Problem 1. ACJC Prelim 9758/2017/01/Q5

The points O, A and B are on a plane such that relative to the point O, the points A and B have non-parallel position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively.

(a) The point C with position vector c is on the plane OAB such that OC bisects the angle AOB. Show that $\left(\frac{\mathbf{a}}{|\mathbf{a}|} - \frac{\mathbf{b}}{|\mathbf{b}|}\right) \cdot \mathbf{c} = 0$.

(b) The lines AB and OC intersect at P. By first verifying that $\overrightarrow{OC}$ is parallel to $\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|}$, show that the ratio of $AP : PB = |\mathbf{a}| : |\mathbf{b}|$.

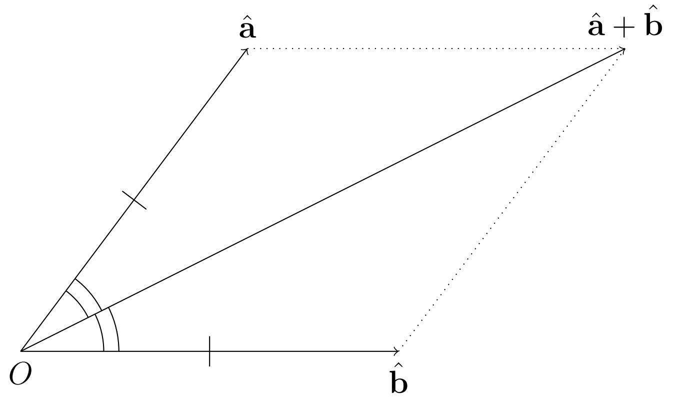
Solution

Part (a)

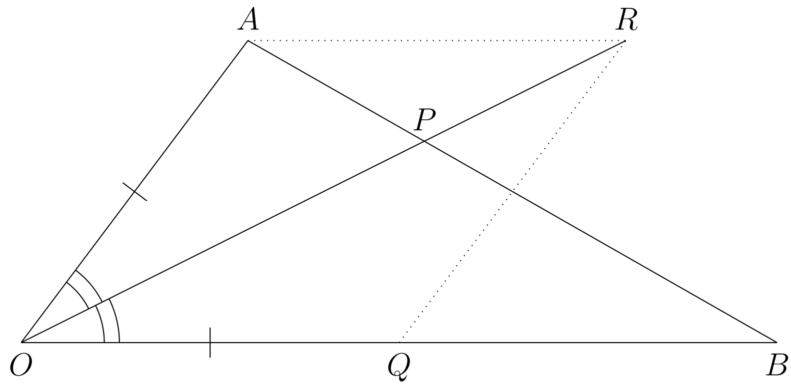
Since OC bisects $\angle AOB$,

$$\begin{aligned}
& \angle AOC = \angle COB \\
\implies \quad & \cos \angle AOC = \cos \angle COB \\
\implies \quad & \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} \\
\implies \quad & \left(\frac{\mathbf{a}}{|\mathbf{a}|}\right) \cdot \mathbf{c} = \left(\frac{\mathbf{b}}{|\mathbf{b}|}\right) \cdot \mathbf{c} \\
\implies \quad & \left(\frac{\mathbf{a}}{|\mathbf{a}|} - \frac{\mathbf{b}}{|\mathbf{b}|}\right) \cdot \mathbf{c} = 0
\end{aligned}$$

Part (b)



Consider the above diagram. Since $|\hat{\mathbf{a}}| = \left|\hat{\mathbf{b}}\right|$, they form a rhombus. Recall that the diagonals of a rhombus bisect opposite angles. Thus, the sum $\hat{\mathbf{a}} + \hat{\mathbf{b}}$ bisects $\angle AOB$ and is hence parallel to $\overrightarrow{OC}$.



Consider the above diagram. We have Q on OB such that $OA = OQ$. We also have R such that $OA \parallel QR$ and $OA = AR$. From the earlier discussion, P is the intersection of OR and AB.

Now observe that $\triangle OBP$ is similar to $\triangle RAP$. Let λ be the scale factor of $\triangle RAP$ with respect to $\triangle OBP$. We hence have

$$\begin{aligned} & |\mathbf{a}| = OA = AR = \lambda OB = \lambda\,|\mathbf{b}| \text{ and } AP = \lambda BP \\ \implies & \frac{|\mathbf{a}|}{|\mathbf{b}|} = \lambda \text{ and } \frac{AP}{BP} = \lambda \end{aligned}$$

Thus, $\dfrac{|\mathbf{a}|}{|\mathbf{b}|} = \dfrac{AP}{BP}$, whence $AP : PB = |\mathbf{a}| : |\mathbf{b}|$.

Problem 2. AJC Prelim 9758/2017/01/Q9

The position vectors of A, B and C with respect to the origin O are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. It is given that $\overrightarrow{AC} = 4\overrightarrow{CB}$ and $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

(a) By considering $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$, show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

(b) Find the length of the projection of $\mathbf{c}$ on $\mathbf{a}$ in terms of $|\mathbf{a}|$.

(c) Given that F is the foot of the perpendicular from C to OA and $\mathbf{f}$ denotes the position vector $\overrightarrow{OF}$, state the geometrical meaning of $|\mathbf{c}\times\mathbf{f}|$.

(d) Two points X and Y move along line segments OA and AB respectively such that

$$\begin{aligned}\overrightarrow{OX} &= (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + \frac{1}{2}\mathbf{k}\\ \overrightarrow{OY} &= (\sin t)\mathbf{i} + (\cos t)\mathbf{j} - 2\mathbf{k}\end{aligned}$$

where t is a real parameter, $0 \le t \le 2\pi$. By expressing the scalar product of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in the form of $p\sin(qt)+r$ where p, q and r are real values to be determined, find the greatest value of the angle XOY.

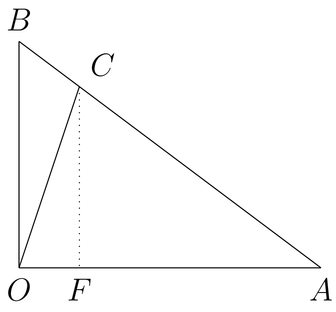
Solution

Part (a)

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} + \mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b}\\ &= |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{b}|^2\\ &= |\mathbf{a}+\mathbf{b}|^2 + 2\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

Since $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}+\mathbf{b}|^2$, we have that $\mathbf{a}\cdot\mathbf{b} = 0$, whence $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

Part (b)



By the Ratio Theorem, $\overrightarrow{OC} = \frac{1}{5}\mathbf{a} + \frac{4}{5}\mathbf{b}$. Since F lies on OA, it has the direction vector $\frac{1}{5}\mathbf{a}$. Thus, OF, the length of projection of $\mathbf{c}$ on $\mathbf{a}$, is $\frac{1}{5}|\mathbf{a}|$.

$$\boxed{\text{The length of projection of } \mathbf{c} \text{ on } \mathbf{a} \text{ is } \frac{1}{5}|\mathbf{a}|.}$$

Part (c)

$|\mathbf{c} \times \mathbf{f}|$ is the area of a parallelogram defined by $\mathbf{c}$ and $\mathbf{f}$.

Part (d)

We have $\overrightarrow{OX} = \begin{pmatrix} \cos 3t \\ \sin 3t \\ 1/2 \end{pmatrix}$ and $\overrightarrow{OY} = \begin{pmatrix} \sin t \\ \cos t \\ -2 \end{pmatrix}$. Hence,

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{OY} &= \begin{pmatrix} \cos 3t \\ \sin 3t \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} \sin t \\ \cos t \\ -2 \end{pmatrix} \\ &= \cos 3t \sin t + \sin 3t \cos t - 1 \\ &= \sin 4t - 1 \end{aligned}$$

From the geometric definition of the scalar product, we have

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{OY} &= \left| \begin{pmatrix} \cos 3t \\ \sin 3t \\ 1/2 \end{pmatrix} \right| \left| \begin{pmatrix} \sin t \\ \cos t \\ -2 \end{pmatrix} \right| \cos \angle XOY \\ \implies \quad \sin 4t - 1 &= \sqrt{\cos^2 3t + \sin^2 3t + \left(\frac{1}{2}\right)^2} \sqrt{\sin^2 t + \cos^2 t + (-2)^2} \cos \angle XOY \\ &= \sqrt{1 + \frac{1}{4}} \sqrt{1+4} \cos \angle XOY \\ &= \frac{5}{2} \cos \angle XOY \\ \implies \cos \angle XOY &= \frac{2}{5}(\sin 4t - 1) \end{aligned}$$

Observe that $\angle XOY \in [0, \pi)$, where $\cos \angle XOY$ is decreasing. Hence, the maximum value of $\angle XOY$ occurs when $\cos \angle XOY$ is at a minimum. Since the minimum of $\sin 4t$ is -1, we have

$$\begin{aligned} \min \cos \angle XOY &= \frac{2}{5}(-1-1) \\ \implies \quad \max \angle XOY &= \arccos\left(-\frac{4}{5}\right) \\ &= 2.50 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{The greatest value of } \angle XOY \text{ is } 2.50.}$$

Problem 3. CJC Prelim 9758/2017/02/Q2

Referred to the origin O, the points A, B, P and Q have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$ and $\mathbf{q}$ respectively, such that $|\mathbf{a}| = 2$, $\mathbf{b}$ is a unit vector, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$.

(a) Give a geometrical interpretation of $|\mathbf{b} \cdot \mathbf{a}|$.

(b) Find $|\mathbf{a} \times \mathbf{b}|$, leaving your answer in exact form.

It is also given that $\mathbf{p} = 3\mathbf{a} + (\mu + 2)\mathbf{b}$ and $\mathbf{q} = (\mu + 3)\mathbf{a} + \mu\mathbf{b}$, where $\mu \in \mathbb{R}$.

(c) Show that $\mathbf{p} \times \mathbf{q} = (\mu^2 + 2\mu + 6)(\mathbf{b} \times \mathbf{a})$.

(d) Hence, find the smallest area of the triangle OPQ as μ varies.

Solution

Part (a)

$|\mathbf{b} \cdot \mathbf{a}|$ is the area of the parallelogram defined by $\mathbf{b}$ and $\mathbf{a}$.

Part (b)

Let $\theta = \frac{\pi}{4}$ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}|\,|\mathbf{b}| \sin\theta \\ &= 2 \cdot 1 \cdot \sin\frac{\pi}{4} \\ &= \sqrt{2} \end{aligned}$$

$$\boxed{|\mathbf{a} \times \mathbf{b}| = \sqrt{2}}$$

Part (c)

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= \Big[3\mathbf{a} + (\mu+2)\mathbf{b}\Big] \times \Big[(\mu+3)\mathbf{a} + \mu\mathbf{b}\Big] \\ &= 3(\mu+3)\mathbf{a} \times \mathbf{a} + 3\mu\mathbf{a} \times \mathbf{b} + (\mu+2)(\mu+3)\mathbf{b} \times \mathbf{a} + \mu(\mu+2)\mathbf{b} \times \mathbf{b} \\ &= 3\mu\mathbf{a} \times \mathbf{b} + (\mu+2)(\mu+3)\mathbf{b} \times \mathbf{a} \\ &= -3\mu\mathbf{b} \times \mathbf{a} + \left(\mu^2 + 5\mu + 6\right)\mathbf{b} \times \mathbf{a} \\ &= \left(\mu^2 + 2\mu + 6\right)\mathbf{b} \times \mathbf{a} \end{aligned}$$

Part (d)

$$\begin{aligned} \min \text{Area}\,\triangle OPQ &= \min \frac{1}{2}\,|\mathbf{p} \times \mathbf{q}| \\ &= \min \frac{1}{2}\left|\mu^2 + 2\mu + 6\right|\,|\mathbf{b} \times \mathbf{a}| \\ &= \min \frac{1}{2}\left|(\mu+1)^2 + 5\right|\sqrt{2} \end{aligned}$$

$$= \frac{1}{2} \cdot 5 \cdot \sqrt{2}$$
$$= \frac{5}{\sqrt{2}}$$

$$\boxed{\text{The smallest area of } \triangle OPQ \text{ is } \frac{5}{\sqrt{2}} \text{ units}^2.}$$

Problem 4. IJC Prelim 9758/2017/01/Q3

The vectors $\mathbf{p}$ and $\mathbf{q}$ are given by $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + a\mathbf{k}$ and $\mathbf{q} = b\mathbf{i} + \mathbf{j}$, where a and b are non-zero constants.

(a) Find $(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})$ in terms of a and b.

Given that the **i**- and **j**− components of the answer to part (a) are equal, find the value of b. Use the value of b you have found to solve parts (b) and (c).

(b) Given that the magnitude of $(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})$ is 80, find the possible exact values of a.

(c) Given instead that $2\mathbf{p} - 5\mathbf{q}$ and $2\mathbf{p} + 5\mathbf{q}$ are perpendicular, find the exact value of $|\mathbf{p}|$.

Solution

Part (a)

We have $\mathbf{p} = \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix}$. Hence,

$$\begin{aligned}
(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q}) &= 4\mathbf{p} \times \mathbf{p} + 10\mathbf{p} \times \mathbf{q} - 10\mathbf{q} \times \mathbf{p} - 25\mathbf{q} \times \mathbf{q} \\
&= 10\mathbf{p} \times \mathbf{q} - 10\mathbf{q} \times \mathbf{p} \\
&= 10\mathbf{p} \times \mathbf{q} + 10\mathbf{p} \times \mathbf{q} \\
&= 20\mathbf{p} \times \mathbf{q} \\
&= 20 \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix} \times \begin{pmatrix} b \\ 1 \\ 0 \end{pmatrix} \\
&= 20 \begin{pmatrix} -a \\ ab \\ 2 - b \end{pmatrix}
\end{aligned}$$

$$\boxed{(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q}) = 20 \begin{pmatrix} -a \\ ab \\ 2 - b \end{pmatrix}}$$

Since the **i**- and **j**-components are equal, we have

$$\begin{aligned}
-a &= ab \\
\implies ab + a &= 0 \\
\implies a(b+1) &= 0
\end{aligned}$$

We thus have $b = -1$. Note that we reject $a = 0$ since a is non-zero.

$$\boxed{b = -1}$$

Part (b)

$$
\begin{aligned}
&& \left|(2\mathbf{p} - 5\mathbf{q}) \times (2\mathbf{p} + 5\mathbf{q})\right| &= 80 \\
\implies && \left|20 \begin{pmatrix} -a \\ -a \\ 3 \end{pmatrix}\right| &= 80 \\
\implies && \left|\begin{pmatrix} -a \\ -a \\ 3 \end{pmatrix}\right| &= 4 \\
\implies && \sqrt{(-a)^2 + (-a)^2 + 3^2} &= 4 \\
\implies && 2a^2 + 9 &= 16 \\
\implies && a^2 &= \frac{7}{2} \\
\implies && a &= \pm\sqrt{\frac{7}{2}}
\end{aligned}
$$

$$\boxed{a = \pm\sqrt{\frac{7}{2}}}$$

Part (c)

Since $2\mathbf{p} - 5\mathbf{q}$ and $2\mathbf{p} + 5\mathbf{q}$, their dot product is 0.

$$
\begin{aligned}
&& (2\mathbf{p} - 5\mathbf{q}) \cdot (2\mathbf{p} + 5\mathbf{q}) &= 0 \\
\implies && 4\mathbf{p} \cdot \mathbf{p} + 10\mathbf{p} \cdot \mathbf{q} - 10\mathbf{q} \cdot \mathbf{p} - 25\mathbf{q} \cdot \mathbf{q} &= 0 \\
\implies && 4\mathbf{p} \cdot \mathbf{p} - 25\mathbf{q} \cdot \mathbf{q} &= 0 \\
\implies && 4\left|\mathbf{p}\right|^2 - 25\left|\mathbf{q}\right|^2 &= 0 \\
\implies && 4\left|\mathbf{p}\right|^2 - 25\left|\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right|^2 &= 0 \\
\implies && 4\left|\mathbf{p}\right|^2 - 25 \cdot 2 &= 0 \\
\implies && \left|\mathbf{p}\right|^2 &= \frac{25}{2} \\
\implies && \left|\mathbf{p}\right| &= \frac{5}{\sqrt{2}}
\end{aligned}
$$

Note that we reject $\left|\mathbf{p}\right| = -\frac{5}{\sqrt{2}}$ since $\left|\mathbf{p}\right| \geq 0$.

$$\boxed{\left|\mathbf{p}\right| = \frac{5}{\sqrt{2}}}$$

Problem 5. JJC Prelim 9758/2017/01/Q6

With respect to the origin O, the position vectors of the points U, V and W are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively. The mid-points of the sides VW, WU and UV of the triangle UVW are M, N and P respectively.

(a) Show that $\overrightarrow{UM} = \frac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u})$.

(b) Find the vector equations of the lines UM and VN. Hence, show that the position vector of the point of intersection, G, of UM and VN is $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$.

Solution

Part (a)

By the Midpoint Theorem,

$$\begin{aligned} & \overrightarrow{OM} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) \\ \implies & \overrightarrow{OU} + \overrightarrow{UM} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) \\ \implies & \overrightarrow{UM} = \frac{1}{2}(\mathbf{v} + \mathbf{w}) - \mathbf{u} \\ & = \frac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u}) \end{aligned}$$

Part (b)

Note that the line UM has direction vector $\mathbf{v} + \mathbf{w} - 2\mathbf{u}$ and passes through U Hence,

$$\boxed{l_{UM} : \mathbf{r} = \mathbf{u} + \lambda(\mathbf{v} + \mathbf{w} - 2\mathbf{u}),\ \lambda \in \mathbb{R}}$$

From the Midpoint Theorem, we have $\overrightarrow{ON} = \frac{1}{2}(\mathbf{w} + \mathbf{u})$. Thus, $\overrightarrow{VN} = \overrightarrow{ON} - \overrightarrow{OV} = \frac{1}{2}(\mathbf{w} + \mathbf{u} - 2\mathbf{v})$. Thus, line VN has direction vector $\mathbf{w} + \mathbf{u} - 2\mathbf{v}$ and passes through V.

$$\boxed{l_{VN} : \mathbf{r} = \mathbf{v} + \mu(\mathbf{w} + \mathbf{u} - 2\mathbf{v}),\ \mu \in \mathbb{R}}$$

Consider $l_{UM} = l_{VN}$.

$$\begin{aligned} & l_{UM} = l_{VN} \\ \implies & \mathbf{u} + \lambda(\mathbf{v} + \mathbf{w} - 2\mathbf{u}) = \mathbf{v} + \mu(\mathbf{w} + \mathbf{u} - 2\mathbf{v}) \\ \implies & (1 - 2\lambda)\mathbf{u} + \lambda\mathbf{v} + \lambda\mathbf{w} = \mu\mathbf{u} + (1 - 2\mu)\mathbf{v} + \mu\mathbf{w} \end{aligned}$$

Comparing coefficients of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ terms, we have the system:

$$\begin{cases} 1 - 2\lambda = \mu \\ \lambda = 1 - 2\mu \\ \lambda = \mu \end{cases}$$

which has solution $\lambda = \mu = \dfrac{1}{3}$. Thus,

$$\begin{aligned}\overrightarrow{OG} &= \mathbf{v} + \frac{1}{3}(\mathbf{w} + \mathbf{u} - \mathbf{v}) \\ &= \frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})\end{aligned}$$

$$\boxed{\overrightarrow{OG} = \frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})}$$

Problem 6. MI Prelim 9740/2017/01/Q5

A line L passes through the points $(3, -1, 0)$ and $B(11, 11, 4)$.

(a) Find the angle between L and the y-axis.

(b) State the geometrical meaning of $\left|\overrightarrow{OB} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|$.

The point $F(2a+1, a, a-1)$ is a point on L, where a is a positive constant. The point P is such that $\overrightarrow{PF} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and the area of the triangle AFP is $\sqrt{\frac{59}{2}}$ units^2.

(c) Determine the value of a.

(d) The point C on L is such that the ratio of the area of triangle AFP to the area of triangle FCP is $2 : 1$. State the ratio $AF : CF$, justifying your answer.

Solution

Part (a)

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 11 \\ 11 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Since L passes through A, it has the vector equation

$$L : \mathbf{r} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Observe that the y-axis has vector equation $\mathbf{r} = \mu \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where $\mu \in \mathbb{R}$. Let θ be the angle between L and the y-axis.

$$\begin{aligned} \cos\theta &= \frac{\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right|} \\ &= \frac{3}{\sqrt{14}} \\ \implies \quad \theta &= \arccos \frac{3}{\sqrt{14}} \\ &= 0.641 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{The angle between } L \text{ and the } y\text{-axis is } 0.641.}$$

Part (b)

$\left| \overrightarrow{OB} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|$ is the length of projection of $\overrightarrow{OB}$ on the z-axis.

Part (c)

Since F is on the line L, we have that

$$\begin{pmatrix} 2a+1 \\ a \\ a-1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

for some $\lambda \in \mathbb{R}$. This gives the system

$$\begin{cases} 2a+1 = 3+2\lambda \\ a \qquad = -1+3\lambda \\ a-1 = \lambda \end{cases}$$

which has solution $a = 2$, $\lambda = 1$.

$$\boxed{a = 2}$$

Part (d)

Since $\triangle AFP$ and $\triangle FCP$ have the same height, the length of the bases of both triangles are in the same ratio as their area. Hence, $AF : CF = \text{Area}\,\triangle AFP : \text{Area}\,\triangle FCP = 2 : 1$.

$$\boxed{AF : CF = 2 : 1}$$

Problem 7. MJC Prelim 9578/2017/01/Q4

(a) The points A and B relative to the origin O have position vectors $3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $-3\mathbf{i} + 2\mathbf{j}$ respectively.

(i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(ii) Hence or otherwise, find the shortest distance from B to line OA.

(b) The points C, D and E relative to the origin O have non-zero and non-parallel position vectors $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$ respectively. Given that $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{e} = 0$, state with reason(s) the relationship between O, C, D and E.

Solution

Part (a)

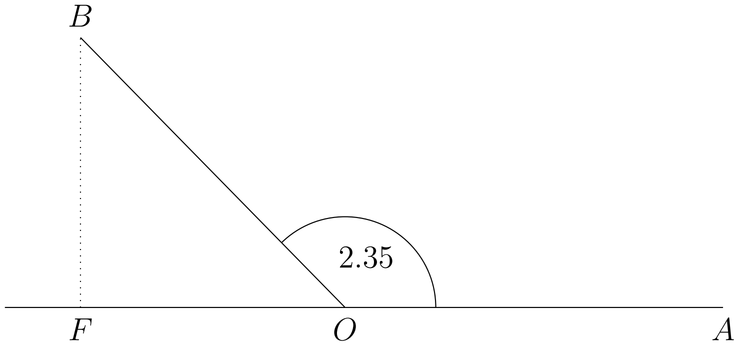
Subpart (i)

We have $\overrightarrow{OA} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$. Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\begin{aligned} \cos\theta &= \frac{\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}}{\left|\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}\right| \left|\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}\right|} \\ &= -\frac{11}{\sqrt{247}} \\ \implies \qquad \theta &= \arccos\left(-\frac{11}{\sqrt{247}}\right) \\ &= 2.35 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{The angle between } \overrightarrow{OA} \text{ and } \overrightarrow{OB} \text{ is } 2.35.}$$

Subpart (ii)



Consider the above diagram, where F is the foot of the perpendicular from B to the line OA. Note that $\angle BOF = \pi - \arccos\left(-\frac{11}{\sqrt{247}}\right)$. Hence,

$$\begin{aligned}
\sin\angle BOF &= \frac{BF}{OB} \\
\implies \qquad BF &= OB \sin\angle BOF \\
&= \sqrt{13}\sin\left(\pi - \arccos\left(-\frac{11}{\sqrt{247}}\right)\right) \\
&= 2.58 \text{ (3 s.f.)}
\end{aligned}$$

$$\boxed{\text{The shortest distance from } B \text{ to the line } OA \text{ is } 2.58 \text{ units.}}$$

Part (b)

Recall that $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{e}$ is the volume of the parallelepiped defined by $\mathbf{c}$, $\mathbf{d}$ and $\mathbf{e}$. Since the volume is 0 and all three vectors are non-zero and non-parallel, they must be coplanar.

Problem 8. NJC Prelim 9758/2017/01/Q1

Given that $\mathbf{p} = 2\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \alpha\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, where α is a real constant and $\mathbf{w}$ is the position vector of a variable point W relative to the origin such that $\mathbf{w} \times \mathbf{p} = \mathbf{q}$.

(a) Find the value of α.

(b) Find the set of vectors $\mathbf{w}$ in the form $\{\mathbf{w} : \mathbf{w} = \mathbf{a} + \lambda\mathbf{b},\ \lambda \in \mathbb{R}\}$.

Solution

Part (a)

We have $\mathbf{p} = \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix}$. Since $\mathbf{w} \times \mathbf{p} = \mathbf{q}$, the vectors $\mathbf{w}$, $\mathbf{p}$ and $\mathbf{q}$ are pairwise orthogonal. Hence,

$$\begin{aligned} \mathbf{p} \cdot \mathbf{q} &= 0 \\ \implies \begin{pmatrix} 2 \\ \alpha \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ 1 \\ 6 \end{pmatrix} &= 0 \\ \implies 2\alpha + \alpha + 6 &= 0 \\ \implies \alpha &= -2 \end{aligned}$$

$$\boxed{\alpha = -2}$$

Part (b)

Let $\mathbf{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

$$\begin{aligned} \mathbf{w} \times \mathbf{p} &= \mathbf{q} \\ \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} \\ \implies \begin{pmatrix} y + 2z \\ 2z - x \\ -2x - 2y \end{pmatrix} &= \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} \end{aligned}$$

This gives the system:

$$\begin{cases} \phantom{-2x - {}} y + 2z = -2 \\ -x \phantom{{}-2y} + 2z = 1 \\ -2x - 2y \phantom{{}+2z} = 6 \end{cases}$$

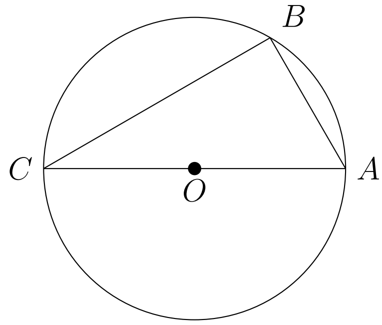
which has solution

$$\begin{cases} x = -1 + 2t \\ y = -2 - 2t \\ z = t \end{cases}$$

Thus, $\mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, where $\lambda \in \mathbb{R}$.

$$\boxed{\left\{ \mathbf{w} : \mathbf{w} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R} \right\}}$$

Problem 9. NJC Prelim 9758/2017/01/Q8



The diagram above shows the cross-section of a sphere containing the centre O of the sphere. The points A, B and C are on the circumference of the cross-section with the line segment AC passing through O. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(a) Find $\overrightarrow{BC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) D is a point on BC such that $\triangle OCD$ is similar to $\triangle ACB$. Find $\overrightarrow{OD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

Point B lies on the x-z plane and has a positive z-component. It is also given that $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\angle OCB = \frac{\pi}{6}$.

(c) Show that $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$.

(d) Hence, determine whether the line passing through O and B and the line $\frac{x-2}{3} = \frac{y}{3} = z - 1$ are skew.

Solution

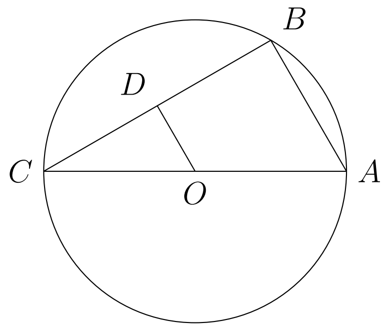
Part (a)

By symmetry, we have $\overrightarrow{OC} = -\overrightarrow{OA}$. Hence,

$$\begin{aligned} && \overrightarrow{OC} &= -\overrightarrow{OA} \\ \implies & \overrightarrow{OB} + & \overrightarrow{BC} &= -\overrightarrow{OA} \\ \implies && \overrightarrow{BC} &= -\overrightarrow{OA} - \overrightarrow{OB} \\ &&&= -\mathbf{a} - \mathbf{b} \end{aligned}$$

$$\boxed{\overrightarrow{BC} = -\mathbf{a} - \mathbf{b}}$$

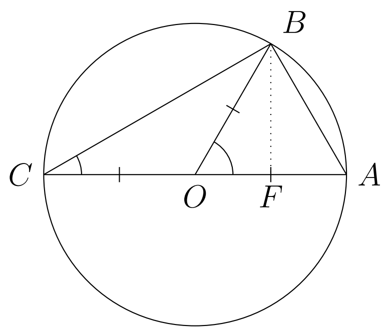
Part (b)



Since $\triangle OCD$ is similar to $\triangle ACB$, we have $\frac{1}{2} = \frac{OC}{AC} = \frac{OD}{AB} \implies OD = \frac{1}{2}AB$. Since $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, we have

$$\boxed{\overrightarrow{OD} = \frac{1}{2}(\mathbf{b} - \mathbf{a})}$$

Part (c)



It is given that $\angle OCB = \frac{\pi}{6}$. Since the angle at the centre is twice the angle at the circumference, we have $\angle AOB = 2\angle OCB = \frac{\pi}{3}$. Since $OB = OA$, it must be that $\triangle OAB$ is equilateral. Let F be the foot of the perpendicular from B to OA. Note that $OB = OC = 2$. Thus, $\cos\angle AOB = \frac{OF}{OB} \implies OF = 1 \implies \overrightarrow{OF} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$. Further, note that $\sin\angle AOB = \frac{FB}{OB} \implies FB = \sqrt{3}$. Since $\overrightarrow{OB}$ has a positive z-component, we have

$$\begin{aligned} \overrightarrow{OB} &= \overrightarrow{OF} + \overrightarrow{FB} \\ &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sqrt{3} \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}}$$

Part (d)

Observe that the line with Cartesian equation $\dfrac{x-2}{3} = \dfrac{y}{3} = z - 1$ has vector equation

$$l_C : \mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$$

Also note that the line OB has equation

$$l_{OB} : \mathbf{r} = \mu \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix},\ \mu \in \mathbb{R}$$

Consider $l_C = l_{OB}$.

$$\begin{aligned} & l_C = l_{OB} \\ \implies \quad & \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \mu \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \\ \implies \mu & \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the system

$$\begin{cases} \mu - 3\lambda = 2 \\ 3\lambda = 0 \\ \sqrt{3}\mu - \lambda = 1 \end{cases}$$

which has no solutions. Hence, the line are skew.

$$\boxed{\text{The lines are skew.}}$$

Problem 10. NYJC Prelim 9758/2017/02/Q1

The position vectors of points A and B with respect to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Point C lies on OA produced such that $4OA = AC$ and point D lies on OB produced such that $OB = BD$. The line BC and AD meet at the point M.

(a) Giving a necessary condition for $\mathbf{a}$ and $\mathbf{b}$, find the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, find the shortest distance of M from the line OC, giving your answer in the form $k\,|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be determined.

Solution

Part (a)

$$\boxed{\mathbf{a} \text{ and } \mathbf{b} \text{ must be non-parallel.}}$$

Note that $\overrightarrow{OC} = 5\mathbf{a}$ and $\overrightarrow{OD} = 2\mathbf{b}$. Hence, $\overrightarrow{AD} = 2\mathbf{b} - \mathbf{a}$ and $\overrightarrow{BC} = 5\mathbf{a} - \mathbf{b}$. Thus, the lines AD and BC have vector equations

$$\begin{aligned} l_{AD} &: \mathbf{r} = \mathbf{a} + \lambda(2\mathbf{b} - \mathbf{a}),\ \lambda \in \mathbb{R} \\ l_{BC} &: \mathbf{r} = \mathbf{b} + \mu(5\mathbf{a} - \mathbf{b}),\ \mu \in \mathbb{R} \end{aligned}$$

Consider $l_{AD} = l_{BC}$.

$$\begin{aligned} & l_{AD} = l_{BC} \\ \implies & \mathbf{a} + \lambda(2\mathbf{b} - \mathbf{a}) = \mathbf{b} + \mu(5\mathbf{a} - \mathbf{b}) \end{aligned}$$

Comparing coefficients of $\mathbf{a}$ and $\mathbf{b}$, we have the system

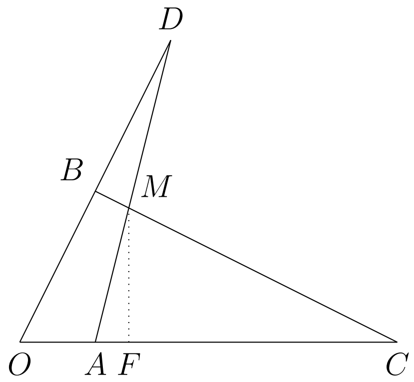
$$\begin{cases} 1 - \lambda = 5\mu \\ 2\lambda = 1 - \mu \end{cases}$$

which has solution $\lambda = \dfrac{4}{9}$ and $\mu = \dfrac{1}{9}$. Thus,

$$\begin{aligned} \overrightarrow{OM} &= \mathbf{b} + \frac{1}{9}(5\mathbf{a} - \mathbf{b}) \\ &= \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b} \end{aligned}$$

$$\boxed{\overrightarrow{OM} = \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}}$$

Part (b)



Let F be the foot of the perpendicular of M to OC. Observe that $\overrightarrow{OM} = \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b} - \mathbf{a} = -\frac{4}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}$ and $\overrightarrow{AC} = 5\mathbf{a} - \mathbf{a} = 4\mathbf{a}$.

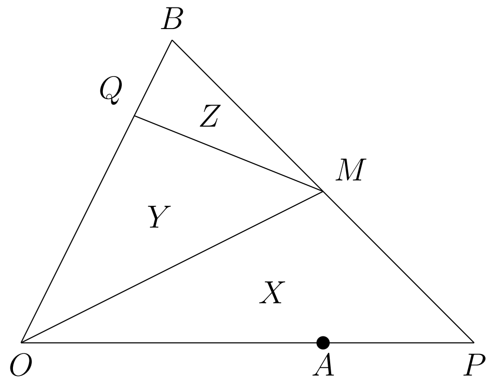
$$\begin{aligned}
& \left|\overrightarrow{AM} \times \overrightarrow{AC}\right| = 2\,\text{Area}\,\triangle AMC \\
\implies \quad & \left|\left(-\frac{4}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}\right) \times 4\mathbf{a}\right| = 2 \cdot \frac{1}{2} \cdot FM \cdot AC \\
\implies & \left|-\frac{16}{9}\mathbf{a} \times \mathbf{a} + 4 \cdot \frac{8}{9}\mathbf{b} \times \mathbf{a}\right| = FM \cdot |4\mathbf{a}| \\
\implies \quad & 4 \cdot \frac{8}{9} \cdot |\mathbf{b} \times \mathbf{a}| = 4 \cdot FM \\
\implies \quad & FM = \frac{8}{9}\,|\mathbf{b} \times \mathbf{a}| \\
& = \frac{8}{9}\,|\mathbf{a} \times \mathbf{b}|
\end{aligned}$$

$$\boxed{\text{The shortest distance of } M \text{ from the line } OC \text{ is } \frac{8}{9}\,|\mathbf{a} \times \mathbf{b}| \text{ units.}}$$

Problem 11. PJC Prelim 9758/2017/02/Q1

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point P lies on OA produced such that $OA : AP = 1 : \lambda$. Point Q lies on OB, between O and B, such that $OQ : QB = 3 : 1$. The mid-point of PB is M. Show that the ratio of the area of triangle OPM to the area of triangle OQM is independent of λ.

Solution



Let the area of $\triangle OPM$, $\triangle OQM$ and $\triangle BQM$ be X, Y and Z respectively. Since $\triangle OPM$ and $\triangle BOM$ share the same height and $BM = MP$, we have

$$X = Y + Z$$

Similarly, since $\triangle OQM$ and $\triangle BQM$ share the same height and $OQ = 3QM$, we have

$$Y = 3Z$$

Thus, $X = Y + \frac{1}{3}Y$, whence $\dfrac{\text{Area}\,\triangle OPM}{\text{Area}\,\triangle OQM} = \dfrac{X}{Y} = \dfrac{4}{3}$. Thus, the required ratio is independent of λ.

Problem 12. RI Prelim 9758/2017/02/Q1

Referred to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively such that

$$\begin{aligned}\mathbf{a} &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}\\ \mathbf{b} &= 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}\\ \mathbf{c} &= 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}\end{aligned}$$

(a) Given that M is the mid-point of AC, use a vector product to find the exact area of triangle ABM.

(b) Find the position vector of the point N on the line AB such that $\overrightarrow{MN}$ is perpendicular to $\overrightarrow{AB}$.

Solution

We have $\mathbf{a} = \begin{pmatrix}2\\3\\-1\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}5\\-2\\3\end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix}4\\1\\-2\end{pmatrix}$.

Part (a)

By the Midpoint Theorem, $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \begin{pmatrix}3\\2\\-3/2\end{pmatrix}$. Thus, $\overrightarrow{AM} = \begin{pmatrix}1\\-1\\-1/2\end{pmatrix}$. We also have $\overrightarrow{AB} = \begin{pmatrix}3\\-5\\4\end{pmatrix}$. Hence,

$$\begin{aligned}\text{Area}\,\triangle ABM &= \frac{1}{2}\left|\overrightarrow{AM} \times \overrightarrow{AB}\right|\\ &= \frac{1}{2}\left|\begin{pmatrix}1\\-1\\-1/2\end{pmatrix} \times \begin{pmatrix}3\\-5\\4\end{pmatrix}\right|\\ &= \frac{1}{4}\left|\begin{pmatrix}-2\\2\\1\end{pmatrix} \times \begin{pmatrix}3\\-5\\4\end{pmatrix}\right|\\ &= \frac{1}{2}\left|\begin{pmatrix}13\\11\\4\end{pmatrix}\right|\\ &= \frac{\sqrt{306}}{4}\\ &= \frac{3\sqrt{34}}{4}\end{aligned}$$

$$\boxed{\text{Area}\,\triangle ABM = \frac{3\sqrt{34}}{4}\text{ units}^2}$$

Part (b)

Note that the line AB has vector equation

$$l_{AB} : \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix},\ \lambda \in \mathbb{R}$$

Since $\overrightarrow{MN}$ is perpendicular to $\overrightarrow{AB}$, we have

$$\begin{aligned}
& & \overrightarrow{MN} \cdot \overrightarrow{AB} &= 0 \\
\implies & & \left(\overrightarrow{ON} - \overrightarrow{OM}\right) \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} &= 0 \\
\implies & & \left[\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ -3/2 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} &= 0 \\
\implies & & \left[\begin{pmatrix} -1 \\ 1 \\ 1/2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} &= 0 \\
\implies & & \begin{pmatrix} -1 \\ 1 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} &= 0 \\
\implies & & -6 + 50\lambda &= 0 \\
\implies & & \lambda &= \frac{3}{25}
\end{aligned}$$

Hence, $\overrightarrow{ON} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \frac{3}{25} \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 59 \\ 60 \\ -13 \end{pmatrix}$.

$$\boxed{\overrightarrow{ON} = \frac{1}{25} \begin{pmatrix} 59 \\ 60 \\ -13 \end{pmatrix}}$$