Problem 1.

Differentiate the following with respect to x.

(a) $\ln \dfrac{x^3}{\sqrt{1+x^2}}$

(b) $\arctan \dfrac{x^2}{2}$

(c) $e^{2x} \sec x$

Solution

Part (a)

$$\frac{d}{dx} \ln \frac{x^3}{\sqrt{1+x^2}} = \frac{d}{dx} 3 \ln x - \frac{1}{2} \ln(1+x^2)$$

$$= 3 \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x$$

$$= \frac{3}{x} - \frac{x}{1+x^2}$$

$$\boxed{\frac{d}{dx} \ln \frac{x^3}{\sqrt{1+x^2}} = \frac{3}{x} - \frac{x}{1+x^2}}$$

Part (b)

$$\frac{d}{dx} \arctan \frac{x^2}{2} = \frac{1}{1+\left(\frac{x^2}{2}\right)} \cdot \frac{1}{2} \cdot 2x$$

$$= \frac{1}{1+\frac{x^4}{4}} \cdot x$$

$$= \frac{4x}{4+x^4}$$

$$\boxed{\frac{d}{dx} \arctan \frac{x^2}{2} = \frac{4x}{4+x^4}}$$

Part (c)

$$\frac{d}{dx} e^{2x} \sec x = e^{2x} \sec x \tan x + \sec x \cdot e^{2x} \cdot 2$$

$$= e^{2x} \sec x \, (\tan x + 2)$$

$$\boxed{\frac{d}{dx} e^{2x} \sec x = e^{2x} \sec x \, (\tan x + 2)}$$

Problem 2.

Find the gradient of the curve $x^3 + xy^2 = 5y$ at the point where $x = 1$ and $0 < y < 1$, leaving your answer to 3 significant figures.

Solution

Substituting $x = 1$ into the given equation,

$$1^4 + 1 \cdot y^2 = 5y$$
$$\implies y^2 - 5y + 1 = 0$$
$$\implies \quad y = \frac{-(-5) \pm \sqrt{5^2 - 4}}{2 \cdot 1}$$
$$= \frac{5 \pm \sqrt{21}}{2}$$

Since $0 < y < 1$, we reject $y = \dfrac{5 + \sqrt{21}}{2}$ and take $y = \dfrac{5 - \sqrt{21}}{2} = 0.20871$ (5 s.f.). Implicitly differentiating the given equation,

$$3x^2 + x \cdot 2y \cdot y' + y^2 = 5y'$$
$$2xy \cdot y' - 5y' = -3x^2 - y^2$$
$$\implies \quad y'(2xy - 5) = -3x^2 - y^2$$
$$\implies \quad y' = \frac{-3x^2 - y^2}{2xy - 5} \tag{2.1}$$

Substituting $x = 1$ and $y = 0.20871$ into Equation 2.1,

$$y' = \frac{-3 \cdot 1^2 - 0.20871^2}{2 \cdot 1 \cdot 0.20871 - 5}$$
$$= 0.664 \text{ (3 s.f.)}$$

$$\boxed{\text{The gradient of the curve is } 0.664.}$$

Problem 3.

A curve C has parametric equations

$$x = \sin^3 \theta, \; y = 3 \sin^2 \theta \cos \theta, \qquad 0 \leq \theta \leq \frac{1}{2}\pi$$

Show that $\dfrac{dy}{dx} = a \cot \theta + b \tan \theta$, where a and b are values to be determined.

Solution

$$\frac{dx}{d\theta} = 3 \sin^2 \theta \cos \theta$$

$$\begin{aligned}
\frac{dy}{d\theta} &= 3 \left(\sin^2 \theta \cdot - \sin \theta + \cos \theta \cdot (2 \sin \theta \cos \theta) \right) \\
&= 3(2 \sin \theta \cos^2 \theta - \sin^3 \theta)
\end{aligned}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \\
&= \frac{dy}{d\theta} \cdot \left(\frac{dx}{d\theta} \right)^{-1} \\
&= \frac{3(2 \sin \theta \cos^2 \theta - \sin^3 \theta)}{3 \sin^2 \theta \cos \theta} \\
&= \frac{2 \cos^2 \theta - \sin^2 \theta}{\sin \theta \cos \theta} \\
&= \frac{2 \cos^2 \theta}{\sin \theta \cos \theta} - \frac{\sin^2 \theta}{\sin \theta \cos \theta} \\
&= \frac{2 \cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \\
&= 2 \cot \theta - \tan \theta
\end{aligned}$$

$$\boxed{a = 2, \; b = -1}$$