Double Math Solutions

https://asdia.dev/projects/doublemath

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Contents

Part I.

Group A

A1. Equations and Inequalities

Tutorial A1

Problem 1. Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

(a) $$\begin{cases} a + 2b - 3c = -5 \\ -2a - 4b - 6c = 10 \\ 3a + 7b - 2c = -13 \end{cases}$$

(b) $$\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$$

(c) $$\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$$

(d) $$\begin{cases} 1/p + 1/q + 1/r = 5 \\ 2/p - 3/q - 4/r = -11 \\ 3/p + 2/q - 1/r = -6 \end{cases}$$

(e) $$\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\gamma = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma = 9 \end{cases}$$ where $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$, and $0 \leq \gamma < \pi$.

Solution.
Part (a). Unique solution: $a = -9$, $b = 2$, $c = 0$.
Part (b). No solution.
Part (c). Infinitely many solutions: $x_1 = 8 - t$, $x_2 = t - 3$, $x_3 = t$.
Part (d). Solving, we obtain

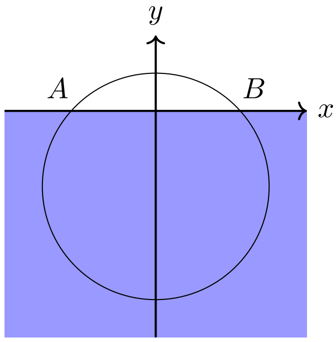
$$\frac{1}{p} = 2, \quad \frac{1}{q} = -3, \quad \frac{1}{r} = 6.$$

There is hence a unique solution: $p = 1/2$, $q = -1/3$, $r = 1/6$.
Part (e). Solving, we obtain

$$\sin\alpha = 1, \quad \cos\beta = -1, \quad \tan\gamma = 0.$$

There is hence a unique solution: $\alpha = \pi/2$, $\beta = \pi$, $\gamma = 0$.

Problem 2. The following figure shows the circular cross-section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0.$$

A and B are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when AB is 40 cm apart horizontally, determine the values of a, b and c by forming a system of linear equations.

Solution. Since $AB = 40$, we have $A(-20, 0)$ and $B(20, 0)$. We also know $(0, 10)$ lies on the circle. Substituting these points into the given equation, we have the following system of equations:

$$\begin{cases} -20a \qquad + c = -400 \\ 20a \qquad + c = -400 \\ \qquad\quad 10b + c = -100 \end{cases}$$

Solving, we obtain $a = 0$, $b = 30$, $c = -400$.

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Problem 3. Find the exact solution set of the following inequalities.

(a) $x^2 - 2 \geq 0$

(b) $4x^2 - 12x + 10 > 0$

(c) $x^2 + 4x + 13 < 0$

(d) $x^3 < 6x - x^2$

(e) $x^2(x-1)(x+3) \geq 0$

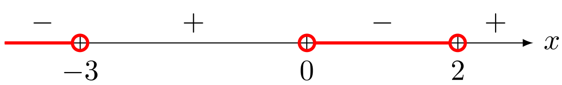
Solution.

Part (a). Note that $x^2 - 2 \geq 0 \implies x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$. The solution set is thus $\left\{x \in \mathbb{R} \colon x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\right\}$.

Part (b). Completing the square, we see that $4x^2 - 12x + 10 > 0 \implies (x - \frac{3}{2})^2 + \frac{19}{4} > 0$. Since $(x - \frac{3}{2})^2 \geq 0$, all $x \in \mathbb{R}$ satisfy the inequality, whence the solution set is $\mathbb{R}$.

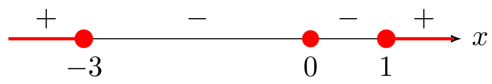
Part (c). Completing the square, we have $x^2 + 4x + 13 < 0 \implies (x+2)^2 + 9 < 0$. Since $(x+2)^2 \geq 0$, there is no solution to the inequality, whence the solution set is $\varnothing$.

Part (d). Note that $x^3 < 6x - x^2 \implies x(x+3)(x-2) < 0$.



The solution set is thus $\{x \in \mathbb{R} \colon x < -3 \text{ or } 0 < x < 2\}$.

Part (e).



The solution set is thus $\{x \in \mathbb{R} \colon x \leq -3 \text{ or } x = 0 \text{ or } x \geq 1\}$.

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Problem 4. Find the exact solution set of the following inequalities.

(a) $|3x + 5| < 4$

(b) $|x - 2| < 2x$

Solution.

Part (a). If $3x + 5 < 4$, then $x < -\frac{1}{3}$. If $-(3x + 5) < 4$, then $x > -3$. Combining both inequalities, we have $-3 < x < -\frac{1}{3}$. Thus, the solution set is $\left\{x \in \mathbb{R} \colon -3 < x < -\frac{1}{3}\right\}$.

Part (b). If $x - 2 < 2x$, then $x > -2$. If $-(x - 2) < 2x$, then $x > \frac{2}{3}$. Combining both inequalities, we have $x > \frac{2}{3}$. Thus, the solution set is $\left\{x \in \mathbb{R} \colon x > \frac{2}{3}\right\}$.

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Problem 5. It is given that $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are constants. Given that the curve with equation $y = p(x)$ is symmetrical about the y-axis, and that it passes through the points with coordinates $(1, 2)$ and $(2, 11)$, find the values of a, b, c and d.

Solution. We know that $(1, 2)$ and $(2, 11)$ lie on the curve. Since $y = p(x)$ is symmetrical about the y-axis, we have that $(-1, 2)$ and $(-2, 11)$ also lie on the curve. Substituting these points into $y = p(x)$, we obtain the following system of equations:

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving, we obtain $a = 0$, $b = -2$, $c = 0$, $d = 3$.

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Problem 6. Mr Mok invested \$50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of \$3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

Solution. Let a, b and c be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a + b + c = 50000 \\ \frac{6}{100}a + \frac{8}{100}b + \frac{10}{100}c = 3700 \\ a = 2c \end{cases}$$

Solving, we have $a = 30000$, $b = 5000$ and $c = 15000$. Thus, Mr Mok invested \$30,000, \$5,000 and \$15,000 in Funds A, B and C respectively.

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Problem 7. Solve the following inequalities with exact answers.

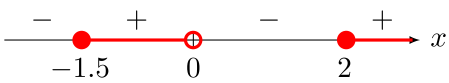
(a) $2x - 1 \geq \frac{6}{x}$

(b) $x - \frac{1}{x} < 1$

(c) $-1 < \frac{2x+3}{x-1} < 1$

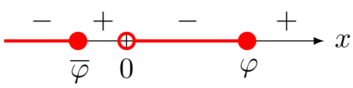
Solution.
Part (a). Note that $x \neq 0$.

$$2x - 1 \geq \frac{6}{x} \implies x^2(2x-1) \geq 6x \implies x\left(2x^2 - x - 6\right) \geq 0 \implies x(2x+3)(x-2) \geq 0.$$



Thus, $-\frac{3}{2} \leq x < 0$ or $x \geq 2$.
Part (b). Note that $x \neq 0$.

$$x - \frac{1}{x} < 1 \implies x^3 - x < x^2 \implies x\left(x^2 - x - 1\right) < 0 \implies x(x-\varphi)(x-\overline{\varphi}) < 0.$$



Thus, $x \leq \bar{\varphi}$ or $0 < x \leq \varphi$.
Part (c).

$$-1 < \frac{2x+3}{x-1} < 1 \implies -3 < \frac{5}{x-1} < -1 \implies -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5} \implies -4 < x < -\frac{2}{3}.$$

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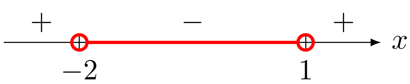
Problem 8. Without using a calculator, solve the inequality $\frac{x^2+x+1}{x^2+x-2} < 0$.

Solution. Observe that $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. The inequality thus reduces to $\frac{1}{x^2+x-2} < 0$.

$$\frac{1}{x^2 + x - 2} < 0 \implies x^2 + x - 2 < 0 \implies (x-1)(x+2) < 0.$$



Hence, $-2 < x < 1$.

Problem 9. Solve the following inequalities using a graphical method.

(a) $|3x+1| < (4x+3)^2$

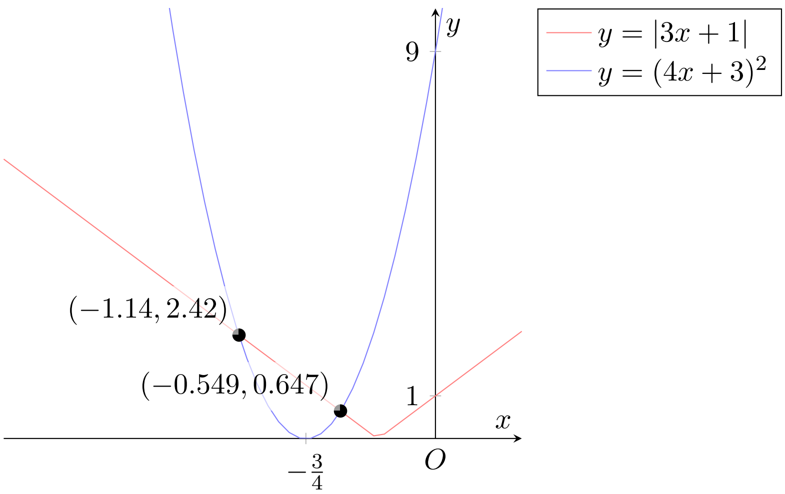
(b) $|3x+1| \geq |2x+7|$

(c) $|x-2| \geq x+|x|$

(d) $5x^2+4x-3 > \ln(x+1)$

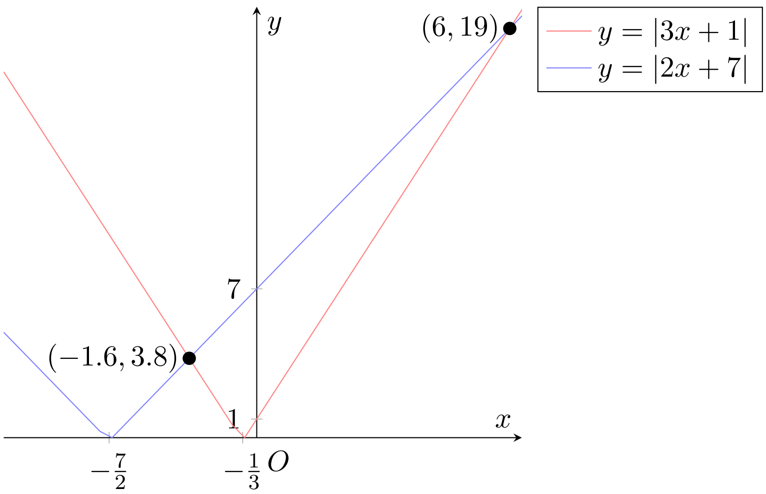
Solution.

Part (a).



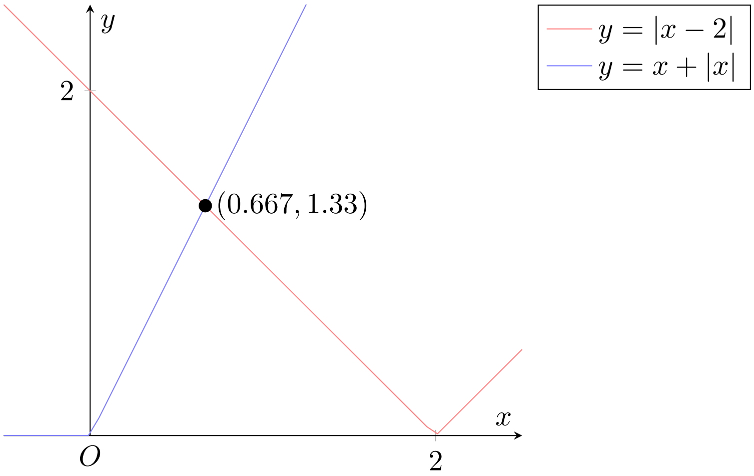
Thus, $x < -1.14$ or $x > -0.549$.

Part (b).



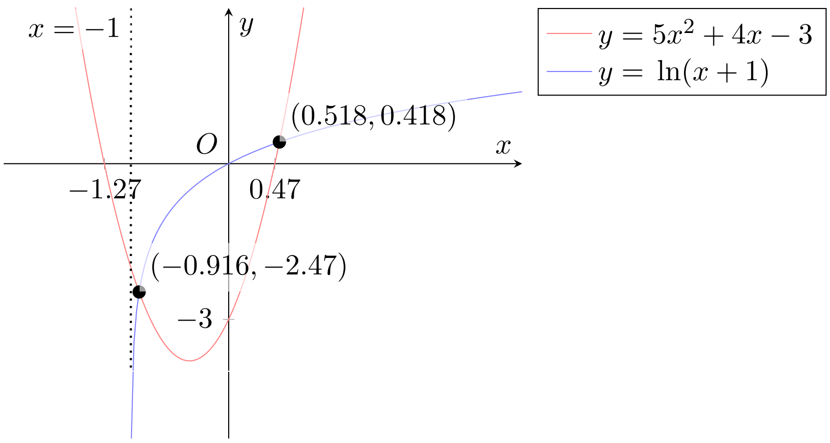
Thus, $x \leq -1.6$ or $x \geq 6$.

Part (c).



Thus, $x \leq 0.667$.

Part (d).



Thus, $-1 < x < -0.916$ or $x > 0.518$.

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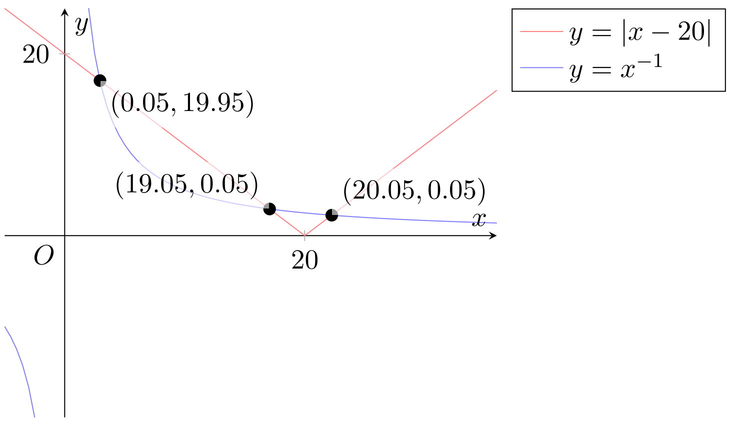
Problem 10. Sketch the graphs of $y = |x - 20|$ and $y = \frac{1}{x}$ on the same diagram. Hence or otherwise, solve the inequality $|x - 20| < \frac{1}{x}$, leaving your answers correct to 2 decimal places.

Solution.



Thus, $0 < x < 0.05$ or $19.95 < x < 20.05$.

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Problem 11. Solve the inequality $\frac{x-9}{x^2-9} \leq 1$. Hence, solve the inequalities

(a) $\frac{|x|-9}{x^2-9} \leq 1$

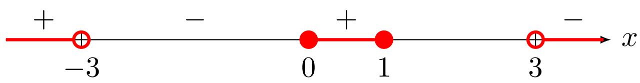
(b) $\frac{x+9}{x^2-9} \geq -1$

Solution. Note that $x^2 - 9 \neq 0 \implies x \neq \pm 3$.

$$\frac{x-9}{x^2-9} \leq 1 \implies (x-9)\left(x^2-9\right) \leq \left(x^2-9\right)^2.$$

Expanding and factoring, we get

$$x^4 - x^3 - 9x^2 + 9x = x(x+3)(x-1)(x-3) \geq 0.$$



Thus, $x < -3$ or $0 \leq x \leq 1$ or $x > 3$.

Part (a). Consider the substitution $x \mapsto |x|$. Then

$$|x| < -3 \text{ or } 0 \leq |x| \leq 1 \text{ or } |x| > 3.$$

This immediately gives us $x < -3$ or $-1 \leq x \leq 1$ or $x > 3$.

Part (b). Consider the substitution $x \mapsto -x$. Then

$$-x < -3 \text{ or } 0 \leq -x \leq 1 \text{ or } -x > 3.$$

This immediately gives us $x < -3$ or $-1 \leq x \leq 0$ or $x > 3$.

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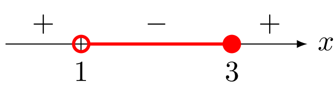
Problem 12. Solve the inequality $\frac{x-5}{1-x} \geq 1$. Hence, solve $0 < \frac{1-\ln x}{\ln x - 5} \leq 1$.

Solution. Note that $x \neq 1$.

$$\frac{x-5}{1-x} \geq 1 \implies (x-5)(1-x) \geq (1-x)^2 \implies 2x^2 - 8x + 6 \leq 0 \implies 2(x-1)(x-3) \leq 0.$$



Thus, $1 < x \leq 3$.

Consider the substitution $x \mapsto \ln x$. Taking reciprocals, we have our desired inequality $0 < \frac{1-\ln x}{\ln x - 5} \leq 1$. Hence,

$$1 < \ln x \leq 3 \implies e < x \leq e^3.$$

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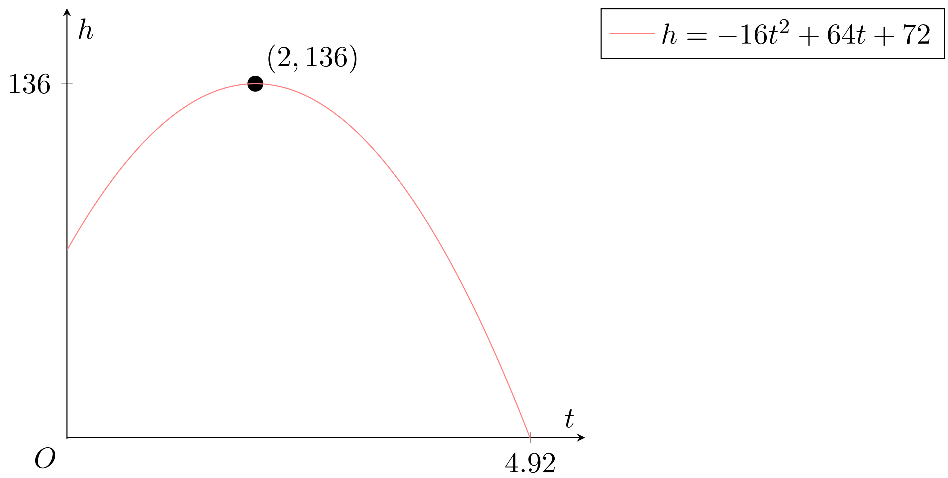
Problem 13. A small rocket is launched from a height of 72 m from the ground. The height of the rocket in metres, h, is represented by the equation $h = -16t^2 + 64t + 72$, where t is the time in seconds after the launch.

(a) Sketch the graph of h against t.

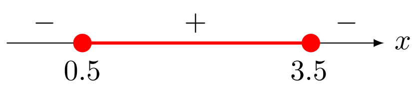
(b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

Solution.

Part (a).



Part (b). Note that $-16t^2 + 64t + 72 \geq 100 \implies -4(2t-1)(2t-7) \geq 0$.



Thus, the rocket will remain at or above 100 m from the ground for 3 seconds.

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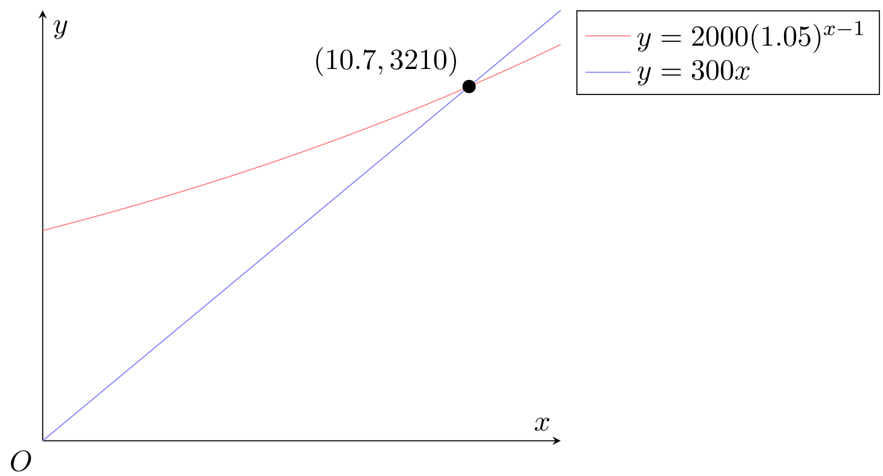
Problem 14. Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of $2000(1.05)^{n-1}$ dollars in the nth quarter, where n is a positive integer. She also gives a regular donation of \$$300n$ in the nth quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

Solution. Consider the curves $y = 2000(1.05)^{x-1}$ and $y = 300x$.



Hence, Xinxin will stop donating in the 11th quarter.

Assignment A1

Problem 1. A traveller just returned from Germany, France and Spain. The amount (in dollars) that he spent each day on housing, food and incidental expenses in each country are shown in the table below.

Country	Housing	Food	Incidental Expenses
Germany	28	30	14
France	23	25	8
Spain	19	22	12

The traveller's records of the trip indicate a total of \$191 spent for housing, \$430 for food and \$180 for incidental expenses. Calculate the number of days the traveller spent in each country.

He did his account again and the amount spent on food is \$337. Is this record correct? Why?

Solution. Let g, f and s represent the number of days the traveller spent in Germany, France and Spain respectively. From the table, we obtain the following system of equations:

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 430 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution $g = 4$, $f = 8$ and $s = 5$. The traveller thus spent 4 days in Germany, 8 days in France and 5 days in Spain.

Consider the scenario where the amount spent on food is \$337.

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 337 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution $g = 66$, $f = -27$ and $s = -44$. The record is hence incorrect as f and s must be positive.

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Problem 2.

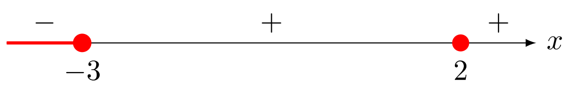
(a) Solve algebraically $x^2 - 9 \geq (x+3)\left(x^2 - 3x + 1\right)$.

(b) Solve algebraically $\frac{7-2x}{3-x^2} \leq 1$.

Solution.

Part (a).

$$\begin{aligned} & x^2 - 9 \geq (x+3)\left(x^2 - 3x + 1\right) \\ \Longrightarrow \quad & (x+3)(x-3) \geq (x+3)\left(x^2 - 3x + 1\right) \\ \Longrightarrow \quad & (x+3)\left(x^2 - 4x + 4\right) \leq 0 \\ \Longrightarrow \quad & (x+3)(x-2)^2 \leq 0 \end{aligned}$$



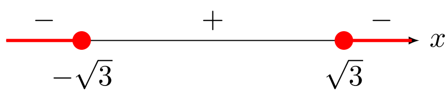
Thus, $x \leq -3$ or $x = 2$.

Part (b). Note that $3 - x^2 \neq 0 \implies x \neq \pm\sqrt{3}$.

$$\begin{aligned} \frac{7-2x}{3-x^2} &\leq 1 \\ \implies \frac{7-2x}{3-x^2} - \frac{3-x^2}{3-x^2} &\leq 0 \\ \implies \frac{x^2-2x+4}{3-x^2} &\leq 0 \end{aligned}$$

Observe that $x^2 - 2x + 4 = (x-1)^2 + 3 > 0$. Dividing through by $x^2 - 2x + 4$, we obtain

$$\begin{aligned} \frac{1}{3-x^2} &\leq 0 \\ \implies 3 - x^2 &\leq 0 \end{aligned}$$



Thus, $x < -\sqrt{3}$ or $x > \sqrt{3}$.

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Problem 3.

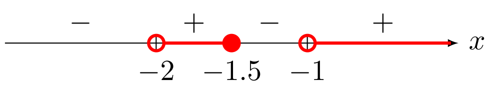
(a) Without using a calculator, solve the inequality $\frac{3x+4}{x^2+3x+2} \geq \frac{1}{x+2}$.

(b) Hence, deduce the set of values of x that satisfies $\frac{3|x|+4}{x^2+3|x|+2} \geq \frac{1}{|x|+2}$.

Solution.

Part (a). Note that $x^2 + 3x + 2 \neq 0$ and $x + 2 \neq 0$, whence $x \neq -1, -2$.

$$\begin{aligned} \frac{3x+4}{x^2+3x+2} &\geq \frac{1}{x+2} \\ \implies \frac{3x+4}{(x+2)(x+1)} &\geq \frac{1}{x+2} \\ \implies (3x+4)(x+2)(x+1) &\geq (x+2)(x+1)^2 \\ \implies (x+2)(x+1)(2x+3) &\geq 0 \end{aligned}$$



Thus, $-2 < x \leq -\frac{3}{2}$ or $x > -1$.

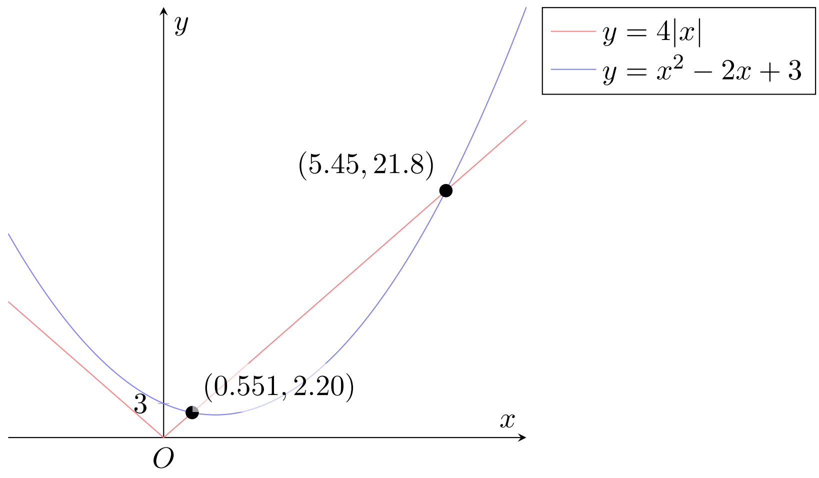
Part (b). Observe that $|x|^2 = x^2$. Hence, with the map $x \mapsto |x|$, we obtain

$$-2 < |x| \leq -\frac{3}{2} \text{ or } |x| > -1.$$

Since $|x| \geq 0$, we have that $|x| > -1$ is satisfied for all real x. Hence, the solution set is $\mathbb{R}$.

Problem 4. On the same diagram, sketch the graphs of $y = 4|x|$ and $y = x^2 - 2x + 3$. Hence or otherwise, solve the inequality $4|x| \geq x^2 - 2x + 3$.

Solution.



From the graph, we see that $0.551 \leq x \leq 5.45$.

A2. Numerical Methods of Finding Roots

Tutorial A2

Problem 1. Without using a graphing calculator, show that the equation $x^3+2x^2-2=0$ has exactly one positive root.

This root is denoted by α and is to be found using two different iterative methods, starting with the same initial approximation in each case.

(a) Show that α is a root of the equation $x = \sqrt{\frac{2}{x+2}}$, and use the iterative formula $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$, with $x_1 = 1$, to find α correct to 2 significant figures.

(b) Use the Newton-Raphson method, with $x_1 = 1$, to find α correct to 3 significant figures.

Solution. Let $f(x) = x^3+2x^2-2$. Observe that for all $x > 0$, we have $f'(x) = 3x^2+4x > 0$. Hence, $f(x)$ is strictly increasing on $(0, \infty)$. Since $f(0)f(1) = (-2)(1) < 0$, it follows that $f(x)$ has exactly one positive root.

Part (a). We know $f(\alpha) = 0$. Hence,

$$\alpha^3 + 2\alpha^2 - 2 = 0 \implies \alpha^2(\alpha+2) = 2 \implies \alpha^2 = \frac{2}{\alpha+2} \implies \alpha = \sqrt{\frac{2}{\alpha+2}}.$$

Note that we reject the negative branch since $\alpha > 0$. We hence see that α is a root of the equation $x = \sqrt{\frac{2}{x+2}}$. Using the iterative formula $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$ with $x_1 = 1$, we have

n	x_n
1	1
2	0.81650
3	0.84268
4	0.83879

Hence, $\alpha = 0.84$ (2 s.f.).

Part (b). Using the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$) with $x_1 = 1$, we have

n	x_n
1	1
2	0.857143
3	0.839545
4	0.839287
5	0.839287

Hence, $\alpha = 0.839$ (3 s.f.).

Problem 2.

(a) Show that the tangent at the point $(\mathrm{e}, 1)$ to the graph $y = \ln x$ passes through the origin, and deduce that the line $y = mx$ cuts the graph $y = \ln x$ in two points provided that $0 < m < 1/\mathrm{e}$.

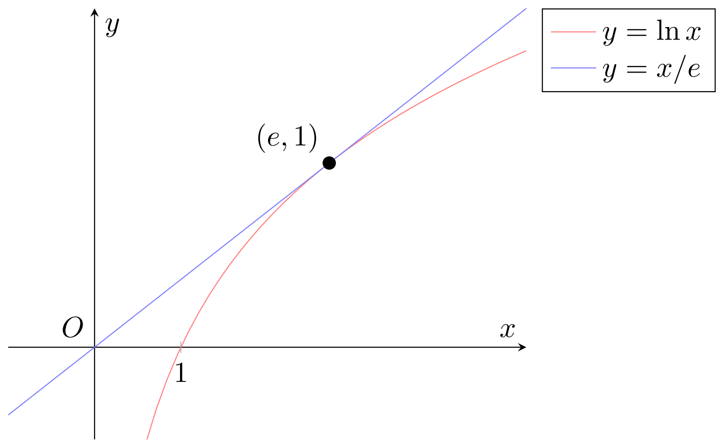
(b) For each root of the equation $\ln x = x/3$, find an integer n such that the interval $n < x < n+1$ contains the root. Using linear interpolation, based on $x = n$ and $x = n+1$, find a first approximation to the smaller root, giving your answer to 1 decimal place. Using your first approximation, obtain, by the Newton-Raphson method, a second approximation to the smaller root, giving your answer to 2 decimal places.

Solution.

Part (a). Note that the derivative of $y = \ln x$ at $x = \mathrm{e}$ is $1/\mathrm{e}$. Using the point slope formula, we see that the equation of the tangent at the point $(\mathrm{e}, 1)$ is given by

$$y - 1 = \frac{x - \mathrm{e}}{\mathrm{e}} \implies y = \frac{x}{\mathrm{e}}.$$

Since $x = 0$, $y = 0$ is clearly a solution, the tangent passes through the origin. From the graph below, it is clear that for $y = mx$ to intersect $y = \ln x$ twice, we must have $0 < m < 1/\mathrm{e}$.



Part (b). Consider $f(x) = x/3 - \ln x$. Let α and β be the smaller and larger root to $f(x) = 0$ respectively. Observe that $f(1)f(2) = (1)(-0.03) < 0$ and $f(4)f(5) = (-0.05)(0.06) < 0$. Thus, for the smaller root α, $n = 1$, while for the larger root β, $n = 4$.

Let x_1 be the first approximation to α. Using linear interpolation, we have

$$x_1 = \frac{f(2) - 2f(1)}{f(2) - f(1)} = 1.9 \text{ (1 d.p.)}$$

Note that $f'(x) = 1/3 - 1/x$. Using the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$), we have

n	x_n
1	1.9
2	1.85585
3	1.85718

Hence, $\alpha = 1.86$ (2 d.p.).

Problem 3. Find the exact coordinates of the turning points on the graph of $y = f(x)$ where $f(x) = x^3 - x^2 - x - 1$. Deduce that the equation $f(x) = 0$ has only one real root α, and prove that α lies between 1 and 2. Use the Newton-Raphson method applied to the equation $f(x) = 0$ to find a second approximation x_2 to α, taking x_1, the first approximation, to be 2. With reference to a graph of $y = f(x)$, explain why all further approximations to α by this process are always larger than α.

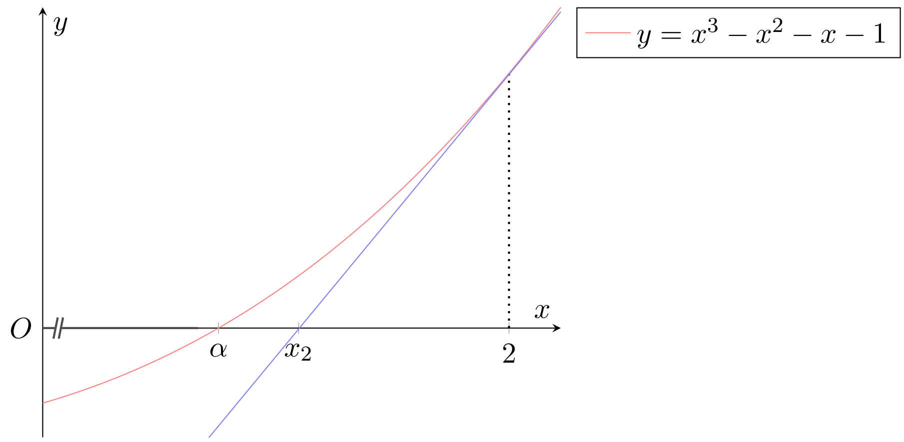
Solution. For turning points, $f'(x) = 0$.

$$f'(x) = 0 \implies 3x^2 - 2x - 1 = 0 \implies (3x+1)(x-1) = 0.$$

Hence, $x = -1/3$ or $x = 1$. When $x = -1/3$, we have $y = -0.815$, giving the coordinate $(-1/3, -0.815)$. When $x = 1$, we have $y = -2$, giving the coordinate $(1, -2)$.

Observe that $f(x)$ is strictly increasing for all $x > 1$. Further, since both turning points have a negative y-coordinate, it follows that $y < 0$ for all $x \leq 1$. Since $f(1)f(2) = (-2)(1) < 0$, the equation $f(x) = 0$ has only one real root.

Using the Newton-Raphson method with $x_1 = 2$, we have $x_2 = x_1 - f(x_1)/f'(x_1) = 13/7$.



Since x_2 lies on the right of α, the Newton-Raphson method gives an over-estimation given an initial approximation of 2. Thus, all further approximations to α will also be larger than α.

* * * * *

Problem 4. A curve C has equation $y = x^5 + 50x$. Find the least value of $\mathrm{d}y/\mathrm{d}x$ and hence give a reason why the equation $x^5 + 50x = 10^5$ has exactly one real root. Use the Newton-Raphson method, with a suitable first approximation, to find, correct to 4 decimal places, the root of the equation $x^5 + 50x = 10^5$. You should demonstrate that your answer has the required accuracy.

Solution. Since $y = x^5 + 50x$, we have $\mathrm{d}y/\mathrm{d}x = 5x^4 + 50$. Since $x^4 \geq 0$ for all real x, the minimum value of $\mathrm{d}y/\mathrm{d}x$ is 50.

Let $f(x) = x^5 + 50x$. Since $\min \mathrm{d}f/\mathrm{d}x = 50 > 0$, it follows that $f(x)$ is strictly increasing. Hence, $f(x)$ will intersect only once with the line $y = 10^5$, whence the equation $x^5 + 50x = 10^5$ has exactly one real root.

Observe that $f(9)f(10) = (-40901)(50) < 0$. Thus, there must be a root in the interval $(9, 10)$. We now use the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$) with $x_1 = 9$ as the first approximation.

n	x_n
1	9
2	10.2178921
3	10.0017491
4	9.9901221
5	9.9899912
6	9.9899900

Thus, the root is 9.9900 (4 d.p.).

Observe that $f(9.98995)f(9.99005) = (-2.00)(3.00) < 0$. Hence, the root lies in the interval $(9.98995, 9.99005)$ whence the calculated root has the required accuracy.

* * * * *

Problem 5.

(a) A function f is such that $f(4) = 1.158$ and $f(5) = -3.381$, correct to 3 decimal places in each case. Assuming that there is a value of x between 4 and 5 for which $f(x) = 0$, use linear interpolation to estimate this value.

For the case when $f(x) = \tan x$, and x is measured in radians, the value of $f(4)$ and $f(5)$ are as given above. Explain, with the aid of a sketch, why linear interpolation using these values does not give an approximation to a solution of the equation $\tan x = 0$.

(b) Show, by means of a graphical argument or otherwise, that the equation $\ln(x-1) = -2x$ has exactly one real root, and show that this root lies between 1 and 2.

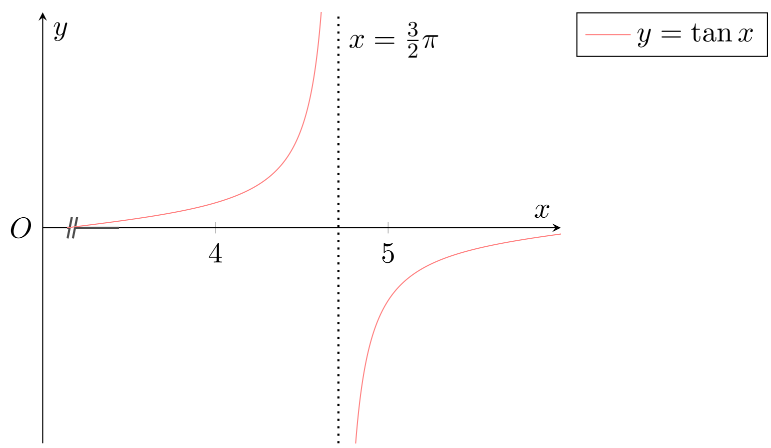
The equation may be written in the form $\ln(x-1) + 2x = 0$. Show that neither $x = 1$ nor $x = 2$ is a suitable initial value for the Newton-Raphson method in this case.

The equation may also be written in the form $x - 1 - \mathrm{e}^{-2x} = 0$. For this form, use two applications of the Newton-Raphson method, starting with $x = 1$, to obtain an approximation to the root, giving 3 decimal places in your answer.

Solution.

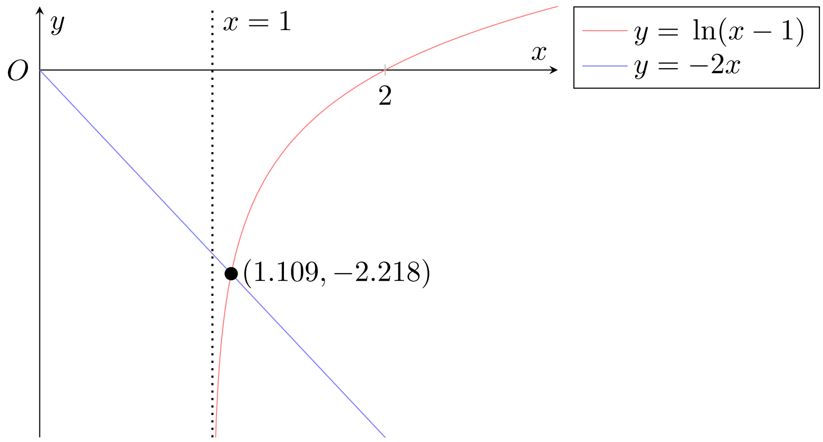
Part (a). Let the root of $f(x) = 0$ be α. Using linear interpolation on the interval $[4, 5]$, we have

$$\alpha = \frac{4f(5) - 5f(4)}{f(5) - f(4)} = 4.255 \text{ (3 d.p.)}.$$



Since $\tan x$ has a vertical asymptote at $x = 3\pi/2$, it is not continuous on $[4, 5]$. Thus, linear interpolation diverges when applied to the equation $\tan x = 0$.

Part (b).



Since there is only one intersection between the graphs $y = \ln(x-1)$ and $y = -2x$, there is only one real root to the equation $\ln(x-1) = -2x$. Furthermore, since $y = -2x$ is negative for all $x > 0$ and $y = \ln(x-1)$ is negative only when $1 < x < 2$, it follows that the root must lie between 1 and 2.

Let $f(x) = \ln(x-1) + 2x$. Then $f'(x) = \frac{1}{x-1} + 2$. Note that the Newton-Raphson method is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Since $f'(1)$ is undefined, an initial approximation of $x_1 = 1$ cannot be used for the Newton-Raphson method, which requires a division by $f'(1)$.

Using the Newton-Raphson method with the initial approximation $x_2 = 2$, we see that $x_2 = 1$. Once again, because $f'(1)$ is undefined, $x_1 = 2$ is also not a suitable initial value.

Let $g(x) = x - 1 - \mathrm{e}^{-2x}$. Then $g'(x) = 1 + 2\mathrm{e}^{-2x}$. Using the Newton-Raphson method with the initial approximation $x_1 = 1$, we have

n	x_n
1	1
2	1.106507
3	1.108857

Hence, $x = 1.109$ (3 d.p.).

* * * * *

Problem 6. The equation $x = 3\ln x$ has two roots α and β, where $1 < \alpha < 2$ and $4 < \beta < 5$. Using the iterative formula $x_{n+1} = F(x_n)$, where $F(x) = 3\ln x$, and starting with $x_0 = 4.5$, find the value of β correct to 3 significant figures. Find a suitable $F(x)$ for computing α.

Solution. Using the iterative formula $x_{n+1} = F(x_n)$, we have

n	x_n	n	x_n
0	4.5	5	4.53175
1	4.51223	6	4.53333
2	4.52038	7	4.53437
3	4.52579	8	4.53506
4	4.52937	9	4.53551

Hence, $\beta = 4.54$ (3 s.f.).

Note that $x = 3\ln x \implies x = \mathrm{e}^{x/3}$. Observe that $\mathrm{d}(\mathrm{e}^{x/3})/\mathrm{d}x = \frac{1}{3}\mathrm{e}^{x/3}$, which is between -1 and 1 for all $1 < x < 2$. Thus, the iterative formula $x_{n+1} = F(x_n)$ will converge, whence $F(x) = \mathrm{e}^{x/3}$ is suitable for computing α.

* * * * *

Problem 7. Show that the cubic equation $x^3 + 3x - 15 = 0$ has only one real root. This root is near $x = 2$. The cubic equation can be written in any one of the forms below:

(a) $x = \frac{1}{3}(15 - x^3)$

(b) $x = \frac{15}{x^2+3}$

(c) $x = (15 - 3x)^{1/3}$

Determine which of these forms would be suitable for the use of the iterative formula $x_{r+1} = F(x_r)$, where $r = 1, 2, 3, \ldots$.

Hence, find the root correct to 3 decimal places.

Solution. Let $f(x) = x^3 + 3x - 15$. Then $f'(x) = 3x^2 + 3 > 0$ for all real x. Hence, f is strictly increasing. Since f is continuous, $f(x) = 0$ has only one real root.

Part (a). Let $g_1(x) = \frac{1}{3}(15 - x^3)$. Then $g_1'(x) = -x^2$. For values of x near 2, $|g_1'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_1(x_n)$ will diverge and $g_1(x)$ is unsuitable.

Part (b). Let $g_2(x) = \frac{15}{x^2+3}$. Then $g_2'(x) = \frac{-30x}{(x^2+3)^2}$. For values of x near 2, $|g_2'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_2(x_n)$ will diverge and $g_2(x)$ is unsuitable.

Part (c). Let $g_3(x) = (15 - 3x)^{1/3}$. Then $g_3'(x) = -(15 - 3x)^{-2/3}$. For values of x near 2, $|g_3'(x)| < 1$. Hence, the iterative formula $x_{n+1} = g_3(x_n)$ will converge and $g_3(x)$ is suitable.

Using the iterative formula $x_{r+1} = g_3(x_r)$, we get

r	x_r
1	2
2	2.080084
3	2.061408
4	2.065793
5	2.064765

Hence, $x = 2.065$ (3 d.p.).

* * * * *

Problem 8. The equation of a curve is $y = f(x)$. The curve passes through the points $(a, f(a))$ and $(b, f(b))$, where $0 < a < b$, $f(a) > 0$ and $f(b) < 0$. The equation $f(x) = 0$ has precisely one root α such that $a < \alpha < b$. Derive an expression, in terms of a, b, $f(a)$ and $f(b)$, for the estimated value of α based on linear interpolation.

Let $f(x) = 3\mathrm{e}^{-x} - x$. Show that $f(x) = 0$ has a root α such that $1 < \alpha < 2$, and that for all x, $f'(x) < 0$ and $f''(x) > 0$. Obtain an estimate of α using linear interpolation to 2 decimal places, and explain by means of a sketch whether the value obtained is an over-estimate or an under-estimate.

Use one application of the Newton-Raphson method to obtain a better estimate of α, giving your answer to 2 decimal places.

Solution. Using the point-slope formula, the equation of the line that passes through both $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(a) - f(b)}{a - b}(x - a).$$

Note that $(\alpha, 0)$ is approximately the solution to the above equation. Thus,

$$0 - f(a) \approx \frac{f(a) - f(b)}{a - b}(\alpha - a) \implies \alpha \approx \frac{bf(a) - af(b)}{f(a) - f(b)}.$$

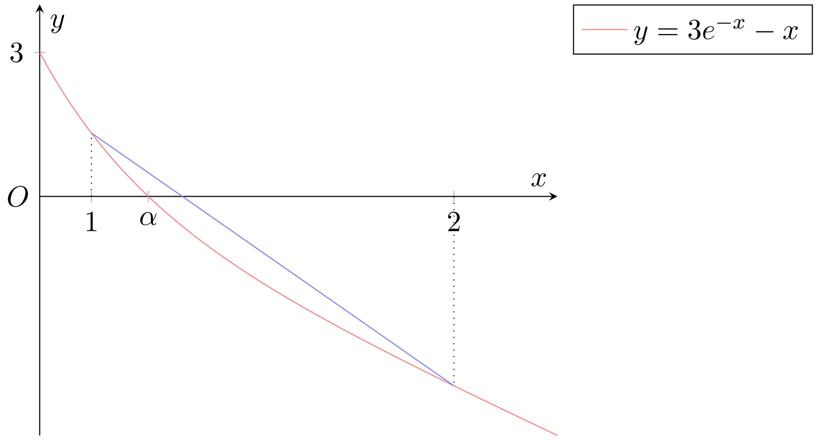
Since $f(x)$ is continuous, and $f(1)f(2) = (0.10)(-1.6) < 0$, there exists a root $\alpha \in (1, 2)$.

Note that $f'(x) = -3\mathrm{e}^{-x} - 1$ and $f''(x) = 3\mathrm{e}^{-x}$. Since $\mathrm{e}^{-x} > 0$ for all x, we have that $f'(x) < 0$ and $f''(x) > 0$ for all x.

Using linear interpolation on the interval $(1, 2)$, we have

$$\alpha = \frac{2f(1) - f(2)}{f(1) - f(2)} = 1.06 \text{ (2 d.p.)}.$$

Since $f'(x) < 0$ and $f''(x) > 0$, we know that $f(x)$ is strictly decreasing and is concave upwards. $f(x)$ hence has the following shape:



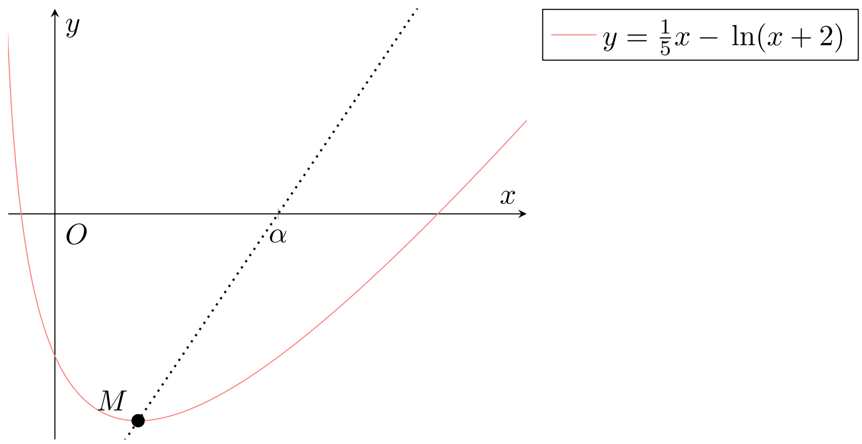
From the graph, we see that the value obtained is an over-estimate.

Using the Newton-Raphson method with the initial approximation $x_1 = 1.06$, we get

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.05 \text{ (2 d.p.)}.$$

* * * * *

Problem 9.



The diagram shows a sketch of the graph $y = x/3 - \ln(x + 2)$. Find the x-coordinate of the minimum point M on the graph, and verify that y is positive when $x = 20$.

Show that the gradient of the curve is always less than $1/5$. Hence, by considering the line through M having gradient $1/5$, show that the positive root of the equation $x/3 - \ln(x+2) = 0$ is greater than 8.

Use linear interpolation, once only, on the interval $[8, 20]$, to find an approximate value a for this positive root, giving your answer to 1 decimal place.

Using a as an initial value, carry out one application of the Newton-Raphson method to obtain another approximation to the positive root, giving your answer to 2 decimal places.

Solution. For stationary points, $y' = 0$.

$$y' = 0 \implies \frac{1}{5} - \frac{1}{x+2} \implies x = 3.$$

By the second derivative test, we see that $y''(x) = \frac{1}{(x+2)^2} > 0$. Hence, the x-coordinate of M is 3. Substituting $x = 20$ into the equation of the curve gives $y = 4 - \ln 22 = 0.909 > 0$.

We know that $y' = 1/5 - 1/(x+2)$, hence $y' < 1/5$ for all $x > -2$. Since the domain of the curve is $x > -2$, y' is always less than $1/5$.

Let $(\alpha, 0)$ be the coordinates of the root of the line through M having gradient $\frac{1}{5}$. We know that the coordinates of M are $(3, 3/5 - \ln 5)$. Taking the gradient of the line segment joining M and $(\alpha, 0)$, we get

$$\frac{(3/5 - \ln 5) - 0}{3 - \alpha} = \frac{1}{5} \implies \alpha = 5 \ln 5 = 8.05 > 8.$$

Since the gradient of the curve is always less than $1/5$, α represents the lowest possible value of the positive root of the curve. Hence, the positive root of the equation $x/5 - \ln(x+2) = 0$ is greater than 8.

Let $f(x) = x/5 - \ln(x+2)$. Using linear interpolation on the interval $[8, 20]$, we have

$$\alpha = \frac{8f(20) - 20f(8)}{f(20) - f(8)} = 13.2 \text{ (1 d.p.)}.$$

Using the Newton-Raphson method with the initial approximation $x_1 = 13.2$, we have

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 13.81 \text{ (2 d.p.)}.$$

* * * * *

Problem 10.

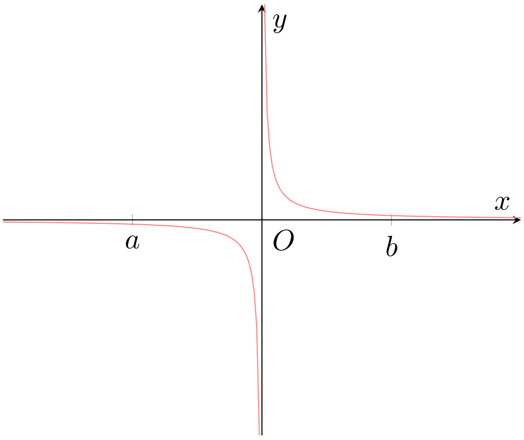
(a) The function f is such that $f(a)f(b) < 0$, where $a < b$. A student concludes that the equation $f(x) = 0$ has exactly one root in the interval (a, b). Draw sketches to illustrate two distinct ways in which the student could be wrong.

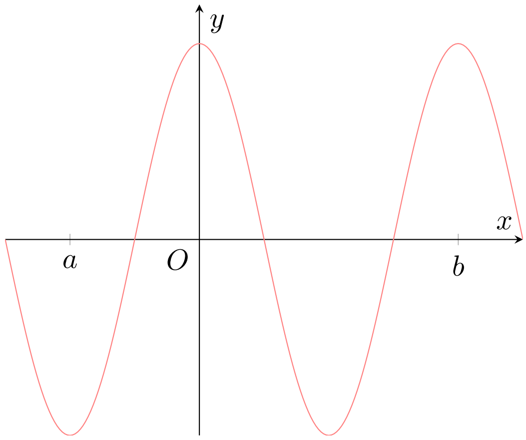
(b) The equation $\sec^2 x - e^2 = 0$ has a root α in the interval $[1.5, 2.5]$. A student uses linear interpolation once on this interval to find an approximation to α. Find the approximation to α given by this method and comment on the suitability of the method in this case.

(c) The equation $\sec^2 x - e^x = 0$ also has a root β in the interval $(0.1, 0.9)$. Use the Newton-Raphson method, with $f(x) = \sec^2 x - e^x$ and initial approximation 0.5, to find a sequence of approximations $\{x_1, x_2, x_3, \ldots\}$ to β. Describe what is happening to x_n for large n, and use a graph of the function to explain why the sequence is not converging to β.

Solution.

Part (a).





Part (b). Let $f(x) = \sec^2 x - \mathrm{e}^x$. Using linear interpolation on the interval $[1.5, 2.5]$,

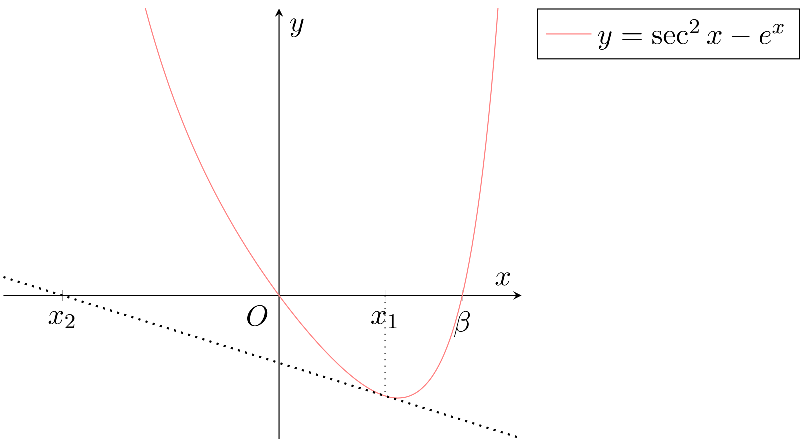
$$a = \frac{1.5f(2.5) - 2.5f(1.5)}{f(2.5) - f(1.5)} = 1.06 \text{ (2 d.p.)}.$$

$\sec^2 x$ is not continuous on the interval $[1.5, 2.5]$ due to the presence of an asymptote at $x = \pi/2$. Hence, linear interpolation is not suitable in this case.

Part (c). We know $f'(x) = 2\sec^2 x \tan x - \mathrm{e}^x$. Using the Newton-Raphson method with the initial approximation $x_1 = 0.5$,

r	x_r
1	0.5
2	-1.02272
3	-0.75526
4	-0.40306
5	-0.09667
6	-0.00466
7	-0.00000

As $n \to \infty$, $x_n \to 0^-$.



From the above graph, we see that the initial approximation of $x_1 = 0.5$ is past the turning point. Hence, all subsequent approximations will converge to the root at 0 instead of the root at β. Thus, the sequence does not converge to β.

Problem 11. The function f is given by $f(x) = \sqrt{1-x^2} + \cos x - 1$ for $0 \leq x \leq 1$. It is known, from graphical work, that the equation $f(x) = 0$ has a single root $x = \alpha$.

(a) Express g(x) in terms of x, where $g(x) = x - \frac{f(x)}{f'(x)}$.

A student attempts to use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to calculate the value of α correct to 3 decimal places.

(b) (i) The student first uses an initial approximation to α of $x_1 = 0$. Explain why this will be unsuccessful in finding a value for α.

(ii) The student next uses an initial approximation to α of $x_1 = 1$. Explain why this will also be unsuccessful in finding a value for α.

(iii) The student then uses an initial approximate to α of $x_1 = 0.5$. Investigate what happens in this case.

(iv) By choosing a suitable value for x_1, use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to determine α correct to 3 decimal places.

Solution.

Part (a). We know $f'(x) = \frac{-x}{\sqrt{1-x^2}} - \sin x$. Hence,

$$g(x) = x - \frac{\sqrt{1-x^2} + \cos x - 1}{\frac{-x}{\sqrt{1-x^2}} - \sin x}.$$

Part (b).

Part (b)(i). Observe that $f'(0) = 0$. Hence, $g(0)$ is undefined. Thus, starting with an initial approximation of $x_1 = 0$ will be unsuccessful in finding a value for α.

Part (b)(ii). Observe that $\sqrt{1-x^2}$ is 0 when $x = 1$. Hence, $f'(0)$ is undefined. Thus, $g(0)$ is also undefined. Hence, starting with an initial approximation of $x_1 = 1$ will also be unsuccessful in finding a value for α.

Part (b)(iii). When $x_1 = 0.5$, we have $x_2 = g(x_1) = 1.20$. Since $g(x)$ is only defined for $0 \leq x \leq 1$, $x_3 = g(x_2)$ is undefined. Hence, an initial approximation of $x_1 = 0.5$ will also be unsuccessful in finding a value for α.

Part (b)(iv). Using the Newton-Raphson method with $x_1 = 0.9$, we have

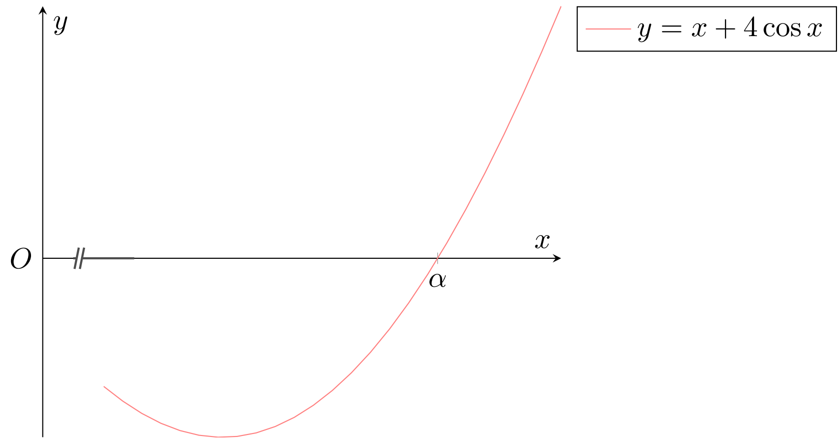
r	x_r
1	0.9
2	0.92019
3	0.91928
4	0.91928

Thus, $\alpha = 0.919$ (3 d.p.).

Assignment A2

Problem 1. By considering the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, or otherwise, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

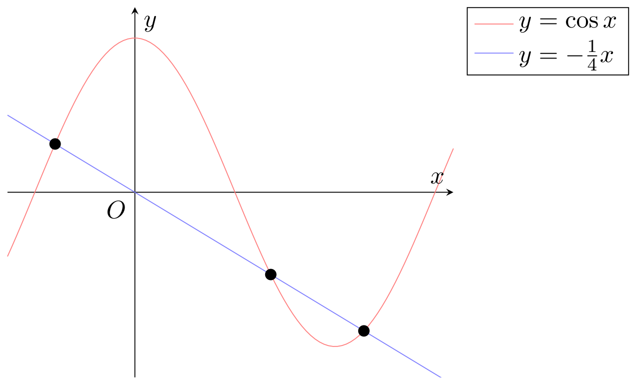
Use linear interpolation, once only, on the interval $[-1.5, 1]$ to find an approximation to the negative root of the equation $x + 4\cos x = 0$ correct to 2 decimal places.



The diagram shows part of the graph of $y = x + 4\cos x$ near the larger positive root, α, of the equation $x + 4\cos x = 0$. Explain why, when using the Newton-Raphson method to find α, an initial approximation which is smaller than α may not be satisfactory.

Use the Newton-Raphson method to find α correct to 2 significant figures. You should demonstrate that your answer has the required accuracy.

Solution.



Note that $x + 4\cos x = 0 \implies \cos x = -\frac{1}{4}x$. Plotting the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, we see that there is one negative root and two positive roots. Hence, the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

Let $f(x) = x + 4\cos x$. Let β be the negative root of the equation $f(x) = 0$. Using linear interpolation on the interval $[-1.5. -1]$,

$$\beta = \frac{-1.5f(-1) - (-1)f(-1.5)}{f(-1) - f(1.5)} = -1.24 \text{ (2 d.p.)}.$$

There is a minimum at $x = m$ such that m is between the two positive roots. Hence, when using the Newton-Raphson method, an initial approximation which is smaller than m would result in subsequent approximations being further away from the desired root α. Hence, an initial approximation that is smaller than α may not be satisfactory.

We know from the above graph that $\alpha \in (\pi, 3\pi/2)$. We hence pick $3\pi/2$ as our initial approximation. Using the Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with $x_1 = 3\pi/2$, we have

r	x_r
1	$\frac{3}{2}\pi$
2	3.7699
3	3.6106
4	3.5955
5	3.5953

Since $f(3.55)f(3.65) = (-0.1)(0.2) < 0$, we have $\alpha \in (3.55, 3.65)$. Hence, $\alpha = 3.6$ (2 s.f.).

* * * * *

Problem 2. Find the coordinates of the stationary points on the graph $y = x^3 + x^2$. Sketch the graph and hence write down the set of values of the constant k for which the equation $x^3 + x^2 = k$ has three distinct real roots.

The positive root of the equation $x^3 + x^2 = 0.1$ is denoted by α.

(a) Find a first approximation to α by linear interpolation on the interval $0 \leq x \leq 1$.

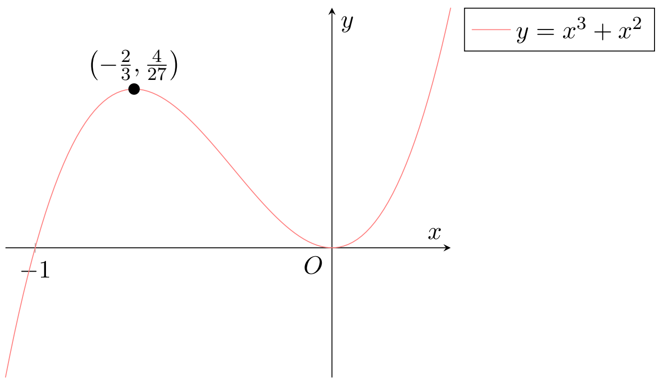
(b) With the aid of a suitable figure, indicate why, in this case, linear interpolation does not give a good approximation to α.

(c) Find an alternative first approximation to α by using the fact that if x is small then x^3 is negligible when compared to x^2.

Solution. For stationary points, $y' = 0$.

$$y' = 0 \implies 3x^2 + 2x = 0 \implies x(3x + 2) = 0.$$

Hence, $x = 0$ or $x = -2.3$. When $x = 0$, $y = 0$. When $x = -2/3$, $y = 4/27$. Thus, the coordinates of the stationary points of $y = x^3 + x^2$ are $(0, 0)$ and $(-2/3, 4/27)$.

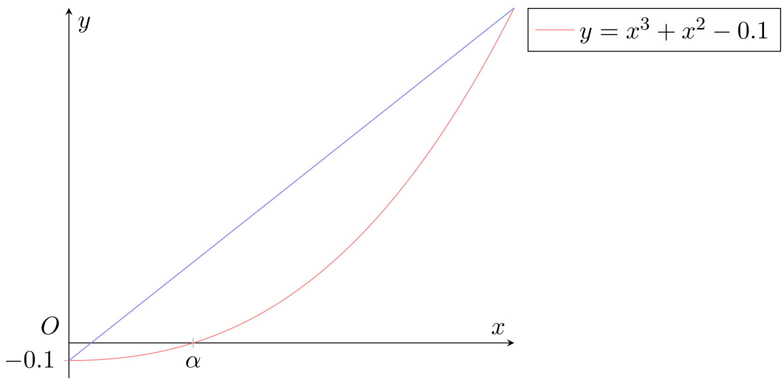


Therefore, $k \in (0, 4/27)$. The solution set of k is thus $\{k \in \mathbb{R} : 0 < k < 4/27\}$.

Part (a). Let $f(x) = x^2 + x^2 - 0.1$. Using linear interpolation on the interval $[0, 1]$,

$$\alpha = \frac{-f(0)}{f(1) - f(0)} = \frac{1}{20}.$$

Part (b).



On the interval $[0, 1]$, the gradient of $y = x^3 + x^2 - 0.1$ changes considerably. Hence, linear interpolation gives an approximation much less than the actual value.

Part (c). For small x, x^3 is negligible when compared to x^2. Consider $g(x) = x^2 - 0.1$. Then the positive root of $g(x) = 0$ is approximately α. Hence, an alternative approximation to α is $\sqrt{0.1} = 0.316$ (3 s.f.).

* * * * *

Problem 3. The equation $2\cos x - x = 0$ has a root α in the interval $[1, 1.2]$. Iterations of the form $x_{n+1} = F(x_n)$ are based on each of the following rearrangements of the equation:

(a) $x = 2\cos x$

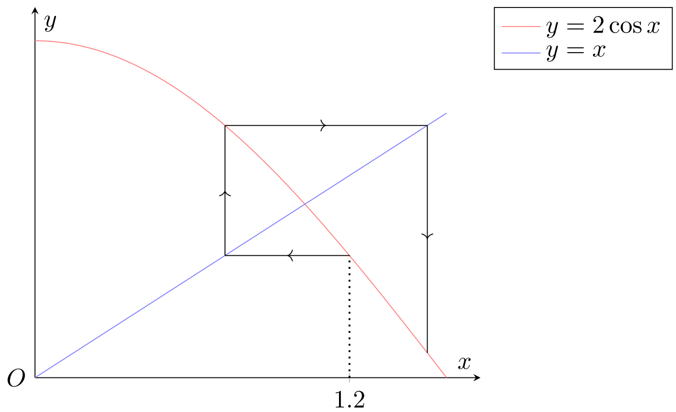
(b) $x = \cos x + \frac{1}{2}x$

(c) $x = \frac{2}{3}(\cos x + x)$

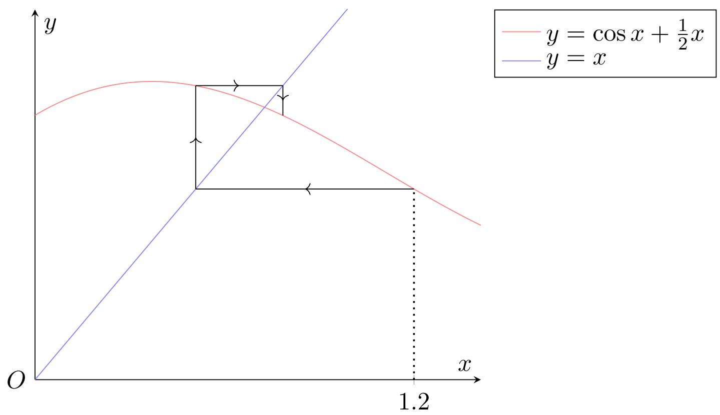
Determine which iteration will converge to α and illustrate your answer by a 'staircase' or 'cobweb' diagram. Use the most appropriate iteration with $x_1 = 1$, to find α to 4 significant figures. You should demonstrate that your answer has the required accuracy.

Solution.

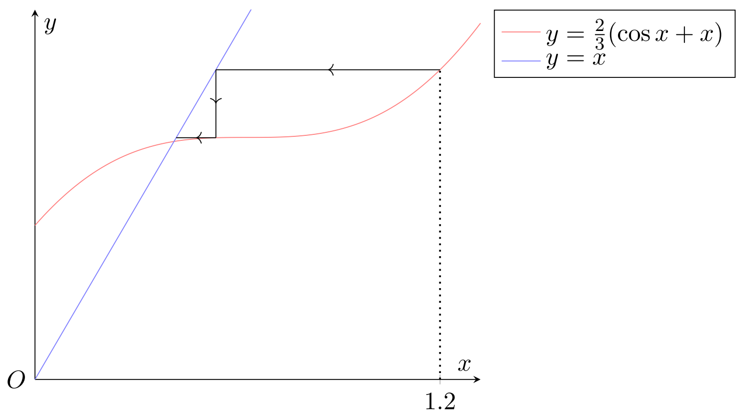
Part (a). Consider $f(x) = 2\cos x$. Then $f'(x) = -2\sin x$. Observe that $\sin x$ is increasing on $[1, 1.2]$. Since $\sin 1 > \frac{1}{2}$, $|f'(x)| > 1$ for all $x \in [1, 1.2]$. Thus, fixed-point iteration fails and will not converge to α.



Part (b). Consider $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} - \left(\sin x - \frac{1}{2}\right)$. Since $0 \leq \sin x \leq 1$ for $x \in \left[0, \frac{\pi}{2}\right]$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we know $-\frac{1}{2} \leq \sin x - \frac{1}{2} \leq \frac{1}{2}$ for $x \in [1, 1.2]$. Thus, $0 \leq \left|\sin x - \frac{1}{2}\right| \leq \frac{1}{2}$ for $x \in [1, 1.2]$. Hence, fixed-point iteration will work and converge to α.



Part (c). Consider $f(x) = \frac{2}{3}(\cos x + x)$. Then $f'(x) = \frac{2}{3}(-\sin x + 1)$. For fixed-point iteration to converge to α, we need $|f'(x)| < 1$ for x near α. It thus suffices to show that $|-\sin x + 1| < \frac{3}{2}$ for all $x \in [1, 1.2]$. Observe that $1 - \sin x$ is strictly decreasing and positive for $x \in \left[0, \frac{\pi}{2}\right]$. Since $1 - \sin 1 < \frac{3}{2}$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we have that $|-\sin x + 1| < \frac{3}{2}$ for all $x \in [1, 1.2]$. Thus, $|f'(x)| < 1$ for x near α. Hence, fixed-point iteration will work and converge to α.



For $x \in [1, 1.2]$, $\left|\frac{2}{3}(-\sin x + 1)\right| < \left|-\sin x + \frac{1}{2}\right| < 1$. Thus, $x_{n+1} = \frac{2}{3}(\cos x_n + x_n)$ is the most suitable iteration as it will converge to α the quickest. Using $F(x_{n+1}) = \frac{2}{3}(\cos x_n + x_n)$ with $x_1 = 1$,

r	x_r
1	1
2	1.02687
3	1.02958
4	1.02984
5	1.02986

Since $F(1.0295) > 1.0295$ and $F(1.0305) < 1.0305$, we have $\alpha \in (1.0295, 1.0305)$. Hence, $\alpha = 1.030$ (4 s.f.).

A3. Sequences and Series I

Tutorial A3

Problem 1. Determine the behaviour of the following sequences.

(a) $u_n = 3\left(\frac{1}{2}\right)^{n-1}$

(b) $v_n = 2 - n$

(c) $t_n = (-1)^n$

(d) $w_n = 4$

Solution.
Part (a). Decreasing, converges to 0.
Part (b). Decreasing, diverges.
Part (c). Alternating, diverges.
Part (d). Constant, converges to 4.

* * * * *

Problem 2. Find the sum of all even numbers from 20 to 100 inclusive.

Solution. The even numbers from 20 to 100 inclusive form an AP with common difference 2, first term 20 and last term 100. Since we are adding a total of $\frac{100-20}{2} + 1 = 41$ terms, we get a sum of $41\left(\frac{20+100}{2}\right) = 2460$.

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Problem 3. A geometric series has first term 3, last term 384 and sum 765. Find the common ratio.

Solution. Let the nth term of the geometric series be ar^{n-1}, where $1 \leq n \leq k$. We hence have $3r^{k-1} = 384$, which gives $r^k = 128r$. Thus,

$$\frac{3(1-r^k)}{1-r} = 765 \implies \frac{3(1-128r)}{1-r} = 765 \implies r = 2.$$

* * * * *

Problem 4.

(a) Find the first four terms of the following sequence $u_{n+1} = \frac{u_n+1}{u_n+2}$, $u_1 = 0$, $n \geq 1$.

(b) Write down the recurrence relation between the terms of these sequences.

(i) $-1, 2, -4, 8, -16, \ldots$

(ii) $1, 3, 7, 15, 31, \ldots$

Solution.
Part (a). Using G.C., the first four terms of u_n are 0, $\frac{1}{2}$, $\frac{3}{5}$ and $\frac{8}{13}$.

Part (b).

Part (b)(i). $u_{n+1} = -2u_n$, $u_1 = -1$, $n \geq 1$.

Part (b)(ii). $u_{n+1} = 2u_n + 1$, $u_1 = 1$, $n \geq 1$.

* * * * *

Problem 5. The sum of the first n terms of a series, S_n, is given by $S_n = 2n(n+5)$. Find the nth term and show that the terms are in arithmetic progression.

Solution. We have

$$u_n = S_n - S_{n-1} = 2n(n+5) - 2(n-1)(n+4) = 4n+8.$$

Observe that $u_n - u_{n-1} = [4n+8] - [4(n-1)+8] = 8$ is a constant. Hence, u_n is in AP.

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Problem 6. The sum of the first n terms, S_n, is given by

$$S_n = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}.$$

(a) Find an expression for the nth term of the series.

(b) Hence or otherwise, show that it is a geometric series.

(c) State the values of the first term and the common ratio.

(d) Give a reason why the sum of the series converges as n approaches infinity and write down its value.

Solution.

Part (a). Note that

$$u_n = S_n - S_{n-1} = \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}\right] - \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n}\right] = \left(\frac{1}{2}\right)^{n+1}.$$

Part (b). Since $\frac{u_{n+1}}{u_n} = \frac{(1/2)^{n+2}}{(1/2)^{n+1}} = \frac{1}{2}$ is constant, u_n is in GP.

Part (c). The first term is $\frac{1}{4}$ and the common ratio is $\frac{1}{2}$.

Part (d). As $n \to \infty$, we clearly have $\left(\frac{1}{2}\right)^{n+1} \to 0$. Hence, $S_\infty = \frac{1}{2}$.

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Problem 7. The first term of an arithmetic series is $\ln x$ and the rth term is $\ln\left(xk^{r-1}\right)$, where k is a real constant. Show that the sum of the first n terms of the series is $S_n = \frac{n}{2}\ln\left(x^2k^{n-1}\right)$. If $k = 1$ and $x \neq 1$, find the sum of the series $\mathrm{e}^{S_1} + \mathrm{e}^{S_2} + \mathrm{e}^{S_3} + \ldots + \mathrm{e}^{S_n}$.

Solution. Let u_n be the nth term in the arithmetic series. Then

$$u_n = \ln\left(xk^{r-1}\right) = \ln x + (r-1)\ln k.$$

We thus see that the arithmetic series has first term $\ln x$ and common difference of $\ln k$. Thus,

$$S_n = n\left(\frac{\ln x + (\ln x + (r-1)\ln k)}{2}\right) = \frac{n}{2}\ln\left(x^2k^{r-1}\right).$$

When $k = 1$, we have $S_n = \ln(x^n)$, whence $\mathrm{e}^{S_n} = x^n$. Thus,

$$\mathrm{e}^{S_1} + \mathrm{e}^{S_2} + \mathrm{e}^{S_3} + \ldots + \mathrm{e}^{S_n} = x + x^2 + x^3 + \ldots x^n = \frac{x(1-x^{n+1})}{1-x}.$$

* * * * *

Problem 8. A baker wants to bake a 1-metre tall birthday cake. It comprises 10 cylindrical cakes each of equal height 10 cm. The diameter of the cake at the lowest layer is 30 cm. The diameter of each subsequent layer is 4% less than the diameter of the cake below. Find the volume of this cake in cm^3, giving your answer to the nearest integer.

Solution. Let the diameter of the nth layer be d_n cm. We have $d_{n+1} = 0.96 d_n$ and $d_1 = 30$, whence $d_n = 30 \cdot 0.96^{n-1}$. Let the nth layer have volume v_n cm^3. Then

$$v_n = 10\pi \left(\frac{d_n}{2}\right)^2 = 10\pi \left(\frac{900 \cdot 0.9216^{n-1}}{4}\right) = 2250\pi \cdot 0.9216^{n-1}.$$

The volume of the cake in cm^3 is thus given by

$$2250\pi \left(\frac{1-0.9216^{10}}{1-0.9216}\right) = 50309.$$

* * * * *

Problem 9. The sum to infinity of a geometric progression is 5 and the sum to infinity of another series is formed by taking the first, fourth, seventh, tenth, ... terms is 4. Find the exact common ratio of the series.

Solution. Let the nth term of the geometric progression be given by ar^{n-1}. Then, we have

$$\frac{a}{1-r} = 5 \implies a = 5(1-r). \tag{1}$$

Note that the first, fourth, seventh, tenth, ... terms forms a new geometric series with common ratio r^3: $a, ar^3, ar^6, ar^9, \ldots$. Thus,

$$\frac{a}{1-r^3} = 4 \implies a = 4(1-r^3). \tag{2}$$

Equating (1) and (2), we have

$$5(1-r) = 4(1-r^3) \implies 4r^3 + 5r + 1 = 0 \implies (r-1)(4r^2+4r-1) = 0.$$

Since $|r| < 1$, we only have $4r^2 + 4r - 1 = 0$, which has solutions $r = \frac{-1+\sqrt{2}}{2}$ or $r = \frac{-1-\sqrt{2}}{2}$. Once again, since $|r| < 1$, we reject $r = \frac{-1-\sqrt{2}}{2}$. Hence, $r = \frac{-1+\sqrt{2}}{2}$.

* * * * *

Problem 10. A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, fourth and sixth terms respectively of the arithmetic series.

(a) Show that $3r^2 - 5r + 2 = 0$

(b) Deduce that the geometric series is convergent and find, in terms of a, the sum of infinity.

(c) The sum of the first n terms of the arithmetic series is denoted by S. Given that $a > 0$, find the set of possible values of n for which S exceeds $4a$.

Solution.

Part (a). Let the nth term of the geometric series be $G_n = G_1 r^{n-1}$. Let the nth term of the arithmetic series be $A_n = a + (n-1)d$.

Since $G_1 = A_1$, we have $G_1 = a$. We can thus write $G_n = ar^{n-1}$. From $G_2 = A_4$, we have $ar = a + 3d$, which gives $a = \frac{3d}{r-1}$. From $G_3 = A_6$, we have $ar^2 = a + 5d$. Thus,

$$\frac{3d}{r-1} \cdot r^2 = \frac{3d}{r-1} + 5d \implies \frac{3r^2}{r-1} = \frac{3}{r-1} + 5 \implies 3r^2 - 5r + 2 = 0.$$

Part (b). Note that the roots to $3r^2 - 5r + 2 = 0$ are $r = 1$ and $r = 2/3$. Clearly, $r \neq 1$ since $a = 3d/(r-1)$ would be undefined. Hence, $r = 2/3$, whence the geometric series is convergent.

Let S_∞ be the sum to infinity of G_n. Then $S_\infty = a/(1-r) = 3a$.

Part (c). Note that $d = a(r-1)/3 = -\frac{a}{9}$. Hence,

$$S = n\left(\frac{a + [a + (n-1)d]}{2}\right) = n\left(\frac{2a + (n-1)\left(-\frac{a}{9}\right)}{2}\right) = \frac{an}{18}(19 - n).$$

Consider $S > 4a$.

$$S > 4a \implies \frac{n}{18}(19 - n) > 4 \implies -n^2 + 19n - 72 > 0.$$

Using G.C., we see that $5.23 < n < 13.8$. Since n is an integer, the set of values that n can take on is $\{n \in \mathbb{Z} : 6 \leq n \leq 13\}$.

* * * * *

Problem 11. Two musical instruments, A and B, consist of metal bars of decreasing lengths.

(a) The first bar of instrument A has length 20 cm and the lengths of the bars form a geometric progression. The 25th bar has length 5 cm. Show that the total length of all the bars must be less than 357 cm, no matter how many bars there are.

Instrument B consists of only 25 bars which are identical to the first 25 bars of instrument A.

(b) Find the total length, L cm, of all the bars of instrument B and the length of the 13th bar.

(c) Unfortunately, the manufacturer misunderstands the instructions and constructs instrument B wrongly, so that the lengths of the bars are in arithmetic progression with a common difference d cm. If the total length of the 25 bars is still L cm and the length of the 25th bar is still 5 cm, find the value of d and the length of the longest bar.

Solution.

Part (a). Let $u_n = u_1 r^{n-1}$ be the length of the nth bar. Since $u_1 = 20$, we have $u_n = 20r^{n-1}$. Since $u_{25} = 5$, we have $r = 4^{-\frac{1}{24}}$. Hence, $u_n = 20 \cdot 4^{-\frac{n-1}{24}}$. Now, consider the sum to infinity of u_n:

$$S_\infty = \frac{u_1}{1-r} = \frac{20}{1 - 4^{-1/24}} = 356.3 < 357.$$

Hence, no matter how many bars there are, the total length of the bars will never exceed 357 cm.

Part (b). We have

$$L = u_1 \left(\frac{1 - r^{25}}{1 - r} \right) = 20 \left(\frac{1 - 4^{-25/24}}{1 - 4^{-1/24}} \right) = 272.26 = 272 \text{ (3 s.f.)}.$$

Note that

$$u_{13} = 20 \cdot \left(4^{-1/24} \right)^{13-1} = 10.$$

The 13th bar is hence 10 cm long.

Part (c). Let $v_n = a + (n-1)d$ be the length of the wrongly-manufactured bars. Since the length of the 25th bar is still 5 cm, we know $v_{25} = a + 24d = 5$. Now, consider the total lengths of the bars, which is still L cm.

$$L = 25 \left(\frac{a + 5}{2} \right) = 272.26.$$

Solving, we see that $a = 16.781$. Hence, $d = \frac{5-a}{24} = -0.491$, and the longest bar is $16.8 =$ cm long.

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Problem 12. A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man a woman both invest money.

(a) The man decides to invest \$$x$ at the beginning of one year and then a further \$$x$ at the beginning of the second and each subsequent year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up.

(i) How much will there be in the account at the end of 1 year, including the interest?

(ii) Show that, at the end of n years, when the interest for the last year has been added, he will have a total of \$$21(1.05^n - 1)x$ in his account.

(iii) After how many complete years will he have, for the first time, at least \$$12x$ in his account?

(b) The woman decides that, to assist her in her everyday expenses, she will withdraw the interest as soon as it has been added. She invests \$$y$ at the beginning of each year. Show that, at the end of n years, she will have received a total of \$$\frac{1}{40}n(n+1)y$ in interest.

Solution.

Part (a).

Part (a)(i). There will be \$$1.05x$ in the account at the end of 1 year.

Part (a)(ii). Let \$$u_n x$ be the amount of money in the account at the end of n years. Then, u_n satisfies the recurrence relation $u_{n+1} = 1.05(1 + u_n)$, with $u_1 = 1.05$. Observe that

$$u_1 = 1.05 \implies u_2 = 1.05 + 1.05^2 \implies u_3 = 1.05 + 1.05^2 + 1.05^3 \implies \cdots.$$

We thus have

$$u_n = 1.05 + 1.05^2 + \cdots + 1.05^n = 1.05 \left(\frac{1 - 1.05^n}{1 - 1.05} \right) = 21 \left(1.05^n - 1 \right).$$

Hence, there will be \$$21(1.05^n - 1)x$ in the account after n years.

Part (a)(iii). Consider the inequality $u_n \geq 12x$.

$$u_n \geq 12x \implies 21(1.05^n - 1) \geq 12 \implies n \geq 9.26.$$

Since n is an integer, the smallest value of n is 10. Hence, after 10 years, he will have at least \$$12x$ in his account for the first time.

Part (b). After n years, the woman will have \$$ny$ in her account. Hence, the interest she gains at the end of the nth year is $\frac{1}{20}ny$. Thus, the total interest she will gain after n years is

$$\frac{y}{20} + \frac{2y}{20} + \cdots + \frac{ny}{20} = \frac{y}{20}(1 + 2 + \cdots + n) = \frac{y}{20} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)y}{40}.$$

* * * * *

Problem 13. The sum, S_n, of the first n terms of a sequence $U_1, U_2, U_3, \ldots$ is given by

$$S_n = \frac{n}{2}(c - 7n),$$

where c is a constant.

(a) Find U_n in terms of c and n.

(b) Find a recurrence relation of the form $U_{n+1} = f(U_n)$.

Solution.

Part (a). Observe that

$$U_n = S_n - S_{n-1} = \frac{n}{2}(c - 7n) - \frac{n-1}{2}(c - 7(n-1)) = -7n + \frac{7+c}{2}.$$

Part (b). Observe that $U_{n+1} - U_n = -7$. Thus,

$$U_{n+1} = U_n - 7, \quad U_1 = \frac{7+c}{2}, \quad n \geq 1.$$

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Problem 14. The positive numbers x_n satisfy the relation

$$x_{n+1} = \sqrt{\frac{9}{2} + \frac{1}{x_n}}$$

for $n = 1, 2, 3, \ldots$.

(a) Given that $n \to \infty$, $x_n \to \theta$, find the exact value of θ.

(b) By considering $x_{n+1}^2 - \theta^2$, or otherwise, show that if $x_n > \theta$, then $0 < x_{n+1} < \theta$.

Solution.

Part (a). Observe that

$$\theta = \lim_{n\to\infty} \sqrt{\frac{9}{2} + \frac{1}{x_n}} = \sqrt{\frac{9}{2} + \frac{1}{\theta}} \implies 2\theta^3 - 9\theta - 2 = 0 \implies (\theta + 2)\left(2\theta^2 - 4\theta - 1\right) = 0.$$

We reject $\theta = -2$ since $\theta > 0$. We thus consider $2\theta^2 - 4\theta - 1 = 0$, which has roots $\theta = 1 + \sqrt{\frac{3}{2}}$ and $\theta = 1 - \sqrt{\frac{3}{2}}$. Once again, we reject $\theta = 1 - \sqrt{\frac{3}{2}}$ since $\theta > 0$. Thus, $\theta = 1 + \sqrt{\frac{3}{2}}$.

Part (b). Suppose $x_n > \theta$. Then

$$x_{n+1}^2 = \frac{9}{2} + \frac{1}{x_n} < \frac{9}{2} + \frac{1}{\theta} = \theta^2 \implies 0 < x_{n+1} < \theta.$$

Assignment A3

Problem 1. A university student has a goal of saving at least \$1 000 000 (in Singapore dollars). He begins working at the start of the year 2019. In order to achieve his goal, he saves 40% of his annual salary at the end of each year. If his annual salary in the year 2019 is \$40800, and it increases by 5% (of his previous year's salary) every year, find

(a) his annual savings in 2027 (to the nearest dollar),

(b) his total savings at the end of n years.

What is the minimum number of complete years for which he has to work in order to achieve his goal?

Solution. Let \$$u_n$ be his annual salary in the nth year after 2019, with $n \in \mathbb{N}$. Then $u_{n+1} = 1.05 \cdot u_n$, with $u_0 = 40800$. Hence, $u_n = 40800 \cdot 1.05^n$. Let \$$v_n$ be the amount saved in the nth year after 2019. Then $v_n = 0.40 \cdot u_n = 16320 \cdot 1.05^n$.

Part (a). In 2027, $n = 8$. Hence, his annual savings in 2027, in dollars, is given by

$$v_8 = 16320 \cdot 1.05^8 = 24112 \text{ (to the nearest integer).}$$

Part (b). His total savings at the end of n years, in dollars, is given by

$$16320\left(1.05^0 + 1.05^1 + \cdots + 1.05^n\right) = 16320\left(\frac{1 - 1.05^n}{1 - 1.05}\right) = 326400\left(1.05^n - 1\right).$$

Consider $326400\left(1.05^n - 1\right) \geq 1000000$. Using G.C., we see that $n \geq 28.7$. Thus, he needs to work for a minimum of 29 complete years to reach his goal.

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Problem 2.

(a) A rope of length 200π cm is cut into pieces to form as many circles as possible, whose radii follow an arithmetic progression with common difference 0.25 cm. Given that the smallest circle has an area of π cm^2, find the area of the largest circle in terms of π.

(b) The sum of the first n terms of a sequence is given by $S_n = \alpha^{-n} - 1$, where α is a non-zero constant, $\alpha \neq 1$.

 (i) Show that the sequence is a geometric progression and state its common ratio in terms of α.

 (ii) Find the set of values of α for which the sum to infinity of the sequence exists.

 (iii) Find the value of the sum to infinity.

Solution.

Part (a). Let the sequence r_n be the radius of the nth smallest circle, in centimetres. Hence, $r_n = \frac{1}{4} + r_{n-1}$. Since the smallest circle has area π cm^2, $r_1 = 1$. Thus, $r_n = 1 + \frac{1}{4}(n-1)$.

Consider the nth partial sum of the circumferences:

$$2\pi r_1 + 2\pi r_2 + \cdots + 2\pi r_n = 2\pi \cdot n\left(\frac{1 + \left[1 + \frac{1}{4}(n-1)\right]}{2}\right) = \frac{\pi(n^2 + 7n)}{4}.$$

Since the rope has length 200π cm, we have the inequality

$$\frac{\pi(n^2+7n)}{4} \leq 200\pi \implies n^2 - 7n - 800 \leq 0 \implies (n+32)(n-25) \leq 0.$$

Hence, $n \leq 25$. Since the rope is cut to form as many circles as possible, $n = 25$. Thus, the largest circle has area $\pi \cdot r_{25}^2 = 49\pi$ cm^2.

Part (b). Let the sequence being summed by $u_1, u_2, \ldots$. Observe that

$$u_n = S_n - S_{n-1} = \left(\alpha^{-n} - 1\right) - \left(\alpha^{-(n-1)} - 1\right) = \alpha^{-n}(1-\alpha).$$

Part (b)(i). Observe that

$$\frac{u_{n+1}}{u_n} = \frac{\alpha^{-(n+1)}(1-\alpha)}{\alpha^{-n}(1-\alpha)} = \alpha^{-1},$$

which is a constant. Thus, u_n is in GP with common ratio α^{-1}.

Part (b)(ii). Consider $S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(\alpha^{-n} - 1)$. For S_∞ to exist, we need $\lim_{n\to\infty} \alpha^{-n}$ to exist. Hence, $\left|\alpha^{-1}\right| < 1$, whence $|\alpha| > 1$. Thus, $\alpha < -1$ or $\alpha > 1$. The solution set of α is thus $\{x \in \mathbb{R} : x < -1 \text{ or } x > 1\}$.

Part (b)(iii). Since $\left|\alpha^{-1}\right| < 1$, we know $\lim_{n\to\infty} \alpha^{-n} = 0$. Hence, $S_\infty = -1$.

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Problem 3. A sequence $u_1, u_2, u_3, \ldots$ is such that $u_{n+1} = 2u_n + An$, where A is a constant and $n \geq 1$.

(a) Given that $u_1 = 5$ and $u_2 = 15$, find A and u_3.

It is known that the nth term of this sequence is given by

$$u_n = a(2^n) + bn + c,$$

where a, b and c are constants.

(b) Find a, b and c.

Solution.

Part (a). Substituting $n = 1$ into the recurrence relation yields $u_2 = 2u_1 + A$. Thus, $A = u_2 - 2u_1 = 5$. Substituting $n = 2$ into the recurrence relation yields $u_3 = 2u_2 + 2A = 40$.

Part (b). Since $u_1 = 5$, $u_2 = 15$ and $u_3 = 40$, we have the following system

$$\begin{cases} 2a + b + c = 5 \\ 4a + 2b + c = 15 \\ 8a + 3b + c = 40 \end{cases}$$

which has the unique solution $a = \frac{15}{2}$, $b = -5$ and $c = -5$

Problem 4. The graphs of $y = 2^x/3$ and $y = x$ intersect at $x = \alpha$ and $x = \beta$ where $\alpha < \beta$. A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3} \cdot 2^{x_n}, \qquad n \geq 1.$$

(a) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(b) By using the graphs of $y = \frac{1}{3} \cdot 2^x$ and $y = x$, prove that

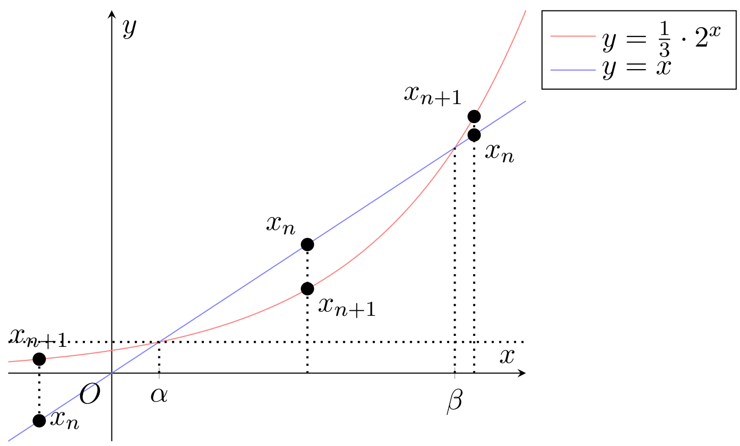
- if $\alpha < x_n < \beta$, then $\alpha < x_{n+1} < x_n$
- if $x_n < \alpha$, then $x_n < x_{n+1} < \alpha$
- if $x_n > \beta$, then $x_n < x_{n+1}$

Describe the behaviour of the sequence for the three cases.

Solution.

Part (a). Let $L = \lim_{n\to\infty} x_n$. Then $L = \frac{1}{3} \cdot 2^L$. Since $y = x$ and $y = \frac{1}{3} \cdot 2^x$ intersect only at $x = \alpha$ and $x = \beta$, then α and β are the only roots of $x = \frac{1}{3} \cdot 2^x$. Since L is also a root of $x = \frac{1}{3} \cdot 2^x$, L must be either α or β.

Part (b).



If $\alpha < x_n < \beta$, then x_n is decreasing and converges to α. If $x_n < \alpha$, then x_n is increasing and converges to α. If $x_n > \beta$, then x_n is increasing and diverges.

A4. Sequences and Series II

Tutorial A4

Problem 1. True or False? Explain your answers briefly.

(a) $\sum_{r=1}^{n}(2r+3) = \sum_{k=1}^{n}(2k+3)$

(b) $\sum_{r=1}^{n}\left(\frac{1}{r}+5\right) = \sum_{r=1}^{n}\frac{1}{r}+5$

(c) $\sum_{r=1}^{n}\frac{1}{r} = 1/\sum_{r=1}^{n} r$

(d) $\sum_{r=1}^{n} c = \sum_{r=0}^{n-1}(c+1)$

Solution.
Part (a). True: A change in index does not affect the sum.
Part (b). False: In general, $\sum_{r=1}^{n} 5$ is not equal to 5.
Part (c). False: In general, $\sum \frac{a}{b} \neq \sum a / \sum b$.
Part (d). False: Since c is a constant, $\sum_{r=1}^{n} c = nc \neq n(c+1) = \sum_{r=0}^{n-1}(c+1)$.

* * * * *

Problem 2. Write the following series in sigma notation twice, with $r = 1$ as the lower limit in the first and $r = 0$ as the lower limit in the second.

(a) $-2+1+4+\ldots+40$

(b) $a^2+a^4+a^6+\ldots+a^{50}$

(c) $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term}$

(d) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ to n terms

(e) $\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30}$

Solution.
Part (a).

$$-2+1+4+\ldots+40 = \sum_{r=1}^{15}(3r-5) = \sum_{r=0}^{14}(3r-2).$$

Part (b).

$$a^2+a^4+a^6+\ldots+a^{50} = \sum_{r=1}^{25} a^{2r} = \sum_{r=0}^{24} a^{2r+2}.$$

Part (c).

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term} = \sum_{r=1}^{n}\frac{1}{2r+1} = \sum_{r=0}^{n-1}\frac{1}{2r+3}.$$

Part (d).

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots \text{ to } n \text{ terms} = \sum_{r=1}^{n}\left(-\frac{1}{2}\right)^{r-1} = \sum_{r=0}^{n-1}\left(-\frac{1}{2}\right)^{r}.$$

Part (e).

$$\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30} = \sum_{r=1}^{27}\frac{1}{(r+1)(r+3)} = \sum_{r=0}^{26}\frac{1}{(r+2)(r+4)}.$$

* * * * *

Problem 3. Without using the G.C., evaluate the following sums.

(a) $\sum_{r=1}^{50}(2r-7)$

(b) $\sum_{r=1}^{a}(1-a-r)$

(c) $\sum_{r=2}^{n}\left(\ln r+3^r\right)$

(d) $\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right)$

Solution.

Part (a).

$$\sum_{r=1}^{50}(2r-7) = 2\sum_{r=1}^{50}r - 7\sum_{r=1}^{50}1 = 2\left(\frac{50\cdot 51}{2}\right)-7(50) = 2200.$$

Part (b).

$$\sum_{r=1}^{a}(1-a-r) = (1-a)\sum_{r=1}^{a}1-\sum_{r=1}^{a}r = (1-a)a-\frac{a(a+1)}{2} = \frac{a}{2}(1-3a).$$

Part (c).

$$\sum_{r=2}^{n}\left(\ln r+3^r\right) = \sum_{r=2}^{n}\ln r+\sum_{r=2}^{n}3^r = \ln n! + 3^2\left(\frac{1-3^{n-2+1}}{1-3}\right) = \ln n! + \frac{9}{2}\left(3^{n-1}-1\right).$$

Part (d).

$$\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right) = \sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^r - \sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r = \frac{2/3}{1-2/3}-\frac{1/3}{1-1/3} = \frac{3}{2}.$$

* * * * *

Problem 4. The nth term of a series is $2^{n-2}+3n$. Find the sum of the first N terms.

Solution.

$$\begin{aligned}\sum_{n=1}^{N}\left(2^{n-2}+3r\right) &= \sum_{n=1}^{N}2^{n-2}+3\sum_{n=1}^{N}n \\ &= 2^{1-2}\left(\frac{\left(2^N-1\right)}{2-1}\right)+3\left(\frac{N(N+1)}{2}\right) \\ &= \frac{1}{2}\left(2^N+3N^2+3N-1\right).\end{aligned}$$

Problem 5. The rth term, u_r, of a series is given by $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$. Express $\sum_{r=1}^{n} u_r$ in the form $A\left(1 - \frac{B}{27^n}\right)$, where A and B are constants. Deduce the sum to infinity of the series.

Solution. Observe that

$$u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1} = 12\left(\frac{1}{3}\right)^{3r} = 12\left(\frac{1}{27}\right)^{r}.$$

Hence,

$$\sum_{r=1}^{n} = 12 \cdot \frac{1}{27}\left(\frac{1-1/27^n}{1-1/27}\right) = \frac{6}{13}\left(1 - \frac{1}{27^n}\right),$$

whence $A = \frac{6}{13}$ and $B = 1$. In the limit as $n \to \infty$, $\frac{1}{27^n} \to 0$. Hence, the sum to infinity is $\frac{6}{13}$.

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Problem 6. The rth term, u_r, of a series is given by $u_r = \ln \frac{r}{r+1}$. Find $\sum_{r=1}^{n} u_r$ in terms of n. Comment on whether the series converges.

Solution. Observe that $u_r = \ln \frac{r}{r+1} = \ln r - \ln(r+1)$. Hence,

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n} (\ln r - \ln(r+1)) \\ &= [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + \cdots + [\ln n - \ln(n+1)] \\ &= \ln 1 - \ln(n+1) = \ln \frac{1}{n+1}.\end{aligned}$$

As $n \to \infty$, $\ln \frac{1}{n+1} \to \ln 0$. Hence, the series diverges to negative infinity.

* * * * *

Problem 7. Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, without using the G.C., find the following sums.

(a) $\sum_{r=0}^{n} [r(r+4)+n]$

(b) $\sum_{r=n+1}^{2n} (2r-1)^2$

(c) $\sum_{r=-15}^{20} r(r-2)$

Solution.

Part (a).

$$\begin{aligned}\sum_{r=0}^{n} [r(r+4)+n] &= \sum_{r=0}^{n} \left(r^2 + 4r + n\right) \\ &= \frac{n}{6}(n+1)(2n+1) + 4\left[\frac{n(n+1)}{2}\right] + n(n+1) \\ &= \frac{n}{6}(n+1)(2n+19).\end{aligned}$$

Part (b).

$$\sum_{r=n+1}^{2n}(2r-1)^2=\sum_{r=1}^{n}(2(r+n)-1)^2=\sum_{r=1}^{n}\left(4r^2+4(2n-1)r+(2n-1)^2\right)$$
$$=4\left[\frac{n}{6}(n+1)(2n+1)\right]+4(2n-1)\left[\frac{n(n+1)}{2}\right]+(2n-1)^2n$$
$$=\frac{1}{3}n\left(28n^2-1\right)$$

Part (c).

$$\sum_{r=-15}^{20}r(r-2)=\sum_{r=1}^{36}(r-16)[(r-16)-2]=\sum_{r=1}^{36}\left(r^2-34r+288\right)$$
$$=\frac{36}{6}\left[(36+1)(2\cdot 36+1)\right]-34\left[\frac{36\cdot 37}{2}\right]+288(36)$$
$$=3930$$

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Problem 8. Let $S=\sum_{r=0}^{\infty}\frac{(x-2)^r}{3^r}$ where $x\neq 2$. Find the range of values of x such that the series S converges. Given that $x=1$, find

(a) the value of S

(b) S_n, in terms of n, where $S_n=\sum_{r=0}^{n-1}\frac{(x-2)^r}{3^r}$

(c) the least value of n for which $|S_n-S|$ is less than 0.001% of S

Solution. Note that
$$S=\sum_{r=0}^{\infty}\frac{(x-2)^r}{3^r}=\sum_{r=0}^{\infty}\left(\frac{x-2}{3}\right)^r.$$
Hence, for S to converge, we must have $\left|\frac{x-2}{3}\right|<1$, which gives $-1<x<5$, $x\neq 2$.

Part (a). When $x=1$, we get
$$S=\sum_{r=0}^{\infty}\left(-\frac{1}{3}\right)^r=\frac{1}{1-(-\frac{1}{3})}=\frac{3}{4}.$$

Part (b). We have
$$S_n=\sum_{r=0}^{n-1}\left(-\frac{1}{3}\right)^r=\frac{1-(-\frac{1}{3})^n}{1-(-\frac{1}{3})}=\frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right].$$

Part (c). Observe that
$$|S_n-S|<0.001\%S\implies\left|\frac{S_n-S}{S}\right|<\frac{1}{100000}\implies\left|\frac{\frac{3}{4}(1-(-\frac{1}{3})^n)}{\frac{3}{4}}-1\right|<\frac{1}{100000}.$$

Using G.C., the least value of n that satisfies the above inequality is 11.

Problem 9. Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$,

(a) write down $\sum_{r=1}^{2k} r^2$ in terms of k

(b) find $2^2 + 4^2 + 6^2 + \ldots + (2k)^2$.

Hence, show that $\sum_{r=1}^{k}(2r-1)^2 = \frac{k}{3}(2k+1)(2k-1)$.

Solution.

Part (a).

$$\sum_{r=1}^{2k} r^2 = \frac{2k}{6}(2k+1)(2(2k)+1) = \frac{k}{3}(2k+1)(4k+1).$$

Part (b).

$$2^2 + 4^2 + 6^2 + \ldots + (2k)^2 = \sum_{r=1}^{k}(2r)^2 = \sum_{r=1}^{k} 4r^2 = \frac{2k}{3}(k+1)(2k+1).$$

From parts (a) and (b), we clearly have

$$\sum_{r=1}^{k}(2r-1)^2 = \sum_{r=1}^{2k} r^2 - \sum_{r=1}^{k}(2r)^2 = \frac{k}{3}(2k+1)(4k+1) - \frac{2k}{3}(k+1)(2k+1) = \frac{k}{3}(2k+1)(2k-1).$$

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Problem 10. Given that $u_n = e^{nx} - e^{(n+1)x}$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. Hence, determine the set of values of x for which the infinite series $u_1 + u_2 + u_3 + \ldots$ is convergent and give the sum to infinity for cases where this exists.

Solution.

$$\sum_{n=1}^{N} u_n = \left(e^x - e^{2x}\right) + \left(e^{2x} - e^{3x}\right) + \cdots + \left(e^{Nx} + e^{(N+1)x}\right) = e^x - e^{(N+1)x}.$$

For the infinite series to converge, we require $|e^x| < 1$. Hence, $x \in \mathbb{R}_0^-$.

We now consider the sum to infinity.

Case 1. Suppose $x = 0$. Then $e^x = 1$, whence the sum to infinity is clearly 0.

Case 2. Suppose $x < 0$. Then $\lim_{N\to\infty} e^{(N+1)x} \to 0$. Thus, the sum to infinity is e^x.

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Problem 11. Given that r is a positive integer and $f(r) = \frac{1}{r^2}$, express $f(r) - f(r+1)$ as a single fraction. Hence, prove that $\sum_{r=1}^{4n}\left(\frac{2r+1}{r^2(r+1)^2}\right) = 1 - \frac{1}{(4n+1)^2}$. Give a reason why the series is convergent and state the sum to infinity. Find $\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right)$.

Solution.

$$f(r) - f(r+1) = \frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.$$

$$\begin{aligned}\sum_{r=1}^{4n}\left(\frac{2r+1}{r^2(r+1)^2}\right) &= \sum_{r=1}^{4n}[f(r) - f(r+1)] \\ &= [f(1) - f(2)] + [f(2) - f(3)] + \cdots + [f(4n) - f(4n-1)] \\ &= f(1) - f(4n+1) = 1 - \frac{1}{(4n+1)^2}\end{aligned}$$

As $n \to \infty$, $\frac{1}{(4n+1)^2} \to 0$. Hence, the series converges to 1.

$$\begin{aligned}\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right) &= \sum_{r=1}^{4n-1}\left(\frac{2r+1}{r^2(r+1)^2}\right) = \sum_{r=1}^{4n-1}[f(r)-f(r+1)]\\ &= [f(1)-f(2)]+[f(2)-f(3)]+\cdots+[f(4n-1)-f(4n)]\\ &= 1-f(4n) = 1-\frac{1}{16n^2}\end{aligned}$$

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Problem 12.

(a) Express $\frac{1}{(2x+1)(2x+3)(2x+5)}$ in partial fractions.

(b) Hence, show that $\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$.

(c) Deduce the sum of $\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \ldots + \frac{1}{41\cdot43\cdot45}$.

Solution.

Part (a). Using the cover-up rule, we obtain

$$\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}.$$

Part (b).

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} &= \sum_{r=1}^{n}\left(\frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)}\right)\\ &= \frac{1}{8}\left[\left(\sum_{r=1}^{n}\frac{1}{2r+1} - \sum_{r=1}^{n}\frac{1}{2r+3}\right) - \left(\sum_{r=1}^{n}\frac{1}{2r+3} - \sum_{r=1}^{n}\frac{1}{2r+5}\right)\right]\end{aligned}$$

Observe that the two terms in brackets clearly telescope, leaving us with

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{8}\left[\left(\frac{1}{3} - \frac{1}{2n+3}\right) - \left(\frac{1}{5} - \frac{1}{2n+5}\right)\right],$$

which simplifies to

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$$

as desired.

Part (c).

$$\begin{aligned}&\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \ldots + \frac{1}{41\cdot43\cdot45}\\ &= \frac{1}{1\cdot3\cdot5} + \sum_{r=1}^{20}\frac{1}{(2r+1)(2r+3)(2r+5)}\\ &= \frac{1}{15} + \left(\frac{1}{60} - \frac{1}{4(2\cdot20+3)(2\cdot20+5)}\right)\\ &= \frac{161}{1935}.\end{aligned}$$

A5. Recurrence Relations

Tutorial A5

Problem 1. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 2u_{n-1}$, for $n \geq 1$, $u_0 = 3$

(b) $u_n = 3u_{n-1} + 7$, for $n \geq 1$, $u_0 = 5$

Solution.
Part (a). $u_n = 2^n \cdot u_0 = 3 \cdot 2^n$.
Part (b). Let k be a constant such that $u_n + k = 3(u_{n-1} + k)$. Then $k = \frac{7}{2}$. Hence,

$$u_n + \frac{7}{2} = 3\left(u_{n-1} + \frac{7}{2}\right) \implies u_n + \frac{7}{2} = 3^n\left(u_0 + \frac{7}{2}\right) \implies u_n = \frac{17}{2} \cdot 3^n - \frac{7}{2}.$$

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Problem 2. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 5u_{n-1} - 6u_{n-2}$, for $n \geq 2$, $u_0 = 1$, $u_1 = 0$

(b) $u_n = 4u_{n-2}$, for $n \geq 2$, $u_0 = 0$, $u_1 = 4$

(c) $u_n = 4u_{n-1} - 4u_{n-2}$, for $n \geq 2$, $u_0 = 6$, $u_1 = 8$

(d) $u_n = -6u_{n-1} - 9u_{n-2}$, for $n \geq 2$, $u_0 = 3$, $u_1 = -3$

(e) $u_n = 2u_{n-1} - 2u_{n-2}$, for $n \geq 2$, $u_0 = 2$, $u_1 = 6$

Solution.
Part (a). Note that the characteristic equation of u_n, $x^2 - 5x + 6 = 0$, has roots 2 and 3. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n.$$

From $u_0 = 1$ and $u_1 = 0$, we have the equations $A + B = 1$ and $2A + 3B = 0$. Solving, we see that $A = 3$ and $B = 2$, whence

$$u_n = 3 \cdot 2^n + 2 \cdot 3^n.$$

Part (b). Note that the characteristic equation of u_n, $x^2 - 4 = 0$, has roots -2 and 2. Thus,

$$u_n = A(-2)^n + B \cdot 2^n.$$

From $u_0 = 0$ and $u_1 = 4$, we get $A + B = 0$ and $-2A + 2B = 4$. Solving, we see that $A = -1$ and $B = 1$, whence

$$u_n = -(-2)^n + 2^n.$$

Part (c). Note that the characteristic equation of u_n, $x^2 - 4x + 4 = 0$, has only one root, 2. Thus,

$$u_n = (A + Bn)2^n.$$

From $u_0 = 6$ and $u_1 = 8$, we obtain $A = 6$ and $A + B = 4$, whence $B = -2$. Thus,

$$u_n = (6 - 2n)2^n.$$

Part (d). Note that the characteristic equation of u_n, $x^2 + 6x + 9 = 0$, has only one root, -3. Thus,

$$u_n = (A + Bn)(-3)^n.$$

From $u_0 = 3$ and $u_1 = -3$, we get $A = 3$ and $A + B = 1$, whence $B = -2$. Thus,

$$u_n = (3 - 2n)2^n.$$

Part (e). Consider the characteristic equation of u_n, $x^2 - 2x + 2 = 0$. By the quadratic formula, this has roots $x = 1 \pm i = \sqrt{2}\exp\left(\pm\frac{i\pi}{4}\right)$. Hence,

$$u_n = A \cdot 2^{\frac{1}{2}n} \cos\left(\frac{n\pi}{4}n\right) + B \cdot 2^{\frac{1}{2}n} \sin\left(\frac{n\pi}{4}\right).$$

From $u_0 = 2$, we obtain $A = 2$. From $u_0 = 6$, we obtain $A + B = 6$, whence $B = 4$. Thus,

$$u_n = 2^{\frac{1}{2}n+1} \cos\left(\frac{n\pi}{4}\right) + 2^{\frac{1}{2}n+2} \sin\left(\frac{n\pi}{4}\right).$$

* * * * *

Problem 3.

(a) A sequence is defined by the formula $b_n = \frac{n!n!}{(2n)!} \cdot 2^n$, where $n \in \mathbb{Z}^+$. Show that the sequence satisfies the recurrence relation $b_{n+1} = \frac{n+1}{2n+1} b_n$.

(b) A sequence is defined recursively by the formula

$$u_{n+1} = 2u_n + 3, \qquad n \in \mathbb{Z}_0^+,\ u_0 = a$$

Show that $u_n = 2^n a + 3\left(2^n - 1\right)$.

Solution.

Part (a).

$$b_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot 2^{n+1} = \frac{2(n+1)^2}{(2n+1)(2n+2)} \cdot \left[\frac{n!n!}{(2n)!} \cdot 2^n\right] = \frac{n+1}{2n+1} b_n.$$

Part (b). Let k be a constant such that $u_{n+1} + k = 2(u_n + k)$. Then $k = 3$. Hence,

$$u_{n+1} + 3 = 2(u_n + 3) \implies u_n + 3 = 2^n(u_0 + 3) \implies u_n = 2^n(a+3) - 3 = 2^n a + 3\left(2^n - 1\right).$$

Problem 4. The volume of water, in litres, in a storage tank decreases by 10% by the end of each day. However, 90 litres of water is also pumped into the tank at the end of each day. The volume of water in the tank at the end of n days is denoted by x_n and x_0 is the initial volume of water in the tank.

(a) Write down a recurrence relation to represent the above situation.

(b) Show that $x_n = 0.9^n(x_0 - 900) + 900$.

(c) Deduce the amount of water in the tank when n becomes very large.

Solution.

Part (a). $x_{n+1} = 0.9x_n + 90$, $n \in \mathbb{N}$

Part (b). Let k be a constant such that $x_{n+1} + k = 0.9(x_n + k)$. Then $k = -900$. Hence,

$$x_{n+1} - 900 = 0.9(x_n - 900) \implies x_n - 900 = 0.9^n(x_0 - 900) \implies x_n = 0.9^n(x_0 - 900) + 900.$$

Part (c). As $n \to \infty$, $0.9^n \to 0$. Hence, the amount of water in the tank will converge to 900 litres.

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Problem 5. A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year, two dividends are awarded and reinvested into the fund. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

(a) Find a recurrence relation $\{P_n\}$ where P_n is the amount at the start of the nth year if no money is ever withdrawn.

(b) How much is in the account after n years if no money is ever withdrawn?

Solution.

Part (a).

$$P_{n+2} = P_{n+1} + 0.2P_{n+1} + 0.45P_n = 1.2P_{n+1} + 0.45P_n.$$

Part (b). Note that the characteristic equation of P_n, $x^2 - 1.2x - 0.45 = 0$, has roots $-\frac{3}{10}$ and $\frac{3}{2}$. Thus,

$$P_n = A\left(-\frac{3}{10}\right)^n + B\left(\frac{3}{2}\right)^n.$$

From $P_0 = 0$ and $P_1 = 100000$, we have $A + B = 0$ and $-\frac{3}{10}A + \frac{3}{2}B = 100000$. Solving, we have $A = -\frac{500000}{9}$ and $B = \frac{500000}{9}$. Thus,

$$P_n = \frac{500000}{9}\left[\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right].$$

Hence, there will be $\$\left\{\frac{500000}{9}\left[\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right]\right\}$ in the account after n years if no money is ever withdrawn

Problem 6. A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbit produces another pair each month.

(a) Find a recurrence relation $\{f_n\}$ where f_n is the total number of pairs of rabbits, assuming that no rabbits ever die.

(b) What is the number of pairs of rabbits at the end of the nth month, assuming that no rabbits ever die?

Solution.
Part (a). $f_{n+2} = f_{n+1} + f_n$, $n \geq 2$, $f_0 = 0, f_1 = 1$
Part (b). Consider the characteristic equation of f_n, $x^2 - x - 1 = 0$. By the quadratic formula, the roots of the characteristic equation are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Hence

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

From $f_0 = 0$, we get $A+B = 0$. From $f_1 = 1$, we get $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$. Solving, we get $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Hence,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

* * * * *

Problem 7. For $n \in \{2^j : j \in \mathbb{Z}, j \geq 1\}$, it is given that $T_n = 3T_{n/2} + 17$, where $T_1 = 4$. By considering the substitution $n = 2^i$ and another suitable substitution, show that the recurrence relation can be expressed in the form

$$t_i = 3t_{i-1} + 17, \qquad i \in \mathbb{Z}^+$$

Hence, find an expression for T_n in terms of n.

Solution. Let $n = 2^i \iff i = \log_2 n$. The given recurrence relation transforms to

$$T_{2^i} = 3T_{2^{i-1}} + 17, T_{2^0} = 4.$$

Let $t_i = T_{2i}$. Then

$$t_i = 3t_{i-1} + 17, t_0 = 4.$$

Let k be a constant such that $t_i + k = 3(t_{i-1} + k)$. Then $k = \frac{17}{2}$. We thus obtain a formula for t_i:

$$t_i + \frac{17}{2} = 3\left(t_{i-1} + \frac{17}{2}\right) \implies t_i + \frac{17}{2} = 3^i\left(t_0 + \frac{17}{2}\right) \implies t_i = \frac{25}{2}\cdot 3^i - \frac{17}{2}.$$

Thus,

$$T_{2i} = \frac{25}{2}\cdot 3^i - \frac{17}{2} \implies T_n = \frac{25}{2}\cdot 3^{\log_2 n} - \frac{17}{2}.$$

Problem 8. Consider the sequence $\{a_n\}$ given by the recurrence relation

$$a_{n+1} = 2a_n + 5^n, \qquad n \geq 1.$$

(a) Given that $a_n = k\,(5^n)$ satisfies the recurrent relation, find the value of the constant k.

(b) Hence, by considering the sequence $\{b_n\}$ where $b_n = a_n - k(5^n)$, find the particular solution to the recurrence relation for which $a_1 = 2$.

Solution.

Part (a).

$$a_{n+1} = 2a_n + 5^n \implies k\left(5^{n+1}\right) = 2 \cdot k\,(5^n) + 5^n \implies 5k = 2k + 1 \implies k = \frac{1}{3}.$$

Part (b).

$$b_n = a_n - \frac{5^n}{3} = \left(2a_{n-1} - 5^{n-1}\right) - \frac{5^n}{3} = 2a_{n-1} - \frac{2}{3} \cdot 5^{n-1} = 2\left(a_{n-1} - \frac{5^{n-1}}{3}\right) = 2b_{n-1}.$$

Hence, $b_n = b_1 \cdot 2^{n-1}$. Note that $b_1 = a_1 - \frac{5}{3} = \frac{1}{3}$. Thus, $b_n = \frac{2^{n-1}}{3}$, which gives

$$b_n = a_n - \frac{5^n}{3} = \frac{2^{n-1}}{3} \implies a_n = \frac{2^n + 2 \cdot 5^n}{6}.$$

* * * * *

Problem 9. The sequence $\{X_n\}$ is given by

$$\sqrt{X_{n+2}} = \frac{X_{n+1}}{X_n^2}, \qquad n \geq 1.$$

By applying the natural logarithm to the recurrence relation, use a suitable substitution to find the general solution of the sequence, expressing your answer in trigonometric form.

Solution. Taking the natural logarithm of the recurrence relation and simplifying, we get

$$\ln X_{n+2} = 2\ln X_{n+1} - 4\ln X_n.$$

Let $L_n = \ln X_n \iff X_n = \exp(L_n)$. Then,

$$L_{n+2} = 2L_{n+1} - 4L_n.$$

Consider the characteristic equation of L_n, $x^2 - 2x + 4 = 0$. By the quadratic formula, this has roots $1 \pm \sqrt{3}i = 2\exp\left(\pm\frac{i\pi}{3}\right)$. Thus, we can express L_n as

$$L_n = A \cdot 2^n \cos\frac{n\pi}{3} + B \cdot 2^n \sin\frac{n\pi}{3} = 2^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right).$$

Thus, X_n has the general solution

$$X_n = \exp\left(2^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right)\right).$$

Problem 10. The sequence $\{X_n\}$ is given by $X_1 = 2$, $X_2 = 15$ and

$$X_{n+2} = 5\left(1 + \frac{1}{n+2}\right) X_{n+1} - 6\left(1 + \frac{2}{n+1}\right) X_n, \qquad n \geq 1.$$

By dividing the recurrence relation throughout by $n + 3$, use a suitable substitution to determine X_n as a function of n.

Solution. Dividing the recurrence relation by $n + 3$, we obtain

$$\frac{X_{n+2}}{n+3} = 5\left(\frac{1}{n+3} + \frac{1}{(n+2)(n+3)}\right) X_{n+1} - 6\left(\frac{1}{n+3} + \frac{2}{(n+1)(n+3)}\right) X_n.$$

Note that $\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$ and $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$. Thus,

$$\frac{X_{n+2}}{n+3} = 5\left(\frac{X_{n+1}}{n+2}\right) - 6\left(\frac{X_n}{n+1}\right).$$

Let $Y_n = \frac{n+1}{X_n} \iff X_n = (n+1)Y_n$. Then,

$$Y_{n+2} = 5Y_{n+1} - 6Y_n.$$

Note that the characteristic equation of Y_n, $x^2 - 5x + 6 = 0$, has roots 2 and 3. Hence,

$$Y_n = A \cdot 2^n + B \cdot 3^n \implies X_n = (n+1)\left(A \cdot 2^n + B \cdot 3^n\right).$$

From $X_1 = 2$ and $X_2 = 15$, we have $2A + 3B = 1$ and $4A + 9B = 5$. Solving, we obtain $A = -1$ and $B = 1$. Thus,

$$X_n = (n+1)\left(3^n - 2^n\right).$$

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Problem 11. A logistics company set up an online platform providing delivery services to users on a monthly paid subscription basis. The company's sales manager models the number of subscribers that the company has at the end of each month. She notes that approximately 10% of the existing subscribers leave each month, and that there will be a constant number k of new subscribers in each subsequent month after the first.

Let T_n, $n \geq 1$, denote the number of subscribers the company has at the end of the nth month after the online platform was set up.

(a) Express T_{n+1} in terms of T_n.

The company has 250 subscribers at the end of the first month.

(b) Find T_n in terms of n and k.

(c) Find the least number of subscribers the company needs to attract in each subsequent month after the first if it aims to have at least 350 subscribers by the end of the 12th month.

Let $k = 50$ for the rest of the question.

The monthly running cost of the company is assumed to be fixed at \$4,000. The monthly subscription fee is \$10 per user which is charged at the end of each month.

(d) Given that the mth month is the first month in which the company's revenue up to and including that month is able to cover its cost up to and including that month, find the value of m.

(e) Using your answer to part (b), determine the long-term behaviour of the number of subscribers that the company has. Hence, explain whether this behaviour is appropriate in terms of long-term prospects for the company's success.

Solution.

Part (a). $T_{n+1} = 0.9T_n + k$

Part (b). Let m be a constant such that $T_{n+1} + m = 0.9\left(T_n + m\right)$. Then $m = -10k$. Hence,

$$T_{n+1} - 10k = 0.9\left(T_n - 10k\right) \implies T_n - 10k = 0.9^{n-1}\left(T_0 - 10k\right).$$

Since $T_0 = 250$, we get

$$T_n = 0.9^{n-1}\left(250 - 10k\right) + 10k.$$

Part (c). Consider $T_{12} \geq 350$.

$$T_{12} \geq 350 \implies 0.9^{12-1}\left(250 - 10k\right) + 10k \geq 350.$$

Using G.C., $k \geq 39.6$. Hence, the company needs to attract at least 40 subscribers in each subsequent month.

Part (d). Since $k = 50$, $T_n = -250 \cdot 0.9^{n-1} + 500$. Let $\$S_m$ be the total revenue for the first m months.

$$\begin{aligned} S_m &= 10\sum_{n=1}^{m} T_n = 10\sum_{n=1}^{m}\left(-250 \cdot 0.9^{n-1} + 500\right) \\ &= 10\left[-250\left(\frac{1 - 0.9^m}{1 - 0.9}\right) + 500m\right] = 25000\left(0.9^m - 1\right) + 5000m. \end{aligned}$$

Note that the total cost for the first m months is $\$4000m$. Hence, the total profit for the first m months is given by $\$(S_m - 4000m)$. Hence, we consider $S_m - 4000m \geq 0$:

$$S_m - 4000m \geq 0 \implies 25000\left(0.9^m - 1\right) + 1000m \geq 0.$$

Using G.C., we obtain $m \geq 22.7$, whence the least value of m is 23.

Part (e). As $n \to \infty$, $0.9^{n-1} \to 0$. Hence, $T_n \to 500$. Hence, as n becomes very large, the profit per month approaches $500 \cdot 10 - 4000 = 1000$ dollars. Thus, this behaviour is appropriate as the business will remain profitable in the long run.

Assignment A5

Problem 1. In an auction at a charity gala dinner, a group of wealthy businessmen are competing with each other to be the highest bidder. Each time one of them makes a bid amount, another counter-bids by 50% more, less a service charge of ten dollars (e.g. If A bids \$1000, then B will bid \$1490). Let u_n be the amount at the nth bid and u_1 be the initial amount.

(a) Write down a recurrence relation that describes the bidding process.

(b) Show that $u_n = \$(1.5^{n-1}(u_1 - 20) + 20)$.

(c) The target amount to be raised is \$1 234 567 and the bidding stops when the bid amount meets or crosses this target amount. Given that $u_1 = 111$,

 (i) state the least number of bids required to meet this amount.

 (ii) find the winning bid amount, correct to the nearest thousand dollars.

Solution.

Part (a). $u_{n+1} = 1.5u_n - 10$.

Part (b). Let k be the constant such that $u_{n+1} + k = 1.5(u_n + k)$. Then $k = -20$. Hence, $u_{n+1} - 20 = 1.5(u_n - 20)$.

$$u_{n+1} - 20 = 1.5(u_n - 20) \implies u_n - 20 = 1.5^{n-1}(u_1 - 20) \implies u_n = 1.5^{n-1}(u_1 - 20) + 20.$$

Part (c).

Part (c)(i). Let m be the least integer such that $u_m \geq 1234567$. Consider $u_m \geq 1234567$:

$$u_m \geq 1234567 \implies 1.5^{m-1}(111 - 20) + 20 \geq 1234567.$$

Using G.C., $m \geq 24.5$. Hence, it takes at least 25 bids to meet this amount.

Part (c)(ii). Since $u_{25} = 1.5^{25-1}(111 - 20) = 1532000$ (to the nearest thousand), the winning bid is \$1 532 000.

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Problem 2. Solve these recurrence relations together with the initial conditions.

(a) $u_{n+2} = -u_n + 2u_{n+1}$, for $n \geq 0$, $u_0 = 5$, $u_1 = -1$.

(b) $4u_n = 4u_{n-1} + u_{n-2}$, for $n \geq 2$, $u_0 = a$, $u_1 = b$, $a, b \in \mathbb{R}$.

Solution.

Part (a). Observe that the characteristic equation of u_n, $x^2 - 2x + 1 = 0$, has only one root, namely $x = 1$. Thus,

$$u_n = (A + Bn) \cdot 1^n = A + Bn.$$

Thus, u_n is in AP. Since $u_0 = 5$ and $u_1 = -1$, it follows that

$$u_n = 5 - 6n.$$

Part (b). Rewriting the given recurrence relation, we have $u_n = u_{n-1} + \frac{1}{4}u_{n-2}$. Thus, the characteristic equation is $x^2 - x - \frac{1}{4} = 0$, which has roots $\frac{1}{2}(1 \pm \sqrt{2})$. Thus,

$$u_n = A\left(\frac{1+\sqrt{2}}{2}\right)^n + B\left(\frac{1-\sqrt{2}}{2}\right)^n.$$

Since $u_0 = a$, we obviously have $A+B = a$. Since $u_1 = b$, we get $A\left(\frac{1+\sqrt{2}}{2}\right)+B\left(\frac{1-\sqrt{2}}{2}\right) = b$. Solving, we get

$$A = \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b, \quad B = \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b.$$

Thus,

$$u_n = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b\right)\left(\frac{1+\sqrt{2}}{2}\right)^n + \left(\frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b\right)\left(\frac{1-\sqrt{2}}{2}\right)^n.$$

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Problem 3. A passcode is generated using the digits 1 to 5, with repetitions allowed. The passcodes are classified into two types. A Type A passcode has an even number of the digit 1, while a Type B passcode has an odd number of the digit 1. For example, a Type A passcode is 1231, and a Type B passcode is 1541213. Let a_n and b_n denote the number of n-digit Type A and Type B passcodes respectively.

(a) State the values of a_1 and a_2.

(b) By considering the relationship between a_n and b_n, show that

$$a_n = xa_{n-1} + y^{n-1}, \qquad n \geq 2$$

where x and y are constants to be determined.

(c) Using the substitution $c_n = za_n + y^n$, where z is a constant to be determined, find a first order linear recurrence relation for c_n. Hence, find the general term formula for a_n.

Solution.

Part (a). $a_1 = 4$, $a_2 = 17$.

Part (b). Let P be an n-digit passcode with Type T, where T is either A or B. Let Type T' be the other type.

By concatenating a digit from 1 to 5 to P, five $(n+1)$-digit passcodes can be created. Let P' denote a new passcode that is created via this process. If the digit 1 is concatenated, then P' is of Type T'. If the digit 1 is not concatenated, then P' is of Type T. There are 4 choices for such a case. This hence gives the recurrence relations

$$\begin{cases} a_n = 4a_{n-1} + b_{n-1} \\ b_n = 4b_{n-1} + a_{n-1} \end{cases}$$

Adding the two equations, we see that $a_n + b_n = 5(a_{n-1} + b_{n-1})$. Thus,

$$a_n + b_n = 5^{n-1}(a_1 + b_1) = 5^{n-1}(4+1) = 5^n.$$

Hence,

$$a_n = 4a_{n-1} + b_{n-1} = 3a_{n-1} + a_{n-1} + b_{n-1} = 3a_{n-1} + 5^{n-1},$$

whence $x = 3$ and $y = 5$.

Part (c). Observe that

$$\begin{aligned} c_n &= za_n + 5^n = z\left(3a_{n-1} + 5^{n-1}\right) + 5^n = 3\left(za_{n-1} + 5^{n-1}\right) + (2+z)5^{n-1} \\ &= 3c_{n-1} + (2+z)5^{n-1}. \end{aligned}$$

Let $z = -2$. Then,

$$c_n = 3c_{n-1} = 3^{n-1}c_1 = 3^{n-1}\left(-2a_1 + 5\right) = -3^n.$$

Note that $a_n = \frac{1}{z}\left(c_n - y^n\right)$. Thus,

$$a_n = \frac{-3^n - 5^n}{-2} = \frac{3^n + 5^n}{2}.$$

A6. Polar Coordinates

Tutorial A6

Problem 1.

(a) Find the rectangular coordinates of the following points.
   (i) $(3, -\frac{\pi}{4})$
   (ii) $(1, \pi)$
   (iii) $(\frac{1}{2}, \frac{3}{2}\pi)$

(b) Find the polar coordinates of the following points.
   (i) $(3, 3)$
   (ii) $(-1, -\sqrt{3})$
   (iii) $(2, 0)$
   (iv) $(4, 2)$

Solution.

Part (a).

Part (a)(i). Note that $r = 3$ and $\theta = -\frac{\pi}{4}$. This gives

$$x = r\cos\theta = \frac{3}{\sqrt{2}}, \quad y = r\sin\theta = -\frac{3}{\sqrt{2}}.$$

Hence, the rectangular coordinate of the point is $(3/\sqrt{2}, -3\sqrt{2})$.

Part (a)(ii). Note that $r = 1$ and $\theta = \pi$. This gives

$$x = r\cos\theta = -1, \quad y = r\sin\theta = 0.$$

Hence, the rectangular coordinate of the point is $(-1, 0)$.

Part (a)(iii). Note that $r = \frac{1}{2}$ and $\theta = \frac{3}{2}\pi$. This gives

$$x = \rho\cos\theta = 0, \quad y = r\sin\theta = -\frac{1}{2}.$$

Hence, the rectangular coordinate of the point is $(0, -1/2)$.

Part (b).

Part (b)(i). Note that $x = 3$ and $y = -3$. This gives

$$r^2 = x^2 + y^2 \implies r = 3\sqrt{2}, \quad \tan\theta = \frac{y}{x} \implies \theta = -\frac{\pi}{4}.$$

Hence, the polar coordinate of the point is $(3\sqrt{2}, -\pi/4)$.

Part (b)(ii). Note that $x = -1$ and $y = -\sqrt{3}$. This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan\theta = \frac{y}{x} \implies \theta = \frac{\pi}{3}.$$

Hence, the polar coordinate of the point is $(2, \pi/3)$.

Part (b)(iii). Note that $x = 2$ and $y = 0$. This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan\theta = \frac{y}{x} \implies \theta = 0.$$

Hence, the polar coordinate of the point is $(2, 0)$.

Part (b)(iv). Note that $x = 4$ and $y = 2$. This gives

$$r^2 = x^2 + y^2 \implies r = 2\sqrt{5}, \quad \tan\theta = \frac{y}{x} \implies \theta = \arctan\frac{1}{2}.$$

Hence, the polar coordinate of the point is $(2\sqrt{5}, \arctan(1/2))$.

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Problem 2. Rewrite the following equations in polar form.

(a) $2x^2 + 3y^2 = 4$

(b) $y = 2x^2$

Solution.

Part (a).

$$2x^2 + 3y^2 = 2(r\cos\theta)^2 + 3(r\sin\theta)^2 = 4 \implies r^2 = \frac{4}{2\cos^2\theta + 3\sin^2\theta} = \frac{4}{2 + \sin^2\theta}.$$

Part (b).

$$y = 2x^2 \implies \frac{y}{x} = 2x \implies \tan\theta = 2r\cos\theta \implies r = \frac{1}{2}\tan\theta\sec\theta.$$

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Problem 3. Rewrite the following equations in rectangular form.

(a) $r = \frac{1}{1-2\cos\theta}$

(b) $r = \sin\theta$

Solution.

Part (a).

$$r = \frac{1}{1 - 2\cos\theta} \implies r - 2r\cos\theta = 1 \implies r = 2x + 1 \implies r^2 = 4x^2 + 4x + 1$$
$$\implies x^2 + y^2 = 4x^2 + 4x + 1 \implies y^2 = 3x^2 + 4x + 1.$$

Part (b).

$$r = \sin\theta \implies r^2 = r\sin\theta \implies x^2 + y^2 = y.$$

Problem 4.

(a) Show that the curve with polar equation $r = 3a\cos\theta$, where a is a positive constant, is a circle. Write down its centre and radius.

(b) By finding the Cartesian equation, sketch the curve whose polar equation is $r = a\sec\left(\theta - \frac{\pi}{4}\right)$, where a is a positive constant.

Solution.

Part (a).

$$r = 3a\cos\theta \implies r^2 = 3ar\cos\theta \implies x^2 + y^2 = 3ax \implies x^2 - 3ax + y^2 = 0.$$

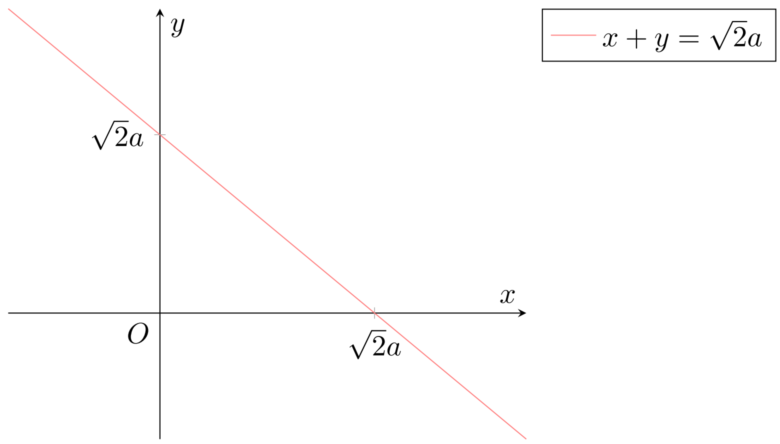
Completing the square, we get

$$\left(x - \frac{3a}{2}\right)^2 + y^2 \left(\frac{3a}{2}\right)^2.$$

Thus, the circle has centre $(3a/2, 0)$ and radius $3a/2$.

Part (b).

$$r = a\sec\left(\theta - \frac{\pi}{4}\right) \implies r\cos\left(\theta - \frac{\pi}{4}\right) = a \implies r(\cos\theta + \sin\theta) = \sqrt{2}a \implies x + y = \sqrt{2}a.$$



Problem 5. Sketch the following polar curves, where r is non-negative and $0 \le \theta \le 2\pi$.

(a) $r = 2$

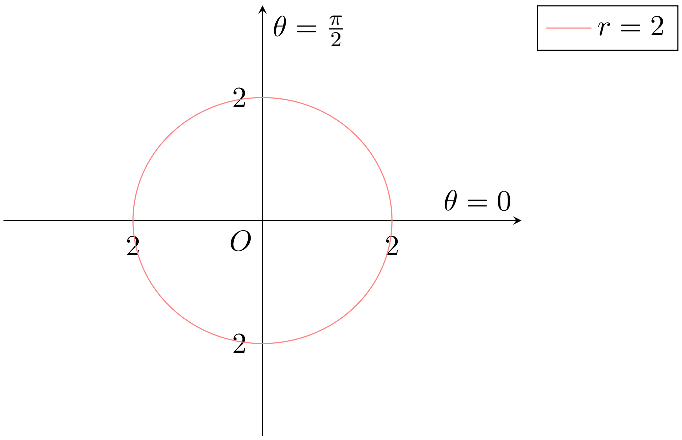
(b) $\theta = \frac{\pi}{4}$

(c) $r = \frac{1}{2}\theta$

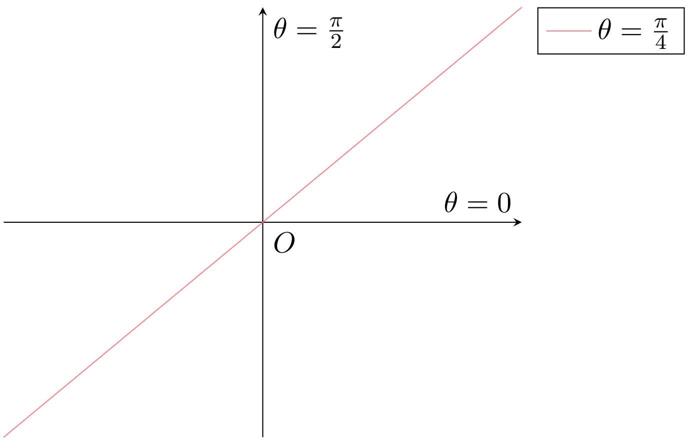
(d) $r = 2\csc\theta$

Solution.

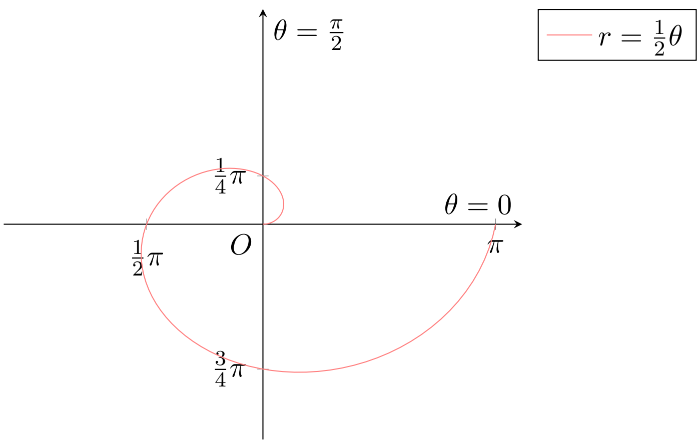
Part (a).



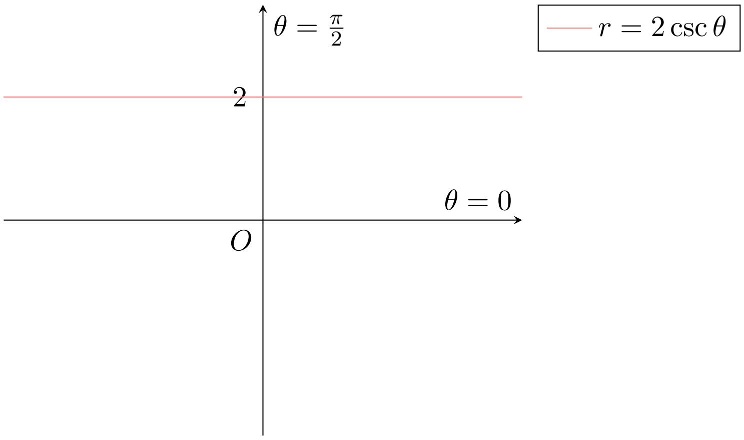
Part (b).



Part (c).



Part (d).



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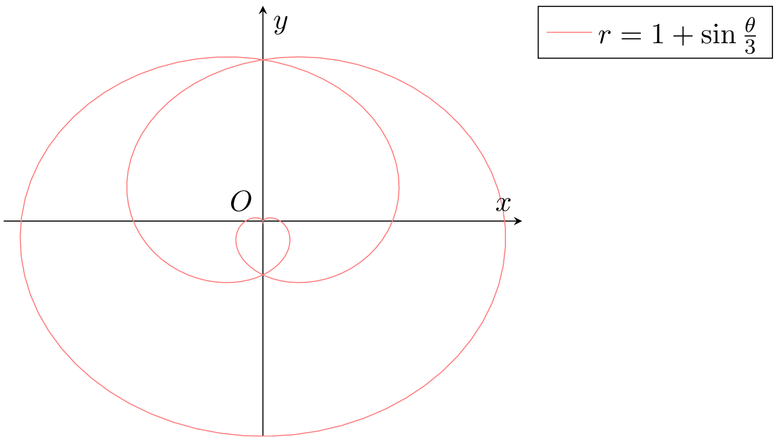
Problem 6. A sketch of the curve $r = 1 + \sin\frac{\theta}{3}$ is shown. Copy the diagram and indicate the x- and y-intercepts.



Solution. Observe that the curve is symmetric about the y-axis. Also observe that $\frac{\theta}{3} \in [0, 2\pi)$, hence we take $\theta \in [0, 6\pi)$.

For x-intercepts, $y = r\sin\theta = 0 \implies \theta = n\pi$, where $n \in \mathbb{Z}$. Due to the symmetry of the curve, we consider only $n = 0, 2, 4$.

Case 1. $n = 0 \implies r = 1 + \sin\frac{0}{3}\pi = 1$.

Case 2. $n = 2 \implies r = 1 + \sin\frac{2}{3}\pi = 1 + \frac{\sqrt{3}}{2}$.

Case 3. $n = 4 \implies r = 1 + \sin\frac{4}{3}\pi = 1 - \frac{\sqrt{3}}{2}$.

Hence, the curve intersects the x-axis at $x = 1, 1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}$. Correspondingly, the curve also intersects the x-axis at $x = -1, -1 - \frac{\sqrt{3}}{2}, -1 + \frac{\sqrt{3}}{2}$.

For y-intercepts, $x = r\cos\theta = 0 \implies \theta = (n + \frac{1}{2})\pi$, where $n \in \mathbb{Z}$. Due to the restriction on θ, we consider $n \in [0, 5)$.

Case 1. $n = 0, r = 1 + \sin\frac{1/2}{3}\pi = \frac{3}{2}$.

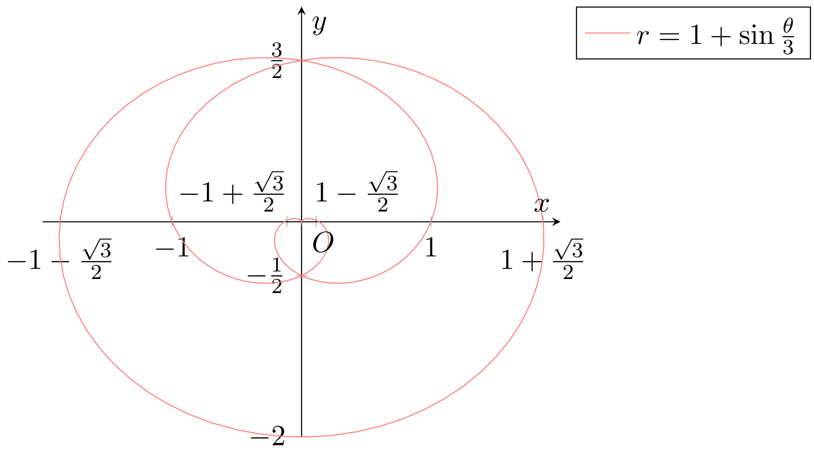
Case 2. $n = 1, r = 1 + \sin\frac{3/2}{3}\pi = 2$.

Case 3. $n = 2, r = 1 + \sin\frac{5/2}{3}\pi = \frac{3}{2}$.

Case 4. $n = 3, r = 1 + \sin\frac{7/2}{3}\pi = \frac{1}{2}$.

Case 5. $n = 4, r = 1 + \sin\frac{9/2}{3}\pi = 0$.

Hence, the curve intersects the y-axis at $y = -2, -\frac{1}{2}, \frac{3}{2}$.



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Problem 7.

(a) A graph has polar equation $r = \frac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha}$, where α is a constant. Express the equation in Cartesian form. Hence, sketch the graph in the case $\alpha = \frac{\pi}{4}$, giving the Cartesian coordinates of the intersection with the axes.

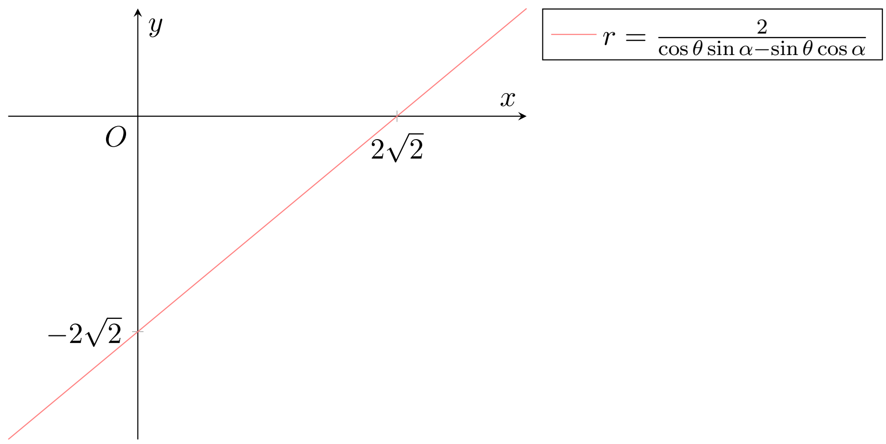
(b) A graph has Cartesian equation $(x^2 + y^2)^2 = 4x^2$. Express the equation in polar form. Hence, or otherwise, sketch the graph.

Solution.

Part (a).

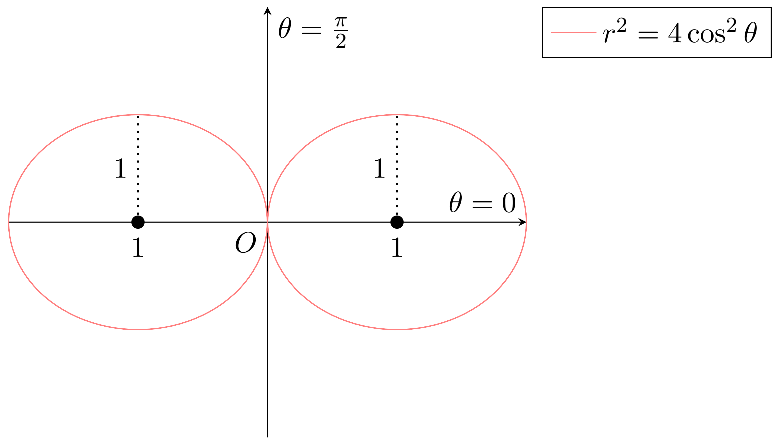
$$r = \frac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha} \implies r\cos\theta\sin\alpha - r\sin\theta\cos\alpha = x\sin\alpha - y\cos\alpha = 2$$
$$\implies y = x\tan\alpha - 2\sec\alpha.$$

When $\alpha = \frac{\pi}{4}$, we have $y = x - 2\sqrt{2}$.



Part (b).

$$(x^2+y^2)^2 = 4x^2 \implies \left(r^2\right)^2 = 4(r\cos\theta)^2 \implies r^4 = 4r^2\cos^2\theta \implies r^2 = 4\cos^2\theta.$$



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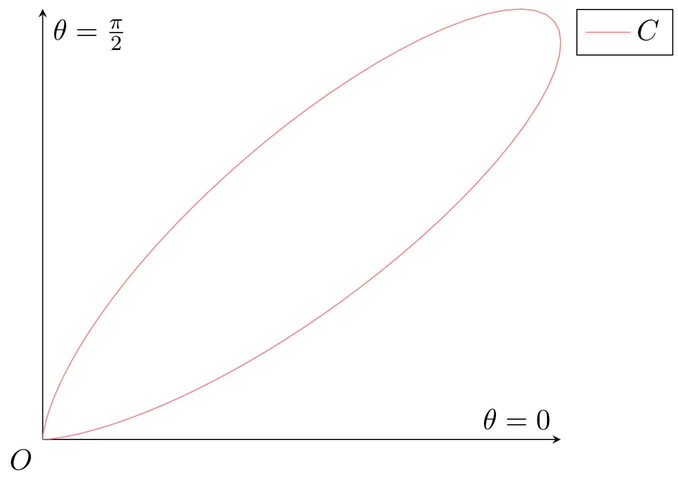
Problem 8. Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \leq \theta \leq \pi/2$.

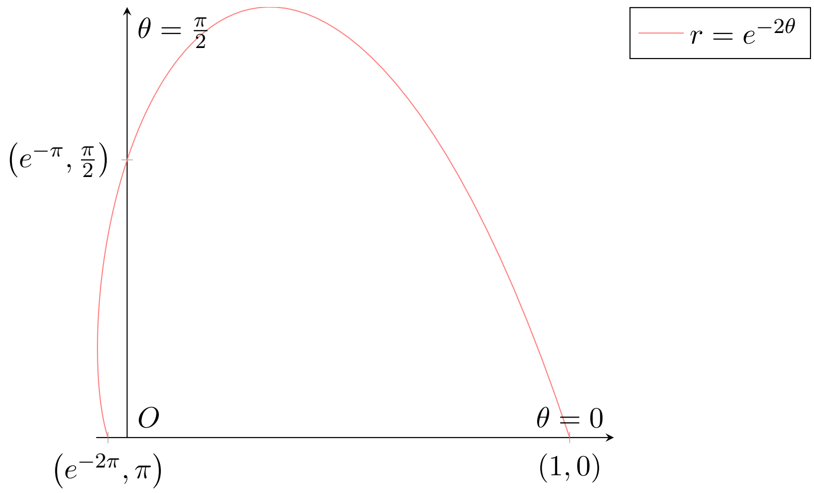
Solution.

$$x^5 + y^5 = 5bx^2y^2 \implies (r\cos\theta)^5 + (r\sin\theta)^5 = 5b(r\cos\theta)^2(r\sin\theta)^2$$
$$\implies r\left(\cos^5\theta + \sin^5\theta\right) = 5b\cos^2\theta\sin^2\theta \implies r = \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}.$$



Problem 9. The equation of a curve, in polar coordinates, is $r = \mathrm{e}^{-2\theta}$, for $0 \le \theta \le \pi$. Sketch the curve, indicating clearly the polar coordinates of any axial intercepts.

Solution.



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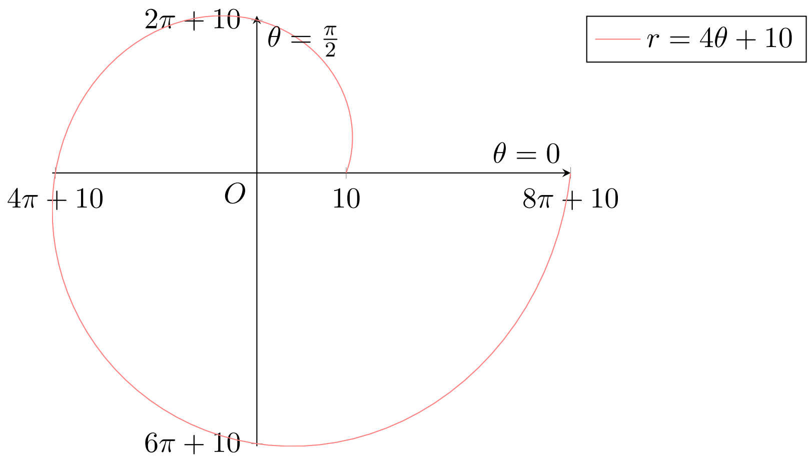
Problem 10. Suppose that a long thin rod with one end fixed at the pole of a polar coordinate system rotates counter-clockwise at the constant rate of 0.5 rad/sec. At time $t = 0$, a bug on the rod is 10 mm from the pole and is moving outward along the rod at a constant speed of 2 mm/sec. Find an equation of the form $r = f(\theta)$ for the part of motion of the bug, assuming that $\theta = 0$ when $t = 0$. Sketch the path of the bug on the polar coordinate system for $0 \le t \le 4\pi$.

Solution. Let $\theta(t)$ and $r(t)$ be functions of time, with $\theta(0) = 0$ and $r(0) = 10$. We know that $\mathrm{d}\theta/\mathrm{d}t = 0.5$ and $\mathrm{d}r/\mathrm{d}t = 2$. Hence,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^{-1} = 2 \cdot (0.5)^{-1} = 4.$$

Thus, $r = 4\theta + r(0) = 4\theta + 10$.

Since $\mathrm{d}\theta/\mathrm{d}t = 0.5$ and $\theta(0) = 0$, we have $\theta = 0.5t$. Hence, $0 \le t \le 4\pi \implies 0 \le \theta \le 2\pi$.



Problem 11. The equation, in polar coordinates, of a curve C is $r = a\mathrm{e}^{\frac{1}{2}\theta}$, $0 \le \theta \le 2\pi$, where a is a positive constant. Write down, in terms of θ, the Cartesian coordinates, x and y, of a general point P on the curve. Show that the gradient at P is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\tan\theta+2}{1-2\tan\theta}$.

Hence, show that the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$. Sketch the curve C.

Solution. Note that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = a\mathrm{e}^{\frac{1}{2}\theta}\cos\theta$ and $y = a\mathrm{e}^{\frac{1}{2}\theta}\sin\theta$. Hence, $P\left(a\mathrm{e}^{\frac{1}{2}\theta}\cos\theta, a\mathrm{e}^{\frac{1}{2}\theta}\sin\theta\right)$.

Observe that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{1}{2}a\mathrm{e}^{\frac{1}{2}\theta} = \frac{1}{2}r$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} = \frac{\frac{1}{2}r\sin\theta + r\cos\theta}{\frac{1}{2}r\cos\theta - r\sin\theta} = \frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} = \frac{\tan\theta + 2}{1 - 2\tan\theta}.$$

Let $\mathbf{t} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ represent the direction of the tangent line. Then

$$\mathbf{t} = \begin{pmatrix} 1 \\ \mathrm{d}y/\mathrm{d}x \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\tan\theta+2}{1-2\tan\theta} \end{pmatrix} = \frac{1}{1-2\tan\theta}\begin{pmatrix} 1 - 2\tan\theta \\ \tan\theta + 2 \end{pmatrix}$$

and

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a\mathrm{e}^{\frac{1}{2}\theta}\cos\theta \\ a\mathrm{e}^{\frac{1}{2}\theta}\sin\theta \end{pmatrix} = a\mathrm{e}^{\frac{1}{2}\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

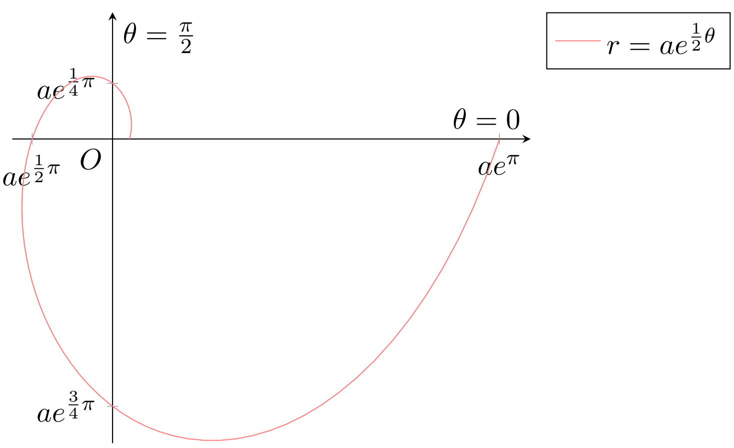
By the definition of the dot-product, we have $\mathbf{t}\cdot\overrightarrow{OP} = |\mathbf{t}|\left|\overrightarrow{OP}\right|\cos\alpha$, whence

$$\cos\alpha = \frac{\mathbf{t}\cdot\overrightarrow{OP}}{|\mathbf{t}|\left|\overrightarrow{OP}\right|} = \frac{(1-2\tan\theta)\cos\theta + (\tan\theta+2)\sin\theta}{\sqrt{(1-2\tan\theta)^2 + (\tan\theta+2)^2}\cdot\sqrt{\cos^2\theta + \sin^2\theta}}$$
$$= \frac{\cos\theta + \tan\theta\sin\theta}{\sqrt{5\tan^2\theta + 5}} = \frac{\cos^2\theta + \sin^2\theta}{\sqrt{5\sin^2\theta + 5\cos^2\theta}} = \frac{1}{\sqrt{5}}.$$

Thus, $\alpha = \arccos\frac{1}{\sqrt{5}}$. Since $\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$,

$$\tan\alpha = \tan\left(\arccos\frac{1}{\sqrt{5}}\right) = \frac{\sqrt{1-(1/\sqrt{5})^2}}{1/\sqrt{5}} = 2.$$

Hence, the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$.



Problem 12. The polar equation of a curve is given by $r = \mathrm{e}^{\theta}$ where $0 \leq \theta \leq \frac{\pi}{2}$. Cartesian axes are taken at the pole O. Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(\mathrm{e}^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$.

Solution. Recall that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = \mathrm{e}^{\theta}\cos\theta$ and $y = \mathrm{e}^{\theta}\sin\theta$. Thus, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \mathrm{e}^{\theta}(\cos\theta - \sin\theta)$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^{\theta}(\cos\theta + \sin\theta)$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{\mathrm{e}^{\theta}(\cos\theta + \sin\theta)}{\mathrm{e}^{\theta}(\cos\theta - \sin\theta)} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

At $\left(\mathrm{e}^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$, we clearly have $x = 0$ and $y = \mathrm{e}^{\pi/2}$. Also, $\mathrm{d}y/\mathrm{d}x = -1$. By the point-slope formula, the equation of the tangent line at $\left(\mathrm{e}^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$ is given by $y = -x + \mathrm{e}^{\frac{\pi}{2}}$.

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Problem 13. A curve C has polar equation $r = a\cot\theta$, $0 < \theta \leq \pi$, where a is a positive constant.

(a) Show that $y = a$ is an asymptote of C.

(b) Find the tangent at the pole.

Hence, sketch C and find the Cartesian equation of C in the form $y^2(x^2 + y^2) = bx^2$, where b is a constant to be determined.

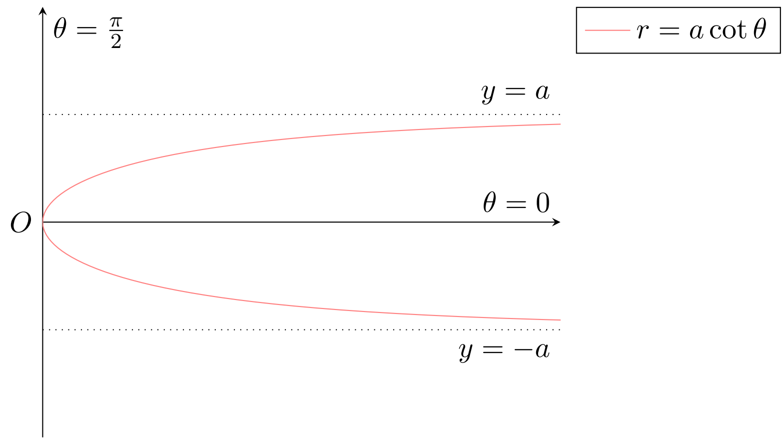
Solution.

Part (a). Note that

$$r = a\cot\theta \implies y = r\sin\theta = a\cos\theta.$$

As $\theta \to 0$, $r \to \infty$. Hence, there is an asymptote at $\theta = 0$. Since $\cos\theta = 1$ when $\theta = 0$, the line $y = a\cos\theta = a$ is an asymptote of C.

Part (b). For tangents at the pole, $r = 0 \implies \cot\theta = 0 \implies \theta = \frac{\pi}{2}$.



Note that

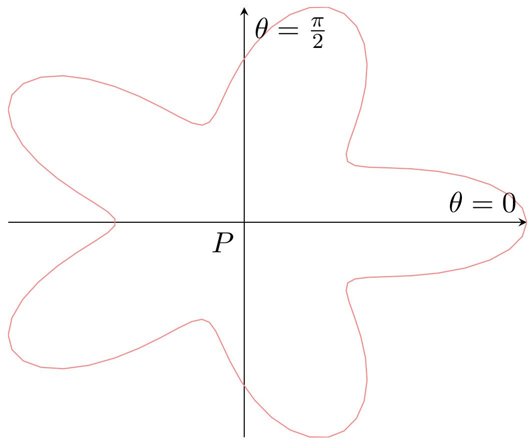
$$r = a\cot\theta = a\left(\frac{r\cos\theta}{r\sin\theta}\right) = a\left(\frac{x}{y}\right).$$

Thus,

$$x^2 + y^2 = r^2 = a^2\left(\frac{x^2}{y^2}\right) \implies y^2\left(x^2 + y^2\right) = a^2x^2,$$

whence $b = a^2$.

Problem 14.



Relative to the pole P and the initial line $\theta = 0$, the polar equation of the curve shown is either

i. $r = a + b \sin n\theta$, or

ii. $r = a + b \cos n\theta$

where a, b and n are positive constants. State, with a reason, whether the equation is (i) or (ii) and state the value of n.

The maximum value of r is $\frac{11}{2}$ and the minimum value of r is $\frac{5}{2}$. Find the values of a and b.

Solution. Since the curve is symmetrical about the horizontal half-line $\theta = 0$, the polar equation of the curve is a function of $\cos n\theta$ only. Hence, the polar equation of the curve is $r = a + b \cos n\theta$, with $n = 5$.

Observe that the maximum value of r is achieved when $\cos 5\theta = 1$, whence $r = a + b$. Thus, $a + b = \frac{11}{2}$. Also observe that the minimum value of r is achieved when $\cos 5\theta = -1$, whence $r = a - b$. Thus, $a - b = \frac{5}{2}$. Solving, we get $a = 4$ and $b = \frac{3}{2}$.

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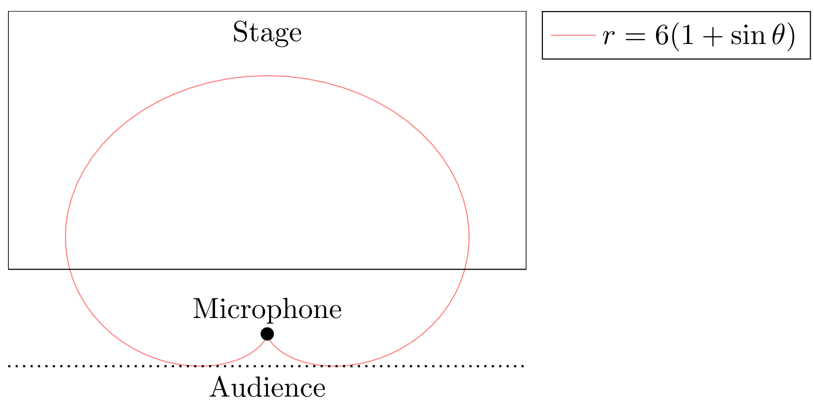
Problem 15.



Sound engineers often use a microphone with a cardioid acoustic pickup pattern to record live performances because it reduces pickup from the audience. Suppose a cardioid microphone is placed 3 metres from the front of the stage, and the boundary of the optimal pickup region is given by the cardioid with polar equation

$$r = 6(1 + \sin\theta)$$

where r is measured in metres and the microphone is at the pole.

Find the minimum distance from the front of the stage the first row of the audience can be seated such that the microphone does not pick up noise from the audience.

Solution. Note that $r = 6(1+\sin\theta) = 6(1+\frac{y}{r})$, whence $r^2 = 6r + 6y$. Thus,

$$r^2 - 6r - 6y = 0 \implies r = 3 \pm \sqrt{9+6y} \implies 9 + 6y = (r-3)^2.$$

Since $9 + 6y = (r-3)^2 \geq 0$, we have $y \geq -1.5$. Thus, the furthest distance the audience has to be from the stage is $|-1.5| + 3 = 4.5$ m.

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Problem 16. To design a flower pendant, a designer starts off with a curve C_1, given by the Cartesian equation

$$\left(x^2+y^2\right)^2 = a^2\left(3x^2 - y^2\right)$$

where a is a positive constant.

(a) Show that a corresponding polar equation of C_1 is $r^2 = a^2(1+2\cos 2\theta)$.

(b) Find the equations of the tangents to C_1 at the pole.

Another curve C_2 is obtained by rotating C_1 anti-clockwise about the origin by $\frac{\pi}{3}$ radians.

(c) State a polar equation of C_2.

(d) Sketch C_1 and C_2 on the same diagram, stating clearly the exact polar coordinates of the points of intersection of the curves with the axes. Find also the exact polar coordinates of the points of intersection with C_1 and C_2.

The curve C_3 is obtained by reflecting C_2 in the line $\theta = \frac{\pi}{2}$.

(e) State a polar equation of C_3.

(f) The designer wishes to enclose the 3 curves inside a circle given by the polar equation $r = r_1$. State the minimum value of r_1 in terms of a.

Solution.

Part (a). Observe that $\left(x^2+y^2\right)^2 = r^4$ and $3x^2 - y^2 = r^2\left(3\cos^2\theta - \sin^2\theta\right)$. Hence,

$$\left(x^2+y^2\right)^2 = a^2\left(3x^2-y^2\right) \implies r^2 = a^2\left(3\cos^2\theta - \sin^2\theta\right).$$

Note that

$$3\cos^2\theta - \sin^2\theta = 1 + 2\cos^2\theta - 2\sin^2\theta = 1 + 2\cos 2\theta.$$

Thus,

$$r^2 = a^2\left(1+2\cos 2\theta\right).$$

Part (b). For tangents at the pole,

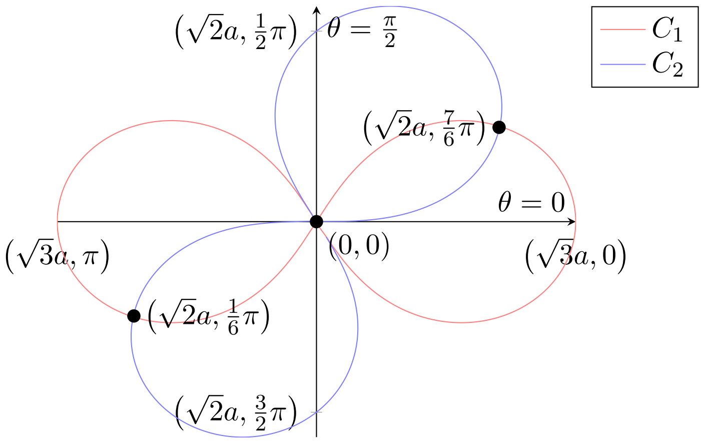
$$r = 0 \implies 1 + 2\cos 2\theta = 0 \implies \cos 2\theta = -\frac{1}{2}.$$

Since $0 \leq 2\theta \leq 2\pi$, we have $\theta = \pi/3, 2\pi/3$. For full lines, we also have $\theta = 4\pi/3$ and $\theta = 5\pi/3$.

Part (c).

$$r^2 = a^2\left[1 + 2\cos\left(2\left(\theta - \frac{\pi}{3}\right)\right)\right] = a^2\left[1 + 2\cos\left(2\theta - \frac{2}{3}\pi\right)\right].$$

Part (d).



Consider the horizontal intercepts of C_1. When $\theta = 0$, $r = \sqrt{3}a$. Hence, by symmetry, C_1 intercepts the horizontal axis at $\left(\sqrt{3}a, 0\right)$ and $\left(\sqrt{3}a, \pi\right)$.

Consider the vertical intercepts of C_2. When $\theta = \pi/2$, $r = \sqrt{2}a$. Hence, by symmetry, C_2 intercepts the vertical axis at $\left(\sqrt{2}a, \pi/2\right)$ and $\left(\sqrt{2}a, 3\pi/2\right)$.

Now consider the intersections between C_1 and C_2. By symmetry, it is obvious that the points of intersections must lie along the half-lines $\pi/6$ and $7\pi/6$, or along the half-lines $4\pi/6$ and $10\pi/6$. By symmetry, we consider only the half-lines $\pi/6$ and $4\pi/6$.

Case 1: $\theta = \pi/6$. Substituting $\theta = \pi/6$ into the equation of C_1, we obtain $r = \sqrt{2}a$. Hence, C_1 and C_2 intersect at $\left(\sqrt{2}a, \pi/6\right)$ and, by symmetry, at $\left(\sqrt{2}a, 7\pi/6\right)$.

Case 2.$\theta = 4\pi/6$ Substituting $\theta = 4\pi/6$ into the equation of C_1, we obtain $r = 0$. Hence, C_1 and C_2 intersect at $(0, 0)$.

Part (e). Reflecting about the line $\theta = \pi/2$ is equivalent to applying the map $\theta \mapsto \theta + \pi/3$ to C_1. Hence,

$$r^2 = a^2\left[1 + 2\cos\left(2\left(\theta + \frac{1}{3}\pi\right)\right)\right] = a^2\left[1 + 2\cos\left(2\theta + \frac{2}{3}\pi\right)\right].$$

Part (f). $r_1 = \sqrt{3}a$.

Assignment A6

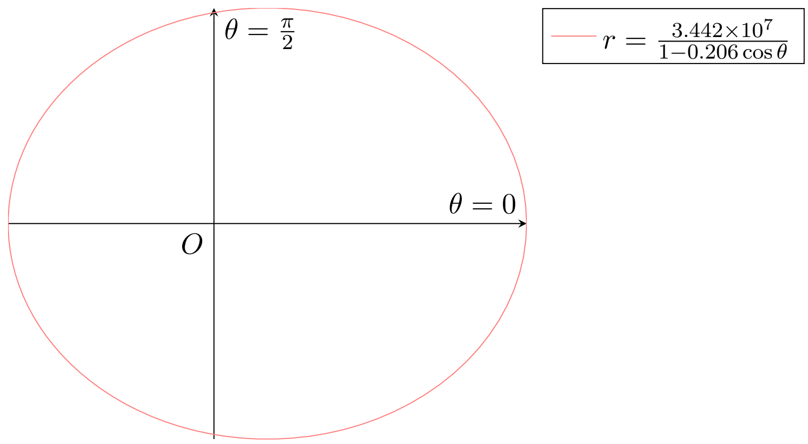
Problem 1. The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cos\theta},$$

where r is measured in miles and the sun is at the pole.

Sketch the orbit and find the distance from Mercury to the sun at the aphelion (the greatest distance from the sun) and at the perihelion (the shortest distance from the sun).

Solution.



Observe that r attains a maximum when $\cos\theta$ is also at its maximum. Since the maximum value of $\cos\theta$ is 1,

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cdot 1} = 4.34 \times 10^7 \text{ (3 s.f.)}.$$

Hence, the distance from Mercury to the sun at the aphelion is 4.34×10^7 miles.

Observe that r attains a minimum when $\cos\theta$ is also at its minimum. Since the minimum value of $\cos\theta$ is -1,

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cdot -1} = 2.85 \times 10^7 \text{ (3 s.f.)}.$$

Hence, the distance from Mercury to the sun at the perihelion is 2.85×10^7 miles.

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Problem 2. A variable point P has polar coordinates (r, θ), and fixed points A and B have polar coordinates $(1, 0)$ and $(1, \pi)$ respectively. Given that P moves so that the product $PA \cdot PB = 2$, show that

$$r^2 = \cos 2\theta + \sqrt{3 + \cos^2 2\theta}.$$

(a) Given that $r \geq 0$ and $0 \leq \theta \leq 2\pi$, find the maximum and minimum values of r, and the values of θ at which they occur.

(b) Verify that the path taken by P is symmetric about the lines $\theta = 0$ and $\theta = \dfrac{\pi}{2}$, giving your reasons.

Solution. Note that A and B have Cartesian coordinates $(1, 0)$ and $(-1, 0)$ respectively. Let $P(x, y)$. Then

$$PA^2 = (x - 1)^2 + y^2, \qquad PB^2 = (x + 1)^2 + y^2.$$

Hence,

$$PA \cdot PB = \left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right) = \left(x^2 + y^2\right)^2 - 2\left(x^2 - y^2\right) + 1.$$

Since $x^2 - y^2 = r^2\left(\cos^2\theta - \sin^2\theta\right) = r^2\cos 2\theta$, the polar equation of the locus of P is

$$r^4 - 2r^2\cos 2\theta + 1 = (PA \cdot PB)^2 = 4 \implies r^4 - 2r^2\cos 2\theta - 3 = 0.$$

By the quadratic formula, we have

$$r^2 = \frac{2\cos 2\theta \pm \sqrt{4\cos^2 2\theta + 12}}{2} = \cos 2\theta \pm \sqrt{\cos^2 2\theta + 3}.$$

Since $\sqrt{\cos^2 2\theta + 3} > \cos 2\theta$ and $r^2 \geq 0$, we reject the negative case. Thus,

$$r^2 = \cos 2\theta + \sqrt{3 + \cos^2 2\theta}.$$

Part (a). Differentiating with respect to θ, we obtain

$$2r\frac{\mathrm{d}r}{\mathrm{d}\theta} = -2\sin 2\theta\left(1 + \frac{1}{2\sqrt{3 + \cos^2 2\theta}}\right).$$

For stationary points, $\mathrm{d}r/\mathrm{d}\theta = 0$. Since $1 + 1/2\sqrt{3 + \cos^2 2\theta} > 0$, we must have $\sin 2\theta = 0$, whence $\theta = 0, \pi/2, \pi, 3\pi/2$. By symmetry, we only consider $\theta = 0$ and $\theta = \pi/2$.

Case 1. When $\theta = 0$, we have $r^2 = 3$, whence $r = \sqrt{3}$.

Case 2. When $\theta = \pi/2$, we have $r^2 = 1$, whence $r = 1$.

Thus, $\max r = \sqrt{3}$ and occurs when $\theta = 0, \pi$, while $\min r = 1$ and occurs when $\theta = \pi/2, 3\pi/2$.

Part (b). Recall that the path taken by P is given by

$$\left((x-1)^2 + y^2\right)\left((x+1)^2 + y^2\right) = 4.$$

Observe that the above equation is invariant under the transformations $x \mapsto -x$ and $y \mapsto -y$. Hence, the path is symmetric about both the x- and y-axes, i.e. the lines $\theta = 0$ and $\theta = \pi/2$.

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Problem 3.

(a) Explain why the curve with equation $x^3 + 2xy^2 - a^2y = 0$ where a is a positive constant lies entirely in the region $|x| \leq 2^{-\frac{3}{4}}a$.

(b) Show that the polar equation of this curve is $r^2 = \dfrac{a^2\tan\theta}{2 - \cos^2\theta}$.

(c) Sketch the curve.

Solution.

Part (a). Consider the discriminant Δ of $x^3 + 2xy^2 - a^2y = 0$ with respect to y:

$$\Delta = \left(-a^2\right)^2 - 4\left(2x\right) = a^4 - 8x^4.$$

For points on the curve, we clearly have $\Delta \geq 0$. Thus,

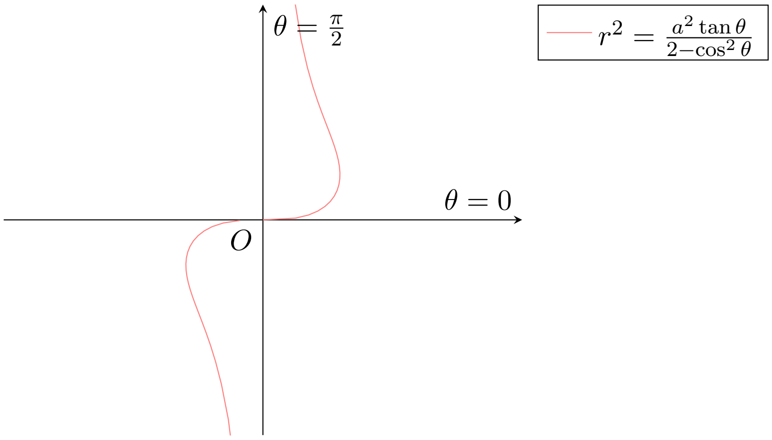
$$a^3 - 8x^4 \geq 0 \implies x^4 \leq 2^{-3}a^4 \implies |x| \leq 2^{-3/4}a.$$

Part (b).

$$x^3 + 2xy^2 - a^2y = 0 \implies 2\left(x^2 + y^2\right) - x^2 - a^2\frac{y}{x} = 0 \implies 2r^2 - r^2\cos^2\theta - a^2\tan\theta = 0$$

$$\implies r^2 = \frac{a^2\tan\theta}{2 - \cos^2\theta}.$$

Part (c).



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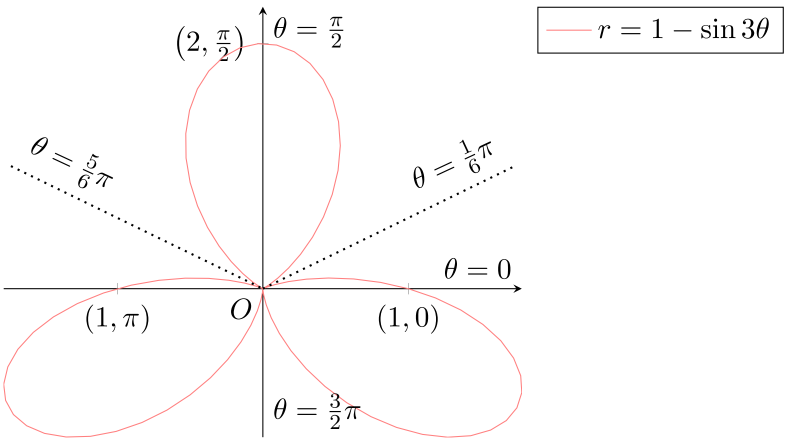
Problem 4. The curve C has polar equation $r = 1 - \sin 3\theta$, where $0 \le \theta \le 2\pi$.

(a) Sketch the curve C, showing the tangents at the pole and the intersections with the axes.

(b) Find the gradient of the curve at the point where $\theta = \dfrac{\pi}{3}$, giving your answer in the form $a + b\sqrt{3}$, where a and b are constants to be determined.

Solution.

Part (a).



When $\theta = 0$ or $\theta = \pi$, we have $r = 1$. Thus, C intersects the horizontal axis at $(1, 0)$ and $(1, \pi)$. When $\theta = \pi/2$, we have $r = 2$. Thus, C intersects the vertical axis at $(2, \pi/2)$. When $\theta = 3\pi/2$, we have $r = 0$. Thus, C passes through the pole.

For tangents at the pole, $r = 0 \implies \sin 3\theta = 1 \implies \theta = \pi/6, 5\pi/6, 3\pi/2$.

Part (b). Note that $\mathrm{d}r/\mathrm{d}\theta = -3\cos 3\theta$ evaluates to 3 when $\theta = \pi/3$. Thus,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta}\right|_{\theta=\frac{\pi}{3}} = \frac{3\sqrt{3}+1}{3-\sqrt{3}} = \frac{12+10\sqrt{3}}{6} = 2 + \frac{5}{3}\sqrt{3}.$$

Hence, when $\theta = \pi/3$, the gradient of the curve is $2 + 5\sqrt{3}/2$.

A7. Vectors I - Basic Properties and Vector Algebra

Tutorial A7

Problem 1. The vector $\mathbf{v}$ is defined by $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. Find the unit vector in the direction of $\mathbf{v}$ and hence find a vector of magnitude 25 which is parallel to $\mathbf{v}$.

Solution.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3^2 + (-4)^2 + 1^2}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \quad 25\hat{\mathbf{v}} = \frac{25}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

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Problem 2. With respect to an origin O, the position vectors of the points A, B, C and D are $4\mathbf{i} + 7\mathbf{j}$, $\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + d\mathbf{j}$ respectively.

(a) Find the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Find the value of d if B, C and D are collinear. State the ratio $\frac{BC}{BD}$.

Solution.

Part (a). Note that

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

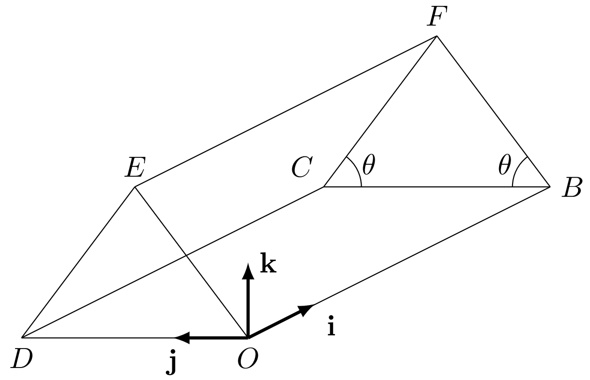
Part (b). If B, C and D are collinear, then $\overrightarrow{BC} = \lambda\overrightarrow{CD}$ for some $\lambda \in \mathbb{R}$.

$$\overrightarrow{BC} = \lambda\overrightarrow{CD} \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda\left(\overrightarrow{OD} - \overrightarrow{OC}\right) = \lambda\left[\begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} \lambda \\ \lambda(d-4) \end{pmatrix}.$$

Hence, $\lambda = 1$ and $\lambda(d-4) = 1$, whence $d = 5$. Also, $\overrightarrow{BC} = \overrightarrow{CD}$. Thus,

$$\frac{BC}{BD} = \frac{BC}{BC + CD} = \frac{BC}{BC + BC} = \frac{1}{2}.$$

Problem 3. The diagram shows a roof, with horizontal rectangular base $OBCD$, where $OB = 10$ m and $BC = 6$ m. The triangular planes ODE and BCF are vertical and the ridge EF is horizontal to the base. The planes $OBFE$ and $DCFE$ are each inclined at an angle θ to the horizontal, where $\tan\theta = 4/3$. The point O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, each of length 1 m, are taken along OB, OD and vertically upwards from O respectively.



Find the position vectors of the points B, C, D, E and F.

Solution. Note that $\overrightarrow{OB} = 10\mathbf{i}$ and $\overrightarrow{BC} = 6\mathbf{j}$. Thus, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = 10\mathbf{i} + 6\mathbf{j}$. Also, note that $\triangle ODE \cong \triangle BCF$. Hence, $\overrightarrow{OD} = \overrightarrow{BC} = 6\mathbf{j}$. Note that $\triangle ODE$ is isosceles. Let G be the mid-point of OD. Since $\tan\theta = 4/3$, we have

$$\frac{EG}{DG} = \frac{4}{3} \implies EG = \frac{4}{3}DG = \frac{2}{3}OD = \frac{2}{3}\cdot 6 = 4 \implies \overrightarrow{GE} = 4\mathbf{k}.$$

Hence,

$$\overrightarrow{OE} = \overrightarrow{OG} + \overrightarrow{GE} = \frac{1}{2}\overrightarrow{OD} + \overrightarrow{GE} = 3\mathbf{j} + 4\mathbf{k}.$$

Hence,

$$\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \overrightarrow{OB} + \overrightarrow{OE} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

Thus,

$$\overrightarrow{OB} = 10\mathbf{i}, \quad \overrightarrow{OC} = 10\mathbf{i} + 6\mathbf{j}, \quad \overrightarrow{OD} = 6\mathbf{j}, \quad \overrightarrow{OE} = 3\mathbf{j} + 4\mathbf{k}, \quad \overrightarrow{OF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

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Problem 4. Find $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\times\mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ given that

(a) $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

(b) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Solution.

Part (a). We have $\mathbf{u} = \langle 1, -1, 1\rangle$ and $\mathbf{v} = \langle 3, 2, 7\rangle$. Hence,

$$\mathbf{u}\cdot\mathbf{v} = (1)(3) + (-1)(2) + (1)(7) = 8, \quad \mathbf{u}\times\mathbf{v} = \begin{pmatrix} (-1)(7) - (2)(1) \\ (1)(3) - (7)(1) \\ (1)(2) - (3)(-1) \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \\ 5 \end{pmatrix}.$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{8}{\sqrt{3}\sqrt{62}} \implies \theta = 54.1^\circ \text{ (1 d.p.)}.$$

Part (b). We have $\mathbf{u} = \langle 2,\, 0,\, -3\rangle$ and $\mathbf{v} = \langle -1,\, 7,\, 2\rangle$. Hence,

$$\mathbf{u}\cdot\mathbf{v} = (2)(-1) + (0)(7) + (-3)(2) = -8, \quad \mathbf{u}\cdot\mathbf{v} = \begin{pmatrix} (0)(2)-(7)(-3) \\ (-3)(-1)-(2)(2) \\ (2)(7)-(-1)(0) \end{pmatrix} = \begin{pmatrix} 21 \\ -1 \\ 14 \end{pmatrix}.$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ.

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{-8}{\sqrt{13}\sqrt{54}} \implies \theta = 107.6^\circ \text{ (1 d.p.)}.$$

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Problem 5. Find $\mathbf{u}\cdot\mathbf{v}$ and $|\mathbf{u}\times\mathbf{v}|$ given that $\mathbf{u} = 2\mathbf{a} - \mathbf{b}$, $\mathbf{v} = -\mathbf{a} + 3\mathbf{b}$, where $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

Solution.

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= (2\mathbf{a}-\mathbf{b})\cdot(-\mathbf{a}+3\mathbf{b}) = -2\mathbf{a}\cdot\mathbf{a} + 6\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} - 3\mathbf{b}\cdot\mathbf{b} \\ &= -2\,|\mathbf{a}|^2 - 3\,|\mathbf{b}|^2 + 7\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta = -2(2)^2 - 3(1)^2 + 7(2)(1)\cos 60^\circ = -4. \\ |\mathbf{u}\times\mathbf{v}| &= |(2\mathbf{a}-\mathbf{b})\times(-\mathbf{a}+3\mathbf{b})| = |-2\mathbf{a}\times\mathbf{a} + 6\mathbf{a}\times\mathbf{b} + \mathbf{b}\times\mathbf{a} - 3\mathbf{b}\times\mathbf{b}| \\ &= |5\mathbf{a}\times\mathbf{b}| = 5\,|\mathbf{a}|\,|\mathbf{b}|\sin\theta = 5(2)(1)\sin 60^\circ = 5\sqrt{3}. \end{aligned}$$

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Problem 6. If $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$, find

(a) a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$,

(b) a vector perpendicular to both $(3\mathbf{b} - 5\mathbf{c})$ and $(7\mathbf{b} + \mathbf{c})$.

Solution.

Part (a). Note that $\mathbf{a}\times\mathbf{b} = \langle 11,\, -4,\, -5\rangle$. Hence, $\widehat{\mathbf{a}\times\mathbf{b}} = \frac{1}{\sqrt{162}}\langle 11,\, -4,\, -5\rangle$.

Part (b). Observe that $(3\mathbf{b} - 5\mathbf{c})\times(7\mathbf{b} + \mathbf{c}) = \lambda\mathbf{b}\times\mathbf{c}$ for some $\lambda\in\mathbb{R}$. It hence suffices to find $\mathbf{b}\times\mathbf{c}$, which works out to be $\langle -3,\, 6,\, 3\rangle$.

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Problem 7. The position vectors of the points A, B and C are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$ respectively. Find

(a) a unit vector parallel to $\overrightarrow{AB}$;

(b) the position vector of the point D such that $ABCD$ is a parallelogram, leaving your answer in terms of λ;

(c) the value of λ if A, B and C are collinear;

(d) the position vector of the point P on AB is $AP : PB = 2 : 1$.

Solution.

Part (a).

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}.$$

Note that $\left|\overrightarrow{AB}\right| = \sqrt{61}$. Hence, the required vector is $\frac{1}{\sqrt{61}} \langle 3, -4, 6 \rangle$.

Part (b). Since $ABCD$ is a parallelogram, we have that $\overrightarrow{AD} = \overrightarrow{BC}$. Thus,

$$\overrightarrow{OD} - \mathbf{a} = \mathbf{c} - \mathbf{b} \implies \overrightarrow{OD} = \mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} = \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix}.$$

Part (c). Given that A, B and C are collinear, we have $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R}$. Hence,

$$\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} = k(\mathbf{c} - \mathbf{b}) = k\left[\begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right] = k\begin{pmatrix} 6 \\ \lambda + 1 \\ 12 \end{pmatrix}.$$

We hence see that $k = 1/2$, whence $\lambda = -9$.

Part (d). By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + 2\mathbf{b}}{2+1} = \frac{1}{3}\left[\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}.$$

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Problem 8. $ABCD$ is a square, and M and N are the midpoints of BC and CD respectively. Express $\overrightarrow{AC}$ in terms of $\mathbf{p}$ and $\mathbf{q}$, where $\overrightarrow{AM} = \mathbf{p}$ and $\overrightarrow{AN} = \mathbf{q}$.

Solution. Let $ABCD$ be a square with side length $2k$ with A at the origin. Then $\mathbf{p} = \overrightarrow{AM} = \langle 2k, -k \rangle$ and $\mathbf{q} = \overrightarrow{AN} = \langle k, -2k \rangle$. Hence, $\mathbf{p} + \mathbf{q} = \langle 3k, -3k \rangle$. Thus, $\overrightarrow{AC} = \langle 2k, -2k \rangle = \frac{2}{3}\langle 3k, -3k \rangle = \frac{2}{3}(\mathbf{p} + \mathbf{q})$.

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Problem 9. The points A, B have position vectors $\mathbf{a}$, $\mathbf{b}$ respectively, referred to an origin O, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel to each other. The point C lies on AB between A and B and is such that $\frac{AC}{CB} = 2$, and D is the mid-point of OC. The line AD produced meets OB at E.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$,

(a) the position vector of C (referred to O),

(b) the vector $\overrightarrow{AD}$. Find the values of $\frac{OE}{EB}$ and $\frac{AE}{ED}$.

Solution.

Part (a). By the ratio theorem,

$$\overrightarrow{OC} = \frac{\mathbf{a} + 2\mathbf{b}}{2+1} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}.$$

Part (b). Since D is the midpoint of OC, we have $\overrightarrow{OD} = \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}$. Hence,

$$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \left(\frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}\right) - \mathbf{a} = -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}.$$

Using Menelaus' theorem on $\triangle BCO$,

$$\frac{BA}{AC}\frac{CD}{DO}\frac{OE}{EB} = 1 \implies \frac{OE}{EB} = \frac{2}{3}.$$

Using Menelaus' theorem on $\triangle BEA$,

$$\frac{BO}{OE}\frac{ED}{DA}\frac{AC}{CB} = 1 \implies \frac{ED}{AD} = \frac{1}{5} \implies \frac{AE}{ED} = \frac{AD + DE}{ED} = 6.$$

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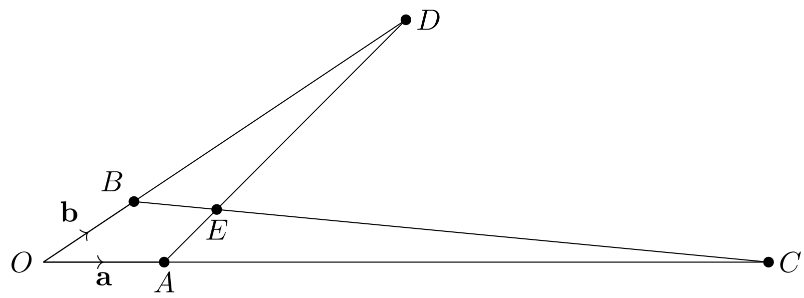
Problem 10.

(a) The angle between the vectors $(3\mathbf{i} - 2\mathbf{j})$ and $(6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k})$ is $\arccos\frac{6}{13}$. Show that $2d^2 - 117d + 333 = 0$.

(b) With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AC} = 5\mathbf{a}$, $\overrightarrow{BD} = 3\mathbf{b}$. The lines AD and BC cross at E.



(i) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) The point F divides the line CD in the ratio $5 : 3$. Show that O, E and F are collinear, and find $OE : EF$.

Solution.

Part (a). Let $\mathbf{a} = \langle 3, -2, 0\rangle$ and $\mathbf{b} = \left\langle 6, d, -\sqrt{7}\right\rangle$. Note that $\mathbf{a}\cdot\mathbf{b} = 18 - 2d$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} \implies \frac{6}{13} = \frac{18 - 2d}{\sqrt{43 + d^2}\sqrt{13}} \implies \frac{9}{13} = \frac{(9-d)^2}{43 + d^2}$$
$$\implies 9(43 + d^2) = 13(d^2 - 18d + 81) \implies 2d^2 - 117d + 333 = 0.$$

Part (b).

Part (b)(i). By Menelaus' theorem,

$$\frac{OC}{CA}\frac{AE}{ED}\frac{DB}{BO} = 1 \implies \frac{AE}{ED} = \frac{5}{18} \implies \overrightarrow{AE} = \frac{5}{23}\overrightarrow{AD} \implies \overrightarrow{OE} = \overrightarrow{OA} + \frac{5}{23}\overrightarrow{AD}.$$

Since $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 4\mathbf{b} - \mathbf{a}$. Thus,

$$\overrightarrow{OE} = \mathbf{a} + \frac{5}{23}(4\mathbf{b} - \mathbf{a}) = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}.$$

Part (b)(ii). By the ratio theorem,

$$\overrightarrow{OF} = \frac{3\mathbf{c}+5\mathbf{d}}{5+3} = \frac{23}{8}\left(\frac{18}{23}\mathbf{a}+\frac{20}{23}\mathbf{b}\right) = \frac{23}{8}\overrightarrow{OE}.$$

Thus, $OE : OF = 8 : 23$.

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Problem 11. Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(a) The point P divides the line AB in the ratio $2 : 1$. Find the coordinates of P.

(b) Show that AB and OP are perpendicular.

(c) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector and give the geometrical meaning of $|\mathbf{a}\cdot\mathbf{c}|$.

(d) Find $\mathbf{a}\times\mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a}\times\mathbf{p}|$. Hence, write down the area of triangle OAP.

Solution.

Part (a). We have $\mathbf{a} = \langle 14, 14, 14\rangle = 14\langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 11, -13, 2\rangle$. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a}+2\mathbf{b}}{2+1} = \frac{1}{3}\left[\begin{pmatrix}14\\14\\14\end{pmatrix} + 2\begin{pmatrix}11\\-13\\2\end{pmatrix}\right] = \begin{pmatrix}12\\-4\\6\end{pmatrix} = 2\begin{pmatrix}6\\-2\\3\end{pmatrix}.$$

Hence, $P(12, -4, 6)$

Part (b). Consider $\overrightarrow{AB}\cdot\overrightarrow{OP}$.

$$\overrightarrow{AB}\cdot\overrightarrow{OP} = \left[\begin{pmatrix}11\\-13\\2\end{pmatrix} - \begin{pmatrix}14\\14\\14\end{pmatrix}\right]\cdot\begin{pmatrix}12\\-4\\6\end{pmatrix} = -3\begin{pmatrix}1\\9\\4\end{pmatrix}\cdot 2\begin{pmatrix}6\\-2\\3\end{pmatrix} = 0.$$

Since $\overrightarrow{AB}\cdot\overrightarrow{OP} = 0$, AB and OP must be perpendicular.

Part (c). We have

$$\mathbf{c} = \frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|} = \frac{1}{\sqrt{6^2+(-2)^2+3^2}}\begin{pmatrix}6\\-2\\3\end{pmatrix} = \frac{1}{7}\begin{pmatrix}6\\-2\\3\end{pmatrix}.$$

$|\mathbf{a}\cdot\mathbf{c}|$ is the length of the projection of $\mathbf{a}$ on $\overrightarrow{OP}$.

Part (d). We have

$$\mathbf{a}\times\mathbf{p} = 14\begin{pmatrix}1\\1\\1\end{pmatrix}\times 2\begin{pmatrix}6\\-2\\3\end{pmatrix} = 28\begin{pmatrix}1\cdot 3-(-2)\cdot 1\\1\cdot 6-3\cdot 1\\1\cdot -2-6\cdot 1\end{pmatrix} = 28\begin{pmatrix}5\\3\\-8\end{pmatrix}.$$

$|\mathbf{a}\times\mathbf{p}|$ is twice the area of $\triangle OAP$.

$$[\triangle OAP] = \frac{1}{2}|\mathbf{a}\times\mathbf{p}| = 14\sqrt{98} = 98\sqrt{2}\text{ units}^2.$$

Problem 12. The points A, B and C have position vectors given by $\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ respectively.

(a) Find the area of the triangle ABC. Hence, find the sine of the angle BAC.

(b) Find a vector perpendicular to the plane ABC.

(c) Find the projection vector of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$.

(d) Find the distance of C to AB.

Solution.

Part (a). We have $\overrightarrow{OA} = \langle 1, -1, 1\rangle$, $\overrightarrow{OB} = \langle 0, 1, -1\rangle$ and $\overrightarrow{OC} = \langle 2, -1, -1\rangle$. Note that $\overrightarrow{AB} = \langle -1, 2, -2\rangle$ and $\overrightarrow{AC} = \langle 1, 0, -2\rangle$. Thus,

$$[\triangle ABC] = \frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \frac{1}{2}\left|\begin{pmatrix}-4\\-4\\-2\end{pmatrix}\right| = \frac{1}{2}\cdot 6 = 3 \text{ units}^2.$$

We have

$$\sin BAC = \frac{\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{6}{3\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

Part (b). $\langle 2, 2, 1\rangle$ is parallel to $\overrightarrow{AB}\times\overrightarrow{AC}$ and is hence perpendicular to the plane ABC.

Part (c). The projection vector of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$ is given by

$$\left(\overrightarrow{AC}\cdot\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right)\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} = \frac{1}{3}\begin{pmatrix}-1\\2\\-2\end{pmatrix}.$$

Part (d). Observe that

$$\left|\overrightarrow{AC}\times\frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}\right| = \frac{1}{3}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 2.$$

Hence, the perpendicular distance between C and AB is 2 units.

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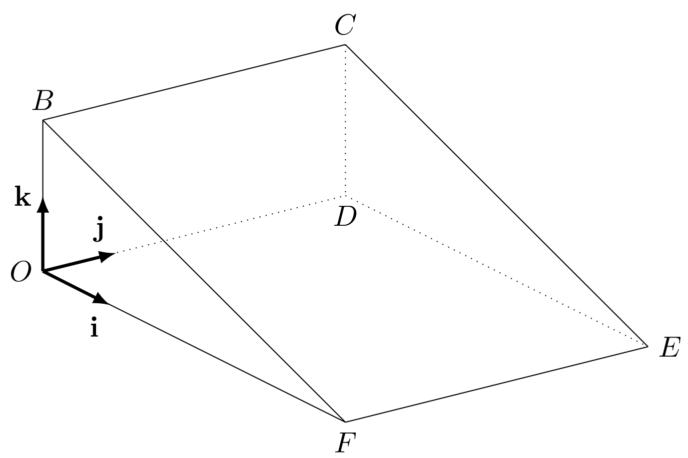
Problem 13.



The diagram shows a vehicle ramp $OBCDEF$ with horizontal rectangular base $ODEF$ and vertical rectangular face $OBCD$. Taking the point O as the origin, the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to the edges OF, OD and OB respectively. The lengths of OF, OD and OB are $2h$ units, 3 units and h units respectively.

(a) Show that $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$.

(b) The point P divides the segment CF in the ratio $2:1$. Find $\overrightarrow{OP}$ in terms of h.

For parts (c) and (d), let $h = 1$.

(c) Find the length of projection of $\overrightarrow{OP}$ onto $\overrightarrow{OC}$.

(d) Using the scalar product, find the angle that the rectangular face $BCEF$ makes with the horizontal base.

Solution.

Part (a). We have

$$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC} = \overrightarrow{OD} + \overrightarrow{OB} = 3\mathbf{j} + h\mathbf{k}.$$

Part (b). By the ratio theorem,

$$\overrightarrow{OP} = \frac{\overrightarrow{OC} + 2\overrightarrow{OF}}{2+1} = \frac{1}{3}\left[\begin{pmatrix}0\\3\\h\end{pmatrix} + 2\begin{pmatrix}2h\\0\\0\end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix}4h\\3\\h\end{pmatrix}.$$

Part (c). The length of projection of $\overrightarrow{OP}$ onto $\overrightarrow{OC}$ is given by

$$\left|\overrightarrow{OP} \cdot \frac{\overrightarrow{OC}}{\left|\overrightarrow{OC}\right|}\right| = \frac{1}{3\sqrt{10}}\left|\begin{pmatrix}4\\3\\1\end{pmatrix} \cdot \begin{pmatrix}0\\3\\1\end{pmatrix}\right| = \frac{\sqrt{10}}{3} \text{ units.}$$

Part (d). Note that $\overrightarrow{OF} = \langle 2, 0, 0\rangle$ and $\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \langle 2, 0, -1\rangle$. Let θ be the angle the rectangular face $BCEF$ makes with the horizontal base.

$$\cos\theta = \frac{\overrightarrow{OF} \cdot \overrightarrow{BF}}{\left|\overrightarrow{OF}\right|\left|\overrightarrow{BF}\right|} = \frac{4}{2\sqrt{5}} \implies \theta = 26.6^\circ \text{ (1 d.p.).}$$

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Problem 14. The position vectors of the points A and B relative to the origin O are $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ respectively. The point P on AB is such that $AP : PB = \lambda : 1 - \lambda$. Show that $\overrightarrow{OP} = (1+\lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8)\mathbf{k}$ where λ is a real parameter.

(a) Find the value of λ for which OP is perpendicular to AB.

(b) Find the value of λ for which angles $\angle AOP$ and $\angle POB$ are equal.

Solution. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\lambda\overrightarrow{OB} + (1-\lambda)\overrightarrow{OA}}{\lambda + (1-\lambda)} = \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix} + (1-\lambda)\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}.$$

Part (a). Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 1,\, -5,\, 8\rangle$. For OP to be perpendicular to AB, we must have $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$.

$$\overrightarrow{OP} \cdot \overrightarrow{AB} = 0 \implies \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = 0 \implies -25 + 90\lambda = 0 \implies \lambda = \frac{5}{18}.$$

Part (b). Suppose $\angle AOP = \angle POB$. Then $\cos\angle AOP = \cos\angle POB$. Thus,

$$\frac{\overrightarrow{OP}\cdot\overrightarrow{OA}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OA}\right|} = \frac{\overrightarrow{OP}\cdot\overrightarrow{OB}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OB}\right|} \implies \overrightarrow{OP}\cdot\left(\frac{1}{3}\overrightarrow{OA} - \frac{1}{7}\overrightarrow{OB}\right) = 0 \implies \overrightarrow{OP}\cdot\left(7\overrightarrow{OA} - 3\overrightarrow{OB}\right) = 0.$$

This gives

$$\begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \left[7\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}\right] = \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 23 \\ -32 \end{pmatrix} = 0.$$

Taking the dot product and simplifying, we see that $111 - 370\lambda = 0$, whence $\lambda = \frac{3}{10}$.

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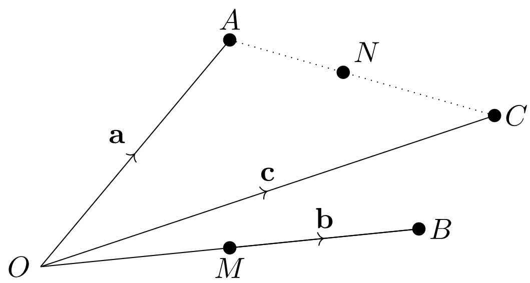
Problem 15.



The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$,

(a) Explain why $\mathbf{c}$ can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ.

The point N is on AC such that $AN : NC = 3 : 4$.

(b) Write down the position vector of N in terms of $\mathbf{a}$ and $\mathbf{c}$.

(c) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms of $\mathbf{a}$ and $\mathbf{b}$, find λ in terms of μ in the case where λ and μ are both positive.

Solution.

Part (a). Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are co-planar and $\mathbf{a}$ is not parallel to $\mathbf{b}$, $\mathbf{c}$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Part (b). By the ratio theorem,

$$\overrightarrow{ON} = \frac{4\mathbf{a} + 3\mathbf{c}}{3+4} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}.$$

Part (c). Let $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$. The area of $\triangle ONC$ is given by

$$[\triangle ONC] = \frac{1}{2}\left|\overrightarrow{ON} \times \hat{\mathbf{c}}\right| = \frac{1}{2}\left|\left[\frac{4}{7}\mathbf{a} + \frac{3}{7}(\lambda\mathbf{a} + \mu\mathbf{b})\right] \times \frac{(\lambda\mathbf{a} + \mu\mathbf{b})}{|\mathbf{c}|}\right| = \frac{2}{7\,|\mathbf{c}|}\,|\mathbf{a} \times \mathbf{b}|\,.$$

Meanwhile, the area of $\triangle OMC$ is given by

$$[\triangle OMC] = \frac{1}{2}\left|\overrightarrow{OM} \times \hat{\mathbf{c}}\right| = \frac{1}{2}\left|\frac{1}{2}\mathbf{b} \times \frac{(\lambda\mathbf{a} + \mu\mathbf{b})}{|\mathbf{c}|}\right| = \frac{\lambda}{4\,|\mathbf{c}|}\,|\mathbf{a} \times \mathbf{b}|\,.$$

Since the two areas are equal,

$$[\triangle ONC] = [\triangle OMC] \implies \frac{2}{7\,|\mathbf{c}|}\,|\mathbf{a} \times \mathbf{b}| = \frac{\lambda}{4\,|\mathbf{c}|}\,|\mathbf{a} \times \mathbf{b}| \implies \lambda = \frac{8}{7}\mu.$$

Assignment A7

Problem 1. The points A and B have position vectors relative to the origin O, denoted by $\mathbf{a}$ and $\mathbf{b}$ respectively, where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. The point P lies on AB such that $AP : PB = \lambda : 1$. The point Q lies on OP extended such that $OP = 2PQ$ and $\overrightarrow{BQ} = \overrightarrow{OA} + \mu\overrightarrow{OB}$. Find the values of the real constants λ and μ.

Solution. By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + \lambda\mathbf{b}}{1+\lambda} \implies \overrightarrow{OQ} = \frac{3}{2}\overrightarrow{OP} = \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1+\lambda}.$$

However, we also have

$$\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = \mathbf{a} + (1+\mu)\mathbf{b}.$$

This gives the equality

$$\frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1+\lambda} = \mathbf{a} + (1+\mu)\mathbf{b}.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel, we can compare the $\mathbf{a}$- and $\mathbf{b}$-components of both vectors separately. This gives us

$$\frac{3}{2} \cdot \frac{1}{1+\lambda} = 1, \quad \frac{3}{2} \cdot \frac{\lambda}{1+\lambda} = 1 + \mu,$$

which has the unique solution $\lambda = 1/2$ and $\mu = -1/2$.

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Problem 2. Given that $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ where $\lambda \in \mathbb{R}$, find the possible value(s) of λ for which the angle between $\mathbf{p}$ and $\mathbf{k}$ is 45°.

Solution. Observe that

$$\mathbf{p} = \lambda\mathbf{a} + (1-\lambda)\mathbf{b} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 - 3\lambda \\ -2 + 3\lambda \\ 6 - 6\lambda \end{pmatrix}.$$

Thus,

$$|\mathbf{p}|^2 = (4-3\lambda)^2 + (-2+3\lambda)^2 + (6-6\lambda)^2 = 54\lambda^2 - 108\lambda + 56.$$

Since the angle between $\mathbf{p}$ and $\mathbf{k}$ is 45°,

$$\cos 45° = \frac{\mathbf{p} \cdot \mathbf{k}}{|\mathbf{p}|\,|\mathbf{k}|} \implies \frac{1}{\sqrt{2}} = \frac{6-6\lambda}{|\mathbf{p}|} \implies \frac{|\mathbf{p}|^2}{2} = (6-6\lambda)^2.$$

We thus obtain the quadratic equation

$$\frac{54\lambda^2 - 108\lambda + 56}{2} = 36\lambda^2 - 72\lambda + 36 \implies 9\lambda^2 - 18\lambda + 8 = 0,$$

which has solutions $\lambda = 2/3$ and $\lambda = 4/3$. However, we must reject $\lambda = 4/3$ since $6 - 6\lambda = |\mathbf{p}|/\sqrt{2} > 0 \implies \lambda < 1$. Thus, $\lambda = 2/3$.

Problem 3.

(a) $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors such that $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$. State the relation between the directions of $\mathbf{a}$ and $\mathbf{b}$, and find $|\mathbf{b}|$.

(b) $\mathbf{a}$ is a non-zero vector such that $\mathbf{a} = \sqrt{3}$ and $\mathbf{b}$ is a unit vector. Given that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel and the angle between them is $5\pi/6$, find the exact value of the length of projection of $\mathbf{a}$ on $\mathbf{b}$. By considering $(2\mathbf{a}+\mathbf{b}) \cdot (2\mathbf{a}+\mathbf{b})$, or otherwise, find the exact value of $|2\mathbf{a}+\mathbf{b}|$.

Solution.

Part (a). $\mathbf{a}$ and $\mathbf{b}$ either have the same or opposite direction. Let $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$.

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} = (\mathbf{a} \cdot \lambda\mathbf{a})\lambda\mathbf{a} = \lambda^2 |\mathbf{a}|^2 \mathbf{a} \implies \lambda^2 |\mathbf{a}|^2 = 1 \implies |\mathbf{b}| = |\lambda |\mathbf{a}|| = 1.$$

Part (b). Note that $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \cos(5\pi/6) = -3/2$. Hence, the length of projection of $\mathbf{a}$ on $\mathbf{b}$ is $\left|\mathbf{a} \cdot \hat{\mathbf{b}}\right| = 3/2$ units.

Observe that

$$|2\mathbf{a}+\mathbf{b}|^2 = (2\mathbf{a}+\mathbf{b}) \cdot (2\mathbf{a}+\mathbf{b}) = 4|\mathbf{a}|^2 + 4(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 = 7.$$

Thus, $|2\mathbf{a}+\mathbf{b}| = \sqrt{7}$.

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Problem 4. The points A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ given by $\mathbf{a} = \mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$, respectively. The point P lies on AB produced such that $AP = 2AB$, and the point Q is the mid-point of AC.

(a) Show that PQ is perpendicular to AQ.

(b) Find the area of the triangle APQ.

(c) Find a vector perpendicular to the plane ABC.

(d) Find the cosine of the angle between $\overrightarrow{AD}$ and $\overrightarrow{BD}$.

Solution. Note that $\overrightarrow{AB} = \langle 0, 0, -1\rangle$, $\overrightarrow{AC} = \langle 2, 0, -2\rangle$ and $\overrightarrow{AD} = \langle 3, -3, -4\rangle$.

Part (a). Note that

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + 2\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

and

$$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

Thus,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Since $\overrightarrow{PQ} \cdot \overrightarrow{AQ} = 0$, the two vectors are perpendicular, whence $PQ \perp AQ$.

Part (b). Note that $\overrightarrow{AP} = \langle 0,\ 0,\ -2\rangle$. Hence,

$$[\triangle APQ] = \frac{1}{2}\left|\overrightarrow{AP} \times \overrightarrow{AQ}\right| = \frac{1}{2}\left|\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right| = 1 \text{ units}^2.$$

Part (c). The vector $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0,\ -2,\ 0\rangle$ is perpendicular to the plane ABC.

Part (d). Let the angle between $\overrightarrow{AD}$ and $\overrightarrow{BD}$ be θ. Note that $\overrightarrow{BD} = -3\,\langle -1,\ 1,\ 1\rangle$. Hence,

$$\cos\theta = \frac{\overrightarrow{AD} \cdot \overrightarrow{BD}}{\left|\overrightarrow{AD}\right| \left|\overrightarrow{BD}\right|} = \frac{30}{\sqrt{34} \cdot 3\sqrt{3}} = \frac{10}{\sqrt{102}}.$$

A8. Vectors II - Lines

Tutorial A8

Problem 1. For each of the following, write down a vector equivalent of the line l and convert it to parametric and Cartesian forms.

(a) l passes through the point with position vector $-\mathbf{i}+\mathbf{k}$ and is parallel to the vector $\mathbf{i}+\mathbf{j}$.

(b) l passes through the points $P(1,-1,3)$ and $Q(2,1,-2)$.

(c) l passes through the origin and is parallel to the line $m : \mathbf{r} = \langle 1,\ -1,\ 3\rangle + \lambda\langle 1,\ 2,\ 3\rangle$, where $\lambda \in \mathbb{R}$.

(d) l is the x-axis.

(e) l passes through the point $C(4,-1,2)$ and is parallel to the z-axis.

Solution.

Part (a).

Form	Expression
Vector	$\mathbf{r} = \langle -1,\ 0,\ 1\rangle + \lambda\langle 1,\ 1,\ 0\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda - 1,\ y = \lambda,\ z = 1$
Cartesian	$x + 1 = y,\ z = 1$

Part (b).

Form	Expression
Vector	$\mathbf{r} = \langle 1,\ -1,\ 3\rangle + \lambda\langle 1,\ 2,\ -5\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda + 1,\ y = 2\lambda - 1,\ z = -5\lambda + 3$
Cartesian	$x - 1 = \frac{y+1}{2} = \frac{3-z}{5}$

Part (c).

Form	Expression
Vector	$\mathbf{r} = \lambda\langle 1,\ 2,\ 3\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda,\ y = 2\lambda,\ z = 3\lambda$
Cartesian	$x = \frac{y}{2} = \frac{z}{3}$

Part (d).

Form	Expression
Vector	$\mathbf{r} = \lambda\langle 1,\ 0,\ 0\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = \lambda,\ y = 0,\ z = 0$
Cartesian	$x \in \mathbb{R},\ y = 0,\ z = 0$

Part (e).

Form	Expression
Vector	$\mathbf{r} = \langle 4,\ -1,\ 2\rangle + \lambda\langle 0,\ 0,\ 1\rangle,\ \lambda \in \mathbb{R}$
Parametric	$x = 4,\ y = -1,\ z = \lambda + 2$
Cartesian	$x = 4,\ y = -1,\ z \in \mathbb{R}$

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Problem 2. For each of the following, determine if l_1 and l_2 are parallel, intersecting or skew. In the case of intersecting lines, find the position vector of the point of intersection. In addition, find the acute angle between the lines l_1 and l_2.

(a) $l_1 : x - 1 = -y = z - 2$ and $l_2 : \frac{x-2}{2} = -\frac{y+1}{2} = \frac{z-4}{2}$

(b) $l_1 : \mathbf{r} = \langle 1,\ 0,\ 0\rangle + \alpha\langle 4,\ -2,\ -3\rangle,\ \alpha \in \mathbb{R}$ and $l_2 : \mathbf{r} = \langle 0,\ 10,\ 1\rangle + \beta\langle 3,\ 8,\ 1\rangle$

(c) $l_1 : \mathbf{r} = (\mathbf{i} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}),\ \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} - 4\mathbf{j} - \mathbf{k}),\ \mu \in \mathbb{R}$

Solution.

Part (a). Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\2\end{pmatrix},\ \mu \in \mathbb{R}.$$

Since $\langle 2,\ -2,\ 2\rangle = 2\langle 1,\ -1,\ 1\rangle$, l_1 and l_2 are parallel ($\theta = 0$). Since $\langle 1,\ 0,\ 2\rangle \neq \langle 2,\ 1,\ 4\rangle + \mu\langle 2,\ -2,\ 2\rangle$ for all real μ, we have that l_1 and l_2 are distinct.

Part (b). Since $\langle 4,\ -2,\ 3\rangle \neq \beta\langle 3,\ 8,\ 1\rangle$ for all real β, it follows that l_1 and l_2 are not parallel.

Consider $l_1 = l_2$.

$$l_1 = l_2 \implies \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix} = \begin{pmatrix}0\\10\\1\end{pmatrix} + \beta\begin{pmatrix}3\\8\\1\end{pmatrix} \implies \alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix} - \beta\begin{pmatrix}3\\8\\1\end{pmatrix} = \begin{pmatrix}-1\\10\\1\end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 4\alpha - 3\beta = -1 \\ -2\alpha - 8\beta = 10 \\ -3\alpha - \ \beta = 1 \end{cases}$$

There are no solutions to the above system. Hence, l_1 and l_2 do not intersect and are thus skew.

Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{|\langle 4,\ -2,\ -3\rangle \cdot \langle 3,\ 8,\ 1\rangle|}{|\langle 4,\ -2,\ -3\rangle|\,|\langle 3,\ 8,\ 1\rangle|} = \frac{7}{\sqrt{2146}} \implies \theta = 81.3^\circ \text{ (1 d.p.).}$$

Part (c). Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}5\\-4\\-1\end{pmatrix},\ \lambda, \mu \in \mathbb{R}.$$

Since $\langle 1,\, -1,\, 1\rangle \neq \mu \langle 5,\, -4,\, -1\rangle$ for all real μ, it follows that l_1 and l_2 are not parallel. Consider $l_1 = l_2$.

$$l_1 = l_2 \implies \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \implies \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} -5\mu + \lambda = 0 \\ 4\mu - \lambda = -1 \\ \mu + \lambda = 6 \end{cases}$$

The above system has the unique solution $\lambda = 5$ and $\mu = 1$. Hence, l_1 and l_2 intersect at $\langle 1,\, 0,\, -5\rangle + 5\,\langle 1,\, -1,\, 1\rangle = \langle 6,\, -5,\, 0\rangle$.

Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{\left|\langle 1,\, -1,\, 1\rangle \cdot \langle 5,\, -4,\, -1\rangle\right|}{\left|\langle 1,\, -1,\, 1\rangle\right| \left|\langle 5,\, -4,\, -1\rangle\right|} = \frac{8}{3\sqrt{14}} \implies \theta = 44.5^\circ \text{ (1 d.p.)}.$$

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Problem 3.

(a) Find the shortest distance from the point $(1, 2, 3)$ to the line with equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$.

(b) Find the length of projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\frac{x+5}{4} = \frac{y-5}{3} = 10 - 2z$.

(c) Find the projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\frac{x+5}{4} = \frac{y-5}{3} = 10 - 2z$.

Solution.

Part (a). Let $\overrightarrow{OP} = \langle 1,\, 2,\, 3\rangle$ and $\overrightarrow{OA} = \langle 3,\, 2,\, 4\rangle$. Note that $\overrightarrow{AP} = \langle -2,\, 0,\, -1\rangle$. The shortest distance between P and the line is thus

$$\text{Shortest distance} = \frac{\left|\langle -2,\, 0,\, -1\rangle \times \langle 1,\, 2,\, 2\rangle\right|}{\left|\langle 1,\, 2,\, 2\rangle\right|} = \frac{\left|\langle 2,\, -3,\, -4\rangle\right|}{3} = \frac{\sqrt{29}}{3} \text{ units.}$$

Part (b). Note that the line has vector form

$$\mathbf{r} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda' \begin{pmatrix} 4 \\ 3 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

The length of projection of $\langle 4,\, -5,\, 6\rangle$ onto the line is thus given by

$$\text{Length of projection} = \frac{\left|\langle 4,\, -5,\, 6\rangle \cdot \langle 8,\, 6,\, -1\rangle\right|}{\left|\langle 8,\, 6,\, -1\rangle\right|} = \frac{4}{\sqrt{101}} \text{ units.}$$

Part (c).

$$\text{Projection} = \left[\frac{\langle 4,\, -5,\, 6\rangle \cdot \langle 8,\, 6,\, -1\rangle}{\left|\langle 8,\, 6,\, -1\rangle\right|}\right] \cdot \frac{\langle 8,\, 6,\, -1\rangle}{\left|\langle 8,\, 6,\, -1\rangle\right|} = \frac{-4}{101} \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}$$

Problem 4. The points P and Q have coordinates $(0, -1, -1)$ and $(3, 0, 1)$ respectively, and the equations of the lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \ \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \ \mu \in \mathbb{R}.$$

(a) Show that P lies on l_1 but not on l_2.

(b) Determine if l_2 passes through Q.

(c) Find the coordinates of the foot of the perpendicular from P to l_2. Hence, or otherwise, find the perpendicular distance from P to l_2.

(d) Find the length of projection of $\overrightarrow{PQ}$ onto l_2.

Solution. We have that $\overrightarrow{OP} = \langle 0,\ -1,\ -1 \rangle$ and $\overrightarrow{OQ} = \langle 3,\ 0,\ 1 \rangle$.

Part (a). When $\lambda = -2$, we have $\langle 0,\ 1,\ -3 \rangle - 2\,\langle 0,\ 1,\ -1 \rangle = \langle 0,\ -1,\ -1 \rangle = \overrightarrow{OP}$. Hence, P lies on l_1.

Observe that all points on l_2 have a z-coordinate of 1. Since P has a z-coordinate of -1, P does not lie on l_2.

Part (b). When $\mu = 3$, we have $\langle -3,\ 3,\ 1 \rangle + 3\,\langle 2,\ -1,\ 0 \rangle = \langle 3,\ 0,\ 1 \rangle = \overrightarrow{OQ}$. Hence, l_2 passes through Q.

Part (c). Let the foot of the perpendicular from P to l_2 be F. Since F is on l_2, we have that $\overrightarrow{OF} = \langle -3,\ 3,\ 1 \rangle + \mu\,\langle 2,\ -1,\ 0 \rangle$ for some real μ. We also have that $\overrightarrow{PF} \cdot \langle 2,\ -1,\ 0 \rangle = 0$. Note that

$$\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 + 2\mu \\ 4 - \mu \\ 2 \end{pmatrix}.$$

Hence,

$$\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \implies \begin{pmatrix} -3 + 2\mu \\ 4 - \mu \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \implies -10 + 5\mu = 0 \implies \mu = 2.$$

Hence, $\overrightarrow{OF} = \langle -3,\ 3,\ 1 \rangle + 2\,\langle 3,\ -1,\ 0 \rangle = \langle 1,\ 1,\ 1 \rangle$. Thus, $F(1, 1, 1)$. The perpendicular distance from P to l_2 is thus $\left|\overrightarrow{PF}\right| = \left|\langle 1,\ 2,\ 2 \rangle\right| = 3$ units.

Part (d). Note that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. The length of projection of $\overrightarrow{PQ}$ onto l_2 is thus given by

$$\text{Length of projection} = \frac{\left|\langle 3,\ 1,\ 2 \rangle \cdot \langle 2,\ -1,\ 0 \rangle\right|}{\left|\langle 2,\ -1,\ 0 \rangle\right|} = \frac{5}{\sqrt{5}} = \sqrt{5} \text{ units.}$$

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Problem 5. The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

respectively. Find the position vectors of the points P on l_1 and Q on l_2 such that O, P and Q are collinear, where O is the origin.

Solution. We have that $\overrightarrow{OP} = \langle 0,\ 1,\ 2\rangle + s\,\langle 1,\ 0,\ 3\rangle$ and $\overrightarrow{OQ} = \langle -2,\ 3,\ 1\rangle + t\,\langle 2,\ 1,\ 0\rangle$ for some $s, t \in \mathbb{R}$. For O, P and Q to be collinear, we need $\overrightarrow{OP} = \lambda\overrightarrow{OQ}$ for some $\lambda \in \mathbb{R}$:

$$\begin{pmatrix}0\\1\\2\end{pmatrix} + s\begin{pmatrix}1\\0\\3\end{pmatrix} = \lambda\left[\begin{pmatrix}-2\\3\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}\right] \implies \begin{pmatrix}s\\1\\2+3s\end{pmatrix} = \lambda\begin{pmatrix}-2+2t\\3+t\\1\end{pmatrix}.$$

This gives us the system:

$$\begin{cases} s = \lambda(-2+2t) \\ 1 = \lambda(3+t) \\ 2+3s = \lambda \end{cases}$$

Substituting the third equation into the first two gives the reduced system:

$$\begin{cases} s = (2+3s)(-2+2t) \\ 1 = (2+3s)(3+t) \end{cases}$$

Subtracting twice of the second equation from the first yields $s - 2 = -8(2+3s)$, whence $s = -14/25$. It quickly follows that $t = 1/8$. Hence,

$$\overrightarrow{OP} = \begin{pmatrix}0\\1\\2\end{pmatrix} - \frac{14}{25}\begin{pmatrix}1\\0\\3\end{pmatrix} = \frac{1}{25}\begin{pmatrix}-14\\25\\8\end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix}-2\\3\\1\end{pmatrix} + \frac{1}{8}\begin{pmatrix}2\\1\\0\end{pmatrix} = \frac{1}{8}\begin{pmatrix}-14\\25\\8\end{pmatrix}.$$

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Problem 6. Relative to the origin O, the points A, B and C have position vectors $5\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$, $-4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $-5\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$ respectively.

(a) Find the Cartesian equation of the line AB.

(b) Find the length of projection of $\overrightarrow{AC}$ onto the line AB. Hence, find the perpendicular distance from C to the line AB.

(c) Find the position vector of the foot N of the perpendicular from C to the line AB.

(d) The point D is such that it is a reflection of point C about the line AB. Find the position vector of D.

Solution. We have that $\overrightarrow{OA} = \langle 5,\ 4,\ 10\rangle$, $\overrightarrow{OB} = \langle -4,\ 4,\ -2\rangle$ and $\overrightarrow{OC} = \langle -5,\ 9,\ 5\rangle$.

Part (a). Note that $\overrightarrow{AB} = \langle -9,\ 0,\ -12\rangle = -3\,\langle 3,\ 0,\ 4\rangle$. The line AB hence has the vector form

$$\mathbf{r} = \begin{pmatrix}5\\4\\10\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\4\end{pmatrix}, \ \lambda \in \mathbb{R}$$

and Cartesian form $\frac{x-5}{3} = \frac{z-10}{4}$, $y = 4$.

Part (b). Note that $\overrightarrow{AC} = \langle -10,\ 5,\ -5\rangle = -5\,\langle 2,\ -1,\ 1\rangle$. Hence, the length of projection of $\overrightarrow{AC}$ onto the line AB is given by

$$\text{Length of projection} = \frac{\left|\overrightarrow{AC}\cdot\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|} = \frac{1}{15}\left|5\begin{pmatrix}2\\-1\\1\end{pmatrix}\cdot 3\begin{pmatrix}3\\0\\4\end{pmatrix}\right| = 10 \text{ units.}$$

Since $\left|\overrightarrow{AC}\right| = 5\sqrt{6}$, the perpendicular distance from C to the line AB is $\sqrt{\left(5\sqrt{6}\right)^2 - 10^2} = 5\sqrt{2}$ units.

Part (c). Let $\overrightarrow{AN} = \lambda \langle -9,\ 0,\ -12\rangle$ for some $\lambda \in \mathbb{R}$ such that $\left|\overrightarrow{AN}\right| = 10$.

$$\left|\overrightarrow{AN}\right| = 10 \implies 15\lambda = 10 \implies \lambda = \frac{2}{3}.$$

Hence, $\overrightarrow{AN} = \frac{2}{3}\langle -9,\ 0,\ -12\rangle = \langle -6,\ 0,\ -8\rangle$. Thus, $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \langle -1,\ 4,\ 2\rangle$.

Part (d). Note that $\overrightarrow{NC} = \overrightarrow{OC} - \overrightarrow{ON} = \langle -4,\ 5,\ 3\rangle$. Since D is the reflection of C about AB, we have that $\overrightarrow{ND} = -\overrightarrow{NC}$. Thus,

$$\overrightarrow{OD} = \overrightarrow{ON} + \overrightarrow{ND} = \overrightarrow{ON} - \overrightarrow{NC} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}.$$

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Problem 7. The points A and B have coordinates $(0, 9, c)$ and $(d, 5, -2)$ respectively, where c and d are constants. The line l has equation $\frac{x+3}{-1} = \frac{y-1}{4} = \frac{z-5}{3}$.

(a) Given that $d = 22/7$ and the line AB intersects l, find the value of c. Find also the coordinates of the foot of the perpendicular from A to l.

(b) Given instead that the lines AB and l are parallel, state the value of c and d and find the shortest distance between the lines AB and l.

Solution. We have that $\overrightarrow{OA} = \langle 0,\ 9,\ c\rangle$ and $\overrightarrow{OB} = \langle d,\ 5,\ -2\rangle$. We also have that the line l is given by the vector $\mathbf{r} = \langle -3,\ 1,\ 5\rangle + \lambda \langle -1,\ 4,\ 3\rangle$ for $\lambda \in \mathbb{R}$.

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle d,\ -4,\ -2-c\rangle$. Hence, the line AB is given by the vector $\mathbf{r}_{AB} = \langle d,\ 5,\ -2\rangle + \mu \langle d,\ -4,\ -2-c\rangle$ for $\mu \in \mathbb{R}$.

Part (a). Consider the direction vectors of AB and l. Since $\langle 22/7,\ -4,\ -2-c\rangle \neq \lambda \langle -1,\ 4,\ 3\rangle$ for all real λ and c, the lines AB and l are not parallel. Hence, AB and l intersect at only one point. Thus, there must be a unique solution to $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{aligned} \mathbf{r} = \mathbf{r}_{AB} &\implies \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 22/7 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 22/7 \\ -4 \\ -2-c \end{pmatrix} \\ &\implies \lambda \begin{pmatrix} -7 \\ 28 \\ 21 \end{pmatrix} - \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} = \begin{pmatrix} 43 \\ 28 \\ -49 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} -\lambda - \qquad\quad 22\mu = 43 \\ 4\lambda + \qquad\quad 28\mu = 28 \\ 3\lambda + (14+7c)\,\mu = -49 \end{cases}$$

Solving the first two equations gives $\lambda = 91/3$ and $\mu = -10/3$. It follows from the third equation that $c = 4$.

Let F be the foot of the perpendicular from A to l. We have that $\overrightarrow{OF} = \langle -3,\ 1,\ 5\rangle + \lambda \langle -1,\ 4,\ 3\rangle$ for some $\lambda \in \mathbb{R}$. We also have that $\overrightarrow{AF} \cdot \langle -1,\ 4,\ 3\rangle = 0$. Note that

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} -3-\lambda \\ -8+4\lambda \\ 1+3\lambda \end{pmatrix}.$$

Hence,

$$\overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3-\lambda \\ -8+4\lambda \\ 1+3\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \implies -26+26\lambda = 0 \implies \lambda = 1.$$

Hence, $\overrightarrow{OF} = \langle -3,\ 1,\ 5\rangle + \langle -1,\ 4,\ 3\rangle = \langle -4,\ 5,\ 8\rangle$. The foot of the perpendicular from A to l hence has coordinates $(-4, 5, 8)$.

Part (b). Given that AB is parallel to l, one of their direction vectors must be a scalar multiple of the other. Hence, for some real λ, $\langle -1,\ 4,\ 3\rangle = \lambda \langle d,\ -4,\ -2-c\rangle$. It is obvious that $\lambda = -1$, whence $c = 1$ and $d = 1$.

Note that the direction vector of l and AB is $\langle -1,\ 4,\ 3\rangle$. Also note that l passes through $(-3, 1, 5)$ and AB passes through $(1, 5, -2)$. Since $\langle 1,\ 5,\ -2\rangle - \langle -3,\ 1,\ 5\rangle = \langle 4,\ 4,\ -7\rangle$, the shortest distance between AB and l is

$$\frac{|\langle -1,\ 4,\ 3\rangle \times \langle 4,\ 4,\ -7\rangle|}{|\langle -1,\ 4,\ 3\rangle|} = \frac{1}{\sqrt{26}} \left| \begin{pmatrix} -40 \\ -5 \\ -20 \end{pmatrix} \right| = \frac{45}{\sqrt{26}} \text{ units.}$$

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Problem 8. The equation of the line L is $\mathbf{r} = \langle 1,\ 3,\ 7\rangle + t\langle 2,\ -1,\ 5\rangle$, $t \in \mathbb{R}$. The points A and B have position vectors $\langle 9,\ 3,\ 26\rangle$ and $\langle 13,\ 9,\ \alpha\rangle$ respectively. The line L intersects the line through A and B at P.

(a) Find α and the acute angle between line L and AB.

The point C has position vector $\langle 2,\ 5,\ 1\rangle$ and the foot of the perpendicular from C to L is Q.

(b) Find the position vector of Q. Hence, find the shortest distance from C to L.

(c) Find the position vector of the point of reflection of the point C about the line L. Hence, find the reflection of the line passing through C and the point $(1, 3, 7)$ about the line L.

Solution.

Part (a). We have that $\overrightarrow{OA} = \langle 9,\ 3,\ 26\rangle$ and $\overrightarrow{OB} = \langle 13,\ 9,\ \alpha\rangle$. Hence, $\overrightarrow{AB} = \langle 4,\ 6,\ \alpha - 26\rangle$. The line AB is thus given by $\mathbf{r}_{AB} = \langle 9,\ 3,\ 26\rangle + u\langle 4,\ 6,\ \alpha-26\rangle$ for $u \in \mathbb{R}$. Note that AB is not parallel to L. Hence, $\overrightarrow{OP}$ is the only solution to the equation $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u\begin{pmatrix} 4 \\ 6 \\ \alpha-26 \end{pmatrix} \implies t\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - u\begin{pmatrix} 4 \\ 6 \\ \alpha-26 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 19 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 2t - 4u = 8 \\ -t - 6u = 0 \\ 5t - (\alpha - 26)\,u = 19 \end{cases}$$

Solving the first two equations gives $t = 3$ and $u = -\frac{1}{2}$. It follows from the third equation that $\alpha = 34$.

Let the acute angle between L and AB be θ.

$$\cos\theta = \frac{|\langle 2,\ -1,\ 5\rangle \cdot \langle 4,\ 6,\ 8\rangle|}{|\langle 2,\ -1,\ 5\rangle|\,|\langle 4,\ 6,\ 8\rangle|} = \frac{42}{\sqrt{30}\sqrt{116}} \implies \theta = 44.6^\circ \text{ (1 d.p.)}.$$

Part (b). Since Q is on L, we have that $\overrightarrow{OQ} = \langle 1,\ 3,\ 7\rangle + t\,\langle 2,\ -1,\ 5\rangle$ for some real t. Further, since $\overrightarrow{CQ} \perp L$, we have that $\overrightarrow{CQ} \cdot \langle 2,\ -1,\ 5\rangle = 0$. Note that

$$\overrightarrow{CQ} = \overrightarrow{OQ} - \overrightarrow{OC} = \begin{pmatrix} -1+2t \\ -2-t \\ 6+5t \end{pmatrix}.$$

Thus,

$$\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0 \implies \begin{pmatrix} -1+2t \\ -2-t \\ 6+5t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0 \implies 30 + 30t = 0 \implies t = 1.$$

Hence, $\overrightarrow{OQ} = \langle 1,\ 3,\ 7\rangle + \langle 2,\ -1,\ 5\rangle = \langle -1,\ 4,\ 2\rangle$. The shortest distance from C to L is thus

$$\left|\overrightarrow{CQ}\right| = \left|\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right| = \left|\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}\right| = \sqrt{11} \text{ units}.$$

Part (c). Let C' be the reflection of C about L. Note that

$$\overrightarrow{OC'} = \overrightarrow{OQ} - \overrightarrow{QC} = \overrightarrow{OQ} + \overrightarrow{CQ} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}.$$

Note that $(1, 3, 7)$ is on L and is hence invariant under a reflection about L. Let the reflection about L of the line passing through C and $(1, 3, 7)$ be L'. Since $\langle -4,\ 3,\ 3\rangle - \langle 1,\ 3,\ 7\rangle = \langle -5,\ 0,\ -4\rangle \parallel \langle 5,\ 0,\ 4\rangle$, L' hence has direction vector $\langle 5,\ 0,\ 4\rangle$. Thus, L' is given by $\mathbf{r}' = \langle 1,\ 3,\ 7\rangle + \lambda\,\langle 5,\ 0,\ 4\rangle$ for $\lambda \in \mathbb{R}$.

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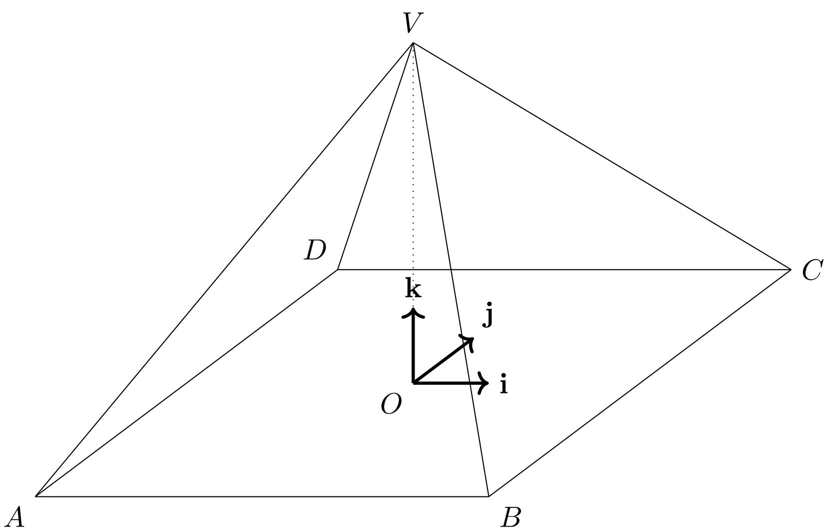
Problem 9.



In the diagram, O is the origin of the square base $ABCD$ of a right pyramid with vertex V. The perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to AB, AD and OV respectively. The length of AB is 4 units and the length of OV is $2h$ units. P, Q, M and N are the mid-points of AB, BC, CV and VA respectively. The point O is taken as the origin for position vectors.

Show that the equation of the line PM may be expressed as $\mathbf{r} = \langle 0,\, -2,\, 0\rangle + t\,\langle 1,\, 3,\, h\rangle$, where t is a parameter.

(a) Find an equation for the line QN.

(b) Show that the lines PM and QN intersect and that the position vector $\overrightarrow{OX}$ of their point of intersection is $\mathbf{r} = \frac{1}{2}\langle 1,\, -1,\, h\rangle$.

(c) Given that OX is perpendicular to VB, find the value of h and calculate the acute angle between PM and QN, giving your answer correct to the nearest $0.1°$.

Solution. We are given that $\overrightarrow{OP} = \langle 0,\, -2,\, 0\rangle$, $\overrightarrow{OC} = \langle 2,\, 2,\, 0\rangle$ and $\overrightarrow{OV} = \langle 0,\, 0,\, 2h\rangle$. Hence, $\overrightarrow{CV} = \overrightarrow{OV} - \overrightarrow{OC} = \langle -2,\, -2,\, 2h\rangle$. Thus, $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CV} = \langle -1,\, -1,\, h\rangle$. Since $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \langle 1,\, 1,\, h\rangle$, we have that $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \langle 1,\, 3,\, h\rangle$. Thus, PM is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix},\ t \in \mathbb{R}.$$

Part (a). Since $\overrightarrow{OM} = \langle 1,\, 1,\, h\rangle$, by symmetry, $\overrightarrow{ON} = \langle -1,\, -1,\, h\rangle$. Given that $\overrightarrow{OQ} = \langle 2,\, 0,\, 0\rangle$, we have that $\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \langle -3,\, -1,\, h\rangle$. Thus, QN is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix},\ u \in \mathbb{R}.$$

Part (b). Consider $PM = QN$.

$$PM = QN \implies \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u\begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} \implies t\begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} - u\begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} t + 3u = 2 \\ 3t + u = 2 \\ ht - hu = 0 \end{cases}$$

From the first two equations, we see that $t = \frac{1}{2}$ and $u = \frac{1}{2}$, which is consistent with the third equation. Hence, $\overrightarrow{OX} = \langle 0,\, -2,\, 0\rangle + \frac{1}{2}\langle 1,\, 3,\, h\rangle = \frac{1}{2}\langle 1,\, -1,\, h\rangle$.

Part (c). Note that $\overrightarrow{OB} = \langle 2,\, -2,\, 6\rangle$, whence $\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \langle 2,\, -2,\, -2h\rangle$. Since OX is perpendicular to VB, we have that $\overrightarrow{OX} \cdot \overrightarrow{VB} = 0$.

$$\overrightarrow{OX} \cdot \overrightarrow{VB} = 0 \implies \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} \cdot 2\begin{pmatrix} 1 \\ -1 \\ -h \end{pmatrix} = 0 \implies h^2 = 2.$$

We hence have that $h = \sqrt{2}$. Note that we reject $h = -\sqrt{2}$ since $h > 0$.

Let the acute angle between PM and QN be θ.

$$\cos\theta = \frac{\left|\overrightarrow{PM} \cdot \overrightarrow{QN}\right|}{\left|\overrightarrow{PM}\right|\left|\overrightarrow{QN}\right|} = \frac{1}{\sqrt{12}\sqrt{12}}\left|\begin{pmatrix} 1 \\ 3 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ \sqrt{2} \end{pmatrix}\right| = \frac{1}{3} \implies \theta = 70.5° \text{ (1 d.p.)}.$$

Assignment A8

Problem 1. Find the position vector of the foot of the perpendicular from the point with position vector $\mathbf{c}$ to the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. Leave your answers in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Solution. Let the foot of the perpendicular be F. We have that $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b}$ for some real λ, and $\overrightarrow{CF} \cdot \mathbf{b} = 0$. Note that $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b} - \mathbf{c}$. Thus,

$$\overrightarrow{CF} \cdot \mathbf{b} = 0 \implies (\mathbf{a} + \lambda\mathbf{b} - \mathbf{c}) \cdot \mathbf{b} = 0 \implies \lambda\,|\mathbf{b}|^2 + (\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 0 \implies \lambda = \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2}.$$

Thus,

$$\overrightarrow{OF} = \mathbf{a} + \left(\frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}.$$

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Problem 2. The point O is the origin, and points A, B, C have position vectors given by $\overrightarrow{OA} = 6\mathbf{i}$, $\overrightarrow{OB} = 3\mathbf{j}$, $\overrightarrow{OC} = 4\mathbf{k}$. The point P is on the line AB between A and B, and is such that $AP = 2PB$. The point Q has position vector given by $\overrightarrow{OQ} = q\mathbf{i}$, where q is a scalar.

(a) Express, in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the vector $\overrightarrow{CP}$.

(b) Show that the line BQ has equation $\mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j})$, where t is a parameter. Give an equation of the line CP in a similar form.

(c) Find the value of q for which the lines CP and BQ are perpendicular.

(d) Find the sine of the acute angle between the lines CP and BQ in terms of q.

Solution. We have that $\overrightarrow{OA} = \langle 6,\, 0,\, 0\rangle$, $\overrightarrow{OB} = \langle 0,\, 3,\, 0\rangle$ and $\overrightarrow{OC} = \langle 0,\, 0,\, 4\rangle$.
Part (a). By the ratio theorem,

$$\overrightarrow{OP} = \frac{2\,\overrightarrow{OB} + \overrightarrow{OA}}{1+2} = \frac{1}{3}\left[2\begin{pmatrix}0\\3\\0\end{pmatrix} + \begin{pmatrix}6\\0\\0\end{pmatrix}\right] = \begin{pmatrix}2\\2\\0\end{pmatrix} \implies \overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \begin{pmatrix}2\\2\\-4\end{pmatrix}.$$

Hence, $\overrightarrow{CP} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$.
Part (b). Note that $\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB} = \langle q,\, -3,\, 0\rangle$. Thus, BQ is given by

$$\mathbf{r} = \begin{pmatrix}0\\3\\0\end{pmatrix} + t\begin{pmatrix}q\\-3\\0\end{pmatrix},\ t \in \mathbb{R} \iff \mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j}),\ t \in \mathbb{R}.$$

Note that $\overrightarrow{CP} = \langle 2,\, 2,\, -4\rangle = 2\,\langle 1,\, 1,\, -2\rangle$. Hence, CP is given by

$$\mathbf{r} = \begin{pmatrix}0\\0\\4\end{pmatrix} + u\begin{pmatrix}1\\1\\-2\end{pmatrix},\ u \in \mathbb{R} \iff \mathbf{r} = 4\mathbf{k} + u(\mathbf{i} + \mathbf{j} - 2\mathbf{k}),\ u \in \mathbb{R}.$$

Part (c). Since CP is perpendicular to BQ, we have $\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0$. Thus,

$$\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0 \implies 2\begin{pmatrix}1\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}q\\-3\\0\end{pmatrix} = 0 \implies q - 3 + 0 = 0 \implies q = 3.$$

Part (d). Let θ be the acute angle between CP and BQ.

$$\sin\theta = \frac{|\langle 1,\ 1,\ -2\rangle \times \langle q,\ -3,\ 0\rangle|}{|\langle 1,\ 1,\ -2\rangle|\,|\langle q,\ -3,\ 0\rangle|} = \frac{|\langle -6,\ 2q,\ 3-q\rangle|}{\sqrt{6}\sqrt{q^2+9}} = \sqrt{\frac{5q^2-6q+45}{6q^2+54}}.$$

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Problem 3. Line l_1 passes through the point A with position vector $3\mathbf{i}-2\mathbf{k}$ and is parallel to $-2\mathbf{i}+4\mathbf{j}-\mathbf{j}$. Line l_2 has Cartesian equation given by $\frac{x-1}{2} = y = z+3$.

(a) Show that the two lines intersect and find the coordinates of their point of intersection.

(b) Find the acute angle between the two lines l_1 and l_2. Hence, or otherwise, find the shortest distance from point A to line l_2.

(c) Find the position vector of the foot N of the perpendicular from A to the line l_2. The point B lies on the line AN produced and is such that N is the mid-point of AB. Find the position vector of B.

Solution. We have

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}, \quad l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix},\ \mu \in \mathbb{R}.$$

Part (a). Consider $l_1 = l_2$.

$$l_1 = l_2 \implies \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \implies \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 2\lambda + 2\mu = 2 \\ -4\lambda + \mu = 0 \\ \lambda + \mu = 1 \end{cases}$$

which has the unique solution $\mu = 4/5$ and $\lambda = 1/5$. Thus, the intersection point P has position vector $\langle 3,\ 0,\ -2\rangle + \frac{1}{5}\langle -2,\ 4,\ -1\rangle = \frac{1}{5}\langle 13,\ 4,\ -11\rangle$ and thus has coordinates $(13/5, 4/5, -11/5)$.

Part (b). Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{|\langle -2,\ 4,\ -1\rangle \cdot \langle 2,\ 1,\ 1\rangle|}{|\langle -2,\ 4,\ -1\rangle|\,|\langle 2,\ 1,\ 1\rangle|} = \frac{1}{\sqrt{126}} \implies \theta = 84.9^\circ \text{ (1 d.p.)}.$$

Note that

$$AP = \sqrt{\left(\frac{17}{5}-3\right)^2 + \left(-\frac{4}{5}-0\right)^2 + \left(-\frac{9}{5}-(-2)\right)^2} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}.$$

Since $\sin\theta = \frac{AN}{AP}$, we have that $AN = AP\sin\theta$. Note that

$$\sin\theta = \sin\arccos\frac{1}{\sqrt{126}} = \frac{\sqrt{\left(\sqrt{126}\right)^2 - 1}}{\sqrt{126}} = \frac{\sqrt{125}}{\sqrt{126}} = \frac{5\sqrt{5}}{\sqrt{6}\sqrt{21}}.$$

Thus,

$$AN = \frac{\sqrt{21}}{5} \cdot \frac{5\sqrt{5}}{\sqrt{6}\sqrt{21}} = \sqrt{\frac{5}{6}}.$$

The shortest distance between A and l_2 is hence $\sqrt{\frac{5}{6}}$ units.

Part (c). Since N is on l_2, we have that $\overrightarrow{ON} = \langle 1, 0, -3 \rangle + \mu \langle 2, 1, 1 \rangle$ for some real μ. Additionally, since $\overrightarrow{AN} \perp l_2$, we have $\overrightarrow{AN} \cdot \langle 2, 1, 1 \rangle = 0$. Note that

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix}.$$

Thus,

$$\overrightarrow{AN} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \implies \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \implies -5 + 6\mu = 0 \implies \mu = \frac{5}{6}.$$

Hence,

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}.$$

Note that $\overrightarrow{ON} = \frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$. Hence,

$$\overrightarrow{OB} = 2\overrightarrow{ON} - \overrightarrow{OA} = \frac{2}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}.$$

A9. Vectors III - Planes

Tutorial A9

Problem 1. A student claims that a unique plane can always be defined based on the given information. True or False? (Whenever a line is mentioned, assume the vector equation is known.)

	Statement	**T/F**
(a)	Any 2 vectors parallel to the plane and a point lying on the plane.	False
(b)	Any 3 distinct points lying on the plane.	False
(c)	A vector perpendicular to the plane and a point lying on the plane.	True
(d)	A line l perpendicular to the plane and a particular point on l lying on the plane.	True
(e)	A line l lying on the plane.	False
(f)	A line l and a point not on l, both lying on the plane.	True
(g)	A pair of distinct, intersecting lines, both lying on the plane.	True
(h)	A pair of distinct, parallel lines, both lying on the plane.	True
(i)	A pair of skew lines both parallel to the plane.	False
(j)	2 intersecting lines both parallel to the plane.	False

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Problem 2. Find the equations of the following planes in parametric, scalar product and Cartesian form:

(a) The plane passes through the point with position vector $7\mathbf{i} + 2\mathbf{j} - 3\mathbf{j}$ and is parallel to $\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{j} - 2\mathbf{k}$.

(b) The plane passes through the points $A(2, 0, 1)$, $B(1, -1, 2)$ and $C(1, 3, 1)$.

(c) The plane passes through the point with position vector $7\mathbf{i}$ and is parallel to the plane $\mathbf{r} = (2 - p + q)\mathbf{i} + (p + 3q)\mathbf{j} + (-2 - 3q)\mathbf{k}$, $p, q \in \mathbb{R}$.

(d) The plane contains the line $l : \mathbf{r} = (-2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$ and is perpendicular to the plane $\pi : \mathbf{r} \cdot (7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = 2$.

Solution.

Part (a). Parametric. Note that $\langle 0,\ 4,\ -2\rangle \parallel \langle 0,\ 2,\ -1\rangle$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\mathbf{n} = \langle 1,\ 3,\ 0\rangle \times \langle 0,\ 2,\ -1\rangle = \langle -3,\ 1,\ 2\rangle \implies d = \langle 7,\ 2,\ -3\rangle \cdot \langle -3,\ 1,\ 2\rangle = -25$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -2.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$-3x + y + 2z = -25.$$

Part (b). Parametric. Since the plane passes through the points A, B and C, it is parallel to both $\overrightarrow{AB} = -\langle 1,\, 1,\, -1\rangle$ and $\overrightarrow{AC} = \langle -1,\, 3,\, 0\rangle$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}-1\\3\\0\end{pmatrix},\ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\mathbf{n} = \langle 1,\, 1,\, -1\rangle \times \langle -1,\, 3,\, 0\rangle = \langle 3,\, 1,\, 4\rangle \implies d = \langle 2,\, 0,\, 1\rangle \cdot \langle 3,\, 1,\, 4\rangle = 10$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix}3\\1\\4\end{pmatrix} = 10.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$3x + y + 4z = 10.$$

Part (c). Parametric. Note that the plane is parallel to $\mathbf{r} = \langle 2,\, 0,\, -1\rangle + p\langle -1,\, 1,\, 0\rangle + q\langle 1,\, 3,\, -3\rangle$ and passes through $(7, 0, 0)$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix}7\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\-3\end{pmatrix},\ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\langle -1,\, 1,\, 0\rangle \times \langle 1,\, 3,\, -3\rangle = \langle -3,\, -3,\, -4\rangle \parallel \langle 3,\, 3,\, 4\rangle$. We hence take $\mathbf{n} = \langle 3,\, 3,\, 4\rangle$, whence $d = \langle 7,\, 0,\, 0\rangle \cdot \langle 3,\, 3,\, 4\rangle = 21$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix}3\\3\\4\end{pmatrix} = 21.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$3x + 3y + 4z = 21.$$

Part (d). Parametric. Since the plane contains the line with equation $\mathbf{r} = \langle -2,\, 5,\, -3\rangle + \lambda\langle 2,\, 1,\, 2\rangle,\ \lambda \in \mathbb{R}$, the plane passes through $(-2, 5, -3)$ and is parallel to the vector $\langle 2,\, 1,\, 2\rangle$. Furthermore, since the plane is perpendicular to the plane with normal $\langle 7,\, 4,\, 5\rangle$, it must be parallel to said vector. Thus, the plane has the following parametric form:

$$\mathbf{r} = \begin{pmatrix}-2\\5\\-3\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\2\end{pmatrix} + \mu\begin{pmatrix}7\\4\\5\end{pmatrix},\ \lambda, \mu \in \mathbb{R}.$$

Scalar Product. Note that $\mathbf{n} = \langle 2,\, 1,\, 2\rangle \times \langle 7,\, 4,\, 5\rangle = \langle -3,\, 4,\, 1\rangle \implies d = \langle -2,\, 5,\, -3\rangle \cdot \langle -3,\, 4,\, 1\rangle = 23$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = 23.$$

Cartesian. Let $\mathbf{r} = \langle x,\, y,\, z\rangle$. From the scalar product form, we have

$$-3x + 4y + z = 23.$$

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Problem 3. The line l passes through the points A and B with coordinates $(1, 2, 4)$ and $(-2, 3, 1)$ respectively. The plane p has equation $3x - y + 2z = 17$. Find

(a) the coordinates of the point of intersection of l and p,

(b) the acute angle between l and p,

(c) the perpendicular distance from A to p, and

(d) the position vector of the foot of the perpendicular from B to p.

The line m passes through the point C with position vector $6\mathbf{i} + \mathbf{j}$ and is parallel to $2\mathbf{j} + \mathbf{k}$.

(e) Determine whether m lies in p.

Solution. Note that $\overrightarrow{OA} = \langle 1,\, 2,\, 4\rangle$ and $\overrightarrow{OB} = \langle -2,\, 3,\, 1\rangle$, whence $\overrightarrow{AB} = -\langle 3,\, -1,\, 3\rangle$. Thus, the line l has vector equation

$$\mathbf{r} = \begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\3\end{pmatrix},\ \lambda \in \mathbb{R}.$$

Also note that the equation of the plane p can be written as

$$\mathbf{r} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17.$$

Part (a). Let the point of intersection of l and p be P. Consider $l = p$.

$$l = p \implies \left[\begin{pmatrix}1\\2\\4\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\3\end{pmatrix}\right] \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17 \implies 9 + 16\lambda = 17 \implies \lambda = \frac{1}{2}.$$

Thus, $\overrightarrow{OP} = \langle 1,\, 2,\, 4\rangle + \frac{1}{2}\langle 3,\, -1,\, 3\rangle = \langle 5/2,\, 3/2,\, 11/2\rangle$, whence $P(5/2, 3/2, 11/2)$.

Part (b). Let θ be the acute angle between l and p.

$$\sin\theta = \frac{|\langle 3,\, -1,\, 3\rangle \cdot \langle 3,\, -1,\, 2\rangle|}{|\langle 3,\, -1,\, 3\rangle|\,|\langle 3,\, -1,\, 2\rangle|} = \frac{16}{\sqrt{266}} \implies \theta = 78.8^\circ \text{ (1 d.p.)}.$$

Part (c). Note that $\overrightarrow{AP} = -\frac{1}{2}\langle 3,\, -1,\, 3\rangle$. The perpendicular distance from A to p is hence

$$\left|\overrightarrow{AP} \cdot \hat{\mathbf{n}}\right| = \frac{\left|-\frac{1}{2}\langle 3,\, -1,\, 3\rangle \cdot \langle 3,\, -1,\, 2\rangle\right|}{|\langle 3,\, -1,\, 2\rangle|} = \frac{8}{\sqrt{14}} \text{ units}.$$

Part (d). Let F be the foot of the perpendicular from B to p. Since F is on p, we have $\overrightarrow{OF} \cdot \langle 3,\, -1,\, 2\rangle = 17$. Furthermore, since BF is perpendicular to p, we have $\overrightarrow{BF} = \lambda\mathbf{n} =$

$\lambda \langle 3, -1, 2\rangle$ for some $\lambda \in \mathbb{R}$. We hence have $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \langle -2, 3, 1\rangle + \lambda \langle 3, -1, 2\rangle$. Thus,

$$\left[\begin{pmatrix}-2\\3\\1\end{pmatrix} + \lambda \begin{pmatrix}3\\-1\\2\end{pmatrix}\right] \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17 \implies -7 + 14\lambda = 17 \implies \lambda = \frac{12}{7}.$$

Hence, $\overrightarrow{OF} = \langle -2, 3, 1\rangle + \frac{12}{7}\langle 3, -1, 2\rangle = \frac{1}{7}\langle 22, 9, 31\rangle$.

Part (e). Note that m has the vector equation

$$\mathbf{r} = \begin{pmatrix}6\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}0\\2\\1\end{pmatrix}, \lambda \in \mathbb{R}.$$

Consider $m \cdot \mathbf{n}$:

$$m \cdot \mathbf{n} = \left[\begin{pmatrix}6\\1\\0\end{pmatrix} + \lambda \begin{pmatrix}0\\2\\1\end{pmatrix}\right] \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17.$$

Since $m \cdot \mathbf{n} = 17$ for all $\lambda \in \mathbb{R}$, it follows that m lies in p.

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Problem 4. A plane contains distinct points P, Q, R and S, of which no 3 points are collinear. What can be said about the relationship between the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$?

Solution. Each of the three vectors can be expressed as a unique linear combination of the other two.

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Problem 5.

(a) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter.

(b) Interpret geometrically the vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents.

(c) Given that $\mathbf{b} \cdot \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \cdot \mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically.

Solution.

Part (a). The vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ represents a line with direction vector $\mathbf{b}$ that passes through the point with position vector $\mathbf{a}$.

Part (b). The vector equation $\mathbf{r} \cdot \mathbf{n} = d$ represents a plane perpendicular to $\mathbf{n}$ that has a perpendicular distance of d units from the origin. Here, a negative value of d corresponds to a plane d units from the origin in the opposite direction of $\mathbf{n}$.

Part (c).

$$\begin{aligned}\mathbf{r} \cdot \mathbf{n} = d &\implies (\mathbf{a} + t\mathbf{b}) \cdot \mathbf{n} = d \implies \mathbf{a} \cdot \mathbf{n} + t\mathbf{b} \cdot \mathbf{n} = d \\ &\implies t = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \implies \mathbf{r} = \mathbf{a} + \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}.\end{aligned}$$

$\mathbf{a} + \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}}\mathbf{b}$ is the position vector of the point of intersection of the line and plane.

Problem 6. The planes p_1 and p_2 have equations $\mathbf{r} \cdot \langle 2,\, -2,\, 1 \rangle = 1$ and $\mathbf{r} \cdot \langle -6,\, 3,\, 2 \rangle = -1$ respectively, and meet in the line l.

(a) Find the acute angle between p_1 and p_2.

(b) Find a vector equation for l.

(c) The point $A(4, 3, c)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of c.

Solution.

Part (a). Let θ the acute angle between p_1 and p_2.

$$\cos\theta = \frac{|\langle 2,\, -2,\, 1 \rangle \cdot \langle -6,\, 3,\, 2 \rangle|}{|\langle 2,\, -2,\, 1 \rangle|\,|\langle -6,\, 3,\, 2 \rangle|} = \frac{16}{21} \implies \theta = 40.4^\circ \text{ (1 d.p.)}.$$

Part (b). Observe that p_1 has the Cartesian equation $2x - 2y + z = 1$ and p_2 has the Cartesian equation $-6x + 3y + 2z = -1$. Consider $p_1 = p_2$. Solving both Cartesian equations simultaneously gives the solution

$$x = -\frac{1}{6} + \frac{7}{6}t, \quad y = -\frac{2}{3} + \frac{5}{3}t, \quad z = t$$

for all $t \in \mathbb{R}$. The line l thus has vector equation

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{6}\begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + t\begin{pmatrix} 7 \\ 10 \\ 6 \end{pmatrix}, \; t \in \mathbb{R}.$$

Part (c). Let Q be the point with position vector $-\frac{1}{6}\langle 1,\, 4,\, 0 \rangle$. Then $\overrightarrow{AQ} = -\frac{1}{6}\langle 25,\, 22,\, 6c \rangle$. Since Q lies on l, it lies on both p_1 and p_2. Since A is equidistant to p_1 and p_2, the perpendicular distances from A to p_1 and p_2 are equal.

The perpendicular distance from A to p_1 is given by:

$$\frac{\left|\overrightarrow{AQ} \cdot \langle 2,\, -2,\, 1 \rangle\right|}{|\langle 2,\, -2,\, 1 \rangle|} = \frac{1}{3}\left|-\frac{1}{6}\begin{pmatrix} 25 \\ 22 \\ 6c \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}\right| = \frac{1}{3}\left|1 + c\right|.$$

Meanwhile, the perpendicular distance from A to p_2 is given by:

$$\frac{\left|\overrightarrow{AQ} \cdot \langle -6,\, 3,\, 2 \rangle\right|}{|\langle -6,\, 3,\, 2 \rangle|} = \frac{1}{7}\left|-\frac{1}{6}\begin{pmatrix} 25 \\ 22 \\ 6c \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix}\right| = \frac{1}{7}\left|-14 + 2c\right|.$$

Equating the two gives

$$\frac{1}{3}\left|1 + c\right| = \frac{1}{7}\left|-14 + 2c\right| \implies \left|7 + 7c\right| = \left|-42 + 6c\right|.$$

This splits into the following two cases:

Case 1. $(7 + 7c)(-42 + 6c) > 0 \implies 7 + 7c = -42 + 6c \implies c = -49.$

Case 2. $(7 + 7c)(-42 + 6c) < 0 \implies 7 + 7c = -(-42 + 6c) \implies c = -35/13.$

Problem 7. A plane Π has equation $\mathbf{r} \cdot (2\mathbf{i} + 3\mathbf{j}) = -6$.

(a) Find, in vector form, an equation for the line passing through the point P with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and normal to the plane Π.

(b) Find the position vector of the foot Q of the perpendicular from P to the plane Π and hence find the position vector of the image of P after the reflection in the plane Π.

(c) Find the sine of the acute angle between OQ and the plane Π.

The plane Π' has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 5$.

(d) Find the position vector of the point A where the planes Π, Π' and the plane with equation $\mathbf{r} \cdot \mathbf{i} = 0$ meet.

(e) Hence, or otherwise, find also the vector equation of the line of intersection of planes Π and Π'.

Solution.

Part (a). Let l be the required line. Since l is normal to Π, it is parallel to the normal vector of Π, $\langle 2, 3, 0\rangle$. Thus, l has vector equation

$$l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}.$$

Part (b). Since Q is on Π, $\overrightarrow{OQ} \cdot \langle 2, 3, 0\rangle = -6$. Furthermore, observe that Q is also on the line l. Thus, $\overrightarrow{OQ} = \langle 2, 1, 4\rangle + \lambda \langle 2, 3, 0\rangle$ for some $\lambda \in \mathbb{R}$. Hence,

$$\overrightarrow{OQ} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6 \implies \left[\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6 \implies 7 + 13\lambda = -6 \implies \lambda = -1.$$

Thus, $\overrightarrow{OQ} = \langle 2, 1, 4\rangle - \langle 2, 3, 0\rangle = \langle 0, -2, 4\rangle$.

Let the reflection of P in Π be P'. Then

$$\overrightarrow{PQ} = \overrightarrow{QP'} \implies \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{OP'} - \overrightarrow{OQ} \implies \overrightarrow{OP'} = 2\overrightarrow{OQ} - \overrightarrow{OP}.$$

Hence, $\overrightarrow{OP'} = 2\langle 0, -2, 4\rangle - \langle 2, 1, 4\rangle = \langle -2, -5, 4\rangle$.

Part (c). Let θ be the acute angle between OQ and Π.

$$\sin\theta = \frac{|\langle 0, -2, 4\rangle \cdot \langle 2, 3, 0\rangle|}{|\langle 0, -2, 4\rangle|\,|\langle 2, 3, 0\rangle|} = \frac{3}{\sqrt{65}}.$$

Part (d). Let $\overrightarrow{OA} = \langle x, y, z\rangle$. We thus have the following system:

$$\begin{cases} \langle x, y, z\rangle \cdot \langle 2, 3, 0\rangle = -6 & \implies 2x + 3y = -6 \\ \langle x, y, z\rangle \cdot \langle 1, 1, 1\rangle = 5 & \implies x + y + z = 5 \\ \langle x, y, z\rangle \cdot \langle 1, 0, 0\rangle = 0 & \implies x = 0 \end{cases}$$

Solving, we obtain $x = 0$, $y = -2$ and $z = 7$, whence $\overrightarrow{OA} = \langle 0, -2, 7\rangle$.

Part (e). Let the line of intersection of Π and Π' be l'. Observe that A is on Π and Π' and thus lies on l'. Hence,

$$l' : \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}.$$

Since l' lies on both Π and Π', $\mathbf{b}$ is perpendicular to the normals of both planes, i.e. $\langle 2, 3, 0 \rangle$ and $\langle 1, 1, 1 \rangle$. Thus, $\mathbf{b} = \langle 2, 3, 0 \rangle \times \langle 1, 1, 1 \rangle = \langle 3, -2, -1 \rangle$ and

$$l' : \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

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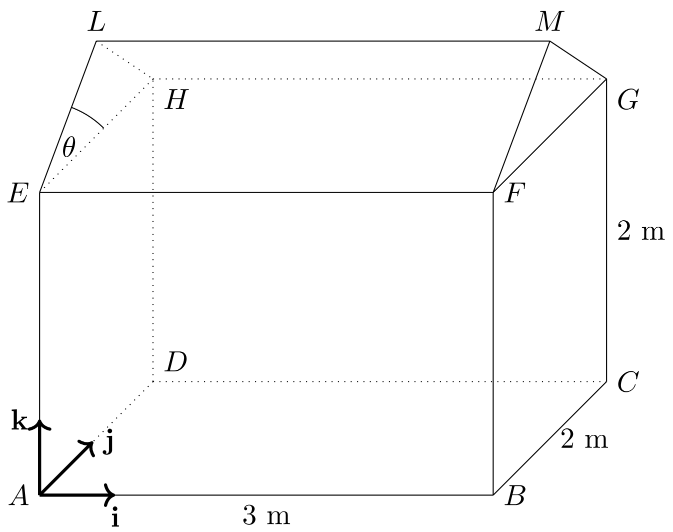
Problem 8.



The diagram shows a garden shed with horizontal base $ABCD$, where $AB = 3$ m and $BC = 2$ m. There are two vertical rectangular walls $ABFE$ and $DCGH$, where $AE = BF = CG = DH = 2$ m. The roof consists of two rectangular planes $EFML$ and $HGML$, which are inclined at an angle θ to the horizontal such that $\tan\theta = \frac{3}{4}$.

The point A is taken as the origin and the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, each of length 1 m, are taken along AB, AD and AE respectively.

(a) Verify that the plane with equation $\mathbf{r} \cdot (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$ passes through B, D and M.

(b) Find the perpendicular distance, in metres, from A to the plane BDM.

(c) Find a vector equation of the straight line EM.

(d) Show that the perpendicular distance from C to the straight line EM is 2.91 m, correct to 3 significant figures.

Solution.

Part (a). We have $\overrightarrow{AB} = \langle 3,\ 0,\ 0\rangle$, $\overrightarrow{BF} = \overrightarrow{AE} = \langle 0,\ 0,\ 2\rangle$ and $\overrightarrow{FG} = \overrightarrow{AD} = \langle 0,\ 2,\ 0\rangle$. Let T be the midpoint of FG. We have $\overrightarrow{FT} = \langle 0,\ 1,\ 0\rangle$ and $TM/FT = \tan\theta = 3/4$, whence $\overrightarrow{TM} = \langle 0,\ 0,\ 3/4\rangle$. Hence,

$$\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FT} + \overrightarrow{TM} = \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\3/4\end{pmatrix} = \frac{1}{4}\begin{pmatrix}12\\4\\11\end{pmatrix}.$$

Consider $\overrightarrow{AB} \cdot \langle 22,\ 33,\ -12\rangle$, $\overrightarrow{AD} \cdot \langle 22,\ 33,\ -12\rangle$ and $\overrightarrow{AM} \cdot \langle 22,\ 33,\ -12\rangle$.

$$\begin{aligned}
\overrightarrow{AB} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= \begin{pmatrix}3\\0\\0\end{pmatrix} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66\\
\overrightarrow{AD} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= \begin{pmatrix}0\\2\\0\end{pmatrix} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66\\
\overrightarrow{AM} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= \frac{1}{4}\begin{pmatrix}12\\4\\11\end{pmatrix} \cdot \begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66
\end{aligned}$$

Since $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AM}$ satisfy the equation $\mathbf{r} \cdot \langle 22,\ 33,\ -12\rangle = 66$, they all lie on the plane with said equation.

Part (b). The perpendicular distance from A to the plane BDM is given by

$$\text{Perpendicular distance} = \left|\overrightarrow{AB} \cdot \hat{\mathbf{n}}\right| = \frac{|\langle 3,\ 0,\ 0\rangle \cdot \langle 22,\ 33,\ -12\rangle|}{|\langle 22,\ 33,\ -12\rangle|} = \frac{66}{\sqrt{1717}} \text{ m}.$$

Part (c). Observe that $\overrightarrow{EM} = \overrightarrow{AM} - \overrightarrow{AE} = \frac{1}{4}\langle 12,\ 4,\ 3\rangle$. Hence, the line EM has vector equation

$$\mathbf{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}12\\4\\3\end{pmatrix},\ \lambda \in \mathbb{R}.$$

Part (d). Note that $\overrightarrow{EC} = \overrightarrow{AC} - \overrightarrow{AE} = \langle 3,\ 2,\ -2\rangle$. The perpendicular distance from C to the line EM is hence given by

$$\frac{\left|\overrightarrow{EC} \times \langle 12,\ 4,\ 3\rangle\right|}{|\langle 12,\ 4,\ 3\rangle|} = \frac{1}{13}\left|\begin{pmatrix}3\\2\\-2\end{pmatrix} \times \begin{pmatrix}12\\4\\3\end{pmatrix}\right| = \frac{1}{13}\left|\begin{pmatrix}14\\-33\\-12\end{pmatrix}\right| = \frac{\sqrt{1429}}{13} = 2.91 \text{ m (3 s.f.)}.$$

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Problem 9. The planes π_1 and π_2 have equations

$$x + y - z = 0 \text{ and } 2x - 4y + z + 12 = 0$$

respectively. The point P has coordinates $(3, 8, 2)$ and O is the origin.

(a) Verify that the vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is parallel to both π_1 and π_2.

(b) Find the equation of the plane which passes through P and is perpendicular to both π_1 and π_2.

(c) Verify that $(0, 4, 4)$ is a point common to both π_1 and π_2, and hence or otherwise, find the equation of the line of intersection of π_1 and π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$.

(d) Find the coordinates of the point in which the line OP meets π_2.

(e) Find the length of projection of OP on π_1.

Solution. Note that π_1 and π_2 have vector equations $\mathbf{r} \cdot \langle 1,\, 1,\, -1\rangle = 0$ and $\mathbf{r} \cdot \langle 2,\, -4,\, 1\rangle = -12$ respectively.

Part (a). Observe that $\langle 1,\, 1,\, 2\rangle \cdot \langle 1,\, 1,\, -1\rangle = \langle 1,\, 1,\, 2\rangle \cdot \langle 2,\, -4,\, 1\rangle = 0$. Thus, the vector $\langle 1,\, 1,\, 2\rangle$ is perpendicular to the normal vectors of both π_1 and π_2 and is hence parallel to them.

Part (b). Let the required plane be π_3. Since π_3 is perpendicular to both π_1 and π_2, its normal vector is parallel to both planes. Thus, $\mathbf{n} = \langle 1,\, 1,\, 2\rangle \implies d = \langle 3,\, 8,\, 2\rangle \cdot \langle 1,\, 1,\, 2\rangle = 15$. π_3 hence has the vector equation

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 15.$$

Part (c). Since $\langle 0,\, 4,\, 4\rangle \cdot \langle 1,\, 1,\, -1\rangle = 0$ and $\langle 0,\, 4,\, 4\rangle \cdot \langle 2,\, -4,\, 1\rangle = -12$, $(0, 4, 4)$ satisfies the vector equation of both π_1 and π_2 and thus lies on both planes.

Let l be the line of intersection of π_1 and π_2. Since $(0, 4, 4)$ is a point common to both planes, l passes through it. Furthermore, since l lies on both π_1 and π_2, it is perpendicular to the normal vector of both planes and hence has direction vector $\langle 1,\, 1,\, -1\rangle \times \langle 2,\, -4,\, 1\rangle = -3\,\langle 1,\, 1,\, 2\rangle$. Thus, l can be expressed as

$$l : \mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

Part (d). Note that the line OP, denoted l_{OP} has equation

$$l_{OP} : \mathbf{r} = \mu \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix},\ \mu \in \mathbb{R}.$$

Consider the intersection between l_{OP} and π_2.

$$\mu \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = -12 \implies -24\mu = -12 \implies \mu = \frac{1}{2}.$$

Hence, OP meets π_2 at $(3/2, 4, 1)$.

Part (e). The length of projection of OP on π_1 is given by

$$\frac{\overrightarrow{OP} \times \langle 1,\, 1,\, -1\rangle}{|\langle 1,\, 1,\, -1\rangle|} = \frac{1}{\sqrt{3}} \left| \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right| = \frac{1}{\sqrt{3}} \left| \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} \right| = \frac{5\sqrt{6}}{\sqrt{3}} = 5\sqrt{2} \text{ units.}$$

* * * * *

Problem 10. The line l_1 passes through the point A, whose position vector is $3\mathbf{i} - 5\mathbf{j} - 4\mathbf{k}$, and is parallel to the vector $3\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, and is parallel to the vector $\mathbf{i} - \mathbf{j} - 4\mathbf{k}$. The point P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The plane Π contains PQ and l_1.

(a) Find a vector parallel to PQ.

(b) Find the equation of Π in the forms $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\lambda, \mu \in \mathbb{R}$ and $\mathbf{r} \cdot \mathbf{n} = D$.

(c) Find the perpendicular distance from B to Π.

(d) Find the acute angle between Π and l_2.

(e) Find the position vectors of P and Q.

Solution.

Part (a). Note that l_1 and l_2 have vector equations

$$\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}, \mu \in \mathbb{R}$$

respectively. Since PQ is perpendicular to both l_1 and l_2, it is parallel to $\langle 3, 4, 2\rangle \times \langle 1, -1, -4\rangle = \langle -14, 14, -7\rangle = -7\,\langle 2, -2, 1\rangle$.

Part (b). Since Π contains PQ and l_1, it is parallel to $\langle 2, -2, 1\rangle$ and $\langle 3, 4, 2\rangle$. Also note that Π contains $\langle 3, -5, -4\rangle$. Thus,

$$\Pi : \mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \lambda, \mu \in \mathbb{R}.$$

Note that $\langle 2, -2, 1\rangle \times \langle 3, 4, 2\rangle = \langle -8, -1, 14\rangle \parallel \langle 8, 1, -14\rangle$. We hence take $\mathbf{n} = \langle 8, 1, -14\rangle$, whence $d = \langle 3, -5, -4\rangle \cdot \langle 8, 1, -14\rangle = 75$. Thus, $|Pi$ is also given by

$$\Pi : \mathbf{r} \cdot \begin{pmatrix} 8 \\ 1 \\ -14 \end{pmatrix} = 75.$$

Part (c). Note that $\overrightarrow{AB} = \langle -1, 8, 9\rangle$. Hence, the perpendicular distance from B to Π is given by

$$\frac{|\langle -1, 8, 9\rangle \cdot \langle 8, 1, -14\rangle|}{|\langle 8, 1, -14\rangle|} = \frac{126}{\sqrt{261}} \text{ units.}$$

Part (d). Let θ be the acute angle between Π and l_2.

$$\sin\theta = \frac{|\langle 1, -1, -4\rangle \cdot \langle 8, 1, -14\rangle|}{|\langle 1, -1, -4\rangle|\,|\langle 8, 1, -14\rangle|} = \frac{7}{\sqrt{58}} \implies \theta = 66.8^\circ \text{ (1 d.p.)}.$$

Part (e). Since P is on l_1, we have $\overrightarrow{OP} = \langle 3, -5, -4\rangle + \lambda\,\langle 3, 4, 2\rangle$ for some $\lambda \in \mathbb{R}$. Similarly, since Q is on l_2, we have $\overrightarrow{OQ} = \langle 2, 3, 5\rangle + \mu\,\langle 1, -1, 4\rangle$ for some $\mu \in \mathbb{R}$. Thus,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}.$$

Recall that PQ is parallel to $\langle 2, -2, 1\rangle$. Hence, $\overrightarrow{PQ}$ can be expressed as $\nu\,\langle 2, -2, 1\rangle$ for some $\nu \in \mathbb{R}$. Equating the two expressions for $\overrightarrow{PQ}$, we obtain

$$\begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \implies \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 3\lambda - \mu + 2\nu = -1 \\ 4\lambda + \mu - 2\nu = 8 \\ 2\lambda + 4\mu + \nu = 9 \end{cases}$$

which has the unique solution $\lambda = 1$, $\mu = 2$ and $\nu = -1$. Thus,

$$\overrightarrow{OP} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -2 \end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}.$$

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Problem 11. The equations of three planes p_1, p_2 and p_3 are

$$\begin{aligned} 2x - 5y + 3z &= 3 \\ 3x + 2y - 5z &= -5 \\ 5x + \lambda y + 17z &= \mu \end{aligned}$$

respectively, where λ and μ are constants. The planes p_1 and p_2 intersect in a line l.

(a) Find a vector equation of l.

(b) Given that all three planes meet in the line l, find λ and μ.

(c) Given instead that the three planes have no point in common, what can be said about the values of λ and μ?

(d) Find the Cartesian equation of the plane which contains l and the point $(1, -1, 3)$.

Solution.

Part (a). Consider the intersection of p_1 and p_2:

$$\begin{cases} 2x - 5y + 3z = 3 \\ 3x + 2y - 5z = -5 \end{cases}$$

The above system has solution

$$x = -1 + t, \quad y = -1 + t, \quad z = t$$

for all $t \in \mathbb{R}$. Thus, the line l has vector equation

$$l : \mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Part (b). Since all three planes meet in the line l, l must satisfy the equation of p_3. Substituting the above solution to the given equation, we have

$$5(-1 + t) + \lambda(-1 + t) + 17t = \mu \implies (22 + \lambda)t - (5 + \lambda + \mu) = 0.$$

Comparing the coefficients of t and the constant terms, we have the following system:

$$\begin{cases} \lambda \qquad + 22 = 0 \\ \lambda + \mu + \ \ 5 = 0 \end{cases}$$

which has the unique solution $\lambda = -22$ and $\mu = 17$.

Part (c). If the three planes have no point in common, we have

$$(22+\lambda)t-(5+\lambda+\mu)\neq 0$$

for all $t\in\mathbb{R}$. To satisfy this relation, we need $22+\lambda=0$ and $5+\lambda+\mu\neq 0$, whence $\lambda=-22$ and $\mu\neq 17$.

Part (d). Note that $\langle -1,\,-1,\,0\rangle$ lies on l and is thus contained on the required plane. Observe that $\langle -1,\,-1,\,0\rangle-\langle 1,\,-1,\,3\rangle=\langle -2,\,0,\,-3\rangle$. Thus, the required plane is parallel to $\langle 1,\,1,\,1\rangle$ and $\langle -2,\,0,\,-3\rangle$ and hence has vector equation

$$\mathbf{r}=\begin{pmatrix}-1\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\1\end{pmatrix}+\mu\begin{pmatrix}-2\\0\\-3\end{pmatrix},\qquad \lambda,\mu\in\mathbb{R}.$$

Observe that $\mathbf{n}=\langle 1,\,1,\,1\rangle\times\langle -2,\,0,\,3\rangle=\langle -3,\,1,\,2\rangle$, whence $d=\langle -1,\,-1,\,0\rangle\cdot\langle -3,\,1,\,2\rangle=2$. The required plane thus has the equation

$$\mathbf{r}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix}=2.$$

Let $\mathbf{r}=\langle x,\,y,\,z\rangle$. It follows that the plane has Cartesian equation

$$-3x+y+2=2.$$

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Problem 12. The planes p_1 and p_2, which meet in line l, have equations $x-2y+2z=0$ and $2x-2y+z=0$ respectively.

(a) Find an equation of l in Cartesian form.

The plane p_3 has equation $(x-2y+2z)+c(2x-2y+z)=d$.

(b) Given that $d=0$, show that all 3 planes meet in the line l for any constant c.

(c) Given instead that the 3 planes have no point in common, what can be said about the value of d?

Solution.

Part (a). Consider the intersection of p_1 and p_2. This gives the system

$$\begin{cases} x-2y+2z=0\\ 2x-2y+z=0\end{cases}$$

which has solution $x=t$, $y=\frac{3}{2}t$ and $z=t$. Thus, l has Cartesian equation

$$x=\frac{2}{3}y=z.$$

Part (b). When $d=0$, p_3 has equation

$$(x-2y+2z)+c(2x-2y+z)=0.$$

Observe that the line l satisfies the equations $x-2y+2z=0$ and $2x-2y+z=0$. Hence, l also satisfies the equation that gives p_3 for all c. Thus, p_3 contains l, implying that all 3 planes meet in the line l.

Part (c). If the 3 planes have no point in common, then l does not have any point in common with p_3. That is, all points on l satisfy the relation

$$(x - 2y + 2z) + c(2x - 2y + z) \neq d.$$

Since $x - 2y + 2z = 0$ and $2x - 2y + z = 0$ for all points on l, the LHS simplifies to 0. Thus, to satisfy the above relation, we require $d \neq 0$.

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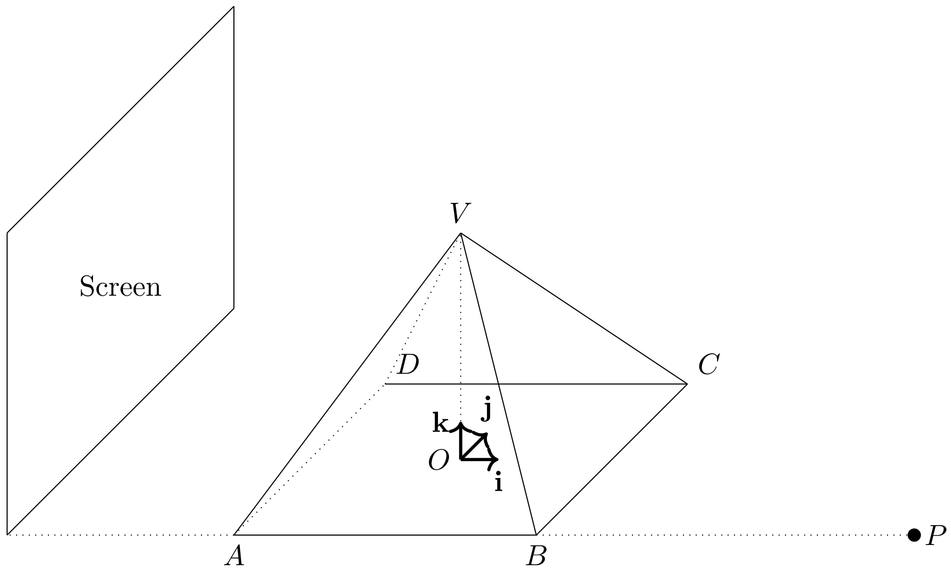
Problem 13.



A right opaque pyramid with square base $ABCD$ and vertex V is placed at ground level for a shadow display, as shown in the diagram. O is the centre of the square base $ABCD$, and the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are in the directions of $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OV}$ respectively. The length of AB is 8 units and the length of OV is $2h$ units.

A point light source for this shadow display is placed at the point $P(20, -4, 0)$ and a screen of height 35 units is placed with its base on the ground such that the screen lies on a plane with vector equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha$, where $\alpha < -4$.

(a) Find a vector equation of the line depicting the path of the light ray from P to V in terms of h.

(b) Find an inequality between α and h so that the shadow of the pyramid cast on the screen will not exceed the height of the screen.

The point light source is now replaced by a parallel light source whose light rays are perpendicular to the screen. It is also given that $h = 10$.

(c) Find the exact length of the shadow cast by the edge VB on the screen.

A mirror is placed on the plane VBC to create a special effect during the display.

(d) Find a vector equation of the plane VBC and hence find the angle of inclination made by the mirror with the ground.

Solution.

Part (a). Note that $\overrightarrow{OV} = \langle 0,\ 0,\ 2h \rangle$ and $\overrightarrow{OP} = \langle 20,\ -4,\ 0 \rangle$, whence $\overrightarrow{PV} = \langle -20,\ 4,\ 2h \rangle = 2 \langle -10,\ 2,\ h \rangle$. Thus, the line from P to V, denoted l_{PV}, has the vector equation

$$l_{PV} : \mathbf{r} = \begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Part (b). Let the point of intersection between l_{PV} and the screen be I.

$$\left[\begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha \implies 20 - 10\lambda = \alpha \implies \lambda = \frac{20 - \alpha}{10}.$$

Hence, $\overrightarrow{OI} = \langle 20,\ -4,\ 0 \rangle + \frac{20-\alpha}{10} \langle -10,\ 2,\ h \rangle$. To prevent the shadow from exceeding the screen, we require the $\mathbf{k}$-component of $\overrightarrow{OI}$ to be less than the height of the screen, i.e. 35 units. This gives the inequality $\frac{20-\alpha}{10} \cdot h \leq 35$, whence we obtain

$$h \leq \frac{350}{20 - \alpha}.$$

Part (c). Since the light rays emitted by the light source are now perpendicular to the screen, the image of some point with coordinates (a, b, c) on the screen is given by (α, b, c). Thus, the image of $B(4, -4, 0)$ and $V(0, 0, 20)$ on the screen have coordinates $(\alpha, -4, 0)$ and $(\alpha, 0, 20)$. The length of the shadow cast by VB is thus

$$\sqrt{(\alpha - \alpha)^2 + (-4 - 0)^2 + (0 - 20)^2} = 4\sqrt{26} \text{ units.}$$

Part (d). Note that $\overrightarrow{BV} = 4 \langle -1,\ 1,\ 5 \rangle$ and $\overrightarrow{BC} = 8 \langle 0,\ 1,\ 0 \rangle$. Hence, the plane VBC is parallel to $\langle -1,\ 1,\ 5 \rangle$ and $\langle 0,\ 1,\ 0 \rangle$. Note that $\langle -1,\ 1,\ 5 \rangle \times \langle 0,\ 1,\ 0 \rangle = - \langle 5,\ 0,\ 1 \rangle$. Thus, $\mathbf{n} = \langle 5,\ 0,\ 1 \rangle$, whence $d = \langle 0,\ 0,\ 20 \rangle \cdot \langle 5,\ 0,\ 1 \rangle = 20$. Thus, the plane VBC has the vector equation

$$\mathbf{r} \cdot \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 20.$$

Observe that the ground is given by the vector equation $\mathbf{r} \cdot \langle 0,\ 0,\ 1 \rangle = 0$. Let θ be the angle of inclination made by the mirror with the ground.

$$\cos\theta = \frac{\langle 5,\ 0,\ 1 \rangle \cdot \langle 0,\ 0,\ 1 \rangle}{|\langle 5,\ 0,\ 1 \rangle|\,|\langle 0,\ 0,\ 1 \rangle|} = \frac{1}{\sqrt{26}} \implies \theta = 78.7^\circ \text{ (1 d.p.).}$$

Assignment A9

Problem 1. The equation of the plane Π_1 is $y+z=0$ and the equation of the line l is $\frac{x-2}{2}=\frac{y-2}{-1}=\frac{z-2}{3}$. Find

(a) the position vector of the point of intersection of l and Π_1,

(b) the length of the perpendicular from the origin to l,

(c) the Cartesian equation for the plane Π_2 which contains l and the origin,

(d) the acute angle between the planes Π_1 and Π_2, giving your answer correct to the nearest $0.1°$.

Solution. Note that Π_1 has equation $\mathbf{r}\cdot\langle 0,\,1,\,1\rangle=0$ and l has equation $\mathbf{r}=\langle 5,\,2,\,2\rangle+\lambda\langle 2,\,-1,\,3\rangle$, $\lambda\in\mathbb{R}$.

Part (a). Let P be the point of intersection of Π_1 and l. Then $\overrightarrow{OP}=\langle 5,\,2,\,2\rangle+\lambda\langle 2,\,-1,\,3\rangle$ for some $\lambda\in\mathbb{R}$. Also, $\overrightarrow{OP}\cdot\langle 0,\,1,\,1\rangle=0$. Hence,

$$\left[\begin{pmatrix}5\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}\right]\cdot\begin{pmatrix}0\\1\\1\end{pmatrix}=0\implies 4+2\lambda=0\implies\lambda=-2.$$

Thus,

$$\overrightarrow{OP}=\begin{pmatrix}5\\2\\2\end{pmatrix}-2\begin{pmatrix}2\\-1\\3\end{pmatrix}=\begin{pmatrix}1\\4\\-4\end{pmatrix}.$$

Part (b). The perpendicular distance from the origin to l is

$$\frac{|\langle 5,\,2,\,2\rangle\times\langle 2,\,-1,\,3\rangle|}{|\langle 2,\,-1,\,3\rangle|}=\frac{1}{\sqrt{14}}\left|\begin{pmatrix}8\\-11\\-9\end{pmatrix}\right|=\frac{\sqrt{266}}{\sqrt{14}}=\sqrt{19}\text{ units.}$$

Part (c). Observe that Π_2 is parallel to $\langle 5,\,2,\,2\rangle$ and $\langle 2,\,-1,\,3\rangle$. Thus, $\mathbf{n}=\langle 5,\,2,\,2\rangle\times\langle 2,\,-1,\,3\rangle=\langle 8,\,-11,\,-9\rangle$. Since Π_2 contains the origin, $d=0$. Hence, Π_2 has vector equation $\mathbf{r}\cdot\langle 8,\,-11,\,-9\rangle=0$, which translates to $8x-11y-9z=0$.

Part (d). Let the acute angle be θ.

$$\cos\theta=\frac{|\langle 0,\,1,\,1\rangle\cdot\langle 8,\,-11,\,-9\rangle|}{|\langle 0,\,1,\,1\rangle|\,|\langle 8,\,-11,\,-9\rangle|}=\frac{20}{\sqrt{2}\sqrt{266}}\implies\theta=29.9°\text{ (1 d.p.).}$$

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Problem 2. The plane Π_1 has equation $\mathbf{r}\cdot(-\mathbf{i}+2\mathbf{k})=-4$ and the points A and P have position vectors $4\mathbf{i}$ and $\mathbf{i}+\alpha\mathbf{j}+\mathbf{k}$ respectively, where $\alpha\in\mathbb{R}$.

(a) Show that A lies on Π_1, but P does not.

(b) Find, in terms of α, the position vector of N, the foot of perpendicular of P on Π_1.

The plane Π_2 contains the points A, P and N.

(c) Show that the equation of Π_2 is $\mathbf{r}\cdot(2\alpha\mathbf{i}+5\mathbf{j}+\alpha\mathbf{k})=8\alpha$ and write down the equation of l, the line of the intersection of Π_1 and Π_2.

The plane Π_3 has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$.

(d) By considering l, or otherwise, find the value of α for which the three planes intersect in a line.

Solution. Note that $\Pi_1 : \mathbf{r} \cdot \langle -1, 0, 2\rangle = -4$, $\overrightarrow{OA} = \langle 4, 0, 0\rangle$ and $\overrightarrow{OP} = \langle 1, \alpha, 1\rangle$.

Part (a). Since $\overrightarrow{OA} \cdot \langle -1, 0, 2\rangle = \langle 4, 0, 0\rangle \cdot \langle -1, 0, 2\rangle = -4$, A lies on Π_1. On the other hand, since $\overrightarrow{OP} \cdot \langle -1, 0, 2\rangle = \langle 1, \alpha, 1\rangle \cdot \langle -1, 0, 2\rangle = 1 \neq -4$, P does not lie on Π_1.

Part (b). Note that $\overrightarrow{NP} = \lambda \langle -1, 0, 2\rangle$ for some $\lambda \in \mathbb{R}$, and $\overrightarrow{ON} \cdot \langle -1, 0, 2\rangle = -4$. Hence,

$$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON} = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} - \overrightarrow{ON} = \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}.$$

Thus,

$$\left[\begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} - \overrightarrow{ON}\right] \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \implies 1 - (-4) = 5\lambda \implies \lambda = 1.$$

Hence, $\overrightarrow{NP} = \langle -1, 0, 2\rangle$, whence $\overrightarrow{ON} = \overrightarrow{OP} - \overrightarrow{NP} = \langle 2, \alpha, -1\rangle$.

Part (c). Note that Π_2 is parallel to $\overrightarrow{NP} = \langle -1, 0, 2\rangle$ and $\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \langle -2, \alpha, -1\rangle$. Since $\langle -1, 0, 2\rangle \times \langle -2, \alpha, -1\rangle = -\langle 2\alpha, 5, \alpha\rangle$, we take $\mathbf{n} = \langle 2\alpha, 5, \alpha\rangle$, whence $d = \langle 4, 0, 0\rangle \cdot \langle 2\alpha, 5, \alpha\rangle = 8\alpha$. Thus, Π_2 has vector equation $\mathbf{r} \cdot \langle 2\alpha, 5, \alpha\rangle = 8\alpha$ which translates to $\mathbf{r} \cdot (2\alpha\mathbf{i} + 5\mathbf{j} + \alpha\mathbf{k}) = 8\alpha$.

Meanwhile, the line of intersection between Π_1 and Π_2 has equation

$$l : \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ \alpha \\ -1 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$

Part (d). If the three planes intersect in a line, they must intersect at l. Hence, l lies on Π_3.

$$\left[\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ \alpha \\ -1 \end{pmatrix}\right] \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 4 \implies 4 + (\alpha - 4)\mu = 4 \implies (\alpha - 4)\mu = 0.$$

Since $(\alpha - 4)\mu = 0$ must hold for all $\mu \in \mathbb{R}$, we must have $\alpha = 4$.

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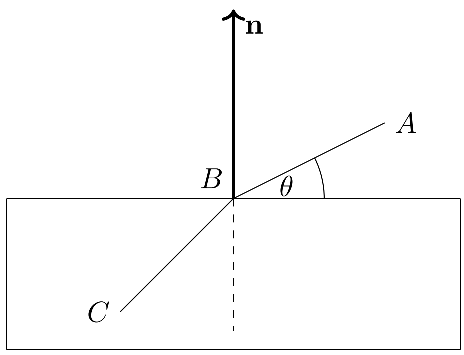
Problem 3. When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1, 2, 2)$ and enters a glass object at point $B(0, 0, 2)$. The surface of the glass object is a plane with normal vector $\mathbf{n}$. The diagram shows a cross-section of the glass object in the plane of the light ray and $\mathbf{n}$.



(a) Find a vector equation of the line AB.

The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the plane.

(b) Calculate the value of θ, giving your answer in degrees.

The line BC makes an angle of $45°$ with the normal to the plane, and BC is parallel to the unit vector $\langle -2/3,\, p,\, q\rangle$.

(c) By considering a vector perpendicular to the plane containing the light ray and $\mathbf{n}$, or otherwise, find the values of p and q.

The light ray leaves the glass object through a plane with equation $3x + 3z = -4$.

(d) Find the exact thickness of the glass object, taking one unit as one cm.

(e) Find the exact coordinates of the point at which the light ray leaves the glass object.

Solution. Let Π_G be the plane representing the surface of the glass object.

Part (a). Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle 0,\, 0,\, 2\rangle - \langle 1,\, 2,\, 2\rangle = -\langle 1,\, 2,\, 0\rangle$. Hence,

$$l_{AB} : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \lambda \in \mathbb{R}.$$

Part (b). Observe that Π_G has equation $\mathbf{r} \cdot \langle 1,\, 0,\, 1\rangle = 2$. Hence,

$$\sin\theta = \frac{|\langle 1,\, 0,\, 1\rangle \cdot \langle 1,\, 2,\, 0\rangle|}{|\langle 1,\, 0,\, 1\rangle|\,|\langle 1,\, 2,\, 0\rangle|} = \frac{1}{\sqrt{2}\sqrt{5}} \implies \theta = 71.6° \text{ (1 d.p.)}.$$

Part (c). Since line BC makes an angle of $45°$ with $\mathbf{n}_G$,

$$\sin 45° = \frac{|\langle 1,\, 0,\, 1\rangle \cdot \langle -2/3,\, p,\, q\rangle|}{|\langle 1,\, 0,\, 1\rangle|\,|\langle -2/3,\, p,\, q\rangle|} \implies \frac{1}{\sqrt{2}} = \frac{|q - 2/3|}{\sqrt{2} \cdot 1} \implies \left| q - \frac{2}{3} \right| = 1.$$

Hence, $q = -1/3$. Note that we reject $q = 5/3$ since $\langle -2/3,\, p,\, q\rangle$ is a unit vector, which implies that $|q| \leq 1$.

Let Π_L be the plane containing the light ray. Note that Π_L is parallel to $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Hence, $\mathbf{n}_L = \langle 1,\, 2,\, 0\rangle \times \langle -2/3,\, p,\, q\rangle = \frac{1}{3}\langle 6q,\, -3q,\, 3p + 4\rangle$. Since Π_L contains $\mathbf{n}_G$, we have that $\mathbf{n}_L \perp \mathbf{n}_G$, whence $\mathbf{n}_L \cdot \mathbf{n}_G = 0$. This gives us

$$\begin{pmatrix} 6q \\ -3q \\ 3p + 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \implies 6q + 3p + 4 = 0 \implies 6\left(-\frac{1}{3}\right) + 3p + 4 = 0 \implies p = -\frac{2}{3}.$$

Part (d). Let Π_G' be the plane with equation $3x + 3z = -4$. Observe that Π_G is parallel to Π_G'. Also note that $(-4/3, 0, 0)$ is a point on Π_G'. Hence, the distance between Π_G and Π_G' is given by

$$\frac{|2 - \langle -4/3,\, 0,\, 0\rangle \cdot \langle 1,\, 0,\, 1\rangle|}{|\langle 1,\, 0,\, 1\rangle|} = \frac{10}{3\sqrt{2}} \text{ cm}.$$

Part (e). Observe that $\langle -2/3,\, p,\, q\rangle = \langle -2/3,\, -2/3,\, -1/3\rangle = -\frac{1}{3}\langle 2,\, 2,\, 1\rangle$, whence the line BC has equation $\mathbf{r} = \langle 0,\, 0,\, 2\rangle + \mu\langle 2,\, 2,\, 1\rangle$, $\mu \in \mathbb{R}$. Let P be the intersection between line BC and Π_G'. Also note that $\overrightarrow{OP} = \langle 0,\, 0,\, 2\rangle + \mu\langle 2,\, 2,\, 1\rangle$ for some $\mu \in \mathbb{R}$, and $\overrightarrow{OP} \cdot \langle 3,\, 0,\, 3\rangle = -4$. Hence,

$$\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} = -4 \implies 6 - 9\mu = -4 \implies \mu = -\frac{10}{9}.$$

Hence, $\overrightarrow{OP} = \langle 0,\, 0,\, 2\rangle - \frac{10}{9}\langle 2,\, 2,\, 1\rangle = \langle -20/9,\, -20/9,\, 8/9\rangle$. The coordinates of the point are hence $(-20/9, -20/9.8/9)$.

A10.1. Complex Numbers - Complex Numbers in Cartesian Form

Tutorial A10.1

Problem 1. Given that $z = 3 - 2\mathrm{i}$ and $w = 1 + 4\mathrm{i}$, express in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$:

(a) $z + 2w$

(b) zw

(c) z/w

(d) $(w - w^*)^3$

(e) z^4

Solution.
Part (a).

$$z + 2w = (3 - 2\mathrm{i}) + 2(1 + 4\mathrm{i}) = 3 - 2\mathrm{i} + 2 + 8\mathrm{i} = 5 + 6\mathrm{i}.$$

Part (b).

$$zw = (3 - 2\mathrm{i})(1 + 4\mathrm{i}) = 3 + 12\mathrm{i} - 2\mathrm{i} + 8 = 11 + 10\mathrm{i}.$$

Part (c).

$$\frac{z}{w} = \frac{3 - 2\mathrm{i}}{1 + 4\mathrm{i}} = \frac{(3 - 2\mathrm{i})(1 - 4\mathrm{i})}{(1 + 4\mathrm{i})(1 - 4\mathrm{i})} = \frac{3 - 12\mathrm{i} - 2\mathrm{i} - 8}{1^2 + 4^2} = \frac{-5 - 14\mathrm{i}}{17} = -\frac{5}{17} - \frac{14}{17}\mathrm{i}.$$

Part (d).

$$(w - w^*)^3 = [2\mathrm{Im}(w)\mathrm{i}]^3 = (8\mathrm{i})^3 = -512\mathrm{i}.$$

Part (e).

$$\begin{aligned} z^4 = (3 - 2\mathrm{i})^4 &= 3^4 + 4 \cdot 3^3(-2\mathrm{i}) + 6 \cdot 3^2(-2\mathrm{i})^2 + 4 \cdot 3(-2\mathrm{i})^3 + (-2\mathrm{i})^4 \\ &= 81 - 216\mathrm{i} - 216 + 96\mathrm{i} + 16 = -119 - 120\mathrm{i}. \end{aligned}$$

* * * * *

Problem 2. Is the following true or false in general?

(a) $\mathrm{Im}(zw) = \mathrm{Im}(z)\mathrm{Im}(w)$

(b) $\mathrm{Re}(zw) = \mathrm{Re}(z)\mathrm{Re}(w)$

Solution. Let $z = a+bi$ and $w = c+di$. Then $zw = (a+bi)(c+di) = (ac-bd)+(ad+bc)i$.
Part (a). Observe that

$$\mathrm{Im}(zw) = ad + bc \neq bd = \mathrm{Im}(z)\mathrm{Im}(w).$$

Hence, the statement is false in general.

Part (b). Observe that

$$\operatorname{Re}(zw) = ac - bd \neq ac = \operatorname{Re}(z)\operatorname{Re}(w).$$

Hence, the statement is false in general.

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Problem 3.

(a) Find the complex number z such that $\frac{z-2}{z} = 1 + \mathrm{i}$.

(b) Given that $u = 2 + \mathrm{i}$ and $v = -2 + 4\mathrm{i}$, find in the form $a + b\mathrm{i}$, where $a, b \in \mathbb{R}$, the complex number z such that $\frac{1}{z} = \frac{1}{u} + \frac{1}{v}$.

Solution.

Part (a).

$$\frac{z-2}{z} = 1 + \mathrm{i} \implies z - 2 = z + \mathrm{i}z \implies \mathrm{i}z = -2 \implies z = -\frac{2}{\mathrm{i}} = 2\mathrm{i}.$$

Part (b).

$$\frac{1}{z} = \frac{1}{u} + \frac{1}{v} \implies z = \frac{1}{1/u + 1/v} = \frac{uv}{u+v} = \frac{(2+\mathrm{i})(-2+4\mathrm{i})}{(2+\mathrm{i}) + (-2+4\mathrm{i})} = \frac{-8+6\mathrm{i}}{5\mathrm{i}} = \frac{6}{5} + \frac{8}{5}\mathrm{i}.$$

* * * * *

Problem 4. The complex numbers z and w are $1 + a\mathrm{i}$ and $b - 2\mathrm{i}$ respectively, where a and b are real and a is negative. Given that $zw^* = 8i$, find the exact values of a and b.

Solution. Note that

$$zw^* = (1 + a\mathrm{i})(b + 2\mathrm{i}) = (b - 2a) + (2 + ab)\mathrm{i}.$$

Comparing real and imaginary parts, we have $b - 2a = 0 \implies b = 2a$ and $2 + ab = 8$. Hence, $2a^2 = 6$, giving $a = -\sqrt{3}$ and $b = -2\sqrt{3}$.

* * * * *

Problem 5. Find, in the form $x + \mathrm{i}y$, the two complex numbers z satisfying both of the equations

$$\frac{z}{z^*} = \frac{3}{5} + \frac{4}{5}\mathrm{i} \quad \text{and} \quad zz^* = 5.$$

Solution. Multiplying both equations together, we have $z^2 = 3 + 4\mathrm{i}$. Let $z = x + \mathrm{i}y$, with $x, y \in \mathbb{R}$. We thus have $z^2 = x^2 - y^2 + 2xy\mathrm{i} = 3 + 4\mathrm{i}$. Comparing real and imaginary parts, we obtain the following system:

$$x^2 - y^2 = 3, \quad 2xy = 4.$$

Squaring the second equation yields $x^2y^2 = 4$. From the first equation, we have $x^2 = 3+y^2$. Thus, $y^2(3 + y^2) = 4 \implies y^2 = 1 \implies y = \pm 1 \implies x = \pm 2$. Hence, $z = 2 + \mathrm{i}$ or $z = -2 - \mathrm{i}$.

Problem 6.

(a) Given that $\mathrm{i}w + 3z = 2 + 4\mathrm{i}$ and $w + (1 - \mathrm{i})z = 2 - \mathrm{i}$, find z and w in the form of $x + \mathrm{i}y$, where x and y are real numbers.

(b) Determine the value of k such that $z = \frac{1-k\mathrm{i}}{\sqrt{3}+\mathrm{i}}$ is purely imaginary, where $k \in \mathbb{R}$.

Solution.

Part (a). Let $w = a + b\mathrm{i}$ and $z = c + d\mathrm{i}$. Then

$$\mathrm{i}w + 3z = \mathrm{i}(a + b\mathrm{i}) + 3(c + d\mathrm{i}) = (-b + 3c) + (a + 3d)\mathrm{i} = 2 + 4\mathrm{i}$$

and

$$w + (1 - \mathrm{i})z = (a + b\mathrm{i}) + (1 - \mathrm{i})(c + d\mathrm{i}) = (a + c + d) + (b - c + d)\mathrm{i} = 2 - \mathrm{i}.$$

Comparing the real and imaginary parts of both equations yields the following system:

$$\begin{cases} -b + 3c = 2 \\ a + 3d = 4 \\ a + c + d = 2 \\ b - c + d = -1 \end{cases}$$

which has the unique solution $a = 1$, $b = -2$, $c = 0$ and $d = 1$. Hence, $w = 1 - 2\mathrm{i}$ and $z = \mathrm{i}$.

Part (b).

$$z = \frac{1 - k\mathrm{i}}{\sqrt{3} + \mathrm{i}} = \frac{(1 - k\mathrm{i})(\sqrt{3} - \mathrm{i})}{\sqrt{3}^2 + 1^2} = \frac{1}{4}(\sqrt{3} - \mathrm{i} - k\sqrt{3}\mathrm{i} - k) = \frac{1}{4}\left[(\sqrt{3} - k) - (1 + k\sqrt{3})\mathrm{i}\right].$$

Since z is purely imaginary, $\mathrm{Re}(z) = 0$. Hence, $\frac{1}{4}(\sqrt{3} - k) = 0 \implies k = \sqrt{3}$.

* * * * *

Problem 7.

(a) The complex number $x + \mathrm{i}y$ is such that $(x + iy)^2 = \mathrm{i}$. Find the possible values of the real numbers x and y, giving your answers in exact form.

(b) Hence, find the possible values of the complex number w such that $w^2 = -\mathrm{i}$.

Solution.

Part (a). Note that $(x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i} = \mathrm{i}$. Comparing real and imaginary parts, we have

$$x^2 - y^2 = 0, \quad 2xy = 1.$$

Note that the second equation implies that both x and y have the same sign. Hence, from the first equation, we have $x = y$. Thus, $x^2 = y^2 = 1/2 \implies x = y = \pm 1/\sqrt{2}$.

Part (b).

$$w^2 = -\mathrm{i} \implies (w^*)^2 = \mathrm{i} \implies w^* = \pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\mathrm{i} \implies w = \pm\frac{1}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}\mathrm{i}.$$

Problem 8.

(a) The roots of the equation $z^2 = -8\mathrm{i}$ are z_1 and z_2. Find z_1 and z_2 in Cartesian form $x + \mathrm{i}y$, showing your working.

(b) Hence, or otherwise, find in Cartesian form the roots w_1 and w_2 of the equation $w^2 + 4w + (4 + 2\mathrm{i}) = 0$.

Solution.

Part (a). Let $z = x + \mathrm{i}y$ where $x, y \in \mathbb{R}$. Then $(x + \mathrm{i}y)^2 = x^2 - y^2 + 2xy\mathrm{i} = -8\mathrm{i}$. Comparing real and imaginary parts, we have the following system:

$$x^2 - y^2 = 0, \quad 2xy = 8.$$

From the second equation, we know that x and y have opposite signs. Hence, from the first equation, we have that $x = -y$. Thus, $x^2 = 4 \implies x = \pm 2 \implies y = \mp 2$. Thus, $z = \pm 2(1 - \mathrm{i})$, whence $z_1 = 2 - 2\mathrm{i}$ and $z_2 = -2 + 2\mathrm{i}$.

Part (b).

$$\begin{gathered} w^2 + 4w + (4 + 2\mathrm{i}) = 0 \implies (w + 2)^2 = -2\mathrm{i} \implies (2w + 4)^2 = -8\mathrm{i} \\ \implies 2w + 4 = \pm 2(1 - \mathrm{i}) \implies w = 2 \pm (1 - \mathrm{i}). \end{gathered}$$

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Problem 9. One of the roots of the equations $2x^3 - 9x^2 + 2x + 30 = 0$ is $3 + \mathrm{i}$. Find the other roots of the equation.

Solution. Let $P(x) = 2x^3 - 9x^2 + 2x + 30$. Since $P(x)$ is a polynomial with real coefficients, by the conjugate root theorem, we have that $(3 + \mathrm{i})^* = 3 - \mathrm{i}$ is also a root of $P(x)$. Let α be the third root of $P(x)$. Then

$$P(x) = 2x^3 - 9x^2 + 2x + 30 = 2(x - \alpha)\left[x - (3 + i)\right]\left[x - (3 - i)\right].$$

Comparing constants,

$$2(-\alpha)(-3 - \mathrm{i})(-3 + \mathrm{i}) = 30 \implies \alpha = -\frac{15}{(-3 - \mathrm{i})(-3 + \mathrm{i})} = -\frac{3}{2}.$$

Hence, the other roots of the equation are $3 - \mathrm{i}$ and $-3/2$.

* * * * *

Problem 10. Obtain a cubic equation having 2 and $\frac{5}{4} - \frac{\sqrt{7}}{4}\mathrm{i}$ as two of its roots, in the form $az^3 + bz^2 + cz + d = 0$, where a, b, c and d are real integral coefficients to be determined.

Solution. Let $P(z) = az^3 + bz^2 + cz + d$. Since $P(z)$ is a polynomial with real coefficients, by the conjugate root theorem, we have that $\left(\frac{5}{4} - \frac{\sqrt{7}}{4}\mathrm{i}\right)^* = \frac{5}{4} + \frac{\sqrt{7}}{4}\mathrm{i}$ is also a root of $P(z)$. We can thus write $P(z)$ as

$$\begin{aligned} P(z) &= k(z - 2)\left[z - \left(\frac{5}{4} - \frac{\sqrt{7}}{4}\mathrm{i}\right)\right]\left[z - \left(\frac{5}{4} + \frac{\sqrt{7}}{4}\mathrm{i}\right)\right] \\ &= k(z - 2)\left[\left(z - \frac{5}{4}\right)^2 + \left(\frac{\sqrt{7}}{4}\right)^2\right] = k(z - 2)\left(z^2 - \frac{5}{2}z + 2\right) \\ &= \frac{1}{2}k(2z^3 - 9z^2 + 14z - 8), \end{aligned}$$

where k is an arbitrary real number. Taking $k = 2$, we have $P(z) = 2z^3 - 9z^2 + 14z - 8$, whence $a = 2$, $b = -9$, $c = 14$ and $d = -8$.

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Problem 11.

(a) Verify that $-1 + 5\mathrm{i}$ is a root of the equation $w^2 + (-1 - 8\mathrm{i})w + (-17 + 7\mathrm{i}) = 0$. Hence, or otherwise, find the second root of the equation in Cartesian form, $p + \mathrm{i}q$, showing your working.

(b) The equation $z^3 - 5z^2 + 16z + k = 0$, where k is a real constant, has a root $z = 1 + a\mathrm{i}$, where a is a positive real constant. Find the values of a and k, showing your working.

Solution.

Part (a). Let $P(w) = w^2 + (-1 - 8\mathrm{i})w + (-17 + 7\mathrm{i})$. Consider $P(-1 + 5\mathrm{i})$.

$$\begin{aligned} P(-1 + 5\mathrm{i}) &= (-1 + 5\mathrm{i})^2 + (-1 - 8\mathrm{i})(-1 + 5\mathrm{i}) + (-17 + 7\mathrm{i}) \\ &= (1 - 10\mathrm{i} - 25) + (1 - 5\mathrm{i} + 8\mathrm{i} + 40) + (-17 + 7\mathrm{i}) = 0. \end{aligned}$$

Hence, $-1 + 5\mathrm{i}$ is a root of $w^2 + (-1 - 8\mathrm{i})w + (-17 + 7\mathrm{i}) = 0$.

Let α be the other root of the equation. By Vieta's formula, we have

$$\alpha + (-1 + 5\mathrm{i}) = -\left(\frac{-1 - 8\mathrm{i}}{1}\right) = 1 + 8\mathrm{i} \implies \alpha = 2 + 3\mathrm{i}.$$

Part (b). Let $P(z) = z^3 - 5z^2 + 16z + k$. Then $P(1 + a\mathrm{i}) = 0$. Note that

$$\begin{aligned} P(1 + a\mathrm{i}) &= (1 + a\mathrm{i})^3 - 5(1 + a\mathrm{i})^2 + 16(1 + a\mathrm{i}) + k \\ &= \left[1 + 3a\mathrm{i} - 3a^2 - a^3\mathrm{i}\right] - 5(1 + 2a\mathrm{i} - a^2) + (16 + 16a\mathrm{i}) + k \\ &= (12 + k + 2a^2) + (9 - a^2)a\mathrm{i}. \end{aligned}$$

Comparing real and imaginary parts, we have $a(9 - a^2) = 0 \implies a = 3$ (since $a > 0$) and $12 + k + 2a^2 = 0 \implies k = -30$.

Assignment A10.1

Problem 1. The complex number w is such that $ww^* + 2w = 3 + 4\mathrm{i}$, where w^* is the complex conjugate of w. Find w in the form $a + \mathrm{i}b$, where a and b are real.

Solution. Note $ww^* = (\operatorname{Re} w)^2 + (\operatorname{Im} w)^2 \in \mathbb{R}$.

Taking the imaginary part of the given equation,

$$\operatorname{Im}(ww^* + 2w) = \operatorname{Im}(3 + 4i) \implies 2 \operatorname{Im} w = 4 \implies \operatorname{Im} w = 2.$$

Taking the real part of the given equation,

$$\begin{aligned} \operatorname{Re}(ww^* + 2w) = \operatorname{Re}(3 + 4i) &\implies \left[(\operatorname{Re} w)^2 + (\operatorname{Im} w)^2\right] + 2 \operatorname{Re} w = 3 \\ \implies (\operatorname{Re} w)^2 + 2\operatorname{Re}(w) + 1 = 0 &\implies (\operatorname{Re} w + 1)^2 = 0 \implies \operatorname{Re}(w) = -1. \end{aligned}$$

Hence, $w = -1 + 2\mathrm{i}$.

* * * * *

Problem 2. Express $(3 - \mathrm{i})^2$ in the form $a + \mathrm{i}b$.

Hence, or otherwise, find the roots of the equation $(z + \mathrm{i})^2 = -8 + 6\mathrm{i}$.

Solution. We have

$$(3 - \mathrm{i})^2 = 3^2 - 6\mathrm{i} + \mathrm{i}^2 = 8 - 6\mathrm{i}.$$

Consider $(z + \mathrm{i})^2 = -8 + 6\mathrm{i}$. Note that $-(z + \mathrm{i})^2 = (\mathrm{i}z - 1)^2$.

$$\begin{aligned} (z + \mathrm{i})^2 = -8 + 6\mathrm{i} &\implies (\mathrm{i}z - 1)^2 = 8 - 6\mathrm{i} \implies \mathrm{i}z - 1 = \pm(3 - \mathrm{i}) \\ \implies z &= \frac{1}{\mathrm{i}}(1 \pm (3 - \mathrm{i})) = -\mathrm{i}(1 \pm (3 - \mathrm{i})) = -1 - 4\mathrm{i} \text{ or } 1 + 2\mathrm{i}. \end{aligned}$$

* * * * *

Problem 3.

(a) It is given that $z_1 = 1 + \sqrt{3}\mathrm{i}$. Find the value of z_1^3, showing clearly how you obtain your answer.

(b) Given that $1 + \sqrt{3}\mathrm{i}$ is a root of the equation

$$2z^3 + az^2 + bz + 4 = 0$$

find the values of the real numbers a and b. Hence, solve the above equation.

Solution.

Part (a). We have

$$z_1^3 = \left(1 + \sqrt{3}\mathrm{i}\right)^3 = 1 + 3\left(\sqrt{3}\mathrm{i}\right) + 3\left(\sqrt{3}\mathrm{i}\right)^2 + \left(\sqrt{3}\mathrm{i}\right)^3 = 1 + 3\sqrt{3}\mathrm{i} - 9 - 3\sqrt{3}\mathrm{i} = -8.$$

Part (b). Since $1 + \sqrt{3}\mathrm{i}$ is a root of the given equation, we have

$$\begin{aligned} &2\left(1 + \sqrt{3}\mathrm{i}\right)^3 + a\left(1 + \sqrt{3}\mathrm{i}\right)^2 + b\left(1 + \sqrt{3}\mathrm{i}\right) + 4 = 0 \\ \implies &-16 + a\left(-2 + 2\sqrt{3}\mathrm{i}\right) + b\left(1 + \sqrt{3}\mathrm{i}\right) + 4 = 0 \implies (-2a + b) + \sqrt{3}(2a + b)\mathrm{i} = 12. \end{aligned}$$

Comparing real and imaginary parts, we obtain $-2a + b = 12$ and $2a + b = 0$, whence $a = -3$ and $b = 6$.

Since the coefficients of $2z^3 + az^2 + bz + 4$ are all real, the second root is $(1+\sqrt{3}\mathrm{i})^* = 1 - \sqrt{3}\mathrm{i}$. Let the third root be α. By Vieta's formula,

$$(1+\sqrt{3}\mathrm{i})(1-\sqrt{3}\mathrm{i})\alpha = -\frac{4}{2} \implies 4\alpha = -2 \implies \alpha = -\frac{1}{2}.$$

The roots of the equation are hence $1+\sqrt{3}\mathrm{i}$, $1-\sqrt{3}\mathrm{i}$ and $-\frac{1}{2}$.

* * * * *

Problem 4. The complex number z is such that $az^2 + bz + a = 0$ where a and b are real constants. It is given that $z = z_0$ is a solution to this equation where $\mathrm{Im}(z_0) \neq 0$.

(a) Verify that $z = \frac{1}{z_0}$ is the other solution. Hence, show that $|z_0| = 1$.

Take $\mathrm{Im}(z_0) = 1/2$ for the rest of the question.

(b) Find the possible complex numbers for z_0.

(c) If $\mathrm{Re}(z_0) > 0$, find b in terms of a.

Solution.

Part (a).

$$a\left(\frac{1}{z_0}\right)^2 + b\left(\frac{1}{z_0}\right) + a = \left(\frac{1}{z_0}\right)^2\left(a + bz_0 + az_0^2\right) = 0$$

Hence, $z = 1/z_0$ is a root of the given equation.

Since $a, b \in \mathbb{R}$, by the conjugate root theorem, $z_0^* = 1/z_0$. Hence,

$$z_0 z_0^* = 1 \implies \mathrm{Re}(z_0)^2 + \mathrm{Im}(z_0)^2 = |z_0|^2 = 1 \implies |z_0| = 1.$$

Part (b). Let $z_0 = x + \frac{1}{2}\mathrm{i}$. Then

$$\left|x + \frac{1}{2}i\right| = 1 \implies x^2 + \left(\frac{1}{2}\right)^2 = 1^2 \implies x^2 = \frac{3}{4} \implies x = \pm\frac{\sqrt{3}}{2}.$$

Hence, $z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$ or $z_0 = -\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$.

Part (c). Since $\mathrm{Re}(z_0) > 0$, we have $z_0 = \frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}$. By Vieta's formula,

$$-\frac{b}{a} = z_0 + \frac{1}{z_0} = z_0 + z_0^* = 2\mathrm{Re}(z_0) = \sqrt{3} \implies b = -\sqrt{3}a.$$

A10.2. Complex Numbers - Complex Numbers in Polar Form

Tutorial A10.2

Problem 1. Is the following true or false in general?

(a) $\left|w^2\right| = |w|^2$

(b) $|z + 2w| = |z| + |2w|$

Solution.

Part (a). Let $w = r\mathrm{e}^{\mathrm{i}\theta}$, where $r, \theta \in \mathbb{R}$. Note that $\left|\mathrm{e}^{\mathrm{i}\theta}\right| = \left|\mathrm{e}^{2\mathrm{i}\theta}\right| = 1$.

$$\left|w^2\right| = \left|r^2\mathrm{e}^{2\mathrm{i}\theta}\right| = r^2\left|\mathrm{e}^{2\mathrm{i}\theta}\right| = r^2 = r^2\left|\mathrm{e}^{\mathrm{i}\theta}\right|^2 = \left|r\mathrm{e}^{\mathrm{i}\theta}\right|^2 = |w|^2.$$

The statement is hence true in general.

Part (b). Take $z = 1$ and $w = -1$.

$$|z + 2w| = |1 - 2| = 1 \neq 3 = |1| + |2(-1)| = |z| + |2w|.$$

The statement is hence false in general.

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Problem 2. Express the following complex numbers z in polar form $r(\cos\theta + \mathrm{i}\sin\theta)$ with exact values.

(a) $z = 2 - 2\mathrm{i}$

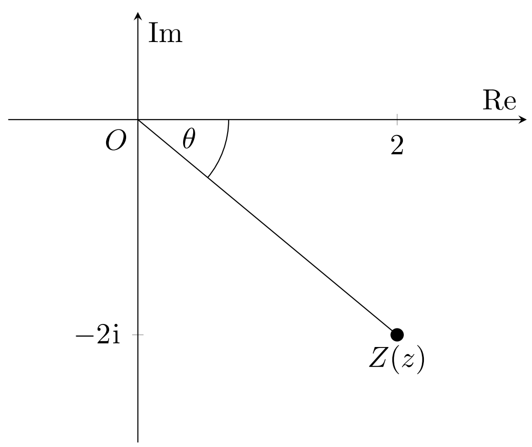
(b) $z = -1 + \mathrm{i}\sqrt{3}$

(c) $z = -5\mathrm{i}$

(d) $z = -2\sqrt{3} - 2\mathrm{i}$

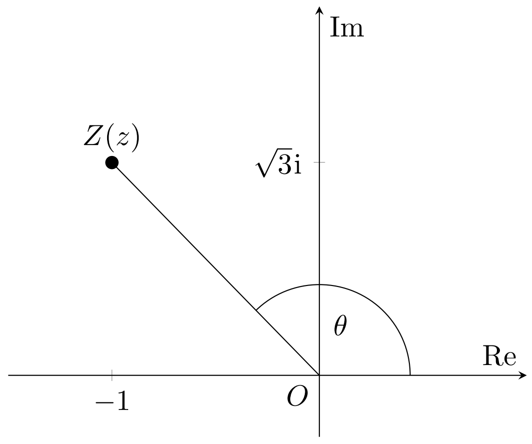
Solution.

Part (a).



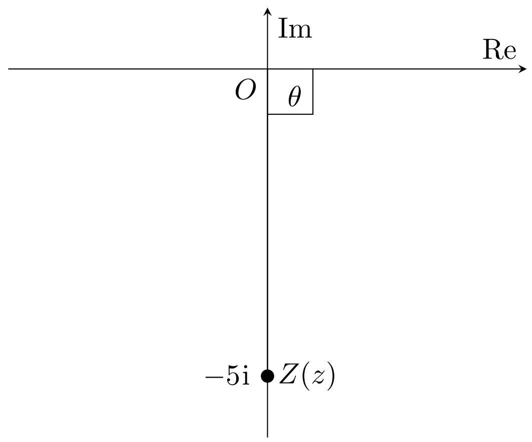
We have $r^2 = 2^2 + (-2)^2 \implies r = 2\sqrt{2}$ and $\tan\theta = -2/2 \implies \theta = -\pi/4$. Hence, $2 - 2\mathrm{i} = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + \mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right]$.

Part (b).



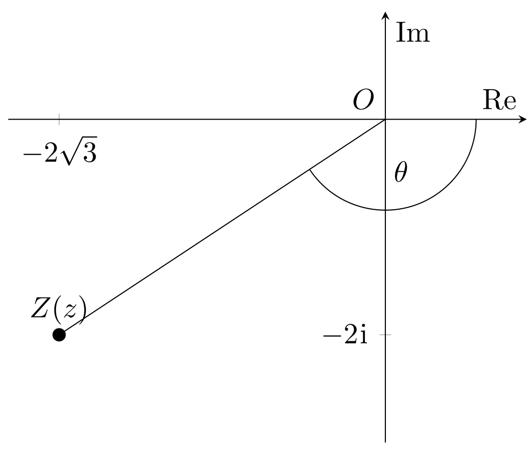
We have $r^2 = (-1)^2 + (\sqrt{3})^2 \implies r = 2$ and $\tan t = \sqrt{3}/(-1) \implies \theta = 2\pi/3$. Hence, $-1 + \sqrt{3}\mathrm{i} = 2\left[\cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)\right]$.

Part (c).



We have $r = 5$ and $\theta = -\pi/2$. Hence, $-5\mathrm{i} = 5\left[\cos\left(-\frac{\pi}{2}\right) + \mathrm{i}\sin\left(-\frac{\pi}{2}\right)\right]$.

Part (d).



We have $r^2 = (-2\sqrt{3})^2 + (-2)^2 \implies r = 4$ and $\tan t = -2/(-2\sqrt{3}) \implies \theta = -5\pi/6$. Hence, $-2\sqrt{3} - 2\mathrm{i} = 4\left[\cos\left(-\frac{5\pi}{6}\right) + \mathrm{i}\sin\left(-\frac{5\pi}{6}\right)\right]$.

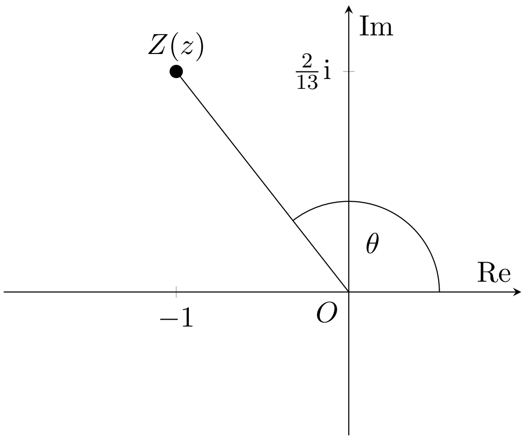
Problem 3. Express the following complex numbers z in exponential form $r\mathrm{e}^{\mathrm{i}\theta}$.

(a) $z = -1 + \frac{2}{13}\mathrm{i}$

(b) $z = \cos 50° - \mathrm{i}\sin 50°$

Solution.

Part (a).



We have $r^2 = (-1)^2 + \left(\frac{2}{13}\right)^2 \implies r = 1.01$ (3 s.f.) and $\tan t = \frac{2/13}{-1} \implies \theta = 2.99$ (3 s.f.). Hence, $-1 + \frac{2}{13}\mathrm{i} = 1.01\mathrm{e}^{2.99\mathrm{i}}$.

Part (b). We have $r = 1$ and $\theta = -50° = -\frac{5}{18}\pi$. Hence, $\cos 50° + i\sin 50° = \mathrm{e}^{-\mathrm{i}\frac{5}{18}\pi}$.

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Problem 4. Express the following complex numbers z in Cartesian form.

(a) $z = 7\mathrm{e}^{1-5\mathrm{i}}$

(b) $z = 6\left(\cos\frac{\pi}{8} - \mathrm{i}\sin\frac{\pi}{8}\right)$

Solution.

Part (a). We have

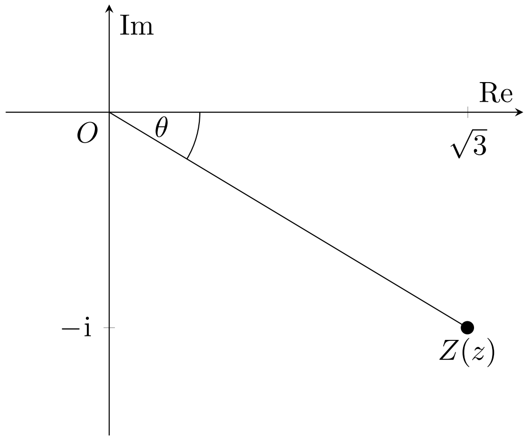
$$z = 7\mathrm{e}^{1-5\mathrm{i}} = 7\mathrm{e}\cdot\mathrm{e}^{-5\mathrm{i}} = 7\mathrm{e}\left[\cos(-5) + \mathrm{i}\sin(-5)\right] = 5.40 + 18.2\mathrm{i} \text{ (3 s.f.)}.$$

Part (b). We have

$$z = 6\left(\cos\frac{\pi}{8} - \mathrm{i}\sin\frac{\pi}{8}\right) = 5.54 - 2.30\mathrm{i} \text{ (3 s.f.)}.$$

Problem 5. Given that $z = \sqrt{3} - \mathrm{i}$, find the exact modulus and argument of z. Hence, find the exact modulus and argument of $1/z^2$ and z^{10}.

Solution.



We have $r^2 = (\sqrt{3})^2 + (-1)^2 \implies r = 2$ and $\tan\theta = -1/\sqrt{3} \implies \theta = -\pi/6$. Hence, $|z| = 2$ and $\arg z = -\pi/6$.

Note that $\left|1/z^2\right| = |z|^{-2} = 1/4$. Also, $\arg\left(1/z^2\right) = -2\arg z = \pi/3$.

Note that$\left|z^{10}\right| = |z|^1 0 = 1024$. Also, $\arg z^{10} = 10 \arg z = -5\pi/3 \equiv \pi/3$.

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Problem 6. If $\arg(z - 1/2) = \pi/5$, determine $\arg(2z - 1)$.

Solution.

$$\arg(2z-1) = \arg\left(\frac{1}{2}\left[z - \frac{1}{2}\right]\right) = \arg\left(z - \frac{1}{2}\right) = \frac{\pi}{5}.$$

* * * * *

Problem 7. In an Argand diagram, points P and Q represent the complex numbers $z = 1 + \mathrm{i}$ and $w = 1 + 2\mathrm{i}$ respectively, and O is the origin.

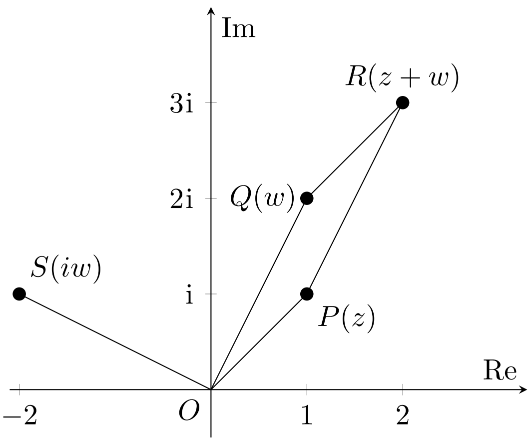
(a) Mark on the Argand diagram the points P and Q, and the points R and S which represent $z + w$ and iw respectively.

(b) What is the geometrical shape of $OPRQ$?

(c) State the angle SOP.

Solution.

Part (a).



Part (b). $OPRQ$ is a parallelogram.

Part (c). $\angle SOP = \pi/2$.

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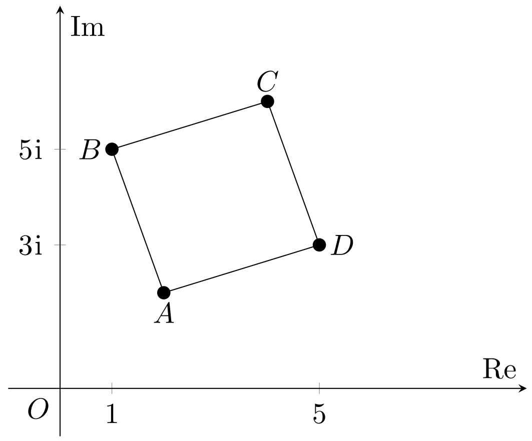
Problem 8. B and D are points in the Argand diagram representing the complex numbers $1+5\mathrm{i}$ and $5+3\mathrm{i}$ respectively. Given that BD is a diagonal of the square $ABCD$, calculate the complex numbers represented by A and C.

Solution.



Let $A(x+\mathrm{i}y)$. Since $AB \perp AD$, we have $b-a=\mathrm{i}(d-a)$.

$$b-a=\mathrm{i}(d-a) \implies (1+5\mathrm{i})-(x+\mathrm{i}y)=i\left[(5+3\mathrm{i})-(x+\mathrm{i}y)\right]$$
$$\implies (1-x)+(5-y)\mathrm{i}=(-3+y)+(5-x)\mathrm{i} \implies (x+y)+(y-x)\mathrm{i}=4.$$

Comparing real and imaginary parts, we obtain $x=y=2$. Hence, $A(2+2\mathrm{i})$.

Let $C(u+\mathrm{i}v)$. Since $CB \perp CD$, we have $d-c=\mathrm{i}(b-c)$.

$$d-c=\mathrm{i}(b-c) \implies (5+3\mathrm{i})-(u+\mathrm{i}v)=\mathrm{i}\left[(1+5\mathrm{i})-(u+\mathrm{i}v)\right]$$
$$\implies (5-u)+(3-v)\mathrm{i}=(-5+v)+(1-u)\mathrm{i} \implies (u+v)+(v-u)\mathrm{i}=10+2\mathrm{i}.$$

Comparing real and imaginary parts, we obtain $u=4$ and $v=6$. Hence, $C(4+6\mathrm{i})$.

* * * * *

Problem 9.

(a) Given that $u=2\left(\cos\frac{\pi}{6}+\mathrm{i}\sin\frac{\pi}{6}\right)$ and $w=4\left(\cos\frac{\pi}{3}-\mathrm{i}\sin\frac{\pi}{3}\right)$, find the modulus and argument of u^*/w^3 in exact form.

(b) Let z be the complex number $-1+\mathrm{i}\sqrt{3}$. Find the value of the real number a such that $\arg\left(z^2+az\right)=-\pi/2$.

Solution.

Part (a). Note that $|u|=2$, $\arg u=\pi/6$, $|w|=4$ and $\arg w=-\pi/3$. Hence,

$$\left|\frac{u^*}{w^3}\right|=\frac{|u^*|}{|w^3|}=\frac{|u|}{|w|^3}=\frac{2}{4^3}=\frac{1}{32}$$

and

$$\arg\frac{u^*}{w^3}=\arg u^*-\arg w^3=-\arg u-3\arg w=-\frac{\pi}{6}-3\left(-\frac{\pi}{3}\right)=\frac{5}{6}\pi.$$

Part (b). Since $\arg\big(z^2+az\big) = -\pi/2$, we have that z^2+az is purely imaginary, with a negative imaginary part. Since

$$z^2 + az = \left(-1+\mathrm{i}\sqrt{3}\right)^2 + a\left(-1+\mathrm{i}\sqrt{3}\right) = \left(-2-2\sqrt{3}\mathrm{i}\right) + a\left(-1+\mathrm{i}\sqrt{3}\right).$$

Hence,

$$\mathrm{Re}\big(z^2+az\big) = 0 \implies -2-a=0 \implies a=-2.$$

* * * * *

Problem 10. The complex number w has modulus r and argument θ, where $0<\theta<\pi/2$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p = w/w^*$. Given that p^5 is real and positive, find the possible values of θ.

Solution. Clearly, $|p|=1$ and $\arg p = 2\theta$.

Since p^5 is real and positive, we have $\arg p^5 = 2\pi n$, where $n\in\mathbb{Z}$. Thus, $\arg p = 2\pi n/5 = 2\theta \implies \theta = \pi n/5$. Since $0<\theta<\pi/2$, the possible values of θ are $\pi/5$ and $2\pi/5$.

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Problem 11. The complex number w has modulus $\sqrt{2}$ and argument $-3\pi/4$, and the complex number z has modulus 2 and argument $-\pi/3$. Find the modulus and argument of wz, giving each answer exactly.

By first expressing w and z in the form $x+\mathrm{i}y$, find the exact real and imaginary parts of wz.

Hence, show that $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Solution. Note that

$$|wz| = |w|\,|z| = 2\sqrt{2}$$

and

$$\arg(wz) = \arg w + \arg z = -\frac{3}{4}\pi - \frac{1}{3}\pi = -\frac{13}{12}\pi \equiv \frac{11}{12}\pi.$$

Also,

$$w = \sqrt{2}\left[\cos\left(-\frac{3}{4}\pi\right) + \mathrm{i}\sin\left(-\frac{3}{4}\pi\right)\right] = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\mathrm{i}\right) = -1-\mathrm{i}$$

and

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + \mathrm{i}\sin\left(-\frac{\pi}{3}\right)\right] = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 1-\sqrt{3}\mathrm{i}.$$

Hence,

$$wz = (-1-\mathrm{i})(1-\sqrt{3}\mathrm{i}) = (-1+\sqrt{3}-\mathrm{i}-\sqrt{3}) = (-1-\sqrt{3}) + (\sqrt{3}-1)\mathrm{i},$$

whence $\mathrm{Re}(wz) = -1-\sqrt{3}$ and $\mathrm{Im}(wz) = \sqrt{3}-1$.

From the first part, we have that $wz = 2\sqrt{2}\left[\cos\left(\frac{11}{12}\pi\right) + \mathrm{i}\sin\left(\frac{11}{12}\pi\right)\right]$. Thus, $\mathrm{Im}(wz) = 2\sqrt{2}\sin\left(\frac{11}{12}\pi\right) = 2\sqrt{2}\sin\frac{\pi}{12}$. Equating the result for $\mathrm{Im}(wz)$ found in the second part, we have

$$2\sqrt{2}\sin\frac{\pi}{12} = \sqrt{3}-1 \implies \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

Problem 12. Given that $\frac{5+z}{5-z} = \mathrm{e}^{\mathrm{i}\theta}$, show that z can be written as $5\mathrm{i}\tan\frac{\theta}{2}$.

Solution. Note that

$$\frac{5+z}{5-z} = \mathrm{e}^{\mathrm{i}\theta} \implies 5+z = \mathrm{e}^{\mathrm{i}\theta}(5-z) \implies z + \mathrm{e}^{\mathrm{i}\theta}z = 5\mathrm{e}^{\mathrm{i}\theta} - 5 \implies z = 5\left(\frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\mathrm{e}^{\mathrm{i}\theta}+1}\right).$$

Hence,

$$z = 5\left(\frac{\mathrm{e}^{\mathrm{i}\theta}-1}{\mathrm{e}^{\mathrm{i}\theta}+1}\right) = 5\left(\frac{\mathrm{e}^{\mathrm{i}\theta/2}-\mathrm{e}^{-\mathrm{i}\theta/2}}{\mathrm{e}^{\mathrm{i}\theta/2}+\mathrm{e}^{-\mathrm{i}\theta/2}}\right) = 5\left(\frac{2\mathrm{i}\sin(\theta/2)}{2\cos(\theta/2)}\right) = 5\mathrm{i}\tan\frac{\theta}{2}.$$

* * * * *

Problem 13. The polynomial $P(z)$ has real coefficients. The equation $P(z) = 0$ has a root $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 < \theta < \pi$.

(a) Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2 - 2rz\cos\theta + r^2$.

(b) Given that 3 roots of the equation $z^6 = -64$ are $2\mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$, $2\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$ and $2\mathrm{e}^{-\mathrm{i}\frac{5\pi}{6}}$, express $z^6 + 64$ as a product of three quadratic factors with real coefficients, giving each factor in non-trigonometric form.

(c) Represent all roots of $z^6 = -64$ on an Argand diagram and interpret the geometrical shape formed by joining the roots.

Solution.

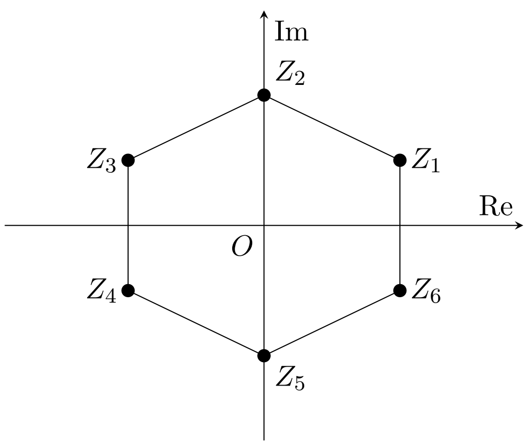
Part (a). Since $P(z)$ has real coefficients, by the conjugate root theorem, $\left(r\mathrm{e}^{\mathrm{i}\theta}\right)^* = r\mathrm{e}^{-\mathrm{i}\theta}$ is also a root of $P(z)$. By the factor theorem, a quadratic factor of $P(z)$ is

$$(z - r\mathrm{e}^{\mathrm{i}\theta})(z - r\mathrm{e}^{-\mathrm{i}\theta}) = z^2 - rz(\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{-\mathrm{i}\theta}) + r^2\mathrm{e}^{\mathrm{i}\theta}\mathrm{e}^{-\mathrm{i}\theta} = z^2 - 2rz\cos\theta + r^2.$$

Part (b). Let $r_1 = r_2 = r_3 = 2$ and $\theta_1 = \pi/6$, $\theta_2 = \pi/2$ and $\theta_3 = -5\pi/6$.

$$\begin{aligned} z^6 + 64 &= \left(z^2 - 2r_1 z\cos\theta_1 + r_1^2\right)\left(z^2 - 2r_2 z\cos\theta_2 + r_2^2\right)\left(z^2 - 2r_3 z\cos\theta_3 + r_3^2\right) \\ &= \left(z^2 - 4z\cos\left(\frac{\pi}{6}\right) + 4\right)\left(z^2 - 4z\cos\left(\frac{\pi}{2}\right) + 4\right)\left(z^2 - 4z\cos\left(-\frac{5}{6}\pi\right) + 4\right) \\ &= \left(z^2 - 2\sqrt{3}z + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right) \end{aligned}$$

Part (c).



The geometrical shape formed is a regular hexagon.

Assignment A10.2

Problem 1. On an Argand diagram, mark and label clearly the points P and Q representing the complex numbers p and q respectively, where

$$p = \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}, \qquad q = 2\cos\frac{\pi}{4} + 2\mathrm{i}\sin\frac{\pi}{4}.$$

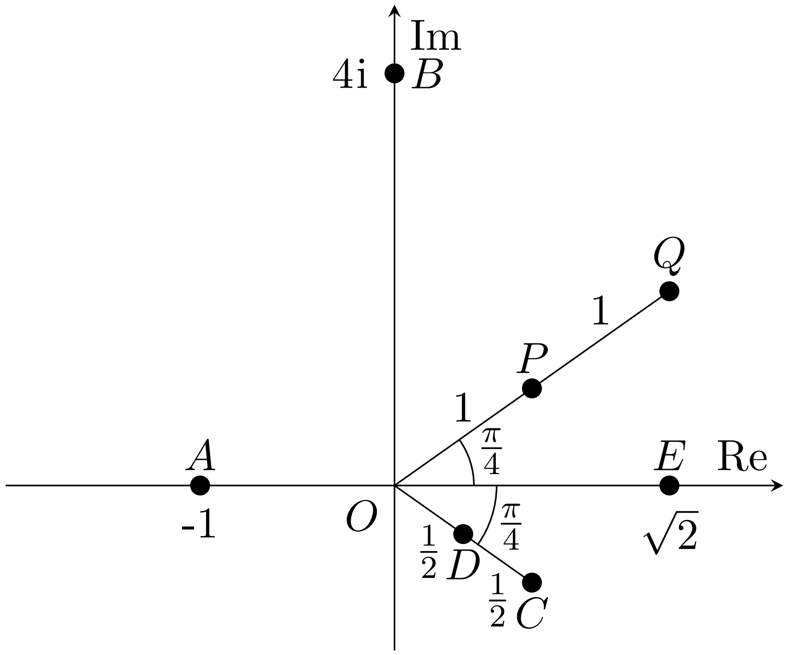
Find the moduli and arguments of the complex numbers a, b, c, d and e, where $a = p^4$, $b = q^2$, $c = -ip$, $d = 1/q$, $e = p + p^*$.

On your Argand diagram, mark and label the points A, B, C, D and E representing these complex numbers.

Find the area of triangle COQ.

Find the modulus and argument of $p^{13/3}q^{45/2}$.

Solution.



Note that $p = \mathrm{e}^{\mathrm{i}\pi/4}$ and $q = 2\mathrm{e}^{\mathrm{i}\pi/4}$.

$$a = p^4 = \left(\mathrm{e}^{\mathrm{i}\pi/4}\right)^4 = \mathrm{e}^{\mathrm{i}\pi}, \quad b = q^2 = \left(2\mathrm{e}^{\mathrm{i}\pi/4}\right)^2 = 4\mathrm{e}^{\mathrm{i}\pi/2}$$

$$c = -\mathrm{i}p = \mathrm{e}^{-\mathrm{i}\pi/2}\mathrm{e}^{\mathrm{i}\pi/4} = \mathrm{e}^{-\mathrm{i}\pi/4}, \quad d = \frac{1}{q} = \frac{1}{2}\mathrm{e}^{-\mathrm{i}\pi/4}$$

$$e = p + p^* = 2\,\mathrm{Re}\,p = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}$$

z	$\lvert z\rvert$	$\arg z$
a	1	π
b	4	$\pi/2$
c	1	$-\pi/4$
d	$1/2$	$-\pi/4$
e	$\sqrt{2}$	0

Since $\angle COQ = \pi/2$, we have $[\triangle COQ] = \frac{1}{2}(2)(1) = 1$ units^2.

We have

$$p^{\frac{13}{3}}q^{\frac{45}{2}} = \left(\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^{\frac{13}{3}}\left(2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}\right)^{\frac{45}{2}} = 2^{\frac{45}{2}}\mathrm{e}^{\mathrm{i}\frac{161\pi}{24}} = 2^{\frac{45}{2}}\mathrm{e}^{\mathrm{i}\frac{17\pi}{24}}.$$

Hence, $\left|p^{13/3}q^{45/2}\right| = \mathrm{e}^{45/2}$ and $\arg\left(p^{13/3}q^{45/2}\right) = \frac{17}{24}\pi$.

* * * * *

Problem 2. The complex number q is given by $q = \frac{\mathrm{e}^{\mathrm{i}2\theta}}{1-\mathrm{e}^{\mathrm{i}2\theta}}$, where $0 < \theta < 2\pi$. In either order,

(a) find the real part of q,

(b) show that the imaginary part of q is $\frac{1}{2}\cot\theta$.

Solution. We have

$$q = \frac{\mathrm{e}^{\mathrm{i}2\theta}}{1-\mathrm{e}^{\mathrm{i}2\theta}} = \frac{\mathrm{e}^{\mathrm{i}\theta}}{\mathrm{e}^{-\mathrm{i}\theta}-\mathrm{e}^{\mathrm{i}\theta}} = \frac{\cos\theta + \mathrm{i}\sin\theta}{-2\mathrm{i}\sin\theta} = -\frac{1}{2} - \frac{1}{2\mathrm{i}}\cot\theta = -\frac{1}{2} + \frac{\mathrm{i}}{2}\cot\theta.$$

Hence, $\operatorname{Re} q = -\frac{1}{2}$ and $\operatorname{Im} q = \frac{1}{2}\cot\theta$.

* * * * *

Problem 3. The complex numbers z and w are such that $z = 4\left(\cos\frac{3}{4}\pi + \mathrm{i}\sin\frac{3}{4}\pi\right)$ and $w = 1 - \mathrm{i}\sqrt{3}$. z^* denotes the conjugate of z.

(a) Find the modulus r and the argument θ of w^2/z^*, where $r > 0$ and $-\pi < \theta < \pi$.

(b) Given that $\left(w^2/z^*\right)^n$ is purely imaginary, find the set of values that n can take.

Solution.
Part (a). Note that $z = 4\mathrm{e}^{\mathrm{i}3\pi/4}$ and $w = 2\left(\frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}\right) = 2\mathrm{e}^{-\mathrm{i}\pi/3}$. Hence,

$$\frac{w^2}{z^*} = \frac{\left(2\mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}\right)^2}{4\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}} = \frac{4\mathrm{e}^{-\mathrm{i}\frac{2\pi}{3}}}{4\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}} = \mathrm{e}^{\mathrm{i}\frac{\pi}{12}}.$$

Thus, $r = 1$ and $\theta = \pi/12$.
Part (b). Note that $\left(w^2/z^*\right)^n = \left(\mathrm{e}^{\mathrm{i}\pi/12}\right)^n = \mathrm{e}^{\mathrm{i}n\pi/12}$. Since $\left(w^2/z^*\right)^n$ is purely imaginary, we have $\arg\left(w^2/z^*\right)^n = \pi/2 + \pi k$, where $k \in \mathbb{Z}$. Thus, $n\pi/12 = \pi/2 + \pi k$, whence $n = 6 + 12k$. Hence, $\{n \in \mathbb{Z} : n = 6 + 12k,\ k \in Z\}$.

* * * * *

Problem 4. The complex number w has modulus $\sqrt{2}$ and argument $\pi/4$ and the complex number z has modulus $\sqrt{2}$ and argument $5\pi/6$.

(a) By first expressing w and z in the form $x + \mathrm{i}y$, find the exact real and imaginary parts of $w + z$.

(b) On the same Argand diagram, sketch the points P, Q, R representing the complex numbers z, w, and $z+w$ respectively. State the geometrical shape of the quadrilateral $OPRQ$.

(c) Referring the Argand diagram in part (b), find $\arg(w + z)$ and show that $\tan\frac{11}{24}\pi = \frac{a+\sqrt{2}}{\sqrt{6}+b}$, where a and b are constants to be determined.

Solution.

Part (a). Note that

$$w = \sqrt{2}\mathrm{e}^{\mathrm{i}\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) = 1 + \mathrm{i}$$

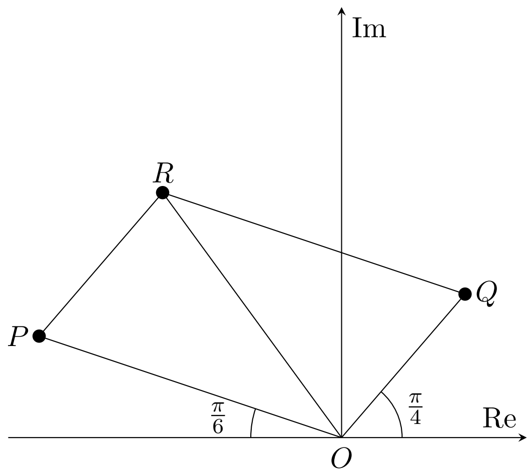
and

$$z = \sqrt{2}\mathrm{e}^{\mathrm{i}5\pi/6} = \sqrt{2}\left(\cos\frac{5}{6}\pi + \mathrm{i}\sin\frac{5}{6}\pi\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) = -\frac{\sqrt{3}}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}.$$

Hence,

$$w + z = (1 + \mathrm{i}) + \left(-\frac{\sqrt{3}}{\sqrt{2}} + \mathrm{i}\frac{1}{\sqrt{2}}\right) = \left(1 - \frac{\sqrt{3}}{\sqrt{2}}\right) + \mathrm{i}\left(1 + \frac{1}{\sqrt{2}}\right).$$

Part (b).



$OPRQ$ is a rhombus.

Part (c). Note that $\angle POQ = \pi - \frac{\pi}{6} - \frac{\pi}{4} = \frac{7}{12}\pi$. Since $|z| = |w|$, we have $OP = OQ$, whence $\angle ROQ = \frac{1}{2} \cdot \frac{7}{12}\pi = \frac{7}{24}\pi$. Hence, $\arg(w + z) = \frac{\pi}{4} + \frac{7}{24}\pi = \frac{13}{24}\pi$. Thus,

$$\tan\left(\frac{13}{24}\pi\right) = \frac{1 + 1/\sqrt{2}}{1 - \sqrt{3}/\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt{2} - \sqrt{3}} = \frac{2 + \sqrt{2}}{2 - \sqrt{6}}$$

However, $\tan\left(\frac{13}{24}\pi\right) = -\tan\left(\pi - \frac{13}{24}\right) = -\tan\left(\frac{11}{24}\pi\right)$. Hence,

$$\tan\left(\frac{11}{24}\pi\right) = -\frac{2 + \sqrt{2}}{2 - \sqrt{6}} = \frac{2 + \sqrt{2}}{\sqrt{6} - 2},$$

whence $a = 2$ and $b = -2$.

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Problem 5. The complex number z is given by $z = 2(\cos\beta + \mathrm{i}\sin\beta)$ where $0 < \beta < \frac{\pi}{2}$.

(a) Show that $\frac{z}{4-z^2} = (k\csc\beta)\mathrm{i}$, where k is positive real constant to be determined.

(b) State the argument of $\frac{z}{4-z^2}$, giving your reasons clearly.

(c) Given the complex number $w = -\sqrt{3} + \mathrm{i}$, find the three smallest positive integer values of n such that $\left(\frac{z}{4-z^2}\right)(w^*)^n$ is a real number.

Solution.

Part (a). Observe that $z = 2\left(\cos\beta + \mathrm{i}\sin\beta\right) = 2\mathrm{e}^{\mathrm{i}\beta}$. Hence,

$$\frac{z}{4-z^2} = \frac{2\mathrm{e}^{\mathrm{i}\beta}}{4-4\mathrm{e}^{\mathrm{i}2\beta}} = \frac{1}{2}\left(\frac{1}{\mathrm{e}^{-\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\beta}}\right) = \frac{1}{2}\left(\frac{1}{-2\mathrm{i}\sin\beta}\right) = \left(\frac{1}{4}\csc\beta\right)\mathrm{i},$$

thus $k = 1/4$.

Part (b). Since $0 < \beta < \pi/2$, we know that $\csc\beta > 0$. Hence, $\operatorname{Im}\left(\frac{z}{4-z^2}\right) > 0$. Furthermore, $\operatorname{Re}\left(\frac{z}{4-z^2}\right) = 0$. Thus, $\arg\left(\frac{z}{4-z^2}\right) = \pi/2$.

Part (c). Note that $w = -\sqrt{3} + \mathrm{i} = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}\mathrm{i}\right) = 2\mathrm{e}^{-\mathrm{i}5\pi/6}$. Hence,

$$\arg\left(\left(\frac{z}{4-z^2}\right)(w^*)^n\right) = \frac{\pi}{2} + n\left(-\frac{5\pi}{6}\right) = \pi\left(\frac{1}{2} - \frac{5n}{6}\right).$$

For $\left(\frac{z}{4-z^2}\right)(w^*)^n$ to be a real number, we require $\arg\left(\left(\frac{z}{4-z^2}\right)(w^*)^n\right) = \pi k$, where $k \in \mathbb{Z}$. Hence,

$$\pi\left(\frac{1}{2} - \frac{5}{6}n\right) = \pi k \implies \frac{1}{2} - \frac{5}{6}n = k \implies 3 - 5n = 6k \implies n \equiv 3 \pmod{6}.$$

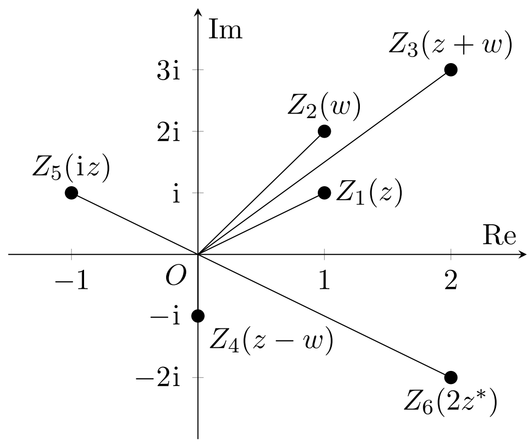
Hence, the three smallest possible values of n are 3, 9 and 15.

A10.3. Complex Numbers - Geometrical Effects and De Moivre's Theorem

Tutorial A10.3

Problem 1. Given that $z = 1 + \mathrm{i}$ and $w = 1 + 2\mathrm{i}$, mark on an Argand diagram, the positions representing: z, w, $z + w$, $z - w$, $\mathrm{i}z$ and $2z^*$.

Solution.



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Problem 2.

(a) Write down the exact values of the modulus and the argument of the complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}$.

(b) The complex numbers z and w satisfy the equation

$$z^2 - zw + w^2 = 0.$$

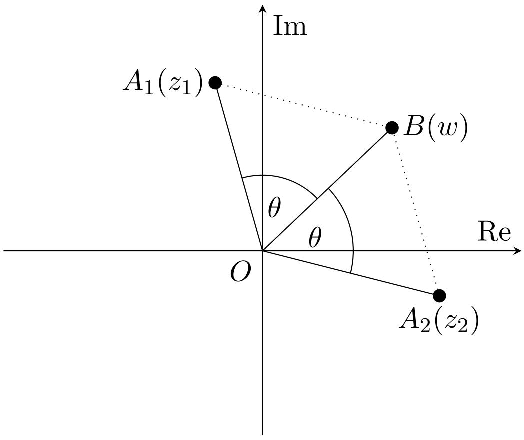
Find z in terms of w. In an Argand diagram, the points O, A and B represent the complex numbers 0, z and w respectively. Show that $\triangle OAB$ is an equilateral triangle.

Solution.

Part (a). We have $r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \implies r = 1$ and $\tan\theta = \frac{\sqrt{3}/2}{1/2} \implies \theta = \frac{\pi}{3}$. Hence, $\left|\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right| = 1$ and $\arg\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i}\right) = \frac{\pi}{3}$.

Part (b). From the quadratic formula, we have

$$z = \frac{w \pm \sqrt{w^2 - 4w^2}}{2} = w\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right).$$



Since $\left|\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right| = 1$, we have that $OB = OA_1 = OA_2$. Further, since $\arg\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}\mathrm{i}\right) = \pm\pi/3$, we know $\angle A_1OB = \angle A_2OB = \pi/3$, whence $\triangle A_1OB$ and $\triangle A_2OB$ are both equilateral.

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Problem 3. Find the exact roots of the equations

(a) $z^3 = 1$

(b) $(z-1)^4 = -16$

in the form $x + \mathrm{i}y$.

Solution.

Part (a). Note that

$$z^3 = 1 = \mathrm{e}^{\mathrm{i}2\pi n} \implies z = \mathrm{e}^{\mathrm{i}2\pi n/3} = \cos\frac{2\pi n}{3} + \mathrm{i}\sin\frac{2\pi n}{3},$$

for $n \in \mathbb{Z}$. Evaluating z in the $n = 0, 1, 2$ cases, we see that the roots of $z^3 = 1$ are

$$z = 1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathrm{i},\ -\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}.$$

Part (b). Note that $(z-1)^4 = -16 = 16\mathrm{e}^{\mathrm{i}\pi(2n+1)}$. Hence,

$$z = 1 + 2\mathrm{e}^{\mathrm{i}\pi(2n+1)/4} = 1 + 2\left[\cos\left(\frac{2n+1}{4}\pi\right) + \mathrm{i}\sin\left(\frac{2n+1}{4}\pi\right)\right],$$

where $n \in \mathbb{Z}$. Evaluating z in the $n = 0, 1, 2, 3$ cases, we see that the roots of $(z-1)^4 = -16$ are

$$z = (1+\sqrt{2}) + \mathrm{i}\sqrt{2},\ (1-\sqrt{2}) + \mathrm{i}\sqrt{2},\ (1-\sqrt{2}) - \mathrm{i}\sqrt{2},\ (1+\sqrt{2}) - \mathrm{i}\sqrt{2}.$$

* * * * *

Problem 4.

(a) Write down the 5 roots of the equation $z^5 - 1 = 0$ in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Show that the roots of the equation $(5+z)^5 - (5-z)^5 = 0$ can be written in the form $5\mathrm{i}\tan\frac{k\pi}{5}$, where $k = 0, \pm1, \pm2$.

Solution.

Part (a). Note that
$$z^5 = 1 = \mathrm{e}^{\mathrm{i}2\pi n} \implies z = \mathrm{e}^{\mathrm{i}2\pi n/5}.$$
Since $-\pi < \theta \leq \pi$, we have
$$z = \mathrm{e}^{-\mathrm{i}4\pi/5},\ \mathrm{e}^{-\mathrm{i}2\pi/5},\ 1,\ \mathrm{e}^{\mathrm{i}2\pi/5},\ \mathrm{e}^{\mathrm{i}4\pi/5}.$$

Part (b). Note that
$$(5+z)^5 - (5-z)^5 = 0 \implies \left(\frac{5+z}{5-z}\right)^5 - 1 = 0 \implies \frac{5+z}{5-z} = \mathrm{e}^{\mathrm{i}2k\pi/5}.$$
Solving for z, we get
$$z = 5\left(\frac{\mathrm{e}^{\mathrm{i}2k\pi/5} - 1}{\mathrm{e}^{\mathrm{i}2k\pi/5} + 1}\right) = 5\left(\frac{\mathrm{e}^{\mathrm{i}k\pi/5} - \mathrm{e}^{-\mathrm{i}k\pi/5}}{\mathrm{e}^{\mathrm{i}k\pi/5} + \mathrm{e}^{-\mathrm{i}k\pi/5}}\right) = 5\left[\frac{2\mathrm{i}\sin(k\pi/5)}{2\cos(k\pi/5)}\right] = 5\mathrm{i}\tan\frac{k\pi}{5}.$$

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Problem 5. De Moivre's theorem for a positive integral exponent states that
$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos n\theta + \mathrm{i}\sin n\theta.$$
Use de Moivre's theorem to show that
$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta.$$
Hence, obtain the roots of the equation
$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$
in the form $\cos q\pi$, where q is a rational number.

Solution. Taking $n = 7$, we have $\cos 7\theta + \mathrm{i}\sin 7\theta = (\cos\theta + \mathrm{i}\sin\theta)^7$, whence $\cos 7\theta = \mathrm{Re}(\cos\theta + \mathrm{i}\sin\theta)^7$. Let $c = \cos\theta$ and $s = \sin\theta$. By the binomial theorem,
$$\cos 7\theta = \mathrm{Re}\,(c + \mathrm{i}s)^7 = \mathrm{Re}\sum_{k=0}^{7}\binom{7}{k}\mathrm{i}^k s^k c^{7-k}.$$
Note that $\mathrm{Re}\, i^k$ is given by
$$\mathrm{Re}\, i^k = \begin{cases} 0, & k = 1, 3 \pmod 4 \\ 1, & k = 0 \pmod 4 \\ -1, & k = 2 \pmod 4 \end{cases}$$
We hence have
$$\begin{aligned}\cos 7\theta &= c^7 - 21c^5s^2 + 35c^3s^4 - 7cs^6 = c^7 - 21c^5\left(1-c^2\right) + 35c^3\left(1-c^2\right)^2 - 7c\left(1-c^2\right)^3 \\ &= 64c^7 - 112c^5 + 56c^3 - 7c = 64\cos^7\theta - 112\cos^5 + 56\cos^3\theta - 7\cos\theta.\end{aligned}$$

Observe that we can manipulate the given equation into
$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0 \implies 64x^7 - 112x^5 + 56x^3 - 7x = -\frac{1}{2}.$$

Under the substitution $x = \cos\theta$, we see that

$$\cos 7\theta = -\frac{1}{2} \implies 7\theta = \frac{2}{3}\pi + 2\pi n \implies \theta = \frac{2\pi}{21}(3n+1),$$

where $n \in \mathbb{Z}$. Taking $0 \leq n < 7$,

$$\begin{aligned} x &= \cos\frac{2\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{20\pi}{21}, \cos\frac{26\pi}{21}, \cos\frac{32\pi}{21}, \cos\frac{38\pi}{21} \\ &= \cos\frac{2\pi}{21}, \cos\frac{4\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{10\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{16\pi}{21}, \cos\frac{20\pi}{21}. \end{aligned}$$

* * * * *

Problem 6. By considering $\sum_{n=1}^{N} z^{2n-1}$, where $z = \mathrm{e}^{\mathrm{i}\theta}$, or by any method, show that

$$\sum_{n=1}^{N} \sin(2n-1)\theta = \frac{\sin^2 N\theta}{\sin\theta},$$

provided $\sin\theta \neq 0$.

Solution. Observe that

$$\sum_{n=1}^{N} \sin(2n-1)\theta = \operatorname{Im}\sum_{n=1}^{N} [\cos(2n-1)\theta + \mathrm{i}\sin(2n-1)\theta] = \operatorname{Im}\sum_{n=1}^{N} z^{2n-1}.$$

Since

$$\begin{aligned} \sum_{n=1}^{N} z^{2n-1} &= \frac{1}{z}\sum_{n=1}^{N} \left(z^2\right)^n = \frac{1}{z}\left(\frac{z^2\left[\left(z^2\right)^N - 1\right]}{z^2 - 1}\right) = \frac{z^{2N} - 1}{z - z^{-1}} \\ &= z^N\left(\frac{z^N - z^{-N}}{z - z^{-1}}\right) = z^N\left(\frac{2\mathrm{i}\sin N\theta}{2\mathrm{i}\sin\theta}\right) = z^N\left(\frac{\sin N\theta}{\sin\theta}\right), \end{aligned}$$

we have

$$\sum_{n=1}^{N} \sin(2n-1)\theta = \left(\frac{\sin N\theta}{\sin\theta}\right)\operatorname{Im}\left(z^N\right) = \left(\frac{\sin N\theta}{\sin\theta}\right)\sin N\theta = \frac{\sin^2 N\theta}{\sin\theta}.$$

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Problem 7. By considering the series $\sum_{n=0}^{N} \left(\mathrm{e}^{2\mathrm{i}\theta}\right)^n$, show that, provided $\sin\theta \neq 0$,

$$\sum_{n=0}^{N} \cos 2n\theta = \frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}$$

and deduce that

$$\sum_{n=0}^{N} \sin^2 n\theta = \frac{N}{2} + \frac{1}{2} - \frac{\sin(N+1)\theta\cos N\theta}{2\sin\theta}.$$

Solution. Let $z = \mathrm{e}^{\mathrm{i}\theta}$. Then

$$\sum_{n=0}^{N} \cos 2n\theta = \operatorname{Re}\sum_{n=0}^{N} (\cos 2n\theta + \mathrm{i}\sin 2n\theta) = \operatorname{Re}\sum_{n=0}^{N} \mathrm{e}^{\mathrm{i}2n\theta} = \operatorname{Re}\sum_{n=0}^{N} \left(z^2\right)^n.$$

Observe that

$$\sum_{n=0}^{N}\left(z^2\right)^n=\frac{\left(z^2\right)^{N+1}-1}{z^2-1}=\frac{z^{N+1}}{z}\left(\frac{z^{N+1}-z^{-(N+1)}}{z-z^{-1}}\right)=z^N\left(\frac{\sin(N+1)\theta}{\sin\theta}\right).$$

Hence,

$$\sum_{n=0}^{N}\cos 2n\theta=\left(\frac{\sin(N+1)\theta}{\sin\theta}\right)\operatorname{Re}\left(z^N\right)=\frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}.$$

Recall that $\cos 2n\theta = 1-2\sin^2 n\theta \implies \sin^2 n\theta=\frac{1}{2}(1-2\cos 2n\theta)$. Thus,

$$\sum_{n=0}^{N}\sin^2 n\theta=\frac{1}{2}\sum_{n=0}^{N}(1-\cos 2n\theta)=\frac{N+1}{2}-\frac{\sin(N+1)\theta\cos N\theta}{2\sin\theta}.$$

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Problem 8. Given that $z=\mathrm{e}^{\mathrm{i}\theta}$, show that $z^k+1/z^k=2\cos k\theta$, $k\in\mathbb{Z}$.

Hence, show that $\cos^8\theta=\frac{1}{128}\left(\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35\right)$.

Find, correct to three decimal places, the values of θ such that $0<\theta<\frac{1}{2}\pi$ and $\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+1=0$.

Solution. Note that

$$\begin{aligned} z^k+\frac{1}{z^k}&=z^k+z^{-k}=\left(\mathrm{e}^{\mathrm{i}\theta}\right)^k+\left(\mathrm{e}^{\mathrm{i}\theta}\right)^{-k}=\mathrm{e}^{\mathrm{i}k\theta}+\mathrm{e}^{-\mathrm{i}k\theta}\\ &=[\cos(k\theta)+\mathrm{i}\sin(k\theta)]+[\cos(-k\theta)+\mathrm{i}\sin(-k\theta)]=2\cos(k\theta). \end{aligned}$$

Observe that

$$\begin{aligned} \cos^8\theta&=\frac{1}{256}(2\cos\theta)^8=\frac{1}{256}(z+z^{-1})^8=\frac{1}{256}z^{-8}\left(z^2+1\right)^8\\ &=\frac{1}{256}\left(z^{-8}+8z^{-6}+28z^{-4}+56z^{-2}+70+56z^2+28z^4+8z^6+z^8\right)\\ &=\frac{1}{128}\left[\left(\frac{z^8+z^{-8}}{2}\right)+8\left(\frac{z^6+z^{-6}}{2}\right)+28\left(\frac{z^4+z^{-4}}{2}\right)+56\left(\frac{z^2+z^{-2}}{2}\right)+\frac{70}{2}\right]\\ &=\frac{1}{128}\left(\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35\right). \end{aligned}$$

Note that we rewrite the equation as

$$\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35=128\cos^8\theta=34.$$

Thus,

$$\cos\theta=\sqrt[8]{\frac{34}{128}}\implies\theta=0.560\ (3\text{ s.f.}).$$

Assignment A10.3

Problem 1.

(a) Solve $z^4 = -4 - 4\sqrt{3}\mathrm{i}$, expressing your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Sketch the roots on an Argand diagram.

(c) Hence, solve $w^4 = -1 + \sqrt{3}\mathrm{i}$, expressing your answers in a similar form.

Solution.

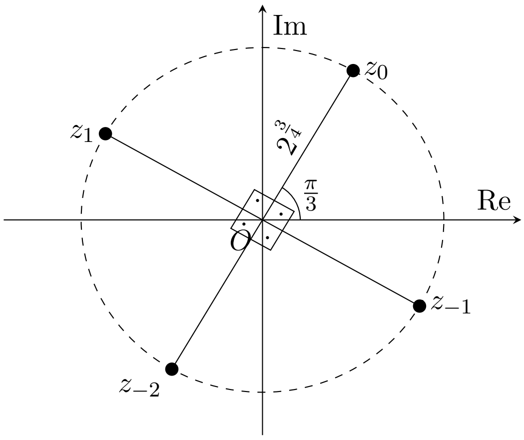
Part (a). Observe that $-4 - 4\sqrt{3}\mathrm{i} = 8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\mathrm{i}\right) = 8\mathrm{e}^{\mathrm{i}\frac{4}{3}\pi + 2k\pi\mathrm{i}}$ for all $k \in \mathbb{Z}$. Hence,

$$z^4 = 8\mathrm{e}^{\mathrm{i}\frac{4}{3}\pi + 2k\pi\mathrm{i}} \implies z = 8^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{3}\pi + \frac{1}{2}k\pi\mathrm{i}} = 2^{\frac{3}{4}}\mathrm{e}^{\mathrm{i}\frac{2+3k}{6}\pi}.$$

Taking $k = -2, -1, 0, 1$, we see that the roots are

$$z_{-2} = 2^{\frac{3}{4}}\mathrm{e}^{-\mathrm{i}\frac{2}{3}\pi}, \quad z_{-1} = 2^{\frac{3}{4}}\mathrm{e}^{-\mathrm{i}\frac{1}{6}\pi}, \quad z_0 = 2^{\frac{3}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{3}\pi}, \quad z_1 = 2^{\frac{3}{4}}\mathrm{e}^{\mathrm{i}\frac{5}{6}\pi}.$$

Part (b).



Part (c). Observe that $w^4 = -1 + \sqrt{3}\mathrm{i} = \frac{1}{4}(-4 + 4\sqrt{3}\mathrm{i}) = 2^{-2}(z^*)^4$. Hence, $w = 2^{-1/2}z^*$. Thus, the roots are

$$w_{-2} = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{2}{3}\pi}, \quad w_{-1} = 2^{\frac{1}{4}}\mathrm{e}^{\mathrm{i}\frac{1}{6}\pi}, \quad w_0 = 2^{\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\frac{1}{3}\pi}, \quad w_1 = 2^{\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\frac{5}{6}\pi}.$$

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Problem 2. Let

$$C = 1 - \binom{2n}{1}\cos\theta + \binom{2n}{2}\cos 2\theta - \binom{2n}{3}\cos 3\theta + \ldots + \cos 2n\theta$$
$$S = -\binom{2n}{1}\sin\theta + \binom{2n}{2}\sin 2\theta - \binom{2n}{3}\sin 3\theta + \ldots + \sin 2n\theta$$

where n is a positive integer.

Show that $C = (-4)^n\cos(n\theta)\sin^{2n}(\theta/2)$, and find the corresponding expression for S.

Solution. Clearly,

$$C = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \cos k\theta, \quad S = \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k \sin k\theta.$$

Hence,

$$\begin{aligned} C + \mathrm{i}S &= \sum_{k=0}^{2n} \binom{2n}{k} (-1)^k (\cos k\theta + \mathrm{i}\sin k\theta) = \sum_{k=0}^{2n} \binom{2n}{k} (-\mathrm{e}^{\mathrm{i}\theta})^k = \left(1 - \mathrm{e}^{\mathrm{i}\theta}\right)^{2n} \\ &= \left(\mathrm{e}^{\mathrm{i}\theta/2}\right)^{2n} \left(\mathrm{e}^{-\mathrm{i}\theta/2} - \mathrm{e}^{\mathrm{i}\theta/2}\right)^{2n} = \mathrm{e}^{\mathrm{i}n\theta} \left(2\mathrm{i}\sin\frac{\theta}{2}\right)^{2n} = \mathrm{e}^{\mathrm{i}n\theta}(-4)^n \sin^{2n}\frac{\theta}{2} \\ &= (\cos n\theta + \mathrm{i}\sin n\theta)(-4)^n \sin^{2n}\frac{\theta}{2}. \end{aligned}$$

Comparing real and imaginary parts, we have

$$C = (-4)^n \cos(n\theta) \sin^{2n}\frac{\theta}{2}, \quad S = (-4)^n \sin(n\theta) \sin^{2n}\frac{\theta}{2}.$$

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Problem 3. Given that $z = \cos\theta + \mathrm{i}\sin\theta$, show that

(a) $z - 1/z = 2\mathrm{i}\sin\theta$,

(b) $z^n + z^{-n} = 2\cos n\theta$.

Hence, show that

$$\sin^6\theta = \frac{1}{32}(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta)$$

Find a similar expression for $\cos^6\theta$, and hence express $\cos^6\theta - \sin^6\theta$ in the form $a\cos 2\theta + b\cos 6\theta$.

Solution.

Part (a). Note that

$$z - \frac{1}{z} = z - z^{-1} = \mathrm{e}^{\mathrm{i}\theta} - \mathrm{e}^{-\mathrm{i}\theta} = [\cos\theta + \mathrm{i}\sin\theta] - [\cos(-\theta) + \mathrm{i}\sin(-\theta)] = 2\mathrm{i}\sin\theta.$$

Part (b). Note that

$$z^n + z^{-n} = \mathrm{e}^{\mathrm{i}n\theta} + e^{-\mathrm{i}n\theta} = [\cos n\theta + \mathrm{i}\sin n\theta] + [\cos(-n\theta) + \mathrm{i}\sin(n\theta)] = 2\cos n\theta.$$

Observe that

$$\begin{aligned} \sin^6\theta &= \frac{1}{(2\mathrm{i})^6}(2\mathrm{i}\sin\theta)^6 = -\frac{1}{64}(z - z^{-1})^6 \\ &= -\frac{1}{64}\left(z^6 - 6z^4 + 15z^2 - 20 + 15z^{-2} - 6z^{-4} + z^{-6}\right) \\ &= -\frac{1}{32}\left[-\frac{20}{2} + 15\left(\frac{z^2 + z^{-2}}{2}\right) - 6\left(\frac{z^4 + z^{-4}}{2}\right) + \left(\frac{z^6 + z^{-6}}{2}\right)\right] \\ &= \frac{1}{32}\left(10 - 15\cos 2\theta + 6\cos 4\theta - \cos 6\theta\right). \end{aligned}$$

Similarly,

$$\begin{aligned}
\cos^6\theta &= \frac{1}{2^6}(2\cos\theta)^6 = \frac{1}{64}(z+z^{-1})^6 \\
&= \frac{1}{64}\left[z^6+6z^4+15z^2+20+15z^{-2}+6z^{-4}+z^{-6}\right] \\
&= \frac{1}{32}\left[\frac{20}{2}+15\left(\frac{z^2+z^{-2}}{2}\right)+6\left(\frac{z^4+z^{-4}}{2}\right)+\left(\frac{z^6+z^{-6}}{2}\right)\right] \\
&= \frac{1}{32}\left(10+15\cos 2\theta+6\cos 4\theta+\cos 6\theta\right).
\end{aligned}$$

Hence,

$$\cos^6\theta-\sin^6\theta = \frac{1}{32}(30\cos 2\theta+2\cos 6\theta) = \frac{15}{16}\cos 2\theta+\frac{1}{16}\cos 6\theta,$$

whence $a = 15/16$ and $b = 1/16$.

A10.4. Complex Numbers - Loci in Argand Diagram

Tutorial A10.4

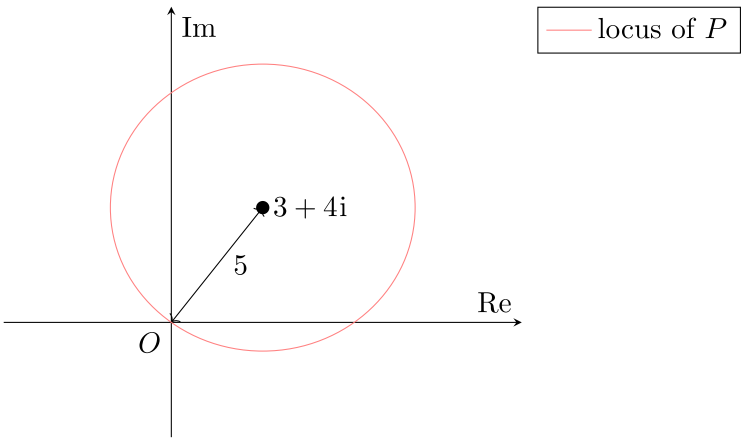
Problem 1. A complex number z is represented in an Argand diagram by the point P. Sketch, on separate Argand diagrams, the locus of P. Describe geometrically the locus of P and determine its Cartesian equation.

(a) $|2z - 6 - 8\mathrm{i}| = 10$

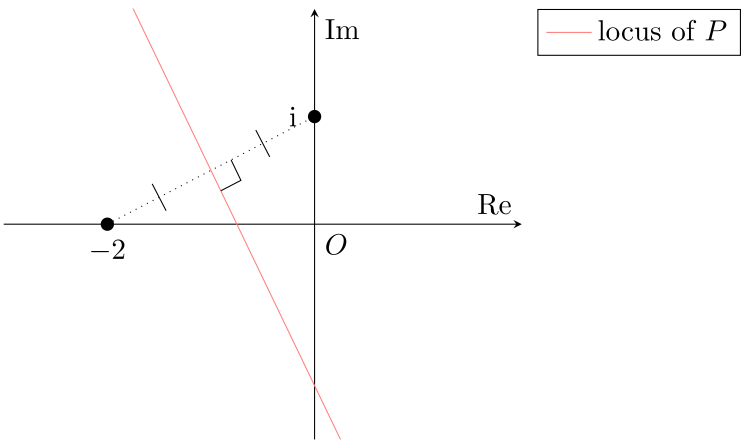
(b) $|z + 2| = |z - \mathrm{i}|$

(c) $\arg(z + 2 - \mathrm{i}) = -\pi/4$

Solution.
Part (a). Note that $|2z - 6 - 8\mathrm{i}| = 10 \implies |z - (3 + 4\mathrm{i})| = 5$.

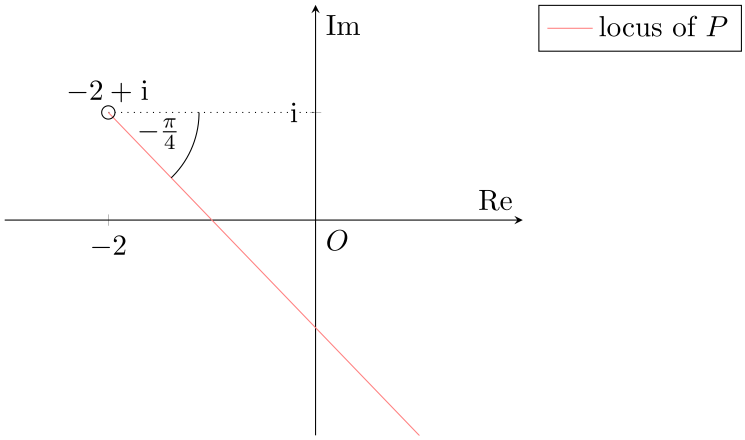


The locus of P is a circle with centre $(3, 4)$ and radius 5. Its Cartesian equation is $(x - 3)^2 + (y - 4)^2 = 5^2$.
Part (b). Note that $|z + 2| = |z - \mathrm{i}| \implies |z - (-2)| = |z - \mathrm{i}|$.



The locus of P is the perpendicular bisector of the line segment joining $(-2,0)$ and $(0,1)$. Its Cartesian equation is $y=-2x-1.5$.

Part (c). Note that $\arg(z+2-\mathrm{i})=-\pi/4 \implies \arg(z-(-2+\mathrm{i}))=-\pi/4$.



The locus of P is the half-line starting from $(-2,1)$ and inclined at an angle $-\pi/4$ to the positive real axis. Its Cartesian equation is $y=-x-1$

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Problem 2. Sketch the following loci on separate Argand diagrams.

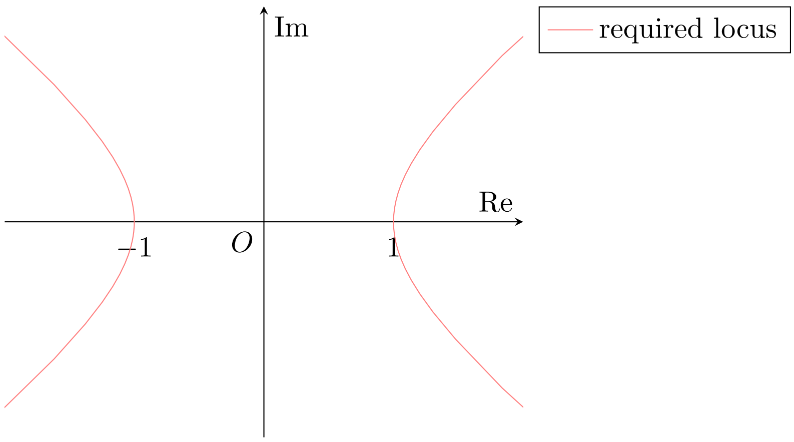
(a) $\operatorname{Re}(z^2)=1$

(b) $|6-\mathrm{i}z|=2$,

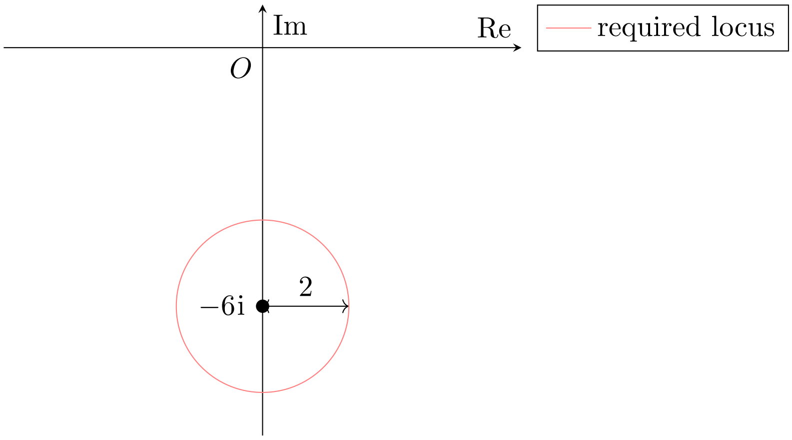
(c) $\arg\left(\frac{\mathrm{i}z}{1-\sqrt{3}\mathrm{i}}\right)=\pi$

Solution.

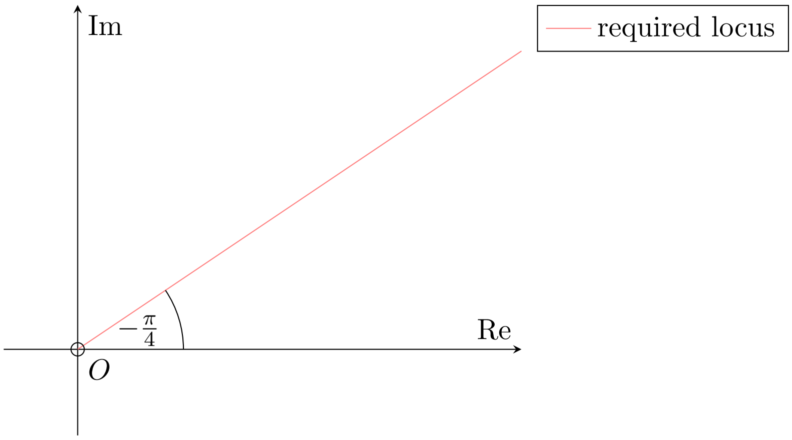
Part (a). Let $z=r(\cos\theta+\mathrm{i}\sin\theta)$. Then $\operatorname{Re}(z^2)=1 \implies r^2\cos 2\theta=1 \implies r^2=\sec 2\theta$.



Part (b). Note $|6-\mathrm{i}z|=2 \implies |-\mathrm{i}(z+6i)|=2 \implies |z+6\mathrm{i}|=2 \implies |z-(6\mathrm{i})|=2$.



Part (c). Note $\arg\left(\frac{\mathrm{i}z}{1-\sqrt{3}\mathrm{i}}\right) = \pi \implies \frac{\pi}{2} + \arg(z) - \left(-\frac{\pi}{3}\right) \implies \arg(z) = \frac{\pi}{6}$.



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Problem 3. Sketch, on separate Argand diagrams, the set of points satisfying the following inequalities.

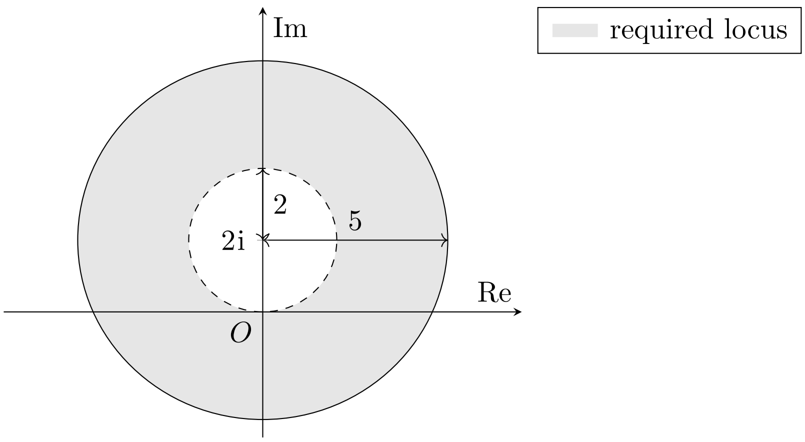
(a) $2 < |z - 2\mathrm{i}| \leq |3 - 4\mathrm{i}|$

(b) $|z + \mathrm{i}| > |z + 1 - \mathrm{i}|$

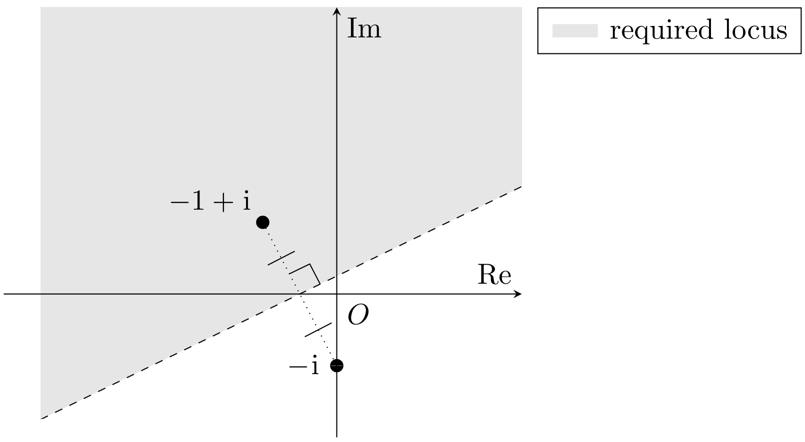
(c) $\frac{\pi}{4} < \arg\left(\frac{1}{z}\right) \leq \frac{\pi}{2}$

Solution.

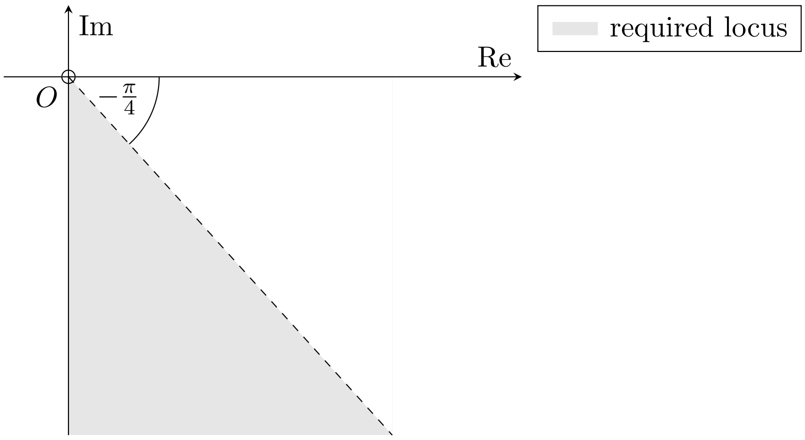
Part (a). Note $2 < |z - 2\mathrm{i}| \leq |3 - 4\mathrm{i}| \implies 2 < |z - 2\mathrm{i}| \leq 5$.



Part (b). Note $|z+\mathrm{i}| > |z+1-\mathrm{i}| \implies |z-(-\mathrm{i})| > |z-(-1+\mathrm{i})|$.



Part (c). Note $\frac{\pi}{4} < \arg\left(\frac{1}{z}\right) \leq \frac{\pi}{2} \implies \frac{\pi}{4} < -\arg(z) \leq \frac{\pi}{2} \implies -\frac{\pi}{2} \geq \arg(z) > -\frac{\pi}{4}$.



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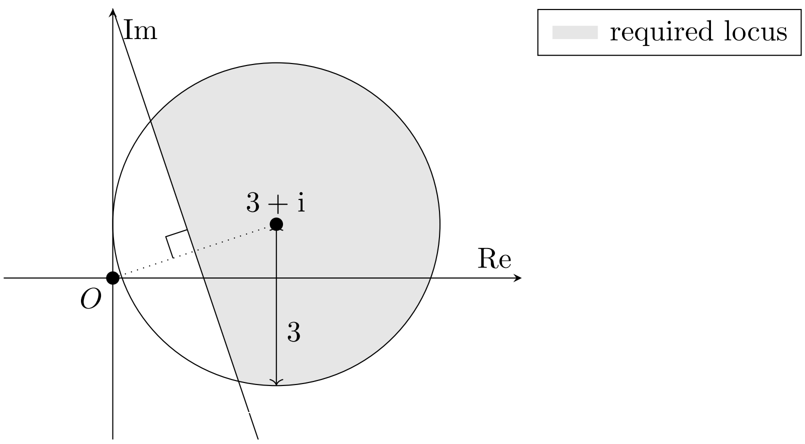
Problem 4. Sketch on separate Argand diagrams for (a) and (b) the set of points representing all complex numbers z satisfying both of the following inequalities.

(a) $|z-3-\mathrm{i}| \leq 3$ and $|z| \geq |z-3-\mathrm{i}|$

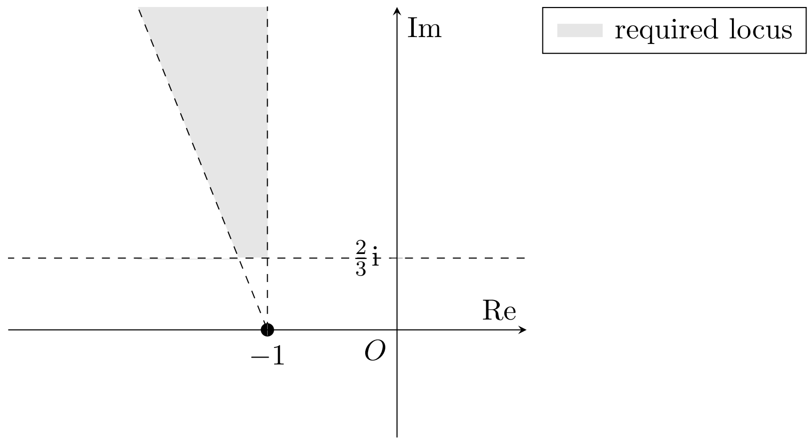
(b) $\frac{\pi}{2} < \arg(z+1) \leq \frac{2}{3}\pi$ and $3\mathrm{Im}(z) > 2$

Solution.

Part (a). Note $|z-3-\mathrm{i}| \leq 3 \implies |z-(3+\mathrm{i})| \leq 3$ and $|z| \geq |z-3-\mathrm{i}| \implies |z| \geq |z-(3+\mathrm{i})|$.



Part (b). Note $\frac{\pi}{2} < \arg(z+1) < \frac{2}{3}\pi \implies \frac{\pi}{2} < \arg(z-(-1)) < \frac{2}{3}\pi$ and $3\mathrm{Im}(z) > 2 \implies \mathrm{Im}(z) > \frac{2}{3}$.



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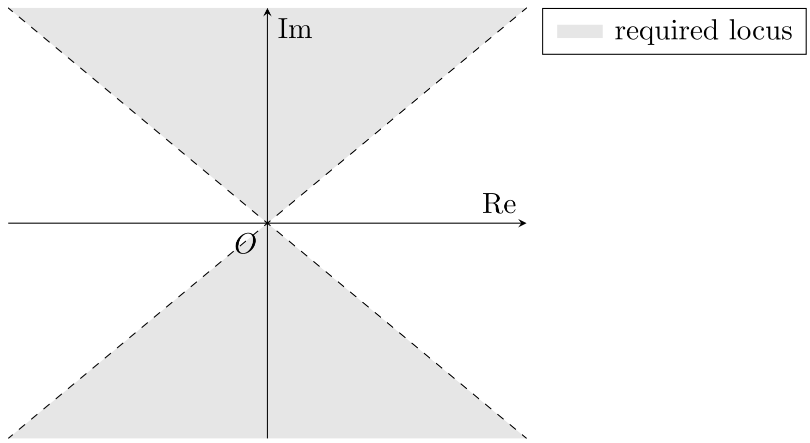
Problem 5. Illustrate, in separate Argand diagrams, the set of points z for which

(a) $\mathrm{Re}(z^2) < 0$

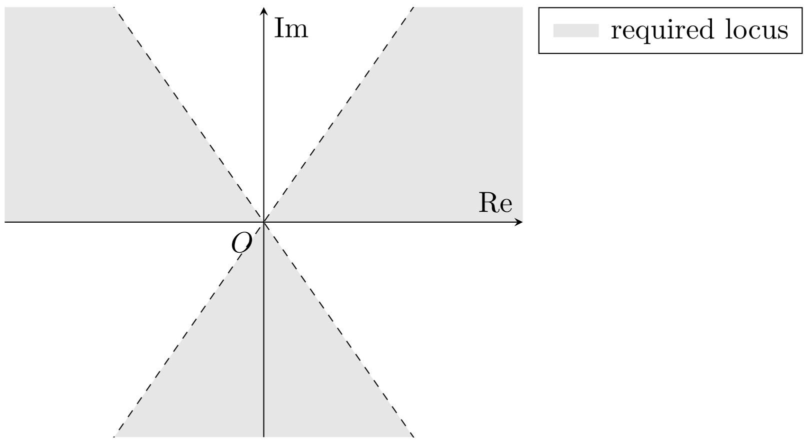
(b) $\mathrm{Im}(z^3) > 0$

Solution.

Part (a). Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $0 \leq \theta < 2\pi$. Then $\mathrm{Re}(z^2) < 0 \implies r^2\cos 2\theta < 0 \implies \cos 2\theta < 0 \implies 2\theta \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right) \cup \left(\frac{5}{2}\pi, \frac{7}{2}\pi\right) \implies \theta \in \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) \cup \left(\frac{5}{4}\pi, \frac{7}{4}\pi\right)$.



Part (b). Let $z = r(\cos\theta + \mathrm{i}\sin\theta)$, $0 \le \theta < 2\pi$. Then $\mathrm{Im}(z^3) > 0 \implies r^3 \sin 3\theta > 0 \implies \sin 3\theta > 0 \implies 3\theta \in (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \implies \theta \in \left(0, \frac{1}{3}\pi\right) \cup \left(\frac{2}{3}\pi, \pi\right) \cup \left(\frac{4}{3}\pi, \frac{5}{3}\pi\right)$.



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Problem 6. The complex number z satisfies $|z + 4 - 4\mathrm{i}| = 3$.

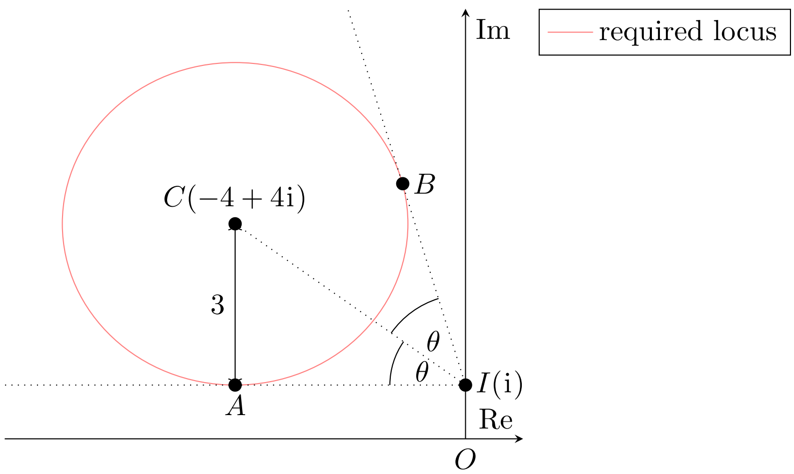
(a) Describe, with the aid of a sketch, the locus of the point which represents z in an Argand diagram.

(b) Find the least possible value of $|z - \mathrm{i}|$.

(c) Find the range of values of $\arg(z - \mathrm{i})$.

Solution.

Part (a). Note $|z + 4 - 4\mathrm{i}| = 3 \implies |z - (-4 + 4\mathrm{i})| = 3$.



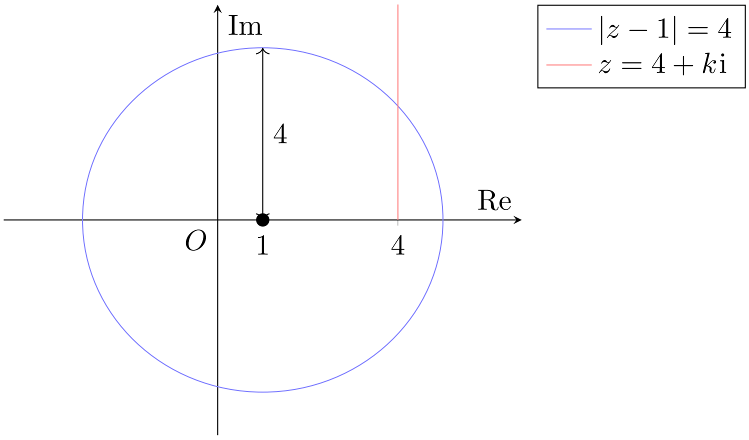
Part (b). Observe that the distance CI is equal to the sum of the radius of the circle and $\min|z - \mathrm{i}|$. Hence,

$$\min|z - \mathrm{i}| = \sqrt{(-4-0)^2 + (4-1)^2} - 3 = 2.$$

Part (c). Let A and B be points on the circle such that AI and BI are tangent to the circle. Let $\angle CIA = \theta$. Then $\tan\theta = \frac{3}{4} \implies \theta = \arctan\frac{3}{4}$. By symmetry, we also have $\angle CIB = \theta$, whence $\angle AIB = 2\theta = 2\arctan\frac{3}{4}$. Hence, $\min \arg(z - \mathrm{i}) = \pi - 2\arctan\frac{3}{4}$ (at B) and $\max \arg(z - \mathrm{i}) = \pi$ (at A). Thus, $\pi - 2\arctan\frac{3}{4} \le \arg(z - \mathrm{i}) \le \pi$.

Problem 7. Sketch, on the same Argand diagram, the two loci representing the complex number z for which $z = 4 + k\mathrm{i}$, where k is a positive real variable, and $|z - 1| = 4$. Write down, in the form $x + \mathrm{i}y$, the complex number satisfying both conditions.

Solution.



Note that z is of the form $4 + k\mathrm{i}$, $k \in \mathbb{R}^+$. Since $|z - 1| = 4$, we have $|3 + k\mathrm{i}| = 4 \implies 3^2 + k^2 = 4 \implies k = \sqrt{7}$. Note that we reject $k = -\sqrt{7}$ since $k > 0$. Thus, $z = 4 + \sqrt{7}\mathrm{i}$.

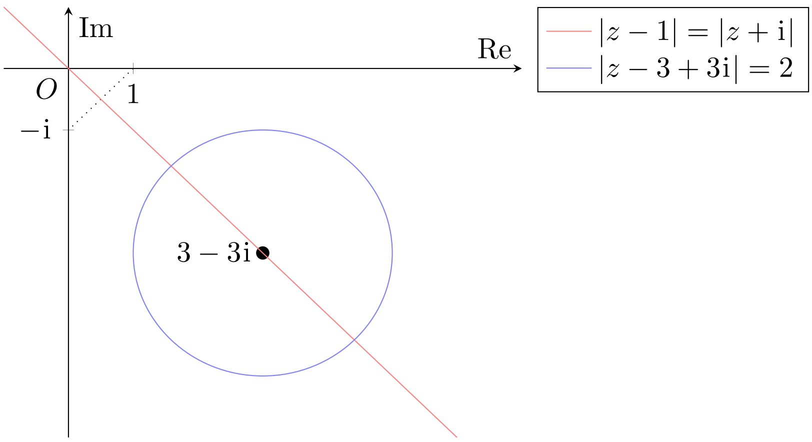
* * * * *

Problem 8. Describe, in geometrical terms, the loci given by $|z - 1| = |z + \mathrm{i}|$ and $|z - 3 + 3\mathrm{i}| = 2$ and sketch both loci on the same diagram.

Obtain, in the form $a + \mathrm{i}b$, the complex numbers representing the points of intersection of the loci, giving the exact values of a and b.

Solution. Note that $|z - 1| = |z + \mathrm{i}| \implies |z - 1| = |z - (-\mathrm{i})|$ and $|z - 3 + 3\mathrm{i}| = 2 \implies |z - (3 - 3\mathrm{i})| = 2$.

The locus given by $|z - 1| = |z + \mathrm{i}|$ is the perpendicular bisector of the line segment joining 1 and $-i$. The locus given by $|z - 3 + 3\mathrm{i}| = 2$ is a circle with centre $3 - 3\mathrm{i}$ and radius 2.



Observe that the locus of $|z - 1| = |z + \mathrm{i}|$ has Cartesian equation $y = -x$ and the locus of $|z - 3 + 3\mathrm{i}| = 2$ has Cartesian equation $(x-3)^2 + (y+3)^2 = 2^2$. Solving both equations simultaneously, we have

$$(x-3)^2 + (y+3)^2 = (x-3)^2 + (3-x)^2 = 2^2 \implies x^2 - 6x + 7 = 0$$
$$\implies x = 3 \pm \sqrt{2} \implies y = -3 \mp \sqrt{2}.$$

Hence, the complex numbers representing the points of intersections of the loci are $(3+\sqrt{2})+(-3-\sqrt{2})\mathrm{i}$ and $(3-\sqrt{2})+(-3+\sqrt{2})\mathrm{i}$.

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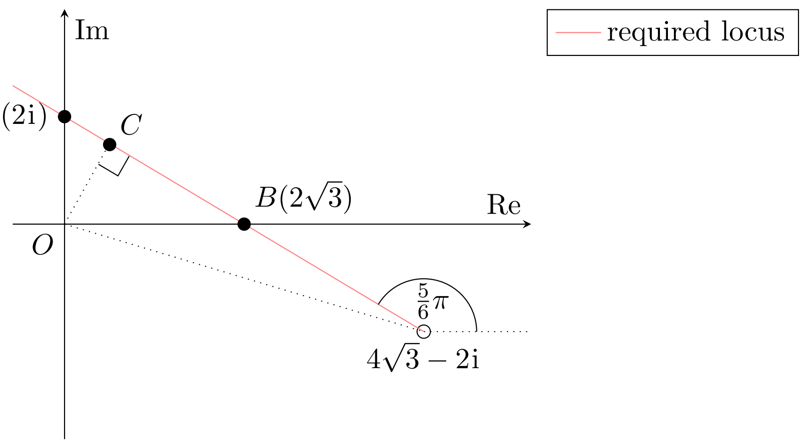
Problem 9. Sketch the locus for $\arg\big(z-(4\sqrt{3}-2\mathrm{i})\big)=\frac{5}{6}\pi$ in an Argand diagram.

(a) Verify that the points $2\mathrm{i}$ and $2\sqrt{3}$ lie on it.

(b) Find the minimum value of $|z|$ and the range of values of $\arg(z)$.

Solution.



Part (a). Note that

$$\arg\Big(2\mathrm{i}-(4\sqrt{3}-2\mathrm{i})\Big)=\arg\Big(-\sqrt{3}+\mathrm{i}\Big)=\arctan\frac{1}{-\sqrt{3}}=\frac{5}{6}\pi$$

and

$$\arg\Big(2\sqrt{3}-(4\sqrt{3}-2\mathrm{i})\Big)=\arg\Big(-\sqrt{3}+\mathrm{i}\Big)=\arctan\frac{1}{-\sqrt{3}}=\frac{5}{6}\pi.$$

Hence, the points $2\mathrm{i}$ and $2\sqrt{3}$ satisfy the equation $\arg\big(z-(4\sqrt{3}-2\mathrm{i})\big)=\frac{5}{6}\pi$ and thus lie on its locus.

Part (b). Let $A(2\mathrm{i})$ and $B(2\sqrt{3})$. Let C be the point on the required locus such that $OC\perp AB$. Observe that $\triangle OAB$, $\triangle COB$ and $\triangle CAO$ are all similar to one another. Hence,

$$\frac{OC}{CB}=\frac{AO}{BO}=\frac{1}{\sqrt{3}}\implies AC=\frac{1}{\sqrt{3}}OC,\quad\frac{OC}{CA}=\frac{BO}{OA}=\frac{\sqrt{3}}{1}\implies BC=\sqrt{3}OC.$$

Hence, $AB=AC+CB=\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right)OC$, whence

$$\min|z|=OC=\frac{AB}{\sqrt{3}+1/\sqrt{3}}=\frac{\sqrt{2^2+(2\sqrt{3})^2}}{\sqrt{3}+1\sqrt{3}}=\frac{4\sqrt{3}}{4}=\sqrt{3}.$$

Observe that $\max\arg(z)=\frac{5}{6}\pi$ and $\min\arg(z)=\min\arg\big(4\sqrt{3}-2\mathrm{i}\big)=\arctan\frac{-2}{4\sqrt{3}}=-\arctan\frac{1}{2\sqrt{3}}$. Thus, $-\arctan\frac{1}{2\sqrt{3}}<\arg(z)\le\frac{5}{6}\pi$.

Problem 10. The complex number z satisfies $|z-3-3\mathrm{i}| \geq |z-1-\mathrm{i}|$ and $\frac{\pi}{6} < \arg(z) \leq \frac{\pi}{3}$.

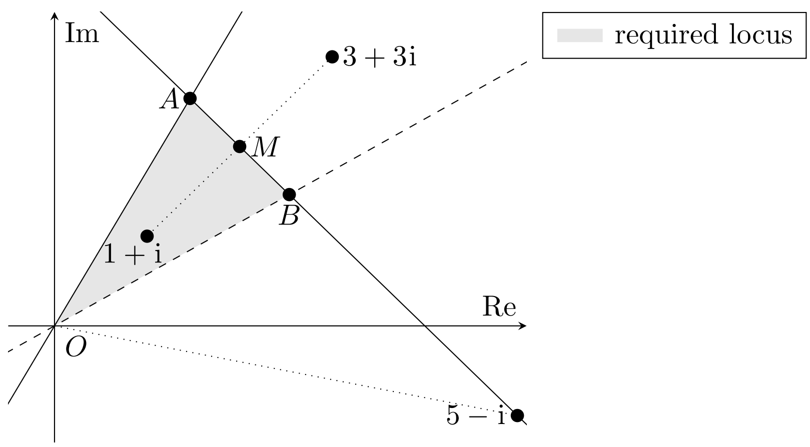
(a) On an Argand diagram, sketch the region in which the point representing z can lie.

(b) Find the area of the region in part (a).

(c) Find the range of values of $\arg(z-5+\mathrm{i})$.

Solution.

Part (a). Note that $|z-3-3\mathrm{i}| \leq |z-1-\mathrm{i}| \implies |z-(3+3\mathrm{i})| \leq |z-(1+\mathrm{i})|$.



Part (b). Note that the locus of $|z-3-3\mathrm{i}| = |z-1-\mathrm{i}|$ has Cartesian equation $y = -x+4$, while the loci of $\frac{\pi}{6} = \arg(z)$ and $\arg(z) = \frac{\pi}{3}$ have Cartesian equations $y = \frac{1}{\sqrt{3}}x$ and $y = \sqrt{3}x$ respectively. Let A and B be the intersections between $y = -x+4$ with $y = \sqrt{3}x$ and $y = \frac{1}{\sqrt{3}}x$ respectively.

At A, we have $y = \sqrt{3}x = -x+4$, whence $A\left(\frac{4}{1+\sqrt{3}}, \frac{4\sqrt{3}}{1+\sqrt{3}}\right)$. Thus,

$$OA = \sqrt{\left(\frac{4}{1+\sqrt{3}}\right)^2 + \left(\frac{4\sqrt{3}}{1+\sqrt{3}}\right)^2} = \frac{8}{1+\sqrt{3}}.$$

By symmetry, we also have $OA = OB$. Finally, since $\angle AOB = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$,

$$[\triangle AOB] = \frac{1}{2}(OA)(OB)\sin\angle AOB = \frac{1}{2}\left(\frac{8}{1+\sqrt{3}}\right)^2 \frac{1}{2} = \frac{16}{\left(1+\sqrt{3}\right)^2} = 4\left(1-\sqrt{3}\right)^2.$$

Part (c). Observe that $\min\arg(z-(5-\mathrm{i})) = \frac{3}{4}\pi$ and $\max\arg(z-(5-\mathrm{i})) = \arctan\frac{-1}{5} + \pi = \pi - \arctan\frac{1}{5}$. Hence, $\frac{3}{4}\pi \leq \arg(z-5+\mathrm{i}) < \pi - \arctan\frac{1}{5}$.

Problem 11. Sketch on an Argand diagram the set of points representing all complex numbers z satisfying both inequalities

$$|\mathrm{i}z - 2\mathrm{i} - 2| \leq 2 \qquad \text{and} \qquad \mathrm{Re}(z) > \left|1 + \sqrt{3}\mathrm{i}\right|$$

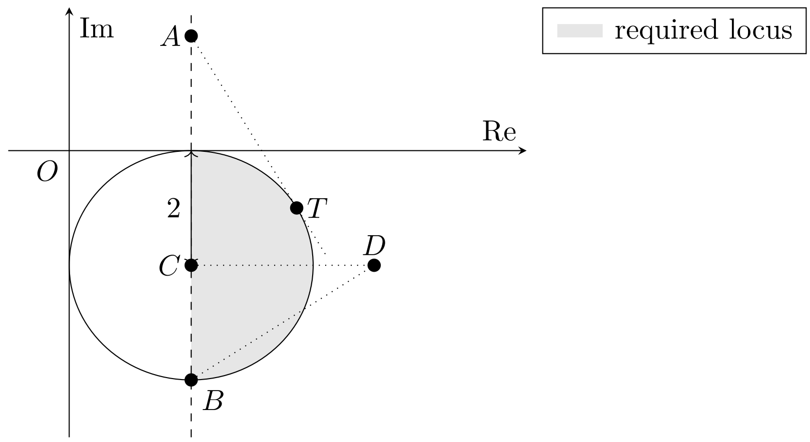
Find

(a) the range of $\arg(z - 2 - 2\mathrm{i})$,

(b) the complex number z where $\arg(z - 2 - 2\mathrm{i})$ is a maximum.

The locus of the complex number w is defined by $|w - 5 + 2\mathrm{i}| = k$, where k is a real and positive constant. Find the range of values of k such that the loci of w and z will intersect.

Solution. Note $|\mathrm{i}z - 2i - 2| \leq 2 \implies |\mathrm{i}(z - 2 + 2i)| \leq 2 \implies |z - (2 - 2\mathrm{i})| \leq 2$ and $\mathrm{Re}(z) > \left|1 + \sqrt{3}\mathrm{i}\right| = 2$.



Part (a). Note $|z - 2 - 2\mathrm{i}| = \arg(z - (2 + 2\mathrm{i}))$. Let $A(2+2\mathrm{i})$ and $C(2-2\mathrm{i})$. Let T be the point at which AT is tangent to the circle. Then $\angle ATC = \frac{\pi}{2}$, $AC = 4$ and $TC = 2$. Hence, $\angle CAT = \arcsin\frac{2}{4} = \frac{\pi}{6}$. Thus, $\min \arg(z - 2 - 2\mathrm{i}) = -\frac{\pi}{2}$ and $\max \arg(z - 2 - 2\mathrm{i}) = \min \arg(z - 2 - 2\mathrm{i}) + \angle CAT = -\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$. Hence, $-\frac{\pi}{2} < \arg(z - 2 - 2i) \leq -\frac{\pi}{3}$.

Part (b). Relative to C, T is given by $2\left(\cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6}\right) = \sqrt{3} + \mathrm{i}$. Thus, $T = (\sqrt{3} + \mathrm{i}) + (2 - 2\mathrm{i}) = 2 + \sqrt{3} - \mathrm{i}$.

Note $|w - 5 + 2\mathrm{i}| = k \implies |w - (5 - 2\mathrm{i})| = k$. Let $D(5 - 2\mathrm{i})$. Observe that CD is given by the sum of the radius of the circle and $\min k$. Hence, $\min k = 3 - 2 = 1$. Let $B(2 - 4\mathrm{i})$. Then $\max k$ is given by the distance between B and D. By the Pythagorean Theorem, we have $\max k = \sqrt{(5-2)^2 + (-2-(-4))^2} = \sqrt{13}$. Thus, $1 \leq k \leq \sqrt{13}$.

Assignment A10.4

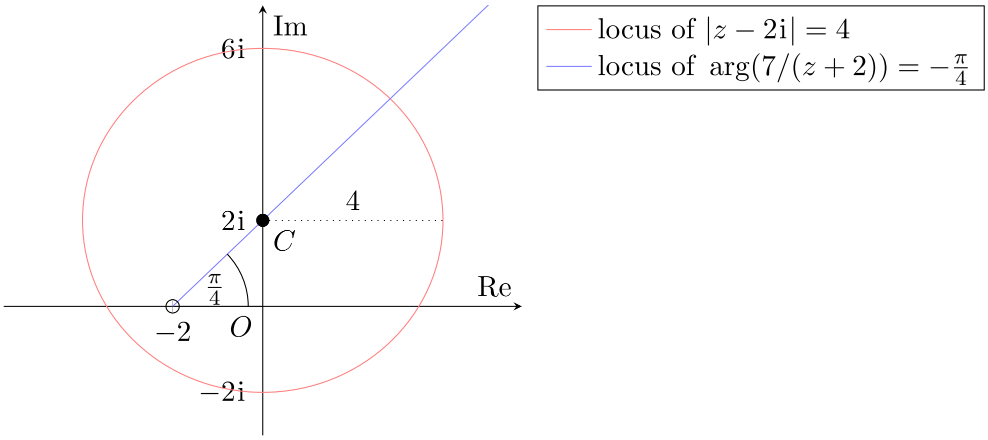
Problem 1. On a single Argand diagram, sketch the following loci.

(a) $|z - 2\mathrm{i}| = 4$.

(b) $\arg\left(\frac{7}{z+2}\right) = -\frac{\pi}{4}$.

Hence, or otherwise, find the exact value of z satisfying both equations in part (a) and (b).

Solution. Note that $\arg\left(\frac{7}{z+2}\right) = -\frac{\pi}{4} \implies \arg(z-(-2)) = \frac{\pi}{4}$.



Solving both equations simultaneously,

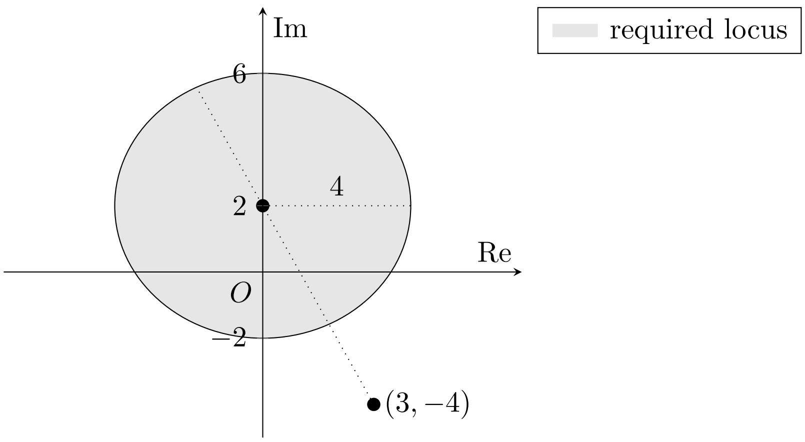
$$z = 2\mathrm{i} + \left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right) = 2\mathrm{i} + \frac{\sqrt{2}}{2} + \mathrm{i}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} + \left(2 + \frac{\sqrt{2}}{2}\right)\mathrm{i}.$$

* * * * *

Problem 2. Given that $|z - 2\mathrm{i}| \le 4$, illustrate the locus of the point representing the complex number z in an Argand diagram.

Hence, find the greatest and least possible value of $|z - 3 + 4\mathrm{i}|$, given that $|z - 2\mathrm{i}| \le 4$.

Solution.



Note that $|z - 3 + 4\mathrm{i}| = |z - (3 - 4\mathrm{i})|$ represents the distance between z and the point $(3, -4)$. By Pythagoras' Theorem, the distance between the centre of the circle $(0, 2)$

and $(3,-4)$ is $\sqrt{(0-3)^2+(2+4)^2} = 3\sqrt{5}$. Hence, $\max|z-3+4\mathrm{i}| = 3\sqrt{5}+4$, while $\min|z-3+4\mathrm{i}| = 3\sqrt{5}-4$. Thus, $\max|z-3+4\mathrm{i}| = 3\sqrt{5}+4$, $\min|z-3+4\mathrm{i}| = 3\sqrt{5}-4$.

* * * * *

Problem 3. The point A on an Argand diagram represents the fixed complex number a, where $0 < \arg a < \frac{\pi}{2}$. The complex numbers z and w are such that $|z-2\mathrm{i}a| = |a|$ and $|w| = |w+\mathrm{i}a|$.

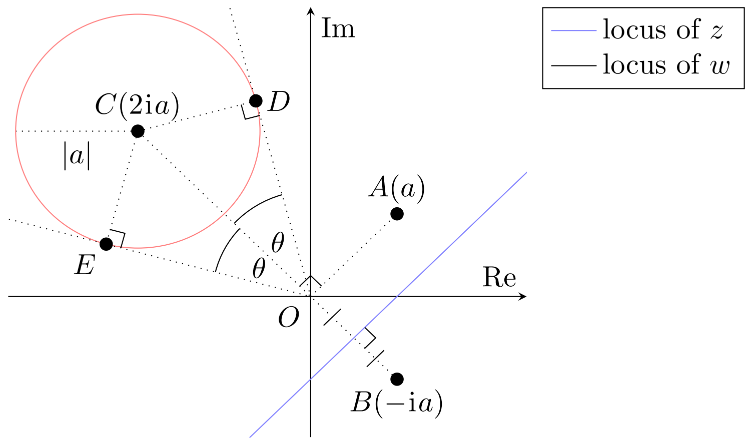
Sketch, on a single diagram, the loci of the point representing z and w.

Find

(a) the minimum value of $|z-w|$ in terms of $|a|$,

(b) the range of values of $\arg\frac{1}{z}$ in terms of $\arg a$.

Solution. Note that $|w| = |w+ia| \implies |w-0| = |w-(-ia)|$.



Part (a). Let $B(-\mathrm{i}a)$ and $C(2\mathrm{i}a)$. Note that $W\left(-\frac{1}{2}\mathrm{i}a\right)$ lies on the locus of w as well as the line passing through OC. Since CW is perpendicular to the locus of w, it follows that the minimum value of $|z-w|$ is given by

$$CW - |a| = \left|2\mathrm{i}a + \frac{1}{2}\mathrm{i}a\right| - |a| = \frac{5}{2}|a|\,|\mathrm{i}| - |a| = \frac{3}{2}|a|.$$

Part (b). Let D and E be such that OD and OE are tangent to the circle given by the locus of z. Let $\angle COD = \theta$. Observe that $\sin\theta = \frac{CD}{CO} = \frac{|a|}{|2\mathrm{i}a|} = \frac{1}{2}$, whence $\theta = \frac{\pi}{6}$. Since $\angle COA = \arg\mathrm{i} = \frac{\pi}{2}$, it follows that $\angle DOA = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arcsin\frac{1}{2} = \frac{\pi}{3}$. Thus, $\min\arg z = \arg a + \angle DOA = \arg a + \frac{\pi}{3}$. Meanwhile, $\angle COE = \angle COD = \theta$, whence $\max\arg z = \arg a + \frac{\pi}{2} + \theta = \arg a + \frac{2}{3}\pi$. Since $\arg\frac{1}{z} = -\arg z$, we thus have $\arg\frac{1}{z} \in \left[-\left(\arg a + \frac{2}{3}\pi\right), -\left(\arg a + \frac{\pi}{3}\right)\right]$.

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Problem 4.

(a) Solve the equation
$$z^7 - (1+\mathrm{i}) = 0,$$
giving the roots in the form $r\mathrm{e}^{\mathrm{i}\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$.

(b) Show the roots on an Argand diagram.

(c) The roots represented by z_1 and z_2 are such that $0 < \arg z_1 < \arg z_2 < \frac{\pi}{2}$. Explain why the locus of all points z such that $|z-z_1| = |z-z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its Cartesian equation.

(d) Describe the transformation that will map the points representing the roots of the equation $z^7 - (1 + \mathrm{i}) = 0$ to the points representing the roots of the equation $(z - 2)^7 - (1 + \mathrm{i}) = 0$ on the Argand diagram.

Solution.

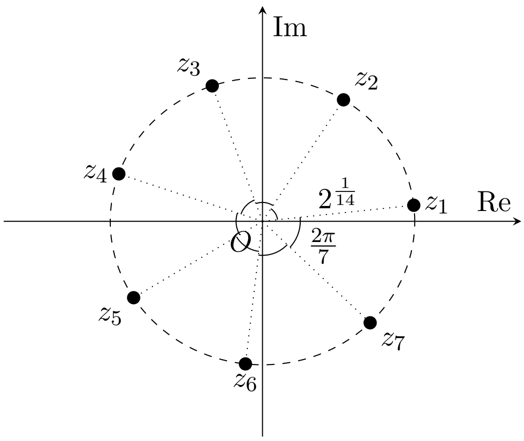
Part (a). Note that $1 + \mathrm{i} = 2^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)}$, where $k \in \mathbb{Z}$. Hence,

$$z^7 = 1 + \mathrm{i} = 2^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)} \implies z = 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi(\frac{1}{4}+2k)/7} = 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi(1+8k)/28}.$$

Taking $k \in \{-3, -2, \ldots, 2, 3\}$, we have

$$z = 2^{\frac{1}{14}} \mathrm{e}^{-\mathrm{i}\pi\frac{23}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{-\mathrm{i}\pi\frac{15}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{-\mathrm{i}\pi\frac{7}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{1}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{9}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{17}{28}},\ 2^{\frac{1}{14}} \mathrm{e}^{\mathrm{i}\pi\frac{25}{28}}.$$

Part (b).



Part (c). Since $|z_1| = |z_2| = 2^{\frac{1}{14}}$, the distance between z_1 and the origin and the distance between z_2 and the origin are equal. Since the locus of $|z - z_1| = |z - z_2|$ represents all points equidistant from z_1 and z_2, it passes through the origin.

Observe that the midpoint of z_1 and z_2 will have argument $\frac{1}{2}\left(\frac{1}{28}\pi + \frac{9}{28}\pi\right) = \frac{5}{28}\pi$. Thus, the Cartesian equation of the locus of z is given by $y = \tan(5\pi/28)x$.

Part (d). Translate the points 2 units in the positive real direction.

A11. Permutations and Combinations

Tutorial A11

Problem 1. In a particular country, the alphabet contains 25 letters. A car registration number consists of two different letters of the alphabet followed by an integer n such that $100 \leq n \leq 999$. Find the number of possible car registration numbers.

Solution. Note that the number of possible n is $999 - 100 + 1 = 900$. Hence, the number of possible car registration numbers is given by ${}^{25}C_2 \cdot 900 = 540000$.

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Problem 2. A girl wishes to phone a friend but cannot remember the exact number. She knows that it is a five-digit number that is even, and that it consists of the digits 2, 3, 4, 5, and 6 in some order. Using this information, find the greatest number of wrong telephone numbers she could try.

Solution. Since the number is odd, there are only 3 possibilities for the last digit. Hence, the maximum wrong numbers she could try is $3 \cdot 4! - 1 = 71$.

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Problem 3. How many ways are there to select a committee of

(a) 3 students

(b) 5 students

out of a group of 8 students?

Solution.
Part (a). There are ${}^{8}C_3 = 56$ ways.
Part (b). There are ${}^{8}C_5 = 56$ ways.

* * * * *

Problem 4. How many ways are there for 2 men, 2 women and 2 children to sit a round table?

Solution. Since the men, women and children are all distinct, there are $(2+2+2-1)! = 120$ ways.

* * * * *

Problem 5. Find the number of different arrangements of the eight letters of the word NONSENSE if

(a) there is no restriction on the arrangement,

(b) the two letters E are together,

(c) the two letters E are not together,

(d) the letters N are all separated,

(e) only two of the letters N are together.

Solution.

Part (a). Note that N, S and E are repeated 3, 2, and 2 times respectively. Thus, the total number of arrangements is given by $\frac{8!}{3!\,2!\,2!} = 1680$.

Part (b). Consider the two E's as one unit. Altogether, there are 7 units. Hence, the required number of arrangements is given by $\frac{7!}{3!\,2!} = 420$.

Part (c). From part (a) and part (b), the required number of arrangements is given by $1680 - 420 = 1260$.

Part (d). There are $\frac{5!}{2!\,2!}$ ways to arrange the non-N letters, and ${}^6C_3$ ways to slot in the 3 N's into the 6 gaps in between the non-N letters. Thus, the required number of arrangements is given by $\frac{5!}{2!\,2!} \cdot {}^6C_3 = 600$.

Part (e). Consider the three N's as one unit. Altogether there are 6 units. Hence, the number of arrangements where all 3 N's are together is given by $\frac{6!}{2!\,2!} = 180$. Thus, from parts (a) and (d), the required number of arrangements is given by $1680-600-180 = 900$.

* * * * *

Problem 6. Find the number of teams of 11 that can be select from a group of 15 players

(a) if there is no restriction on choice,

(b) if the youngest two players and at most one of the oldest two players are to be included.

Solution.

Part (a). The number of teams is given by ${}^{15}C_{11} = 1365$.

Part (b). Given that the youngest two players are always included, we are effectively finding the number of teams of 9 from a group of 13 players with the restriction that at most one of the oldest two players are to be included.

Disregarding the restriction, the total number of teams is given by ${}^{13}C_9 = 715$.

Consider now that number of teams where both of the 2 oldest players are included. This is given by ${}^{11}C_7 = 330$.

Thus, the required number of teams is $715 - 330 = 385$.

* * * * *

Problem 7. A ten-digit number is formed by writing down the digits 0, 1, ..., 9 in some order. No number is allowed to start with 0. Find how many such numbers are

(a) odd,

(b) less than 2 500 000 000.

Solution.

Part (a). Since the number is odd, there are 5 possibilities for the last digit. Furthermore, since no number is allowed to start with 0, there are $10 - 2 = 8$ possibilities for the first digit. The remaining 8 digits are free. Hence, the required number of numbers is $5 \cdot 8 \cdot 8! = 1612800$.

Part (b). *Case 1: Number starts with 1.* Since there are no further restrictions, the number of valid numbers in this case is 9!.

Case 2: Number starts with 2. Given the restriction that the number be less than 2 500 000 000, the second digit must be strictly less than 5, thus giving 4 possibilities for the second digit. The remaining 8 digits are free, for a total number of valid numbers of $4 \cdot 8!$.

Thus, the required number of numbers is $9! + 4 \cdot 8! = 524160$.

* * * * *

Problem 8. Eleven cards each bear a single letter, and together, they can be made to spell the word "EXAMINATION".

(a) Three cards are selected from the eleven cards, and the order of selection is not relevant. Find how many possible selections can be made

(i) if the three cards all bear different letters,

(ii) if two of the three cards bear the same letter.

(b) Two cards bearing the letter N have been taken away. Find the number of different arrangements for the remaining cards that can be made with no two adjacent letters the same.

Solution.

Part (a).

Part (a)(i). Observe that there are 8 distinct letters in "EXAMINATION". Hence, the number of possible selections is ${}^8C_3 = 56$.

Part (a)(ii). Note that there are 3 letters that appear twice in "EXAMINATION". Hence, the number of possible selections is given by ${}^3C_1 \cdot {}^7C_1 = 21$.

Part (b). Note that there are now 2 letters that appear twice, namely A and I. Hence, the total number of possible arrangements is $\frac{9!}{2!\,2!}$.

Consider "AA" and "II" as one unit each. Altogether, there are 7 units. The number of arrangements with two pairs of adjacent letters that are the same is hence given by 7!.

Consider "AA" as one unit, and suppose the two I's are not adjacent to each other. Observe that the non-I letters comprise 6 units, hence giving 6! ways of arranging them. Also observe that there are ${}^7C_2$ ways to slot in the two I's (which guarantee that they are not adjacent to each other). There are hence $6! \cdot {}^7C_2$ possible arrangements in this case. A similar argument follows for the case where the two I's are adjacent but the A's are not.

From the above discussion, it follows that the required number of arrangements is given by $\frac{9!}{2!\,2!} - 7! - 2 \cdot 6! \cdot {}^7C_2 = 55440$.

* * * * *

Problem 9. Find how many three-letter code words can be formed from the letters of the word:

(a) PEAR.

(b) APPLE.

(c) BANANA.

Solution.

Part (a). Since all 4 letters are distinct, the number of code-words is given by ${}^4P_3 = 24$.

Part (b). Tally of letters: 2 'P', 1 'A', 1 'L', 1 'E' (5 letters, 4 distinct).

Case 1: *All letters distinct.* Since there are 4 distinct letters, the number of code-words in this case is ${}^4P_3 = 24$.

Case 2: *2 letters the same, 1 different.* Note that 'P' is the only letter repeated more than once. Reserving two spaces for 'P' leaves one space left for three remaining letters. Hence, there are ${}^1C_1 \cdot {}^3C_1 = 3$ different combinations that can be formed, with $\frac{3!}{2!} = 3$ ways to arrange each combination. Hence, the number of code-words in this case is $3 \cdot 3 = 9$.

Thus, the total number of code-words is $24 + 9 = 33$.

Part (c). Tally of letters: 3 'A', 2 'N', 1 'B' (6 letters, 3 distinct).

Case 1: *All letters distinct.* Since there are only 3 distinct letters, the number of code-words in this case is ${}^3P_3 = 6$.

Case 2: *2 letters the same, 1 different.* Observe that both 'A' and 'N' are repeated more than once. Reserving 2 spaces for either letter leaves one space left for the two remaining letters. Hence, there are ${}^2C_1 \cdot {}^2C_1 = 4$ different combinations that can be formed, with $\frac{3!}{2!} = 3$ ways to arrange each combination. Hence, the number of code-words in this case is $4 \cdot 3 = 12$.

Case 3: *All letters the same.* Observe that 'A' is the only letter repeated thrice. Hence, the number of code-words in this case is 1.

Altogether, the total number of code-words is $6 + 12 + 1 = 19$.

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Problem 10. A group of diplomats is to be chosen to represent three islands, K, L and M. The group is to consist of 8 diplomats and is chosen from a set of 12 diplomats consisting of 3 from K, 4 from L and 5 from M. Find the number of ways in which the group can be chosen if it includes

(a) 2 diplomats from K, 3 from L and 3 from M,

(b) diplomats from L and M only,

(c) at least 4 diplomats from M,

(d) at least 1 diplomat from each island.

Solution.

Part (a). Note that there are ${}^3C_2$ ways to select 2 diplomats from K, ${}^4C_3$ ways to select 3 diplomats from L, and ${}^5C_3$ ways to select 3 diplomats from M. Thus, the number of possible groups is given by ${}^3C_2 \cdot {}^4C_3 \cdot {}^5C_3 = 120$.

Part (b). There are a total of 9 diplomats from L and M. Hence, the number of possible groups is ${}^9C_8 = 9$.

Part (c). *Case 1*: *4 diplomats from M.* Note that there are ${}^5C_4$ combinations for the 4 diplomats from M. Furthermore, since M contributes 4 diplomats, K and L must contribute the other 4 diplomats. Since K and L have a total of 7 diplomats, this gives a total of ${}^5C_4 \cdot {}^7C_4$ possibilities.

Case 2: *5 diplomats from M.* Since M has 5 diplomats, there is only one way for M to send 5 diplomats (all of them have to be chosen). Meanwhile, K and L must contribute the other 3 diplomats from a pool of 7. This gives a total of ${}^7C_3$ possibilities.

Altogether, there are ${}^5C_4 \cdot {}^7C_4 + {}^7C_3 = 210$ total possibilities.

Part (d). Observe that K and M have a total of 8 diplomats. Hence, there is only one possibility where the group only consists of diplomats from K and M.

Since K and L have a total of 7 diplomats, it is impossible for the group to only come from K and L.

From part (b), we know that there are 9 ways where the group consists only of diplomats from L and M.

Note that there are a total of ${}^{12}C_8$ possible ways to choose the group.

Altogether, the required number of possibilities is given by ${}^{12}C_8 - 9 - 1 = 485$.

Problem 11. Alisa and Bruce won a hamper at a competition. The hamper comprises 9 different items.

(a) How many ways can the 9 items be divided among Alisa and Bruce if each of them gets at least one item each?

(b) How many ways can a set of 3 or more items be selected from the 9 items?

Solution.

Part (a). Note that the total number of ways to distribute the items is given by $2^9 = 512$. Also note that the only way either of them does not receive an item is when the other party gets all the items. This can only occur twice (once when Alisa receives nothing, and once when Bruce receives nothing). Thus, the number of ways where both of them gets at least one item each is $512 - 2 = 510$.

Part (b). Observe that the number of ways to choose a set of n items from the original 9 is given by ${}^9C_n$. Hence, the required number of ways is given by $512-({}^9C_0+{}^9C_1+{}^9C_2) = 466$.

* * * * *

Problem 12. In how many ways can 12 different books be distributed among students A, B, C and D

(a) if A gets 5, B gets 4, C gets 2 and D gets 1?

(b) if each student gets 3 books each?

Solution.

Part (a). At the start, A gets to pick 5 books from the 12 available books. There are ${}^{12}C_5$ ways to do so. Next, B gets to pick 4 books from the $12 - 5 = 7$ remaining books. There are ${}^7C_4$ ways to do so. Similarly, there are ${}^3C_2$ ways for C to pick his book, and ${}^1C_1$ ways for D to pick his. Hence, there are a total of ${}^{12}C_5 \cdot {}^7C_4 \cdot {}^3C_2 \cdot {}^1C_1 = 83160$ ways for the 12 books to be distributed.

Part (b). Following a similar argument as in part (a), the number of ways the 12 books can be distributed is given by ${}^{12}C_3 \cdot {}^9C_3 \cdot {}^6C_3 \cdot {}^3C_3 = 369600$.

* * * * *

Problem 13. 3 men, 2 women and 2 children are arranged to sit around a round table with 7 non-distinguishable seats. Find the number of ways if

(a) (i) the 3 men are to be together,

(ii) the 3 men are to be together, and the seats are numbered,

(b) no 2 men are to be adjacent to each other,

(c) only 2 men are adjacent to each other.

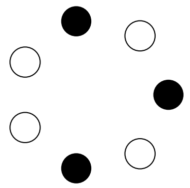
Solution.

Part (a).

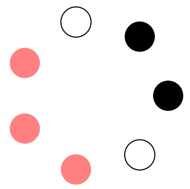
Part (a)(i). Consider the 3 men as one unit. Altogether, there are a total of 5 units, which gives a total of $(5 - 1)! = 4!$ ways for the 5 units to be arranged around the table. Since there are $3!$ ways to arrange the men, there are a total of $4! \cdot 3! = 144$ arrangements.

Part (a)(ii). Since there are a total of 7 distinguishable seats, the total number of arrangements is 7 times that of the number of arrangements with non-distinguishable seats. From part (a), this gives $144 \cdot 7 = 1008$ total arrangements.

Part (b). Observe that there is only one possible layout for no 2 men to be adjacent to each other (as shown in the diagram below). Since there are 4! ways to arrange the non-men, and 3! ways to arrange the men, there are a total of $4! \cdot 3! = 144$ arrangements.



Part (c). Observe that there are 3 possible layouts for only 2 men to be adjacent to each other (as shown in the diagram below). Since there are 4! ways to arrange the non-men, and 3! ways to arrange the men, there are a total of $3 \cdot 4! \cdot 3! = 432$ arrangements.



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Problem 14. Find the number of ways for 4 men and 4 boys to be seated alternately if they sit

(a) in a row,

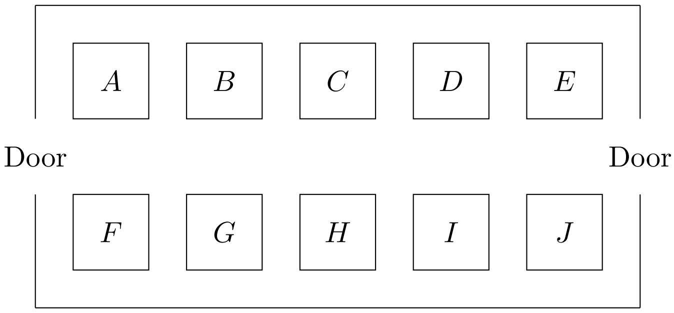
(b) at a round table.

Solution.

Part (a). Note that there are 2 possible layouts: one where a man sits at the start of the row, and one where a boy sits at the start of the row. Since there are 4! ways to arrange both the men and boys, there are a total of $2 \cdot 4! \cdot 4! = 1152$ arrangements.

Part (b). Given the rotational symmetry of the circle, there is now only one possible layout. Fixing one man, there are 3! ways to arrange the other men and 4! ways to arrange the boys, giving a total of $3! \cdot 4! = 144$ arrangements.

Problem 15. A rectangular shed, with a door at each end, contains ten fixed concrete bases marked A, B, C, $\ldots$, J, five on each side (see diagram). Ten canisters, each containing a different chemical, are placed with one canister on each base. In how many ways can the canisters be placed on the bases?



Find the number of ways in which the canisters can be placed

(a) if 2 particular canisters must not be placed on any of the 4 bases A, E, F and J next to a door,

(b) if 2 particular canisters must not be placed next to each other on the same side.

Solution. There are $10! = 3628800$ ways to place the canisters on the bases.

Part (a). Observe that there are ${}^6P_2$ possible placements for the two particular canisters. Since the other 8 canisters have no restrictions, the total number of ways to place the canisters is given by ${}^6P_2 \cdot 8! = 1209600$.

Part (b). Consider the number of ways the two particular canisters can be placed adjacently. There are $2 \cdot (5 - 1) = 8$ possible arrangements per side, giving a total of $2 \cdot 8 = 16$ possible arrangements. Since the other 8 canisters have no restrictions, the total number of ways to place the canisters is given by $16 \cdot 8! = 645120$. The required number of ways is thus given by $3628800 - 645120 = 2983680$.

Assignment A11

Problem 1. Find the number of different arrangements of seven letters in the word ADVANCE. Find the number of these arrangements which begin and end with "A" and in which "C" and "D" are always together.

Find the number of 4-letter code words that can be made from the letters of the word ADVANCE, using

(a) neither of the "A"s,

(b) both of the "A"s.

Solution. Tally of letters: 2 "A"s, 1 "D", 1 "V", 1 "N", 1 "C", 1 "E" (7 total, 6 distinct)

$$\text{Number of different arrangements} = \frac{7!}{2!} = 2520.$$

Since both "A"s are at the extreme ends, we are effectively finding the number of arrangements of the word "DVNCE" such that "C" and "D" are always together.

Let "C" and "D" be one unit. Altogether, there are 4 units. Hence,

$$\text{Required number of arrangements} = 4! \cdot 2 = 48.$$

Part (a). Without both "A"s, there are only 5 available letters to form the code words. This gives ${}^5C_4$ ways to select the 4 letters of the code word. Since each of the 5 remaining letters are distinct, there are 4! possible ways to arrange each word. This gives ${}^5C_4{\cdot}4! = 120$ such code words.

Part (b). With both "A"s included, we need another 2 letters from the 5 non-"A" letters. This gives ${}^5C_2$ ways to select the 4 letters of the code word. Since the 2 non-"A" letters are distinct, but the "A"s are repeated, there are $\frac{4!}{2!}$ possible ways to arrange each code word. This gives ${}^5C_2 \cdot \frac{4!}{2!} = 120$ such code words.

* * * * *

Problem 2. A box contains 8 balls, of which 3 are identical (and so are indistinguishable from one another) and the other 5 are different from each other. 3 balls are to be picked out of the box; the order in which they are picked out does not matter. Find the number of different possible selections of 3 balls.

Solution. Note that there are 6 distinct balls in the box.

Case 1: *No indentical balls chosen.* No. of selections $= {}^6C_3$

Case 2: *2 identical balls chosen.* No. of selections $= {}^5C_1$

Case 3: *3 identical balls chosen.* No. of selections $= {}^3C_3$

Hence, the total number of selections is given by ${}^6C_3 + {}^5C_1 + {}^3C_3 = 26$.

* * * * *

Problem 3. The management board of a company conists of 6 men and 4 women. A chairperson, a secretary and a treasurer are chosen from the 10 members of the board. Find the number of ways the chairperson, the secretary and the treasurer can be chosen so that

(a) they are all women,

(b) at least one is a woman and at least one is a man.

The 10 members of the board sit at random around a round table. Find the number of ways that

(c) the chairperson, the secretary and the treasurer sit in three adjacent places.

(d) the chairperson, the secretary and the treasurer are all separated from each other by at least one other person.

(Extension) What if the seats around the table are numbered? Try parts (c) and (d) again.

Solution.

Part (a). Since there are 4 women and 3 distinct roles, the required number of ways is given by ${}^4P_3 = 24$.

Part (b). Note that the number of ways that all three positions are men is given by ${}^6P_3$, while the number of ways to choose without restriction is given by ${}^{10}P_3$. Hence, the required number of ways is given by ${}^{10}P_3 - {}^6P_3 - 24 = 576$.

Part (c). Consider the three positions as one unit. This gives 8 units altogether. There are hence $(8-1)! \cdot 3! = 30240$ ways.

Part (d). Seat the seven other people first. There are $(7-1)!$ ways to do so. Then, slot in the three positions in the 7 slots. There are ${}^7C_3 \cdot 3!$ ways to do so. Hence, the required number of ways is given by $(7-1)! \cdot {}^7C_3 \cdot 3! = 151200$.

Extension. Since the seats are numbered, the number of ways scales up by the number of seats, i.e. 10. Hence, the number of ways becomes 302400 and 1512000.

A12. Probability

Tutorial A12

Problem 1. A and B are two independent events such that $\mathrm{P}(A) = 0.2$ and $\mathrm{P}(B) = 0.15$. Evaluate the following probabilities.

(a) $\mathrm{P}(A \mid B)$,

(b) $\mathrm{P}(A \cap B)$,

(c) $\mathrm{P}(A \cup B)$.

Solution.
Part (a). Since A and B are independent, $\mathrm{P}(A \mid B) = \mathrm{P}(A) = 0.2$.
Part (b). Since A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = 0.2 \cdot 0.15 = 0.03$.
Part (c). $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.2 + 0.15 - 0.03 = 0.32$.

* * * * *

Problem 2. Two events A and B are such that $\mathrm{P}(A) = \frac{8}{15}$, $\mathrm{P}(B) = \frac{1}{3}$ and $\mathrm{P}(A \mid B) = \frac{1}{5}$. Calculate the probabilities that

(a) both events occur,

(b) only one of the two events occurs,

(c) neither event occurs.

Determine if event A and B are mutually exclusive or independent.

Solution.
Part (a).

$$\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A \mid B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}.$$

Part (b).

$$\begin{aligned}\mathrm{P}(\text{only one occurs}) &= \mathrm{P}(A \cup B) - \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - 2\mathrm{P}(A \cap B) \\ &= \frac{8}{15} + \frac{1}{3} - 2\left(\frac{1}{15}\right) = \frac{11}{15}.\end{aligned}$$

Part (c).

$$\mathrm{P}(\text{neither occurs}) = 1 - \mathrm{P}(\text{at least one occurs}) = 1 - \left(\frac{1}{15} - \frac{11}{15}\right) = \frac{1}{5}.$$

Since $\mathrm{P}(A) = \frac{8}{15} \neq \frac{1}{5} = \mathrm{P}(A \mid B)$, it follows that A and B are not independent. Also, since $\mathrm{P}(A \cap B) = \frac{1}{15} \neq 0$, the two events are also not mutually exclusive.

Problem 3. Two events A and B are such that $\mathrm{P}(A) = \mathrm{P}(B) = p$ and $\mathrm{P}(A \cup B) = \frac{5}{9}$.

(a) Given that A and B are independent, find a quadratic equation satisfied by p.

(b) Hence, find the value of p and the value of $\mathrm{P}(A \cap B)$.

Solution.

Part (a). Since A and B are independent, we have $\mathrm{P}(A \mid B) = \mathrm{P}(A) = p$. Hence,

$$p = \mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)}{\mathrm{P}(B)} = \frac{p + p - 5/9}{p} = 2 - \frac{5}{9p}$$
$$\implies 9p^2 = 18p - 5 \implies 9p^2 - 18p + 5 = 0.$$

Part (b). Observe that $9p^2 - 18p + 5 = (3p-1)(3p-5)$. Thus, $p = \frac{1}{3}$. Note that $p \neq \frac{5}{3}$ since $0 < p \leq 1$.

Since A and B are independent, $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$.

* * * * *

Problem 4. Two players A and B regularly play each other at chess. When A has the first move in a game, the probability of A winning that game is 0.4 and the probability of B winning that game is 0.2. When B has the first move in a game, the probability of B winning that game is 0.3 and the probability of A winning that game is 0.2. Any game of chess that is not won by either player ends in a draw.

(a) Given that A and B toss a fair coin to decide who has the first move in a game, find the probability of the game ending in a draw.

(b) To make their games more enjoyable, A and B agree to change the procedure for deciding who has the first move in a game. As a result of their new procedure, the probability of A having the first move in any game is p. Find the value of p which gives A and B equal chances of winning each game.

Solution.

Part (a).

$$\begin{aligned}\mathrm{P}(\text{draw}) &= \mathrm{P}(A \text{ first})\mathrm{P}(\text{draw} \mid A \text{ first}) + \mathrm{P}(B \text{ first})\mathrm{P}(\text{draw} \mid B \text{ first}) \\ &= 0.5 \cdot (1 - 0.4 - 0.2) + 0.5 \cdot (1 - 0.3 - 0.2) = 0.45.\end{aligned}$$

Part (b). Observe that

$$\begin{aligned}\mathrm{P}(A \text{ wins}) &= \mathrm{P}(A \text{ first})\mathrm{P}(A \text{ wins} \mid A \text{ first}) + \mathrm{P}(B \text{ first})\mathrm{P}(A \text{ wins} \mid B \text{ first}) \\ &= p \cdot 0.4 + (1-p) \cdot 0.2 = 0.2p + 0.2\end{aligned}$$

and

$$\begin{aligned}\mathrm{P}(B \text{ wins}) &= \mathrm{P}(A \text{ first})\mathrm{P}(B \text{ wins} \mid A \text{ first}) + \mathrm{P}(B \text{ first})\mathrm{P}(B \text{ wins} \mid B \text{ first}) \\ &= p \cdot 0.2 + (1-p) \cdot 0.3 = -0.1p + 0.3\end{aligned}$$

Consider $\mathrm{P}(A \text{ wins}) = \mathrm{P}(B \text{ wins})$. Then $0.2p + 0.2 = -0.1p + 0.3 \implies p = \frac{1}{3}$.

Problem 5. Two fair dices are thrown, and events A, B and C are defined as follows:

- A: the sum of the two scores is odd,
- B: at least one of the two scores is greater than 4,
- C: the two scores are equal.

Find, showing your reasons clearly in each case, which two of these three events are

(a) mutually exclusive,

(b) independent.

Find also $\mathrm{P}(C \mid B)$, making your method clear.

Solution.

Part (a). Let the scores of the first and second die be p and q respectively. Suppose A occurs. Then p and q are of different parities (e.g. p even $\implies$ q odd). Thus, p and q cannot be equal. Hence, C cannot occur, whence A and C are mutually exclusive.

Part (b). Let the scores of the first and second die be p and q respectively. Observe that p is independent of q, and vice versa. Hence, the parity of q is not affected by the parity of p. Thus, $\mathrm{P}(A) = \mathrm{P}(p \text{ even})\mathrm{P}(q \text{ odd}) + \mathrm{P}(p \text{ odd})\mathrm{P}(q \text{ even}) = \frac{3}{6} \cdot \frac{3}{6} + \frac{3}{6} \cdot \frac{3}{6} = \frac{1}{2}$.

We also have $\mathrm{P}(B) = 1 - \mathrm{P}(\text{neither } p \text{ nor } q \text{ is greater than } 4) = 1 - \left(\frac{4}{6}\right)^2 = \frac{20}{36}$.

$p\backslash q$	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

We now consider $\mathrm{P}(A \cap B)$. From the table of outcomes above, it is clear that $\mathrm{P}(A \cap B) = \frac{10}{36} = \mathrm{P}(A)\mathrm{P}(B)$. Hence, A and B are independent.

* * * * *

Problem 6. For events A and B, it is given that $\mathrm{P}(A) = 0.7$, $\mathrm{P}(B) = 0.6$ and $\mathrm{P}(A \mid B') = 0.8$. Find

(a) $\mathrm{P}(A \cap B')$,

(b) $\mathrm{P}(A \cup B)$,

(c) $\mathrm{P}(B' \mid A)$.

For a third event C, it is given that $\mathrm{P}(C) = 0.5$ and that A and C are independent.

(d) Find $\mathrm{P}(A' \cap C)$.

(e) Hence find an inequality satisfied by $\mathrm{P}(A' \cap B \cap C)$ in the form

$$p \leq \mathrm{P}\big(A' \cap B \cap C\big) \leq q,$$

where p and q are constants to be determined.

Solution.

Part (a).

$$\mathrm{P}\big(A \cap B'\big) = \mathrm{P}\big(B'\big)\mathrm{P}\big(A \mid B'\big) = (1 - 0.6) \cdot 0.8 = 0.32.$$

Part (b).

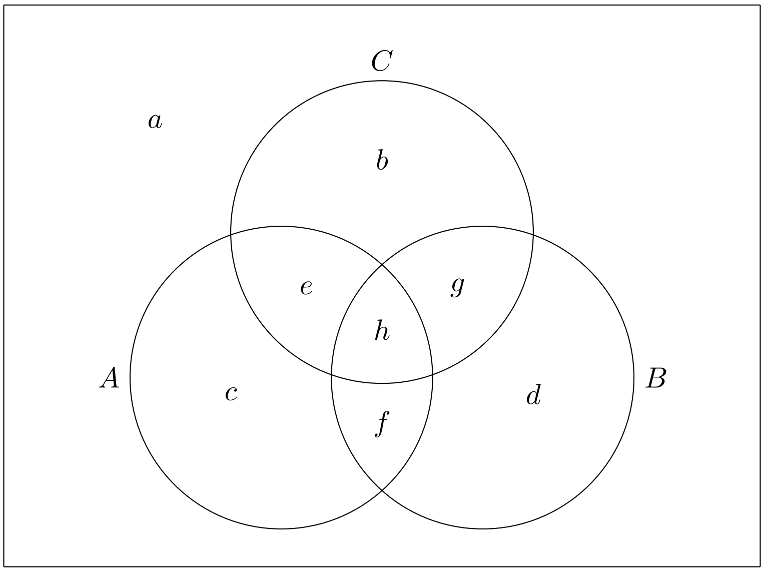
$$\begin{aligned}\mathrm{P}(A \cup B) &= \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \big[\mathrm{P}(A) - \mathrm{P}\big(A \cap B'\big)\big] \\ &= 0.7 + 0.6 - (0.7 - 0.32) = 0.92.\end{aligned}$$

Part (c).

$$\mathrm{P}\big(B' \mid A\big) = \frac{\mathrm{P}(B' \cap A)}{\mathrm{P}(A)} = \frac{0.32}{0.7} = \frac{16}{35}.$$

Part (d). Since A and C are independent, $\mathrm{P}(A \cap C) = \mathrm{P}(A)\mathrm{P}(C)$. Hence, $\mathrm{P}(A' \cap C) = \mathrm{P}(C) - \mathrm{P}(A \cap C) = 0.5 - 0.7 \cdot 0.5 = 0.15$.

Part (e). Consider the following Venn diagram.



Note that $\mathrm{P}(A' \cap B \cap C) = g$. Firstly, from part (d), we have $b + g = \mathrm{P}(A' \cap C) = 0.15$. Hence, $g \leq 0.15$. Secondly, from part (b), we have $a + b = 1 - \mathrm{P}(A \cup B) = 1 - 0.92 = 0.08$. Hence, $b \leq 0.08 \implies g \geq 0.07$. Lastly, we know that $\mathrm{P}(A' \cap B) = \mathrm{P}(A \cup B) - \mathrm{P}(A) = 0.92 - 0.7 = 0.22$. Hence, $d + g = 0.22 \implies g \leq 0.22$.

Thus, $0.07 \leq g \leq 0.15$, whence $0.07 \leq \mathrm{P}(A' \cap B \cap C) \leq 0.15$.

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Problem 7. Camera lenses are made by two companies, A and B. 60% of all lenses are made by A and the remaining 40% by B. 5% of the lenses made by A are faulty. 7% of the lenses made by B are faulty.

(a) One lens is selected at random. Find the probability that

  (i) it is faulty,

  (ii) it was made by A, given that it is faulty.

(b) Two lenses are selected at random. Find the probability that both were made by A, given that exactly one is faulty.

(c) Ten lenses are selected at random. Find the probability that exactly two of them are faulty.

Solution.

Part (a).

Part (a)(i).

$$\mathrm{P}(\text{faulty}) = \mathrm{P}(A \cup \text{faulty}) + \mathrm{P}(B \cup \text{faulty}) = 0.6 \cdot 0.05 + 0.4 \cdot 0.07 = 0.058.$$

Part (a)(ii).

$$\mathrm{P}(A \mid \text{faulty}) = \frac{\mathrm{P}(A \cap \text{faulty})}{\mathrm{P}(\text{faulty})} = \frac{0.6 \cdot 0.05}{0.058} = \frac{15}{19}.$$

Part (b).

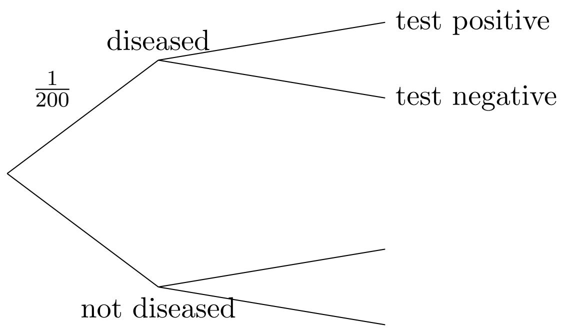
$$\mathrm{P}(\text{both } A \mid \text{one faulty}) = \frac{\mathrm{P}(\text{both } A \cup \text{one faulty})}{\mathrm{P}(\text{one faulty})} = \frac{[0.6 \cdot 0.05] \cdot [0.6 \cdot (1 - 0.05)]}{0.058 \cdot (1 - 0.058)} = \frac{1425}{4553}.$$

Part (c).

$$\mathrm{P}(\text{two faulty}) = 0.058^2 (1 - 0.058)^8 \cdot \frac{10!}{2!8!} = 0.0939 \text{ (3 s.f.)}$$

* * * * *

Problem 8. A certain disease is present in 1 in 200 of the population. In a mass screening programme a quick test fo the disease is used, but the test is not totally reliable. For someone who does have the disease there is a probability of 0.9 that the test will prove positive, whereas for someone who does not have the disease there is a probability of 0.02 that the test will prove positive.



(a) One person is selected at random and test.

   (i) Copy and complete the tree diagram, which illustrates one application of the test.

   (ii) Find the probability that the person has the disease and the test is positive.

   (iii) Find the probability that the test is negative.

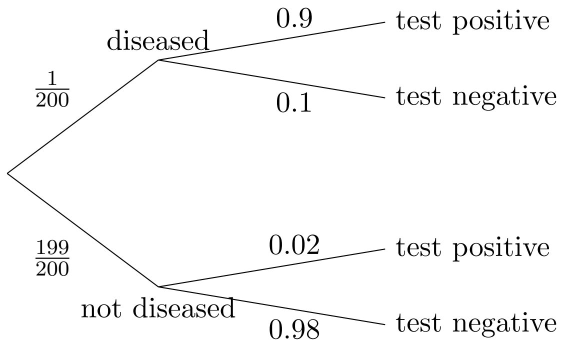
   (iv) Given that the test is positive, find the probability that the person has the disease.

(b) People for whom the test proves positive are recalled and re-tested. Find the probability that a person has the disease if the second test also proves positive.

Solution.

Part (a).

Part (a)(i).



Part (a)(ii).

$$\mathrm{P}(\text{diseased} \cap \text{positive}) = \frac{1}{200} \cdot 0.9 = 0.0045.$$

Part (a)(iii).

$$\mathrm{P}(\text{negative}) = \frac{1}{200} \cdot 0.1 + \frac{199}{200} \cdot 0.98 = 0.9756.$$

Part (a)(iv).

$$\mathrm{P}(\text{diseased} \mid \text{positive}) = \frac{\mathrm{P}(\text{diseased} \cap \text{positive})}{\mathrm{P}(\text{positive})} = \frac{0.0045}{1 - 0.9756} = 0.184.$$

Part (b).

$$\begin{aligned}
\text{Required probability} &= \frac{\mathrm{P}(\text{diseased} \cap \text{both positive})}{\mathrm{P}(\text{both positive})} \\
&= \frac{\mathrm{P}(\text{diseased} \cap \text{both positive})}{\mathrm{P}(\text{diseased} \cap \text{both positive}) + \mathrm{P}(\text{not diseased} \cap \text{both positive})} \\
&= \frac{1/200 \cdot 0.9^2}{1/200 \cdot 0.9^2 + 199/200 \cdot 0.02^2} = \frac{2025}{2224}.
\end{aligned}$$

* * * * *

Problem 9. In a probability experiment, three containers have the following contents.

- A jar contains 2 white dice and 3 black dice.
- A white box contains 5 red balls and 3 green balls.
- A black box contains 4 red balls and 3 green balls.

One die is taken at random from the jar. If the die is white, two balls are taken from the white box, at random and without replacement. If the die is black, two balls are taken from the black box, at random and without replacement. Events W and M are defined as follows:

- W: A white die is taken from the jar.
- M: One red ball and one green ball are obtained.

Show that $\mathrm{P}(M \mid W) = \frac{15}{28}$.

Find, giving each of your answers as an exact fraction in its lowest terms,

(a) $\mathrm{P}(M \cap W)$,

(b) $\mathrm{P}(W \mid M)$,

(c) $\mathrm{P}(W \cup M)$.

All the dice and balls are now placed in a single container, and four objects are taken at random, each object being replaced before the next one is taken. Find the probability that one object of each colour is obtained.

Solution. Since W has occurred, both red and green balls must come from the white box. Note that there are two ways for M to occur: first a red then a green, or first a green then a red. Hence, $\mathrm{P}(M \mid W) = \frac{5}{8} \cdot \frac{3}{7} + \frac{3}{8} \cdot \frac{5}{7} = \frac{15}{28}$ as desired.

Part (a).

$$\mathrm{P}(M \cup W) = \mathrm{P}(W)\mathrm{P}(M \mid W) = \frac{2}{5} \cdot \frac{15}{28} = \frac{3}{14}.$$

Part (b). Let B represent the event that a black die is taken from the jar. Then

$$\begin{aligned}\mathrm{P}(M) = \mathrm{P}(M \cap W) + \mathrm{P}(M \cap B) &= \mathrm{P}(M \cap W) + \mathrm{P}(B)\mathrm{P}(M \mid B) \\ &= \frac{3}{14} + \frac{3}{5}\left(\frac{4}{7} \cdot \frac{3}{6} + \frac{3}{7} \cdot \frac{4}{6}\right) = \frac{39}{70}.\end{aligned}$$

Hence, $\mathrm{P}(W \mid M) = \frac{\mathrm{P}(W \cap M)}{\mathrm{P}(M)} = \frac{3/14}{39/70} = \frac{5}{13}$.

Part (c).

$$\mathrm{P}(W \cup M) = \mathrm{P}(W) + \mathrm{P}(M) - \mathrm{P}(W \cap M) = \frac{2}{5} + \frac{39}{70} - \frac{3}{14} = \frac{26}{35}.$$

Note that the container has 2 white objects, 3 black objects, 9 red objects and 6 green objects, for a total of 20 objects. The probability that one object of each colour is taken is thus given by

$$\frac{2}{20} \cdot \frac{3}{20} \cdot \frac{9}{20} \cdot \frac{6}{20} \cdot 4! = \frac{243}{5000}.$$

* * * * *

Problem 10. A man writes 5 letters, one each to A, B, C, D and E. Each letter is placed in a separate envelope and sealed. He then addresses the envelopes, at random, one each to A, B, C, D and E.

(a) Find the probability that the letter to A is in the correct envelope and the letter to B is in an incorrect envelope.

(b) Find the probability that the letter to A is in the correct envelope, given that the letter to B is in an incorrect envelope.

(c) Find the probability that both the letters to A and B are in incorrect envelopes.

Solution.

Part (a).

$$\mathrm{P}(A \text{ correct} \cap B \text{ incorrect}) = \frac{1}{5} \times \frac{3}{4} = \frac{3}{20}.$$

Part (b).

$$\mathrm{P}(A \text{ correct} \mid B \text{ incorrect}) = \frac{\mathrm{P}(A \text{ correct} \cap B \text{ incorrect})}{\mathrm{P}(B \text{ incorrect})} = \frac{3/20}{4/5} = \frac{3}{16}.$$

Part (c).

$$\begin{aligned}\mathrm{P}(A \text{ incorrect} \cap B \text{ incorrect}) &= \mathrm{P}(B \text{ incorrect})\mathrm{P}(A \text{ incorrect} \mid B \text{ incorrect}) \\ &= \frac{4}{5}\left(1 - \frac{3}{16}\right) = \frac{13}{20}.\end{aligned}$$

* * * * *

Problem 11. A bag contains 4 red counters and 6 green counters. Four counters are drawn at random from the bag, without replacement. Calculate the probability that

(a) all the counters drawn are green,

(b) at least one counter of each colour is drawn,

(c) at least two green counters are drawn,

(d) at least two green counters are drawn, given that at least one counter of each colour is drawn.

State with a reason whether the events "at least two green counters are drawn" and "at least one counter of each colour is drawn" are independent.

Solution.

Part (a).

$$\mathrm{P}(\text{all green}) = \frac{{}^6C_4}{10!/(4!\,6!)} = \frac{1}{14}.$$

Part (b).

$$\mathrm{P}(\text{one of each colour}) = 1 - \mathrm{P}(\text{all green}) - \mathrm{P}(\text{all red}) = 1 - \frac{1}{14} - \frac{{}^4C_4}{10!/(4!\,6!)} = \frac{97}{105}.$$

Part (c).

$$\mathrm{P}(\text{at least 2 green}) = 1 - \mathrm{P}(\text{no green}) - \mathrm{P}(\text{one green}) = 1 - \frac{1}{210} - \frac{{}^6C_1 \cdot {}^4C_3}{10!/(4!\,6!)} = \frac{37}{42}.$$

Part (d).

$$\mathrm{P}(\text{at least 2 green} \mid \text{one of each colour}) = \frac{{}^6C_3 \cdot {}^4C_1 + {}^6C_2 \cdot {}^4C_2}{10!/(4!\,6!) - {}^6C_4 - {}^4C_4} = \frac{85}{97}.$$

Since $\mathrm{P}(\text{at least 2 green}) = \frac{37}{42} \neq \frac{85}{97} = \mathrm{P}(\text{at least 2 green} \mid \text{one of each colour})$, the two events are not independent.

Problem 12. A group of fifteen people consists of one pair of sisters, one set of three brothers and ten other people. The fifteen people are arranged randomly in a line.

(a) Find the probability that the sisters are next to each other.

(b) Find the probability that the brother are not all next to one another.

(c) Find the probability that either the sisters are next to each other or the brothers are all next to one another or both.

(d) Find the probability that the sisters are next to each other given that the brothers are not all next to one another.

Solution.

Part (a). Let the two sisters be one unit. There are hence 14 units altogether, giving $14! \cdot 2!$ arrangements with the restriction. Since there are a total of $15!$ arrangements without the restriction, the required probability is $\frac{14! \cdot 2!}{15!} = \frac{2}{15}$.

Part (b). Consider the case where all brothers are next to one another. Counting the brothers as one unit gives 13 units altogether. There are hence $13! \cdot 3!$ arrangements with this restriction. Since there are a total of $15!$ arrangements without the restriction, the probability that all three brothers are not together is given by $\frac{13! \cdot 3!}{15!} = \frac{34}{35}$.

Part (c). Consider the case where both the sisters are adjacent, and all three brothers are next to one another. Counting the sisters as one unit, and counting the brothers as one unit gives 12 units altogether. There are hence $12! \cdot 2! \cdot 3!$ arrangements with this restriction. Since there are a total of $15!$ arrangements without the restriction, we have

$$\mathrm{P}(\text{sisters together} \cap \text{brothers together}) = \frac{12! \cdot 2! \cdot 3!}{15!} = \frac{2}{455}.$$

Hence,

$$\begin{aligned} &\mathrm{P}(\text{sisters together} \cup \text{brothers together}) \\ &= \mathrm{P}(\text{sisters together}) + \mathrm{P}(\text{brothers together}) - \mathrm{P}(\text{sisters together} \cap \text{brothers together}) \\ &= \frac{2}{15} + \left(1 - \frac{1}{35}\right) - \frac{2}{455} = \frac{43}{273}. \end{aligned}$$

Part (d). Note that

$$\begin{aligned} &\mathrm{P}(\text{sisters together} \cap \text{brothers not together}) \\ &= \mathrm{P}(\text{sisters together}) - \mathrm{P}(\text{sisters together} \cap \text{brothers together}) \\ &= \frac{2}{15} - \frac{2}{455} = \frac{176}{1365}. \end{aligned}$$

Hence, the required probability can be calculated as

$$\begin{aligned} \mathrm{P}(\text{sisters together} \mid \text{brothers not together}) &= \frac{\mathrm{P}(\text{sisters together} \cap \text{brothers not together})}{\mathrm{P}(\text{brothers not together})} \\ &= \frac{176/1365}{34/35} = \frac{88}{663}. \end{aligned}$$

Assignment A12

Problem 1.

(a) Events A and B are such that $\mathrm{P}(A) = 0.4$, $\mathrm{P}(B) = 0.3$ and $\mathrm{P}(A \cup B) = 0.5$.

(i) Determine whether A and B are mutually exclusive.

(ii) Determine whether A and B are independent.

(b) In a competition, 2 teams (A and B) will play each other in the best of 3 games. That is, the first team to win 2 games will be the winner and the competition will end. In the first game, both teams have equal chances of winning. In subsequent games, the probability of team A winning team B given that team A won in the previous game is p and the probability of team A winning team B given that team A lost in the previous game is $\frac{1}{3}$.

(i) Illustrate the information with an appropriate tree diagram.

(ii) Find the value of p such that team A has equal chances of winning and losing the competition.

Solution.

Part (a).

Part (a)(i). Note that

$$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B) = 0.4 + 0.3 - 0.5 = 0.2.$$

Since $\mathrm{P}(A \cap B) = 0.2 \neq 0$, A and B are not mutually exclusive.

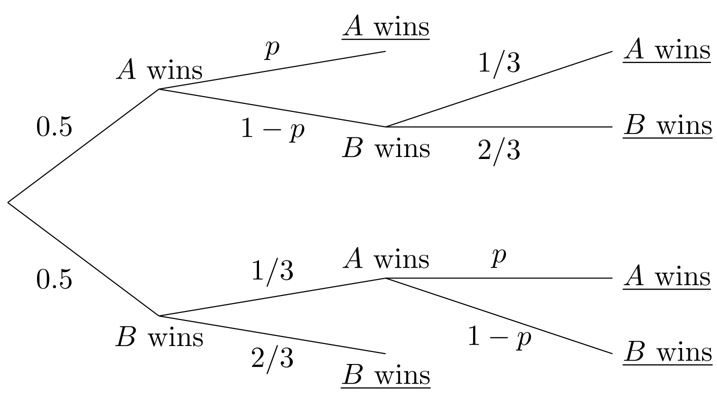
Part (a)(ii). Note that

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{0.2}{0.3} = \frac{2}{3}.$$

Since $\mathrm{P}(A) = 0.4 \neq \frac{2}{3} = \mathrm{P}(A \mid B)$, A and B are not independent.

Part (b).

Part (b)(i).



Part (b)(ii). Consider

$$\mathrm{P}(A \text{ wins competition}) = \left[\frac{1}{2} \cdot p\right] + \left[\frac{1}{2} \cdot (1-p) \cdot \frac{1}{3}\right] + \left[\frac{1}{2} \cdot \frac{1}{3} \cdot p\right] = \frac{p}{2} + \frac{1}{6} = \frac{1}{2}.$$

We hence need $p = \frac{2}{3}$ for A to have equal chances of winning and losing.

Problem 2. A Personal Identification Number (PIN) consists of 4 digits in order, where each digit ranges from 0 to 9. Susie has difficulty remembering her PIN. She tries to remember her PIN and writes down what she thinks it is. The probability that the first digit is correct is 0.8 and the probability that the second digit is correct is 0.86. The probability that the first two digits are both correct is 0.72. Find

(a) the probability that the second digit is correct given that the first digit is correct,

(b) the probability that the first digit is correct, and the second digit is incorrect,

(c) the probability that the second digit is incorrect given that the first digit is incorrect.

Solution. Let $1D$ be the event that the first digit is correct, and $2D$ be the event that the second digit is correct. We have $\mathrm{P}(1D) = 0.8$, $\mathrm{P}(2D) = 0.86$, and $\mathrm{P}(1D \cap 2D) = 0.72$.

Part (a).

$$\mathrm{P}(2D \mid 1D) = \frac{\mathrm{P}(2D \cap 1D)}{\mathrm{P}(1D)} = \frac{0.72}{0.8} = 0.9.$$

Part (b).

$$\mathrm{P}\big(1D \cap 2D'\big) = \mathrm{P}(1D) - \mathrm{P}(1D \cap 2D) = 0.8 - 0.72 = 0.08.$$

Part (c).

$$\mathrm{P}\big(2D' \mid 1D'\big) = \frac{\mathrm{P}(2D' \cap 1D')}{\mathrm{P}(1D')} = \frac{1 - \mathrm{P}(1D \cup 2D)}{1 - \mathrm{P}(1D)}$$

$$= \frac{1 - [\mathrm{P}(1D) + \mathrm{P}(2D) - \mathrm{P}(1D \cap 2D)]}{1 - \mathrm{P}(1D)} = \frac{1 - (0.8 + 0.86 - 0.72)}{1 - 0.8} = 0.3.$$

* * * * *

Problem 3. An international tour group consists of the following seventeen people: a pair of twin sisters and their boyfriends, all from Canada; three policewomen from China; a married couple and their two daughters from Singapore, and a large family from Indonesia, consisting of a man, his wife, his parents and his two sons.

Four people from the group are randomly chosen to play a game. Find the probability that

(a) the four people are all of different nationalities,

(b) the four people are all the same gender,

(c) the four people are all of different nationalities, given that they are all the same gender.

Solution.

TALLY	Male	Female	SUBTOTAL
Canada	2	2	4
China	0	3	3
Singapore	1	3	4
Indonesia	4	2	6
SUBTOTAL	7	10	17

Part (a).

$$\mathrm{P}(\text{all different nationalities}) = \frac{4}{17} \cdot \frac{3}{16} \cdot \frac{4}{15} \cdot \frac{6}{14} \cdot 4! = \frac{72}{595}.$$

Part (b).

$$\mathrm{P}(\text{all same gender}) = \frac{{}^{7}C_4 + {}^{10}C_4}{{}^{17}C_4} = \frac{7}{68}.$$

Part (c).

$$\mathrm{P}(\text{all different nationalities} \mid \text{all female}) = \frac{2}{17} \cdot \frac{3}{16} \cdot \frac{3}{15} \cdot \frac{2}{14} \cdot 4! = \frac{9}{595}$$

Note that $\mathrm{P}(\text{all different nationalities} \mid \text{all male})$ since there are no males from China, whence

$$\begin{aligned}
&\mathrm{P}(\text{all different nationalities} \mid \text{all same gender}) \\
&= \frac{\mathrm{P}(\text{all different nationalities} \cap \text{all same gender})}{\mathrm{P}(\text{all same gender})} = \frac{9/595 + 0}{7/68} = \frac{36}{245}.
\end{aligned}$$

A14. Discrete Random Variables

Tutorial A14A

Problem 1. Alfred and Bertie play a game, each starting with cash amounting to \$100. Two dice are thrown. If the total score if 5 or more, then Alfred pays $\$x$, where $0 < x \leq 8$, to Bertie. If the total score if 4 or less, then Bertie pays $\$(x+8)$ to Alfred.

(a) Show that the expectation of Alfred's cash after the first game is $\$\frac{1}{3}(304 - 2x)$.

(b) Find the expectation of Alfred's cash after six games.

(c) Find the value of x for the game to be fair.

(d) Given that $x = 3$, find the variance of Alfred's cash after the first game.

Solution.

Part (a). Note that

$$\mathrm{P}(\text{score} < 5) = \frac{3+2+1}{6^2} = \frac{1}{6} \implies \mathrm{P}(\text{score} \geq 5) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Let $\$a_n$ be the expectation of Alfred's cash after n games. Suppose Alfred and Bertie play one more game (i.e. $n+1$ total games). Then

$$a_{n+1} = \frac{5}{6}(a_n - x) + \frac{1}{6}(a_n + x + 8) = a_n + \frac{2}{3}(2 - x).$$

a_n is in AP with common difference $\frac{2}{3}(2-x)$ and is thus given by

$$a_n = a_0 + n\left[\frac{2}{3}(2-x)\right] = 100 + \frac{2n}{3}(2-x).$$

Hence, the expectation of Alfred's cash after the first game is

$$a_1 = 100 + \frac{2 \cdot 1}{3}(2-x) = \frac{1}{3}(304 - 2x).$$

Part (b). The expectation of Alfred's cash after six games is

$$a_6 = 100 + \frac{2 \cdot 6}{3}(2-x) = 108 - 4x.$$

Part (c). For the game to be fair, $a_0 = a_1 = a_2 = \cdots$, i.e. the common difference is 0. Hence, $x = 2$.

Part (d). Let the random variable X be Alfred's cash after one game. Since the payouts are unaffected by a_0, we take $a_0 = 0$. When $x = 3$, $\mathrm{E}(X) = -\frac{2}{3}$. Hence,

$$\mathrm{Var}(X) = \frac{5}{6}\left(3 - \frac{2}{3}\right)^2 + \frac{1}{6}\left(3 + 8 + \frac{2}{3}\right)^2 = \frac{245}{9}.$$

Tutorial A14B

Problem 1. In a computer game, a bug moves from left to right through a network of connected paths. The bug starts at S and, at each junction, randomly takes the left fork with probability p or the right fork with probability q, where $q = 1 - p$. The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A - I (see diagram).

(a) Show that the probability that the bug finishes its journey at D is $56p^5q^3$.

(b) Given that the probability that the bug finishes its journey at D is greater than the probability at any one of the other endpoints, find exactly the possible range of values of p.

In another version of the game, the probability that, at each junction, the bug takes the left fork is $0.9p$, the probability that the bug takes the right fork is $0.9q$ and the probability that the bug is swallowed up by a 'black hole' is 0.1.

(c) Find the probability that, in this version of the game, the bug reaches one of the endpoints A - I, without being swallowed up by a black hole.

Solution.

Part (a). Relabel each endpoint from A - I to 0 - 8. Let the random variable X be the end-point that the bug ends up at. Clearly, to reach endpoint i, the bug must take i right forks and $8 - i$ left forks. Hence, $X \sim \mathrm{B}(8, q)$ and the probability that the bug reaches endpoint 3 (i.e. endpoint D) is

$$\mathrm{P}(X = 3) = \binom{8}{3} q^3 (1-q)^{8-3} = 56p^5q^3.$$

Part (b). Since X follows a binomial distribution, it suffices to find the range of values of p that satisfy $\mathrm{P}(X = 2) < \mathrm{P}(X = 3) > \mathrm{P}(X = 4)$.

Case 1: $\mathrm{P}(X = 2) < \mathrm{P}(X = 3)$. Note that $\mathrm{P}(X = 2) = \binom{8}{2} q^2(1-q)^{8-2} = 28p^6q^2$.

$$\mathrm{P}(X = 2) < \mathrm{P}(X = 3) \implies 28p^6q^2 < 56p^5q^3 \implies 28p < 56(1-p) \implies p < \frac{2}{3}.$$

Case 2: $\mathrm{P}(X = 3) > \mathrm{P}(X = 4)$. Note that $\mathrm{P}(X = 4) = \binom{8}{4} q^4(1-q)^{8-4} = 70p^4q^4$.

$$\mathrm{P}(X = 3) > \mathrm{P}(X = 4) \implies 56p^5q^3 > 70p^4q^4 \implies 56p > 70(1-p) \implies p > \frac{5}{9}.$$

Hence, $\frac{5}{9} < p < \frac{2}{3}$.

Part (c). Note that the bug most take a total of 8 forks. Since the probability of not getting swallowed by a black hole at each fork is 0.9, the desired probability is clearly $0.9^8 = 0.430$ (3 s.f.).

Part II.

Group B

B1. Graphs and Transformations I

Tutorial B1A

Problem 1. Without using a calculator, sketch the following graphs and determine their symmetries.

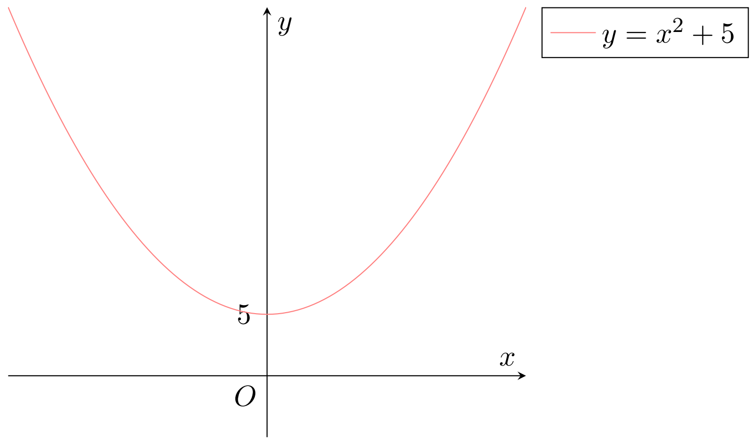
(a) $y = x^2 + 5$

(b) $y = 2x - x^3$

(c) $y = x^2 - 4x + 3$

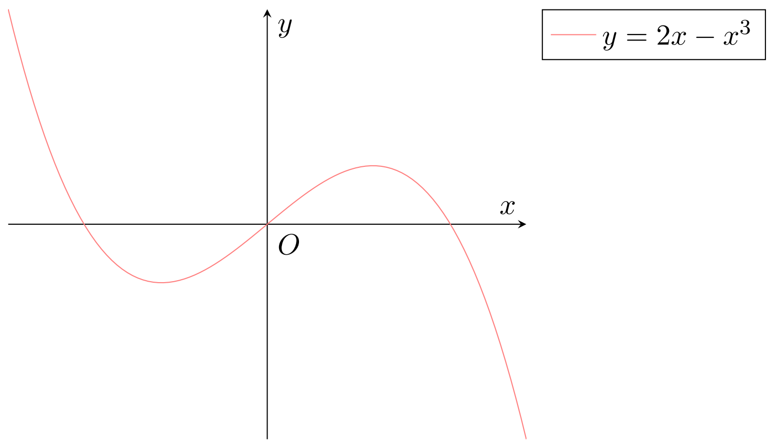
Solution.

Part (a).



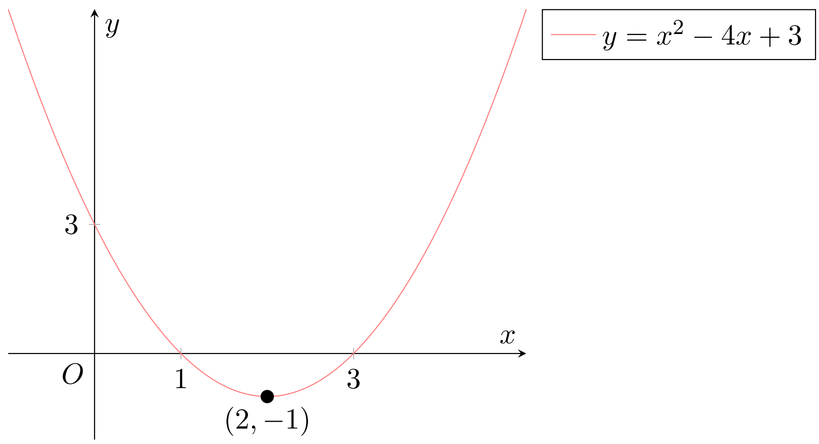
Symmetry: $x = 0$.

Part (b).



Symmetry: $(0, 0)$.

Part (c).



Symmetry: $x = 2$.

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Problem 2. Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

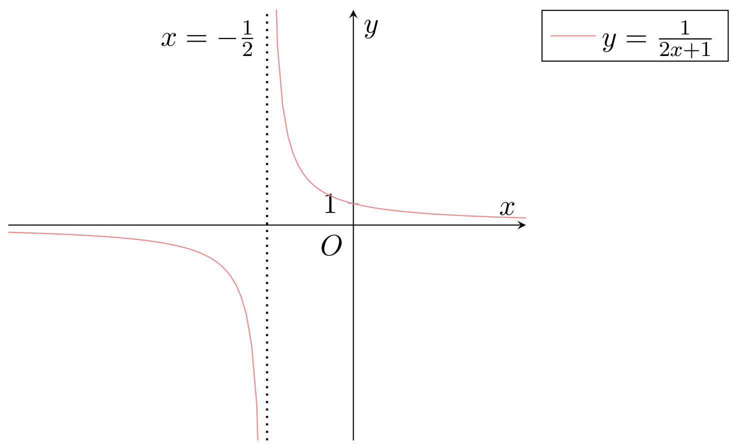
(a) $y = \frac{1}{2x+1}$

(b) $y = \frac{3x}{x-2}$

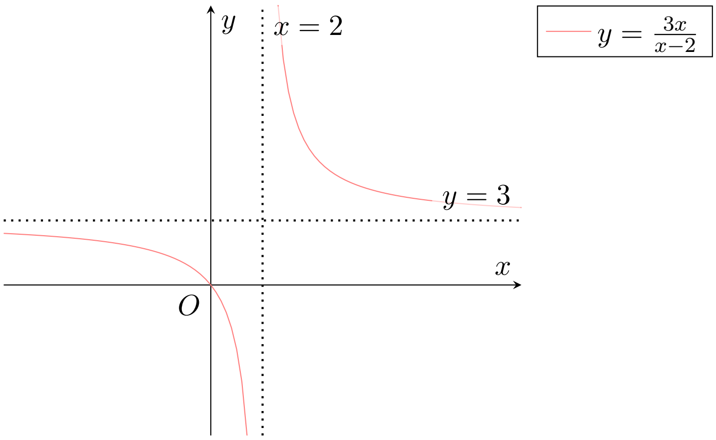
(c) $y = \frac{x^2+x-6}{2x-2}$

Solution.

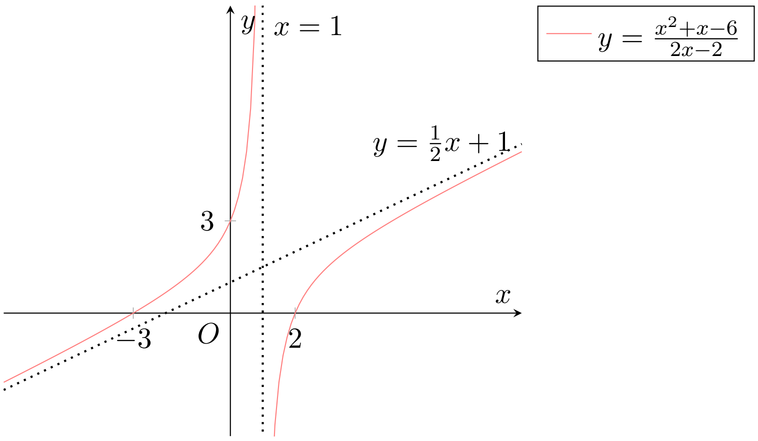
Part (a).



Part (b).



Part (c).



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Problem 3. Sketch the following graphs

(a) $x^2 + 2x + 2y + 4 = 0$

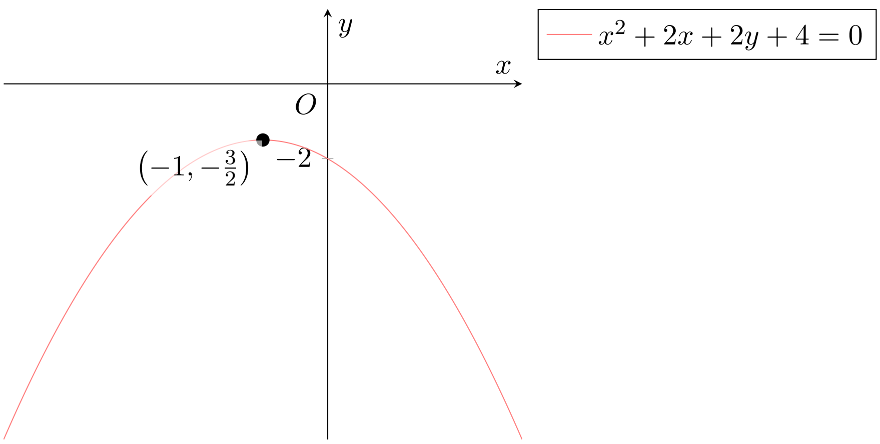
(b) $y^2 = x - 9$

(c) $y^2 = (x - 2)^4 + 5$

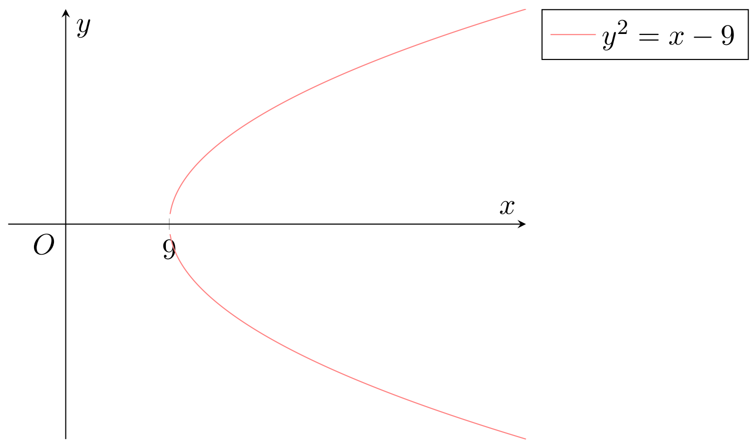
(d) $y = \tan\left(\frac{1}{2}x\right)$, $-2\pi \leq x \leq 2\pi$

Solution.

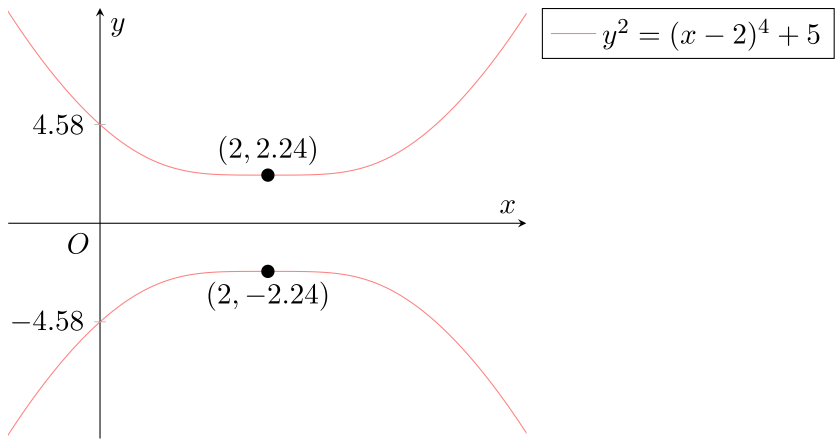
Part (a).



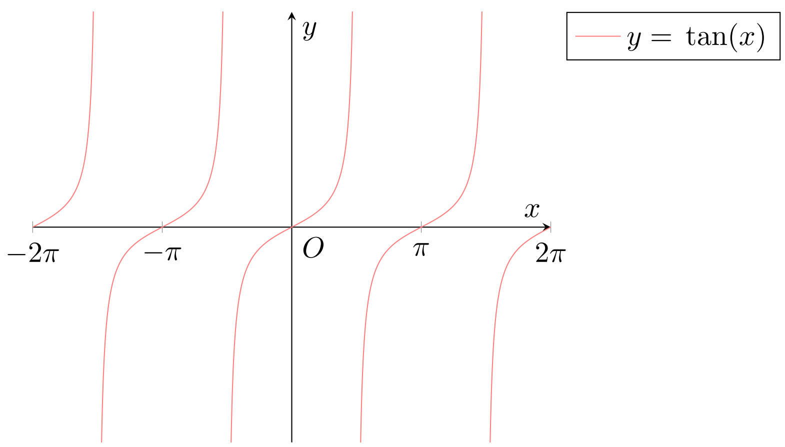
Part (b).



Part (c).



Part (d).



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Problem 4. Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

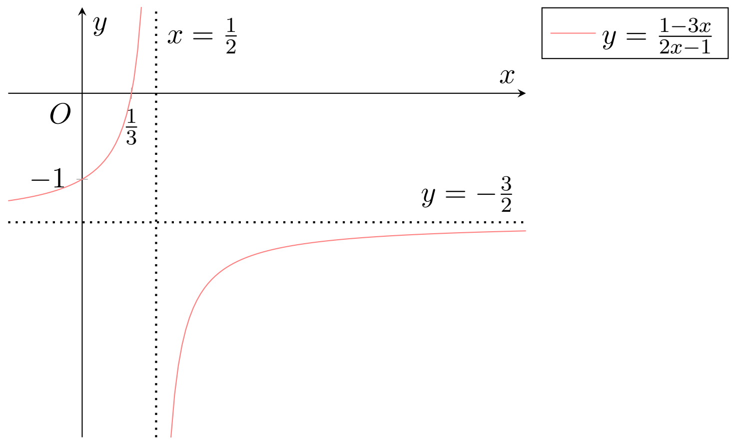
(a) $y = \frac{1-3x}{2x-1}$

(b) $y = \frac{ax}{x-a}$, $a < 0$

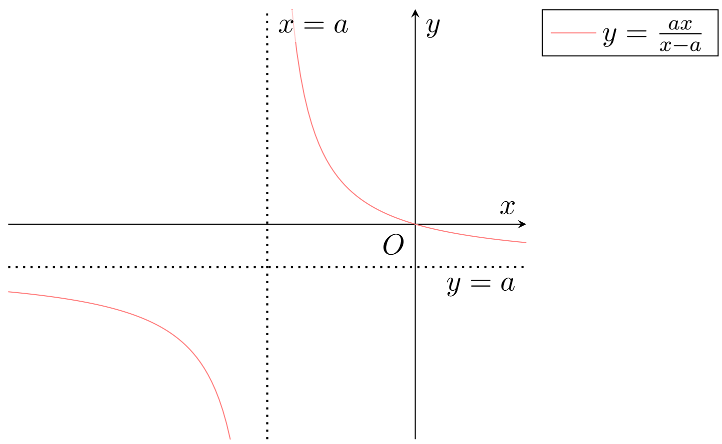
(c) $y = -\frac{b(x+3a)}{x+a}$, $a, b > 0$

Solution.

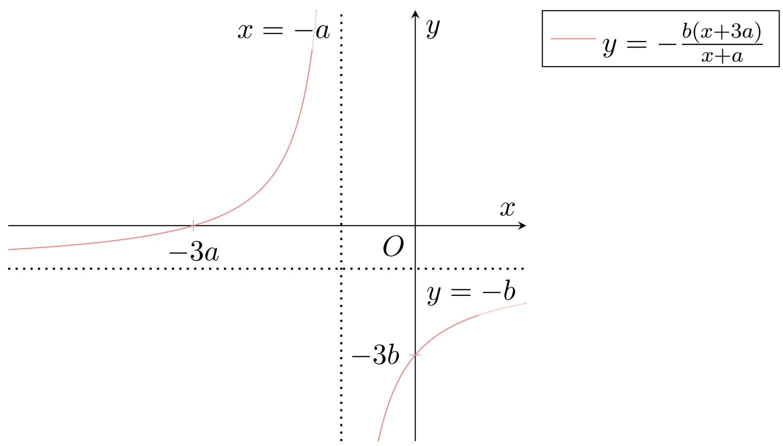
Part (a).



Part (b).



Part (c).



Problem 5. Sketch the following curves and find the coordinates of any turning points on the curves.

(a) $y = x + 2\sin x$, $0 \leq x \leq 2\pi$

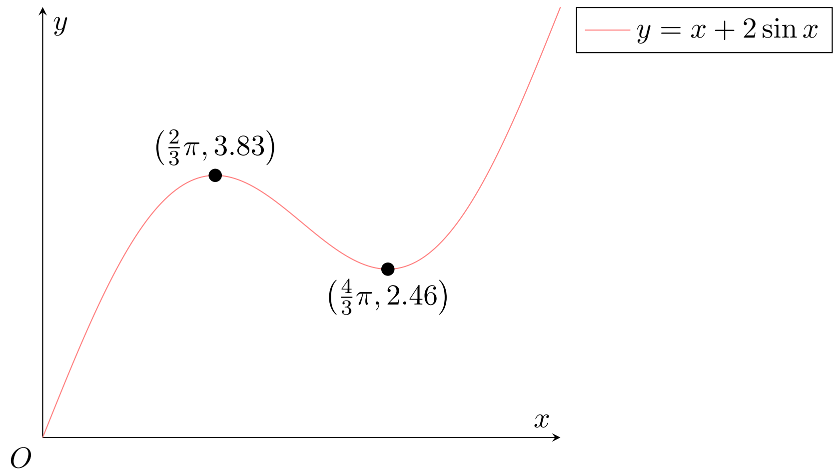
(b) $y = \frac{x}{\ln x}$, $x > 0$, $x \neq 1$

(c) $y = x\mathrm{e}^{-x}$

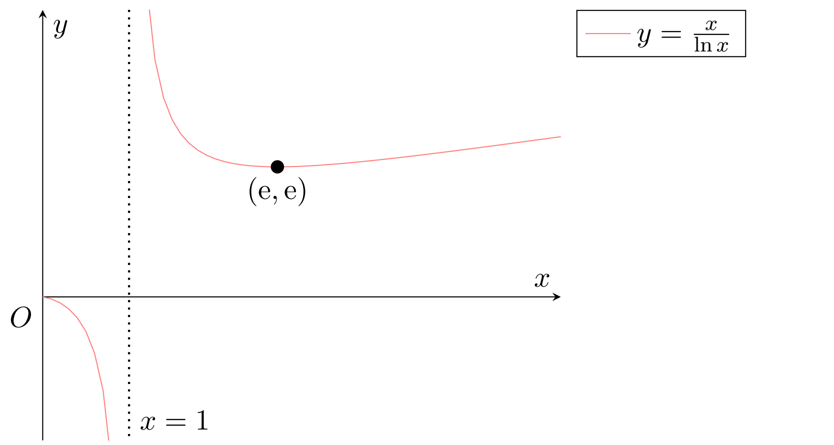
(d) $y = x\mathrm{e}^{-x^2}$

Solution.

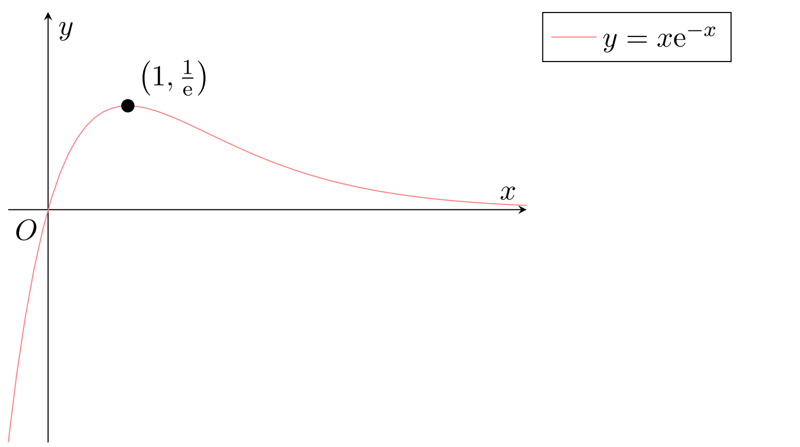
Part (a).



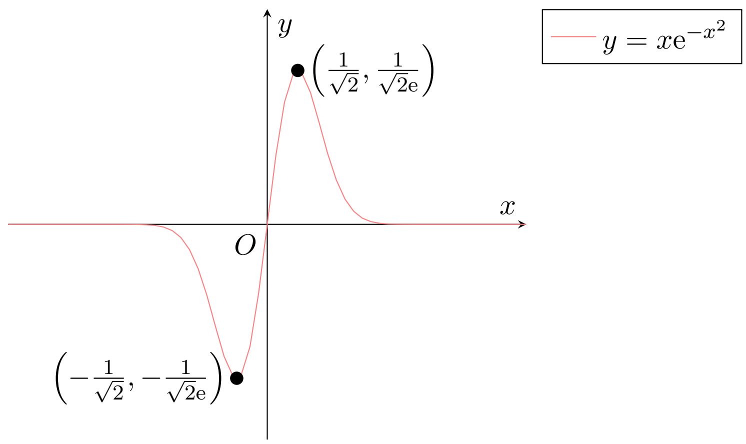
Part (b).



Part (c).



Part (d).



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Problem 6. The equation of a curve C is $y = 1 + \frac{6}{x-3} - \frac{24}{x+3}$.

(a) Explain why $y = 1$ and $x = 3$ are asymptotes to the curve.

(b) Find the coordinates of the points where C meets the axes.

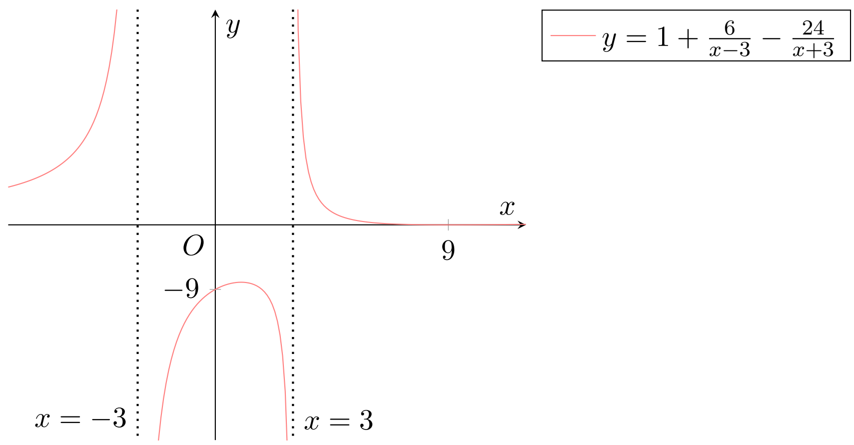
(c) Sketch C.

Solution.

Part (a). As $x \to \pm\infty$, $y \to 1$. Hence, $y = 1$ is an asymptote to C. As $x \to 3^{\pm}$, $y \to \pm\infty$. Hence, $x = 3$ is an asymptote to C.

Part (b). When $x = 0$, $y = -9$. When $y = 0$, $x = 9$. Hence, C meets the axes at $(0, -9)$ and $(9, 0)$.

Part (c).



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Problem 7. The curve C has equation $y = \frac{ax^2+bx}{x+2}$, where $x \neq -2$. It is given that C has an asymptote $y = 1 - 2x$.

(a) Show (do not verify) that $a = -2$ and $b = -3$.

(b) Using an algebraic method, find the set of values that y can take.

(c) Sketch C, showing clearly the positions of any axial intercept(s), asymptote(s) and stationary point(s).

(d) Deduce that the equation $x^4+2x^3+2x^2+3x=0$ has exactly one real non-zero root.

Solution.

Part (a).

$$y=\frac{ax^2+bx}{x+2}=\frac{(ax+b-2a)(x+2)-2(b-2a)}{x+2}=ax+b-2a-\frac{2(b-2a)}{x+2}.$$

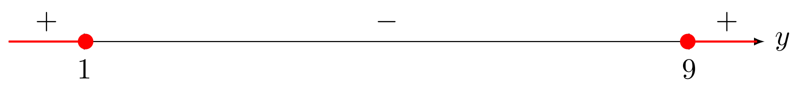
Since C has an asymptote $y=1-2x$, we have $a=-2$ and $b-2a=1$, whence $b=-3$.

Part (b).

$$y=\frac{-2x^2+-3x}{x+2}\implies y(x+2)=-2x^2-3x\implies 2x^2+(3+y)x+2y=0.$$

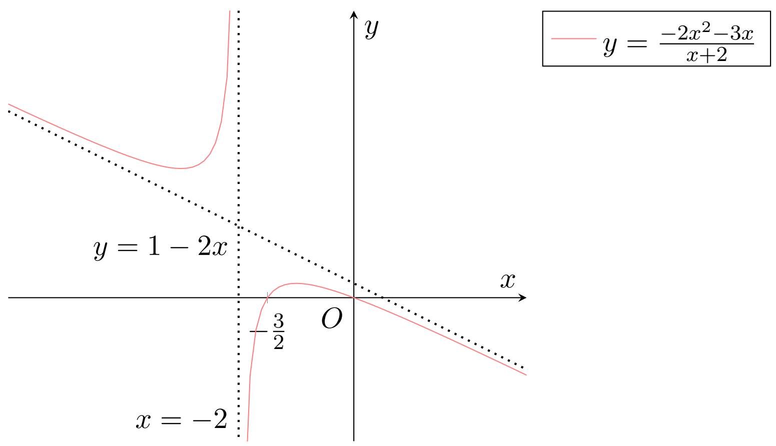
For all values that y can take on, there exists a solution to $2x^2+(3+y)x+2y=0$. Hence, $\Delta\geq 0$.

$$(3+y)^2-4(2)(2y)\geq 0\implies y^2-10y+9\geq 0\implies (y-1)(y-9)\geq 0.$$



Thus, $\{y\in\mathbb{R}\colon y\leq 1 \text{ or } y\geq 9\}$.

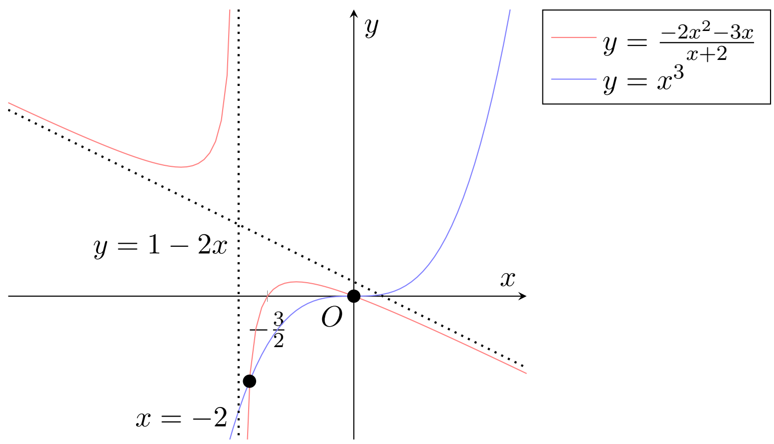
Part (c).



Part (d). Observe that

$$x^4+2x^3+2x^2+3x=0\implies x^3(x+2)=-2x^2-3x\implies x^3=\frac{-2x^2-3x}{x+2}.$$

This motivates us to plot $y=x^3$ and $y=\frac{-2x^2-3x}{x+2}$ on the same graph.



We thus see that $y = x^3$ intersects $y = \frac{-2x^2-3x}{x+2}$ twice, with one intersection point being the origin. Thus, there is only one real non-zero root to $x^4 + 2x^3 + 2x^2 + 3x = 0$.

* * * * *

Problem 8. The curve C is defined by the equation $y = \frac{x}{x^2-5x+4}$.

(a) Write down the equations of the asymptotes.

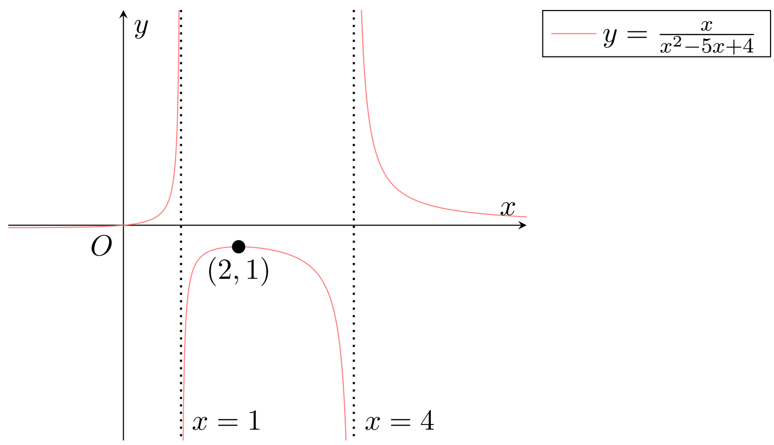
(b) Sketch C, indicating clearly the axial intercept(s), asymptote(s) and turning point(s).

(c) Find the positive value k such that the equation $\frac{x}{x^2-5x+4} = kx$ has exactly 2 distinct real roots.

Solution.

Part (a). As $x \to \pm\infty$, $y \to 0$. Hence, $y = 0$ is an asymptote. Observe that $x^2 - 5x + 4 = (x-1)(x-4)$. Hence, $x = 1$ and $x = 4$ are also asymptotes.

Part (b).



Part (c). Note that $x = 0$ is always a root of $\frac{x}{x^2-5x+4} = kx$. We thus aim to find the value of k such that $\frac{x}{x^2-5x+4} = kx$ has only one non-zero root.

We observe that if $k > 0$, $y = kx$ will intersect with $y = \frac{x}{x^2-5x+4}$ at least twice: before $x = 1$ and after $x = 4$. In order to have only one non-zero root, we must force the intersection point that comes before $x = 1$ to be at the origin $(0, 0)$. Hence, k is tangential to C at $(0, 0)$, thus giving $k = \mathrm{d}C/\mathrm{d}x|_{x=0}$.

$$k = \left.\frac{\mathrm{d}C}{\mathrm{d}x}\right|_{x=0} = \left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x^2-5x+4}\right)\right|_{x=0} = \left.\frac{3x^2-10x+4}{(x^2-5x+4)^2}\right|_{x=0} = \frac{1}{4}.$$

Assignment B1A

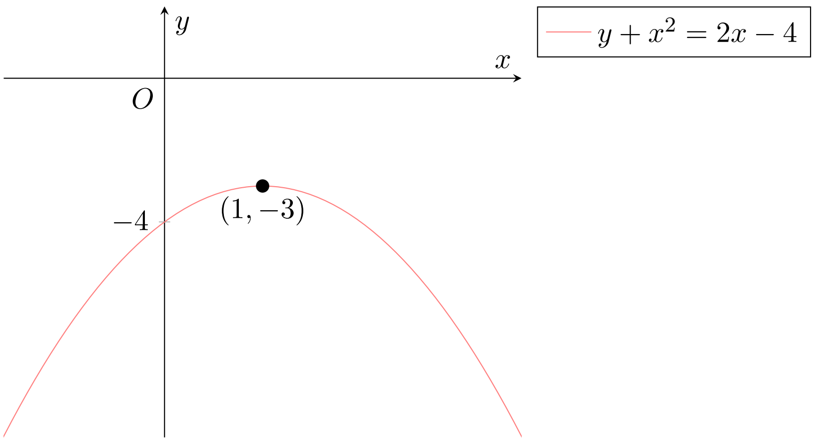
Problem 1. Sketch clearly labelled diagrams of each of the following curves, giving exact values of axial intercepts, stationary points and equations of asymptotes, if any.

(a) $y + x^2 = 2x - 4$

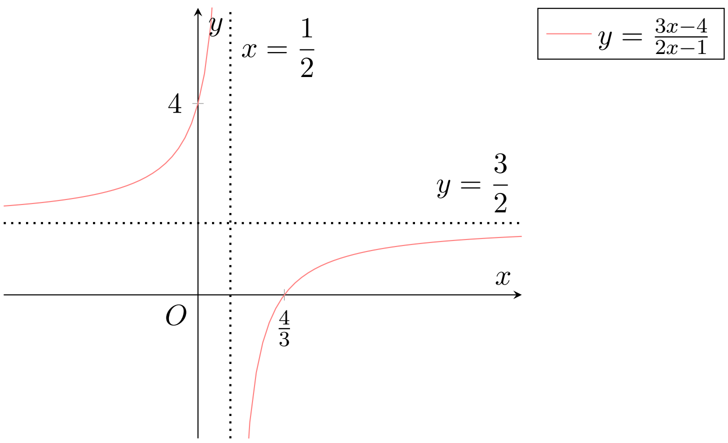
(b) $y = \frac{3x-4}{2x-1}$

Solution.

Part (a).



Part (b).



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Problem 2. On separate diagrams, sketch the graphs of

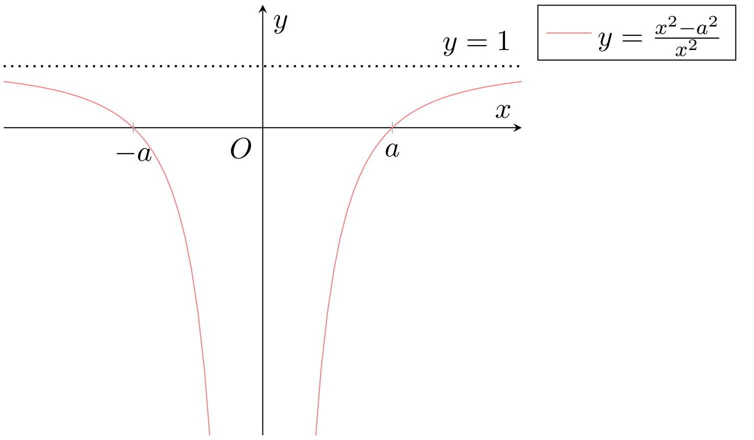
(a) $y = \frac{x^2-a^2}{x^2}$, $a > 0$

(b) $y = \frac{x-1}{2x(x+3)}$

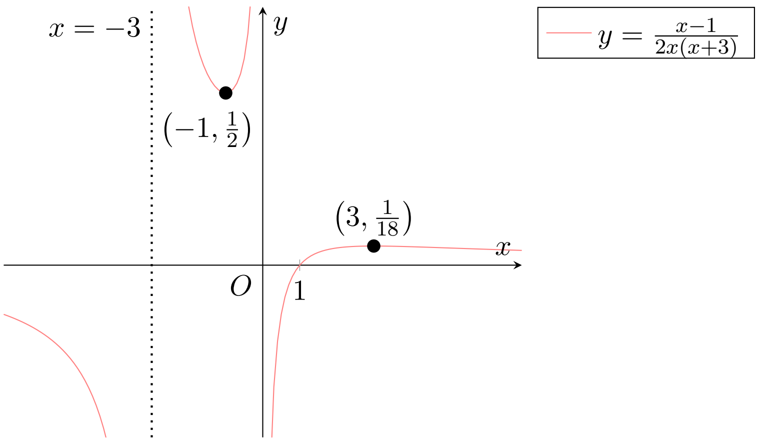
Indicate clearly the coordinates of axial intercepts, stationary points and equations of asymptotes, if any.

Solution.

Part (a).



Part (b).



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Problem 3. The curve C has equation $y = \frac{ax^2+bx-2}{x+4}$, where a and b are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(a) Find the values of a and b.

(b) Sketch C.

(c) Using an algebraic method, find the set of values that y cannot take.

(d) By drawing a sketch of another suitable curve in the same diagram as your sketch of C in part (b), deduce the number of distinct real roots of the equation $x^3+6x^2+3x-2 = 0$.

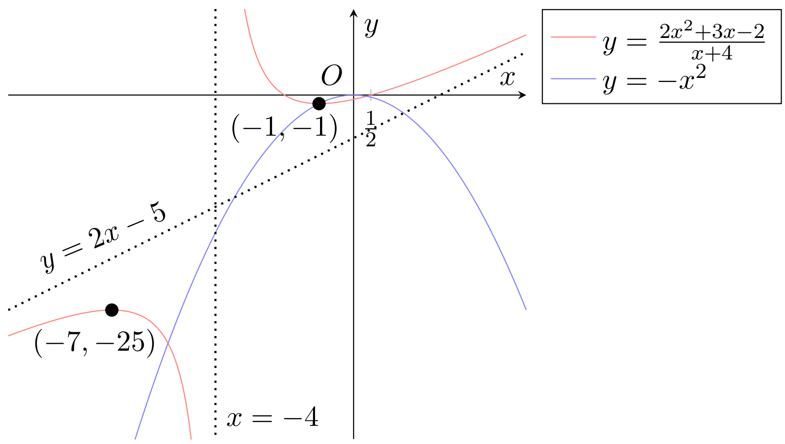
Solution.

Part (a). Since $y = 2x - 5$ is an asymptote of C, $\frac{ax^2+bx-2}{x+4}$ can be expressed in the form $2x - 5 + \frac{k}{x+4}$, where k is a constant.

$$\frac{ax^2 + bx - 2}{x + 4} = 2x - 5 + \frac{k}{x + 4} \implies ax^2 + bx - 2 = (2x - 5)(x + 4) + k = 2x^2 + 3x - 20 + k.$$

Comparing coefficients of x^2, x and constant terms, we have $a = 2$, $b = 3$ and $k = 18$.

Part (b).

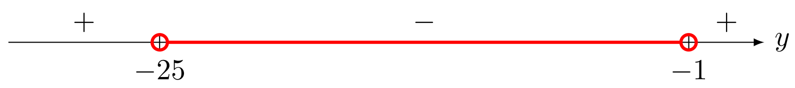


Part (c).

$$y = \frac{2x^2 + 3x - 2}{x + 4} \implies (x+4)y = 2x^2 + 3x - 2 \implies 2x^2 + (3-y)x - (2+4y) = 0.$$

For values that y cannot take on, there exist no solutions to $2x^2 + (3-y)x - (2+4y) = 0$. Hence, $\Delta < 0$. Hence,

$$(3-y)^2 - 4(2)\left(-(2+4y)\right) < 0 \implies y^2 + 26y + 25 < 0 \implies (y+25)(y+1) < 0.$$



Thus, the set of values that y cannot take is $\{y \in \mathbb{R} \colon -25 < y < -1\}$.

Part (d).

$$x^3 + 6x^2 + 3x - 2 = 0 \implies \frac{x^3 + 4x^2}{x+4} + \frac{2x^2 + 3x - 2}{x+4} = x^2 + \frac{2x^2 + 3x - 2}{x+4} = 0$$
$$\implies \frac{2x^2 + 3x - 2}{x+4} = -x^2.$$

Plotting $y = -x^2$ on the same digram, we see that there are 3 intersections between $y = x^2$ and C. Hence, there are 3 distinct real roots to $x^3 + 6x^2 + 3x - 2 = 0$.

Tutorial B1B

Problem 1. Without using a calculator, sketch the following graphs of conics.

(a) $y^2 - 4x = 12$

(b) $(x+1)^2 + y^2 = 4$

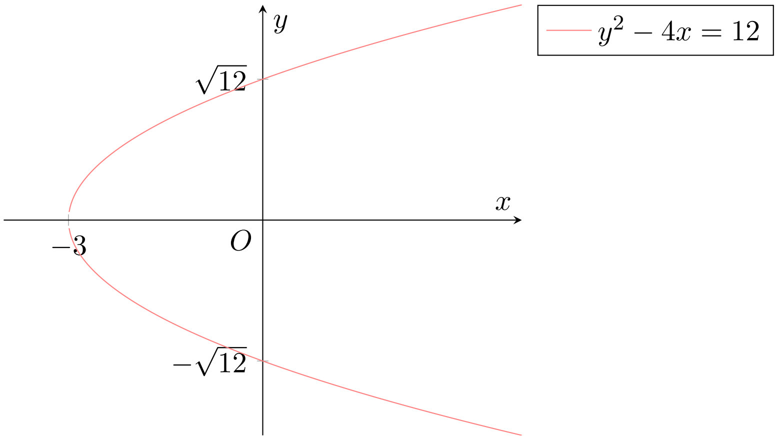
(c) $\frac{(x-3)^2}{9} + \frac{y^2}{2} = 1$

(d) $4x^2 + y^2 = 4$

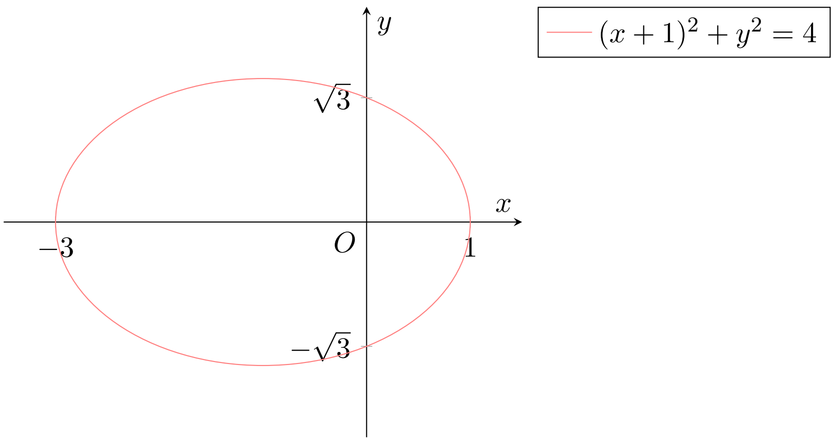
(e) $8y^2 - 2x^2 = 16$

Solution.

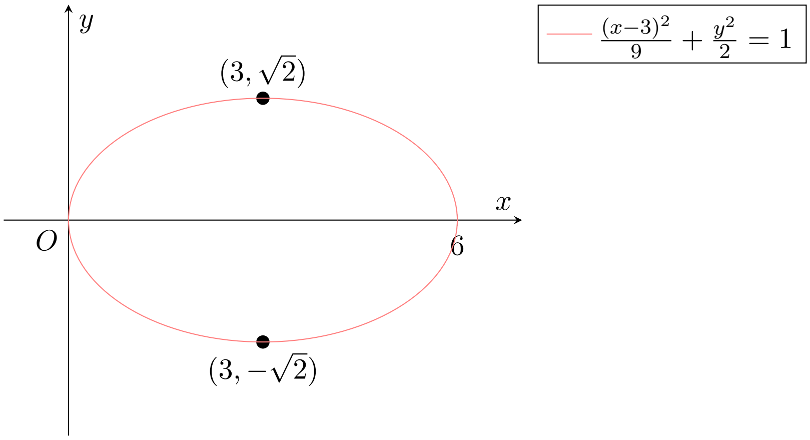
Part (a).



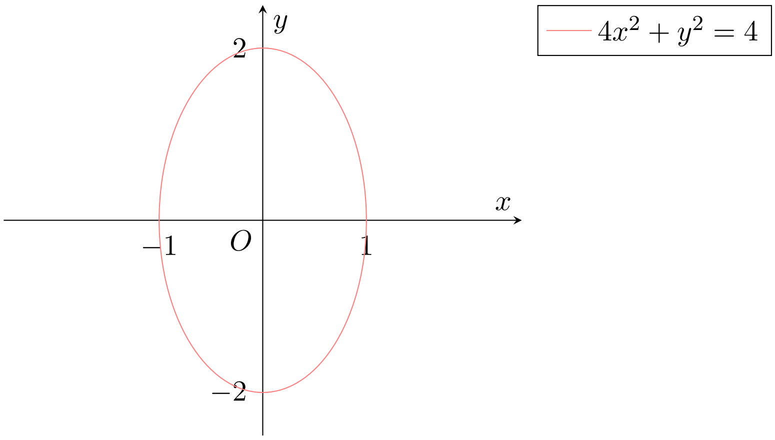
Part (b).



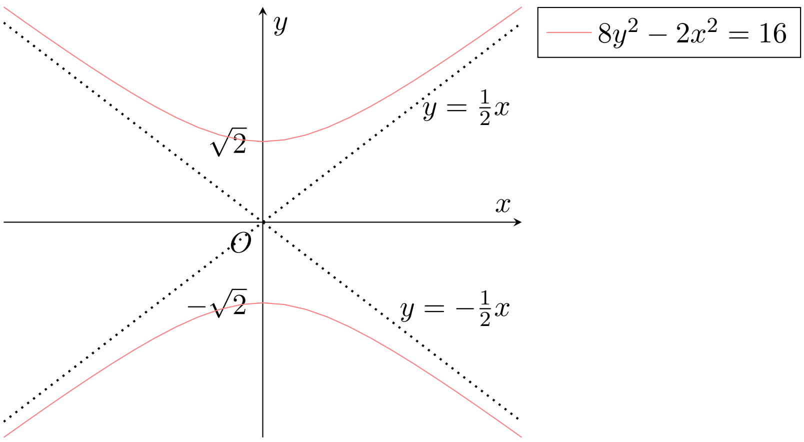
Part (c).



Part (d).



Part (e).



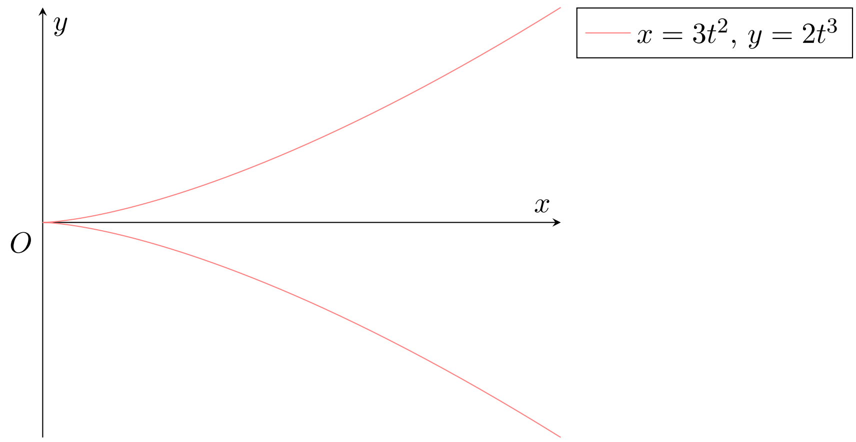
Problem 2. Sketch the curves defined by the following parametric equations, indicating the coordinates of any intersection with the axes.

(a) $x = 3t^2$, $y = 2t^3$

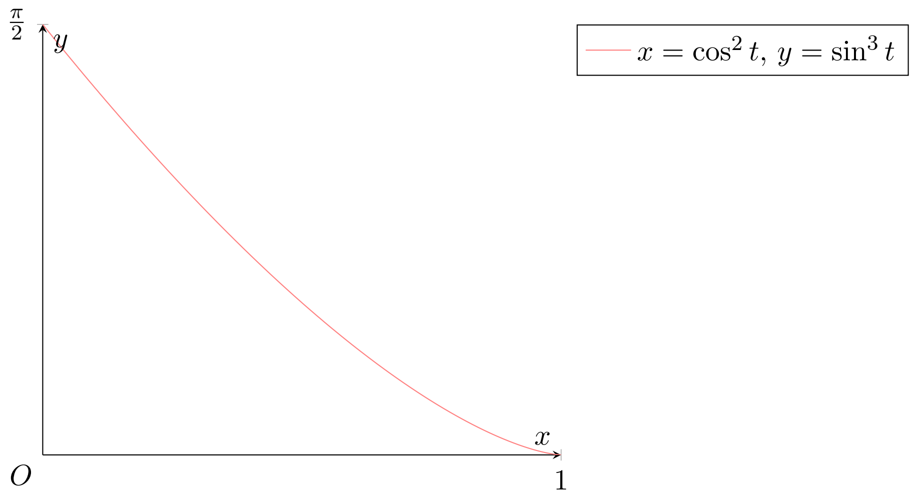
(b) $x = \cos^2 t$, $y = \sin^3 t$, $0 \leq t \leq \frac{\pi}{2}$

Solution.

Part (a).



Part (b).



Problem 3. Without using a calculator, sketch the following graphs of conics.

(a) $y^2 + 4y + x = 0$

(b) $x^2 + y^2 - 4x - 4y = 0$

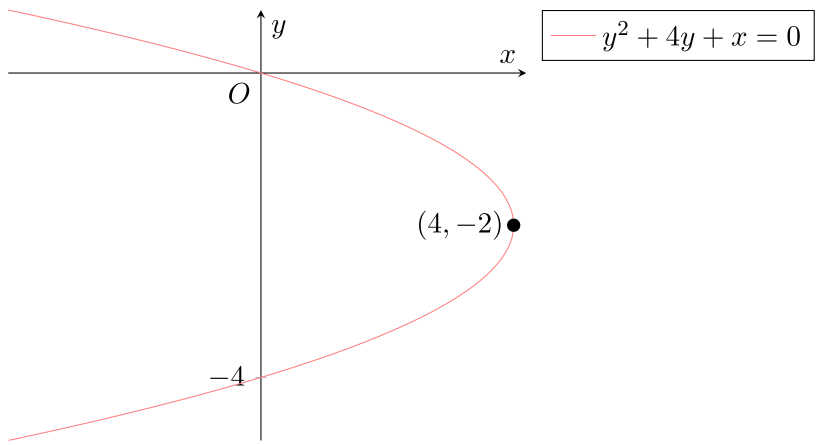
(c) $x^2 + 4y^2 - 2x - 24y + 33 = 0$

(d) $4x^2 - y^2 - 8x + 4y = 1$

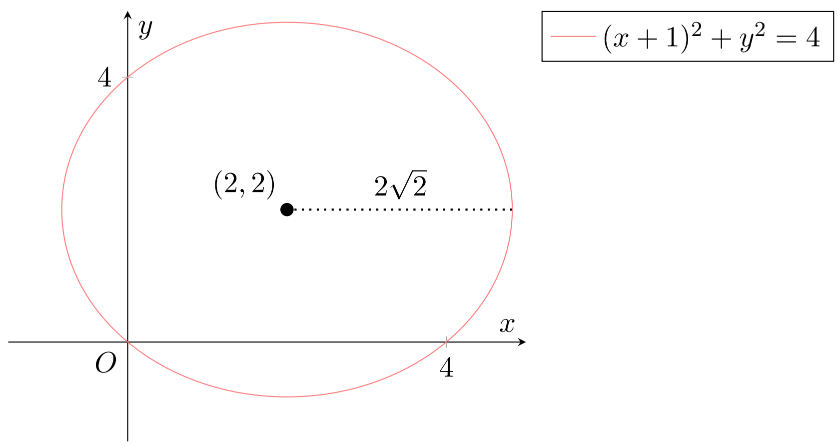
(e) $x = -\sqrt{17 - y^2}$

Solution.

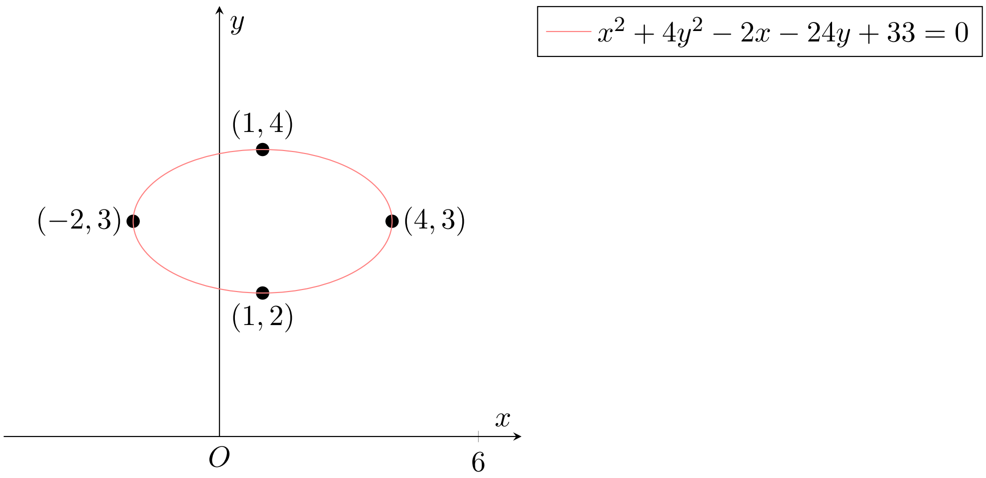
Part (a).



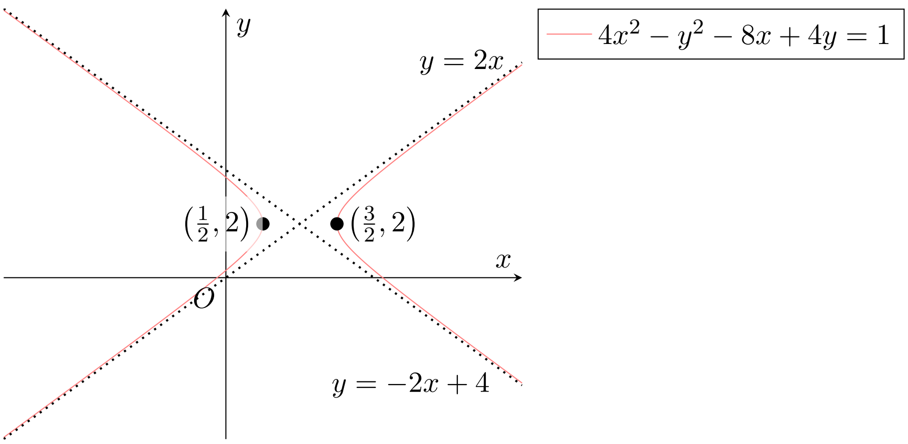
Part (b).



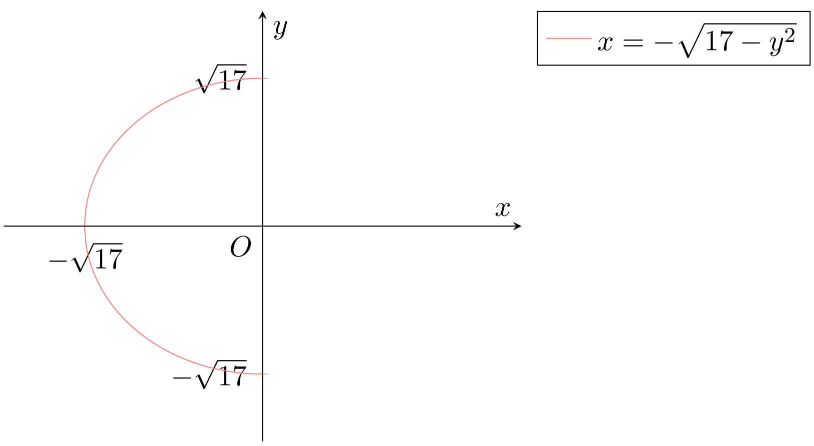
Part (c).



Part (d).



Part (e).



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Problem 4. Sketch the curves defined by the following parametric equations. Find also their respective Cartesian equations.

(a) $x = 4t + 3$, $y = 16t^2 - 9$, $t \in \mathbb{R}$

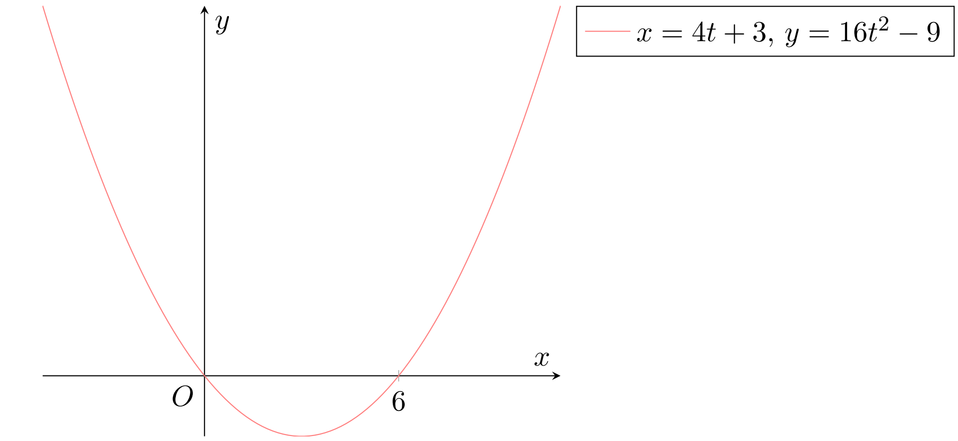
(b) $x = t^2$, $y = 2 \ln t$, $t \geq 1$

(c) $x = 1 + 2\cos\theta$, $y = 2\sin\theta - 1$, $0 \leq \theta \leq \frac{\pi}{2}$

(d) $x = t^2$, $y = \frac{2}{t}$, $t \neq 0$

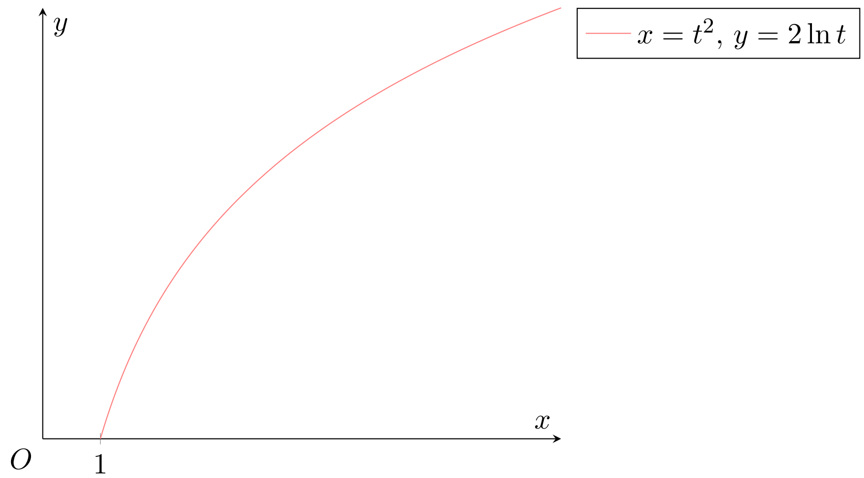
Solution.

Part (a).



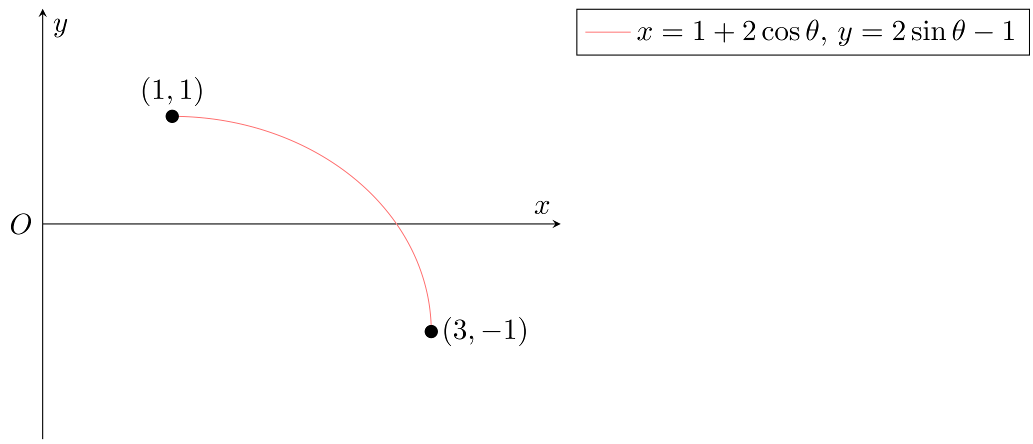
Since $x = 4t + 3$, we have $t = \frac{1}{4}(x - 3)$. Thus, $y = 16\left(\frac{1}{4}(x-3)\right)^2 - 9 = (x-3)^2 - 9$.

Part (b).



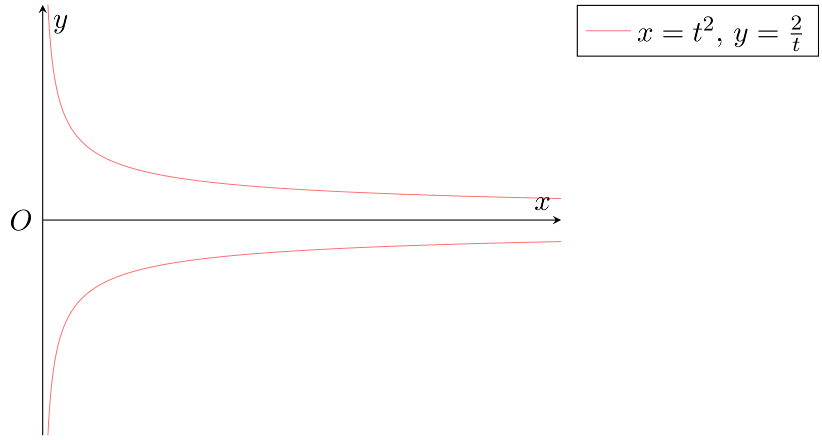
Since $x = t^2$ and $t \geq 1 > 0$, we have $t = \sqrt{x}$. Thus, $y = 2\ln(t) = 2\ln(\sqrt{x}) = \ln(x)$.

Part (c).



We have $2\cos\theta = x-1$ and $2\sin\theta = y+1$. Squaring both equations and adding them, we obtain $4\cos^2\theta + 4\sin^2\theta = (x-1)^2 + (y+1)^2$, which simplifies to $(x-1)^2 + (y+1)^2 = 4$.

Part (d).



Since $x = t^2$, we have $t = \pm\sqrt{x}$. Hence, $y = \pm\frac{2}{\sqrt{x}}$.

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Problem 5. The curve C_1 has equation $y = \frac{x-2}{x+2}$. The curve C_2 has equation $\frac{x^2}{6} + \frac{y^2}{3} = 1$.

(a) Sketch C_1 and C_2 on the same diagram, stating the exact coordinates of any points of intersections with the axes and the equations of any asymptotes.

(b) Show algebraically that the x-coordinates of the points of intersection of C_1 and C_2 satisfy the equation $2(x-2)^2 = (x+2)^2\left(6-x^2\right)$.

(c) Use your calculator to find these x-coordinates.

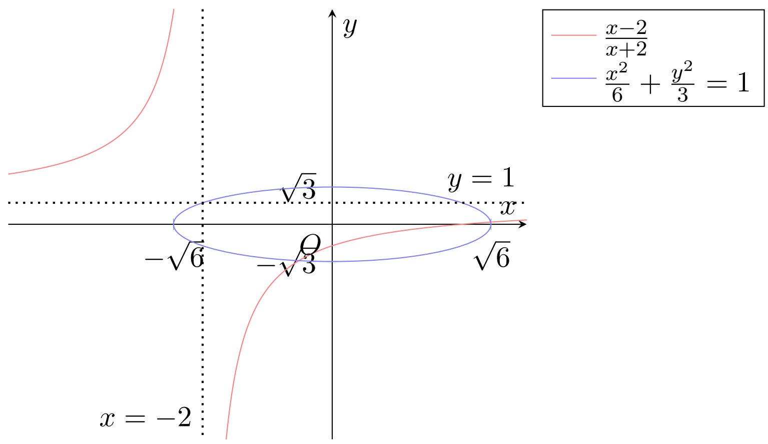
Another curve is defined parametrically by

$$x = \sqrt{6}\cos\theta,\ y = \sqrt{3}\sin\theta,\ -\pi \leq \theta \leq \pi.$$

(d) Find the Cartesian equation of this curve and hence determine the number of roots to the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta - 2}{\sqrt{6}\cos\theta + 2}$ for $-\pi \leq \theta \leq \pi$.

Solution.

Part (a).



Part (b). From C_1, we have $y(x+2) = x-2$. Hence,

$$y^2(x+2)^2 = (x-2)^2.$$

From C_2, we have $x^2+2y^2 = 6$. Hence,

$$y^2 = \frac{6-x^2}{2}.$$

Putting both equations together, we have

$$(x-2)^2 = \frac{\left(6-x^2\right)(x+2)^2}{2} \implies 2(x-2)^2 = \left(6-x^2\right)(x+2)^2.$$

Part (c). The x-coordinates are $x = -0.515$ or $x = 2.45$.

Part (d). Since $x = \sqrt{6}\cos\theta$ and $y = \sqrt{3}\sin\theta$, we have $x^2 = 6\cos^2\theta$ and $2y^2 = 6\sin^2\theta$. Adding both equations together, we have

$$x^2 + 2y^2 = 6\cos^2\theta + 6\sin^2\theta = 6 \implies \frac{x^2}{6} + \frac{y^2}{3} = 1.$$

This is the equation that gives C_1. We further observe that the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta-2}{\sqrt{6}\cos\theta+2}$ simplifies to $y = \frac{x-2}{x+2}$. This is the equation that gives C_2. Since there are two intersections between C_1 and C_2, there are thus two roots to the equation $\sqrt{3}\sin\theta = \frac{\sqrt{6}\cos\theta-2}{\sqrt{6}\cos\theta+2}$.

Assignment B1B

Problem 1. Without using a calculator, sketch the graphs of the conics in parts (a), (b) and c.

(a) $3x^2 + 2y^2 = 6$

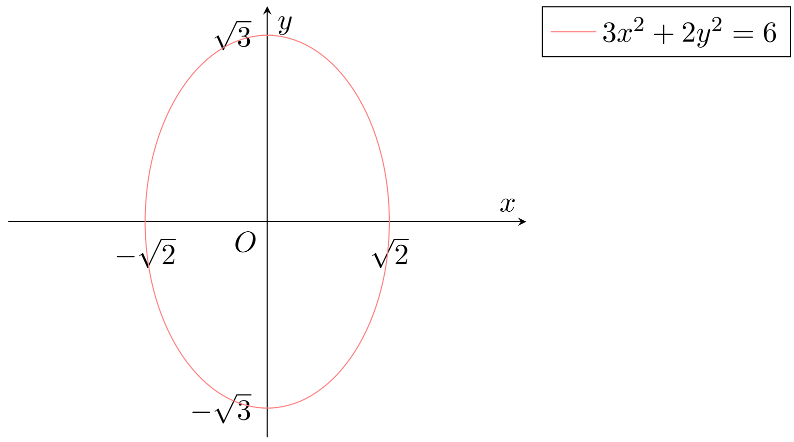
(b) $x^2 + y^2 + 4x - 2y - 20 = 0$

(c) $4(y-1)^2 - x^2 = 4$

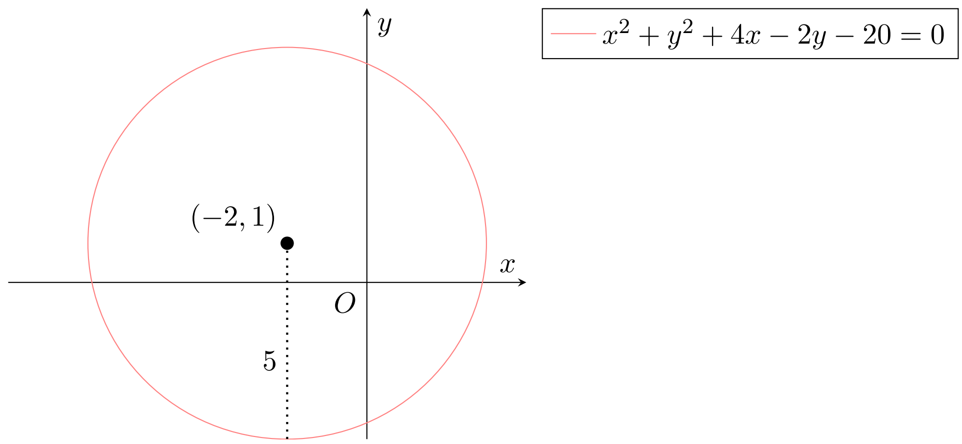
State a transformation that will transform the graph of (a) to a circle with centre $(0, 0)$ and radius $\sqrt{3}$.

Solution.

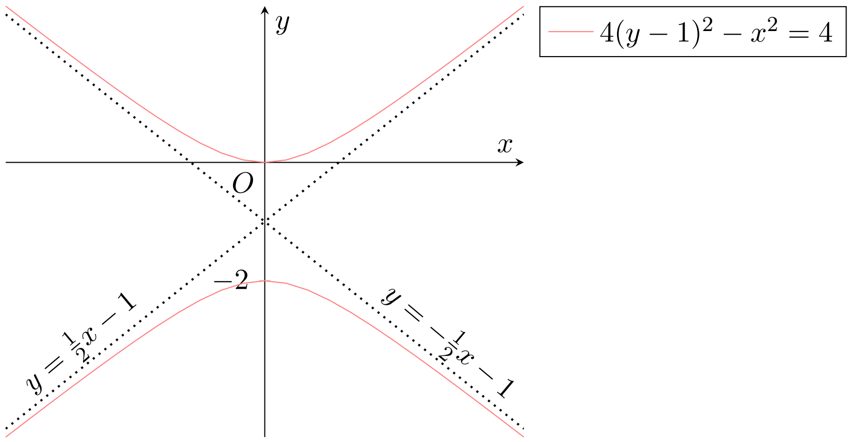
Part (a).



Part (b).



Part (c).



The transformation is $x \mapsto \sqrt{\frac{2}{3}}x$.

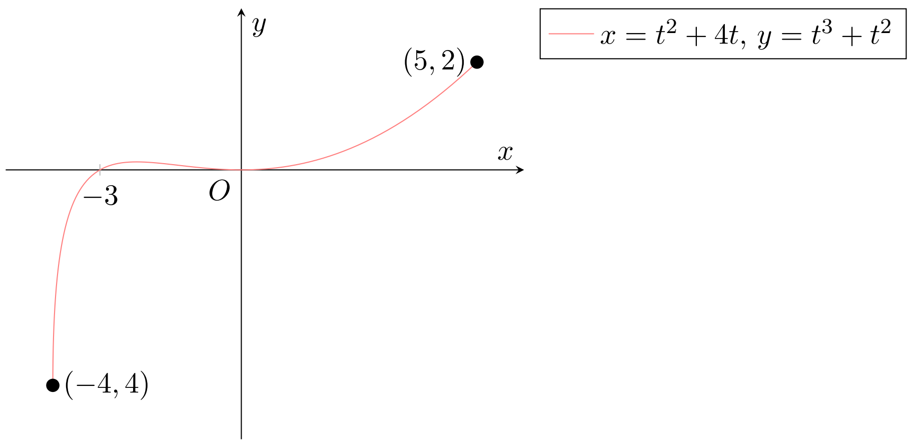
* * * * *

Problem 2. The curve C has parametric equations

$$x = t^2 + 4t,\ y = t^3 + t^2.$$

Sketch the curve for $-2 \leq t \leq 1$, stating the axial intercepts.

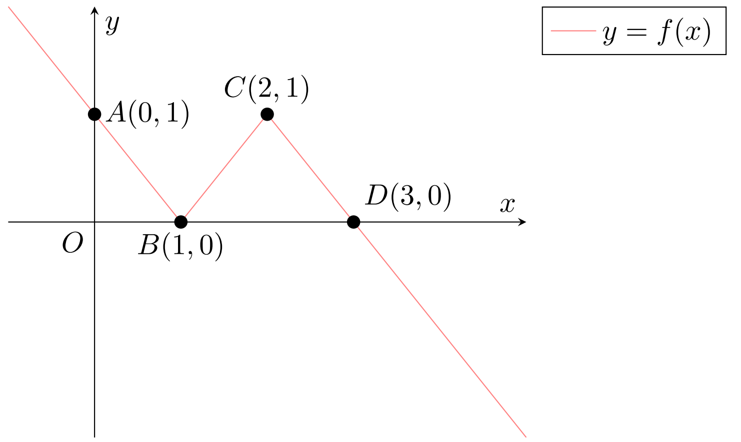
Solution.



B2. Graphs and Transformations II

Tutorial B2

Problem 1.



The graph of $y = f(x)$ is shown here. The points A, B, C and D have coordinates $(0, 1)$, $(1, 0)$, $(2, 1)$ and $(3, 0)$ respectively. Sketch, separately, the graphs of

(a) $y = f(2x)$

(b) $y = f(x + 3)$

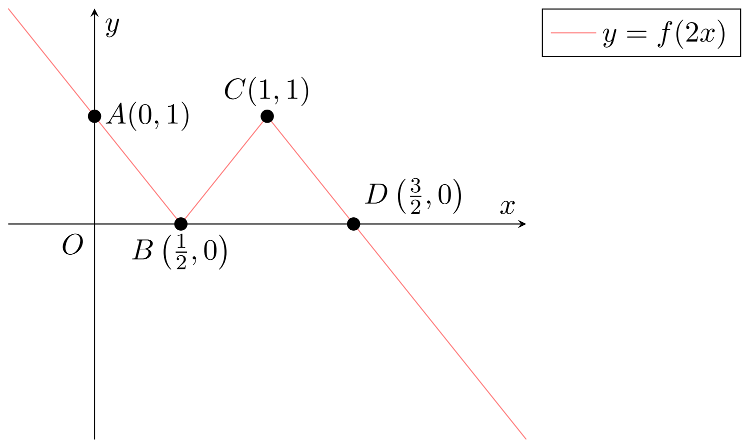
(c) $y = 1 - f(x)$

(d) $y = 3f\left(\frac{x}{2} - 1\right)$

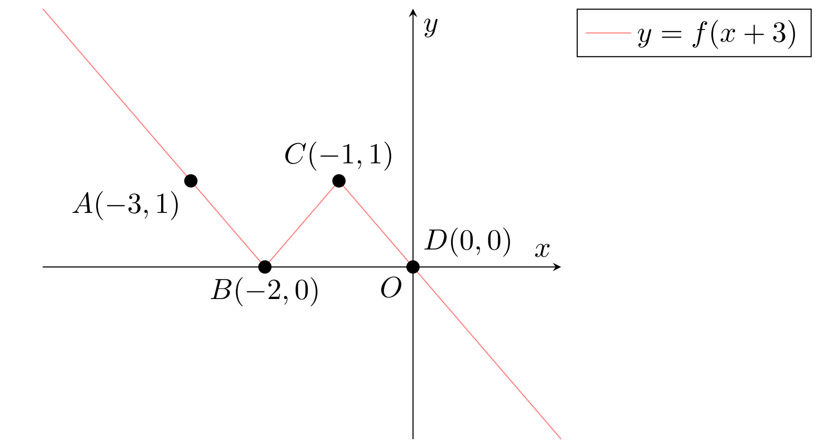
stating, in each case, the coordinates of the points corresponding to A, B, C and D.

Solution.

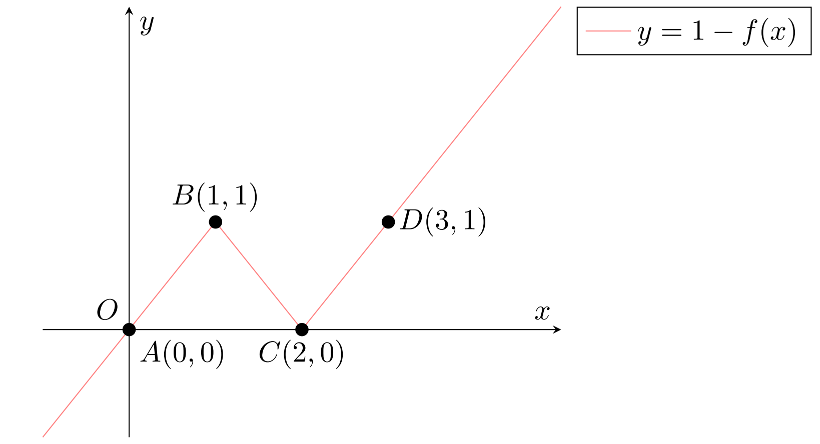
Part (a).



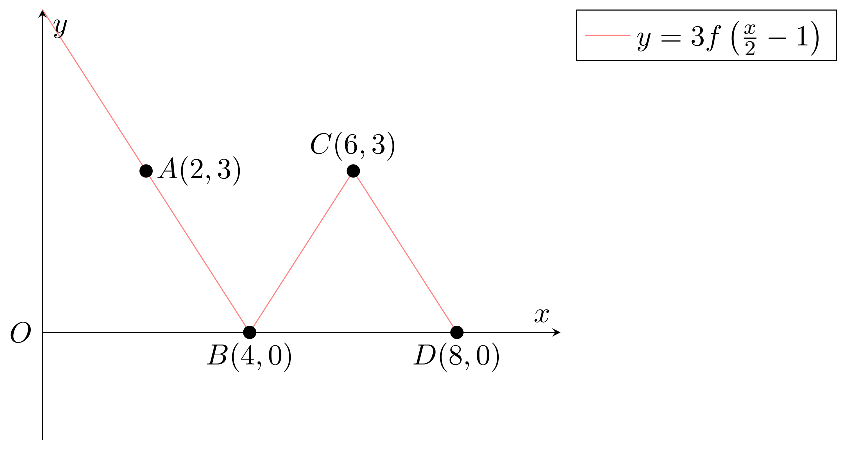
Part (b).



Part (c).



Part (d).



* * * * *

Problem 2. Sketch, on a single clear diagram, the graphs of

(a) $y = x^2$

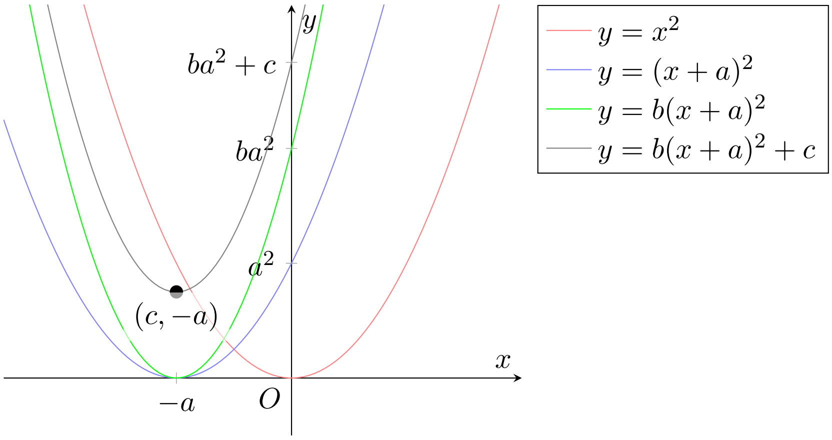
(b) $y = (x+a)^2$

(c) $y = b(x+a)^2$

(d) $y = b(x+a)^2 + c$

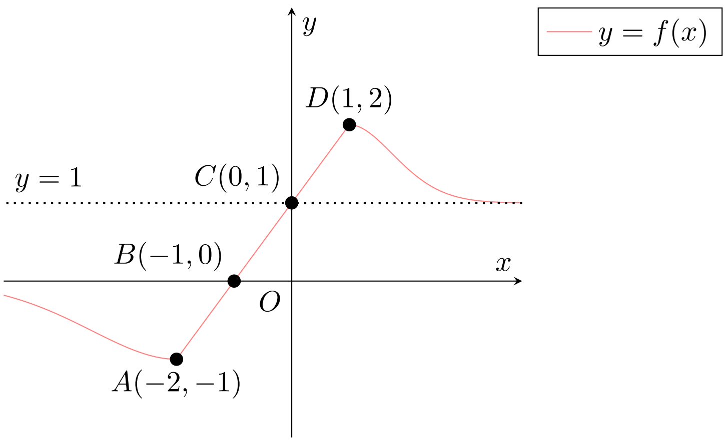
Assume constants $a > 0$, $c > 0$ and $b > 1$.

Solution.



* * * * *

Problem 3. The graph below has equation $y = f(x)$. It has asymptotes $y = 1$ and $y = 0$, a maximum point at $D(1, 2)$, a minimum point at $A(-2, -1)$, cuts the x-axis at $B(-1.0)$ and cuts the y-axis at $C(0, 1)$.



Sketch on separate diagrams the graphs of the following curves, labelling each curve clearly, indicating the horizontal asymptotes and showing the coordinates of the points corresponding to the points A, B, C and D.

(a) $y = f(x+1)$

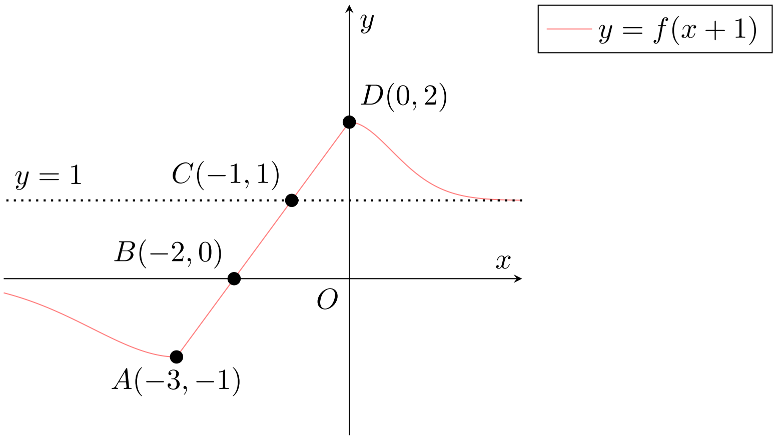
(b) $y = f\left(\frac{x}{2}\right)$

(c) $y = 2f(x) - 2$

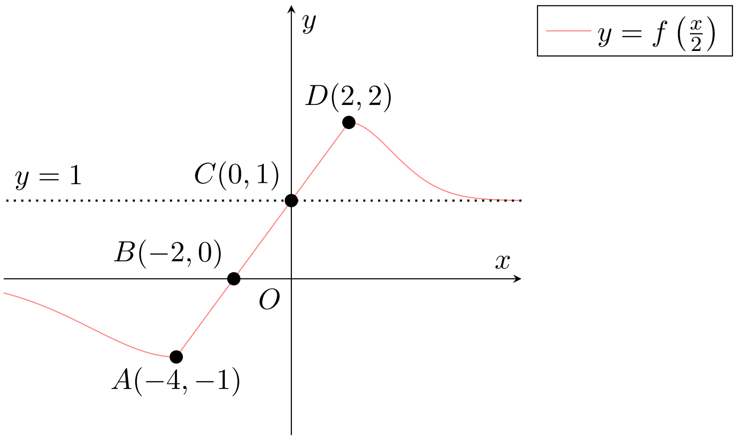
Find the number of solutions to the equation $f(x) = af(x)$ where $a \geq 2$.

Solution.

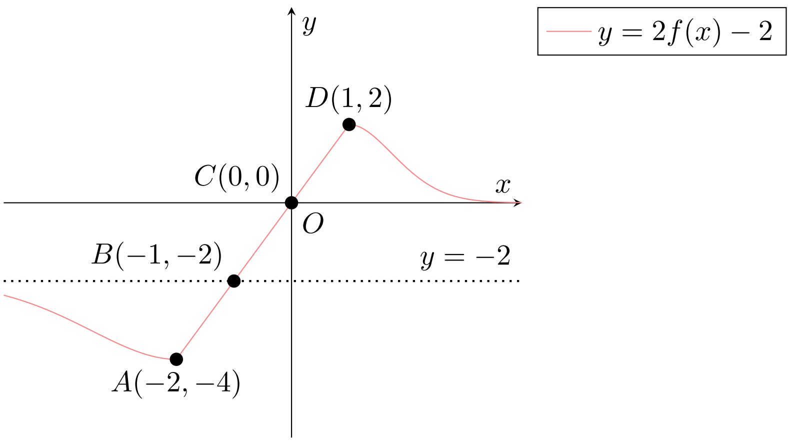
Part (a).



Part (b).



Part (c).



All points with a y-coordinate of 0 are invariant under the transformation $f(x) \mapsto af(x)$. Since there is only one such point $(B(-1,0))$, there is only 1 solution to the equation $f(x) = af(x)$, where $a \geq 2$.

Problem 4. The curve with equation $y = x^2$ is transformed by a translation of 2 units in the positive x-direction, followed by a stretch with scale factor $\frac{1}{2}$ parallel to the y-axis, followed by a translation of 6 units in the negative y-direction. Find the equation of the new curve in the form $y = f(x)$ and the exact coordinates of the points where this curve crosses the x- and y-axes.

Solution.

$$\begin{array}{ccc} & y = x^3 \xrightarrow{\quad x \mapsto x-2 \quad} & y = (x-2)^3 \\ & & \Big\downarrow {\scriptstyle y \mapsto 2y} \\ 2(y+6) = (x-2)^3 & \xleftarrow{\quad y \mapsto y+6 \quad} & 2y = (x-2)^3 \end{array}$$

Hence, $y = \frac{1}{2}(x-2)^3 - 6$

When $x = 0$, $y = -10$. When $y = 0$, $x = 2 + \sqrt[3]{12}$. Thus, the curve crosses the x-axis at $(2 + \sqrt[3]{12}, 0)$ and the y-axis at $(0, -10)$.

* * * * *

Problem 5. Find the values of the constants A and B such that $\frac{x^2-4x}{(x-2)^2} = A + \frac{B}{(x-2)^2}$ for all values of x except $x = 2$.

Hence, state a sequence of transformations by which the graph of $y = \frac{x^2-4x}{(x-2)^2}$ may be obtained from the graph of $y = \frac{1}{x^2}$.

Solution.

$$\frac{x^2-4x}{(x-2)^2} = \frac{(x-2)^2-4}{(x-2)^2} = 1 + \frac{-4}{(x-2)^2}.$$

Thus, $A = 1$ and $B = -4$.

$$\begin{array}{ccccc} y = \dfrac{1}{x^2} & \xrightarrow{\quad x \mapsto x-2 \quad} & y = \dfrac{1}{(x-2)^2} & \xrightarrow{\quad y \mapsto \frac{1}{4}y \quad} & y = \dfrac{4}{(x-2)^2} \\ & & & & \Big\downarrow {\scriptstyle y \mapsto -y} \\ & & y = 1 + \dfrac{-4}{(x-2)^2} & \xleftarrow{\quad y \mapsto y-1 \quad} & y = \dfrac{-4}{(x-2)^2} \end{array}$$

1 Translate the curve 2 units in the positive x-direction.

2 Stretch the curve with a scale factor of 4 parallel to the y-axis.

3 Reflect the curve about the x-axis.

4 Translate the curve 1 unit in the positive y-direction.

* * * * *

Problem 6. The transformations A, B, C and D are given as follows:

- A: A reflection about the y-axis.
- B: A translation of 2 units in the positive x-direction.
- C: A scaling parallel to the y-axis by a factor of 3.

- D: A translation of 1 unit in the positive y-direction.

A curve undergoes the transformations A, B, C and D in succession, and the equation of the resulting curve is $y = 3\sqrt{2-x}+1$. Determine the equation of the curve before the transformations were effected.

Solution.

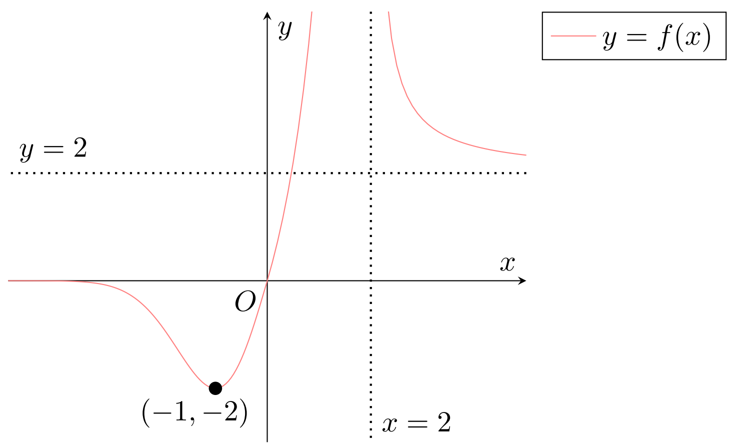
$$\begin{aligned}
A\colon x &\mapsto -x & \implies A^{-1}\colon x &\mapsto -x \\
B\colon x &\mapsto x-2 & \implies B^{-1}\colon x &\mapsto x+2 \\
C\colon y &\mapsto \frac{1}{3}y & \implies C^{-1}\colon y &\mapsto 3y \\
D\colon y &\mapsto y-1 & \implies D^{-1}\colon y &\mapsto y+1
\end{aligned}$$

$$\begin{gathered}
y = 3\sqrt{2-x}+1 \\
\big\downarrow D^{-1} \\
y+1 = 3\sqrt{2-x}+1 \\
\big\downarrow C^{-1} \\
3y+1 = 3\sqrt{2-x}+1 \\
\big\downarrow B^{-1} \\
3y+1 = 3\sqrt{2-(x+2)}+1 \\
\big\downarrow A^{-1} \\
3y+1 = 3\sqrt{2-(-x+2)}+1
\end{gathered}$$

Thus, the original curve has equation $y = \sqrt{x}$.

* * * * *

Problem 7.



The diagram shows the graph of $y = f(x)$. The curve passes through the origin and has minimum point $(-1,-2)$. The asymptotes are $x = 2$, $y = 0$ and $y = 2$.

Sketch, on separate diagrams, the graphs of

(a) $y = f(x-1)$

(b) $y = f(|x|)$

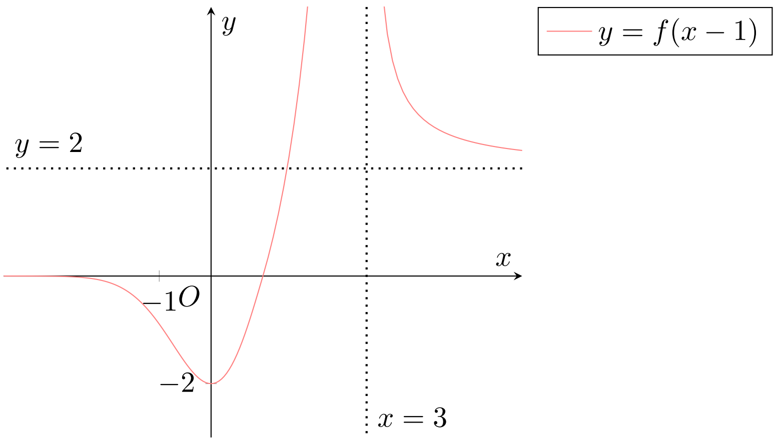
(c) $y = f(|x-1|)$

(d) $y = |f(x)|$

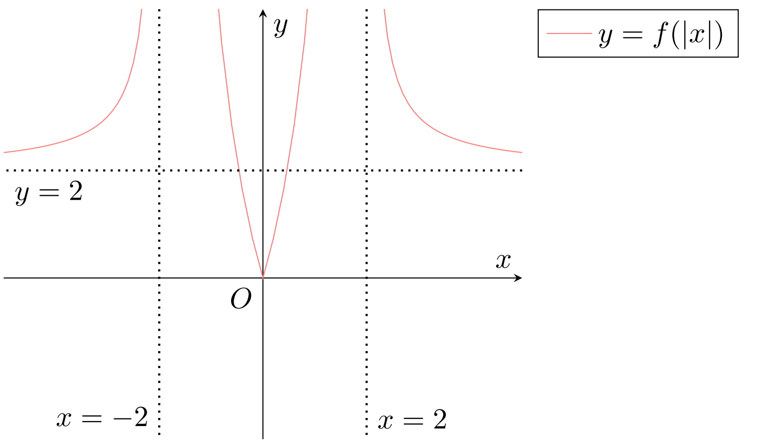
(e) $y = \frac{1}{f(x)}$

Solution.

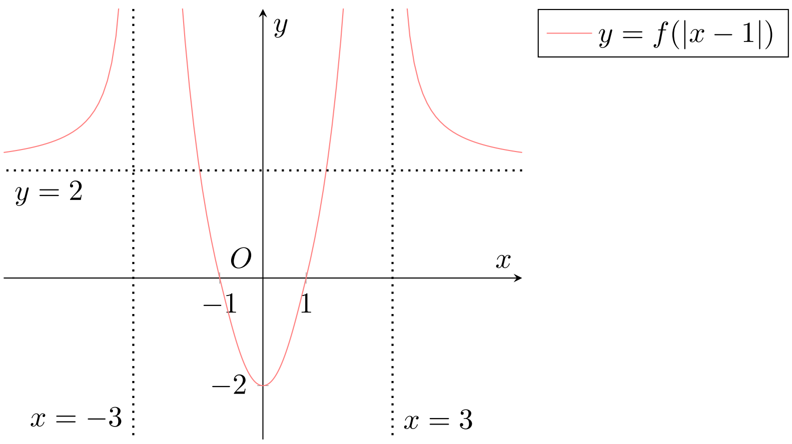
Part (a).



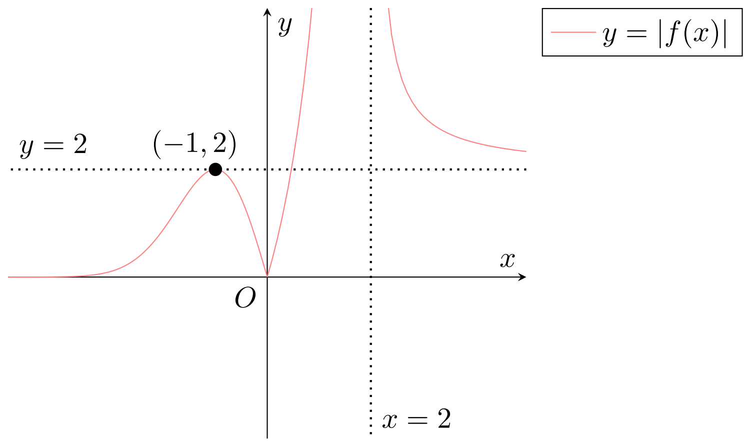
Part (b).



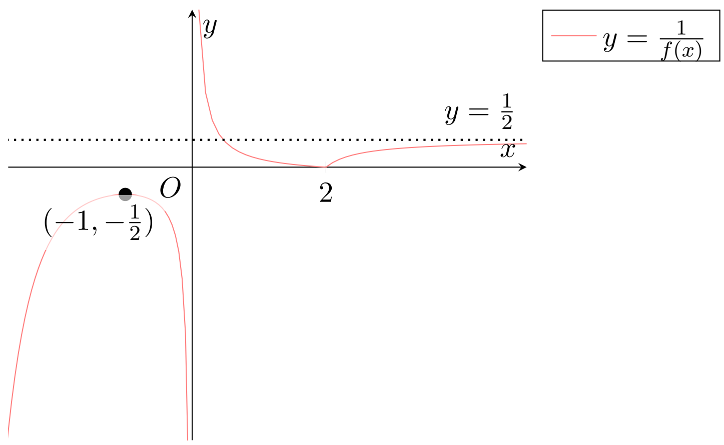
Part (c).



Part (d).



Part (e).



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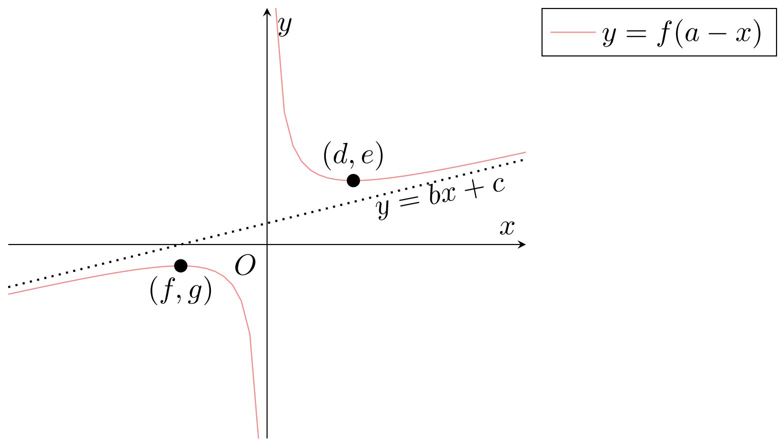
Problem 8.



The graph of $y = f(a-x)$ is shown in the figure, where $a > 0$. The curve has asymptotes $x = 0$, $y = bx + c$, a minimum point at (d, e) and a maximum point at (f, g).

Given $a > d$, sketch separately, the graphs of

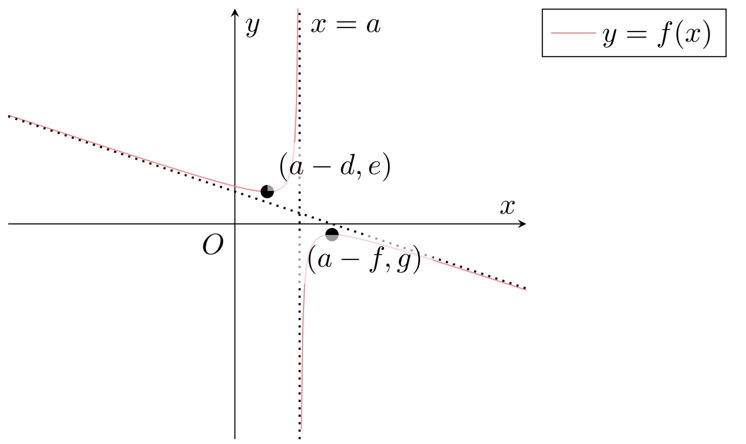
(a) $y = f(x)$

(b) $y = f(|x|)$

(c) $y = \frac{1}{f(x)}$

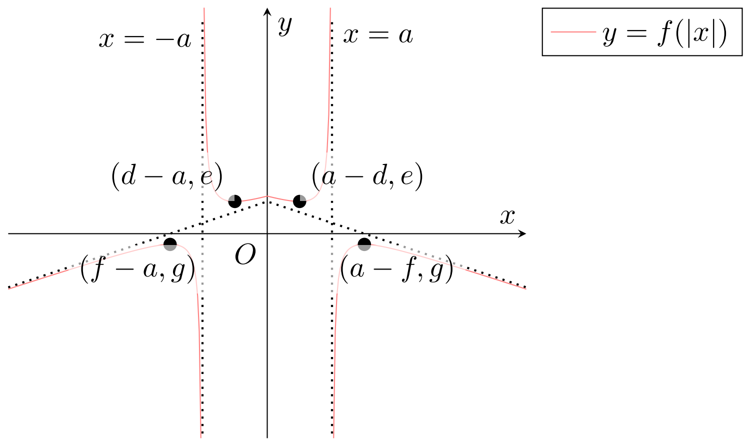
Solution.

Part (a).



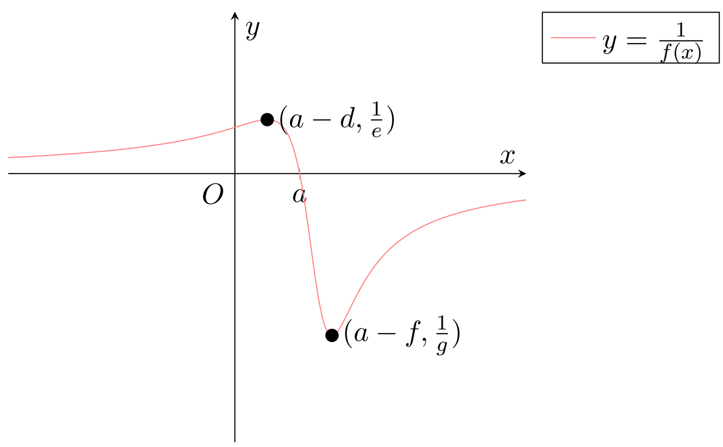
Equation of asymptote: $y = b(a-x) + c$

Part (b).



Equation of asymptotes: $y = b(a+x) + c$, $y = b(a-x) + c$

Part (c).



Problem 9. A curve C_1 is defined by the parametric equations

$$x = t(t+2),\ y = 2(t+1).$$

(a) Find the axial intercepts of the curve.

(b) Sketch C_1.

(c) A curve C_2 is defined by the parametric equations $x = t(t+2)$, $y = t+1$. Describe a geometrical transformation which maps C_1 to C_2. Hence, sketch the curve C_2 in the same diagram as C_1.

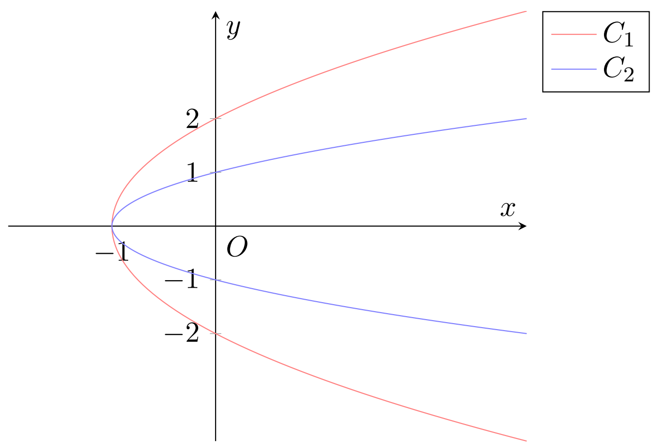
(d) Show that the Cartesian equation of the curve C_1 is given by $y^2 = 4(x+1)$.

Solution.

Part (a). Consider $x = 0$. Then $t(t+2) = 0$, whence $t = 0$ or $t = -2$. When $t = 0$, $y = 2$. When $t = -2$, $y = -2$. Hence, the curve intercepts the y-axis at $(0, 2)$ and $(0, -2)$.

Consider $y = 0$. Then $t = -1$, whence $x = -1$. Hence, the curve intercepts the x-axis at $(-1, 0)$.

Part (b).



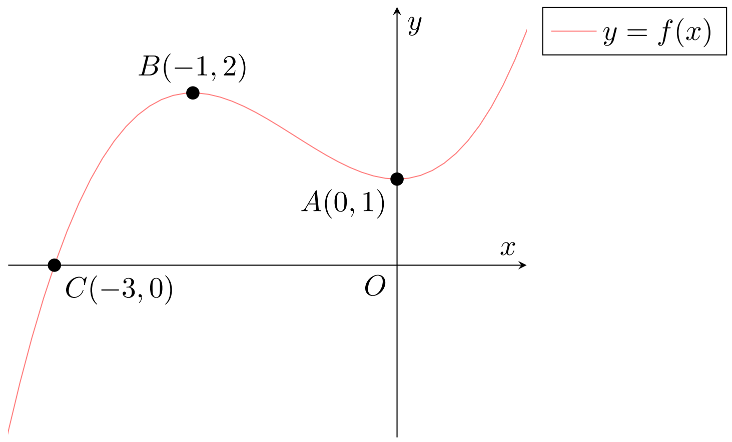
Part (c). Scale by a factor of $\frac{1}{2}$ parallel to the y-axis.

Part (d).

$$y^2 = (2(t+1))^2 = 4(t^2+2t+1) = 4(t(t+1)+1) = 4(x+1).$$

Assignment B2

Problem 1.



The diagram shows the graph of $y = f(x)$. The points A, B and C have coordinates $(0, 1)$, $(-1, 2)$ and $(-3, 0)$ respectively. Sketch, separately, the graphs of

(a) $y = f(-x)$

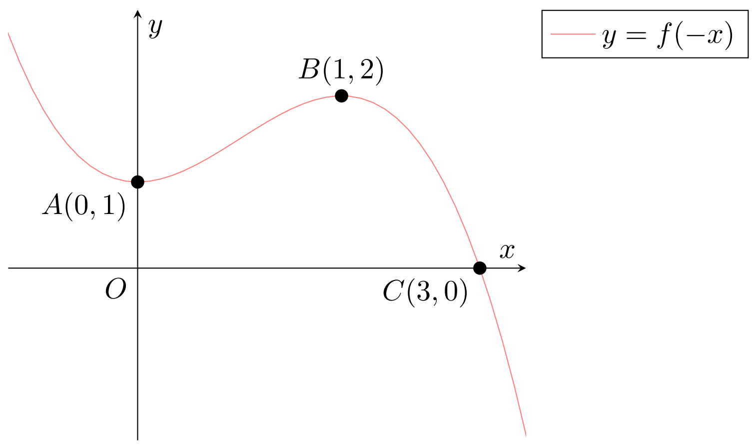
(b) $y = 3 - 2f(x)$

(c) $y = 3f\left(\frac{x}{2} + 1\right)$

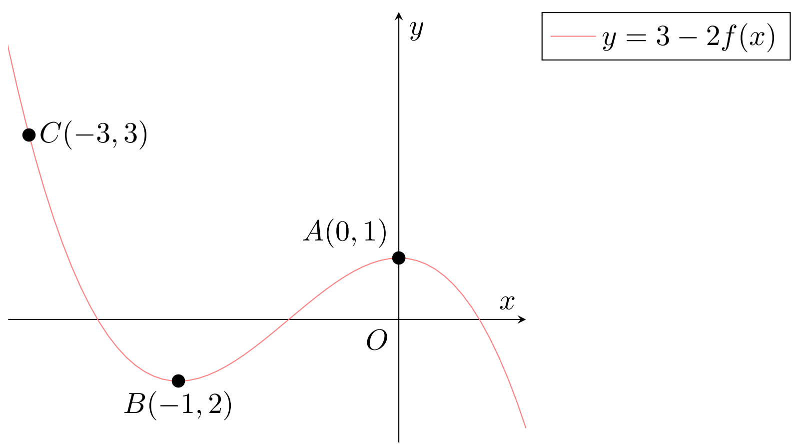
showing in each case the coordinates of the points corresponding to A, B and C.

Solution.

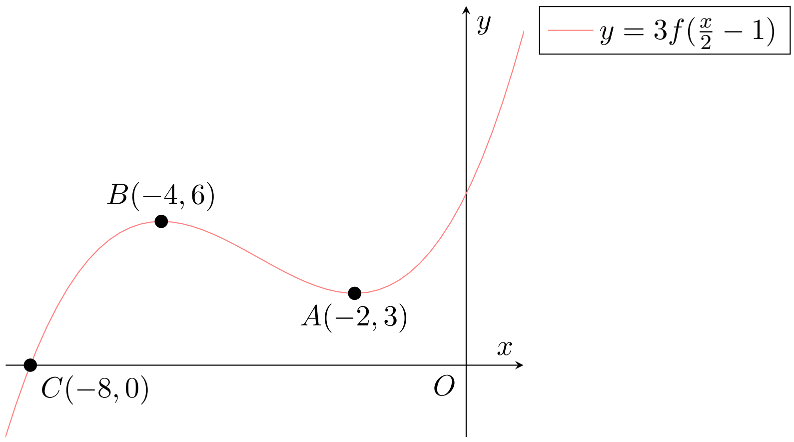
Part (a).



Part (b).



Part (c).



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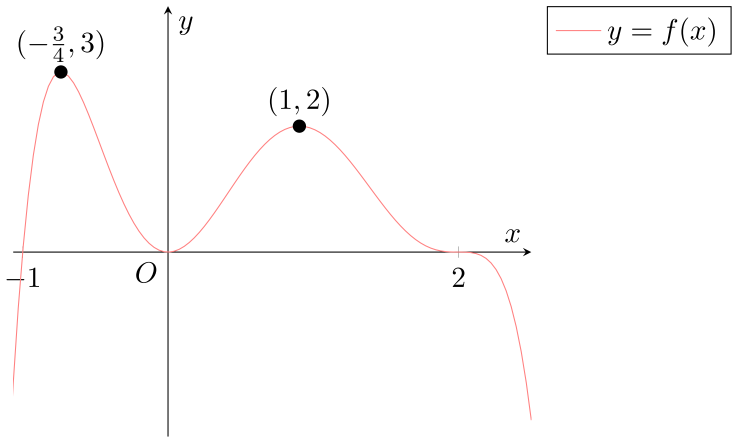
Problem 2.



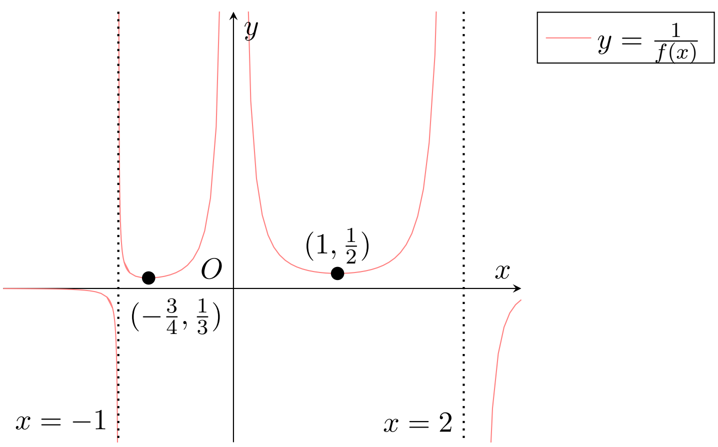
The curve shown is the graph of $y = f(x)$. The x-axis is a tangent at the origin and at $(2, 0)$. The curve has two maximum points at $\left(-\frac{3}{4}, 3\right)$ and $(1, 2)$. On two separate diagrams, sketch the graphs of the following equations. Show clearly the shapes of the graphs where they meet the x-axis and any asymptotes.

(a) $y = \frac{1}{f(x)}$, $x \neq -1, 0, 2$

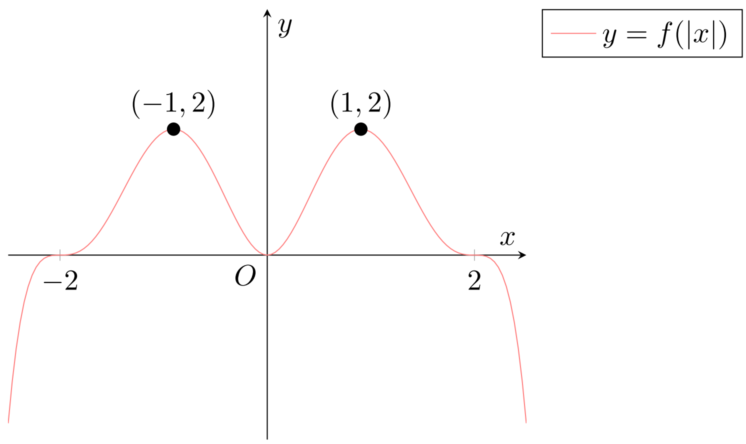
(b) $y = f(|x|)$

Solution.

Part (a).



Part (b).



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Problem 3. A graph with equation $y = f(x)$ undergoes transformation A followed by transformation B where A and B are described as follows:

- A: a translation of 1 unit in the positive direction of the x-axis
- B: a scaling parallel to the x-axis by a factor $\frac{1}{2}$

The resulting equation is $y = 4x^2 - 4x + 1$. Find the equation $y = f(x)$.

Solution. Note that

$$\begin{aligned} &A\colon x \mapsto x - 1 &&\implies A^{-1}\colon x \mapsto x + 1 \\ &B\colon x \mapsto 2x &&\implies B^{-1}\colon x \mapsto \frac{1}{2}x. \end{aligned}$$

Hence,

$$\begin{gathered} y = 4x^2 - 4x + 1 \\ \Big\downarrow B^{-1} \\ y = 4\left(\frac{1}{2}x\right)^2 - 4\left(\frac{1}{2}x\right) + 1 \\ \Big\downarrow A^{-1} \\ y = 4\left[\frac{1}{2}(x+1)\right]^2 - 4\left[\frac{1}{2}(x+1)\right] + 1 \end{gathered}$$

Observe that y simplifies to

$$y = 4\left[\frac{1}{2}(x+1)\right]^2 - 4\left[\frac{1}{2}(x+1)\right] + 1 = (x+1)^2 - 2(x+1) + 1 = x^2 + 2x + 1 - 2x - 2 + 1 = x^2.$$

B3. Functions

Tutorial B3

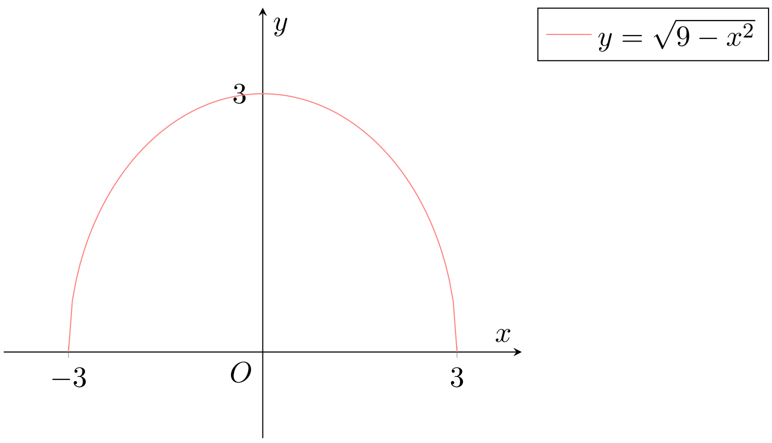
Problem 1. Sketch the following graphs and determine whether each graph represents a function for the given domain.

(a) $y = \sqrt{9 - x^2}$, $-3 \leq x \leq 3$

(b) $x = (y - 4)^2$, $y \in \mathbb{R}$

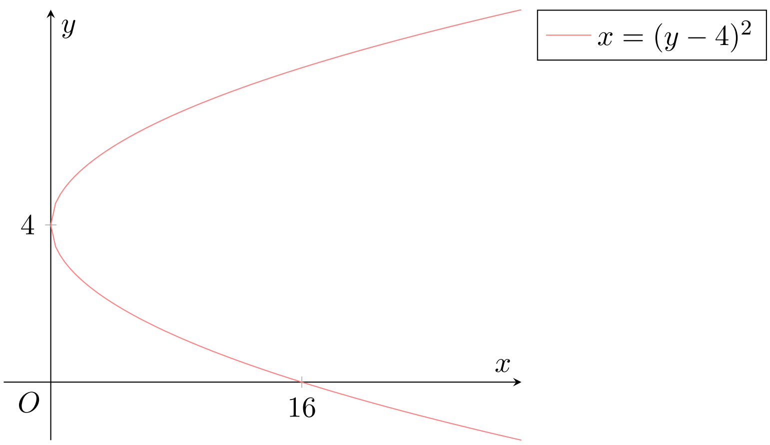
Solution.

Part (a).



$y = \sqrt{9 - x^2}$ passes the vertical line test for $-3 \leq x \leq 3$ and is hence a function.

Part (b).



$x = (y - 4)^2$ does not pass the vertical line test for $y \in \mathbb{R}$ and is hence not a function.

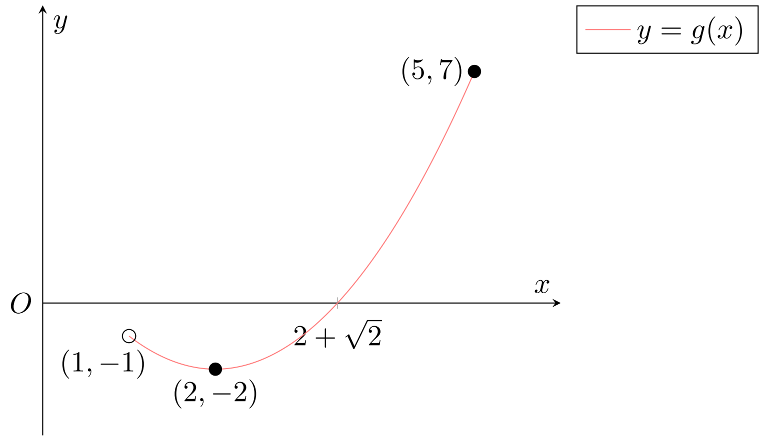
Problem 2. Sketch the graph and find the range for each the following functions.

(a) $g\colon x \mapsto x^2 - 4x + 2,\ 1 < x \leq 5$

(b) $h\colon x \mapsto |2x - 3|\,,\ x < 3$

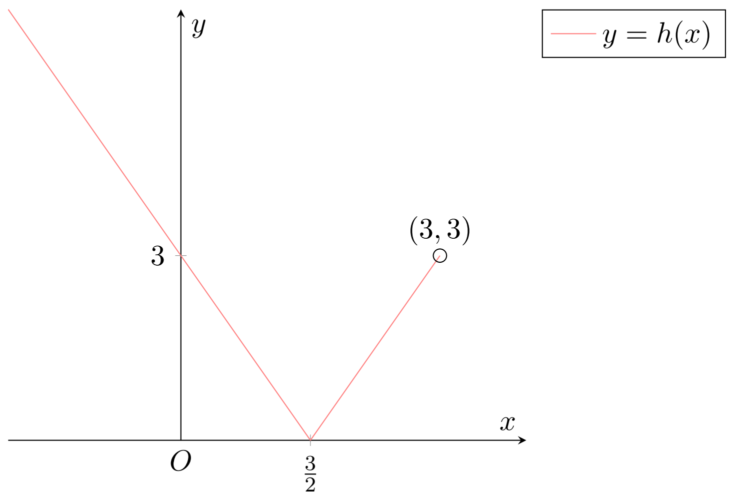
Solution.

Part (a).



From the graph, $R_g = [-2, 7)$.

Part (b).



From the graph, $R_h = [0, \infty)$.

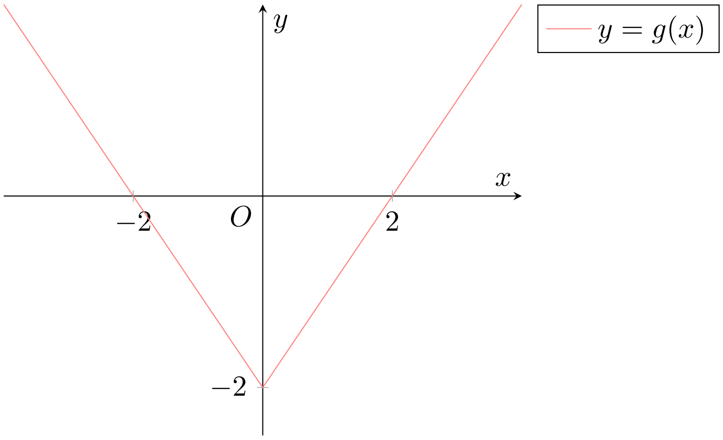
Problem 3. For each of the following functions, sketch its graph and determine if the function is one-one. If it is, find its inverse in a similar form.

(a) $g\colon x \mapsto |x| - 2,\ x \in \mathbb{R}$

(b) $h\colon x \mapsto x^2 + 2x + 5,\ x \le -2$

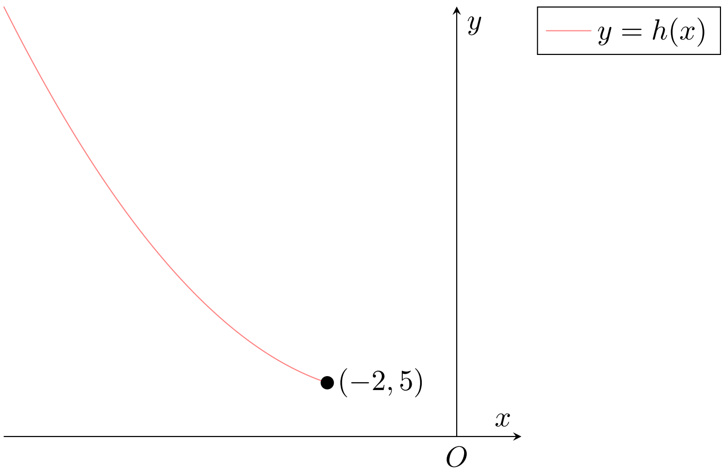
Solution.

Part (a).



$y = g(x)$ does not pass the horizontal line test. Hence, g is not one-one.

Part (b).



$y = h(x)$ passes the horizontal line test. Hence, h is one-one.
Note that $y = h(x) \implies x = h^{-1}(y)$. Now consider $y = h(x)$.

$$y = h(x) = x^2 + 2x + 5 = (x+1)^2 + 4 \implies x = -1 \pm \sqrt{y-4}.$$

Since $x \le -2$, we reject $x = -1 + \sqrt{y-4}$. Note that $D_{h^{-1}} = R_h = [5, \infty)$. Hence,

$$h^{-1}\colon x \mapsto -1 - \sqrt{x-4},\ x \ge 5.$$

Problem 4. The function f is defined by

$$f\colon x \mapsto x + \frac{1}{x},\ x \neq 0.$$

(a) Sketch the graph of f and explain why f^{-1} does not exist.

(b) The function h is defined by $h\colon x \mapsto f(x)$, $x \in \mathbb{R}$, $x \geq \alpha$, where $\alpha \in \mathbb{R}^+$. Find the smallest value of α such that the inverse function of h exists.

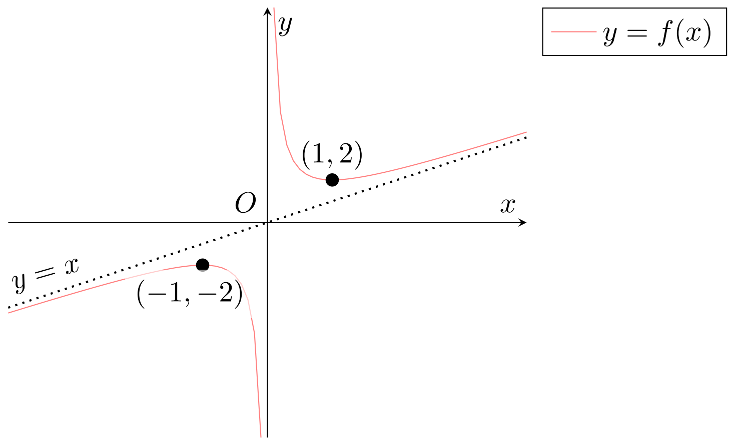
Using this value of α,

(c) State the range of h.

(d) Express h^{-1} in a similar form and sketch on a single diagram, the graphs of h and h^{-1}, showing clearly their geometrical relationship.

Solution.

Part (a).



$y = f(x)$ does not pass the horizontal line test. Hence, f is not one-one. Hence, f^{-1} does not exist.

Part (b). Consider $f'(x) = 0$ for $x > 0$.

$$f'(x) = 1 - \frac{1}{x^2} = 0 \implies x^2 = 1 \implies x = 1.$$

Note that we reject $x = -1$ since $x > 0$.

Looking at the graph of $y = f(x)$, we see that $f(x)$ achieves a minimum at $x = 1$. Hence, f is increasing for all $x \geq 1$. Thus, the smallest value of α is 1.

Part (c). Note $f(1) = 2$. Hence, from the graph, $R_h = [2, \infty)$.

Part (d). Note that $y = h(x) \implies x = h^{-1}(y)$. Now consider $y = h(x)$.

$$y = x + \frac{1}{x} \implies xy = x^2 + 1 \implies x^2 - yx + 1 = 0 \implies x = \frac{1}{2}\left(y \pm \sqrt{y^2 - 4}\right).$$

Note that $f(2) = \frac{5}{2}$. Since $2 = \frac{1}{2}\left(\frac{5}{2} + \sqrt{\left(\frac{5}{2}\right)^2 - 4}\right)$ and $2 \neq \frac{1}{2}\left(\frac{5}{2} - \sqrt{\left(\frac{5}{2}\right)^2 - 4}\right)$, we reject $x = \frac{1}{2}(y - \sqrt{y^2 - 4})$. Since $D_{f^{-1}} = R_f = [2, \infty)$, we thus have

$$h^{-1}\colon x \mapsto \frac{1}{2}\left(x + \sqrt{x^2 - 4}\right),\ x \geq 2.$$

Problem 5. The function f is defined as follows:

$$f : x \mapsto x^3 + x - 7,\ x \in \mathbb{R}.$$

(a) By using a graphical method or otherwise, show that the inverse of f exists.

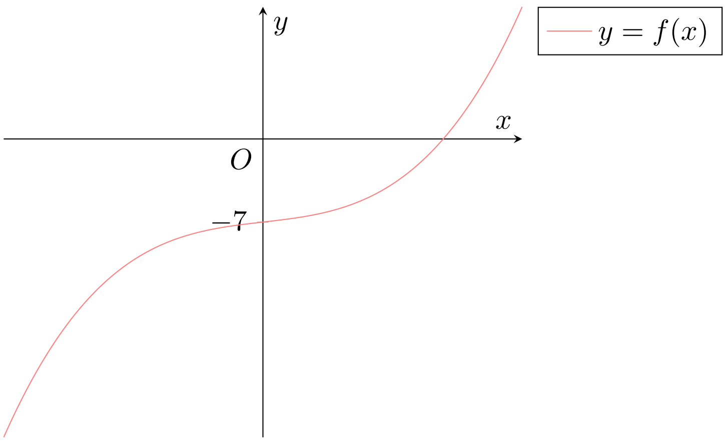
(b) Solve exactly the equation $f^{-1}(x) = 0$. Sketch the graph of f^{-1} together with the graph of f on the same diagram.

(c) Find, in exact form, the coordinates of the intersection point(s) of the graphs of f and f^{-1}.

(d) Given that the gradient of the tangent to the curve with equation $y = f^{-1}(x)$ is $\frac{1}{4}$ at the point with $x = p$, find the possible values of p.

Solution.

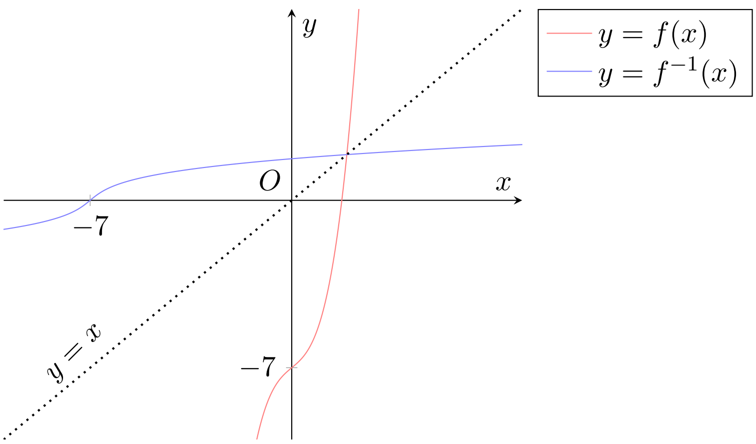
Part (a).



$y = f(x)$ passes the horizontal line test. Hence, f is one-one. Thus, f^{-1} exists.

Part (b). We have

$$f^{-1}(x) = 0 \implies x = f(0) = -7.$$



Part (c). Let (α, β) be the coordinates of the intersection between $f(x)$ and f^{-1}. From the graph, we see that $\alpha = \beta$, hence $f(\alpha) = \alpha$. Hence,

$$f(\alpha) = \alpha^3 + \alpha - 7 = \alpha \implies \alpha^3 = 7 \implies \alpha = \sqrt[3]{7}.$$

The coordinates are thus $\left(\sqrt[3]{7}, \sqrt[3]{7}\right)$.

Part (d). Note that

$$[f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))}.$$

Evaluating at $x = p$, we obtain

$$\frac{1}{4} = \left.\frac{1}{f'(f^{-1}(x))}\right|_{x=p} \implies f'(f^{-1}(x))\big|_{x=p} = 4.$$

Since $f'(x) = 3x^2 + 1$,

$$3f^{-1}(p)^2 + 1 = 4 \implies f^{-1}(p)^2 = 1 \implies f^{-1}(p) = \pm 1.$$

Case 1: $f^{-1}(p) = 1$. Then $p = f(1) = -5$.
Case 2: $f^{-1}(p) = -1$. Then $p = f(-1) = -9$.
Hence, $p = -5$ or $p = -9$.

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Problem 6. The functions g and h are defined as follows:

$$\begin{aligned} g &\colon x \mapsto \ln(x+2), && x \in (-1, 1) \\ h &\colon x \mapsto x^2 - 2x - 1, && x \in \mathbb{R}^+ \end{aligned}$$

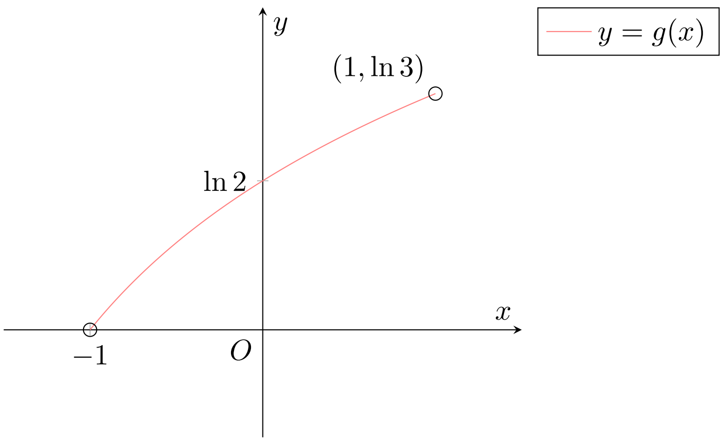
(a) Sketch, on separate diagrams, the graphs of g and h.

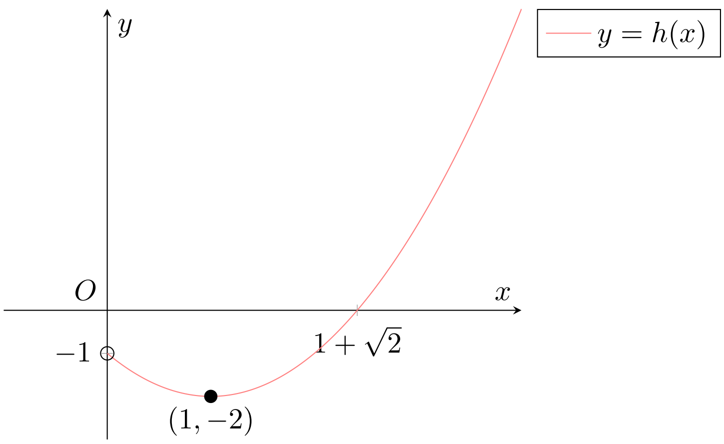
(b) Determine whether the composite function gh exists.

(c) Give the rule and domain of the composite function hg and find its range.

(d) The image of a under the composite function hg is -1.5. Find the value of a.

Solution.
Part (a).





Part (b). Observe that $R_h = [-2, \infty)$ and $D_g = (-1, 1)$. Hence, $R_h \nsubseteq D_g$. Thus, gh does not exist.

Part (c).

$$hg(x) = h(\ln(x+2)) = \ln(x+2)^2 - 2\ln(x+2) - 1.$$

Also note that $D_{hg} = D_g = (-1, 1)$. Hence,

$$hg\colon x \mapsto \ln(x+2)^2 - 2\ln(x+2) - 1,\ x \in (-1, 1).$$

Observe that h is decreasing on the interval $(0, 1]$ and increasing on the interval $[1, \infty)$. Note that $R_g = (0, \ln 3)$. Hence,

$$R_{hg} = [-2, \max\{h(0), h(\ln 3)\}) = [-2, -1).$$

Part (d). Note that $h(x) = (x-1)^2 - 2$. Hence, $h^{-1}(x) = 1 + \sqrt{x+2}$ (we reject $h^{-1}(x) = 1 - \sqrt{x+2}$ since $R_{h^{-1}} = D_h = \mathbb{R}^+$). Also note that $g^{-1} = \mathrm{e}^x - 2$. Thus,

$$\begin{aligned} hg(a) = -1.5 &\implies g(a) = h^{-1}(-1.5) = 1 + \sqrt{-1.5+2} = 1 + \frac{1}{\sqrt{2}} \\ &\implies a = g^{-1}\left(1 + \frac{1}{\sqrt{2}}\right) = \mathrm{e}^{1+\frac{1}{\sqrt{2}}} - 2. \end{aligned}$$

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Problem 7. The functions f and g are defined as follows:

$$\begin{aligned} f\colon x &\mapsto 3 - x, && x \in \mathbb{R} \\ g\colon x &\mapsto \frac{4}{x}, && x \in \mathbb{R},\ x \neq 0 \end{aligned}$$

(a) Show that the composite function fg exists and express the definition of fg in a similar form. Find the range of fg.

(b) Find, in similar form, g^2 and g^3, and deduce g^{2017}.

(c) Find the set of values of x for which $g(x) = g^{-1}(x)$.

Solution.

Part (a). Note that $R_g = \mathbb{R} \setminus \{0\}$ and $D_g = \mathbb{R}$. Hence, $R_g \subseteq D_g$. Thus, fg exists.

$$fg(x) = f\left(\frac{4}{x}\right) = 3 - \frac{4}{x}.$$

Observe that $D_{fg} = D_g = \mathbb{R} \setminus \{0\}$. Thus,

$$fg\colon x \mapsto 3 - \frac{4}{x},\ x \in \mathbb{R} \setminus \{0\}.$$

Since $\frac{4}{x}$ can take on any value except 0, then $fg(x) = 3 - \frac{4}{x}$ can take on any value except 3. Thus,

$$R_{fg} = \mathbb{R} \setminus \{3\}.$$

Part (b). We have

$$g^2(x) = g\left(\frac{4}{x}\right) = \frac{4}{4/x} = x.$$

Hence,

$$g^2\colon x \mapsto x,\ x \in \mathbb{R} \setminus \{0\}.$$

We have

$$g^3(x) = g(g^2(x)) = g(x) = \frac{4}{x}.$$

Hence,

$$g^3\colon x \mapsto \frac{4}{x},\ x \in \mathbb{R} \setminus \{0\}.$$

Thus,

$$g^{2017} = g^{2016}(g(x)) = \left(g^2\right)^{1008} \circ g(x) = g(x) = \frac{4}{x}.$$

Hence,

$$g^{2017}\colon x \mapsto \frac{4}{x},\ x \in \mathbb{R} \setminus \{0\}.$$

Part (c). Note that $g(x) = g^{-1}(x) \implies g^2(x) = x$. From the definition of $g^2(x)$, we know that $g^2(x) = x$ for all x in D_{g^2}. Hence, the solution set is $\mathbb{R} \setminus \{0\}$.

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Problem 8. The function f is defined by

$$f(x) = \begin{cases} 2x+1, & 0 \leq x < 2 \\ (x-4)^2+1, & 2 \leq x < 4. \end{cases}$$

It is further given that $f(x) = f(x+4)$ for all real values of x.

(a) Find the values of $f(1)$ and $f(5)$ and hence explain why f is not one-one.

(b) Sketch the graph of $y = f(x)$ for $-4 \leq x < 8$.

(c) Find the range of f for $-4 \leq x < 8$.

Solution.

Part (a). We have

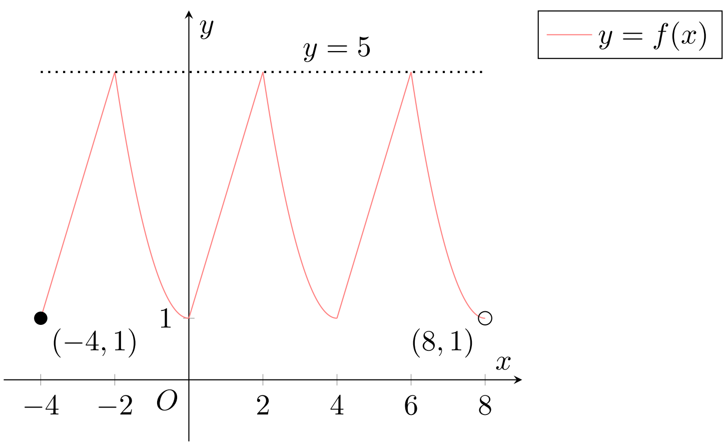
$$f(1) = 2(1) + 1 = 3$$

and

$$f(5) = f(1+4) = f(1) = 3.$$

Since $f(1) = f(5)$, but $1 \neq 5$, f is not one-one.

Part (b).



Part (c). From the graph, $R_f = [1, 5]$.

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Problem 9.

(a) The function f is given by $f \colon x \mapsto 1 + \sqrt{x}$ for $x \in \mathbb{R}^+$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}.

(ii) Find $f^2(x)$ and the range of f^2.

(iii) Show that if $f^2(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence, find the value of x for which $f^2(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$.

(b) The function g, with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1, & n = 0 \\ 2 + g\left(\frac{1}{2}n\right), & n \text{ even} \\ 1 + g(n-1), & n \text{ odd} \end{cases}$$

(i) Find $g(4)$, $g(7)$ and $g(12)$.

(ii) Does g have an inverse? Justify your answer.

Solution.

Part (a).

Part (a)(i). Let $y = f(x)$. Then $x = f^{-1}(y)$.

$$y = f(x) = 1 + \sqrt{x} \implies \sqrt{x} = y - 1 \implies x = (y-1)^2.$$

Hence, $f^{-1}(x) = (x-1)^2$.

Observe that $D_{f^{-1}} = R_f = (1, \infty)$. Thus, $D_{f^{-1}} = (1, \infty)$.

Part (a)(ii). We have

$$f^2(x) = f(1 + \sqrt{x}) = 1 + \sqrt{1 + \sqrt{x}}.$$

Observe that $\sqrt{1+\sqrt{x}} > 1$. Hence, $1 + \sqrt{1 + \sqrt{x}} > 1 + 1 = 2$, whence $R_{f^2} = (2, \infty)$.

Part (a)(iii). Note that $f^2(x) = x \implies 1 + \sqrt{1 + \sqrt{x}} = x$, whence x satisfies the recursion $1 + \sqrt{x} = x$. Hence,

$$1 + \sqrt{x} = x \implies \sqrt{x} = x - 1 \implies x = x^2 - 2x + 1 \implies x^2 - 3x + 1 = 0.$$

We can manipulate this to form the desired cubic equation:

$$0 = x\left(x^2 - 3x + 1\right) - \left(x^2 - 3x + 1\right) = x^3 - 4x^2 + 4x - 1.$$

Solving the initial quadratic equation yields $x = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$. Observe that $\frac{3-\sqrt{5}}{2} < 2$ and $\frac{3+\sqrt{5}}{2} > 2$. Thus, the sole solution is $x = \frac{3+\sqrt{5}}{2}$.

Consider $f(x) = f^{-1}(x)$. Applying f on both sides of the equation, we have $f^2(x) = f(x)$. Since $x = \frac{3+\sqrt{5}}{2}$ satisfies $f^2(x) = f(x)$, it also satisfies $f(x) = f^{-1}(x)$.

Part (b).

Part (b)(i). We have

$$g(4) = 2 + g(2) = 2 + 2 + g(1) = 4 + 1 + g(0) = 5 + 1 = 6,$$

$$g(7) = 1 + g(6) = 1 + 2 + g(3) = 3 + 1 + g(2) = 4 + (g(4) - 2) = 2 + 6 = 8,$$

and

$$g(12) = 2 + g(6) = 2 + (g(7) - 1) = 1 + 8 = 9.$$

Part (b)(ii). Consider $g(5)$ and $g(6)$.

$$g(5) = 1 + g(4) = 1 + 6 = 7, \quad g(6) = g(7) - 1 = 8 - 1 = 7.$$

Since $g(5) = g(6)$, but $5 \neq 6$, g is not one-one. Hence, g^{-1} does not exist.

Assignment B3

Problem 1. Functions f and g are defined as follows:

$$\begin{aligned} f\colon x &\mapsto (x-3)^2+6, && x\in\mathbb{R},\ x\leq 2\\ g\colon x &\mapsto \ln(x-2), && x\in\mathbb{R},\ x>3 \end{aligned}$$

(a) Show that f^{-1} exists and define f^{-1} in a similar form.

(b) Sketch, on the same diagram, the graphs of f, f^{-1} and ff^{-1}.

(c) Find fg and gf if they exist, and find their ranges (where applicable).

Solution.

Part (a). Note that $f' = 2(x-3) < 0$ for all $x \leq 2$. Thus, f is strictly decreasing. Since f is also continuous, f is one-one. Thus, f^{-1} exists.

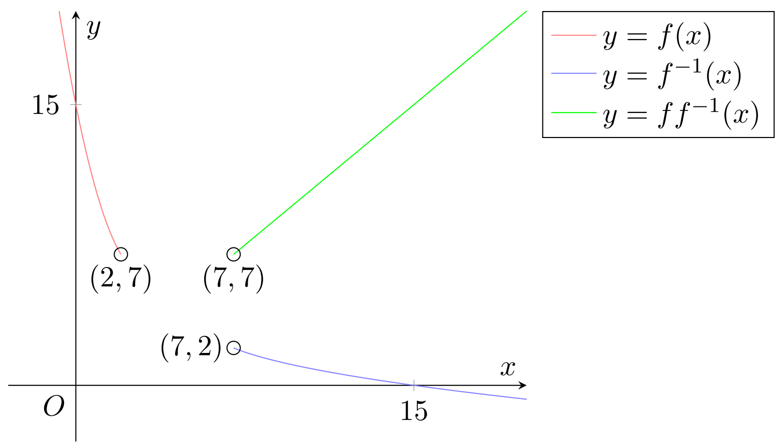
Let $y = f(x) \implies x = f^{-1}(y)$.

$$y = f(x) = (x-3)^2+6 \implies x = 3 \pm \sqrt{y-6}.$$

Since $x < 3$, we reject $x = 3+\sqrt{y-6}$. Lastly, observe that $D_{f^{-1}} = R_f = [f(2), \infty) = [7, \infty)$. Thus,

$$f^{-1}\colon x \mapsto 3-\sqrt{x-6},\ x\in\mathbb{R},\ x\geq 7.$$

Part (b).



Part (c). Note that $R_g = (0,\infty)$ and $D_f = (-\infty, 2]$. Hence, $R_g \not\subseteq D_f$. Thus, fg does not exist. Note that $R_f = [7,\infty)$ and $D_g = (3,\infty)$. Hence, $R_f \subseteq D_g$. Thus, gf exists.

Since $\ln x$ is a strictly increasing function, we have that g is also strictly increasing. Hence, $R_{gf} = [\ln(7-2), \infty) = [\ln 5, \infty)$.

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Problem 2. The function f is defined as follows:

$$f\colon x\mapsto \frac{1}{x^2-1}, \qquad x\in\mathbb{R},\ x\neq -1,\ x\neq 1.$$

(a) Sketch the graph of $y = f(x)$.

(b) If the domain of f is further restricted to $x \geq k$, state with a reason the least value of k for which the function f^{-1} exists.

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \neq -1$, $x \neq 1$, as originally defined.

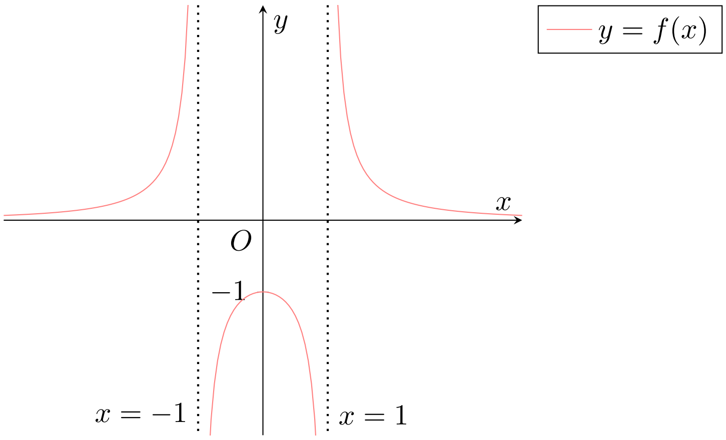
The function g is defined as follows:

$$g\colon x \mapsto \frac{1}{x-3}, \qquad x \in \mathbb{R},\ x \neq 2,\ x \neq 3, x \neq 4.$$

(c) Find the range of fg.

Solution.

Part (a).



Part (b). If the domain of f is further restricted to $x \geq 0$, f would pass the horizontal line test, whence f^{-1} would exist. Hence, $\min k = 0$.

Part (c). Observe that $R_g = \mathbb{R} \setminus \{g(2), g(4)\} = \mathbb{R} \setminus \{-1, 1\}$. Hence, $R_{fg} = R_f = \mathbb{R} \setminus (-1, 0]$.

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Problem 3. The function f is defined by

$$f\colon x \mapsto \frac{x}{x^2-1}, \qquad x \in \mathbb{R},\ x \neq -1,\ x \neq 1.$$

(a) Explain why f does not have an inverse.

(b) The function f has an inverse if the domain is restricted to $x \leq k$. State the largest value of k.

The function g is defined by

$$g\colon x \mapsto \ln x - 1, \qquad x \in \mathbb{R},\ 0 < x < 1.$$

(c) Find an expression for $h(x)$ for each of the following cases:

(i) $gh(x) = x$

(ii) $hg(x) = x^2 + 1$

Solution.

Part (a). Observe that $f(1/2) = -2/3$ and $f(-2) = -2/3$. Hence, $f(1/2) = f(-2)$. Since $1/2 \neq -2$, f is not one-one. Thus, f does not have an inverse.

Part (b). Clearly, $\max k = 0$.

Part (c).

Part (c)(i). Note that $gh(x) = x \implies h(x) = g^{-1}(x)$. Hence, consider $y = g(x) \implies x = h(y)$.

$$y = g(x) = \ln x - 1 \implies \ln x = y + 1 \implies x = \mathrm{e}^{y+1}.$$

Hence, $h(x) = \mathrm{e}^{x+1}$.

Part (c)(ii). Let $h = h_2 \circ h_1$ such that $h_1 g(x) = x \implies h_1(x) = g^{-1}(x) \implies h_1(x) = \mathrm{e}^{x+1}$. Then

$$hg(x) = x^2 + 1 \implies h_2 h_1 g(x) = x^2 + 1 \implies h_2(x) = x^2 + 1.$$

Hence, $h(x) = h_2 h_1(x) = h_2(\mathrm{e}^{x+1}) = \left(\mathrm{e}^{x+1}\right)^2 + 1 = \mathrm{e}^{2x+2} + 1$.

B4. Differentiation

Tutorial B4

Problem 1. Evaluate the following limits.

(a) $\lim\limits_{x\to 5}(6x+7)$

(b) $\lim\limits_{x\to 1}\frac{x^3-1}{1-x}$

(c) $\lim\limits_{x\to\infty}\frac{3x}{2x^2-5}$

Solution.

Part (a).

$$\lim_{x\to 5}(6x+7)=6(5)+7=37.$$

Part (b).

$$\lim_{x\to 1}\frac{x^3-1}{1-x}=\lim_{x\to 1}\frac{(x-1)\left(x^2+x+1\right)}{1-x}=\lim_{x\to 1}-\left(x^2+x+1\right)=-(1^2+1+1)=-3.$$

Part (c).

$$\lim_{x\to\infty}\frac{3x}{2x^2-5}=\lim_{x\to\infty}\frac{3}{2x-5/x}.$$

Note that as $x\to\infty$, $2x-\frac{5}{x}\to\infty$. Hence, $\lim\limits_{x\to\infty}\frac{1}{2x-5/x}=0$.

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Problem 2. Differentiate the following with respect to x from first principles.

(a) $3x+4$

(b) x^3

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x+4)=\lim_{h\to 0}\frac{[3(x+h)+4]-[3x+4]}{h}=\lim_{h\to 0}\frac{3h}{h}=\lim_{h\to 0}3=3.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}x^3=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{3hx^2+3h^2x+h^3}{h}=\lim_{h\to 0}\left(3x^2+3hx+h^2\right)=3x^2.$$

Problem 3. Differentiate each of the following with respect to x, simplifying your answer.

(a) $\left(x^2+4\right)^2\left(2x^3-1\right)$

(b) $\frac{x^2}{\sqrt{4-x^2}}$

(c) $\sqrt{1+\sqrt{x}}$

(d) $\left(\frac{x^3-1}{2x^3+1}\right)^4$

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+4\right)^2\left(2x^3-1\right)=\left(2x^3-1\right)\left[4x\left(x^2+4\right)\right]+\left(x^2+4\right)^2\left(6x^2\right)$$
$$=2x\left(x^2+4\right)\left[2\left(2x^3-1\right)+3x\left(x^2+4\right)\right]=2x\left(x^2+4\right)\left(7x^3+12x-2\right).$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{\sqrt{4-x^2}}=\frac{\sqrt{4-x^2}\,(2x)-x^2\left(\frac{-2x}{2\sqrt{4-x^2}}\right)}{4-x^2}=\frac{2x\left(4-x^2\right)+x^3}{(4-x^2)^{3/2}}=\frac{x\left(8-x^2\right)}{(4-x^2)^{3/2}}.$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\sqrt{x}}=\frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{4\sqrt{x(1+\sqrt{x})}}.$$

Part (d). Note that

$$\frac{x^3-1}{2x^3+1}=\frac{1}{2}\left(\frac{2x^3-2}{2x^3+1}\right)=\frac{1}{2}\left(1-\frac{3}{2x^3+1}\right)=\frac{1}{2}-\frac{3}{2}\left(\frac{1}{2x^3+1}\right).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3-1}{2x^3+1}=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}-\frac{3}{2}\left(\frac{1}{2x^3+1}\right)\right]=-\frac{3}{2}\left[\frac{-6x^2}{(2x^3+1)^2}\right]=\frac{9x^2}{(2x^3+1)^2}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-1}{2x^3+1}\right)^4=4\left(\frac{x^3-1}{2x^3+1}\right)^3\frac{9x^2}{\left(2x^3+1\right)^2}=\frac{36x^2\left(x^3-1\right)^3}{\left(2x^3+1\right)^5}.$$

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Problem 4. Using a graphing calculator, find the derivative of $\frac{\mathrm{e}^{2x}}{x^2+1}$ when $x=1.5$.

Solution.

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{2x}}{x^2+1}\right)\right|_{x=1.5}=6.66.$$

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Problem 5. Find the derivative with respect to x of

(a) $\cos x^\circ$

(b) $\cot\left(1-2x^2\right)$

(c) $\tan^3(5x)$

(d) $\frac{\sec x}{1+\tan x}$

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x^\circ = \frac{\mathrm{d}}{\mathrm{d}x}\cos\Bigl(\frac{\pi}{180}x\Bigr) = -\frac{\pi}{180}\sin\Bigl(\frac{\pi}{180}x\Bigr).$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\cot\bigl(1-2x^2\bigr) = 4x\csc\bigl(1-2x^2\bigr).$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan^3(5x) = 15\tan^2(5x)\sec^2(5x).$$

Part (d). Note that

$$\frac{\sec x}{1+\tan x} = \frac{1}{\sin x+\cos x} = \frac{1}{\sqrt{2}\sin(x+\pi/4)} = \frac{1}{\sqrt{2}}\csc\Bigl(x+\frac{\pi}{4}\Bigr).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sec x}{1+\tan x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{\sqrt{2}}\csc\Bigl(x-\frac{\pi}{4}\Bigr) = -\frac{1}{\sqrt{2}}\csc\Bigl(x+\frac{\pi}{4}\Bigr)\cot\Bigl(x+\frac{\pi}{4}\Bigr).$$

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Problem 6. Find the derivative with respect to x of

(a) $y = \mathrm{e}^{1+\sin 3x}$

(b) $y = x^2\mathrm{e}^{\frac{1}{x}}$

(c) $y = \ln\Bigl(\frac{1-x}{\sqrt{1+x^2}}\Bigr)$

(d) $y = \frac{\ln(2x)}{x}$

(e) $y = \log_2(3x^4 - \mathrm{e}^x)$

(f) $y = 3^{\ln\sin x}$

(g) $y = a^{2\log_a x}$

(h) $y = \sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}}$, $x > 0$

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{1+\sin 3x} = 3\mathrm{e}^{1+\sin 3x}\cos(3x).$$

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}x^2\mathrm{e}^{1/x} = -x^2\left(-\frac{\mathrm{e}^{1/x}}{x^2}\right) + \mathrm{e}^{1/x}(2x) = \mathrm{e}^{1/x}(2x-1).$$

Part (c). Note that

$$y = \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right) = \ln(1-x) - \frac{1}{2}\ln\left(1+x^2\right).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(1-x) - \frac{1}{2}\ln\left(1+x^2\right)\right] = -\frac{1}{1-x} - \frac{x}{1+x^2} = -\frac{1+x}{(1-x)\left(1+x^2\right)}.$$

Part (d).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln(2x)}{x} = \frac{x\left(\frac{1}{x}\right) - \ln(2x)(1)}{x^2} = \frac{1-\ln(2x)}{x^2}.$$

Part (e). Note that

$$y = \log_2\left(3x^4 - \mathrm{e}^x\right) \implies 2^y = 3x^4 - \mathrm{e}^x.$$

Implicitly differentiating with respect to x,

$$2^y \ln 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 - \mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12x^3 - \mathrm{e}^x}{2^y \ln 2} = \frac{12x^3 - \mathrm{e}^x}{\left(3x^4 - \mathrm{e}^x\right)\ln 2}.$$

Part (f). Note that

$$y = 3^{\ln\sin x} \implies \log_3 y = \frac{\ln y}{\ln 3} = \ln\sin x \implies \ln y = \ln 3 \ln\sin x.$$

Implicitly differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3\left(\frac{\cos x}{\sin x}\right) = \ln 3 \cdot \cot x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3 \cdot y\cot x = \ln 3 \cdot \cot(x)\cdot 3^{\ln\sin x}.$$

Part (g). Observe that

$$y = a^{2\log_a x} = a^{\log_a x^2} = x^2 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}x^2 = 2x.$$

Part (h). Note that

$$y = \sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}} \implies \left(x^2+1\right)y^3 = \mathrm{e}^x(x+1).$$

Implicitly differentiating with respect to x,

$$\left(x^2+1\right)\left(3y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right) + y^3\,(2x) = \mathrm{e}^x + (x+1)\mathrm{e}^x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x(x+2) - 2xy^3}{3\left(x^2+1\right)y^2}.$$

Now observe that

$$\frac{\mathrm{e}^x(x+2)}{3\left(x^2+1\right)y^2} = \frac{\mathrm{e}^x(x+1)(x+2)}{3\left(x^2+1\right)(x+1)y^2} = \frac{y^3(x+2)}{3(x+1)y^2} = y\left(\frac{x+2}{3(x+1)}\right).$$

Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left(\frac{x+2}{3(x+1)}\right) - y\left(\frac{2x}{3\left(x^2+1\right)}\right) = \frac{1}{3}\sqrt[3]{\frac{\mathrm{e}^x(x+1)}{x^2+1}}\left(\frac{x+2}{x+1} - \frac{2x}{x^2+1}\right).$$

Problem 7. Find the derivative with respect to x of

(a) $\arccos\left(\frac{x}{10}\right)$

(b) $\arctan\left(\frac{1}{1-x}\right)$

(c) $\arcsin(\tan x)$

Solution.

Part (a).

$$\frac{\mathrm{d}}{\mathrm{d}x} \arccos \frac{x}{10} = -\frac{1}{10\sqrt{1-\left(\frac{x}{10}\right)^2}} = -\frac{1}{\sqrt{100-x^2}}.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan\left(\frac{1}{1-x}\right) = \frac{1}{1+\left(\frac{1}{1-x}\right)^2}\left(\frac{1}{(1-x)^2}\right) = \frac{1}{(1-x)^2+1}.$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin(\tan x) = \frac{\sec^2 x}{1-\tan^2 x}.$$

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Problem 8. Find an expression for $\mathrm{d}y/\mathrm{d}x$ in terms of x and y.

(a) $(y-x)^2 = 2a(y+x)$, where a is a constant

(b) $y^2 = \mathrm{e}^{2x}y + x\mathrm{e}^x$

(c) $y = x^y$

(d) $\sin x \cos y = \frac{1}{2}$

Solution.

Part (a). Implicitly differentiating with respect to x,

$$(y-x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = a\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a+y-x}{y-x-a}.$$

Part (b). Implicitly differential with respect to x,

$$2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\mathrm{e}^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + 2ye^{2x}\right) + (x\mathrm{e}^x + \mathrm{e}^x) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x\,(2y\mathrm{e}^x + x + 1)}{2y - \mathrm{e}^{2x}}.$$

Part (c). Note that

$$y = x^y \implies \ln y = y \ln x.$$

Implicitly differentiating with respect to x,

$$\frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \ln x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y/x}{1/y - \ln x} = \frac{y^2}{x - xy\ln x}.$$

Part (d). Note that

$$\sin x \cos y = \frac{1}{2} \implies \cos y = \frac{1}{2}\csc x.$$

Implicitly differentiating with respect to x,

$$-\sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\csc x \cot x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\csc x \cot x}{2\sin y}.$$

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Problem 9. It is given that x and y satisfy the equation

$$\arctan x + \arctan y + \arctan(xy) = \frac{7}{12}\pi.$$

(a) Find the exact value of y when $x = 1$.

(b) Express $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(xy)$ in terms of x, y and y'.

(c) Show that, when $x = 1$, $y' = -\frac{1}{3} - \frac{1}{2\sqrt{3}}$.

Solution.
Part (a). Substituting $x = 1$ into the given equation,

$$\frac{1}{4}\pi + 2\arctan y = \frac{7\pi}{12} \implies \arctan y = \frac{\pi}{6} \implies y = \frac{1}{\sqrt{3}}.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(xy) = \frac{xy' + y}{1 + (xy)^2}.$$

Part (c). Differentiating the given equation with respect to x,

$$\frac{1}{1+x^2} + \frac{y'}{1+y^2} + \frac{xy'+y}{1+(xy)^2} = 0.$$

Substituting $x = 1$,

$$\frac{1}{2} + \frac{3y'}{4} + \frac{3}{4}\left(y' + \frac{1}{\sqrt{3}}\right) = 0 \implies y' = \frac{2}{3}\left(-\frac{3}{4\sqrt{3}} - \frac{1}{2}\right) = -\frac{1}{2\sqrt{3}} - \frac{1}{3}.$$

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Problem 10. Find $\mathrm{d}y/\mathrm{d}x$ for

(a) $x = \frac{1}{1+t^2}$, $y = \frac{t}{1+t^2}$

(b) $x = \frac{1}{2}(\mathrm{e}^t - \mathrm{e}^{-t})$, $y = \frac{1}{2}(\mathrm{e}^t + \mathrm{e}^{-t})$

(c) $x = a\sec\theta$, $y = a\tan\theta$

(d) $x = \mathrm{e}^{3\theta}\cos(3\theta)$, $y = \mathrm{e}^{3\theta}\sin(3\theta)$

Solution.
Part (a). Observe that $y = xt$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{\mathrm{d}t}{\mathrm{d}x}\right) + t = x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} + t = \frac{1}{1+t^2}\left(-\frac{2t}{(1+t^2)^2}\right)^{-1} + t = \frac{t^2-1}{2t}.$$

Part (b).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{\frac{1}{2}\left(\mathrm{e}^t - \mathrm{e}^{-t}\right)}{\frac{1}{2}\left(\mathrm{e}^t + \mathrm{e}^{-t}\right)} = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}}.$$

Part (c). Recall that $\tan^2\theta + 1 = \sec^2\theta$. Hence, $y^2 + a^2 = x^2$. Implicitly differentiating with respect to x, we have

$$2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 2x \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} = \frac{a\sec\theta}{a\tan\theta} = \csc\theta.$$

Part (d).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}(3\theta)}{\mathrm{d}x/\mathrm{d}(3\theta)} = \frac{\mathrm{e}^{3\theta}\cos 3\theta + \mathrm{e}^{3\theta}\sin 3\theta}{-\mathrm{e}^{3\theta}\sin 3\theta + \mathrm{e}^{3\theta}\cos 3\theta} = \frac{\cos 3\theta + \sin 3\theta}{\cos 3\theta - \sin 3\theta}.$$

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Problem 11. A curve is defined by the parametric equation

$$x = 120t - 4t^2,\ y = 60t - 6t^2.$$

Find the value of $\mathrm{d}y/\mathrm{d}x$ at each of the points where the curve cross the x-axis.

Solution. The curve crosses the x-axis when $y = 0$:

$$y = 60t - 6t^2 = 6t(10 - t) = 0.$$

Hence, $t = 0$ or $t = 10$. Now, consider the derivative with respect to x of the curve.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{60 - 12t}{120 - 8t}.$$

Case 1: $t = 0$.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0} = \frac{60 - 12(0)}{120 - 8(0)} = \frac{1}{2}.$$

Case 2: $t = 10$.

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=10} = \frac{60 - 12(10)}{120 - 8(10)} = -\frac{3}{2}.$$

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Problem 12. A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln t^2$, where $t > 0$. Find the value of t at the point on the curve at which the gradient is 2.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{2t - 2/t}{2 - 1/t} = \frac{2t^2 - 2}{2t - 1}.$$

Consider $\mathrm{d}y/\mathrm{d}x = 2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2t^2 - 2}{2t - 1} = 2 \implies \frac{t^2 - 1}{2t - 1} = 1 \implies t^2 - 1 = 2t - 1 \implies t^2 - 2t = t(t - 2) = 0.$$

Hence, $t = 0$ or $t = 2$. Since $t > 0$, we reject $t = 0$. Thus, $t = 2$.

Problem 13. If $y = \ln\left(\sin^3 2x\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and prove that $3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 = 0$.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6\sin^2 2x\cos 2x}{\sin^3 2x} = 6\cot 2x.$$

Hence,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -12\csc^2 2x = -12\left(1+\cot^2 2x\right) = -12 - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2.$$

Thus, we clearly have

$$3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 = 0.$$

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Problem 14. Given that $y = \mathrm{e}^{\arcsin(2x)}$, show that $\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 4y$. Differentiate this result further to obtain a differential equation for $\frac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Solution. Note that

$$y = \mathrm{e}^{\arcsin(2x)} \implies \ln y = \arcsin(2x).$$

Implicitly differentiating with respect to x,

$$\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{\sqrt{1-4x^2}} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{\sqrt{1-4x^2}}.$$

Implicitly differentiating with respect to x once again,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\sqrt{1-4x^2}\left(2\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2y\left(\frac{-4x}{\sqrt{1-4x^2}}\right)}{1-4x^2}.$$

Now observe that

$$2\sqrt{1-4x^2}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\left(\frac{2y}{\sqrt{1-4x^2}}\right) = 4y + 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Hence,

$$\left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4y + 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \implies \left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 4y.$$

Implicitly differentiating with respect to x once again,

$$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 8x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\left(x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 4\cdot\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Rearranging,

$$\left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 12x\cdot\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Assignment B4

Problem 1. Differentiate the following with respect to x.

(a) $\ln \frac{x^3}{\sqrt{1+x^2}}$

(b) $\arctan\left(\frac{x^2}{2}\right)$

(c) $\mathrm{e}^{2x} \sec x$

Solution.

Part (a).

$$\ln \frac{x^3}{\sqrt{1+x^2}} = 3\ln x - \frac{1}{2}\ln\left(1+x^2\right) \implies \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln \frac{x^3}{\sqrt{1+x^2}}\right) = \frac{3}{x} - \frac{x}{1+x^2}.$$

Part (b).

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan\left(\frac{x^2}{2}\right) = \frac{x}{1+x^4/4} = \frac{4x}{4+x^4}.$$

Part (c).

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{2x}\sec x = \mathrm{e}^{2x}\left(\sec x \tan x\right) + \sec x\left(2\mathrm{e}^{2x}\right) = \mathrm{e}^{2x}\sec x\left(\tan x + 2\right).$$

* * * * *

Problem 2. Find the gradient of the curve $x^3 + xy^2 = 5y$ at the point where $x = 1$ and $0 < y < 1$, leaving your answer to 3 significant figures.

Solution. Substituting $x = 1$ into the given equation,

$$y^2 - 5y + 1 = 0 \implies y = \frac{5 \pm \sqrt{21}}{2}.$$

Since $0 < y < 1$, we reject $y = \frac{1}{2}\left(5 + \sqrt{21}\right)$ and take $y = \frac{1}{2}\left(5 - \sqrt{21}\right) = 0.20871$ (5 s.f.).
Implicitly differentiating the given equation,

$$3x^2 + 2xy \cdot y' + y^2 = 5y' \implies y' = \frac{3x^2 - y^2}{5 - 2xy}.$$

Substituting $x = 1$ and $y = 0.20871$ into the above equation,

$$y' = \frac{3(1)^2 - (0.20871)^2}{2(1)(0.20871) - 5} = 0.664 \text{ (3 s.f.)}.$$

Hence, the gradient of the curve is 0.664.

* * * * *

Problem 3. A curve C has parametric equations

$$x = \sin^3\theta,\ y = 3\sin^2\theta\cos\theta, \qquad 0 \leq \theta \leq \frac{\pi}{2}.$$

Show that $\mathrm{d}y/\mathrm{d}x = a\cot\theta + b\tan\theta$, where a and b are values to be determined.

Solution. Note that

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin^2\theta\cos\theta, \quad \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)}{3\sin^2\theta\cos\theta} = \frac{2\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta} = 2\cot\theta - \tan\theta.$$

Thus, $a = 2$ and $b = -1$.

B5. Applications of Differentiation

Tutorial B5A

Problem 1. The equation of a curve is $y = 2x^3 + 3x^2 + 6x + 4$. Find $\mathrm{d}y/\mathrm{d}x$ and hence show that y is increasing for all real values of x.

Solution.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 6x + 6 = 6\left(x + \frac{1}{2}\right)^2 + \frac{18}{4}.$$

For all $x \in \mathbb{R}$, we have $\left(x + \frac{1}{2}\right)^2 \geq 0$. Hence, $\mathrm{d}y/\mathrm{d}x > 0$. Thus, y is increasing for all real values of x.

* * * * *

Problem 2. Find, by differentiation, the x-coordinates of all the stationary points on the curve $y = \frac{x^3}{(x+1)^2}$ stating, with reasons, the nature of each point.

Solution.

$$y = \frac{x^3}{(x+1)^2} \implies (x+1)^2 y = x^3 \implies y'(x+1)^2 + 2y(x+1) = 3x^2.$$

For stationary points, $y' = 0$. Thus,

$$2y(x+1) = \frac{2x^3}{x+1} = 3x^2 \implies 2x^3 - 3x^2(x+1) = x^2\left(-x - 3\right) = 0.$$

Hence, $x = 0$ or $x = -3$.

x	0^-	0	0^+	$(-3)^-$	-3	$(-3)^+$
$\mathrm{d}y/\mathrm{d}x$	+ve	0	+ve	+ve	0	-ve

By the first derivative test, there is a stationary point of inflexion at $x = 0$ and a maximum point at $x = -3$.

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Problem 3. Differentiate $f(x) = 8\sin(x/2) - \sin x - 4x$ with respect to x and deduce that $f(x) < 0$ for $x > 0$.

Solution.

$$f'(x) = 4\cos\frac{x}{2} - \cos x - 4 = 4\cos\frac{x}{2} - \left(2\cos^2\frac{x}{2} - 1\right) - 4 = -2\left(\cos\frac{x}{2} - 1\right)^2 - 1.$$

Observe that for all $x \in \mathbb{R}$, $\left(\cos\frac{x}{2} - 1\right)^2 \geq 0$. Hence, $f'(x) < 0$ for all real values of x. Thus, $f(x)$ is strictly decreasing on $\mathbb{R}$.

Note that $f(0) = 0$. Since $f(x)$ is strictly decreasing, for all $x > 0$, $f(x) < f(0) = 0$.

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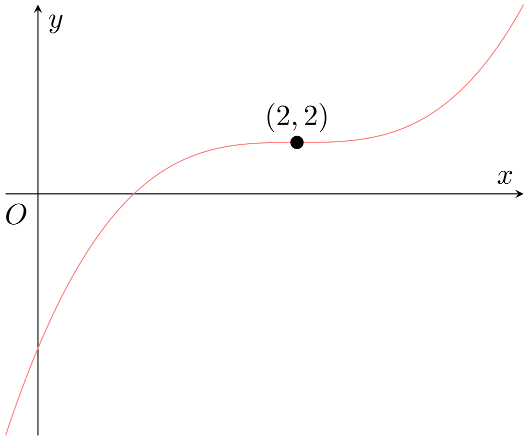
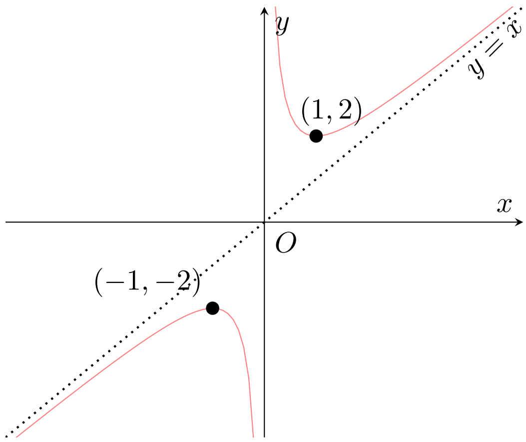
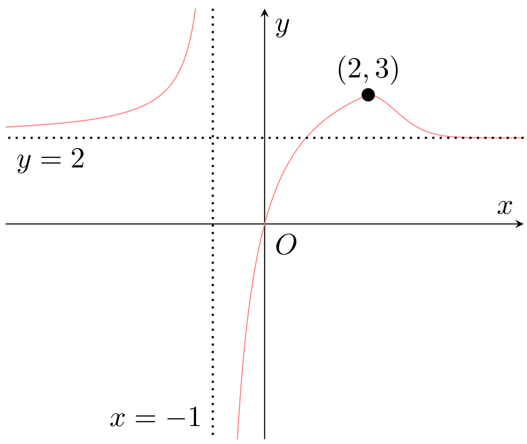
Problem 4. Sketch the graphs of the derivative functions for each of the graphs of the following functions below. In each graph, the point(s) labelled in coordinate form are stationary points.



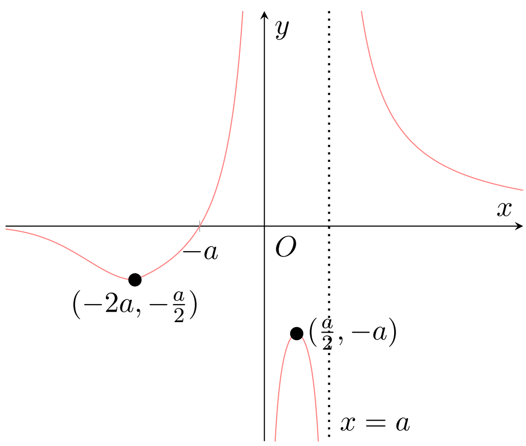
(a)



(b)



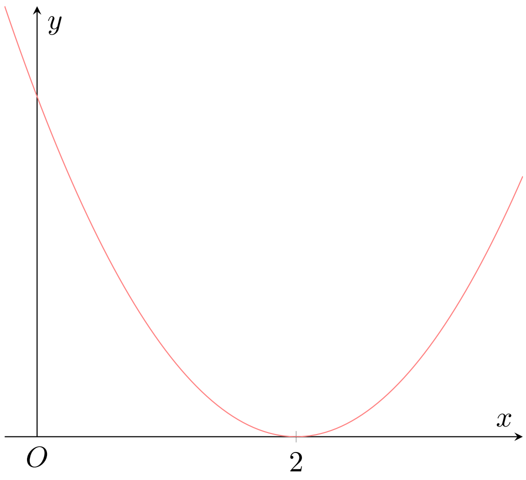
(c)



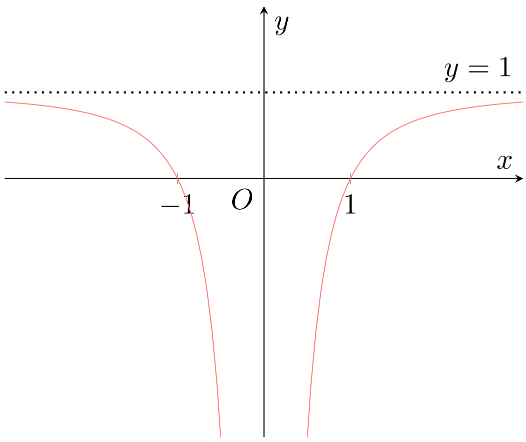
(d)

Solution.

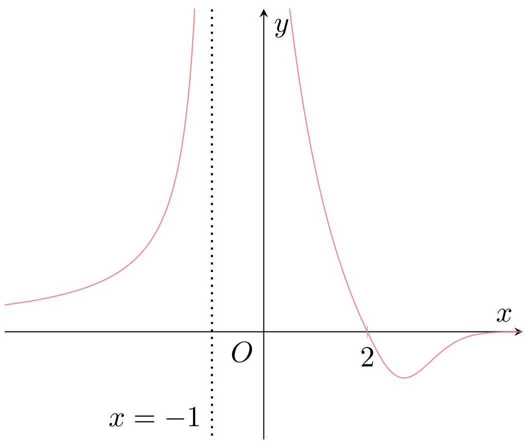
Part (a).



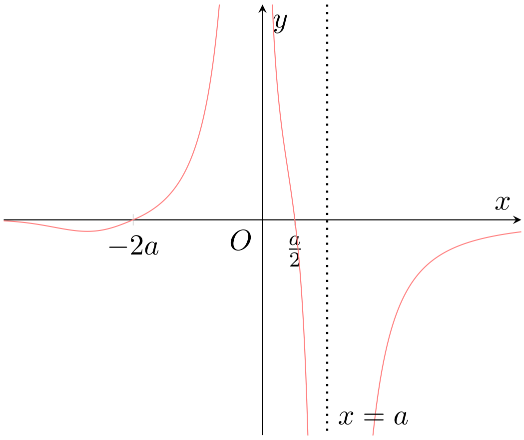
Part (b).



Part (c).



Part (d).



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Problem 5.

(a) Given that $y = ax\sqrt{x+2}$ where $a > 0$, find $\mathrm{d}y/\mathrm{d}x$, expressing your answer as a single algebraic fraction. Hence, show that the curve $y = ax\sqrt{x+2}$ has only one turning point, and state its coordinates in exact form.

(b) Sketch the graph of $y = f'(x)$, where $f(x) = ax\sqrt{x+2}$, where $a > 0$.

Solution.

Part (a).

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a\left(\frac{x}{2\sqrt{x+2}} + \frac{2(x+2)}{2\sqrt{x+2}}\right) = \frac{a(3x+4)}{2\sqrt{x+2}}.$$

Consider the stationary points of $y = ax\sqrt{x+2}$. For stationary points, $\mathrm{d}y/\mathrm{d}x = 0$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a(3x+4)}{2\sqrt{x+2}} = 0 \implies a(3x+4) = 0.$$

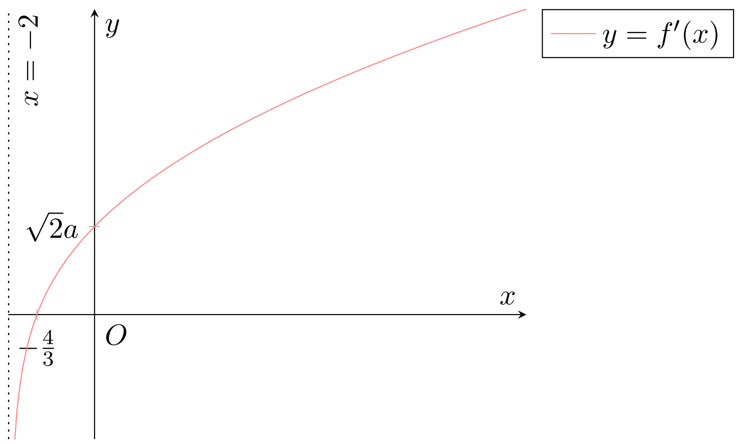
Since $a > 0$, we have $3x + 4 = 0$, whence $x = -4/3$.

x	$(-4/3)^-$	$-4/3$	$(-4/3)^+$
$\mathrm{d}y/\mathrm{d}x$	-ve	0	+ve

Hence, by the first derivative test, there is a turning point (minimum point) at $x = -4/3$. Thus, $y = ax\sqrt{x+2}$ has only one turning point.

Substituting $x = -4/3$ into $y = ax\sqrt{x+2}$, we see that the coordinate of the turning point is $(-\frac{4}{3}, -\frac{4a}{3}\sqrt{\frac{2}{3}})$.

Part (b).



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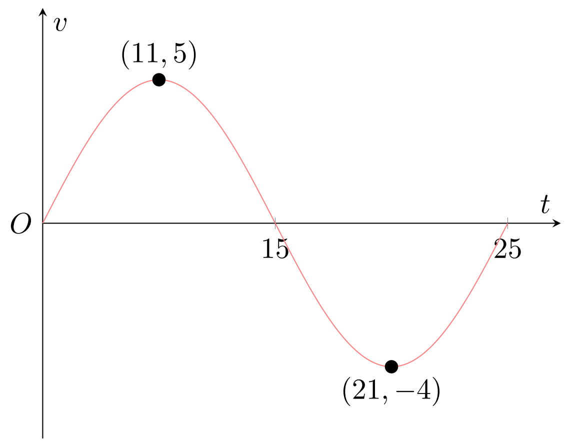
Problem 6. A particle P moves along the x-axis. Initially, P is at the origin O. At time t s, the velocity is v ms^{-1} and the acceleration is a ms^{-2}. Below is the velocity-time graph of the particle for $0 \leq t \leq 25$.



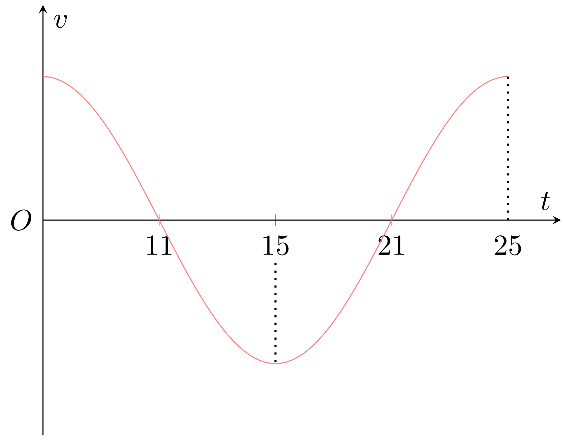
(a) Describe the motion of the particle for $0 \leq t \leq 25$.

(b) Sketch the acceleration-time graph of the particle P.

Solution.

Part (a). From $t = 0$ to $t = 11$, P speeds up and reaches a top speed of 5 ms^{-1}. From $t = 11$ to $t = 15$, P slows down. At $t = 15$, P is instantaneously at rest. From $t = 15$ to $t = 21$, P speeds up and moves in the opposite direction, reaching a top speed of 4 ms^{-1}. From $t = 21$ to $t = 25$, P slows down. At $t = 25$, P is instantaneously at rest.

Part (b).



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Problem 7. The function f defined by $f(x) = \ln x - 2(x - 1/2)$, where $x \in \mathbb{R}, x > 0$. Find $f'(x)$ and show that the function is decreasing for $x > 1/2$. Hence, show that for $x > 1/2$, $2(x - 1/2) - \ln x > \ln 2$.

Solution. Observe that $f'(x) = 1/x - 2 < 0$ for $x > 1/2$. Thus, $f(x)$ is decreasing for all $x > 1/2$. Since $f(1/2) = -\ln 2$, it follows that

$$\left(\forall x > \frac{1}{2}\right): \quad -\ln 2 = f(1/2) > f(x) = \ln x - 2\left(x - \frac{1}{2}\right) \implies 2\left(x - \frac{1}{2}\right) - \ln x > \ln 2.$$

Assignment B5A

Problem 1.

(a) Show, algebraically, that the derivative of the function

$$\ln(1+x) - \frac{2x}{x+2}$$

is never negative.

(b) Hence, show that $\ln(1+x) \geq \frac{2x}{x+2}$ when $x \geq 0$.

Solution. Let

$$f(x) = \ln(1+x) - \frac{2x}{x+2} = \ln(1+x) - 2 + \frac{4}{x+2}.$$

Part (a).

$$f'(x) = \frac{1}{1+x} - \frac{4}{(x+2)^2} = \frac{x^2}{(1+x)(x+2)^2}.$$

Given that $\ln(1+x)$ is defined, it must be that $1+x > 0$. We also know that $x^2 \geq 0$ and $(x+2)^2 \geq 0$. Hence, $f'(x) \geq 0$ for all x in the domain of f and is thus never negative.

Part (b). Note that $f(0) = 0$. Since $f'(x) \geq 0$ for all $x \geq 0$,

$$\ln(1+x) - \frac{2x}{x+2} = f(x) \geq f(0) = 0 \implies \ln(1+x) \geq \frac{2x}{x+2}.$$

* * * * *

Problem 2. The equation of a curve is $y = ax^2 - 2bx + c$, where a, b and c are constants, with $a > 0$.

(a) Using differentiation, find the coordinates of the turning point on the curve, in terms of a, b and c. State whether it is a maximum point or a minimum point.

(b) Given that the turning point of the curve lies on the line $y = x$, find an expression for c in terms of a and b. Show that in this case, whatever the value of b, $c \geq -1/4a$.

(c) Find the numerical values of a, b and c when the curve passes through the point $(0, 6)$ and has a turning point at $(2, 2)$.

Solution.

Part (a). For stationary points, $\mathrm{d}y/\mathrm{d}x = 0$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2ax - 2b = 0 \implies x = \frac{b}{a} \implies y = a\left(\frac{b}{a}\right)^2 - 2b\left(\frac{b}{a}\right) + c = -\frac{b^2}{a} + c.$$

Since $a > 0$, the graph of y is concave upwards. Thus, there is a maximum point at $\left(\frac{b}{a}, -\frac{b^2}{a} + c\right)$.

Part (b). Since the turning point $\left(\frac{b}{a}, -\frac{b^2}{a} + c\right)$ lies on the line $y = x$,

$$\frac{b}{a} = -\frac{b^2}{a} + c \implies c = \frac{b + b^2}{a} = \frac{(b+1/2)^2 - 1/4}{a}.$$

Since $(b+1/2)^2 \geq 0$, it follows that $c \geq -1/4a$.

Part (c). Since the curve passes through $(0, 6)$, it is obvious to see that $c = 6$. Furthermore, since the curve has a turning point at $(2, 2)$, we know that $\frac{b}{a} = 2$ and $-\frac{b^2}{a} + c = 2$. Hence,

$$-\frac{b^2}{a} = 2 - c = -4 \implies b\left(\frac{b}{a}\right) = 4 \implies b = 2 \implies a = 1.$$

Thus, $a = 1$, $b = 2$, and $c = 6$.

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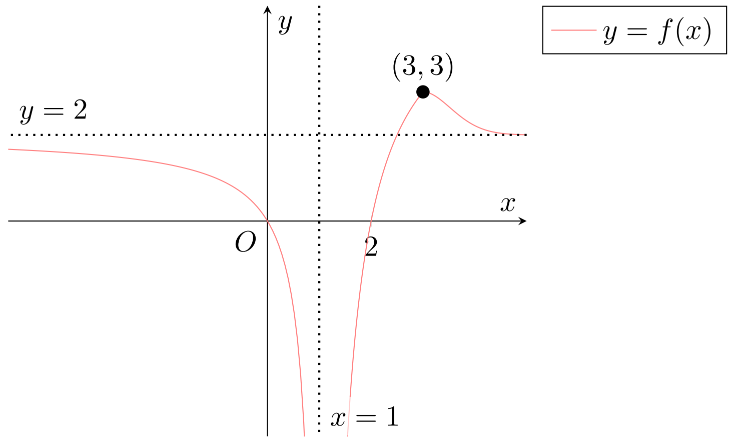
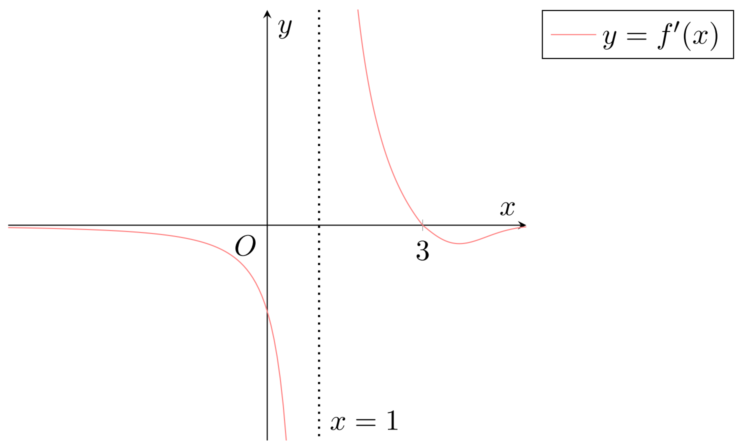
Problem 3. The diagram below shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.



Solution.



Problem 4. The curve C has equation

$$x - y = (x + y)^2.$$

It is given that C has only one turning point.

(a) Show that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x+2y+1}$.

(b) Hence, or otherwise, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

(c) Hence, state, with a reason, whether the turning point is a maximum or a minimum.

Solution.

Part (a). Implicitly differentiating the given equation,

$$1 - \frac{\mathrm{d}y}{\mathrm{d}x} = 2(x+y)\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - (2x+2y)}{2x+2y+1} \implies 1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x+2y+1}.$$

Part (b). Implicitly differentiating the above equation,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -\frac{2\left(2 + 2 \cdot \frac{\mathrm{d}y}{\mathrm{d}x}\right)}{(2x+2y+1)^2} = -\left(\frac{2}{2x+2y+1}\right)^2\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^3.$$

Part (c). For turning points, $\mathrm{d}y/\mathrm{d}x = 0$. Hence, $\mathrm{d}^2y/\mathrm{d}x^2 = -1 < 0$. Thus, the turning point is a maximum.

Tutorial B5B

Problem 1. The equation of a closed curve is $(x+2y)^2+3(x-y)^2=27$.

(a) Show, by differentiation, that the gradient at the point (x, y) on the curve may be expressed in the form $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y-4x}{7y-x}$.

(b) Find the equations of the tangents to the curve that are parallel to

(i) the x-axis,

(ii) the y-axis.

Solution.

Part (a). Implicitly differentiating the given equation,

$$(x+2y)\left(1+2y'\right)+3(x-y)(1-y')=(-x+7y)y'+4x-y=0 \implies y'=\frac{y-4x}{7y-x}.$$

Part (b).

Part (b)(i). When the tangent to the curve is parallel to the x-axis, $y'=0$, whence $y=4x$. Substituting $y=4x$ into the given equation,

$$(9x)^2+3(-3x)^2=27 \implies 108x^2=27 \implies x^2=\frac{1}{4} \implies x=\pm\frac{1}{2} \implies y=\pm 2.$$

The equations of the tangents are hence $y=\pm 2$.

Part (b)(ii). When the tangent to the curve is parallel to the y-axis, y' is undefined. Hence, $7y-x=0 \implies x=7y$. Substituting $x=7y$ into the given equation,

$$(9y)^2+3(6y)^2=27 \implies 189y^2=27 \implies y^2=\frac{1}{7} \implies y=\pm\frac{1}{\sqrt{7}} \implies x=\pm\sqrt{7}.$$

The equations of the tangents are hence $x=\pm\sqrt{7}$.

* * * * *

Problem 2. A piece of wire of length 8 cm is cut into two pieces, one of length x cm, the other of length $(8-x)$ cm. The piece of length x cm is bent to form a circle with circumference x cm. The other piece is bent to form a square with perimeter $(8-x)$ cm. Show that, as x varies, the sum of the areas enclosed by these two pieces of wire is a minimum when the radius of the circle is $\frac{4}{4+\pi}$ cm.

Solution. Let the radius of the circle be r cm. Then we have $x=2\pi r \implies r=x/2\pi$. Let the side length of the square be s cm. Then we have $8-x=4s \implies s=2-x/4$. Let the total area enclosed by the circle and the square be $A(x)$.

$$A(x)=\pi r^2+s^2=\pi\left(\frac{x}{2\pi}\right)^2+\left(2-\frac{x}{4}\right)^2=\left(\frac{1}{4\pi}+\frac{1}{16}\right)x^2-x+4.$$

Consider the stationary points of $A(x)$. For stationary points, $A'(x)=0$.

$$A'(x)=\left(\frac{1}{2\pi}+\frac{1}{8}\right)x-1=0 \implies x=\frac{1}{\frac{1}{2\pi}+\frac{1}{8}}=\frac{8\pi}{4+\pi}.$$

x	$\left(\frac{8\pi}{4+\pi}\right)^-$	$\frac{8\pi}{4+\pi}$	$\left(\frac{8\pi}{4+\pi}\right)^+$
$\mathrm{d}A/\mathrm{d}x$	$-$ve	0	$+$ve

Hence, by the first derivative test, the minimum value of $A(x)$ is achieved when $x = \frac{8\pi}{4+\pi}$, whence

$$r = \frac{1}{2\pi} \cdot \frac{8\pi}{4+\pi} = \frac{4}{4+\pi} \text{ cm.}$$

* * * * *

Problem 3. A spherical balloon is being inflated in such a way that its volume is increasing at a constant rate of 150 cm^3s^{-1}. At time t seconds, the radius of the balloon is r cm.

(a) Find $\mathrm{d}r/\mathrm{d}t$ when $r = 50$.

(b) Find the rate of increase of the surface area of the balloon when its radius is 50 cm.

Solution. Let the volume of the balloon be $V(r) = \frac{4}{3}\pi r^3$ cm^3.
Part (a). Note that $\frac{\mathrm{d}V}{\mathrm{d}t} = 150$ and $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$.

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r/\mathrm{d}V}{\mathrm{d}t/\mathrm{d}V} = \frac{\mathrm{d}V/\mathrm{d}t}{\mathrm{d}V/\mathrm{d}r} = \frac{150}{4\pi r^2} = \frac{75}{2\pi r^2}.$$

Evaluating $\frac{\mathrm{d}r}{\mathrm{d}t}$ at $r = 50$,

$$\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=50} = \frac{75}{2\pi \cdot 50^2} = \frac{3}{200\pi}.$$

Part (b). Let the surface area of the balloon be $A(r) = 4\pi r^2$. Observe that $\frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{r=50} = (8\pi \cdot 50)\left(\frac{3}{200\pi}\right) = 6.$$

Thus, the rate of increase of the surface area of the balloon when its radius is 50 cm is 6 cm/s.

* * * * *

Problem 4. A curve has parametric equations $x = 5\sec\theta$, $y = 3\tan\theta$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Find the exact coordinates of the point on the curve at which the normal is parallel to the line $y = x$.

Solution. Observe that $x^2 = 25\sec^2\theta$ and $\frac{25}{9}y^2 = 25\tan^2\theta$. Using the identity $\tan^2\theta + 1 = \sec^2\theta$, we get

$$\frac{25}{9}y^2 + 25 = x^2. \qquad (*)$$

Implicitly differentiating with respect to x, we get

$$\frac{25}{9}y \cdot y' = x.$$

Since the normal is parallel to $y = x$, the tangent is parallel to $y = -x$, whence $y' = -1$. Thus,

$$y = -\frac{9}{25}x.$$

Substituting $y = -\frac{9}{25}x$ into $(*)$,

$$\frac{25}{9}\left(-\frac{9}{25}x\right)^2 + 25 = x^2 \implies \frac{16}{25}x^2 = 25 \implies \frac{4}{5}x = \pm 5 \implies x = \pm\frac{25}{4}.$$

Observe that for $-\pi/2 < \theta < \pi/2$, $x = 5\sec\theta \geq 5$. We thus take $x = 25/4$, whence $y = -9/4$. The coordinate of the required point is thus $(25/4, -9/4)$.

Problem 5. The parametric equations of a curve are

$$x = t^2,\ y = \frac{2}{t}.$$

(a) Find the equation of the tangent to the curve at the point $(p^2, 2/p)$, simplifying your answer.

(b) Hence, find the coordinates of the points Q and R where this tangent meets the x- and y-axes respectively.

(c) The point F is the mid-point of QR. Find a Cartesian equation of the curve traced by F as p varies.

Solution.

Part (a). Observe that $\mathrm{d}x/\mathrm{d}t = 2t$ and $\mathrm{d}y/\mathrm{d}t = -2/t^2$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-2/t^2}{2t} = -\frac{1}{t^3}.$$

Using the point-slope formula, the tangent to the curve at $(p^2, 2/p)$ is given by the equation

$$y - \frac{2}{p} = -\frac{1}{p^3}\left(x - p^2\right) \implies y = \frac{3}{p} - \frac{1}{p^3}x.$$

Part (b). Consider the case where $y = 0$:

$$0 = \frac{3}{p} - \frac{1}{p^3}x \implies x = 3p^2 \implies Q\left(3p^2, 0\right).$$

Consider the case where $x = 0$:

$$y = \frac{3}{p} \implies R\left(0, \frac{3}{p}\right).$$

Part (c). Note that

$$F = \left(\frac{3}{2}p^2, \frac{3}{2p}\right).$$

As p varies, F traces a curve given by the parametric equations $x = 3p^2/2$, $y = 3/2p$. Hence,

$$p^2 = \frac{2}{3}x = \frac{9}{4y^2} \implies y^2 = \frac{27}{8x}.$$

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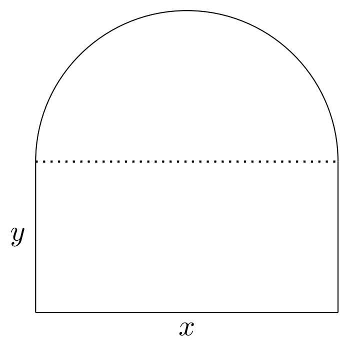
Problem 6.



A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle x m by y m together with a semicircle of diameter x m, as shown in the diagram. A low wall will be built around the flowerbed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of x and y which give a flower-bed of maximum area.

Solution. Observe that the length of the straight parts is $(2y+x)$ m, while the length of the semicircular part is $\frac{1}{2}\pi x$ m. Since a total time of 180 hours is taken to build the wall,

$$3(2y+x)+9\left(\frac{1}{2}\pi x\right)=180 \implies 4y+2x+3\pi x=120 \implies x=\frac{120-4y}{2+3\pi}.$$

Differentiating with respect to y, we get $x'=-4/(2+3\pi)$. Let $A(y)$ be the total area enclosed by the garden, in m^2. Observe that

$$A(y)=xy+\frac{1}{2}\pi\left(\frac{x}{2}\right)^2=xy+\frac{\pi}{8}x^2.$$

Consider the stationary points of $A(y)$. For stationary points, $A'(y)=0$.

$$A'(y)=\left(x'y+x\right)+\frac{\pi}{4}x\cdot x'=0.$$

Substituting in our values of x and x', we get

$$\left[y\left(-\frac{4}{2+3\pi}\right)+\frac{120-4y}{2+3\pi}\right]+\left[\frac{\pi}{4}\left(\frac{120-4y}{2+3\pi}\right)\left(-\frac{4}{2+3\pi}\right)\right]=0.$$

Using G.C., we get $y=12.6$ (3 s.f.), whence $x=6.09$ (3 s.f.).

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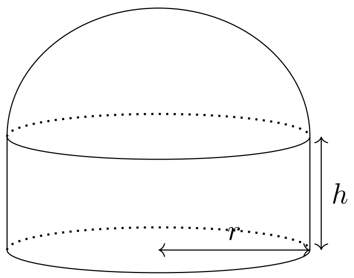
Problem 7.



A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface of a hemisphere of radius r cm.
- The walls are modelled by the curved surface of a cylinder of radius r cm and height h cm.
- The floor is modelled by a circular disc of radius r cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

(a) It is given that the volume of the model is a fixed value k cm^3, and the external surface area is a minimum. Use differentiation to find the values of r and h in terms of k. Simplify your answers.

(b) It is given instead that the volume of the model is 200 cm^3 and its external surface area is 180 cm^2. Show that there are two possible values of r. Given also that $r < h$, find the value of r and the value of h.

Solution.

Part (a). Let the volume of the model be V cm^3. Then

$$V = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right) + \pi r^2 h = k \implies h = \frac{k}{\pi r^2} - \frac{2}{3}r. \tag{1}$$

Let the external surface area of the model be A cm^2. Then

$$A = \frac{4\pi r^2}{2} + 2\pi r h + \pi r^2 = 3\pi r^2 + 2\pi r\left(\frac{k}{\pi r^2} - \frac{2}{3}r\right) = \frac{5\pi}{3}r^2 + \frac{2k}{r}. \tag{2}$$

Consider the stationary points of A. For stationary points, $\mathrm{d}A/\mathrm{d}r = 0$.

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \frac{10\pi}{3}r - \frac{2k}{r^2} = 0 \implies r^3 = \frac{3k}{5\pi} \implies r = \sqrt[3]{\frac{3k}{5\pi}}.$$

r	$\sqrt[3]{\frac{3k}{5\pi}}^-$	$\sqrt[3]{\frac{3k}{5\pi}}$	$\sqrt[3]{\frac{3k}{5\pi}}^+$
$\mathrm{d}A/\mathrm{d}r$	$-$ve	0	+ve

Hence, by the first derivative test, A is at a minimum when $r = \sqrt[3]{\frac{3k}{5\pi}}$.
Substituting $r^3 = \frac{3k}{5\pi}$ into (1),

$$\frac{2}{3}\pi\left(\frac{3k}{5\pi}\right) + \pi r^2 h = \frac{2}{5}k + \pi r^2 h = k \implies r^2 h = \frac{3k}{5\pi} = r^3 \implies h = r = \sqrt[3]{\frac{3k}{5\pi}}.$$

Part (b). From (2), we have

$$\frac{5\pi}{3}r^2 + \frac{2(200)}{r} = 180 \implies \pi r^3 - 108r + 240 = 0.$$

Let $f(r) = \pi r^3 - 108r + 240$. Consider the stationary points of $f(r)$. For stationary points, $f'(r) = 0$.

$$f'(r) = 3\pi r^2 - 108 = 0 \implies r^2 = \frac{36}{\pi} \implies r = \pm\frac{6}{\sqrt{\pi}}.$$

Since $f(r)$ is a cubic with two turning points, it follows that there is exactly one root in each of the following three intervals:

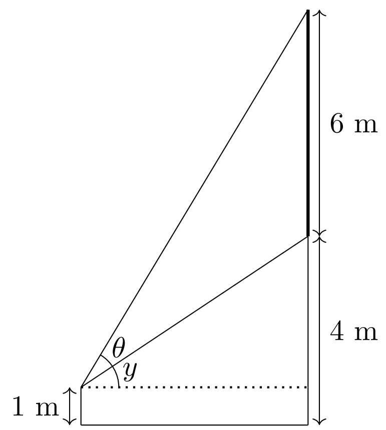
$$\left(-\infty, -\frac{6}{\sqrt{\pi}}\right), \quad \left(-\frac{6}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}\right), \quad \left(\frac{6}{\sqrt{\pi}}, \infty\right).$$

We now show that the root in the interval $\left(-\frac{6}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}\right)$ is positive. Since $f(r)$ has a positive leading coefficient, it must be decreasing in the interval $\left(-\frac{6}{\sqrt{\pi}}, \frac{6}{\sqrt{\pi}}\right)$. Since $f(0) = 240 > 0$, the root in said interval must be positive. Hence, $f(r) = 0$ has two positive roots. Using G.C., the roots are $r = 3.04$ and $r = 3.72$. From (1), we know that

$$h = \frac{200}{\pi r^2} - \frac{2}{3}r.$$

When $r = 3.04$, $h = 4.88 > r$. When $r = 3.72$, $h = 2.12 < r$. Thus, given that $r < h$, we have $r = 3.04$ and $h = 4.88$.

Problem 8.



A movie screen on a vertical wall is 6 m high and 4 m above the horizontal floor. A boy who is standing at x m away from the wall has eye level at 1 m above the floor as shown in the diagram.

The viewing angle of the boy at that position is θ and the angle of elevation of the bottom of the screen is y.

(a) Express y in terms of x.

(b) By expressing θ in terms of x or otherwise, find the stationary value of θ, giving your answers in exact form. Determine if the value is a maximum or minimum value, showing your working clearly.

Solution.

Part (a). Observe that $\tan y = 3/x$, whence $y = \arctan(3/x)$.

Part (b). Observe that $\tan(y + \theta) = 9/x$. Hence,

$$\tan(y+\theta) = \frac{\tan y + \tan\theta}{1 - \tan y \tan\theta} = \frac{3/x + \tan\theta}{1 - (3/x)\tan\theta} = \frac{3 + x\tan\theta}{x - 3\tan\theta} = \frac{9}{x} \implies \tan\theta = \frac{6x}{x^2+27}.$$

Hence,

$$\theta = \arctan\left(\frac{6x}{x^2+27}\right).$$

Differentiating with respect to x,

$$\frac{\mathrm{d}\theta}{\mathrm{d}x} = \frac{1}{1 + \left(\frac{6x}{x^2+27}\right)^2}\left[\frac{6\left(x^2+27\right) - 6x(2x)}{\left(x^2+27\right)^2}\right] = \frac{-6x^2+162}{36x^2 + \left(x^2+27\right)^2}.$$

For stationary points, $\mathrm{d}\theta/\mathrm{d}x = 0$. Hence,

$$-6x^2 + 162 = 0 \implies x^2 = 27 \implies x = \pm 3\sqrt{3}.$$

Since $x > 0$, we only take $x = 3\sqrt{3}$. Thus,

$$\theta = \arctan\left(\frac{6\left(3\sqrt{3}\right)}{27+27}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}.$$

x	$3\sqrt{3}^-$	$3\sqrt{3}$	$3\sqrt{3}^+$
$\mathrm{d}\theta/\mathrm{d}x$	+ve	0	−ve

Thus, by the first derivative test, $\theta = \frac{\pi}{6}$ is a maximum value.

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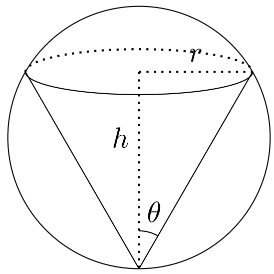
Problem 9.



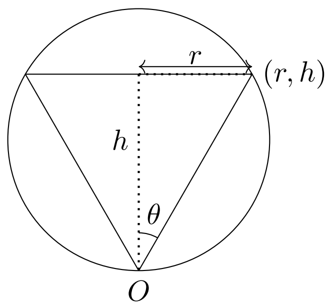
The diagram shows a right inverted cone of radius r, height h and semi-vertical angle θ, which is inscribed in a sphere of radius 1 unit.

Prove that $r^2 = 2h - h^2$.

(a) As r and h varies, determine the exact maximum volume of the cone.

(b) Show that $h = 2\cos^2\theta$. The volume of the cone is increasing at a rate of 6 unit^3/s when $h = \frac{3}{2}$. Determine the rate of change of θ at this instant, leaving your answer in an exact form.

Solution. Consider the following diagram of the cone and sphere.



Let the origin be the tip of the cone. Since the sphere has radius 1 unit, the circle is given by the equation $x^2 + (y-1)^2 = 1$. Since the point (r, h) lies on the circle,

$$r^2 + (h-1)^2 = 1 \implies r^2 = 2h - h^2. \tag{$*$}$$

Part (a). Implicitly differentiating $(*)$ with respect to r,

$$2r = 2 \cdot \frac{\mathrm{d}h}{\mathrm{d}r} - 2h \cdot \frac{\mathrm{d}h}{\mathrm{d}r} \implies \frac{\mathrm{d}h}{\mathrm{d}r} = \frac{r}{1-h}.$$

Let the volume of the cone be $V(r)$ units^3. Then

$$V(r) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(2h - h^2\right)h = \frac{1}{3}\pi\left(2h^2 - h^3\right).$$

Differentiating $V(r)$ with respect to r,

$$V'(r) = \frac{1}{3}\pi\left(4h \cdot \frac{\mathrm{d}h}{\mathrm{d}r} - 3h^2 \cdot \frac{\mathrm{d}h}{\mathrm{d}r}\right) = \frac{1}{3}\left(\frac{\pi r h}{1-h}\right)(4 - 3h).$$

Consider the stationary values of $V(r)$. For stationary values, $V'(r) = 0$, whence $h = 4/3$. Substituting this into $(*)$, we obtain

$$r^2 = 2\left(\frac{4}{3}\right) - \left(\frac{4}{3}\right)^2 = \frac{8}{9} \implies r = \sqrt{\frac{8}{9}}.$$

Note that we reject $r = -\sqrt{8/9}$ as $r > 0$.

r	$\sqrt{8/9}^-$	$\sqrt{8/9}$	$\sqrt{8/9}^+$
$V'(r)$	+ve	0	−ve

Hence, the maximum volume is achieved when $r = \sqrt{8/9}$. Note that

$$V\left(\sqrt{\frac{8}{9}}\right) = \frac{1}{3}\pi\left(\frac{8}{9}\right)\left(\frac{4}{3}\right) = \frac{32}{81}\pi.$$

The maximum volume of the cone is hence $32\pi/81$ units^3.

Part (b). From the diagram, we have

$$\cos\theta = \frac{h}{\sqrt{r^2+h^2}} \implies 2\cos^2\theta = \frac{2h^2}{r^2+h^2} = \frac{2h^2}{2h - h^2 + h^2} = h.$$

Observe that

$$V = \frac{\pi}{3}\left(2h^2 - h^3\right) = \frac{\pi}{3}\left(8\cos^4\theta - 8\cos^6\theta\right) = \frac{8\pi}{3}\left(\cos^4\theta - \cos^6\theta\right).$$

Differentiating with respect to θ, we get

$$\frac{\mathrm{d}V}{\mathrm{d}\theta} = \frac{8\pi}{3}\left(-4\cos^3\theta\sin\theta + 6\cos^5\theta\sin\theta\right) = \frac{16\pi}{3}\cos^3\theta\sin\theta\left(-2+3\cos^2\theta\right).$$

Since $2\cos^2\theta = h = 3/2$, we clearly have $\theta = \pi/6$. Thus,

$$\left.\frac{\mathrm{d}V}{\mathrm{d}\theta}\right|_{h=3/2} = \frac{16\pi}{3}\cos^3\frac{\pi}{6}\sin\frac{\pi}{6}\left(-2+3\cos^2\frac{\pi}{6}\right) = \frac{\sqrt{3}\pi}{4}.$$

Hence,

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_{h=3/2} = \left.\left(\frac{\mathrm{d}\theta}{\mathrm{d}V}\cdot\frac{\mathrm{d}V}{\mathrm{d}t}\right)\right|_{h=3/2} = \frac{6}{\sqrt{3}\pi/4} = \frac{8\sqrt{3}}{\pi}.$$

θ is thus increasing at a rate of $8\sqrt{3}/\pi$ radians per second when $h = \frac{3}{2}$.

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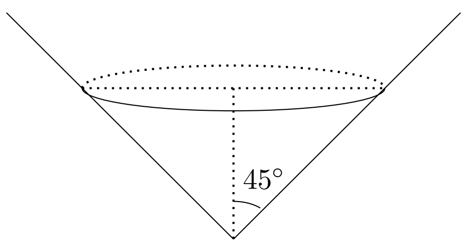
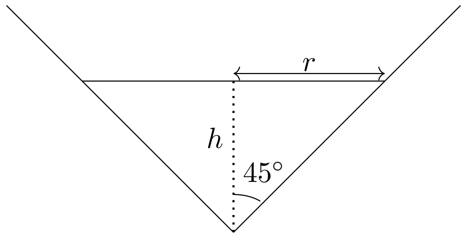
Problem 10.



A hollow cone of semi-vertical angle 45° is held with its axis vertical and vertex downwards. At the beginning of an experiment, it is filled with 390 cm^3 of liquid. The liquid runs out through a small hole at the vertex at a constant rate of 2 cm^3/s.

Find the rate at which the depth of the liquid is decreasing 3 minutes after the start of the experiment.

Solution. Consider the following diagram.



Let the volume of liquid be $V = \frac{1}{3}\pi r^2 h$ cm^3. From the diagram, we have $r = h$. Thus,

$$V = \frac{1}{3}\pi h^3.$$

Differentiating V with respect to h,

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{1}{3}\pi \cdot 3h^2 = \pi h^2.$$

Let t be the time since the start of the experiment in seconds. Consider $\mathrm{d}h/\mathrm{d}t$.

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}V} \cdot \frac{\mathrm{d}V}{\mathrm{d}t} = \left(\frac{\mathrm{d}h}{\mathrm{d}V}\right)^{-1} \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{-2}{\pi h^2}.$$

When $t = 180$, there is $390 - 180(2) = 30$ cm^3 of liquid left in the cone. Thus,

$$V = \frac{1}{3}\pi h^3 = 30 \implies h^3 = \frac{90}{\pi} \implies h = \sqrt[3]{\frac{90}{\pi}}.$$

Evaluating $\mathrm{d}h/\mathrm{d}t$ at $t = 180$,

$$\left.\frac{\mathrm{d}h}{\mathrm{d}t}\right|_{t=180} = \frac{-2}{\pi\left(\sqrt[3]{\frac{90}{\pi}}\right)^2} = -0.0680 \text{ (3 s.f.)}.$$

Thus, the depth of the liquid is decreasing at a rate of 0.0680 cm/s 3 minutes after the start of the experiment.

* * * * *

Problem 11. A particle is projected from the origin O, and it moves freely under gravity in the x-y plane. At time t s after projection, the particle is at the point (x, y) where $x = 30t$ and $y = 20t - 5t^2$, with x and y measured in metres.

(a) Given that the particle passes through two points A and B which are at a distance 15 m above the x-axis, find the time taken for the particle to travel from A to B. Find also the distance AB.

(b) It is known that the particle always travels in a direction tangential to its path. Show that, when $x = 10$, the particle is travelling at an angle of $\arctan(5/9)$ above the horizontal.

The speed of the particle is given by $\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}$. Find the speed of the particle when $x = 10$.

(c) Show that the equation of trajectory is $y = \frac{2}{3}x - \frac{1}{180}x^2$.

Solution.

Part (a). Consider $y = 15$.

$$y = 20t - 5t^2 = 15 \implies t^2 - 4t + 3 = (t-1)(t-3) = 0.$$

Hence, $t = 1$ or $t = 3$. Thus, the particle takes $3 - 1 = 2$ seconds to travel from A to B.

Note that $x = 30$ when $t = 1$, and $x = 90$ when $t = 3$. Hence, $A(30, 15)$ and $B(90, 15)$, whence $AB = 60$ m.

Part (b). Note that $\mathrm{d}x/\mathrm{d}t = 30$ and $\mathrm{d}y/\mathrm{d}t = 20 - 10t$. Thus,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{20-10t}{30} = \frac{2-t}{3}.$$

When $x = 10$, $t = 1/3$. Evaluating $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t = 1/3$,

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=\frac{1}{3}} = \frac{2 - 1/3}{3} = \frac{5}{9}.$$

Hence, the line tangent to the curve at $x = 10$ has gradient $5/9$. Thus, the particle is travelling at an angle of $\arctan(5/9)$ above the horizontal when $x = 10$.

Note that

$$\left.\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\right|_{t=\frac{1}{3}} = \sqrt{30^2 + \left(20 - \frac{10}{3}\right)^2} = 34.3 \text{ (3 s.f.)}.$$

Hence, the particle is travelling at a speed of 34.3 m/s when $x = 10$.

Part (c). Note that $t = x/30$. Hence,

$$y = 20t - 5t^2 = 20\left(\frac{x}{30}\right) - 5\left(\frac{x}{30}\right)^2 = \frac{2}{3}x - \frac{1}{180}x^2.$$

Assignment B5B

Problem 1. Sketch the curve with parametric equations

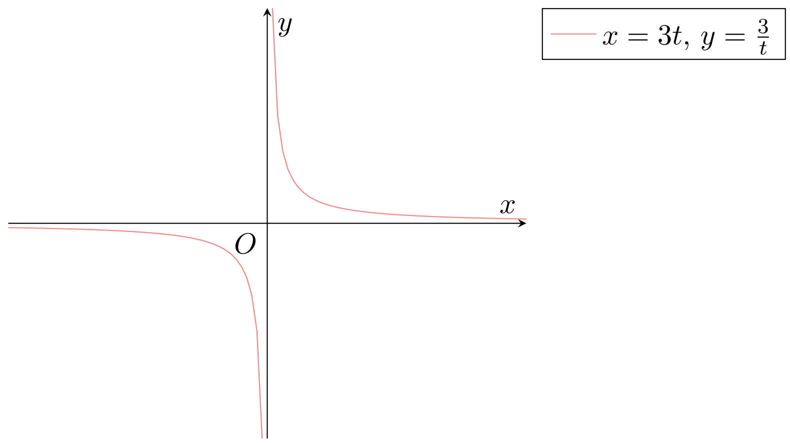
$$x = 3t,\ y = \frac{3}{t}.$$

The point P on the curve has parameter $t = 2$. The normal at P meets the curve again at the point Q.

(a) Show that the normal at P has equation $2y = 8x - 45$.

(b) Find the value of t at Q.

Solution.



Part (a). Consider $\mathrm{d}y/\mathrm{d}x$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = \frac{-3/t^2}{3} = -\frac{1}{t^2}.$$

Hence, the tangent to the curve has gradient $-1/t^2$, whence the normal to the curve has gradient $\frac{-1}{-1/t^2} = t^2$. Thus, the normal to the curve at P has gradient $2^2 = 4$. Note that P has coordinates $(6, 3/2)$. Using the point-slope formula, the normal at P has equation

$$y - \frac{3}{2} = 4(x - 6) \implies 2y = 8(x - 6) + 3 = 8x - 45.$$

Part (b). Observe that

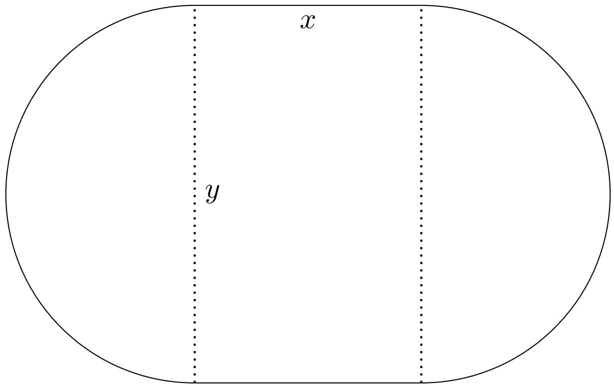
$$x = 3t \implies t = \frac{x}{3} \implies y = \frac{3}{x/3} = \frac{9}{x}.$$

Substituting $y = 9/x$ into the equation of the normal at P,

$$2\left(\frac{9}{x}\right) = 8x - 45 \implies 8x^2 - 45x - 18 = (x - 6)(8x + 3) = 0.$$

Hence, the x-coordinate of Q is $-3/8$ (note that we reject $x = 6$ since that corresponds to P). Thus, $t = -1/8$ at Q.

Problem 2.



A pond with a constant depth of 1 m is being designed for a park. The pond comprises a rectangle x m by y m and two semicircles of diameter y m, as shown in the diagram. The cost to build a boundary around the pond is \$30 per metre for straight parts and \$60 per metre for the semicircular parts. Given that the budget for the boundary is fixed at \$6000, using differentiation or otherwise, find in terms of π, the exact values of x and y which give the pond a maximum volume.

Solution. Observe that the total length of the straight parts is $2x$ m and the total length of the semicircular parts is πy m. Hence,

$$30(2x) + 60(\pi y) = 6000 \implies x + \pi y = 100 \implies x = 100 - \pi y.$$

Let $V(y)$ m^3 be the volume of the pond.

$$V(y) = \pi \left(\frac{y}{2}\right)^2 + xy = \frac{\pi}{4}y^2 + (100 - \pi y)\, y = -\frac{3\pi}{4}y^2 + 100y.$$

Consider the stationary points of $V(y)$. For stationary points, $V'(y) = 0$.

$$V'(y) = -\frac{3\pi}{2}y + 100 = 0 \implies y = \frac{200}{3\pi}.$$

y	$\left(\frac{200}{3\pi}\right)^-$	$\frac{200}{3\pi}$	$\left(\frac{200}{3\pi}\right)^+$
$V'(y)$	+ve	0	−ve

By the first derivative test, the maximum volume of the pond is achieved when $y = 200/3\pi$. Thus, $x = 100 - \pi y = 100/3$.

* * * * *

Problem 3. A circular cylinder is expanding in such a way that, at time t seconds, the length of the cylinder is $20x$ cm and the area of the cross-section is x cm^2. Given that, when $x = 5$, the area of the cross-section is increasing at a rate of 0.025 cm^{2}s^{-1}, find the rate of increase at this instant of

(a) the length of the cylinder,

(b) the volume of the cylinder,

(c) the radius of the cylinder.

Solution. Let $A = x$ cm^2 be the cross-sectional area of the cylinder. Then

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

and

$$\left.\frac{\mathrm{d}A}{\mathrm{d}t}\right|_{x=5} = 0.025.$$

Part (a). Let $L = 20x$ cm be the length of the cylinder. Then

$$\frac{\mathrm{d}L}{\mathrm{d}t} = 20 \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}L}{\mathrm{d}t}\right|_{x=5} = 20\,(0.025) = 0.5.$$

Thus, the length of the cylinder is increasing at a rate of 0.5 cm/s.

Part (b). Let $V = AL = 20x^2$ cm^3 be the volume of the cylinder. Then

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 40x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{x=5} = 40(5)(0.025) = 5.$$

Thus, the volume of the cylinder is increasing at a rate of 5 cm^3/s.

Part (c). Let R cm be the radius of the cylinder. Observe that

$$\pi R^2 = A = x \implies R = \sqrt{\frac{x}{\pi}} = \frac{\sqrt{x}}{\sqrt{\pi}}.$$

Hence,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{2\sqrt{x}} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} \implies \left.\frac{\mathrm{d}R}{\mathrm{d}t}\right|_{x=5} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{5}}\right)(0.025) = 0.00315 \text{ (3 s.f.)}.$$

Thus, the radius of the cylinder is increasing at a rate of 0.00315 cm/s.

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Problem 4. The curve C has equation $2^{-y} = x$. The point A on C has x-coordinate a where $a > 0$. Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{a \ln 2}$ at A and find the equation of the tangent to C at A. Hence, find the equation of the tangent to C which passes through the origin.

The straight line $y = mx$ intersects C at 2 distinct points. Write down the range of values of m.

Solution. Observe that

$$2^{-y} = x \implies y = -\log_2 x = -\frac{\ln x}{\ln 2} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x \ln 2}.$$

At A, $x = a$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{a \ln 2}.$$

Also, we clearly have $A(a, -\ln a / \ln 2)$. Using the point-slope formula, the tangent to C at A has equation

$$y - \left(-\frac{\ln a}{\ln 2}\right) = -\frac{1}{a \ln 2}(x - a) \implies y = -\frac{x}{a \ln 2} + \frac{1 - \ln a}{\ln 2}.$$

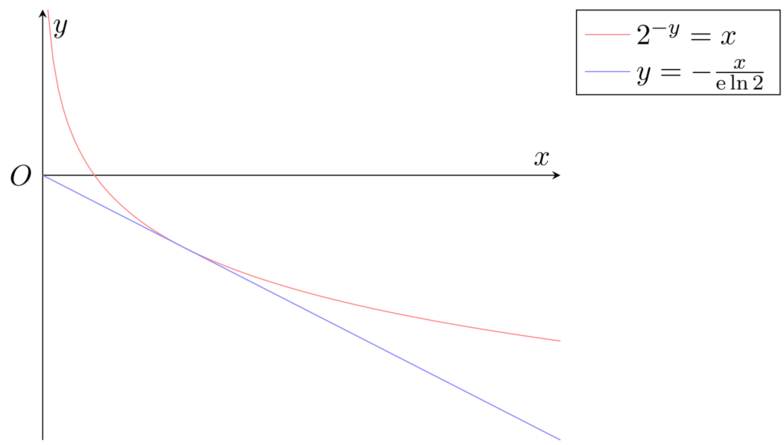
Consider the straight line $y = mx$ that is tangent to C and passes through the origin.

$$0 = -\frac{0}{a \ln 2} + \frac{1 - \ln a}{\ln 2} \implies 1 - \ln a = 0 \implies a = \mathrm{e}.$$

Hence, the equation of the tangent to C that passes through the origin is

$$y = -\frac{x}{\mathrm{e} \ln 2}.$$

Consider the graph of $2^{-y} = x$.



Hence, $m \in (-1/\mathrm{e}\ln 2, 0)$.

Part III.

Examinations