

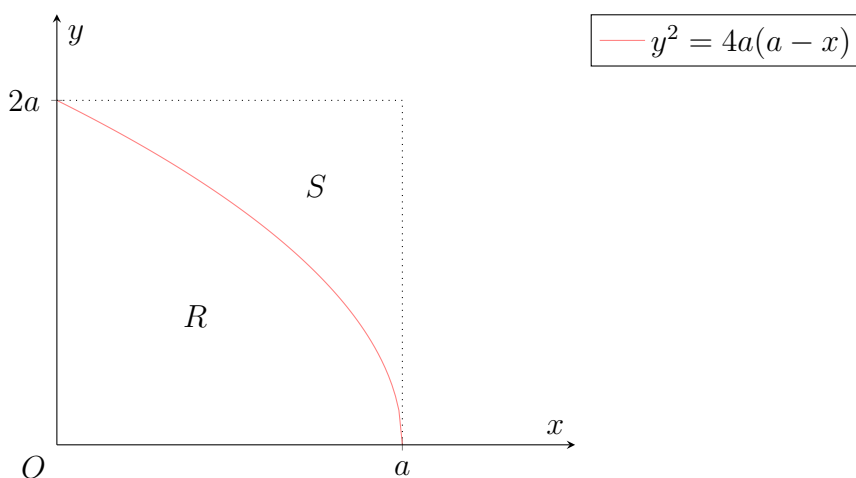
**Problem 1.**

The diagram shows the region  $R$ , which is bounded by the axes and the part of the curve  $y^2 = 4a(a - x)$  lying in the first quadrant.

Find, in terms of  $a$ , the volume,  $V_x$ , of the solid formed when  $R$  is rotated completely about the  $x$ -axis.

The volume of the solid formed when  $R$  is rotated completely about the  $y$ -axis is  $V_y$ . Show that  $V_y = \frac{8}{15}V_x$ .

The region  $S$ , lying in the first quadrant, is bounded by the curve  $y^2 = 4a(a - x)$  and the lines  $x = a$  and  $y = 2a$ . Find, in terms of  $a$ , the volume of the solid formed when  $S$  is rotated completely about the  $y$ -axis.

**Solution**

$$\begin{aligned}
 V_x &= \pi \int_0^a y^2 \, dx \\
 &= \pi \int_0^a 4a(a - x) \, dx \\
 &= 4\pi a \left[ ax - \frac{1}{2}x^2 \right]_0^a \\
 &= 2\pi a^3
 \end{aligned}$$

$$V_x = 2\pi a^3 \text{ units}^3$$

Note that  $x = a - \frac{y^2}{4a} \implies x^2 = \left(a - \frac{y^2}{4a}\right)^2 = a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4$ . Hence,

$$\begin{aligned}
 V_y &= \pi \int_0^{2a} x^2 \, dy \\
 &= \pi \int_0^{2a} \left( a^2 - \frac{1}{2}y^2 + \frac{1}{16a^2}y^4 \right) dy
 \end{aligned}$$

$$\begin{aligned} &= \pi \left[ a^2 y - \frac{1}{2} \cdot \frac{1}{3} y^3 + \frac{1}{16a^2} \cdot \frac{1}{5} y^5 \right]_0^{2a} \\ &= \pi \left( 2a^3 - \frac{8a^3}{6} + \frac{32a^5}{90a^2} \right) \\ &= \frac{16}{15} \pi a^3 \\ &= \frac{8}{15} (2\pi a^3) \\ &= \frac{8}{15} V_x \end{aligned}$$

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Volume  $S$  = Volume of Cylinder  $- V_y$

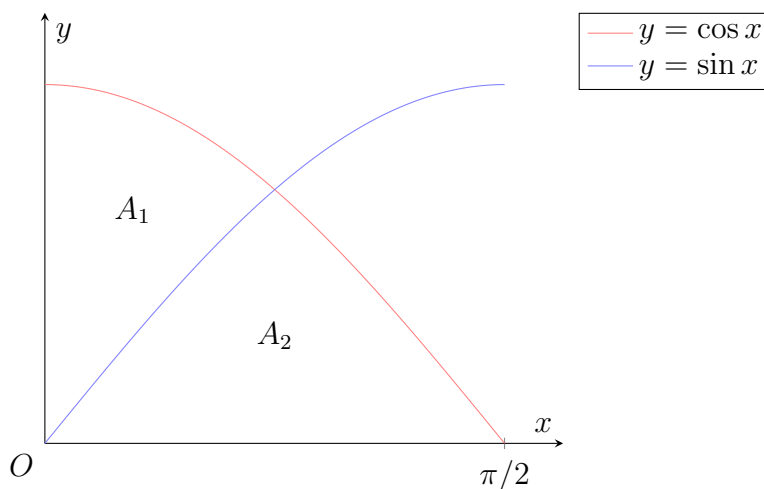
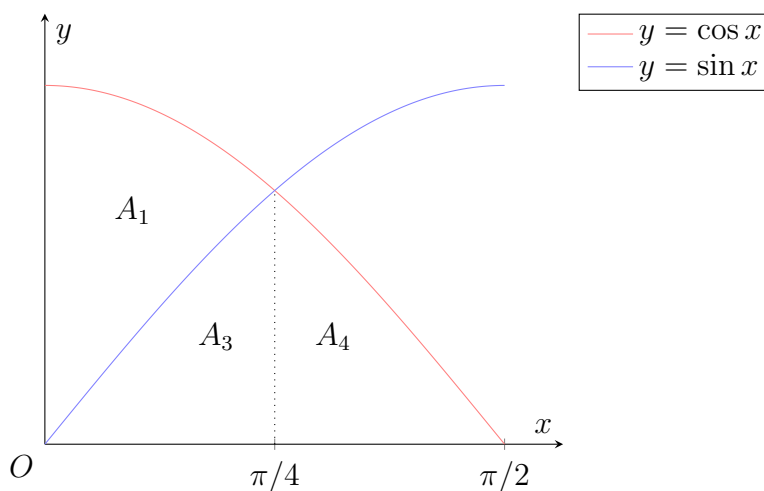
$$\begin{aligned} &= \pi \cdot a^2 \cdot 2a - \frac{16}{15} \pi a^3 \\ &= \frac{14}{15} \pi a^3 \end{aligned}$$

The volume required is $\frac{14}{15} \pi a^3$ units <sup>3</sup> .
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**Problem 2.**

The region bounded by the axes and the curve  $y = \cos x$  from  $x = 0$  to  $x = \frac{1}{2}\pi$  is divided into two parts, of areas  $A_1$  and  $A_2$ , by the curve  $y = \sin x$ .

- (a) Prove that  $A_2 = \sqrt{2}A_1$ .
- (b) Find the volume of the solid obtained when the region with area  $A_2$  is rotated about the  $y$ -axis through  $2\pi$  radians. Give your answer in exact form.

**Solution****Part (a)**

Let  $A_3$  and  $A_4$  be the areas as defined on the diagram above. By the symmetry of  $y = \sin x$  and  $y = \cos x$  about  $x = \pi/4$ , we have  $A_3 = A_4$ .

$$\begin{aligned}
 A_3 &= \int_0^{\pi/4} \sin x \, dx \\
 &= [-\cos x]_0^{\pi/4} \\
 &= 1 - \frac{\sqrt{2}}{2}
 \end{aligned}$$

$$\begin{aligned}
A_1 &= \int_0^{\pi/4} \cos x \, dx - A_3 \\
&= [\sin x]_0^{\pi/4} - \left(1 - \frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{2}}{2} - 1 + \frac{\sqrt{2}}{2} \\
&= \sqrt{2} - 1 \\
\implies A_2 &= 2A_3 \\
&= 2 \left(1 - \frac{\sqrt{2}}{2}\right) \\
&= 2 - \sqrt{2} \\
&= \sqrt{2}(\sqrt{2} - 1) \\
&= \sqrt{2}A_1
\end{aligned}$$

**Part (b)**

Let  $V_3$  and  $V_4$  be the volumes of the solids obtained when  $A_3$  and  $A_4$  are rotated about the  $y$ -axis through  $2\pi$  radians, respectively.

$$\begin{aligned}
V_3 &= 2\pi \int_0^{\pi/4} xy \, dx \\
&= 2\pi \int_0^{\pi/4} x \sin x \, dx
\end{aligned}$$

	$D$	$I$
+	$x$	$\sin x$
-	$1$	$-\cos x$
+	$0$	$-\sin x$

$$\begin{aligned}
&= 2\pi [-x \cos x + \sin x]_0^{\pi/4} \\
&= 2\pi \left[ \left(-\frac{\pi}{4} \cos \frac{\pi}{4} + \sin \frac{\pi}{4}\right) - (0 + \sin 0) \right] \\
&= 2\pi \left( -\frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \\
&= 2\pi \cdot \frac{\sqrt{2}}{2} \left( 1 - \frac{\pi}{4} \right) \\
&= \sqrt{2}\pi \left( 1 - \frac{\pi}{4} \right)
\end{aligned}$$

$$V_4 = 2\pi \int_{\pi/4}^{\pi/2} xy \, dx$$

$$= 2\pi \int_{\pi/4}^{\pi/2} x \cos x \, dx$$

	$D$	$I$
+	$x$	$\cos x$
-	$1$	$\sin x$
+	$0$	$-\cos x$

$$\begin{aligned}
&= 2\pi [x \sin x + \cos x]_{\pi/4}^{\pi/2} \\
&= 2\pi \left( \left[ \frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2} \right] - \left[ \frac{\pi}{4} \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \right] \right) \\
&= 2\pi \left[ \frac{\pi}{2} - \left( \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \right) \right] \\
&= 2\pi \left[ \frac{\pi}{2} - \frac{\sqrt{2}}{2} \left( 1 + \frac{\pi}{4} \right) \right] \\
&= \pi^2 - \sqrt{2}\pi \left( 1 + \frac{\pi}{4} \right)
\end{aligned}$$

$$\Rightarrow \text{Required volume} = V_3 + V_4$$

$$\begin{aligned}
&= \sqrt{2}\pi \left( 1 - \frac{\pi}{4} \right) + \pi^2 - \sqrt{2}\pi \left( 1 + \frac{\pi}{4} \right) \\
&= \pi^2 - 2 \cdot \sqrt{2}\pi \cdot \frac{\pi}{4} \\
&= \pi^2 - \frac{\sqrt{2}}{2} \pi^2
\end{aligned}$$

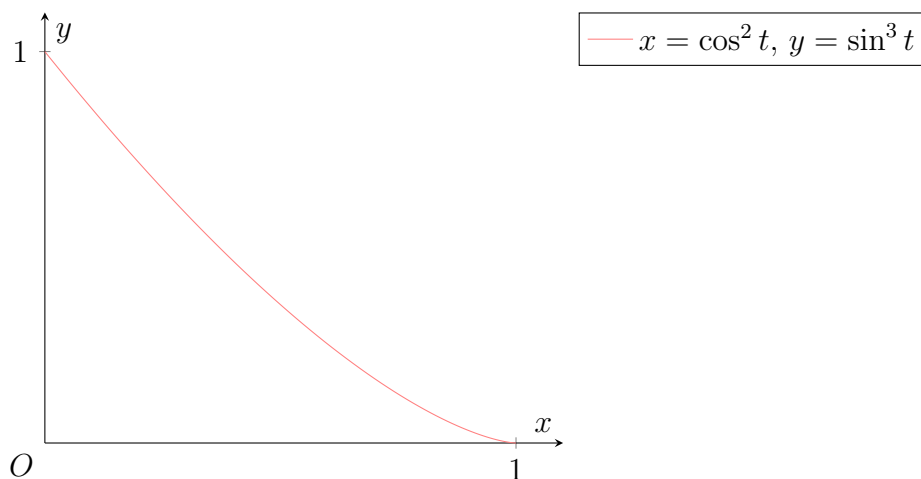
The required volume is  $\left( \pi^2 - \frac{\sqrt{2}}{2} \pi^2 \right)$  units<sup>3</sup>.

**Problem 3.**

A curve has parametric equations

$$x = \cos^2 t, y = \sin^3 t, 0 \leq t \leq \frac{1}{2}\pi$$

- (a) Sketch the curve.
- (b) Show that the area under the curve for  $0 \leq t \leq \frac{1}{2}\pi$  is  $2 \int_0^{\pi/2} \cos t \sin^4 t \, dt$ , and find the exact value of the area.
- (c) Find the volume of the solid obtained when the region in (b) is rotated about the  $y$ -axis through  $2\pi$  radians.

**Solution****Part (a)****Part (b)**

Note that  $x = 0 \implies t = \frac{\pi}{2}$  and  $x = 1 \implies t = 0$ . Hence,

$$\begin{aligned} \text{Area} &= \int_0^1 y \, dx \\ &= \int_{\pi/2}^0 y \frac{dx}{dt} \, dt \\ &= \int_{\pi/2}^0 \sin^3 t \cdot (-2 \cos t \sin t) \, dt \\ &= 2 \int_0^{\pi/2} \cos t \sin^4 t \, dt \\ &= B(5/2, 1) \\ &= \frac{\Gamma(5/2) \Gamma(1)}{\Gamma(5/2 + 1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(5/2) \cdot 1}{5/2 \cdot \Gamma(5/2)} \\
&= \frac{2}{5}
\end{aligned}$$

The area under the curve is  $\frac{2}{5}$  units<sup>2</sup>.

**Part (c)**

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^1 xy \, dx \\
&= 2\pi \int_{\pi/2}^0 \cos^2 t \sin^3 t \cdot (-2 \cos t \sin t) \, dt \\
&= 2\pi \cdot 2 \int_0^{\pi/2} \cos^3 t \sin^4 t \, dt \\
&= 2\pi \cdot B(5/2, 2) \\
&= 2\pi \cdot \frac{\Gamma(5/2) \Gamma(2)}{\Gamma(5/2 + 2)} \\
&= 2\pi \cdot \frac{\Gamma(5/2) \cdot 1}{7/2 \cdot 5/2 \cdot \Gamma(5/2)} \\
&= 2\pi \cdot \frac{2}{7} \cdot \frac{2}{5} \\
&= \frac{8}{35}\pi
\end{aligned}$$

The required volume is  $\frac{8}{35}\pi$  units<sup>3</sup>.

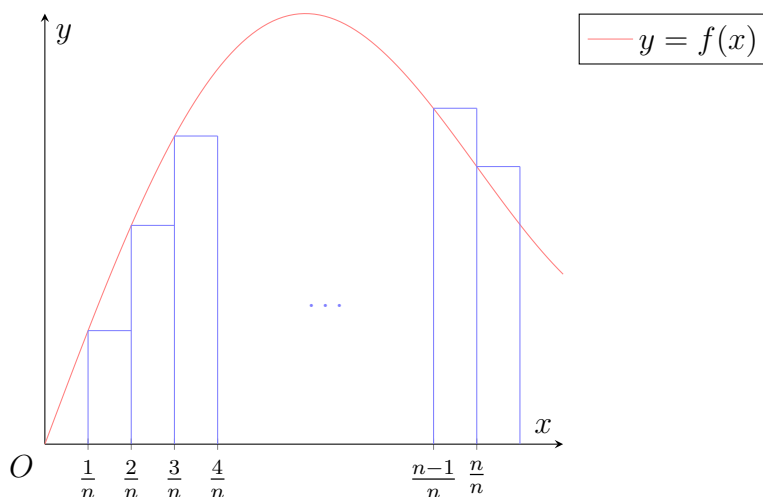
**Problem 4.**

- (a) Given that  $f$  is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right]$$

is  $\int_0^1 f(x) \, dx$ .

- (b) Hence, evaluate  $\lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{\sqrt[3]{1} + \sqrt[3]{2} + \dots + \sqrt[3]{n}}{\sqrt[3]{n}} \right)$ .

**Solution****Part (a)**

The area of the rectangles in the above figure is given by

$$\frac{1}{n} \left[ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right]$$

This gives an approximation of the signed area under the curve from  $x = \frac{1}{n}$  to  $x = \frac{n}{n} = 1$ . As  $n \rightarrow \infty$ , the widths of the rectangles become smaller and the approximation becomes exact. Hence,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left[ f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right) \right] = \int_0^1 f(x) \, dx$$

**Part (b)**

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{\sqrt[3]{1} + \sqrt[3]{2} + \dots + \sqrt[3]{n}}{\sqrt[3]{n}} \right) &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \dots + \sqrt[3]{\frac{n}{n}} \right] \\ &= \int_0^1 \sqrt[3]{x} \, dx \end{aligned}$$



$$\begin{aligned} &= \left[ \frac{1}{1/3 + 1} x^{1/3+1} \right]_0^1 \\ &= \frac{3}{4} \end{aligned}$$

$$\boxed{\lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{\sqrt[3]{1} + \sqrt[3]{2} + \dots + \sqrt[3]{n}}{\sqrt[3]{n}} \right) = \frac{3}{4}}$$

**Problem 5.**

The function  $f$  satisfies  $f'(x) > 0$  for  $a \leq x \leq b$ , and  $g$  is the inverse of  $f$ . By making a suitable change of variable, prove that

$$\int_a^b f(x) \, dx = b\beta - a\alpha - \int_\alpha^\beta g(y) \, dy$$

where  $\alpha = f(a)$  and  $\beta = f(b)$ . Interpret this formula geometrically by means of a sketch where  $\alpha$  and  $a$  are positive. Verify this result for the case where  $f(x) = e^{2x}$ ,  $a = 0$ ,  $b = 1$ .

Prove similarly and interpret geometrically the formula

$$2\pi \int_a^b x f(x) \, dx = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2 \, dy$$

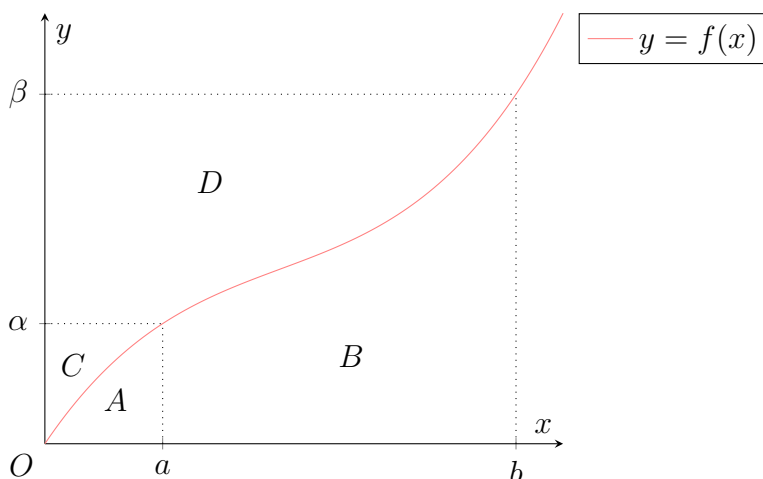
**Solution**

$$\begin{aligned} \int_\alpha^\beta g(y) \, dy &= \int_a^b f^{-1}(f(x)) f'(x) \, dx \\ &= \int_a^b x f'(x) \, dx \end{aligned}$$

$$\begin{aligned} y &= f(x) \\ \Rightarrow dy &= f'(x) \, dx \end{aligned}$$

	$D$	$I$
+	$x$	$f'(x)$
-	$1$	$f(x)$

$$\begin{aligned} &= [x f(x)]_a^b - \int_a^b f(x) \, dx \\ &= b\beta - a\alpha - \int_a^b f(x) \, dx \\ \Rightarrow \int_a^b f(x) \, dx &= b\beta - a\alpha - \int_\alpha^\beta g(y) \, dy \end{aligned}$$



Consider the above diagram. We clearly have  $\text{Area}(A + C) = a\alpha$ ,  $\text{Area}(A + B + C + D) = b\beta$ ,  $\text{Area } B = \int_a^b f(x) \, dx$  and  $\text{Area } D = \int_\alpha^\beta g(y) \, dy$ . Thus,

$$\begin{aligned} \int_a^b f(x) \, dx &= \text{Area } B \\ &= \text{Area}(A + B + C + D) - \text{Area}(A + C) - \text{Area } D \\ &= b\beta - a\alpha - \int_\alpha^\beta g(y) \, dy \end{aligned}$$


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**Standard Way.**

$$\int_0^1 e^{2x} \, dx = \left[ \frac{1}{2} e^{2x} \right]_0^1 = \frac{1}{2} e^2 - \frac{1}{2}$$

**Via Formula.** Let  $f(x) = e^{2x}$ . Then  $g(x) = \frac{1}{2} \ln x$ . Hence,  $\alpha = g(0) = 1$  and  $\beta = g(1) = e^2$ . Invoking the above formula,

$$\begin{aligned} \int_0^1 e^{2x} \, dx &= 1 \cdot e^2 - 0 \cdot 1 - \int_1^{e^2} \frac{1}{2} \ln x \, dx \\ &= e^2 - \frac{1}{2} [x \ln x - x]_1^{e^2} \\ &= e^2 - \frac{1}{2} [(e^2 \ln e^2 - e^2) - (\ln 1 - 1)] \\ &= e^2 - \frac{1}{2} (e^2 + 1) \\ &= \frac{1}{2} e^2 - \frac{1}{2} \end{aligned}$$

Hence, the formula holds for the above case.

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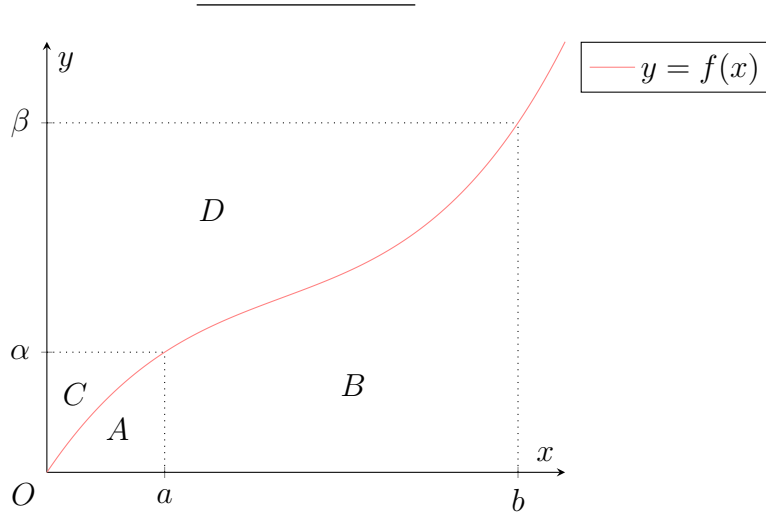
$$\begin{aligned} \int_\alpha^\beta [g(y)]^2 \, dy &= \int_\alpha^\beta [f^{-1}(f(x))]^2 f'(x) \, dx \\ &= \int_a^b x^2 f'(x) \, dx \end{aligned}$$

$$\begin{aligned} y &= f(x) \\ \implies dy &= f'(x) \, dx \end{aligned}$$

	$D$	$I$
+	$x^2$	$f'(x)$
-	$2x$	$f(x)$

$$= [x^2 f(x)]_a^b - 2 \int_a^b x f(x) \, dx$$

$$\begin{aligned}
&= b^2\beta - a^2\alpha - 2 \int_a^b x f(x) \, dx \\
\Rightarrow 2 \int_a^b x f(x) \, dx &= b^2\beta - a^2\alpha - \int_\alpha^\beta [g(y)]^2 \, dy \\
\Rightarrow 2\pi \int_a^b x f(x) \, dx &= \pi (b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2 \, dy
\end{aligned}$$



Let Volume  $R$  represent the volume of the solid obtained when the region  $R$  is rotated completely about the  $y$ -axis.

We clearly have  $\text{Volume}(A + B + C + D) = \pi b^2\beta$ ,  $\text{Volume}(A + C) = \pi a^2\alpha$ ,  $\text{Volume } B = 2\pi \int_a^b x f(x) \, dx$  (using the shell method), and  $\text{Volume } D = \pi \int_\alpha^\beta [g(y)]^2 \, dy$  (using the disc method). Thus,

$$\begin{aligned}
2\pi \int_a^b x f(x) \, dx &= \text{Volume } B \\
&= \text{Volume}(A + B + C + D) - \text{Volume}(A + C) - \text{Volume } D \\
&= \pi b^2\beta - \pi a^2\alpha - \pi \int_\alpha^\beta [g(y)]^2 \, dy \\
&= \pi (b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2 \, dy
\end{aligned}$$