in\theta} \\
&= (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(n\theta)) \\
&= (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta) \\
&= 2\cos n\theta
\end{aligned}$$

$$\begin{aligned}
\sin^6\theta &= \frac{1}{(2i)^6}(2i\sin\theta)^6 \\
&= -\frac{1}{64}(z - z^{-1})^6 \\
&= -\frac{1}{64}\left[\binom{6}{0}z^6 - \binom{6}{1}z^4 + \binom{6}{2}z^2 - \binom{6}{3}z^0 + \binom{6}{4}z^{-2} - \binom{6}{5}z^{-4} + \binom{6}{6}z^{-6}\right] \\
&= -\frac{1}{64}\left[-20 + 15(z^2 + z^{-2}) - 6(z^4 + z^{-4}) + (z^6 + z^{-6})\right] \\
&= -\frac{1}{64}\left[-20 + 15(2\cos 2\theta) - 6(2\cos 4\theta) + 2\cos 6\theta\right] \\
&= \frac{1}{32}\left(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\right)
\end{aligned}$$

$$\begin{aligned}
\cos^6\theta &= \frac{1}{2^6}(2\cos\theta)^6 \\
&= \frac{1}{64}(z+z^{-1})^6 \\
&= \frac{1}{64}\left[\binom{6}{0}z^6 + \binom{6}{1}z^4 + \binom{6}{2}z^2 + \binom{6}{3}z^0 + \binom{6}{4}z^{-2} + \binom{6}{5}z^{-4} + \binom{6}{6}z^{-6}\right] \\
&= \frac{1}{64}\left[20 + 15(z^2+z^{-2}) + 6(z^4+z^{-4}) + (z^6+z^{-6})\right] \\
&= \frac{1}{64}\left[20 + 15(2\cos 2\theta) + 6(2\cos 4\theta) + 2\cos 6\theta\right] \\
&= \frac{1}{32}\left(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta\right)
\end{aligned}$$

$$\begin{aligned}
\cos^6\theta - \sin^6\theta &= \left[\frac{1}{32}\left(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta\right)\right] \\
&\quad - \left[\frac{1}{32}\left(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\right)\right] \\
&= \frac{1}{32}(30\cos 2\theta + 2\cos 6\theta) \\
&= \frac{15}{16}\cos 2\theta + \frac{1}{16}\cos 6\theta
\end{aligned}$$

$$\boxed{a = \frac{15}{16},\, b = \frac{1}{16}}$$