Problem 1.

Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

(a) $$\begin{cases} a+2b-3c=-5 \\ -2a-4b-6c=10 \\ 3a+7b-2c=-13 \end{cases}$$

(b) $$\begin{cases} x-y+3z=3 \\ 4x-8y+32z=24 \\ 2x-3y+11z=4 \end{cases}$$

(c) $$\begin{cases} x_1+x_2=5 \\ 2x_1+x_2+x_3=13 \\ 4x_1+3x_2+x_3=23 \end{cases}$$

(d) $$\begin{cases} \frac{1}{p}+\frac{1}{q}+\frac{1}{r}=5 \\ \frac{2}{p}-\frac{3}{q}-\frac{4}{r}=-11 \\ \frac{3}{p}+\frac{2}{q}-\frac{1}{r}=-6 \end{cases}$$

(e) $$\begin{cases} 2\sin\alpha-\cos\beta+3\tan\gamma=3 \\ 4\sin\alpha+2\cos\beta-2\tan\gamma=2 \\ 6\sin\alpha-3\cos\beta+\tan\gamma=9 \end{cases}$$, where $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$, and $0 \leq \gamma < \pi$.

Solution

Part (a)

$$\boxed{\text{Unique solution: } a=-9,\ b=2,\ c=0}$$

Part (b)

$$\boxed{\text{No solution.}}$$

Part (c)

$$\boxed{\text{Infinitely many solutions: } x_1=8-t,\ x_2=t-3,\ x_3=t}$$

Part (d)

$$\frac{1}{p}=2,\ \frac{1}{q}=-3,\ \frac{1}{r}=6$$

$$\boxed{\text{Unique solution: } p=\frac{1}{2},\ q=-\frac{1}{3},\ r=\frac{1}{6}}$$

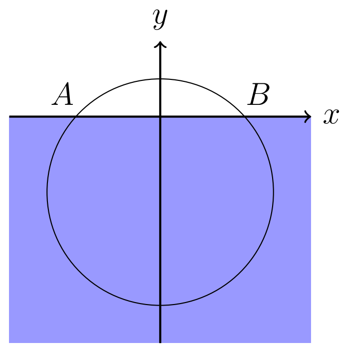
Part (e)

$$\sin\alpha = 1,\ \cos\beta = -1,\ \tan\gamma = 0$$

$$\boxed{\text{Unique solution: } \alpha = \frac{\pi}{2},\ \beta = \pi,\ \gamma = 0}$$

Problem 2.

The following figure shows the circular cross section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0 \tag{2.1}$$

A and B are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when AB is 40 cm apart horizontally, determine the values of a, b and c by forming a system of linear equations.

Solution

Since $AB = 40$, we have $A(-20, 0)$ and $B(20, 0)$. We also know $(0, 10)$ lies on the circle. Substituting these points into Equation 2.1,

$$\begin{cases} -20a + c = -400 \\ 20a + c = -400 \\ 10b + c = -100 \end{cases}$$

Solving the system of equations,

$$\boxed{a = 0,\ b = 30,\ c = -400}$$

Problem 3.

Find the exact solution set of the following inequalities.

(a) $x^2 - 2 \geq 0$

(b) $4x^2 - 12x + 10 > 0$

(c) $x^2 + 4x + 13 < 0$

(d) $x^3 < 6x - x^2$

(e) $x^2(x-1)(x+3) \geq 0$

Solution

Part (a)

$$\begin{aligned} & x^2 - 2 \geq 0 \\ \implies \quad & x^2 \geq 2 \\ \implies \quad & x \leq -\sqrt{2} \vee x \geq \sqrt{2} \end{aligned}$$

$$\boxed{\text{Solution set: } \left\{x \in \mathbb{R} \colon x \leq -\sqrt{2} \vee x \geq \sqrt{2}\right\}}$$

Part (b)

$$\begin{aligned} & 4x^2 - 12x + 10 > 0 \\ \implies \quad & x^2 - 3x + \frac{5}{2} > 0 \\ \implies \quad & x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{2} > 0 \\ \implies \quad & \left(x - \frac{3}{2}\right)^2 + \frac{19}{4} > 0 \end{aligned}$$

Since $\left(x - \frac{3}{2}\right)^2 \geq 0$, all $x \in \mathbb{R}$ satisfy the inequality.

$$\boxed{\text{Solution set: } \mathbb{R}}$$

Part (c)

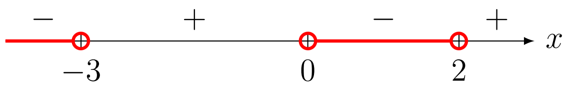
$$\begin{aligned} & x^2 + 4x + 13 < 0 \\ \implies \quad & x^2 + 4x + 4 + 9 < 0 \\ \implies \quad & (x+2)^2 + 9 < 0 \end{aligned}$$

Since $(x+2)^2 \geq 0$, there is no solution to the inequality.

$$\boxed{\text{Solution set: } \varnothing}$$

Part (d)

$$
\begin{aligned}
& x^3 < 6x - x^2 \\
\implies & x(x+3)(x-2) < 0
\end{aligned}
$$

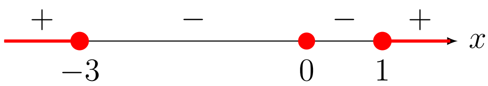


$$\implies x < -3 \lor 0 < x < 2$$

$$\boxed{\text{Solution set: } \{x \in \mathbb{R} \colon x < -3 \lor 0 < x < 2\}}$$

Part (e)

$$x^2(x-1)(x+3) \geq 0$$



$$\implies x \leq -3 \lor x = 0 \lor x \geq 1$$

$$\boxed{\text{Solution set: } \{x \in \mathbb{R} \colon x \leq -3 \lor x = 0 \lor x \geq 1\}}$$

Problem 4.

Find the exact solution set of the following inequalities.

(a) $|3x + 5| < 4$

(b) $|x - 2| < 2x$

Solution

Part (a)

$$|3x + 5| < 4$$

Case 1: $3x + 5 < 4$

$$\begin{aligned} & 3x + 5 < 4 \\ \implies & 3x < -1 \\ \implies & x < -\frac{1}{3} \end{aligned}$$

Case 2: $-(3x + 5) < 4$

$$\begin{aligned} & -(3x + 5) < 4 \\ \implies & -3x - 5 < 4 \\ \implies & -3x < 9 \\ \implies & x > -3 \end{aligned}$$

Combining both inequalities, we have $-3 < x < -\frac{1}{3}$.

$$\boxed{\text{Solution set: } \left\{x \in \mathbb{R} \colon -3 < x < -\frac{1}{3}\right\}}$$

Part (b)

Case 1: $x - 2 < 2x$

$$\begin{aligned} & x - 2 < 2x \\ \implies & x > -2 \end{aligned}$$

Case 2: $-(x - 2) < 2x$

$$\begin{aligned} & -(x - 2) < 2x \\ \implies & -x + 2 < 2x \\ \implies & 3x > 2 \\ \implies & x > \frac{2}{3} \end{aligned}$$

Combining both inequalities, we have $x > \frac{2}{3}$.

$$\boxed{\text{Solution set: } \left\{x \in \mathbb{R} \colon x > \frac{2}{3}\right\}}$$

Problem 5.

It is given that $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are constants. Given that the curve with equation $y = p(x)$ is symmetrical about the y-axis, and that it passes through the points with coordinates $(1, 2)$ and $(2, 11)$, find the values of a, b, c and d.

Solution

We know that $(1, 2)$ and $(2, 11)$ lie on the curve. Since $y = p(x)$ is symmetrical about the y-axis, we have that $(-1, 2)$ and $(-2, 11)$ also lie on the curve. Substituting these points into $y = p(x)$, we obtain the following system of equations.

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving the system of equations,

$$\boxed{a = 0,\ b = -2,\ c = 0,\ d = 3}$$

Problem 6.

Mr Mok invested \$50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of \$3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

Solution

Let a, b and c be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a+b+c=50000 \\ \frac{6}{100}a+\frac{8}{100}b+\frac{10}{100}c=3700 \\ a=2c \end{cases}$$

Solving the system of equations, we have $a = 30000$, $b = 5000$ and $c = 15000$.

$\boxed{\text{Mr Mok invested \$30,000, \$5,000 and \$15,000 in Funds A, B and C respectively.}}$

Problem 7.

Solve the following inequalities with exact answers.

(a) $2x - 1 \geq \dfrac{6}{x}$

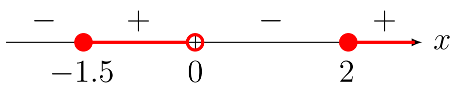
(b) $x - \dfrac{1}{x} < 1$

(c) $-1 < \dfrac{2x+3}{x-1} < 1$

Solution

Part (a)

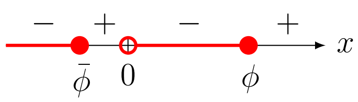
$$\begin{aligned} && 2x - 1 &\geq \frac{6}{x}, \qquad x \neq 0 \\ \implies && x^2(2x-1) &\geq 6x \\ \implies && x^2(2x-1) - 6x &\geq 0 \\ \implies && x(x(2x-1) - 6) &\geq 0 \\ \implies && x\left(2x^2 - x - 6\right) &\geq 0 \\ \implies && x(2x+3)(x-2) &\geq 0 \end{aligned}$$



$$\boxed{-\frac{3}{2} \leq x < 0 \lor x \geq 2}$$

Part (b)

$$\begin{aligned} && x - \frac{1}{x} &< 1, \qquad x \neq 0 \\ \implies && x^3 - x &< x^2 \\ \implies && x^3 - x^2 - x &< 0 \\ \implies && x\left(x^2 - x - 1\right) &< 0 \\ \implies && x(x-\phi)(x-\bar{\phi}) &< 0 \end{aligned}$$



$$\boxed{x \leq \bar{\phi} \lor 0 < x \leq \phi}$$

Part (c)

$$\begin{aligned}
& -1 < \frac{2x+3}{x-1} < 1 \\
\implies & -1 < \frac{(2x-2)+5}{x-1} < 1 \\
\implies & -1 < 2 + \frac{5}{x-1} < 1 \\
\implies & -3 < \frac{5}{x-1} < -1 \\
\implies & -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5} \\
\implies & -5 < x-1 < -\frac{5}{3} \\
\implies & -4 < x < -\frac{2}{3}
\end{aligned}$$

$$\boxed{-4 < x < -\frac{2}{3}}$$

Problem 8.

Without using a calculator, solve the inequality $\dfrac{x^2+x+1}{x^2+x-2} < 0$.

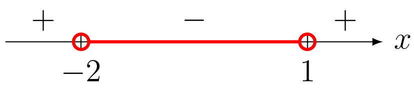
Solution

Observe that

$$\begin{aligned} x^2+x+1 &= x^2+x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1 \\ &= \left(x+\frac{1}{2}\right)^2+\frac{3}{4} \\ &> 0 \end{aligned}$$

Thus, the inequality reduces to $\dfrac{1}{x^2+x-2} < 0$

$$\begin{aligned} & \frac{1}{x^2+x-2} < 0 \\ \implies & \quad x^2+x-1 < 0 \\ \implies & (x-1)(x+2) < 0 \end{aligned}$$



$$\boxed{-2 < x < 1}$$

Problem 9.

Solve the following inequalities using a graphical method.

(a) $|3x+1| < (4x+3)^2$

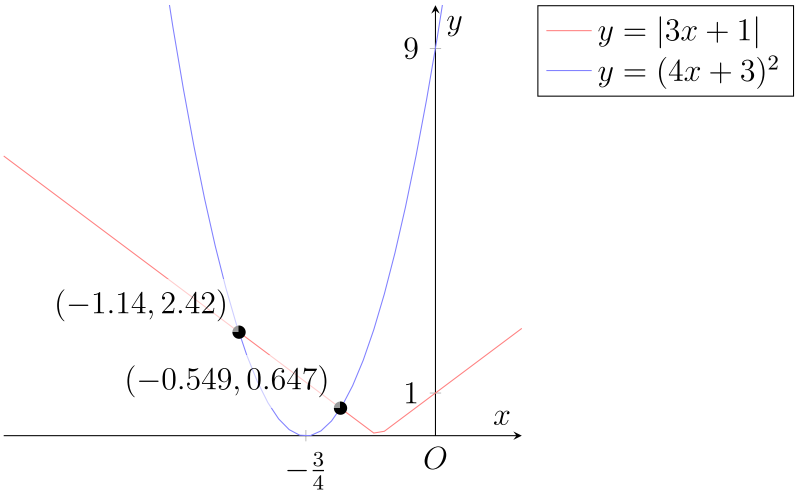
(b) $|3x+1| \geq |2x+7|$

(c) $|x-2| \geq x+|x|$

(d) $5x^2+4x-3 > \ln(x+1)$

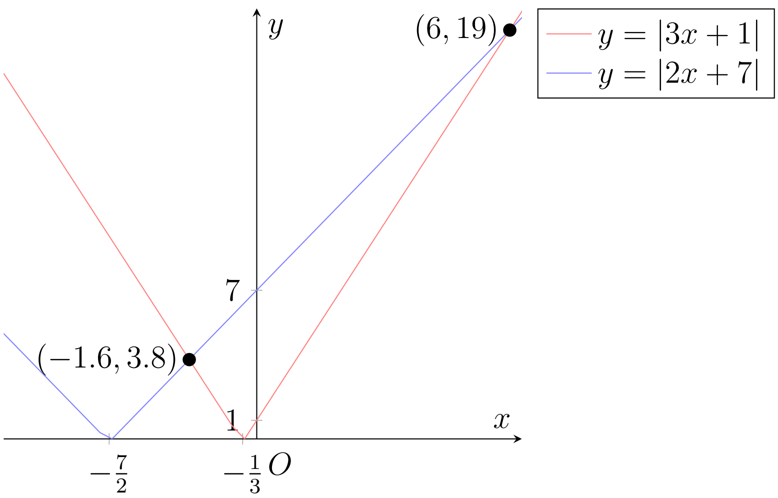
Solution

Part (a)



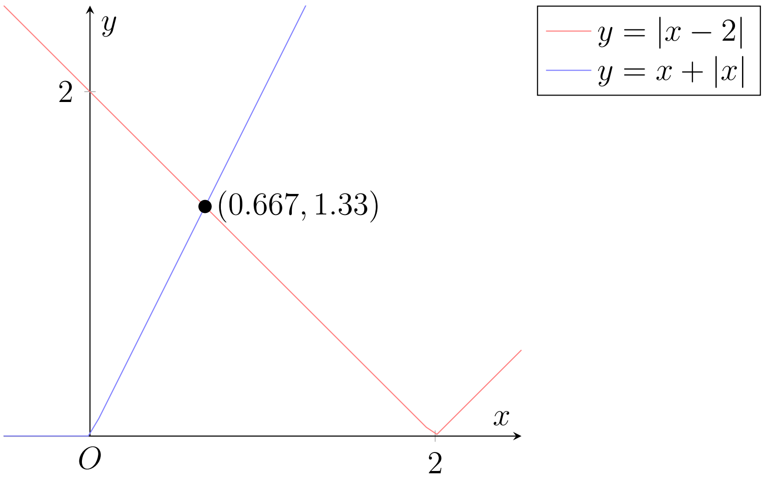
$$\boxed{x < -1.14 \lor x > -0.549}$$

Part (b)



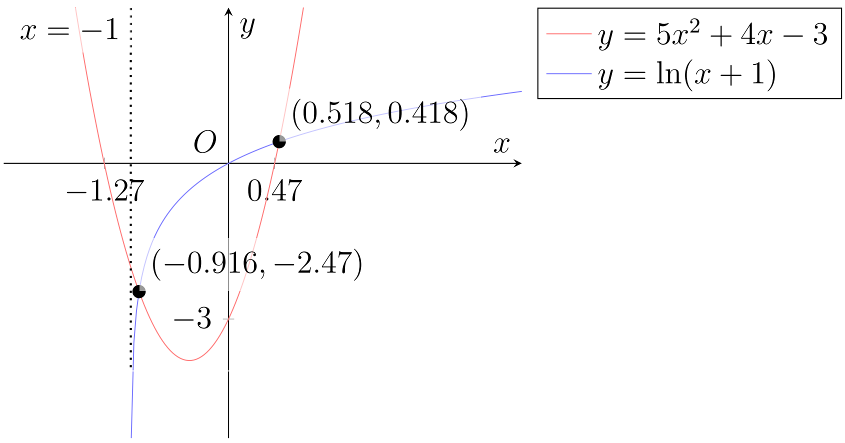
$$\boxed{x \leq -1.6 \lor x \geq 6}$$

Part (c)



$$\boxed{x \leq 0.667}$$

Part (d)

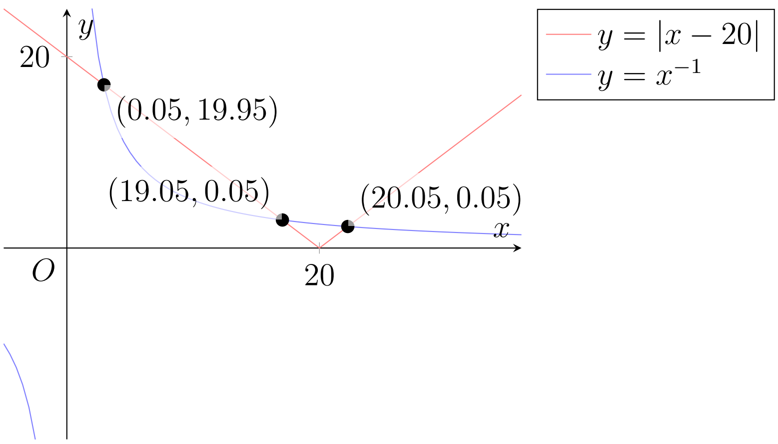


$$\boxed{-1 < x < -0.916 \vee x > 0.518}$$

Problem 10.

Sketch the graphs of $y = |x - 20|$ and $y = \dfrac{1}{x}$ on the same diagram. Hence or otherwise, solve the inequality $|x - 20| < \dfrac{1}{x}$, leaving your anwers correct to 2 decimal places.

Solution



$$\boxed{0 < x < 0.05 \lor 19.95 < x < 20.05}$$

Problem 11.

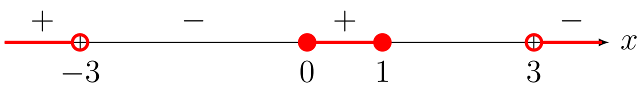
Solve the inequality $\dfrac{x-9}{x^2-9} \leq 1$. Hence, solve the inequalities

(a) $\dfrac{|x|-9}{x^2-9} \leq 1$

(b) $\dfrac{x+9}{x^2-9} \geq -1$

Solution

$$\begin{aligned}
& \frac{x-9}{x^2-9} \leq 1 \\
\implies & (x-9)\left(x^2-9\right) \leq \left(x^2-9\right)^2 \\
\implies & x^3-9x^2-9x+81 \leq x^4-18x^2+81 \\
\implies & x^4-x^3-9x^2+9x \geq 0 \\
\implies & (x+3)x(x-1)(x-3) \geq 0
\end{aligned}$$



$$\boxed{x < -3 \vee 0 \leq x \leq 1 \vee x > 3}$$

Part (a)

Consider the substitution $x \mapsto |x|$ on the inequality $\dfrac{x-9}{x^2-9} \leq 1$. This yields our desired inequality $\dfrac{|x|-9}{x^2-9} \leq 1$. Hence,

$$|x| < -3 \vee 0 \leq |x| \leq 1 \vee |x| > 3$$

Case 1: $|x| < -3$
No solution.

Case 2: $0 \leq |x| \leq 1$

$$-1 \leq x \leq 1$$

Case 3: $|x| > 3$

$$x < -3 \vee x > 3$$

Combining all three cases, we finally have

$$\boxed{x < -3 \vee -1 \leq x \leq 1 \vee x > 3}$$

Part (b)

Consider the substitution $x \mapsto -x$ on the inequality $\frac{x-9}{x^2-9} \leq 1$. This yields $\frac{-x-9}{x^2-9} \leq 1$, which is equivalent to our desired inequality $\frac{x+9}{x^2-9} \geq -1$. Hence,

$$-x < -3 \vee 0 \leq -x \leq 1 \vee -x > 3$$

Case 1: $-x < -3$

$$x > 3$$

Case 2: $0 \leq -x \leq 1$

$$-1 \leq x \leq 0$$

Case 3: $-x > 3$

$$x < -3$$

Combining all three cases, we finally have

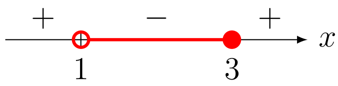
$$\boxed{x < -3 \vee -1 \leq x \leq 0 \vee x > 3}$$

Problem 12.

Solve the inequality $\dfrac{x-5}{1-x} \geq 1$. Hence, solve $0 < \dfrac{1-\ln x}{\ln x - 5} \leq 1$.

Solution

$$\begin{aligned} \frac{x-5}{1-x} &\geq 1, \qquad x \neq 1 \\ \implies (x-5)(1-x) &\geq (1-x)^2 \\ \implies 2x^2 - 8x + 6 &\leq 0 \\ \implies 2(x-1)(x-3) &\leq 0 \end{aligned}$$



$$\boxed{1 < x \leq 3}$$

Consider the substitution $x \mapsto \ln x$ on the inequality $\dfrac{x-5}{1-x} \geq 1$. This yields our desired inequality $0 < \dfrac{1-\ln x}{\ln x - 5} \leq 1$. Hence,

$$\begin{aligned} 1 < \ln x &\leq 3 \\ \implies e < x &\leq e^3 \end{aligned}$$

$$\boxed{e < x \leq e^3}$$

Problem 13.

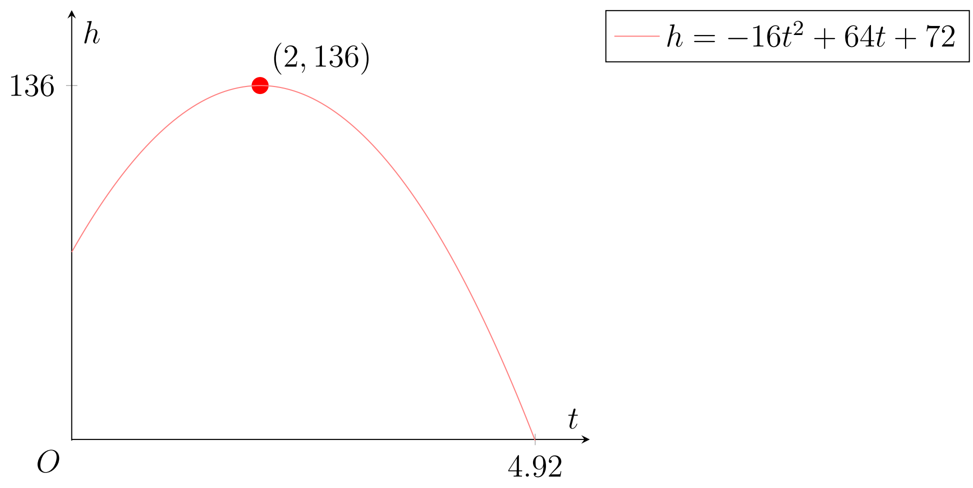
A small rocket is launched from a height of 72 m from the ground. The heigh of the rocket in metres, h, is represented by the equation $h = -16t^2 + 64t + 72$, where t is the time in seconds after the launch.

(a) Sketch the graph of h against t.

(b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

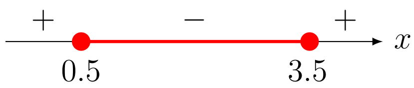
Solution

Part (a)



Part (b)

$$\begin{aligned} & h \geq 100 \\ \implies & -16t^2 + 64t + 72 \geq 100 \\ \implies & -16t^2 + 64t - 28 \geq 0 \\ \implies & \quad 16t^2 - 64t + 28 \geq 0 \\ \implies & 4(2t-1)(2t-7) \leq 0 \end{aligned}$$



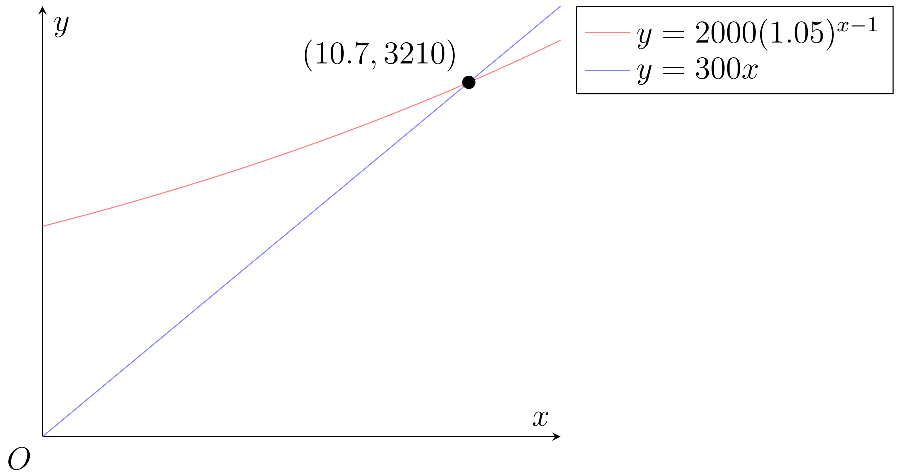
$$\boxed{\text{The rocket will remain at or above 100 m from the ground for } \frac{7}{2} - \frac{1}{2} = 3 \text{ seconds.}}$$

Problem 14.

Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of $2000(1.05)^{n-1}$ dollars in the nth quarter, where n is a positive integer. She also gives a regular donation of \$$300n$ in the nth quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

Solution

Consider the curves $y = 2000(1.05)^{x-1}$ and $y = 300x$.



$$\boxed{\text{Xinxin will stop donating in the } \lceil 10.7 \rceil = 11\text{th quarter.}}$$