Problem 1.

Calculate the exact length of each of the arcs of the following curves.

(a) $y^3 = x^2$ for $-1 \leq x \leq 1$.

(b) $x = t^2 - 1$, $y = t^3 + 1$ from $t = 0$ to $t = 1$.

(c) $r = a\cos\theta$ from $\theta = 0$ to $\theta = \pi/2$.

Solution

Part (a)

Note that $y^3 = x^2 \implies y = x^{2/3} \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{3}x^{-1/3}$.

$$\begin{aligned}
\text{Length} &= \int_{-1}^{1} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\
&= \int_{-1}^{1} \sqrt{1 + \left(\frac{2}{3}x^{-1/3}\right)^2}\,\mathrm{d}x \\
&= \int_{-1}^{1} \sqrt{1 + \frac{4}{9}x^{-2/3}}\,\mathrm{d}x \\
&= 2\int_{0}^{1} \sqrt{1 + \frac{4}{9}x^{-2/3}}\,\mathrm{d}x \\
&= 2\int_{0}^{1} \sqrt{x^{-2/3}\left(x^{2/3} + \frac{4}{9}\right)}\,\mathrm{d}x \\
&= 2\int_{0}^{1} x^{-1/3}\sqrt{x^{2/3} + \frac{4}{9}}\,\mathrm{d}x \\
&= 2\int_{4/9}^{13/9} \sqrt{u}\cdot\frac{3}{2}\,\mathrm{d}u \\
&= 3\left[\frac{2}{3}u^{3/2}\right]_{4/9}^{13/9} \\
&= \frac{2}{27}\left(13\sqrt{13} - 8\right)
\end{aligned}$$

$$u = x^{2/3} + \tfrac{4}{9}$$
$$\mathrm{d}u = \tfrac{2}{3}x^{-1/3}\,\mathrm{d}x$$

$$\boxed{\text{The arc length of the curve is } \frac{2}{27}\left(13\sqrt{13} - 8\right) \text{ units.}}$$

Part (b)

Since the arc length of a curve is invariant under translation, it suffices to find the arc length of the curve with parametric equations $x = t^2, y = t^3$, $0 \leq t \leq 1$. The Cartesian equation of this curve is $y = x^{3/2}$, $0 \leq x \leq 1$, which is the inverse of $y = x^{2/3}$, $0 \leq x \leq 1$. From part (a), the required arc length is $\dfrac{1}{2}\cdot\dfrac{2}{27}\left(13\sqrt{13} - 8\right) = \dfrac{1}{27}\left(13\sqrt{13} - 8\right)$.

$$\boxed{\text{The arc length of the curve is } \frac{1}{27}\left(13\sqrt{13}-8\right) \text{ units.}}$$

Part (c)

Since $r = a\cos\theta$, $0 \leq \theta \leq \dfrac{\pi}{2}$ describes the top half of a circle with centre $\left(\dfrac{a}{2}, 0\right)$ and diameter a, the arc length of the curve is $\dfrac{1}{2} \cdot \pi a = \dfrac{\pi}{2}a$.

$$\boxed{\text{The arc length of the curve is } \frac{\pi}{2}a \text{ units.}}$$

Problem 2.

Find the exact areas of the surfaces generated by completely rotating the following arcs about the (i) x-axis and (ii) y-axis.

(a) The line $2y = x$ between the origin and the point $(4, 2)$.

(b) The curve $x = t^3 - 3t + 2$, $y = 3\left(t^2 - 1\right)$, $t \in \mathbb{R}$ from $t = 1$ to $t = 2$.

Solution

Part (a)

Subpart (i)

When rotated about the x-axis, the curve forms a cone with slant height $\sqrt{4^2 + 2^2} = 2\sqrt{5}$ and radius 2. Hence, the required surface area is $\pi \cdot 2 \cdot 2\sqrt{5} = 4\sqrt{5}\pi$.

$$\boxed{\text{The surface area is } 4\sqrt{5}\pi \text{ units}^2.}$$

Subpart (ii)

When rotated about the y-axis, the curve forms a cone with slant height $\sqrt{4^2 + 2^2} = 2\sqrt{5}$ and radius 4. Hence, the required surface area is $\pi \cdot 4 \cdot 2\sqrt{5} = 8\sqrt{5}\pi$.

$$\boxed{\text{The surface area is } 8\sqrt{5}\pi \text{ units}^2.}$$

Part (b)

Note that $x = t^3 - 3t + 2 \implies \dfrac{\mathrm{d}x}{\mathrm{d}t} = 3t^2 - 3$ and $y = 3\left(t^2 - 1\right) \implies \dfrac{\mathrm{d}y}{\mathrm{d}t} = 6t$.

Subpart (i)

$$\begin{aligned}
\text{Area} &= 2\pi \int_1^2 y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} \,\mathrm{d}t \\
&= 2\pi \int_1^2 3\left(t^2 - 1\right) \sqrt{\left(3t^2 - 3\right)^2 + (6t)^2} \,\mathrm{d}t \\
&= 6\pi \int_1^2 \left(t^2 - 1\right) \sqrt{9t^4 + 18t^2 + 9} \,\mathrm{d}t \\
&= 6\pi \int_1^2 \left(t^2 - 1\right) \sqrt{\left(3t^2 + 3\right)^2} \,\mathrm{d}t \\
&= 18\pi \int_1^2 \left(t^2 - 1\right)\left(t^2 + 1\right) \mathrm{d}t \\
&= 18\pi \int_1^2 \left(t^4 - 1\right) \mathrm{d}t \\
&= 18\pi \left[\frac{1}{5}t^5 - t\right]_1^2 \\
&= \frac{468}{5}\pi
\end{aligned}$$

$$\boxed{\text{The surface area is } \frac{468}{5}\pi \text{ units}^2.}$$

Subpart (ii)

$$\begin{aligned}
\text{Area} &= 2\pi \int_1^2 x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= 2\pi \int_1^2 \left(t^3 - 3t + 2\right) \sqrt{\left(3t^2 - 3\right)^2 + (6t)^2}\,\mathrm{d}t \\
&= 6\pi \int_1^2 \left(t^3 - 3t + 2\right) \left(t^2 + 1\right) \mathrm{d}t \\
&= 6\pi \int_1^2 \left(t^5 - 2t^3 + 2t^2 - 3t + 2\right) \mathrm{d}t \\
&= 6\pi \left[\frac{1}{6}t^6 - \frac{2}{4}t^4 - \frac{2}{3}t^3 - \frac{3}{2}t^2 + 2t\right]_1^2 \\
&= 31\pi
\end{aligned}$$

$$\boxed{\text{The surface area is } 31\pi \text{ units}^2.}$$

Problem 3.

The section of the curve $y = e^x$ between $x = 0$ and $x = 1$ is rotated through one revolution about

(a) the x-axis.

(b) the y-axis.

Find the numerical values of the areas of the surfaces obtained.

Solution

Part (a)

$$\begin{aligned}
\text{Area} &= 2\pi \int_0^1 y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \,\mathrm{d}x \\
&= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} \,\mathrm{d}x \\
&= 22.9 \text{ (3 s.f.)}
\end{aligned}$$

$$\boxed{\text{The surface area is 22.9 units.}}$$

Part (b)

Note that $y = e^x \implies x = \ln y$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^x \implies \dfrac{\mathrm{d}x}{\mathrm{d}y} = e^{-x}$.

$$\begin{aligned}
\text{Area} &= 2\pi \int_1^e x \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2} \,\mathrm{d}y \\
&= 2\pi \int_0^1 \ln y \sqrt{1 + e^{-2x}} \,\mathrm{d}x \\
&= 7.05 \text{ (3 s.f.)}
\end{aligned}$$

$$\boxed{\text{The surface area is 7.05 units.}}$$

Problem 4.

The curve $y^2 = \dfrac{1}{3}x(1-x)^2$ has a loop between $x = 0$ and $x = 1$. Prove that the total length of the loop is $\dfrac{4\sqrt{3}}{3}$.

Solution

Since the curve is even in y, it is symmetric about the x-axis. We thus only consider the part of the curve above the x-axis, i.e. $y \geq 0$, where $y = \sqrt{\dfrac{1}{3}x(1-x)^2}$. Differentiating,

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}\left[\frac{1}{3}x(1-x)^2\right]^{-1/2} \cdot \frac{1}{3}\left[(1-x)^2 + x \cdot 2(1-x) \cdot -1\right] \\
&= \frac{1}{6}\left[\frac{1}{3}x(1-x)^2\right]^{-1/2}\left(3x^2 - 4x + 1\right) \\
&= \frac{1}{6}\left[\frac{1}{3}x(1-x)^2\right]^{-1/2}(3x-1)(x-1) \\
&= \frac{\sqrt{3}}{6}x^{-1/2}(x-1)^{-1}(3x-1)(x-1) \\
&= \frac{3x-1}{2\sqrt{3x}}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\text{Length} &= 2\int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\
&= 2\int_0^1 \sqrt{1 + \left(\frac{3x-1}{2\sqrt{3x}}\right)^2}\,\mathrm{d}x \\
&= 2\int_0^1 \sqrt{1 + \frac{(3x-1)^2}{12x}}\,\mathrm{d}x \\
&= \frac{2}{\sqrt{12}}\int_0^1 \sqrt{\frac{12x + 9x^2 - 6x + 1}{x}}\,\mathrm{d}x \\
&= \frac{\sqrt{3}}{3}\int_0^1 \sqrt{\frac{9x^2 + 6x + 1}{x}}\,\mathrm{d}x \\
&= \frac{\sqrt{3}}{3}\int_0^1 \sqrt{\frac{(3x+1)^2}{x}}\,\mathrm{d}x \\
&= \frac{\sqrt{3}}{3}\int_0^1 \frac{3x+1}{\sqrt{x}}\,\mathrm{d}x \\
&= \frac{\sqrt{3}}{3}\left[\frac{3}{3/2}x^{3/2} + \frac{1}{1/2}x^{1/2}\right]_0^1 \\
&= \frac{4\sqrt{3}}{3}
\end{aligned}
$$

Problem 5.

The tangent at a point P on the curve $x = a\left(t - \frac{1}{3}t^3\right)$, $y = at^2$ cuts the x-axis at T. Prove that the distance of the point T from the origin O is half the length of the arc OP.

Solution

Let P be the point on the curve where $t = t_P$. Note that $x = a\left(t - \frac{1}{3}t^3\right) \implies \frac{\mathrm{d}x}{\mathrm{d}t} = a\left(1 - t^2\right)$ and $y = at^2 \implies \frac{\mathrm{d}y}{\mathrm{d}t} = 2at$.

$$\begin{aligned}
\text{Length of arc } OP &= \int_0^{t_P} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= \int_0^{t_P} \sqrt{\left[a\left(1-t^2\right)\right]^2 + (2at)^2}\,\mathrm{d}t \\
&= a\int_0^{t_P} \sqrt{t^4 + 2t^2 + 1}\,\mathrm{d}t \\
&= a\int_0^{t_P} \sqrt{\left(t^2+1\right)^2}\,\mathrm{d}t \\
&= a\int_0^{t_P} \left(t^2+1\right)\mathrm{d}t \\
&= a\left[\frac{1}{3}t + t\right]_0^{t_P} \\
&= a\left(\frac{1}{3}t_P^3 + t_P\right)
\end{aligned}$$

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{2at}{a\left(1-t^2\right)} = \frac{2t}{1-t^2}$. Hence, the equation of the tangent at P is given by

$$y - at_P^2 = \frac{2t_P}{1-t_P^2}\left[x - a\left(t_P - \frac{1}{3}t_P^3\right)\right]$$

At T, $x = OT$ and $y = 0$. Hence,

$$\begin{aligned}
-at_P^2 &= \frac{2t_P}{1-t_P^2}\left[OT - a\left(t_P - \frac{1}{3}t_P^3\right)\right] \\
\implies -at_P &= \frac{2}{1-t_P^2}\left[OT - a\left(t_P - \frac{1}{3}t_P^3\right)\right] \\
\implies \quad OT &= \frac{-at_P\left(1-t_P^2\right)}{2} + a\left(t_P - \frac{1}{3}t_P^3\right) \\
&= \frac{1}{2}a\left(t_P^3 - t_P + 2t_P - \frac{2}{3}t_P^3\right) \\
&= \frac{1}{2}a\left(\frac{1}{3}t_P^3 + t_P\right) \\
&= \frac{1}{2} \cdot \text{ length of arc } OP
\end{aligned}$$

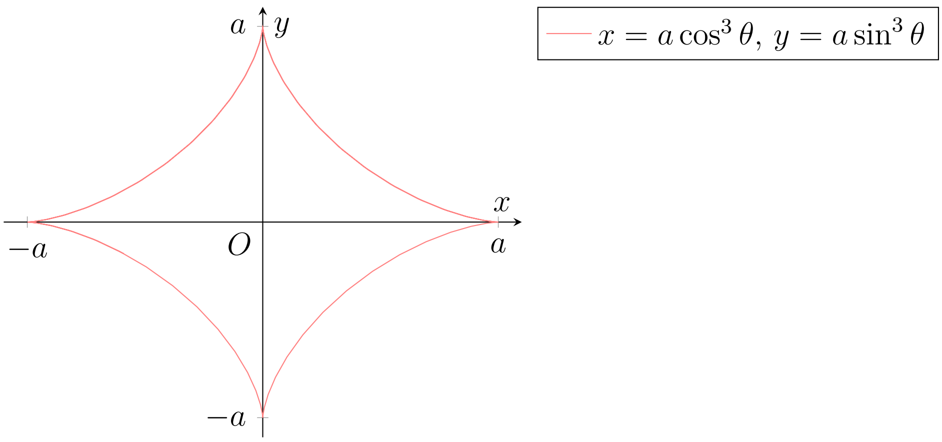
Problem 6.

Sketch the curve whose parametric equations are $x = a\cos^3\theta$, $y = a\sin^3\theta$, $a > 0$.

(a) Find the total length of the curve.

(b) The portion of the curve in the first quadrant is revolved through four right angles about the x-axis. Prove that the area of the surface thus formed is $\frac{6}{5}\pi a^2$.

Solution



Part (a)

By symmetry, we only consider the length of the curve in the first quadrant. Note that $x = 0 \implies \theta = \frac{\pi}{2}$ and $x = a \implies \theta = 0$. Also, $x = a\cos^3\theta \implies \frac{\mathrm{d}x}{\mathrm{d}\theta} = -3a\cos^2\theta\sin\theta$ and $y = a\sin^3\theta \implies \frac{\mathrm{d}y}{\mathrm{d}\theta} = 3a\sin^2\theta\cos\theta$.

$$\begin{aligned}
\text{Length} &= 4\int_0^{\pi/2}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\
&= 4\int_0^{\pi/2}\sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\,\mathrm{d}\theta \\
&= 12a\int_0^{\pi/2}\sqrt{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta}\,\mathrm{d}\theta \\
&= 12a\int_0^{\pi/2}\sqrt{\cos^2\theta\sin^2\theta(\cos^2\theta + \sin^2\theta)}\,\mathrm{d}\theta \\
&= 12a\int_0^{\pi/2}\cos\theta\sin\theta\,\mathrm{d}\theta \\
&= 12a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \\
&= 6a
\end{aligned}$$

$$\boxed{\text{The total length of the curve is } 6a \text{ units.}}$$

Part (b)

$$\begin{aligned}
\text{Area} &= 2\pi \int_0^{\pi/2} x \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\
&= 2\pi \int_0^{\pi/2} a\cos^3\theta \cdot 3a\cos\theta\sin\theta\,\mathrm{d}t \\
&= 3\pi a^2 \cdot 2\int_0^{\pi/2} \sin\theta\cos^4\theta\,\mathrm{d}\theta \\
&= 3\pi a^2 B(1, 5/2) \\
&= 3\pi a^2 \cdot \frac{\Gamma(1)\Gamma(5/2)}{\Gamma(1+5/2)} \\
&= 3\pi a^2 \cdot \frac{\Gamma(5/2)}{\Gamma(5/2)\cdot 5/2} \\
&= 3\pi a^2 \cdot \frac{2}{5} \\
&= \frac{6}{5}\pi a^2
\end{aligned}$$

Problem 7.

The parametric equations of a curve are given by

$$x = \tan t - t,\ y = \ln \sec t,\ t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

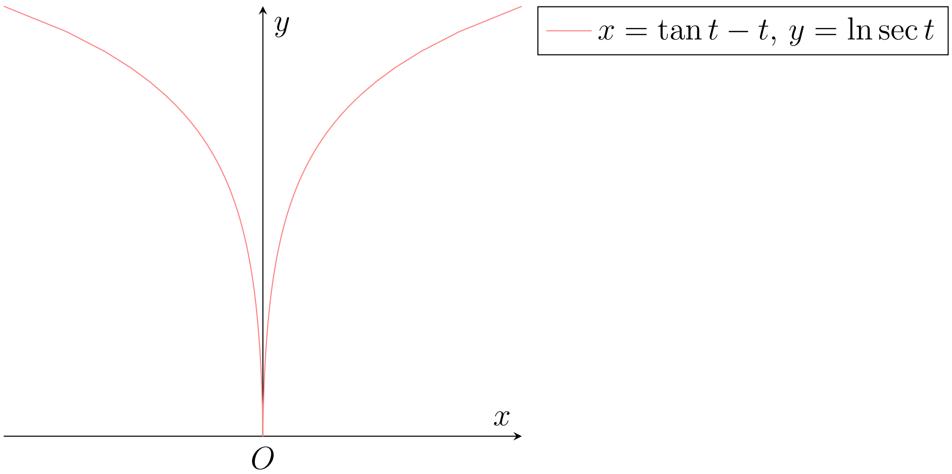
(a) Sketch the curve.

(b) Prove that the arc length of the curve measured from the origin to the point $\left(1 - \frac{\pi}{4}, \frac{1}{2}\ln 2\right)$ is $\sqrt{2} - 1$.

(c) The arc in (b) is rotated about the x-axis through an angle of 360°. Find the exact surface area formed.

Solution

Part (a)



Part (b)

Note that $x = 0 \implies t = 0$ and $x = 1 - \frac{\pi}{4} \implies = t = \frac{\pi}{4}$. Further, $x = \tan t - t \implies \frac{\mathrm{d}x}{\mathrm{d}t} = \sec^2 t - 1 = \tan^2 t$ and $y = \ln \sec t \implies \frac{\mathrm{d}y}{\mathrm{d}t} = \tan t$.

$$\begin{aligned}
\text{Length} &= \int_0^{\pi/4} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= \int_0^{\pi/4} \sqrt{\left(\tan^2 t\right)^2 + (\tan t)^2}\,\mathrm{d}t \\
&= \int_0^{\pi/4} \tan t\sqrt{\tan^2 t + 1}\,\mathrm{d}t \\
&= \int_0^{\pi/4} \tan t \sec t\,\mathrm{d}t \\
&= [\sec t]_0^{\pi/4} \\
&= \sqrt{2} - 1
\end{aligned}$$

Part (c)

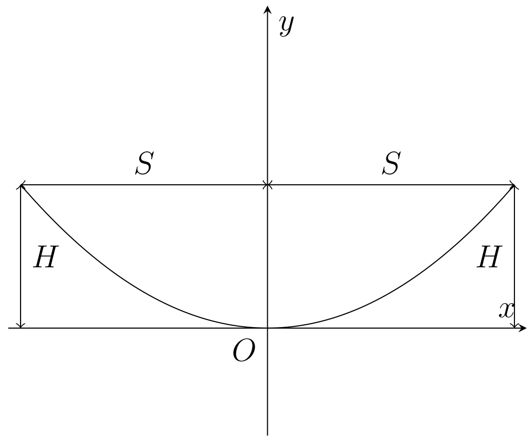
$$\begin{aligned}
\text{Area} &= 2\pi \int_0^{\pi/4} y \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\
&= 2\pi \int_0^{\pi/4} \ln\sec t \cdot \tan t \sec t \,\mathrm{d}t
\end{aligned}$$

	D	I
$+$	$\ln\sec t$	$\tan t \sec t$
$-$	$\tan t$	$\sec t$

$$\begin{aligned}
&= 2\pi \left[[\sec t \ln\sec t]_0^{\pi/4} - \int_0^{\pi/4} \tan t \sec t \,\mathrm{d}t \right] \\
&= 2\pi \left[\sqrt{2} \cdot \frac{1}{2} \ln 2 - (\sqrt{2} - 1) \right] \\
&= \sqrt{2}\pi \left(\ln 2 - 2 + \sqrt{2} \right)
\end{aligned}$$

$\boxed{\text{The surface area is } \sqrt{2}\pi \left(\ln 2 - 2 + \sqrt{2}\right) \text{ units}^2.}$

Problem 8.



The diagram shows a cable for a suspension bridge, which has the shape of a parabola with equation $y = kx^2$. The suspension bridge has a total span $2S$ and the height of the cable relative to the lowest point is H at each end. Show that the total length of the cable is $L = 2\int_0^S \sqrt{1 + \frac{4H^2}{S^4}x^2}\,\mathrm{d}x$.

(a) Engineers from country A proposed a suspension bridge across a strait of 8 km wide to country B. The plan included suspension towers 380 m high at each end. Find the length of the parabolic cable for this proposed bridge to the nearest metre.

(b) By using the result $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(x + \sqrt{a^2 + x^2}\right) = \frac{1}{\sqrt{a^2 + x^2}}$ or otherwise, find L in terms of S and H.

Solution

By symmetry, we only need to consider the length of the curve where $x \geq 0$. Since (S, H) is on the curve, $H = kS^2 \implies k = \frac{H}{S^2}$. Note that $y = kx^2 \implies \frac{\mathrm{d}y}{\mathrm{d}x} = 2kx = \frac{2H}{S^2}x$. Hence,

$$\begin{aligned} L &= 2\int_0^S \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \\ &= 2\int_0^S \sqrt{1 + \frac{4H^2}{S^4}x^2}\,\mathrm{d}x \end{aligned}$$

Part (a)

Note that $2S = 8000 \implies S = 4000$ and $H = 380$. Hence,

$$\begin{aligned} L &= 2\int_0^{4000} \sqrt{1 + \frac{4(380)^2}{(4000)^4}x^2}\,\mathrm{d}x \\ &= 8048 \text{ (to nearest integer)} \end{aligned}$$

$$\boxed{\text{The cable is 8048 m long.}}$$

Part (b)

Consider the integral $I = \displaystyle\int \sqrt{1+(kx)^2}\,\mathrm{d}x$.

$$\begin{aligned} \tan\theta &= kx \\ \sec^2\theta\,\mathrm{d}\theta &= k\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} I &= \int \sqrt{1+(kx)^2}\,\mathrm{d}x \\ &= \frac{1}{k}\int \sqrt{1+\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta \\ &= \frac{1}{k}\int \sec^3\theta\,\mathrm{d}\theta \end{aligned}$$

	D	I
$+$	$\sec\theta$	$\sec^2\theta$
$-$	$\sec\theta\tan\theta$	$\tan t$

$$\begin{aligned} &= \frac{1}{k}\left(\sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,\mathrm{d}\theta\right) \\ &= \frac{1}{k}\left(\sec\theta\tan\theta - \int \sec\theta(\sec^2\theta - 1)\,\mathrm{d}\theta\right) \\ &= \frac{1}{k}\left(\sec\theta\tan\theta - \int \sec^3\theta\,\mathrm{d}\theta + \int \sec\theta\,\mathrm{d}\theta\right) \\ &= \frac{1}{k}\left(\sec\theta\tan\theta - I + \ln|\sec\theta + \tan t|\right) \\ \implies 2I &= \frac{1}{k}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C \\ \implies I &= \frac{1}{2k}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right) + C \\ &= \frac{1}{2k}\left[\sqrt{(kx)^2+1}\cdot kx + \ln\left|\sqrt{(kx)^2+1} + kx\right|\right] + C \end{aligned}$$

$$\begin{aligned} \sec\theta &= \sqrt{\tan^2\theta + 1} \\ &= \sqrt{(kx)^2+1} \end{aligned}$$

In our case, $k = \dfrac{2H}{S^2} > 0$.

$$\begin{aligned} L &= 2\left[\frac{1}{2}\cdot\frac{S^2}{2H}\left(\sqrt{\left(\frac{2H}{S^2}x\right)^2+1}\cdot\frac{2H}{S^2}x + \ln\left(\sqrt{\left(\frac{2H}{S^2}x\right)^2+1} + \frac{2H}{S^2}x\right)\right)\right]_0^S \\ &= \frac{S^2}{2H}\left(\sqrt{\left(\frac{2H}{S^2}S\right)^2+1}\cdot\frac{2H}{S^2}S + \ln\left(\sqrt{\left(\frac{2H}{S^2}S\right)^2+1} + \frac{2H}{S^2}S\right)\right) \\ &= \sqrt{\left(\frac{2H}{S^2}S\right)^2+1}\cdot S + \frac{S^2}{2H}\ln\left(\sqrt{\left(\frac{2H}{S^2}S\right)^2+1} + \frac{2H}{S^2}S\right) \\ &= \sqrt{\left(\frac{2H}{S}\right)^2+1}\cdot S + \frac{S^2}{2H}\ln\left(\sqrt{\left(\frac{2H}{S}\right)^2+1} + \frac{2H}{S}\right) \end{aligned}$$

$$= \sqrt{\frac{4H^2}{S^2} + 1} \cdot S + \frac{S^2}{2H} \ln\left(\sqrt{\frac{4H^2}{S^2} + 1} + \frac{2H}{S}\right)$$

$$= \sqrt{4H^2 + S^2} + \frac{S^2}{2H} \ln\left(\frac{\sqrt{4H^2 + S^2} + 2H}{S}\right)$$

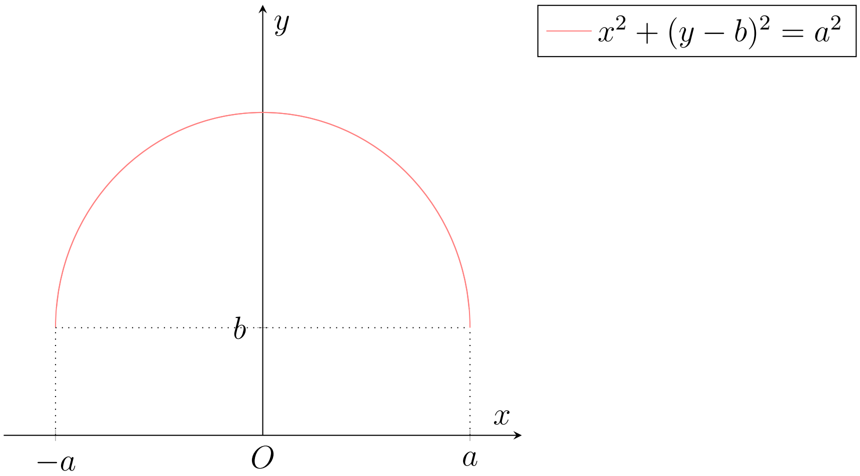
$$\boxed{L = \sqrt{4H^2 + S^2} + \frac{S^2}{2H} \ln\left(\frac{\sqrt{4H^2 + S^2} + 2H}{S}\right)}$$

Problem 9.

Sketch the semicircle with equation $x^2 + (y-b)^2 = a^2$, $y \geq b$ where a and b are positive constants.

A solid is formed by rotating the region bounded by the semicircle and its diameter on the line $y = b$ about the x-axis through 4 right angles. Find the total surface area of the solid.

Solution



Note that $x^2 + (y-b)^2 = a^2 \implies y = b + \sqrt{a^2 - x^2}$ since $y \geq b \implies y - b \geq 0$. Hence, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2\sqrt{a^2-x^2}} \cdot -2x = -\dfrac{x}{\sqrt{a^2-x^2}}$.

$$\begin{aligned}
\text{Area} &= 2\pi \int_{-a}^{a} y\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x + 2\pi \cdot b \cdot 2a \\
&= 2\pi \int_{-a}^{a} (b + \sqrt{a^2-x^2})\sqrt{1 + \left(-\frac{x}{\sqrt{a^2-x^2}}\right)^2}\,\mathrm{d}x + 4\pi ab \\
&= 4\pi \int_{0}^{a} (b + \sqrt{a^2-x^2})\sqrt{\frac{(a^2-x^2)+x^2}{a^2-x^2}}\,\mathrm{d}x + 4\pi ab \\
&= 4\pi \int_{0}^{a} (b + \sqrt{a^2-x^2})\sqrt{\frac{a^2}{a^2-x^2}}\,\mathrm{d}x + 4\pi ab \\
&= 4\pi a \int_{0}^{a} (b + \sqrt{a^2-x^2})\frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x + 4\pi ab \\
&= 4\pi a \left(b\int_{0}^{a} \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x + \int_{0}^{a}\mathrm{d}x\right) + 4\pi ab \\
&= 4\pi a \left(b\left[\arcsin\frac{x}{a}\right]_0^a + [x]_0^a\right) + 4\pi ab \\
&= 4\pi a \left(b \cdot \frac{\pi}{2} + a\right) + 4\pi ab \\
&= 2\pi^2 ab + 4\pi a^2 + 4\pi ab
\end{aligned}$$

$$\boxed{\text{The total surface area is } 2\pi^2 ab + 4\pi a^2 + 4\pi ab \text{ units}^2.}$$

Problem 10.

Using polar coordinates with pole O, the curve C has the equation $r = ae^{\theta/k}$, where a and k are positive constants and $0 \leq \theta \leq 2\pi$. The points A and B on the curve corresponds to $\theta = 0$ and $\theta = \beta$ respectively where $0 < \beta < \pi$. The length of the arc AB is denoted by q and the area of the sector OAB is denoted by Q.

(a) Show that $Q = \frac{1}{4}ka^2\left(e^{2\beta/k} - 1\right)$.

(b) Show that $q = a(1+k^2)^{1/2}\left(e^{\beta/k} - 1\right)$.

(c) Deduce from the results of parts (a) and (b) that, for large values of k, $\frac{Q}{q} \approx \frac{1}{2}a$.

(d) Draw a sketch of C for the case where k is large and explain how the result in part (c) can be deduced from the sketch.

Solution

Part (a)

$$\begin{aligned}
Q &= \frac{1}{2}\int_0^\beta r^2 \,\mathrm{d}\theta \\
&= \frac{1}{2}\int_0^\beta \left(ae^{\theta/k}\right)^2 \mathrm{d}\theta \\
&= \frac{1}{2}a^2\int_0^\beta e^{2\theta/k} \,\mathrm{d}\theta \\
&= \frac{1}{2}a^2\left[\frac{e^{2\theta/k}}{2/k}\right]_0^\beta \\
&= \frac{1}{4}a^2k\left(e^{2\beta/k} - 1\right)
\end{aligned}$$

Part (b)

Note that $r = ae^{\theta/k} \implies \frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{ae^{\theta/k}}{k} = \frac{r}{k}$.

$$\begin{aligned}
q &= \int_0^\beta \sqrt{r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2} \,\mathrm{d}\theta \\
&= \int_0^\beta \sqrt{r^2 + \frac{r^2}{k^2}} \,\mathrm{d}\theta \\
&= \sqrt{1+k^{-2}}\int_0^\beta r \,\mathrm{d}\theta \\
&= \sqrt{1+k^{-2}}\int_0^\beta ae^{\theta/k} \,\mathrm{d}\theta \\
&= a\sqrt{1+k^{-2}}\left[\frac{e^{\theta/k}}{1/k}\right]_0^\beta
\end{aligned}$$

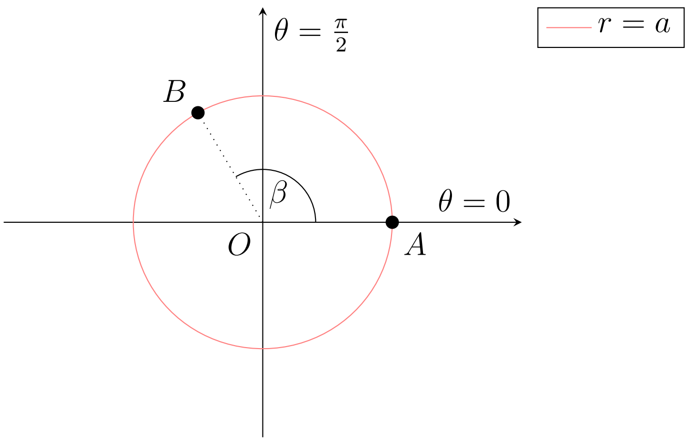
$$\begin{aligned}
&= ak\sqrt{1+k^{-2}}\left(e^{\beta/k}-1\right) \\
&= a\sqrt{k^2+1}\left(e^{\beta/k}-1\right)
\end{aligned}$$

Part (c)

$$\begin{aligned}
\lim_{k\to\infty}\frac{Q}{q} &= \lim_{k\to\infty}\frac{\frac{1}{4}a^2k\left(e^{2\beta/k}-1\right)}{a\sqrt{k^2+1}\left(e^{\beta/k}-1\right)} \\
&= \frac{1}{4}a\lim_{k\to\infty}\left(\frac{k}{\sqrt{k^2+1}}\cdot\frac{e^{2\beta/k}-1}{e^{\beta/k}-1}\right) \\
&= \frac{1}{4}a\lim_{k\to\infty}\left(\frac{k}{\sqrt{k^2+1}}\right)\lim_{k\to\infty}\left(\frac{e^{2\beta/k}-1}{e^{\beta/k}-1}\right) \\
&= \frac{1}{4}a\cdot 1\cdot\lim_{k\to\infty}\left(\frac{2\cdot e^{\beta/k}\cdot \mathrm{d}/\mathrm{d}k\left(e^{\beta/k}\right)}{\mathrm{d}/\mathrm{d}k\left(e^{\beta/k}\right)}\right) \\
&= \frac{1}{2}a\cdot\lim_{k\to\infty}e^{\beta/k} \\
&= \frac{1}{2}a
\end{aligned}$$

Part (d)

Note that $\lim_{k\to\infty} r = \lim_{k\to\infty} ae^{\theta/k} = a$.



As $k \to \infty$, the curve becomes a circle. Hence, Q is the area of a sector with angle β, and q is the arc length of a sector with angle β. Thus, $\frac{Q}{q} = \left(\frac{\beta}{2\pi}\cdot\pi a^2\right)\Big/\left(\frac{\beta}{2\pi}\cdot 2\pi a\right) = \frac{1}{2}a$.