# Problem 1.

(a) Find $\displaystyle\int \frac{6x^3+2}{x^2+1}\,\mathrm{d}x$.

(b) Evaluate $\displaystyle\int_2^4 x\ln x\,\mathrm{d}x$ exactly.

## Solution

### Part (a)

$$\begin{aligned}
\int \frac{6x^3+2}{x^2+1}\,\mathrm{d}x &= 6\int \frac{x^3}{x^2+1}\,\mathrm{d}x + 2\int \frac{1}{x^2+1}\,\mathrm{d}x \\
&= 6\int \frac{x^3}{x^2+1}\,\mathrm{d}x + 2\arctan x + C \\
&= 6\int \frac{x^2}{x^2+1}\cdot x\,\mathrm{d}x + 2\arctan x + C && \begin{aligned} u &= x^2+1 \\ \mathrm{d}u &= 2x\,\mathrm{d}x \end{aligned} \\
&= 6\cdot\frac{1}{2}\int \frac{u-1}{u}\,\mathrm{d}u + 2\arctan x + C \\
&= 3\int \left(1-\frac{1}{u}\right)\mathrm{d}u + 2\arctan x + C \\
&= 3\left(u-\ln|u|\right) + 2\arctan x + C \\
&= 3\left(x^2+1\right) - \ln\left(x^2+1\right) + 2\arctan x + C \\
&= 3x^2 - \ln\left(x^2+1\right) + 2\arctan x + C
\end{aligned}$$

$$\boxed{\int \frac{6x^3+2}{x^2+1}\,\mathrm{d}x = 3x^2 - \ln\left(x^2+1\right) + 2\arctan x + C}$$

### Part (b)

Note that $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}x\ln x = x\cdot\frac{1}{x} + \ln x\cdot 1 = 1+\ln x$.

| | $D$ | $I$ |
|---|---|---|
| $+$ | $x\ln x$ | $1$ |
| $-$ | $1+\ln x$ | $x$ |

Let $I = \int x\ln x\,\mathrm{d}x$.

$$\begin{aligned}
I &= \int x\ln x\,\mathrm{d}x \\
&= x^2\ln x - \int (x + x\ln x)\,\mathrm{d}x \\
&= x^2\ln x - \frac{1}{2}x^2 - I \\
\implies 2I &= x^2\ln x - \frac{1}{2}x^2 + C
\end{aligned}$$

$$\implies I = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C$$

Evaluating $I$ from $x = 2$ to 4,

$$\begin{aligned}\int_2^4 x \ln x \,\mathrm{d}x &= \left[\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2\right]_2^4 \\ &= 14 \ln 2 - 3\end{aligned}$$

$$\boxed{\int_2^4 x \ln x \,\mathrm{d}x = 14 \ln 2 - 3}$$

# Problem 2.

(a) Use the derivative of $\cos\theta$ to show that $\dfrac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta = \sec\theta\tan\theta$.

(b) Use the substitution $x = \sec\theta - 1$ to find the exact value of $\displaystyle\int_{\sqrt{2}-1}^{1} \frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathrm{d}x$.

## Solution

### Part (a)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta &= \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{\cos\theta} \\
&= \frac{1}{\cos^2\theta}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta}\cos\theta \\
&= \frac{1}{\cos^2\theta}\cdot\sin\theta \\
&= \frac{1}{\cos\theta}\cdot\frac{\sin\theta}{\cos\theta} \\
&= \sec\theta\tan\theta
\end{aligned}$$

### Part (b)

Consider the substitution $x = \sec\theta - 1 \implies \mathrm{d}x = \sec\theta\tan\theta\,\mathrm{d}\theta$. When $x = 1$, we have $\theta = \frac{\pi}{3}$. When $x = \sqrt{2}-1$, we have $\theta = \frac{\pi}{4}$. Also note that $x+1 = \sec\theta$. Consider $x^2+2x$.

$$\begin{aligned}
x^2+2x &= (\sec\theta-1)^2 + 2(\sec\theta-1) \\
&= \sec^2\theta - 2\sec\theta + 1 + 2\sec\theta - 2 \\
&= \sec^2\theta - 1 \\
&= \tan^2\theta
\end{aligned}$$

Hence, $\sqrt{x^2+2x} = \tan\theta$. Hence,

$$\begin{aligned}
\int_{\sqrt{2}-1}^{1}\frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathrm{d}x &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,\mathrm{d}\theta \\
&= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\mathrm{d}\theta \\
&= \frac{\pi}{3} - \frac{\pi}{4} \\
&= \frac{\pi}{12}
\end{aligned}$$

$$\boxed{\int_{\sqrt{2}-1}^{1}\frac{1}{(x+1)\sqrt{x^2+2x}}\,\mathrm{d}x = \frac{\pi}{12}}$$

# Problem 3.

The expression $\dfrac{x^2}{9-x^2}$ can be written in the form $A + \dfrac{B}{3-x} + \dfrac{C}{3+x}$.

(a) Find the values of constants $A$, $B$ and $C$.

(b) Show that $\displaystyle\int_0^2 \frac{x^2}{9-x^2}\,\mathrm{d}x = \frac{3}{2}\ln 5 - 2$.

(c) Hence find the value of $\displaystyle\int_0^2 \ln(9-x^2)\,\mathrm{d}x$, giving your answer in terms of $\ln 5$.

## Solution

### Part (a)

$$\begin{aligned}
\frac{x^2}{9-x^2} &= \frac{-(9-x^2)+9}{9-x^2} \\
&= -1 + \frac{9}{9-x^2} \\
&= -1 + \frac{9}{(3-x)(3+x)} \\
&= -1 + \frac{9/6}{3-x} + \frac{9/6}{3+x} \\
&= -1 + \frac{3/2}{3-x} + \frac{3/2}{3+x}
\end{aligned}$$

$$\boxed{A = -1,\ B = \frac{3}{2},\ C = \frac{3}{2}}$$

### Part (b)

$$\begin{aligned}
\int_0^2 \frac{x^2}{9-x^2}\,\mathrm{d}x &= \int_0^2 \left(-1 + \frac{3/2}{3-x} + \frac{3/2}{3+x}\right)\mathrm{d}x \\
&= \left[-x - \frac{3}{2}\ln(3-x) + \frac{3}{2}\ln(3+x)\right]_0^2 \\
&= \frac{3}{2}\ln 5 - 2
\end{aligned}$$

### Part (c)

| | $D$ | $I$ |
|---|---|---|
| $+$ | $\ln(9-x^2)$ | $1$ |
| $-$ | $-\dfrac{2x}{9-x^2}$ | $x$ |

$$
\begin{aligned}
\int_0^2 \ln\left(9-x^2\right) \mathrm{d}x &= \left[x \ln\left(9-x^2\right)\right]_0^2 + 2\int_0^2 \frac{x^2}{9-x^2} \mathrm{d}x \\
&= 2\ln 5 + 2\left(\frac{3}{2}\ln 5 - 2\right) \\
&= 5\ln 5 - 4
\end{aligned}
$$

$$
\boxed{\int_0^2 \ln\left(9-x^2\right) \mathrm{d}x = 5\ln 5 - 4}
$$