Double Math Solutions

https://asdia.dev/projects/doublemath

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v0.1.20241121

Contents

Part I.

Group A

A1. Equations and Inequalities

Tutorial A1

Problem 1. Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

(a) $\begin{cases} a + 2b - 3c = -5 \\ -2a - 4b - 6c = 10 \\ 3a + 7b - 2c = -13 \end{cases}$

(b) $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$

(c) $\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$

(d) $\begin{cases} 1/p + 1/q + 1/r = 5 \\ 2/p - 3/q - 4/r = -11 \\ 3/p + 2/q - 1/r = -6 \end{cases}$

(e) $\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\gamma = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma = 9 \end{cases}$, where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

Solution.
Part (a). Unique solution: $a = -9$, $b = 2$, $c = 0$.
Part (b). No solution.
Part (c). Infinitely many solutions: $x_1 = 8 - t$, $x_2 = t - 3$, $x_3 = t$.
Part (d). Solving, we obtain

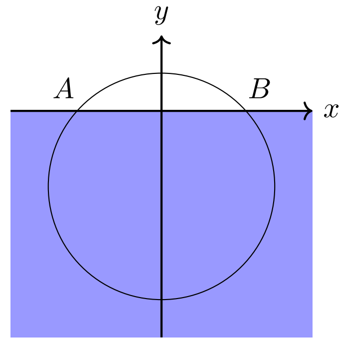
$$\frac{1}{p} = 2, \quad \frac{1}{q} = -3, \quad \frac{1}{r} = 6.$$

There is hence a unique solution: $p = \frac{1}{2}$, $q = -\frac{1}{3}$, $r = \frac{1}{6}$.
Part (e). Solving, we obtain

$$\sin\alpha = 1, \quad \cos\beta = -1, \quad \tan\gamma = 0.$$

There is hence a unique solution: $\alpha = \frac{\pi}{2}$, $\beta = \pi$, $\gamma = 0$.

Problem 2. The following figure shows the circular cross-section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0.$$

A and B are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when AB is 40 cm apart horizontally, determine the values of a, b and c by forming a system of linear equations.

Solution. Since $AB = 40$, we have $A(-20, 0)$ and $B(20, 0)$. We also know $(0, 10)$ lies on the circle. Substituting these points into the given equation, we have the following system of equations:

$$\begin{cases} -20a \qquad + c = -400 \\ \;\;20a \qquad + c = -400 \\ \qquad\quad 10b + c = -100 \end{cases}$$

Solving, we obtain $a = 0$, $b = 30$, $c = -400$.

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Problem 3. Find the exact solution set of the following inequalities.

(a) $x^2 - 2 \geq 0$

(b) $4x^2 - 12x + 10 > 0$

(c) $x^2 + 4x + 13 < 0$

(d) $x^3 < 6x - x^2$

(e) $x^2(x - 1)(x + 3) \geq 0$

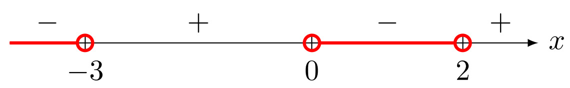
Solution.

Part (a). Note that $x^2 - 2 \geq 0 \implies x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$. The solution set is thus $\{x \in \mathbb{R} : x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\}$.

Part (b). Completing the square, we see that $4x^2 - 12x + 10 > 0 \implies (x - \frac{3}{2})^2 + \frac{19}{4} > 0$. Since $(x - \frac{3}{2})^2 \geq 0$, all $x \in \mathbb{R}$ satisfy the inequality, whence the solution set is $\mathbb{R}$.

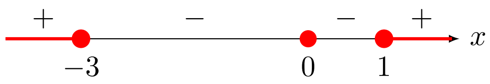
Part (c). Completing the square, we have $x^2 + 4x + 13 < 0 \implies (x + 2)^2 + 9 < 0$. Since $(x + 2)^2 \geq 0$, there is no solution to the inequality, whence the solution set is $\varnothing$.

Part (d). Note that $x^3 < 6x - x^2 \implies x(x + 3)(x - 2) < 0$.



The solution set is thus $\{x \in \mathbb{R} \colon x < -3 \text{ or } 0 < x < 2\}$.

Part (e).



The solution set is thus $\{x \in \mathbb{R} \colon x \leq -3 \text{ or } x = 0 \text{ or } x \geq 1\}$.

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Problem 4. Find the exact solution set of the following inequalities.

(a) $|3x + 5| < 4$

(b) $|x - 2| < 2x$

Solution.

Part (a). If $3x + 5 < 4$, then $x < -\frac{1}{3}$. If $-(3x + 5) < 4$, then $x > -3$. Combining both inequalities, we have $-3 < x < -\frac{1}{3}$. Thus, the solution set is $\left\{x \in \mathbb{R} \colon -3 < x < -\frac{1}{3}\right\}$.

Part (b). If $x - 2 < 2x$, then $x > -2$. If $-(x - 2) < 2x$, then $x > \frac{2}{3}$. Combining both inequalities, we have $x > \frac{2}{3}$. Thus, the solution set is $\left\{x \in \mathbb{R} \colon x > \frac{2}{3}\right\}$.

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Problem 5. It is given that $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c and d are constants. Given that the curve with equation $y = p(x)$ is symmetrical about the y-axis, and that it passes through the points with coordinates $(1, 2)$ and $(2, 11)$, find the values of a, b, c and d.

Solution. We know that $(1, 2)$ and $(2, 11)$ lie on the curve. Since $y = p(x)$ is symmetrical about the y-axis, we have that $(-1, 2)$ and $(-2, 11)$ also lie on the curve. Substituting these points into $y = p(x)$, we obtain the following system of equations:

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving, we obtain $a = 0$, $b = -2$, $c = 0$, $d = 3$.

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Problem 6. Mr Mok invested \$50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of \$3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

Solution. Let a, b and c be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a + b + c = 50000 \\ \frac{6}{100}a + \frac{8}{100}b + \frac{10}{100}c = 3700 \\ a = 2c \end{cases}$$

Solving, we have $a = 30000$, $b = 5000$ and $c = 15000$. Thus, Mr Mok invested \$30,000, \$5,000 and \$15,000 in Funds A, B and C respectively.

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Problem 7. Solve the following inequalities with exact answers.

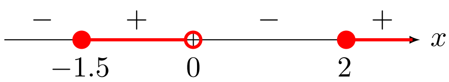
(a) $2x - 1 \geq \frac{6}{x}$

(b) $x - \frac{1}{x} < 1$

(c) $-1 < \frac{2x+3}{x-1} < 1$

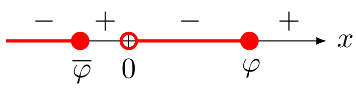
Solution.
Part (a). Note that $x \neq 0$.

$$2x - 1 \geq \frac{6}{x} \implies x^2(2x-1) \geq 6x \implies x\left(2x^2 - x - 6\right) \geq 0 \implies x(2x+3)(x-2) \geq 0.$$



Thus, $-\frac{3}{2} \leq x < 0$ or $x \geq 2$.
Part (b). Note that $x \neq 0$.

$$x - \frac{1}{x} < 1 \implies x^3 - x < x^2 \implies x\left(x^2 - x - 1\right) < 0 \implies x(x-\varphi)(x-\overline{\varphi}) < 0.$$



Thus, $x \leq \bar{\varphi}$ or $0 < x \leq \varphi$.
Part (c).

$$-1 < \frac{2x+3}{x-1} < 1 \implies -3 < \frac{5}{x-1} < -1 \implies -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5} \implies -4 < x < -\frac{2}{3}.$$

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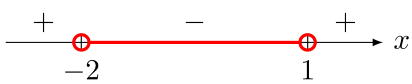
Problem 8. Without using a calculator, solve the inequality $\frac{x^2+x+1}{x^2+x-2} < 0$.

Solution. Observe that $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0$. The inequality thus reduces to $\frac{1}{x^2+x-2} < 0$.

$$\frac{1}{x^2 + x - 2} < 0 \implies x^2 + x - 2 < 0 \implies (x-1)(x+2) < 0.$$



Hence, $-2 < x < 1$.

Problem 9. Solve the following inequalities using a graphical method.

(a) $|3x+1| < (4x+3)^2$

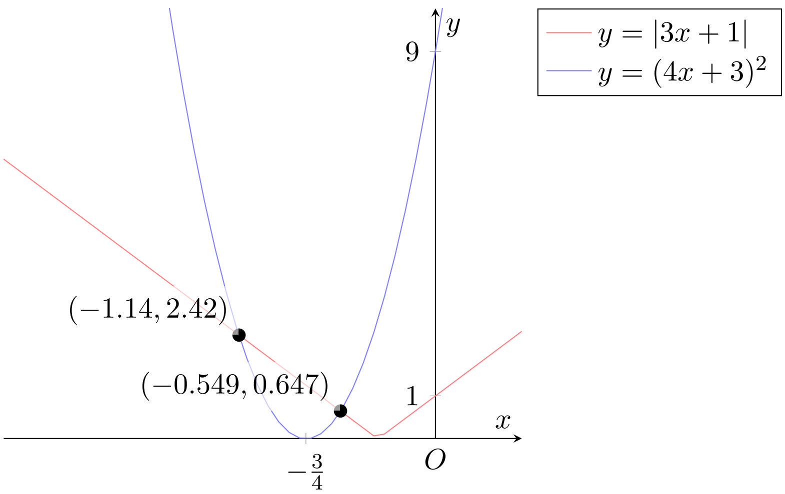
(b) $|3x+1| \geq |2x+7|$

(c) $|x-2| \geq x + |x|$

(d) $5x^2 + 4x - 3 > \ln(x+1)$

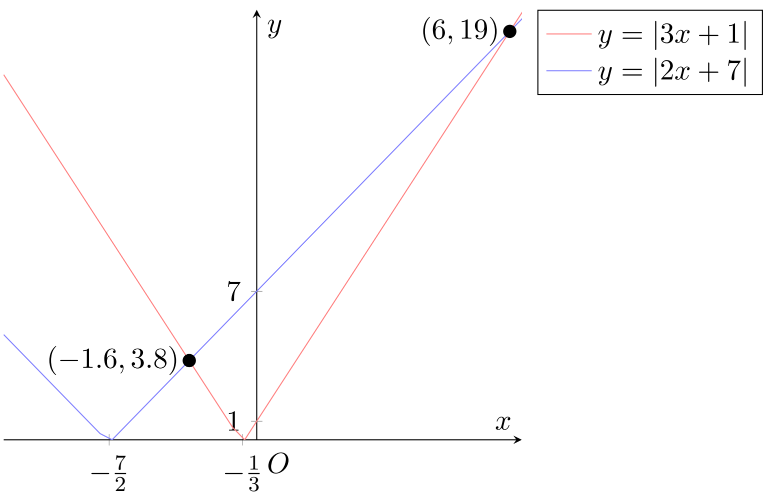
Solution.

Part (a).



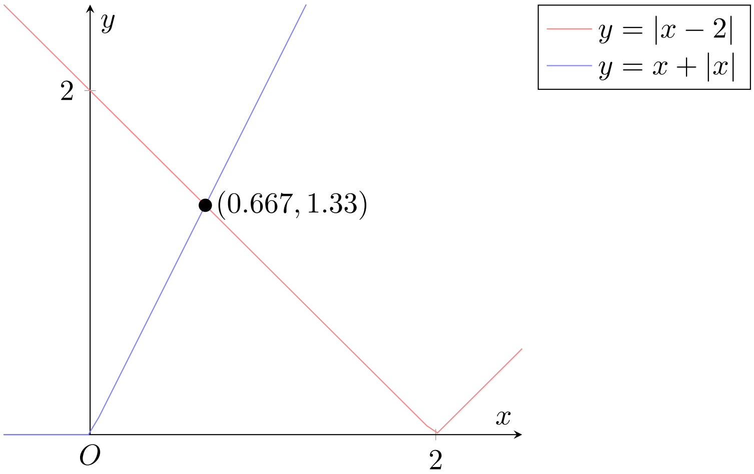
Thus, $x < -1.14$ or $x > -0.549$.

Part (b).



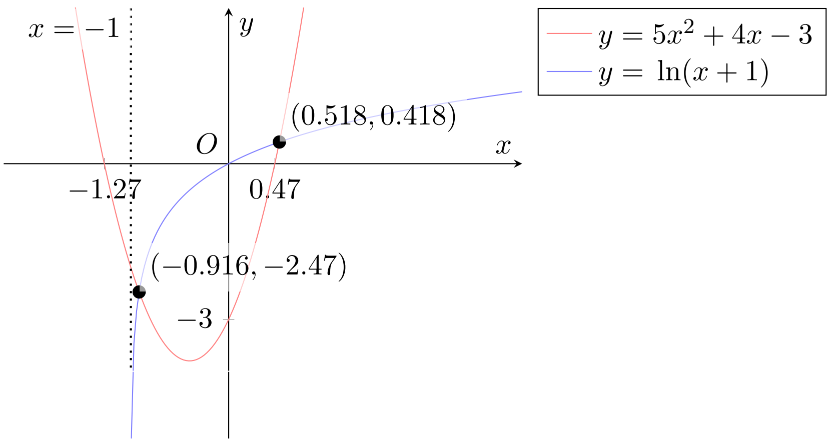
Thus, $x \leq -1.6$ or $x \geq 6$.

Part (c).



Thus, $x \leq 0.667$.

Part (d).



Thus, $-1 < x < -0.916$ or $x > 0.518$.

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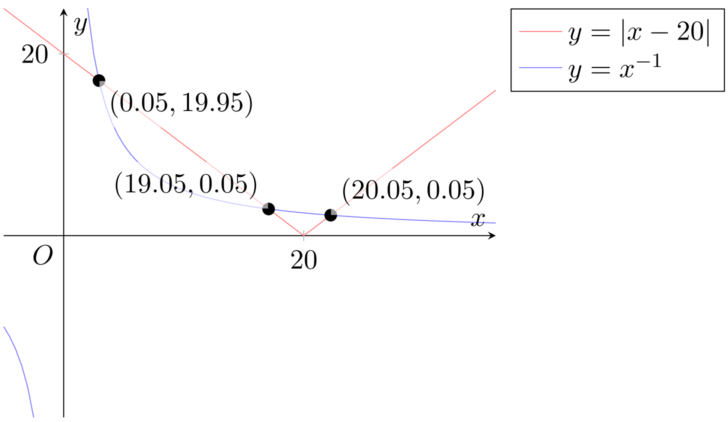
Problem 10. Sketch the graphs of $y = |x - 20|$ and $y = \frac{1}{x}$ on the same diagram. Hence or otherwise, solve the inequality $|x - 20| < \frac{1}{x}$, leaving your answers correct to 2 decimal places.

Solution.



Thus, $0 < x < 0.05$ or $19.95 < x < 20.05$.

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Problem 11. Solve the inequality $\frac{x-9}{x^2-9} \leq 1$. Hence, solve the inequalities

(a) $\frac{|x|-9}{x^2-9} \leq 1$

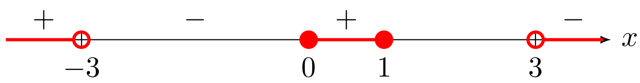
(b) $\frac{x+9}{x^2-9} \geq -1$

Solution. Note that $x^2 - 9 \neq 0 \implies x \neq \pm 3$.

$$\frac{x-9}{x^2-9} \leq 1 \implies (x-9)\left(x^2-9\right) \leq \left(x^2-9\right)^2.$$

Expanding and factoring, we get

$$x^4 - x^3 - 9x^2 + 9x = x(x+3)(x-1)(x-3) \geq 0.$$



Thus, $x < -3$ or $0 \leq x \leq 1$ or $x > 3$.

Part (a). Consider the substitution $x \mapsto |x|$. Then

$$|x| < -3 \text{ or } 0 \leq |x| \leq 1 \text{ or } |x| > 3.$$

This immediately gives us $x < -3$ or $-1 \leq x \leq 1$ or $x > 3$.

Part (b). Consider the substitution $x \mapsto -x$. Then

$$-x < -3 \text{ or } 0 \leq -x \leq 1 \text{ or } -x > 3.$$

This immediately gives us $x < -3$ or $-1 \leq x \leq 0$ or $x > 3$.

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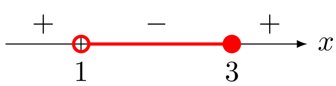
Problem 12. Solve the inequality $\frac{x-5}{1-x} \geq 1$. Hence, solve $0 < \frac{1-\ln x}{\ln x - 5} \leq 1$.

Solution. Note that $x \neq 1$.

$$\frac{x-5}{1-x} \geq 1 \implies (x-5)(1-x) \geq (1-x)^2 \implies 2x^2 - 8x + 6 \leq 0 \implies 2(x-1)(x-3) \leq 0.$$



Thus, $1 < x \leq 3$.

Consider the substitution $x \mapsto \ln x$. Taking reciprocals, we have our desired inequality $0 < \frac{1-\ln x}{\ln x - 5} \leq 1$. Hence,

$$1 < \ln x \leq 3 \implies e < x \leq e^3.$$

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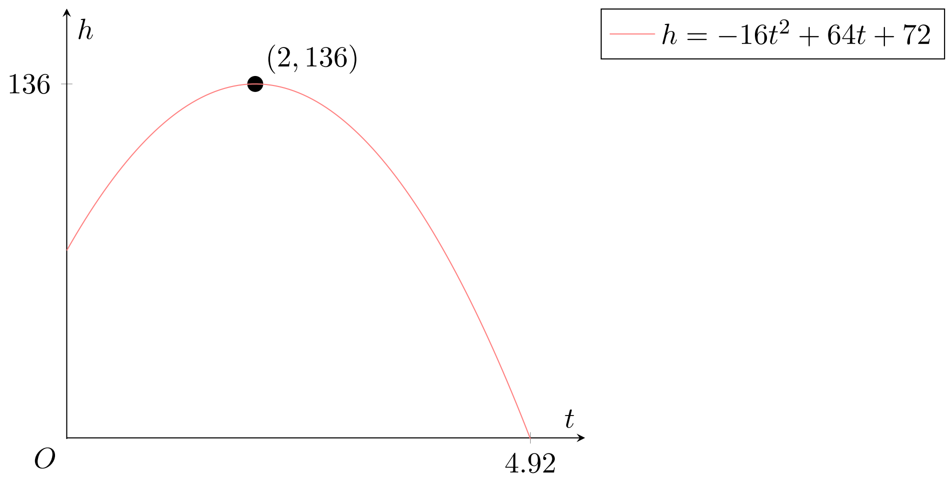
Problem 13. A small rocket is launched from a height of 72 m from the ground. The height of the rocket in metres, h, is represented by the equation $h = -16t^2 + 64t + 72$, where t is the time in seconds after the launch.

(a) Sketch the graph of h against t.

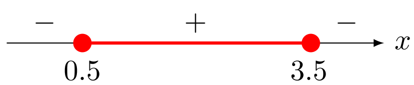
(b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

Solution.

Part (a).



Part (b). Note that $-16t^2 + 64t + 72 \geq 100 \implies -4(2t-1)(2t-7) \geq 0$.



Thus, the rocket will remain at or above 100 m from the ground for 3 seconds.

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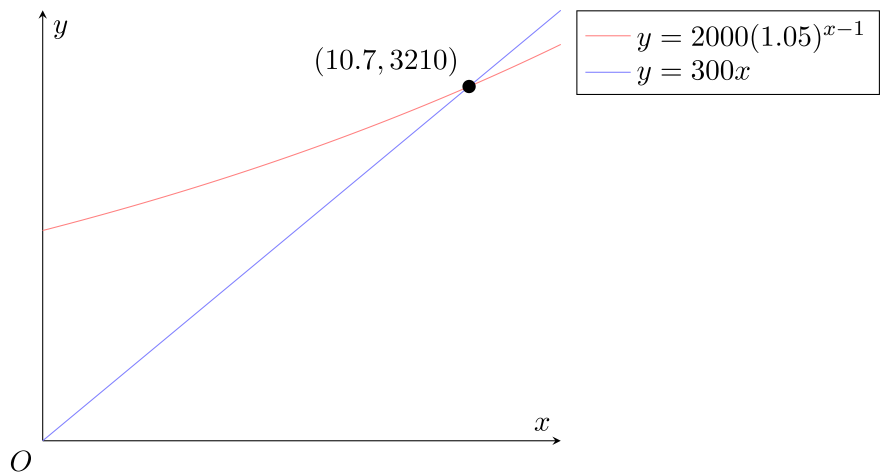
Problem 14. Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of $2000(1.05)^{n-1}$ dollars in the nth quarter, where n is a positive integer. She also gives a regular donation of \$$300n$ in the nth quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

Solution. Consider the curves $y = 2000(1.05)^{x-1}$ and $y = 300x$.



Hence, Xinxin will stop donating in the 11th quarter.

Assignment A1

Problem 1. A traveller just returned from Germany, France and Spain. The amount (in dollars) that he spent each day on housing, food and incidental expenses in each country are shown in the table below.

Country	Housing	Food	Incidental Expenses
Germany	28	30	14
France	23	25	8
Spain	19	22	12

The traveller's records of the trip indicate a total of \$191 spent for housing, \$430 for food and \$180 for incidental expenses. Calculate the number of days the traveller spent in each country.

He did his account again and the amount spent on food is \$337. Is this record correct? Why?

Solution. Let g, f and s represent the number of days the traveller spent in Germany, France and Spain respectively. From the table, we obtain the following system of equations:

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 430 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution $g = 4$, $f = 8$ and $s = 5$. The traveller thus spent 4 days in Germany, 8 days in France and 5 days in Spain.

Consider the scenario where the amount spent on food is \$337.

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 337 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution $g = 66$, $f = -27$ and $s = -44$. The record is hence incorrect as f and s must be positive.

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Problem 2.

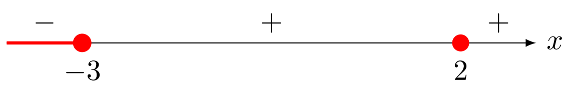
(a) Solve algebraically $x^2 - 9 \geq (x+3)\left(x^2 - 3x + 1\right)$.

(b) Solve algebraically $\frac{7-2x}{3-x^2} \leq 1$.

Solution.

Part (a).

$$\begin{aligned} & x^2 - 9 \geq (x+3)\left(x^2 - 3x + 1\right) \\ \Longrightarrow \quad & (x+3)(x-3) \geq (x+3)\left(x^2 - 3x + 1\right) \\ \Longrightarrow \quad & (x+3)\left(x^2 - 4x + 4\right) \leq 0 \\ \Longrightarrow \quad & (x+3)(x-2)^2 \leq 0 \end{aligned}$$



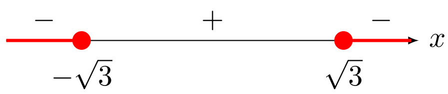
Thus, $x \leq -3$ or $x = 2$.

Part (b). Note that $3 - x^2 \neq 0 \implies x \neq \pm\sqrt{3}$.

$$\begin{aligned} & \frac{7-2x}{3-x^2} \leq 1 \\ \implies & \frac{7-2x}{3-x^2} - \frac{3-x^2}{3-x^2} \leq 0 \\ \implies & \frac{x^2-2x+4}{3-x^2} \leq 0 \end{aligned}$$

Observe that $x^2 - 2x + 4 = (x-1)^2 + 3 > 0$. Dividing through by $x^2 - 2x + 4$, we obtain

$$\begin{aligned} & \frac{1}{3-x^2} \leq 0 \\ \implies & 3 - x^2 \leq 0 \end{aligned}$$



Thus, $x < -\sqrt{3}$ or $x > \sqrt{3}$.

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Problem 3.

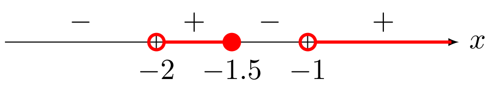
(a) Without using a calculator, solve the inequality $\frac{3x+4}{x^2+3x+2} \geq \frac{1}{x+2}$.

(b) Hence, deduce the set of values of x that satisfies $\frac{3|x|+4}{x^2+3|x|+2} \geq \frac{1}{|x|+2}$.

Solution.

Part (a). Note that $x^2 + 3x + 2 \neq 0$ and $x + 2 \neq 0$, whence $x \neq -1, -2$.

$$\begin{aligned} & \frac{3x+4}{x^2+3x+2} \geq \frac{1}{x+2} \\ \implies & \frac{3x+4}{(x+2)(x+1)} \geq \frac{1}{x+2} \\ \implies & (3x+4)(x+2)(x+1) \geq (x+2)(x+1)^2 \\ \implies & (x+2)(x+1)(2x+3) \geq 0 \end{aligned}$$



Thus, $-2 < x \leq -\frac{3}{2}$ or $x > -1$.

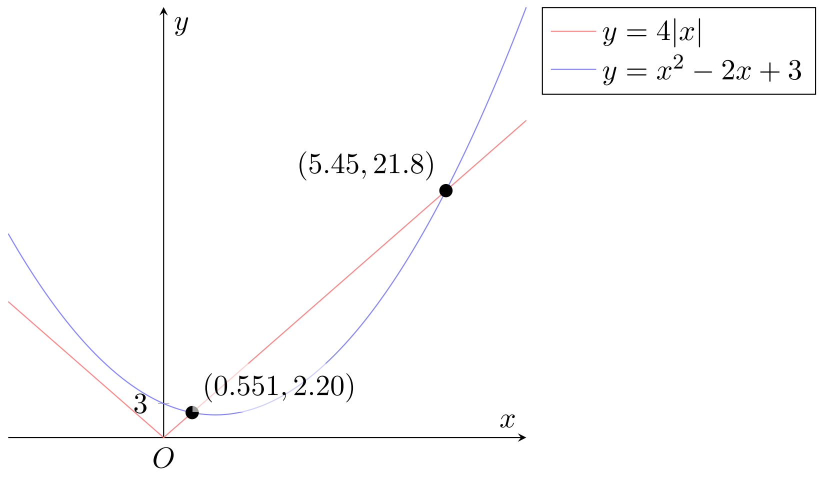
Part (b). Observe that $|x|^2 = x^2$. Hence, with the map $x \mapsto |x|$, we obtain

$$-2 < |x| \leq -\frac{3}{2} \text{ or } |x| > -1.$$

Since $|x| \geq 0$, we have that $|x| > -1$ is satisfied for all real x. Hence, the solution set is $\mathbb{R}$.

Problem 4. On the same diagram, sketch the graphs of $y = 4|x|$ and $y = x^2 - 2x + 3$. Hence or otherwise, solve the inequality $4|x| \geq x^2 - 2x + 3$.

Solution.



From the graph, we see that $0.551 \leq x \leq 5.45$.

A2. Numerical Methods of Finding Roots

Tutorial A2

Problem 1. Without using a graphing calculator, show that the equation $x^3+2x^2-2=0$ has exactly one positive root.

This root is denoted by α and is to be found using two different iterative methods, starting with the same initial approximation in each case.

(a) Show that α is a root of the equation $x = \sqrt{\frac{2}{x+2}}$, and use the iterative formula $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$, with $x_1 = 1$, to find α correct to 2 significant figures.

(b) Use the Newton-Raphson method, with $x_1 = 1$, to find α correct to 3 significant figures.

Solution. Let $f(x) = x^3+2x^2-2$. Observe that for all $x > 0$, we have $f'(x) = 3x^2+4x > 0$. Hence, $f(x)$ is strictly increasing on $(0, \infty)$. Since $f(0)f(1) = (-2)(1) < 0$, it follows that $f(x)$ has exactly one positive root.

Part (a). We know $f(\alpha) = 0$. Hence,

$$\alpha^3 + 2\alpha^2 - 2 = 0 \implies \alpha^2(\alpha+2) = 2 \implies \alpha^2 = \frac{2}{\alpha+2} \implies \alpha = \sqrt{\frac{2}{\alpha+2}}.$$

Note that we reject the negative branch since $\alpha > 0$. We hence see that α is a root of the equation $x = \sqrt{\frac{2}{x+2}}$. Using the iterative formula $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$ with $x_1 = 1$, we have

n	x_n
1	1
2	0.81650
3	0.84268
4	0.83879

Hence, $\alpha = 0.84$ (2 s.f.).

Part (b). Using the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$) with $x_1 = 1$, we have

n	x_n
1	1
2	0.857143
3	0.839545
4	0.839287
5	0.839287

Hence, $\alpha = 0.839$ (3 s.f.).

Problem 2.

(a) Show that the tangent at the point $(e, 1)$ to the graph $y = \ln x$ passes through the origin, and deduce that the line $y = mx$ cuts the graph $y = \ln x$ in two points provided that $0 < m < \frac{1}{e}$.

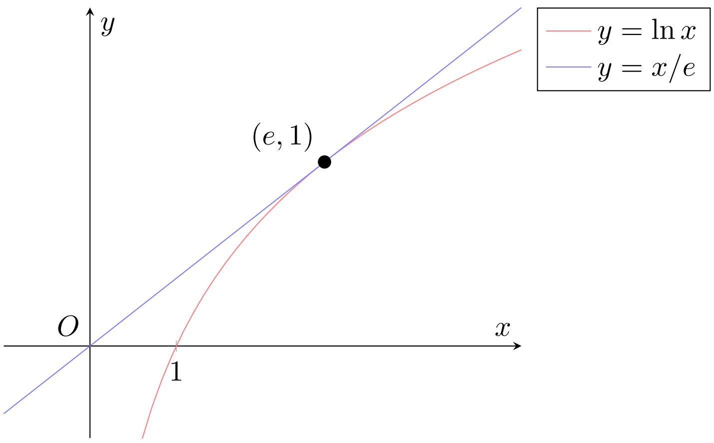
(b) For each root of the equation $\ln x = \frac{1}{3}x$, find an integer n such that the interval $n < x < n+1$ contains the root. Using linear interpolation, based on $x = n$ and $x = n+1$, find a first approximation to the smaller root, giving your answer to 1 decimal place. Using your first approximation, obtain, by the Newton-Raphson method, a second approximation to the smaller root, giving your answer to 2 decimal places.

Solution.

Part (a). Note that the derivative of $y = \ln x$ at $x = e$ is $\frac{1}{e}$. Using the point slope formula, we see that the equation of the tangent at the point $(e, 1)$ is given by

$$y - 1 = \frac{x - e}{e} \implies y = \frac{x}{e}.$$

Since $x = 0$, $y = 0$ is clearly a solution, the tangent passes through the origin. From the graph below, it is clear that for $y = mx$ to intersect $y = \ln x$ twice, we must have $0 < m < \frac{1}{e}$.



Part (b). Consider $f(x) = \frac{1}{3}x - \ln x$. Let α and β be the smaller and larger root to $f(x) = 0$ respectively. Observe that $f(1)f(2) = (1)(-0.03) < 0$ and $f(4)f(5) = (-0.05)(0.06) < 0$. Thus, for the smaller root α, $n = 1$, while for the larger root β, $n = 4$.

Let x_1 be the first approximation to α. Using linear interpolation, we have

$$x_1 = \frac{1f(2) - 2f(1)}{f(2) - f(1)} = 1.9 \text{ (1 d.p.)}$$

Note that $f'(x) = \frac{1}{3} - \frac{1}{x}$. Using the Newton-Raphson method ($x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$), we have

n	x_n
1	1.9
2	1.85585
3	1.85718

Hence, $\alpha = 1.86$ (2 d.p.).

Problem 3. Find the exact coordinates of the turning points on the graph of $y = f(x)$ where $f(x) = x^3 - x^2 - x - 1$. Deduce that the equation $f(x) = 0$ has only one real root α, and prove that α lies between 1 and 2. Use the Newton-Raphson method applied to the equation $f(x) = 0$ to find a second approximation x_2 to α, taking x_1, the first approximation, to be 2. With reference to a graph of $y = f(x)$, explain why all further approximations to α by this process are always larger than α.

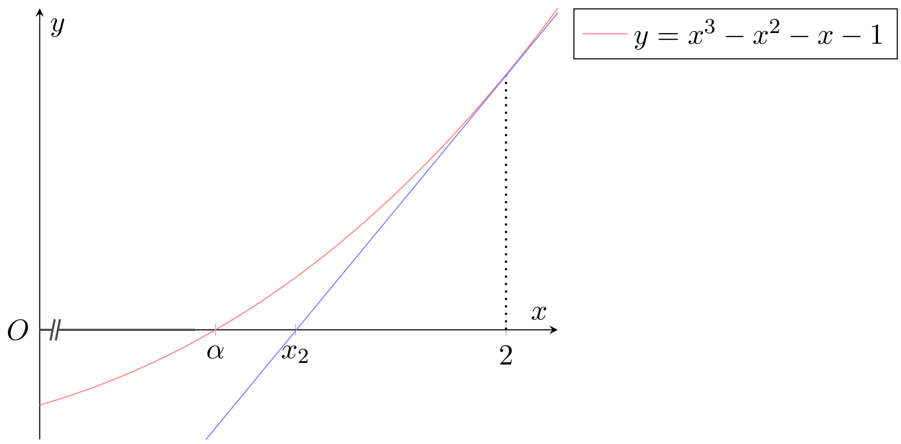
Solution. For turning points, $f'(x) = 0$.

$$f'(x) = 0 \implies 3x^2 - 2x - 1 = 0 \implies (3x+1)(x-1) = 0.$$

Hence, $x = -\frac{1}{3}$ or $x = 1$. When $x = -\frac{1}{3}$, we have $y = -0.815$, giving the coordinate $\left(-\frac{1}{3}, -0.815\right)$. When $x = 1$, we have $y = -2$, giving the coordinate $(1, -2)$.

Observe that $f(x)$ is strictly increasing for all $x > 1$. Further, since both turning points have a negative y-coordinate, it follows that $y < 0$ for all $x \leq 1$. Since $f(1)f(2) = (-2)(1) < 0$, the equation $f(x) = 0$ has only one real root.

Using the Newton-Raphson method with $x_1 = 2$, we have $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{13}{7}$.



Since x_2 lies on the right of α, the Newton-Raphson method gives an over-estimation given an initial approximation of 2. Thus, all further approximations to α will also be larger than α.

* * * * *

Problem 4. A curve C has equation $y = x^5 + 50x$. Find the least value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence give a reason why the equation $x^5 + 50x = 10^5$ has exactly one real root. Use the Newton-Raphson method, with a suitable first approximation, to find, correct to 4 decimal places, the root of the equation $x^5 + 50x = 10^5$. You should demonstrate that your answer has the required accuracy.

Solution. Since $y = x^5 + 50x$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4 + 50$. Since $x^4 \geq 0$ for all real x, the minimum value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is 50.

Let $f(x) = x^5 + 50x$. Since $\min \frac{\mathrm{d}f}{\mathrm{d}x} = 50 > 0$, it follows that $f(x)$ is strictly increasing. Hence, $f(x)$ will intersect only once with the line $y = 10^5$, whence the equation $x^5 + 50x = 10^5$ has exactly one real root.

Observe that $f(9)f(10) = (-40901)(50) < 0$. Thus, there must be a root in the interval $(9, 10)$. We now use the Newton-Raphson method $\left(x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\right)$ with $x_1 = 9$ as the first approximation.

n	x_n
1	9
2	10.2178921
3	10.0017491
4	9.9901221
5	9.9899912
6	9.9899900

Thus, the root is 9.9900 (4 d.p.).

Observe that $f(9.98995)f(9.99005) = (-2.00)(3.00) < 0$. Hence, the root lies in the interval $(9.98995, 9.99005)$ whence the calculated root has the required accuracy.

* * * * *

Problem 5.

(a) A function f is such that $f(4) = 1.158$ and $f(5) = -3.381$, correct to 3 decimal places in each case. Assuming that there is a value of x between 4 and 5 for which $f(x) = 0$, use linear interpolation to estimate this value.

For the case when $f(x) = \tan x$, and x is measured in radians, the value of $f(4)$ and $f(5)$ are as given above. Explain, with the aid of a sketch, why linear interpolation using these values does not give an approximation to a solution of the equation $\tan x = 0$.

(b) Show, by means of a graphical argument or otherwise, that the equation $\ln(x-1) = -2x$ has exactly one real root, and show that this root lies between 1 and 2.

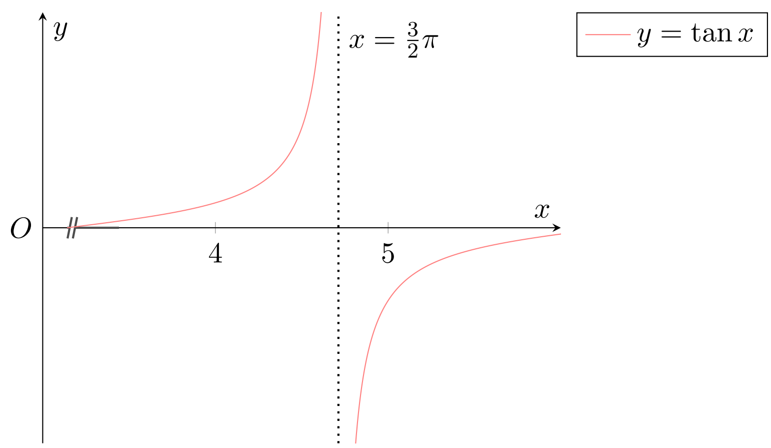
The equation may be written in the form $\ln(x-1) + 2x = 0$. Show that neither $x = 1$ nor $x = 2$ is a suitable initial value for the Newton-Raphson method in this case.

The equation may also be written in the form $x - 1 - e^{-2x} = 0$. For this form, use two applications of the Newton-Raphson method, starting with $x = 1$, to obtain an approximation to the root, giving 3 decimal places in your answer.

Solution.

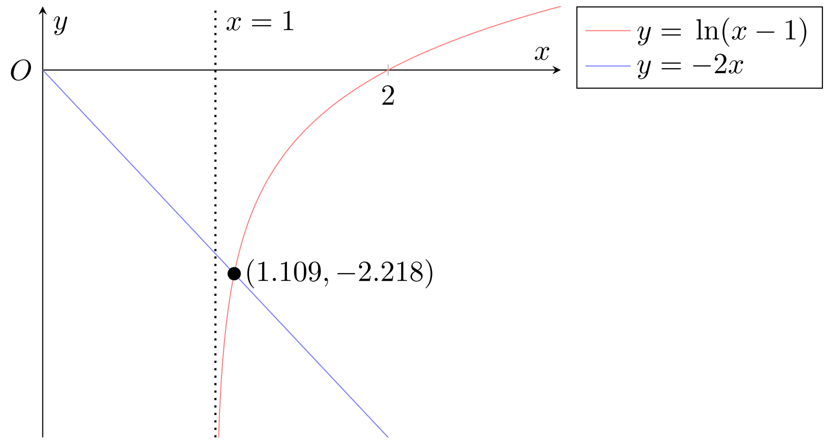
Part (a). Let the root of $f(x) = 0$ be α. Using linear interpolation on the interval $[4, 5]$, we have

$$\alpha = \frac{4f(5) - 5f(4)}{f(5) - f(4)} = 4.255 \text{ (3 d.p.)}.$$



Since $\tan x$ has a vertical asymptote at $x = \frac{3}{2}\pi$, it is not continuous on $[4, 5]$. Thus, linear interpolation diverges when applied to the equation $\tan x = 0$.

Part (b).



Since there is only one intersection between the graphs $y = \ln(x-1)$ and $y = -2x$, there is only one real root to the equation $\ln(x-1) = -2x$. Furthermore, since $y = -2x$ is negative for all $x > 0$ and $y = \ln(x-1)$ is negative only when $1 < x < 2$, it follows that the root must lie between 1 and 2.

Let $f(x) = \ln(x-1) + 2x$. Then $f'(x) = \frac{1}{x-1} + 2$. Note that the Newton-Raphson method is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Since $f'(1)$ is undefined, an initial approximation of $x_1 = 1$ cannot be used for the Newton-Raphson method, which requires a division by $f'(1)$.

Using the Newton-Raphson method with the initial approximation $x_2 = 2$, we see that $x_2 = 1$. Once again, because $f'(1)$ is undefined, $x_1 = 2$ is also not a suitable initial value.

Let $g(x) = x - 1 - e^{-2x}$. Then $g'(x) = 1 + 2e^{-2x}$. Using the Newton-Raphson method with the initial approximation $x_1 = 1$, we have

n	x_n
1	1
2	1.106507
3	1.108857

Hence, $x = 1.109$ (3 d.p.).

* * * * *

Problem 6. The equation $x = 3\ln x$ has two roots α and β, where $1 < \alpha < 2$ and $4 < \beta < 5$. Using the iterative formula $x_{n+1} = F(x_n)$, where $F(x) = 3\ln x$, and starting with $x_0 = 4.5$, find the value of β correct to 3 significant figures. Find a suitable $F(x)$ for computing α.

Solution. Using the iterative formula $x_{n+1} = F(x_n)$, we have

n	x_n	n	x_n
0	4.5	5	4.53175
1	4.51223	6	4.53333
2	4.52038	7	4.53437
3	4.52579	8	4.53506
4	4.52937	9	4.53551

Hence, $\beta = 4.54$ (3 s.f.).

Note that $x = 3\ln x \implies x = e^{x/3}$. Observe that $\frac{\mathrm{d}}{\mathrm{d}x}(e^{x/3}) = \frac{1}{3}e^{x/3}$, which is between -1 and 1 for all $1 < x < 2$. Thus, the iterative formula $x_{n+1} = F(x_n)$ will converge, whence $F(x) = e^{x/3}$ is suitable for computing α.

* * * * *

Problem 7. Show that the cubic equation $x^3 + 3x - 15 = 0$ has only one real root. This root is near $x = 2$. The cubic equation can be written in any one of the forms below:

(a) $x = \frac{1}{3}(15 - x^3)$

(b) $x = \frac{15}{x^2+3}$

(c) $x = (15 - 3x)^{1/3}$

Determine which of these forms would be suitable for the use of the iterative formula $x_{r+1} = F(x_r)$, where $r = 1, 2, 3, \ldots$.

Hence, find the root correct to 3 decimal places.

Solution. Let $f(x) = x^3 + 3x - 15$. Then $f'(x) = 3x^2 + 3 > 0$ for all real x. Hence, f is strictly increasing. Since f is continuous, $f(x) = 0$ has only one real root.

Part (a). Let $g_1(x) = \frac{1}{3}(15 - x^3)$. Then $g_1'(x) = -x^2$. For values of x near 2, $|g_1'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_1(x_n)$ will diverge and $g_1(x)$ is unsuitable.

Part (b). Let $g_2(x) = \frac{15}{x^2+3}$. Then $g_2'(x) = \frac{-30x}{(x^2+3)^2}$. For values of x near 2, $|g_2'(x)| > 1$. Hence, the iterative formula $x_{n+1} = g_2(x_n)$ will diverge and $g_2(x)$ is unsuitable.

Part (c). Let $g_3(x) = (15 - 3x)^{1/3}$. Then $g_3'(x) = -(15 - 3x)^{-2/3}$. For values of x near 2, $|g_3'(x)| < 1$. Hence, the iterative formula $x_{n+1} = g_3(x_n)$ will converge and $g_3(x)$ is suitable.

Using the iterative formula $x_{r+1} = g_3(x_r)$, we get

r	x_r
1	2
2	2.080084
3	2.061408
4	2.065793
5	2.064765

Hence, $x = 2.065$ (3 d.p.).

* * * * *

Problem 8. The equation of a curve is $y = f(x)$. The curve passes through the points $(a, f(a))$ and $(b, f(b))$, where $0 < a < b$, $f(a) > 0$ and $f(b) < 0$. The equation $f(x) = 0$ has precisely one root α such that $a < \alpha < b$. Derive an expression, in terms of a, b, $f(a)$ and $f(b)$, for the estimated value of α based on linear interpolation.

Let $f(x) = 3e^{-x} - x$. Show that $f(x) = 0$ has a root α such that $1 < \alpha < 2$, and that for all x, $f'(x) < 0$ and $f''(x) > 0$. Obtain an estimate of α using linear interpolation to 2 decimal places, and explain by means of a sketch whether the value obtained is an over-estimate or an under-estimate.

Use one application of the Newton-Raphson method to obtain a better estimate of α, giving your answer to 2 decimal places.

Solution. Using the point-slope formula, the equation of the line that passes through both $(a, f(a))$ and $(b, f(b))$ is

$$y - f(a) = \frac{f(a) - f(b)}{a - b}(x - a).$$

Note that $(\alpha, 0)$ is approximately the solution to the above equation. Thus,

$$0 - f(a) \approx \frac{f(a) - f(b)}{a - b}(\alpha - a) \implies \alpha \approx \frac{bf(a) - af(b)}{f(a) - f(b)}.$$

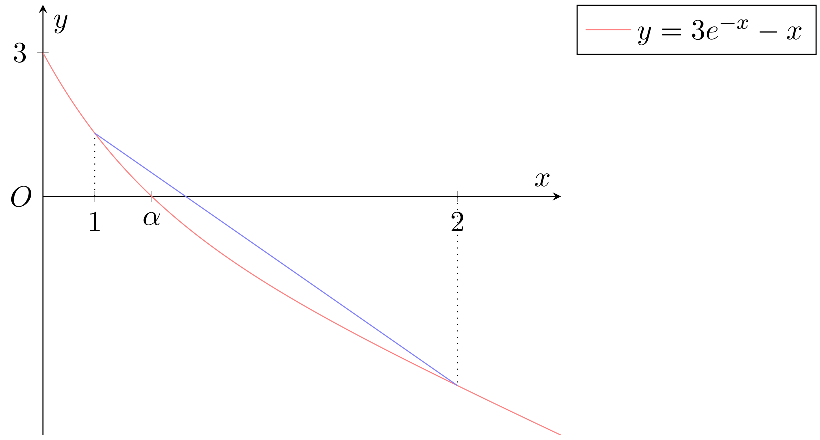
Since $f(x)$ is continuous, and $f(1)f(2) = (0.10)(-1.6) < 0$, there exists a root $\alpha \in (1, 2)$.

Note that $f'(x) = -3e^{-x} - 1$ and $f''(x) = 3e^{-x}$. Since $e^{-x} > 0$ for all x, we have that $f'(x) < 0$ and $f''(x) > 0$ for all x.

Using linear interpolation on the interval $(1, 2)$, we have

$$\alpha = \frac{2 \cdot f(1) - 1 \cdot f(2)}{f(1) - f(2)} = 1.06 \text{ (2 d.p.)}.$$

Since $f'(x) < 0$ and $f''(x) > 0$, we know that $f(x)$ is strictly decreasing and is concave upwards. $f(x)$ hence has the following shape:



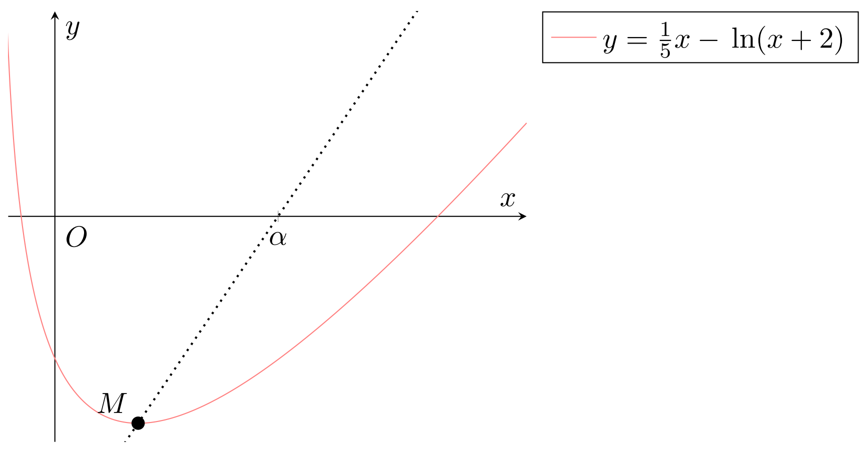
From the graph, we see that the value obtained is an over-estimate.

Using the Newton-Raphson method with the initial approximation $x_1 = 1.06$, we get

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.05 \text{ (2 d.p.)}.$$

* * * * *

Problem 9.



The diagram shows a sketch of the graph $y = \frac{1}{3}x - \ln(x + 2)$. Find the x-coordinate of the minimum point M on the graph, and verify that y is positive when $x = 20$.

Show that the gradient of the curve is always less than $\frac{1}{5}$. Hence, by considering the line through M having gradient $\frac{1}{5}$, show that the positive root of the equation $\frac{1}{3}x - \ln(x+2) = 0$ is greater than 8.

Use linear interpolation, once only, on the interval $[8, 20]$, to find an approximate value a for this positive root, giving your answer to 1 decimal place.

Using a as an initial value, carry out one application of the Newton-Raphson method to obtain another approximation to the positive root, giving your answer to 2 decimal places.

Solution. For stationary points, $y' = 0$.

$$y' = 0 \implies \frac{1}{5} - \frac{1}{x+2} \implies x = 3.$$

By the second derivative test, we see that $y''(x) = \frac{1}{(x+2)^2} > 0$. Hence, the x-coordinate of M is 3. Substituting $x = 20$ into the equation of the curve gives $y = 4 - \ln 22 = 0.909 > 0$.

We know that $y' = \frac{1}{5} - \frac{1}{x+2}$, hence $y' < \frac{1}{5}$ for all $x > -2$. Since the domain of the curve is $x > -2$, y' is always less than $\frac{1}{5}$.

Let $(\alpha, 0)$ be the coordinates of the root of the line through M having gradient $\frac{1}{5}$. We know that the coordinates of M are $\left(3, \frac{3}{5} - \ln 5\right)$. Taking the gradient of the line segment joining M and $(\alpha, 0)$, we get

$$\frac{\left(\frac{3}{5} - \ln 5\right) - 0}{3 - \alpha} = \frac{1}{5} \implies \alpha = 5 \ln 5 = 8.05 > 8.$$

Since the gradient of the curve is always less than $\frac{1}{5}$, α represents the lowest possible value of the positive root of the curve. Hence, the positive root of the equation $\frac{1}{5}x - \ln(x+2) = 0$ is greater than 8.

Let $f(x) = \frac{1}{5}x - \ln(x+2)$. Using linear interpolation on the interval $[8, 20]$, we have

$$\alpha = \frac{8f(20) - 20f(8)}{f(20) - f(8)} = 13.2 \text{ (1 d.p.)}.$$

Using the Newton-Raphson method with the initial approximation $x_1 = 13.2$, we have

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 13.81 \text{ (2 d.p.)}.$$

* * * * *

Problem 10.

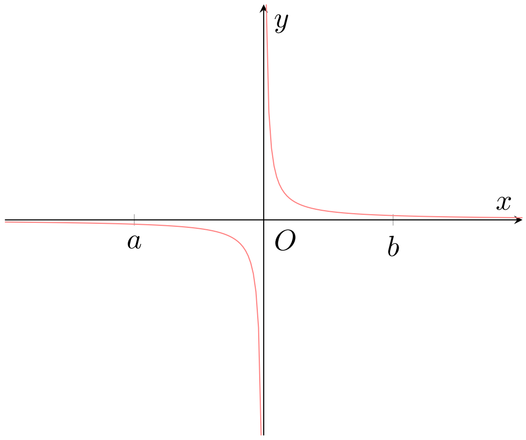
(a) The function f is such that $f(a)f(b) < 0$, where $a < b$. A student concludes that the equation $f(x) = 0$ has exactly one root in the interval (a, b). Draw sketches to illustrate two distinct ways in which the student could be wrong.

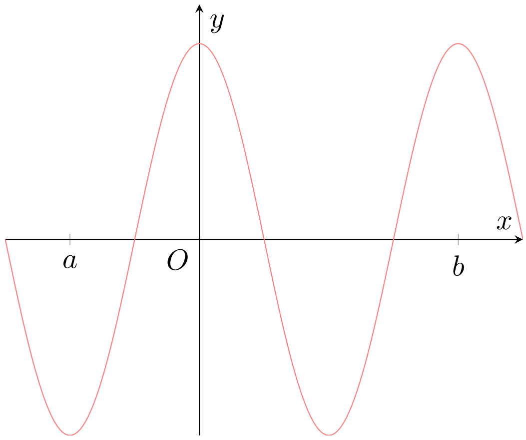
(b) The equation $\sec^2 x - e^2 = 0$ has a root α in the interval $[1.5, 2.5]$. A student uses linear interpolation once on this interval to find an approximation to α. Find the approximation to α given by this method and comment on the suitability of the method in this case.

(c) The equation $\sec^2 x - e^x = 0$ also has a root β in the interval $(0.1, 0.9)$. Use the Newton-Raphson method, with $f(x) = \sec^2 x - e^x$ and initial approximation 0.5, to find a sequence of approximations $\{x_1, x_2, x_3, \ldots\}$ to β. Describe what is happening to x_n for large n, and use a graph of the function to explain why the sequence is not converging to β.

Solution.

Part (a).





Part (b). Let $f(x) = \sec^2 x - e^x$. Using linear interpolation on the interval $[1.5, 2.5]$,

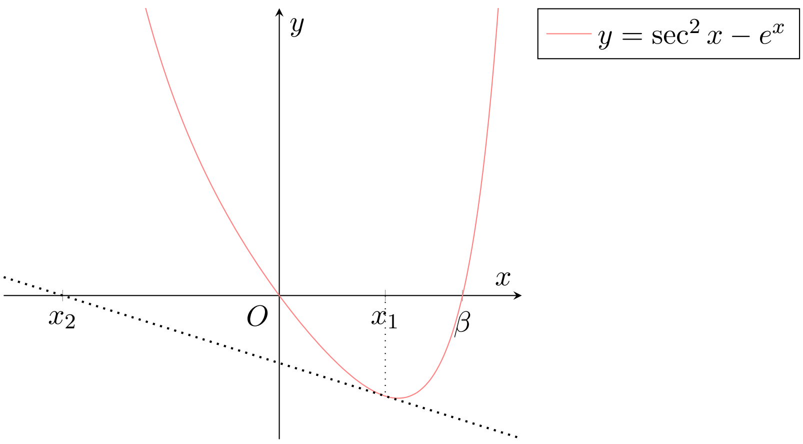
$$a = \frac{1.5f(2.5) - 2.5f(1.5)}{f(2.5) - f(1.5)} = 1.06 \text{ (2 d.p.)}.$$

$\sec^2 x$ is not continuous on the interval $[1.5, 2.5]$ due to the presence of an asymptote at $x = \frac{\pi}{2}$. Hence, linear interpolation is not suitable in this case.

Part (c). We know $f'(x) = 2\sec^2 x \tan x - e^x$. Using the Newton-Raphson method with the initial approximation $x_1 = 0.5$,

r	x_r
1	0.5
2	-1.02272
3	-0.75526
4	-0.40306
5	-0.09667
6	-0.00466
7	-0.00000

As $n \to \infty$, $x_n \to 0^-$.



From the above graph, we see that the initial approximation of $x_1 = 0.5$ is past the turning point. Hence, all subsequent approximations will converge to the root at 0 instead of the root at β. Thus, the sequence does not converge to β.

Problem 11. The function f is given by $f(x) = \sqrt{1-x^2} + \cos x - 1$ for $0 \le x \le 1$. It is known, from graphical work, that the equation $f(x) = 0$ has a single root $x = \alpha$.

(a) Express g(x) in terms of x, where $g(x) = x - \frac{f(x)}{f'(x)}$.

A student attempts to use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to calculate the value of α correct to 3 decimal places.

(b) (i) The student first uses an initial approximation to α of $x_1 = 0$. Explain why this will be unsuccessful in finding a value for α.

(ii) The student next uses an initial approximation to α of $x_1 = 1$. Explain why this will also be unsuccessful in finding a value for α.

(iii) The student then uses an initial approximate to α of $x_1 = 0.5$. Investigate what happens in this case.

(iv) By choosing a suitable value for x_1, use the Newton-Raphson method, based on the form $x_{n+1} = g(x_n)$, to determine α correct to 3 decimal places.

Solution.

Part (a). We know $f'(x) = \frac{-x}{\sqrt{1-x^2}} - \sin x$. Hence,

$$g(x) = x - \frac{\sqrt{1-x^2} + \cos x - 1}{\frac{-x}{\sqrt{1-x^2}} - \sin x}.$$

Part (b).

Part (b)(i). Observe that $f'(0) = 0$. Hence, $g(0)$ is undefined. Thus, starting with an initial approximation of $x_1 = 0$ will be unsuccessful in finding a value for α.

Part (b)(ii). Observe that $\sqrt{1-x^2}$ is 0 when $x = 1$. Hence, $f'(0)$ is undefined. Thus, $g(0)$ is also undefined. Hence, starting with an initial approximation of $x_1 = 1$ will also be unsuccessful in finding a value for α.

Part (b)(iii). When $x_1 = 0.5$, we have $x_2 = g(x_1) = 1.20$. Since $g(x)$ is only defined for $0 \le x \le 1$, $x_3 = g(x_2)$ is undefined. Hence, an initial approximation of $x_1 = 0.5$ will also be unsuccessful in finding a value for α.

Part (b)(iv). Using the Newton-Raphson method with $x_1 = 0.9$, we have

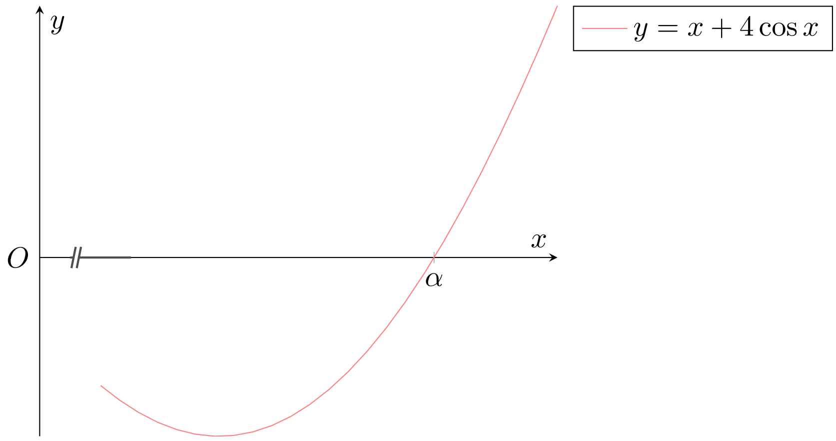
r	x_r
1	0.9
2	0.92019
3	0.91928
4	0.91928

Thus, $\alpha = 0.919$ (3 d.p.).

Assignment A2

Problem 1. By considering the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, or otherwise, show that the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

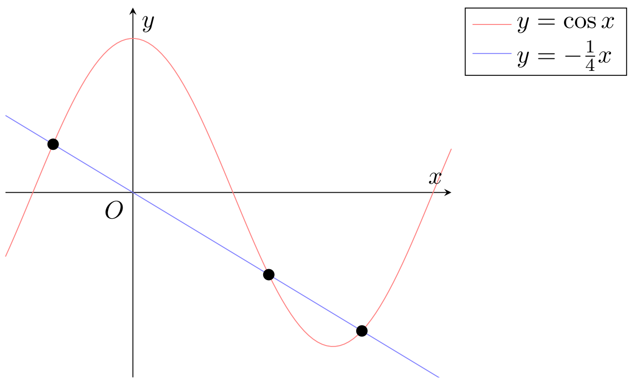
Use linear interpolation, once only, on the interval $[-1.5, 1]$ to find an approximation to the negative root of the equation $x + 4\cos x = 0$ correct to 2 decimal places.



The diagram shows part of the graph of $y = x + 4\cos x$ near the larger positive root, α, of the equation $x + 4\cos x = 0$. Explain why, when using the Newton-Raphson method to find α, an initial approximation which is smaller than α may not be satisfactory.

Use the Newton-Raphson method to find α correct to 2 significant figures. You should demonstrate that your answer has the required accuracy.

Solution.



Note that $x + 4\cos x = 0 \implies \cos x = -\frac{1}{4}x$. Plotting the graphs of $y = \cos x$ and $y = -\frac{1}{4}x$, we see that there is one negative root and two positive roots. Hence, the equation $x + 4\cos x = 0$ has one negative root and two positive roots.

Let $f(x) = x + 4\cos x$. Let β be the negative root of the equation $f(x) = 0$. Using linear interpolation on the interval $[-1.5. -1]$,

$$\beta = \frac{-1.5f(-1) - (-1)f(-1.5)}{f(-1) - f(1.5)} = -1.24 \text{ (2 d.p.)}.$$

There is a minimum at $x = m$ such that m is between the two positive roots. Hence, when using the Newton-Raphson method, an initial approximation which is smaller than m would result in subsequent approximations being further away from the desired root α. Hence, an initial approximation that is smaller than α may not be satisfactory.

We know from the above graph that $\alpha \in \left(\pi, \frac{3}{2}\pi\right)$. We hence pick $\frac{3}{2}\pi$ as our initial approximation. Using the Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with $x_1 = \frac{3}{2}\pi$, we have

r	x_r
1	$\frac{3}{2}\pi$
2	3.7699
3	3.6106
4	3.5955
5	3.5953

Since $f(3.55)f(3.65) = (-0.1)(0.2) < 0$, we have $\alpha \in (3.55, 3.65)$. Hence, $\alpha = 3.6$ (2 s.f.).

* * * * *

Problem 2. Find the coordinates of the stationary points on the graph $y = x^3 + x^2$. Sketch the graph and hence write down the set of values of the constant k for which the equation $x^3 + x^2 = k$ has three distinct real roots.

The positive root of the equation $x^3 + x^2 = 0.1$ is denoted by α.

(a) Find a first approximation to α by linear interpolation on the interval $0 \leq x \leq 1$.

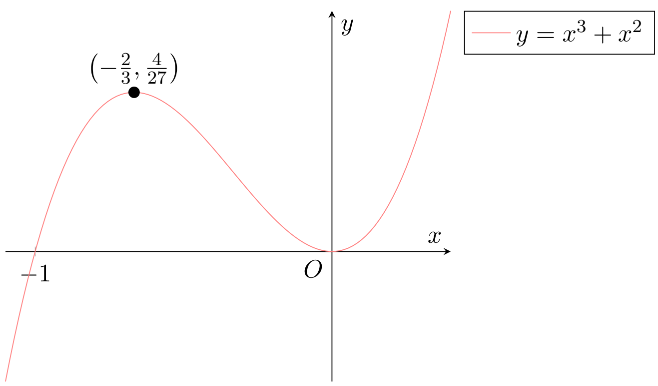
(b) With the aid of a suitable figure, indicate why, in this case, linear interpolation does not give a good approximation to α.

(c) Find an alternative first approximation to α by using the fact that if x is small then x^3 is negligible when compared to x^2.

Solution. For stationary points, $y' = 0$.

$$y' = 0 \implies 3x^2 + 2x = 0 \implies x(3x+2) = 0.$$

Hence, $x = 0$ or $x = -\frac{2}{3}$. When $x = 0$, $y = 0$. When $x = -\frac{2}{3}$, $y = \frac{4}{27}$. Thus, the coordinates of the stationary points of $y = x^3 + x^2$ are $(0, 0)$ and $\left(-\frac{2}{3}, \frac{4}{27}\right)$.

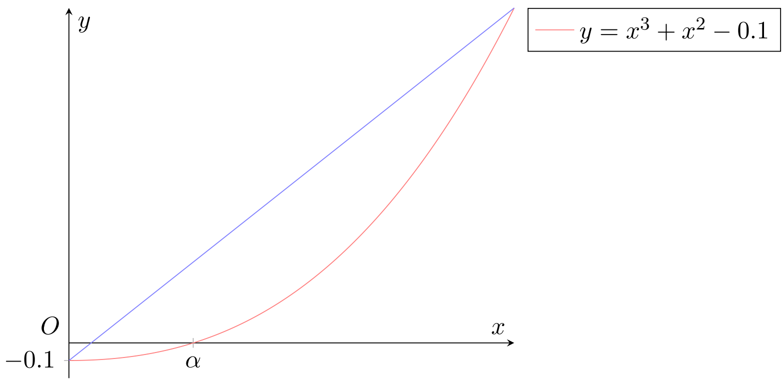


Therefore, $k \in \left(0, \frac{4}{27}\right)$. The solution set of k is thus $\left\{k \in \mathbb{R} : 0 < k < \frac{4}{27}\right\}$.

Part (a). Let $f(x) = x^2 + x^2 - 0.1$. Using linear interpolation on the interval $[0, 1]$,

$$\alpha = \frac{0f(1) - 1f(0)}{f(1) - f(0)} = \frac{1}{20}.$$

Part (b).



On the interval $[0, 1]$, the gradient of $y = x^3 + x^2 - 0.1$ changes considerably. Hence, linear interpolation gives an approximation much less than the actual value.

Part (c). For small x, x^3 is negligible when compared to x^2. Consider $g(x) = x^2 - 0.1$. Then the positive root of $g(x) = 0$ is approximately α. Hence, an alternative approximation to α is $\sqrt{0.1} = 0.316$ (3 s.f.).

* * * * *

Problem 3. The equation $2\cos x - x = 0$ has a root α in the interval $[1, 1.2]$. Iterations of the form $x_{n+1} = F(x_n)$ are based on each of the following rearrangements of the equation:

(a) $x = 2\cos x$

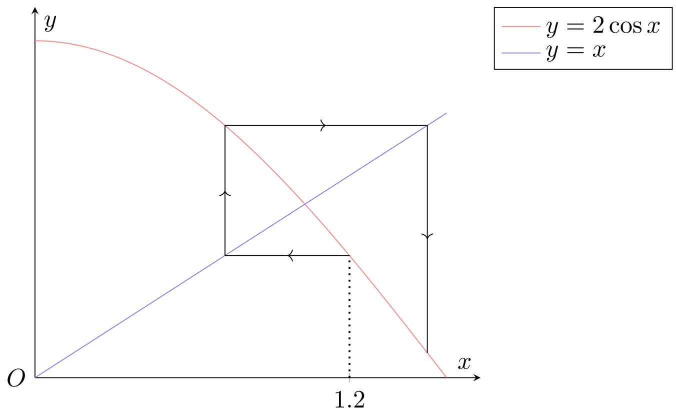
(b) $x = \cos x + \frac{1}{2}x$

(c) $x = \frac{2}{3}(\cos x + x)$

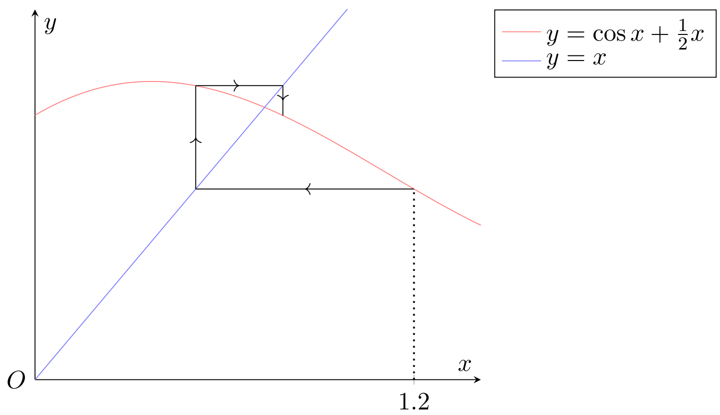
Determine which iteration will converge to α and illustrate your answer by a 'staircase' or 'cobweb' diagram. Use the most appropriate iteration with $x_1 = 1$, to find α to 4 significant figures. You should demonstrate that your answer has the required accuracy.

Solution.

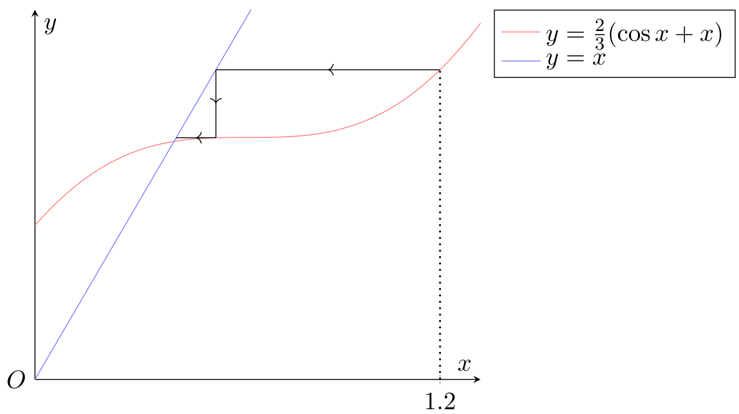
Part (a). Consider $f(x) = 2\cos x$. Then $f'(x) = -2\sin x$. Observe that $\sin x$ is increasing on $[1, 1.2]$. Since $\sin 1 > \frac{1}{2}$, $|f'(x)| > 1$ for all $x \in [1, 1.2]$. Thus, fixed-point iteration fails and will not converge to α.



Part (b). Consider $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} - \left(\sin x - \frac{1}{2}\right)$. Since $0 \leq \sin x \leq 1$ for $x \in \left[0, \frac{\pi}{2}\right]$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we know $-\frac{1}{2} \leq \sin x - \frac{1}{2} \leq \frac{1}{2}$ for $x \in [1, 1.2]$. Thus, $0 \leq |\sin x - \frac{1}{2}| \leq \frac{1}{2}$ for $x \in [1, 1.2]$. Hence, fixed-point iteration will work and converge to α.



Part (c). Consider $f(x) = \frac{2}{3}(\cos x + x)$. Then $f'(x) = \frac{2}{3}(-\sin x + 1)$. For fixed-point iteration to converge to α, we need $|f'(x)| < 1$ for x near α. It thus suffices to show that $|-\sin x + 1| < \frac{3}{2}$ for all $x \in [1, 1.2]$. Observe that $1 - \sin x$ is strictly decreasing and positive for $x \in \left[0, \frac{\pi}{2}\right]$. Since $1 - \sin 1 < \frac{3}{2}$, and $[1, 1.2] \subset \left[0, \frac{\pi}{2}\right]$, we have that $|-\sin x + 1| < \frac{3}{2}$ for all $x \in [1, 1.2]$. Thus, $|f'(x)| < 1$ for x near α. Hence, fixed-point iteration will work and converge to α.



For $x \in [1, 1.2]$, $|\frac{2}{3}(-\sin x + 1)| < |-\sin x + \frac{1}{2}| < 1$. Thus, $x_{n+1} = \frac{2}{3}(\cos x_n + x_n)$ is the most suitable iteration as it will converge to α the quickest. Using $F(x_{n+1}) = \frac{2}{3}(\cos x_n + x_n)$ with $x_1 = 1$,

r	x_r
1	1
2	1.02687
3	1.02958
4	1.02984
5	1.02986

Since $F(1.0295) > 1.0295$ and $F(1.0305) < 1.0305$, we have $\alpha \in (1.0295, 1.0305)$. Hence, $\alpha = 1.030$ (4 s.f.).

A3. Sequences and Series I

Tutorial A3

Problem 1. Determine the behaviour of the following sequences.

(a) $u_n = 3\left(\frac{1}{2}\right)^{n-1}$

(b) $v_n = 2 - n$

(c) $t_n = (-1)^n$

(d) $w_n = 4$

Solution.
Part (a). Decreasing, converges to 0.
Part (b). Decreasing, diverges.
Part (c). Alternating, diverges.
Part (d). Constant, converges to 4.

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Problem 2. Find the sum of all even numbers from 20 to 100 inclusive.

Solution. The even numbers from 20 to 100 inclusive form an AP with common difference 2, first term 20 and last term 100. Since we are adding a total of $\frac{100-20}{2} + 1 = 41$ terms, we get a sum of $41\left(\frac{20+100}{2}\right) = 2460$.

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Problem 3. A geometric series has first term 3, last term 384 and sum 765. Find the common ratio.

Solution. Let the nth term of the geometric series be ar^{n-1}, where $1 \leq n \leq k$. We hence have $3r^{k-1} = 384$, which gives $r^k = 128r$. Thus,

$$\frac{3(1-r^k)}{1-r} = 765 \implies \frac{3(1-128r)}{1-r} = 765 \implies r = 2.$$

* * * * *

Problem 4.

(a) Find the first four terms of the following sequence $u_{n+1} = \frac{u_n+1}{u_n+2}$, $u_1 = 0$, $n \geq 1$.

(b) Write down the recurrence relation between the terms of these sequences.

  (i) $-1, 2, -4, 8, -16, \ldots$

  (ii) $1, 3, 7, 15, 31, \ldots$

Solution.
Part (a). Using G.C., the first four terms of u_n are 0, $\frac{1}{2}$, $\frac{3}{5}$ and $\frac{8}{13}$.

Part (b).

Part (b)(i). $u_{n+1} = -2u_n$, $u_1 = -1$, $n \geq 1$.

Part (b)(ii). $u_{n+1} = 2u_n + 1$, $u_1 = 1$, $n \geq 1$.

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Problem 5. The sum of the first n terms of a series, S_n, is given by $S_n = 2n(n+5)$. Find the nth term and show that the terms are in arithmetic progression.

Solution. We have

$$u_n = S_n - S_{n-1} = 2n(n+5) - 2(n-1)(n+4) = 4n + 8.$$

Observe that $u_n - u_{n-1} = [4n+8] - [4(n-1)+8] = 8$ is a constant. Hence, u_n is in AP.

* * * * *

Problem 6. The sum of the first n terms, S_n, is given by

$$S_n = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}.$$

(a) Find an expression for the nth term of the series.

(b) Hence or otherwise, show that it is a geometric series.

(c) State the values of the first term and the common ratio.

(d) Give a reason why the sum of the series converges as n approaches infinity and write down its value.

Solution.

Part (a). Note that

$$u_n = S_n - S_{n-1} = \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}\right] - \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n}\right] = \left(\frac{1}{2}\right)^{n+1}.$$

Part (b). Since $\frac{u_{n+1}}{u_n} = \frac{(1/2)^{n+2}}{(1/2)^{n+1}} = \frac{1}{2}$ is constant, u_n is in GP.

Part (c). The first term is $\frac{1}{4}$ and the common ratio is $\frac{1}{2}$.

Part (d). As $n \to \infty$, we clearly have $\left(\frac{1}{2}\right)^{n+1} \to 0$. Hence, $S_\infty = \frac{1}{2}$.

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Problem 7. The first term of an arithmetic series is $\ln x$ and the rth term is $\ln\left(xk^{r-1}\right)$, where k is a real constant. Show that the sum of the first n terms of the series is $S_n = \frac{n}{2}\ln\left(x^2k^{n-1}\right)$. If $k = 1$ and $x \neq 1$, find the sum of the series $e^{S_1} + e^{S_2} + e^{S_3} + \ldots + e^{S_n}$.

Solution. Let u_n be the nth term in the arithmetic series. Then

$$u_n = \ln\left(xk^{r-1}\right) = \ln x + (r-1)\ln k.$$

We thus see that the arithmetic series has first term $\ln x$ and common difference of $\ln k$. Thus,

$$S_n = n\left(\frac{\ln x + (\ln x + (r-1)\ln k)}{2}\right) = \frac{n}{2}\ln\left(x^2k^{r-1}\right).$$

When $k = 1$, we have $S_n = \ln(x^n)$, whence $e^{S_n} = x^n$. Thus,

$$e^{S_1} + e^{S_2} + e^{S_3} + \ldots + e^{S_n} = x + x^2 + x^3 + \ldots x^n = \frac{x(1 - x^{n+1})}{1 - x}.$$

* * * * *

Problem 8. A baker wants to bake a 1-metre tall birthday cake. It comprises 10 cylindrical cakes each of equal height 10 cm. The diameter of the cake at the lowest layer is 30 cm. The diameter of each subsequent layer is 4% less than the diameter of the cake below. Find the volume of this cake in cm^3, giving your answer to the nearest integer.

Solution. Let the diameter of the nth layer be d_n cm. We have $d_{n+1} = 0.96 d_n$ and $d_1 = 30$, whence $d_n = 30 \cdot 0.96^{n-1}$. Let the nth layer have volume v_n cm^3. Then

$$v_n = 10\pi \left(\frac{d_n}{2}\right)^2 = 10\pi \left(\frac{900 \cdot 0.9216^{n-1}}{4}\right) = 2250\pi \cdot 0.9216^{n-1}.$$

The volume of the cake in cm^3 is thus given by

$$2250\pi \left(\frac{1 - 0.9216^{10}}{1 - 0.9216}\right) = 50309.$$

* * * * *

Problem 9. The sum to infinity of a geometric progression is 5 and the sum to infinity of another series is formed by taking the first, fourth, seventh, tenth, … terms is 4. Find the exact common ratio of the series.

Solution. Let the nth term of the geometric progression be given by ar^{n-1}. Then, we have

$$\frac{a}{1 - r} = 5 \implies a = 5(1 - r). \tag{1}$$

Note that the first, fourth, seventh, tenth, … terms forms a new geometric series with common ratio r^3: $a, ar^3, ar^6, ar^9, \ldots$. Thus,

$$\frac{a}{1 - r^3} = 4 \implies a = 4(1 - r^3). \tag{2}$$

Equating (1) and (2), we have

$$5(1 - r) = 4(1 - r^3) \implies 4r^3 + 5r + 1 = 0 \implies (r - 1)(4r^2 + 4r - 1) = 0.$$

Since $|r| < 1$, we only have $4r^2 + 4r - 1 = 0$, which has solutions $r = \frac{-1+\sqrt{2}}{2}$ or $r = \frac{-1-\sqrt{2}}{2}$. Once again, since $|r| < 1$, we reject $r = \frac{-1-\sqrt{2}}{2}$. Hence, $r = \frac{-1+\sqrt{2}}{2}$.

* * * * *

Problem 10. A geometric series has common ratio r, and an arithmetic series has first term a and common difference d, where a and d are non-zero. The first three terms of the geometric series are equal to the first, fourth and sixth terms respectively of the arithmetic series.

(a) Show that $3r^2 - 5r + 2 = 0$

(b) Deduce that the geometric series is convergent and find, in terms of a, the sum of infinity.

(c) The sum of the first n terms of the arithmetic series is denoted by S. Given that $a > 0$, find the set of possible values of n for which S exceeds $4a$.

Solution.

Part (a). Let the nth term of the geometric series be $G_n = G_1 r^{n-1}$. Let the nth term of the arithmetic series be $A_n = a + (n-1)d$.

Since $G_1 = A_1$, we have $G_1 = a$. We can thus write $G_n = ar^{n-1}$. From $G_2 = A_4$, we have $ar = a + 3d$, which gives $a = \frac{3d}{r-1}$. From $G_3 = A_6$, we have $ar^2 = a + 5d$. Thus,

$$\frac{3d}{r-1} \cdot r^2 = \frac{3d}{r-1} + 5d \implies \frac{3r^2}{r-1} = \frac{3}{r-1} + 5 \implies 3r^2 - 5r + 2 = 0.$$

Part (b). Note that the roots to $3r^2 - 5r + 2 = 0$ are $r = 1$ and $r = \frac{2}{3}$. Clearly, $r \neq 1$ since $a = \frac{3d}{r-1}$ would be undefined. Hence, $r = \frac{2}{3}$, whence the geometric series is convergent.

Let S_∞ be the sum to infinity of G_n. Then $S_\infty = \frac{a}{1-r} = 3a$.

Part (c). Note that $d = \frac{1}{3}a(r-1) = -\frac{a}{9}$. Hence,

$$S = n\left(\frac{a + [a + (n-1)d]}{2}\right) = n\left(\frac{2a + (n-1)\left(-\frac{a}{9}\right)}{2}\right) = \frac{an}{18}(19 - n).$$

Consider $S > 4a$.

$$S > 4a \implies \frac{n}{18}(19 - n) > 4 \implies -n^2 + 19n - 72 > 0.$$

Using G.C., we see that $5.23 < n < 13.8$. Since n is an integer, the set of values that n can take on is $\{n \in \mathbb{Z} : 6 \leq n \leq 13\}$.

* * * * *

Problem 11. Two musical instruments, A and B, consist of metal bars of decreasing lengths.

(a) The first bar of instrument A has length 20 cm and the lengths of the bars form a geometric progression. The 25th bar has length 5 cm. Show that the total length of all the bars must be less than 357 cm, no matter how many bars there are.

Instrument B consists of only 25 bars which are identical to the first 25 bars of instrument A.

(b) Find the total length, L cm, of all the bars of instrument B and the length of the 13th bar.

(c) Unfortunately, the manufacturer misunderstands the instructions and constructs instrument B wrongly, so that the lengths of the bars are in arithmetic progression with a common difference d cm. If the total length of the 25 bars is still L cm and the length of the 25th bar is still 5 cm, find the value of d and the length of the longest bar.

Solution.

Part (a). Let $u_n = u_1 r^{n-1}$ be the length of the nth bar. Since $u_1 = 20$, we have $u_n = 20r^{n-1}$. Since $u_{25} = 5$, we have $r = 4^{-\frac{1}{24}}$. Hence, $u_n = 20 \cdot 4^{-\frac{n-1}{24}}$. Now, consider the sum to infinity of u_n:

$$S_\infty = \frac{u_1}{1-r} = \frac{20}{1 - 4^{-1/24}} = 356.3 < 357.$$

Hence, no matter how many bars there are, the total length of the bars will never exceed 357 cm.

Part (b). We have

$$L = u_1 \left(\frac{1-r^{25}}{1-r} \right) = 20 \left(\frac{1-4^{-25/24}}{1-4^{-1/24}} \right) = 272.26 = 272 \text{ (3 s.f.)}.$$

Note that

$$u_{13} = 20 \cdot \left(4^{-1/24}\right)^{13-1} = 10.$$

The 13th bar is hence 10 cm long.

Part (c). Let $v_n = a + (n-1)d$ be the length of the wrongly-manufactured bars. Since the length of the 25th bar is still 5 cm, we know $v_{25} = a + 24d = 5$. Now, consider the total lengths of the bars, which is still L cm.

$$L = 25 \left(\frac{a+5}{2} \right) = 272.26.$$

Solving, we see that $a = 16.781$. Hence, $d = \frac{5-a}{24} = -0.491$, and the longest bar is $16.8 =$ cm long.

* * * * *

Problem 12. A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man a woman both invest money.

(a) The man decides to invest $\$x$ at the beginning of one year and then a further $\$x$ at the beginning of the second and each subsequent year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up.

  (i) How much will there be in the account at the end of 1 year, including the interest?

  (ii) Show that, at the end of n years, when the interest for the last year has been added, he will have a total of $\$21(1.05^n - 1)x$ in his account.

  (iii) After how many complete years will he have, for the first time, at least $\$12x$ in his account?

(b) The woman decides that, to assist her in her everyday expenses, she will withdraw the interest as soon as it has been added. She invests $\$y$ at the beginning of each year. Show that, at the end of n years, she will have received a total of $\$\frac{1}{40}n(n+1)y$ in interest.

Solution.

Part (a).

Part (a)(i). There will be $\$1.05x$ in the account at the end of 1 year.

Part (a)(ii). Let $\$u_n x$ be the amount of money in the account at the end of n years. Then, u_n satisfies the recurrence relation $u_{n+1} = 1.05(1 + u_n)$, with $u_1 = 1.05$. Observe that

$$u_1 = 1.05 \implies u_2 = 1.05 + 1.05^2 \implies u_3 = 1.05 + 1.05^2 + 1.05^3 \implies \cdots.$$

We thus have

$$u_n = 1.05 + 1.05^2 + \cdots + 1.05^n = 1.05 \left(\frac{1-1.05^n}{1-1.05} \right) = 21\left(1.05^n - 1\right).$$

Hence, there will be $\$21(1.05^n - 1)x$ in the account after n years.

Part (a)(iii). Consider the inequality $u_n \geq 12x$.

$$u_n \geq 12x \implies 21(1.05^n - 1) \geq 12 \implies n \geq 9.26.$$

Since n is an integer, the smallest value of n is 10. Hence, after 10 years, he will have at least \$$12x$ in his account for the first time.

Part (b). After n years, the woman will have \$$ny$ in her account. Hence, the interest she gains at the end of the nth year is $\frac{1}{20}ny$. Thus, the total interest she will gain after n years is

$$\frac{y}{20} + \frac{2y}{20} + \cdots + \frac{ny}{20} = \frac{y}{20}(1 + 2 + \cdots + n) = \frac{y}{20} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)y}{40}.$$

* * * * *

Problem 13. The sum, S_n, of the first n terms of a sequence $U_1, U_2, U_3, \ldots$ is given by

$$S_n = \frac{n}{2}(c - 7n),$$

where c is a constant.

(a) Find U_n in terms of c and n.

(b) Find a recurrence relation of the form $U_{n+1} = f(U_n)$.

Solution.

Part (a). Observe that

$$U_n = S_n - S_{n-1} = \frac{n}{2}(c - 7n) - \frac{n-1}{2}(c - 7(n-1)) = -7n + \frac{7+c}{2}.$$

Part (b). Observe that $U_{n+1} - U_n = -7$. Thus,

$$U_{n+1} = U_n - 7, \quad U_1 = \frac{7+c}{2}, \quad n \geq 1.$$

* * * * *

Problem 14. The positive numbers x_n satisfy the relation

$$x_{n+1} = \sqrt{\frac{9}{2} + \frac{1}{x_n}}$$

for $n = 1, 2, 3, \ldots$.

(a) Given that $n \to \infty$, $x_n \to \theta$, find the exact value of θ.

(b) By considering $x_{n+1}^2 - \theta^2$, or otherwise, show that if $x_n > \theta$, then $0 < x_{n+1} < \theta$.

Solution.

Part (a). Observe that

$$\theta = \lim_{n\to\infty} \sqrt{\frac{9}{2} + \frac{1}{x_n}} = \sqrt{\frac{9}{2} + \frac{1}{\theta}} \implies 2\theta^3 - 9\theta - 2 = 0 \implies (\theta + 2)\left(2\theta^2 - 4\theta - 1\right) = 0.$$

We reject $\theta = -2$ since $\theta > 0$. We thus consider $2\theta^2 - 4\theta - 1 = 0$, which has roots $\theta = 1 + \sqrt{\frac{3}{2}}$ and $\theta = 1 - \sqrt{\frac{3}{2}}$. Once again, we reject $\theta = 1 - \sqrt{\frac{3}{2}}$ since $\theta > 0$. Thus, $\theta = 1 + \sqrt{\frac{3}{2}}$.

Part (b). Suppose $x_n > \theta$. Then

$$x_{n+1}^2 = \frac{9}{2} + \frac{1}{x_n} < \frac{9}{2} + \frac{1}{\theta} = \theta^2 \implies 0 < x_{n+1} < \theta.$$

Assignment A3

Problem 1. A university student has a goal of saving at least \$1 000 000 (in Singapore dollars). He begins working at the start of the year 2019. In order to achieve his goal, he saves 40% of his annual salary at the end of each year. If his annual salary in the year 2019 is \$40800, and it increases by 5% (of his previous year's salary) every year, find

(a) his annual savings in 2027 (to the nearest dollar),

(b) his total savings at the end of n years.

What is the minimum number of complete years for which he has to work in order to achieve his goal?

Solution. Let $\$u_n$ be his annual salary in the nth year after 2019, with $n \in \mathbb{N}$. Then $u_{n+1} = 1.05 \cdot u_n$, with $u_0 = 40800$. Hence, $u_n = 40800 \cdot 1.05^n$. Let $\$v_n$ be the amount saved in the nth year after 2019. Then $v_n = 0.40 \cdot u_n = 16320 \cdot 1.05^n$.

Part (a). In 2027, $n = 8$. Hence, his annual savings in 2027, in dollars, is given by

$$v_8 = 16320 \cdot 1.05^8 = 24112 \text{ (to the nearest integer)}.$$

Part (b). His total savings at the end of n years, in dollars, is given by

$$16320\left(1.05^0 + 1.05^1 + \cdots + 1.05^n\right) = 16320\left(\frac{1 - 1.05^n}{1 - 1.05}\right) = 326400\left(1.05^n - 1\right).$$

Consider $326400\left(1.05^n - 1\right) \geq 1000000$. Using G.C., we see that $n \geq 28.7$. Thus, he needs to work for a minimum of 29 complete years to reach his goal.

* * * * *

Problem 2.

(a) A rope of length 200π cm is cut into pieces to form as many circles as possible, whose radii follow an arithmetic progression with common difference 0.25 cm. Given that the smallest circle has an area of π cm^2, find the area of the largest circle in terms of π.

(b) The sum of the first n terms of a sequence is given by $S_n = \alpha^{-n} - 1$, where α is a non-zero constant, $\alpha \neq 1$.

(i) Show that the sequence is a geometric progression and state its common ratio in terms of α.

(ii) Find the set of values of α for which the sum to infinity of the sequence exists.

(iii) Find the value of the sum to infinity.

Solution.

Part (a). Let the sequence r_n be the radius of the nth smallest circle, in centimetres. Hence, $r_n = \frac{1}{4} + r_{n-1}$. Since the smallest circle has area π cm^2, $r_1 = 1$. Thus, $r_n = 1 + \frac{1}{4}(n-1)$.

Consider the nth partial sum of the circumferences:

$$2\pi r_1 + 2\pi r_2 + \cdots + 2\pi r_n = 2\pi \cdot n\left(\frac{1 + \left[1 + \frac{1}{4}(n-1)\right]}{2}\right) = \frac{\pi(n^2 + 7n)}{4}.$$

Since the rope has length 200π cm, we have the inequality

$$\frac{\pi(n^2+7n)}{4} \leq 200\pi \implies n^2 - 7n - 800 \leq 0 \implies (n+32)(n-25) \leq 0.$$

Hence, $n \leq 25$. Since the rope is cut to form as many circles as possible, $n = 25$. Thus, the largest circle has area $\pi \cdot r_{25}^2 = 49\pi$ cm^2.

Part (b). Let the sequence being summed by $u_1, u_2, \ldots$. Observe that

$$u_n = S_n - S_{n-1} = \left(\alpha^{-n} - 1\right) - \left(\alpha^{-(n-1)} - 1\right) = \alpha^{-n}(1-\alpha).$$

Part (b)(i). Observe that

$$\frac{u_{n+1}}{u_n} = \frac{\alpha^{-(n+1)}(1-\alpha)}{\alpha^{-n}(1-\alpha)} = \alpha^{-1},$$

which is a constant. Thus, u_n is in GP with common ratio α^{-1}.

Part (b)(ii). Consider $S_\infty = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(\alpha^{-n} - 1)$. For S_∞ to exist, we need $\lim_{n\to\infty} \alpha^{-n}$ to exist. Hence, $|\alpha^{-1}| < 1$, whence $|\alpha| > 1$. Thus, $\alpha < -1$ or $\alpha > 1$. The solution set of α is thus $\{x \in \mathbb{R} : x < -1 \text{ or } x > 1\}$.

Part (b)(iii). Since $|\alpha^{-1}| < 1$, we know $\lim_{n\to\infty} \alpha^{-n} = 0$. Hence, $S_\infty = -1$.

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Problem 3. A sequence $u_1, u_2, u_3, \ldots$ is such that $u_{n+1} = 2u_n + An$, where A is a constant and $n \geq 1$.

(a) Given that $u_1 = 5$ and $u_2 = 15$, find A and u_3.

It is known that the nth term of this sequence is given by

$$u_n = a(2^n) + bn + c,$$

where a, b and c are constants.

(b) Find a, b and c.

Solution.

Part (a). Substituting $n = 1$ into the recurrence relation yields $u_2 = 2u_1 + A$. Thus, $A = u_2 - 2u_1 = 5$. Substituting $n = 2$ into the recurrence relation yields $u_3 = 2u_2 + 2A = 40$.

Part (b). Since $u_1 = 5$, $u_2 = 15$ and $u_3 = 40$, we have the following system

$$\begin{cases} 2a + b + c = 5 \\ 4a + 2b + c = 15 \\ 8a + 3b + c = 40 \end{cases}$$

which has the unique solution $a = \frac{15}{2}$, $b = -5$ and $c = -5$

Problem 4. The graphs of $y = 2^x/3$ and $y = x$ intersect at $x = \alpha$ and $x = \beta$ where $\alpha < \beta$. A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3} \cdot 2^{x_n}, \qquad n \geq 1.$$

(a) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(b) By using the graphs of $y = \frac{1}{3} \cdot 2^x$ and $y = x$, prove that

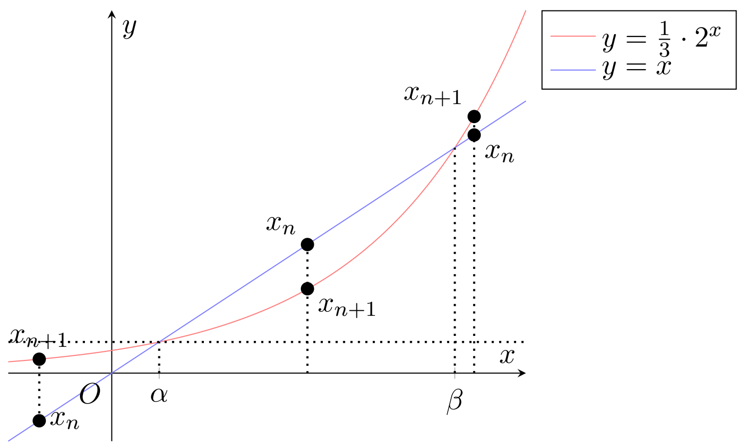
- if $\alpha < x_n < \beta$, then $\alpha < x_{n+1} < x_n$
- if $x_n < \alpha$, then $x_n < x_{n+1} < \alpha$
- if $x_n > \beta$, then $x_n < x_{n+1}$

Describe the behaviour of the sequence for the three cases.

Solution.

Part (a). Let $L = \lim_{n\to\infty} x_n$. Then $L = \frac{1}{3} \cdot 2^L$. Since $y = x$ and $y = \frac{1}{3} \cdot 2^x$ intersect only at $x = \alpha$ and $x = \beta$, then α and β are the only roots of $x = \frac{1}{3} \cdot 2^x$. Since L is also a root of $x = \frac{1}{3} \cdot 2^x$, L must be either α or β.

Part (b).



If $\alpha < x_n < \beta$, then x_n is decreasing and converges to α. If $x_n < \alpha$, then x_n is increasing and converges to α. If $x_n > \beta$, then x_n is increasing and diverges.

A4. Sequences and Series II

Tutorial A4

Problem 1. True or False? Explain your answers briefly.

(a) $\sum_{r=1}^{n}(2r+3) = \sum_{k=1}^{n}(2k+3)$

(b) $\sum_{r=1}^{n}\left(\frac{1}{r}+5\right) = \sum_{r=1}^{n}\frac{1}{r}+5$

(c) $\sum_{r=1}^{n}\frac{1}{r} = 1/\sum_{r=1}^{n} r$

(d) $\sum_{r=1}^{n} c = \sum_{r=0}^{n-1}(c+1)$

Solution.
Part (a). True: A change in index does not affect the sum.
Part (b). False: In general, $\sum_{r=1}^{n} 5$ is not equal to 5.
Part (c). False: In general, $\sum \frac{a}{b} \neq \sum a/\sum b$.
Part (d). False: Since c is a constant, $\sum_{r=1}^{n} c = nc \neq n(c+1) = \sum_{r=0}^{n-1}(c+1)$.

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Problem 2. Write the following series in sigma notation twice, with $r=1$ as the lower limit in the first and $r=0$ as the lower limit in the second.

(a) $-2+1+4+\ldots+40$

(b) $a^2+a^4+a^6+\ldots+a^{50}$

(c) $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n$th term

(d) $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots$ to n terms

(e) $\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30}$

Solution.
Part (a).

$$-2+1+4+\ldots+40 = \sum_{r=1}^{15}(3r-5) = \sum_{r=0}^{14}(3r-2).$$

Part (b).

$$a^2+a^4+a^6+\ldots+a^{50} = \sum_{r=1}^{25} a^{2r} = \sum_{r=0}^{24} a^{2r+2}.$$

Part (c).

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+n\text{th term} = \sum_{r=1}^{n}\frac{1}{2r+1} = \sum_{r=0}^{n-1}\frac{1}{2r+3}.$$

Part (d).

$$1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots \text{ to } n \text{ terms} = \sum_{r=1}^{n}\left(-\frac{1}{2}\right)^{r-1} = \sum_{r=0}^{n-1}\left(-\frac{1}{2}\right)^{r}.$$

Part (e).

$$\frac{1}{2\cdot 4}+\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\ldots+\frac{1}{28\cdot 30} = \sum_{r=1}^{27}\frac{1}{(r+1)(r+3)} = \sum_{r=0}^{26}\frac{1}{(r+2)(r+4)}.$$

* * * * *

Problem 3. Without using the G.C., evaluate the following sums.

(a) $\sum_{r=1}^{50}(2r-7)$

(b) $\sum_{r=1}^{a}(1-a-r)$

(c) $\sum_{r=2}^{n}\left(\ln r+3^r\right)$

(d) $\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right)$

Solution.

Part (a).

$$\sum_{r=1}^{50}(2r-7) = 2\sum_{r=1}^{50}r - 7\sum_{r=1}^{50}1 = 2\left(\frac{50\cdot 51}{2}\right) - 7(50) = 2200.$$

Part (b).

$$\sum_{r=1}^{a}(1-a-r) = (1-a)\sum_{r=1}^{a}1 - \sum_{r=1}^{a}r = (1-a)a - \frac{a(a+1)}{2} = \frac{a}{2}(1-3a).$$

Part (c).

$$\sum_{r=2}^{n}\left(\ln r+3^r\right) = \sum_{r=2}^{n}\ln r + \sum_{r=2}^{n}3^r = \ln n! + 3^2\left(\frac{1-3^{n-2+1}}{1-3}\right) = \ln n! + \frac{9}{2}\left(3^{n-1}-1\right).$$

Part (d).

$$\sum_{r=1}^{\infty}\left(\frac{2^r-1}{3^r}\right) = \sum_{r=1}^{\infty}\left(\frac{2}{3}\right)^r - \sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r = \frac{2/3}{1-2/3} - \frac{1/3}{1-1/3} = \frac{3}{2}.$$

* * * * *

Problem 4. The nth term of a series is $2^{n-2}+3n$. Find the sum of the first N terms.

Solution.

$$\begin{aligned}\sum_{n=1}^{N}\left(2^{n-2}+3r\right) &= \sum_{n=1}^{N}2^{n-2} + 3\sum_{n=1}^{N}n \\ &= 2^{1-2}\left(\frac{\left(2^N-1\right)}{2-1}\right) + 3\left(\frac{N(N+1)}{2}\right) \\ &= \frac{1}{2}\left(2^N+3N^2+3N-1\right).\end{aligned}$$

Problem 5. The rth term, u_r, of a series is given by $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$. Express $\sum_{r=1}^{n} u_r$ in the form $A\left(1 - \frac{B}{27^n}\right)$, where A and B are constants. Deduce the sum to infinity of the series.

Solution. Observe that

$$u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1} = 12\left(\frac{1}{3}\right)^{3r} = 12\left(\frac{1}{27}\right)^{r}.$$

Hence,

$$\sum_{r=1}^{n} = 12 \cdot \frac{1}{27}\left(\frac{1-1/27^n}{1-1/27}\right) = \frac{6}{13}\left(1 - \frac{1}{27^n}\right),$$

whence $A = \frac{6}{13}$ and $B = 1$. In the limit as $n \to \infty$, $\frac{1}{27^n} \to 0$. Hence, the sum to infinity is $\frac{6}{13}$.

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Problem 6. The rth term, u_r, of a series is given by $u_r = \ln \frac{r}{r+1}$. Find $\sum_{r=1}^{n} u_r$ in terms of n. Comment on whether the series converges.

Solution. Observe that $u_r = \ln \frac{r}{r+1} = \ln r - \ln(r+1)$. Hence,

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n}(\ln r - \ln(r+1)) \\ &= [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + \cdots + [\ln n - \ln(n+1)] \\ &= \ln 1 - \ln(n+1) = \ln \frac{1}{n+1}.\end{aligned}$$

As $n \to \infty$, $\ln \frac{1}{n+1} \to \ln 0$. Hence, the series diverges to negative infinity.

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Problem 7. Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$, without using the G.C., find the following sums.

(a) $\sum_{r=0}^{n}[r(r+4)+n]$

(b) $\sum_{r=n+1}^{2n}(2r-1)^2$

(c) $\sum_{r=-15}^{20} r(r-2)$

Solution.

Part (a).

$$\begin{aligned}\sum_{r=0}^{n}[r(r+4)+n] &= \sum_{r=0}^{n}\left(r^2+4r+n\right) \\ &= \frac{n}{6}(n+1)(2n+1) + 4\left[\frac{n(n+1)}{2}\right] + n(n+1) \\ &= \frac{n}{6}(n+1)(2n+19).\end{aligned}$$

Part (b).

$$\begin{aligned}\sum_{r=n+1}^{2n}(2r-1)^2 &= \sum_{r=1}^{n}(2(r+n)-1)^2 = \sum_{r=1}^{n}\left(4r^2+4(2n-1)r+(2n-1)^2\right)\\ &= 4\left[\frac{n}{6}(n+1)(2n+1)\right]+4(2n-1)\left[\frac{n(n+1)}{2}\right]+(2n-1)^2n\\ &= \frac{1}{3}n\left(28n^2-1\right)\end{aligned}$$

Part (c).

$$\begin{aligned}\sum_{r=-15}^{20}r(r-2) &= \sum_{r=1}^{36}(r-16)[(r-16)-2] = \sum_{r=1}^{36}\left(r^2-34r+288\right)\\ &= \frac{36}{6}\left[(36+1)(2\cdot 36+1)\right]-34\left[\frac{36\cdot 37}{2}\right]+288(36)\\ &= 3930\end{aligned}$$

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Problem 8. Let $S=\sum_{r=0}^{\infty}\frac{(x-2)^r}{3^r}$ where $x\neq 2$. Find the range of values of x such that the series S converges. Given that $x=1$, find

(a) the value of S

(b) S_n, in terms of n, where $S_n=\sum_{r=0}^{n-1}\frac{(x-2)^r}{3^r}$

(c) the least value of n for which $|S_n-S|$ is less than 0.001% of S

Solution. Note that

$$S=\sum_{r=0}^{\infty}\frac{(x-2)^r}{3^r}=\sum_{r=0}^{\infty}\left(\frac{x-2}{3}\right)^r.$$

Hence, for S to converge, we must have $|\frac{x-2}{3}|<1$, which gives $-1<x<5$, $x\neq 2$.

Part (a). When $x=1$, we get

$$S=\sum_{r=0}^{\infty}\left(-\frac{1}{3}\right)^r=\frac{1}{1-(-\frac{1}{3})}=\frac{3}{4}.$$

Part (b). We have

$$S_n=\sum_{r=0}^{n-1}\left(-\frac{1}{3}\right)^r=\frac{1-(-\frac{1}{3})^n}{1-(-\frac{1}{3})}=\frac{3}{4}\left[1-\left(-\frac{1}{3}\right)^n\right].$$

Part (c). Observe that

$$|S_n-S|<0.001\%S \implies |\frac{S_n-S}{S}|<\frac{1}{100000} \implies |\frac{\frac{3}{4}(1-(-\frac{1}{3})^n)}{\frac{3}{4}}-1|<\frac{1}{100000}.$$

Using G.C., the least value of n that satisfies the above inequality is 11.

Problem 9. Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$,

(a) write down $\sum_{r=1}^{2k} r^2$ in terms of k

(b) find $2^2 + 4^2 + 6^2 + \ldots + (2k)^2$.

Hence, show that $\sum_{r=1}^{k}(2r-1)^2 = \frac{k}{3}(2k+1)(2k-1)$.

Solution.

Part (a).

$$\sum_{r=1}^{2k} r^2 = \frac{2k}{6}(2k+1)(2(2k)+1) = \frac{k}{3}(2k+1)(4k+1).$$

Part (b).

$$2^2 + 4^2 + 6^2 + \ldots + (2k)^2 = \sum_{r=1}^{k}(2r)^2 = \sum_{r=1}^{k} 4r^2 = \frac{2k}{3}(k+1)(2k+1).$$

From parts (a) and (b), we clearly have

$$\sum_{r=1}^{k}(2r-1)^2 = \sum_{r=1}^{2k} r^2 - \sum_{r=1}^{k}(2r)^2 = \frac{k}{3}(2k+1)(4k+1) - \frac{2k}{3}(k+1)(2k+1) = \frac{k}{3}(2k+1)(2k-1).$$

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Problem 10. Given that $u_n = e^{nx} - e^{(n+1)x}$, find $\sum_{n=1}^{N} u_n$ in terms of N and x. Hence, determine the set of values of x for which the infinite series $u_1 + u_2 + u_3 + \ldots$ is convergent and give the sum to infinity for cases where this exists.

Solution.

$$\sum_{n=1}^{N} u_n = \left(e^x - e^{2x}\right) + \left(e^{2x} - e^{3x}\right) + \cdots + \left(e^{Nx} + e^{(N+1)x}\right) = e^x - e^{(N+1)x}.$$

For the infinite series to converge, we require $|e^x| < 1$. Hence, $x \in \mathbb{R}_0^-$.

We now consider the sum to infinity.

Case 1. Suppose $x = 0$. Then $e^x = 1$, whence the sum to infinity is clearly 0.

Case 2. Suppose $x < 0$. Then $\lim_{N\to\infty} e^{(N+1)x} \to 0$. Thus, the sum to infinity is e^x.

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Problem 11. Given that r is a positive integer and $f(r) = \frac{1}{r^2}$, express $f(r) - f(r+1)$ as a single fraction. Hence, prove that $\sum_{r=1}^{4n}\left(\frac{2r+1}{r^2(r+1)^2}\right) = 1 - \frac{1}{(4n+1)^2}$. Give a reason why the series is convergent and state the sum to infinity. Find $\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right)$.

Solution.

$$f(r) - f(r+1) = \frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.$$

$$\begin{aligned}
\sum_{r=1}^{4n}\left(\frac{2r+1}{r^2(r+1)^2}\right) &= \sum_{r=1}^{4n}[f(r) - f(r+1)] \\
&= [f(1) - f(2)] + [f(2) - f(3)] + \cdots + [f(4n) - f(4n-1)] \\
&= f(1) - f(4n+1) = 1 - \frac{1}{(4n+1)^2}
\end{aligned}$$

As $n \to \infty$, $\frac{1}{(4n+1)^2} \to 0$. Hence, the series converges to 1.

$$\begin{aligned}\sum_{r=2}^{4n}\left(\frac{2r-1}{r^2(r-1)^2}\right) = \sum_{r=1}^{4n-1}\left(\frac{2r+1}{r^2(r+1)^2}\right) &= \sum_{r=1}^{4n-1}[f(r)-f(r+1)]\\ &= [f(1)-f(2)]+[f(2)-f(3)]+\cdots+[f(4n-1)-f(4n)]\\ &= 1-f(4n) = 1-\frac{1}{16n^2}\end{aligned}$$

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Problem 12.

(a) Express $\frac{1}{(2x+1)(2x+3)(2x+5)}$ in partial fractions.

(b) Hence, show that $\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$.

(c) Deduce the sum of $\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \ldots + \frac{1}{41\cdot43\cdot45}$.

Solution.

Part (a). Using the cover-up rule, we obtain

$$\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}.$$

Part (b).

$$\begin{aligned}\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} &= \sum_{r=1}^{n}\left(\frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)}\right)\\ &= \frac{1}{8}\left[\left(\sum_{r=1}^{n}\frac{1}{2r+1} - \sum_{r=1}^{n}\frac{1}{2r+3}\right) - \left(\sum_{r=1}^{n}\frac{1}{2r+3} - \sum_{r=1}^{n}\frac{1}{2r+5}\right)\right]\end{aligned}$$

Observe that the two terms in brackets clearly telescope, leaving us with

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{8}\left[\left(\frac{1}{3}-\frac{1}{2n+3}\right)-\left(\frac{1}{5}-\frac{1}{2n+5}\right)\right],$$

which simplifies to

$$\sum_{r=1}^{n}\frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$$

as desired.

Part (c).

$$\begin{aligned}&\frac{1}{1\cdot3\cdot5} + \frac{1}{3\cdot5\cdot7} + \frac{1}{3\cdot5\cdot7\cdot9} + \ldots + \frac{1}{41\cdot43\cdot45}\\ &= \frac{1}{1\cdot3\cdot5} + \sum_{r=1}^{20}\frac{1}{(2r+1)(2r+3)(2r+5)}\\ &= \frac{1}{15} + \left(\frac{1}{60} - \frac{1}{4(2\cdot20+3)(2\cdot20+5)}\right)\\ &= \frac{161}{1935}.\end{aligned}$$

A5. Recurrence Relations

Tutorial A5

Problem 1. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 2u_{n-1}$, for $n \geq 1$, $u_0 = 3$

(b) $u_n = 3u_{n-1} + 7$, for $n \geq 1$, $u_0 = 5$

Solution.
Part (a). $u_n = 2^n \cdot u_0 = 3 \cdot 2^n$.
Part (b). Let k be a constant such that $u_n + k = 3(u_{n-1} + k)$. Then $k = \frac{7}{2}$. Hence,

$$u_n + \frac{7}{2} = 3\left(u_{n-1} + \frac{7}{2}\right) \implies u_n + \frac{7}{2} = 3^n\left(u_0 + \frac{7}{2}\right) \implies u_n = \frac{17}{2} \cdot 3^n - \frac{7}{2}.$$

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Problem 2. Solve these recurrence relations together with the initial conditions.

(a) $u_n = 5u_{n-1} - 6u_{n-2}$, for $n \geq 2$, $u_0 = 1$, $u_1 = 0$

(b) $u_n = 4u_{n-2}$, for $n \geq 2$, $u_0 = 0$, $u_1 = 4$

(c) $u_n = 4u_{n-1} - 4u_{n-2}$, for $n \geq 2$, $u_0 = 6$, $u_1 = 8$

(d) $u_n = -6u_{n-1} - 9u_{n-2}$, for $n \geq 2$, $u_0 = 3$, $u_1 = -3$

(e) $u_n = 2u_{n-1} - 2u_{n-2}$, for $n \geq 2$, $u_0 = 2$, $u_1 = 6$

Solution.
Part (a). Note that the characteristic equation of u_n, $x^2 - 5x + 6 = 0$, has roots 2 and 3. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n.$$

From $u_0 = 1$ and $u_1 = 0$, we have the equations $A + B = 1$ and $2A + 3B = 0$. Solving, we see that $A = 3$ and $B = 2$, whence

$$u_n = 3 \cdot 2^n + 2 \cdot 3^n.$$

Part (b). Note that the characteristic equation of u_n, $x^2 - 4 = 0$, has roots -2 and 2. Thus,

$$u_n = A(-2)^n + B \cdot 2^n.$$

From $u_0 = 0$ and $u_1 = 4$, we get $A + B = 0$ and $-2A + 2B = 4$. Solving, we see that $A = -1$ and $B = 1$, whence

$$u_n = -(-2)^n + 2^n.$$

Part (c). Note that the characteristic equation of u_n, $x^2 - 4x + 4 = 0$, has only one root, 2. Thus,

$$u_n = (A + Bn)2^n.$$

From $u_0 = 6$ and $u_1 = 8$, we obtain $A = 6$ and $A + B = 4$, whence $B = -2$. Thus,

$$u_n = (6 - 2n)2^n.$$

Part (d). Note that the characteristic equation of u_n, $x^2 + 6x + 9 = 0$, has only one root, -3. Thus,

$$u_n = (A + Bn)(-3)^n.$$

From $u_0 = 3$ and $u_1 = -3$, we get $A = 3$ and $A + B = 1$, whence $B = -2$. Thus,

$$u_n = (3 - 2n)2^n.$$

Part (e). Consider the characteristic equation of u_n, $x^2 - 2x + 2 = 0$. By the quadratic formula, this has roots $x = 1 \pm i = \sqrt{2}\exp\left(\pm\frac{i\pi}{4}\right)$. Hence,

$$u_n = A \cdot 2^{\frac{1}{2}n}\cos\left(\frac{n\pi}{4}n\right) + B \cdot 2^{\frac{1}{2}n}\sin\left(\frac{n\pi}{4}\right).$$

From $u_0 = 2$, we obtain $A = 2$. From $u_0 = 6$, we obtain $A + B = 6$, whence $B = 4$. Thus,

$$u_n = 2^{\frac{1}{2}n+1}\cos\left(\frac{n\pi}{4}\right) + 2^{\frac{1}{2}n+2}\sin\left(\frac{n\pi}{4}\right).$$

* * * * *

Problem 3.

(a) A sequence is defined by the formula $b_n = \frac{n!n!}{(2n)!} \cdot 2^n$, where $n \in \mathbb{Z}^+$. Show that the sequence satisfies the recurrence relation $b_{n+1} = \frac{n+1}{2n+1} b_n$.

(b) A sequence is defined recursively by the formula

$$u_{n+1} = 2u_n + 3, \qquad n \in \mathbb{Z}_0^+,\ u_0 = a$$

Show that $u_n = 2^n a + 3\left(2^n - 1\right)$.

Solution.

Part (a).

$$b_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot 2^{n+1} = \frac{2(n+1)^2}{(2n+1)(2n+2)} \cdot \left[\frac{n!n!}{(2n)!} \cdot 2^n\right] = \frac{n+1}{2n+1} b_n.$$

Part (b). Let k be a constant such that $u_{n+1} + k = 2(u_n + k)$. Then $k = 3$. Hence,

$$u_{n+1} + 3 = 2(u_n + 3) \implies u_n + 3 = 2^n(u_0 + 3) \implies u_n = 2^n(a + 3) - 3 = 2^n a + 3\left(2^n - 1\right).$$

Problem 4. The volume of water, in litres, in a storage tank decreases by 10% by the end of each day. However, 90 litres of water is also pumped into the tank at the end of each day. The volume of water in the tank at the end of n days is denoted by x_n and x_0 is the initial volume of water in the tank.

(a) Write down a recurrence relation to represent the above situation.

(b) Show that $x_n = 0.9^n(x_0 - 900) + 900$.

(c) Deduce the amount of water in the tank when n becomes very large.

Solution.

Part (a). $x_{n+1} = 0.9x_n + 90$, $n \in \mathbb{N}$

Part (b). Let k be a constant such that $x_{n+1} + k = 0.9(x_n + k)$. Then $k = -900$. Hence,

$$x_{n+1} - 900 = 0.9(x_n - 900) \implies x_n - 900 = 0.9^n(x_0 - 900) \implies x_n = 0.9^n(x_0 - 900) + 900.$$

Part (c). As $n \to \infty$, $0.9^n \to 0$. Hence, the amount of water in the tank will converge to 900 litres.

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Problem 5. A deposit of $100,000 is made to an investment fund at the beginning of a year. On the last day of each year, two dividends are awarded and reinvested into the fund. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

(a) Find a recurrence relation $\{P_n\}$ where P_n is the amount at the start of the nth year if no money is ever withdrawn.

(b) How much is in the account after n years if no money is ever withdrawn?

Solution.

Part (a).

$$P_{n+2} = P_{n+1} + 0.2P_{n+1} + 0.45P_n = 1.2P_{n+1} + 0.45P_n.$$

Part (b). Note that the characteristic equation of P_n, $x^2 - 1.2x - 0.45 = 0$, has roots $-\frac{3}{10}$ and $\frac{3}{2}$. Thus,

$$P_n = A\left(-\frac{3}{10}\right)^n + B\left(\frac{3}{2}\right)^n.$$

From $P_0 = 0$ and $P_1 = 100000$, we have $A + B = 0$ and $-\frac{3}{10}A + \frac{3}{2}B = 100000$. Solving, we have $A = -\frac{500000}{9}$ and $B = \frac{500000}{9}$. Thus,

$$P_n = \frac{500000}{9}\left[\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right].$$

Hence, there will be $\$\left\{\frac{500000}{9}\left[\left(\frac{3}{2}\right)^n - \left(-\frac{3}{10}\right)^n\right]\right\}$ in the account after n years if no money is ever withdrawn

Problem 6. A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbit produces another pair each month.

(a) Find a recurrence relation $\{f_n\}$ where f_n is the total number of pairs of rabbits, assuming that no rabbits ever die.

(b) What is the number of pairs of rabbits at the end of the nth month, assuming that no rabbits ever die?

Solution.

Part (a). $f_{n+2} = f_{n+1} + f_n$, $n \geq 2$, $f_0 = 0, f_1 = 1$

Part (b). Consider the characteristic equation of f_n, $x^2 - x - 1 = 0$. By the quadratic formula, the roots of the characteristic equation are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$. Hence

$$f_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

From $f_0 = 0$, we get $A+B = 0$. From $f_1 = 1$, we get $A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right) = 1$. Solving, we get $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$. Hence,

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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Problem 7. For $n \in \{2^j : j \in \mathbb{Z}, j \geq 1\}$, it is given that $T_n = 3T_{n/2} + 17$, where $T_1 = 4$. By considering the substitution $n = 2^i$ and another suitable substitution, show that the recurrence relation can be expressed in the form

$$t_i = 3t_{i-1} + 17, \qquad i \in \mathbb{Z}^+$$

Hence, find an expression for T_n in terms of n.

Solution. Let $n = 2^i \iff i = \log_2 n$. The given recurrence relation transforms to

$$T_{2^i} = 3T_{2^{i-1}} + 17, T_{2^0} = 4.$$

Let $t_i = T_{2i}$. Then

$$t_i = 3t_{i-1} + 17, t_0 = 4.$$

Let k be a constant such that $t_i + k = 3(t_{i-1} + k)$. Then $k = \frac{17}{2}$. We thus obtain a formula for t_i:

$$t_i + \frac{17}{2} = 3\left(t_{i-1} + \frac{17}{2}\right) \implies t_i + \frac{17}{2} = 3^i\left(t_0 + \frac{17}{2}\right) \implies t_i = \frac{25}{2}\cdot 3^i - \frac{17}{2}.$$

Thus,

$$T_{2i} = \frac{25}{2}\cdot 3^i - \frac{17}{2} \implies T_n = \frac{25}{2}\cdot 3^{\log_2 n} - \frac{17}{2}.$$

Problem 8. Consider the sequence $\{a_n\}$ given by the recurrence relation

$$a_{n+1} = 2a_n + 5^n, \qquad n \geq 1$$

(a) Given that $a_n = k\left(5^n\right)$ satisfies the recurrent relation, find the value of the constant k.

(b) Hence, by considering the sequence $\{b_n\}$ where $b_n = a_n - k(5^n)$, find the particular solution to the recurrence relation for which $a_1 = 2$.

Solution.

Part (a).

$$a_{n+1} = 2a_n + 5^n \implies k\left(5^{n+1}\right) = 2 \cdot k\left(5^n\right) + 5^n \implies 5k = 2k + 1 \implies k = \frac{1}{3}.$$

Part (b).

$$b_n = a_n - \frac{5^n}{3} = \left(2a_{n-1} - 5^{n-1}\right) - \frac{5^n}{3} = 2a_{n-1} - \frac{2}{3} \cdot 5^{n-1} = 2\left(a_{n-1} - \frac{5^{n-1}}{3}\right) = 2b_{n-1}.$$

Hence, $b_n = b_1 \cdot 2^{n-1}$. Note that $b_1 = a_1 - \frac{5}{3} = \frac{1}{3}$. Thus, $b_n = \frac{2^{n-1}}{3}$, which gives

$$b_n = a_n - \frac{5^n}{3} = \frac{2^{n-1}}{3} \implies a_n = \frac{2^n + 2 \cdot 5^n}{6}.$$

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Problem 9. The sequence $\{X_n\}$ is given by

$$\sqrt{X_{n+2}} = \frac{X_{n+1}}{X_n^2}, \qquad n \geq 1.$$

By applying the natural logarithm to the recurrence relation, use a suitable substitution to find the general solution of the sequence, expressing your answer in trigonometric form.

Solution. Taking the natural logarithm of the recurrence relation and simplifying, we get

$$\ln X_{n+2} = 2\ln X_{n+1} - 4\ln X_n.$$

Let $L_n = \ln X_n \iff X_n = \exp(L_n)$. Then,

$$L_{n+2} = 2L_{n+1} - 4L_n.$$

Consider the characteristic equation of L_n, $x^2 - 2x + 4 = 0$. By the quadratic formula, this has roots $1 \pm \sqrt{3}i = 2\exp\left(\pm\frac{i\pi}{3}\right)$. Thus, we can express L_n as

$$L_n = A \cdot 2^n \cos\frac{n\pi}{3} + B \cdot 2^n \sin\frac{n\pi}{3} = 2^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right).$$

Thus, X_n has the general solution

$$X_n = \exp\left(2^n\left(A\cos\frac{n\pi}{3} + B\sin\frac{n\pi}{3}\right)\right).$$

Problem 10. The sequence $\{X_n\}$ is given by $X_1 = 2$, $X_2 = 15$ and

$$X_{n+2} = 5\left(1 + \frac{1}{n+2}\right) X_{n+1} - 6\left(1 + \frac{2}{n+1}\right) X_n, \qquad n \geq 1.$$

By dividing the recurrence relation throughout by $n+3$, use a suitable substitution to determine X_n as a function of n.

Solution. Dividing the recurrence relation by $n+3$, we obtain

$$\frac{X_{n+2}}{n+3} = 5\left(\frac{1}{n+3} + \frac{1}{(n+2)(n+3)}\right) X_{n+1} - 6\left(\frac{1}{n+3} + \frac{2}{(n+1)(n+3)}\right) X_n.$$

Note that $\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$ and $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$. Thus,

$$\frac{X_{n+2}}{n+3} = 5\left(\frac{X_{n+1}}{n+2}\right) - 6\left(\frac{X_n}{n+1}\right).$$

Let $Y_n = \frac{n+1}{X_n} \iff X_n = (n+1)Y_n$. Then,

$$Y_{n+2} = 5Y_{n+1} - 6Y_n.$$

Note that the characteristic equation of Y_n, $x^2 - 5x + 6 = 0$, has roots 2 and 3. Hence,

$$Y_n = A \cdot 2^n + B \cdot 3^n \implies X_n = (n+1)\left(A \cdot 2^n + B \cdot 3^n\right).$$

From $X_1 = 2$ and $X_2 = 15$, we have $2A + 3B = 1$ and $4A + 9B = 5$. Solving, we obtain $A = -1$ and $B = 1$. Thus,

$$X_n = (n+1)\left(3^n - 2^n\right).$$

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Problem 11. A logistics company set up an online platform providing delivery services to users on a monthly paid subscription basis. The company's sales manager models the number of subscribers that the company has at the end of each month. She notes that approximately 10% of the existing subscribers leave each month, and that there will be a constant number k of new subscribers in each subsequent month after the first.

Let T_n, $n \geq 1$, denote the number of subscribers the company has at the end of the nth month after the online platform was set up.

(a) Express T_{n+1} in terms of T_n.

The company has 250 subscribers at the end of the first month.

(b) Find T_n in terms of n and k.

(c) Find the least number of subscribers the company needs to attract in each subsequent month after the first if it aims to have at least 350 subscribers by the end of the 12th month.

Let $k = 50$ for the rest of the question.

The monthly running cost of the company is assumed to be fixed at \$4,000. The monthly subscription fee is \$10 per user which is charged at the end of each month.

(d) Given that the mth month is the first month in which the company's revenue up to and including that month is able to cover its cost up to and including that month, find the value of m.

(e) Using your answer to part (b), determine the long-term behaviour of the number of subscribers that the company has. Hence, explain whether this behaviour is appropriate in terms of long-term prospects for the company's success.

Solution.

Part (a). $T_{n+1} = 0.9T_n + k$

Part (b). Let m be a constant such that $T_{n+1} + m = 0.9\,(T_n + m)$. Then $m = -10k$. Hence,

$$T_{n+1} - 10k = 0.9\,(T_n - 10k) \implies T_n - 10k = 0.9^{n-1}\,(T_0 - 10k)\,.$$

Since $T_0 = 250$, we get

$$T_n = 0.9^{n-1}\,(250 - 10k) + 10k.$$

Part (c). Consider $T_{12} \geq 350$.

$$T_{12} \geq 350 \implies 0.9^{12-1}\,(250 - 10k) + 10k \geq 350.$$

Using G.C., $k \geq 39.6$. Hence, the company needs to attract at least 40 subscribers in each subsequent month.

Part (d). Since $k = 50$, $T_n = -250 \cdot 0.9^{n-1} + 500$. Let $\$S_m$ be the total revenue for the first m months.

$$\begin{aligned} S_m = 10\sum_{n=1}^{m} T_n &= 10\sum_{n=1}^{m}\left(-250 \cdot 0.9^{n-1} + 500\right) \\ &= 10\left[-250\left(\frac{1-0.9^m}{1-0.9}\right) + 500m\right] = 25000\,(0.9^m - 1) + 5000m. \end{aligned}$$

Note that the total cost for the first m months is $\$4000m$. Hence, the total profit for the first m months is given by $\$(S_m - 4000m)$. Hence, we consider $S_m - 4000m \geq 0$:

$$S_m - 4000m \geq 0 \implies 25000\,(0.9^m - 1) + 1000m \geq 0.$$

Using G.C., we obtain $m \geq 22.7$, whence the least value of m is 23.

Part (e). As $n \to \infty$, $0.9^{n-1} \to 0$. Hence, $T_n \to 500$. Hence, as n becomes very large, the profit per month approaches $500 \cdot 10 - 4000 = 1000$ dollars. Thus, this behaviour is appropriate as the business will remain profitable in the long run.

Assignment A5

Problem 1. In an auction at a charity gala dinner, a group of wealthy businessmen are competing with each other to be the highest bidder. Each time one of them makes a bid amount, another counter-bids by 50% more, less a service charge of ten dollars (e.g. If A bids \$1000, then B will bid \$1490). Let u_n be the amount at the nth bid and u_1 be the initial amount.

(a) Write down a recurrence relation that describes the bidding process.

(b) Show that $u_n = \$(1.5^{n-1}(u_1 - 20) + 20)$.

(c) The target amount to be raised is \$1 234 567 and the bidding stops when the bid amount meets or crosses this target amount. Given that $u_1 = 111$,

  (i) state the least number of bids required to meet this amount.

  (ii) find the winning bid amount, correct to the nearest thousand dollars.

Solution.

Part (a). $u_{n+1} = 1.5u_n - 10$.

Part (b). Let k be the constant such that $u_{n+1} + k = 1.5(u_n + k)$. Then $k = -20$. Hence, $u_{n+1} - 20 = 1.5(u_n - 20)$.

$$u_{n+1} - 20 = 1.5(u_n - 20) \implies u_n - 20 = 1.5^{n-1}(u_1 - 20) \implies u_n = 1.5^{n-1}(u_1 - 20) + 20.$$

Part (c).

Part (c)(i). Let m be the least integer such that $u_m \geq 1234567$. Consider $u_m \geq 1234567$:

$$u_m \geq 1234567 \implies 1.5^{m-1}(111 - 20) + 20 \geq 1234567.$$

Using G.C., $m \geq 24.5$. Hence, it takes at least 25 bids to meet this amount.

Part (c)(ii). Since $u_{25} = 1.5^{25-1}(111 - 20) = 1532000$ (to the nearest thousand), the winning bid is \$1 532 000.

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Problem 2. Solve these recurrence relations together with the initial conditions.

(a) $u_{n+2} = -u_n + 2u_{n+1}$, for $n \geq 0$, $u_0 = 5$, $u_1 = -1$.

(b) $4u_n = 4u_{n-1} + u_{n-2}$, for $n \geq 2$, $u_0 = a$, $u_1 = b$, $a, b \in \mathbb{R}$.

Solution.

Part (a). Observe that the characteristic equation of u_n, $x^2 - 2x + 1 = 0$, has only one root, namely $x = 1$. Thus,

$$u_n = (A + Bn) \cdot 1^n = A + Bn.$$

Thus, u_n is in AP. Since $u_0 = 5$ and $u_1 = -1$, it follows that

$$u_n = 5 - 6n.$$

Part (b). Rewriting the given recurrence relation, we have $u_n = u_{n-1} + \frac{1}{4}u_{n-2}$. Thus, the characteristic equation is $x^2 - x - \frac{1}{4} = 0$, which has roots $\frac{1}{2}(1 \pm \sqrt{2})$. Thus,

$$u_n = A\left(\frac{1+\sqrt{2}}{2}\right)^n + B\left(\frac{1-\sqrt{2}}{2}\right)^n.$$

Since $u_0 = a$, we obviously have $A+B = a$. Since $u_1 = b$, we get $A\left(\frac{1+\sqrt{2}}{2}\right)+B\left(\frac{1-\sqrt{2}}{2}\right) = b$. Solving, we get

$$A = \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b, \quad B = \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b.$$

Thus,

$$u_n = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b\right)\left(\frac{1+\sqrt{2}}{2}\right)^n + \left(\frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b\right)\left(\frac{1-\sqrt{2}}{2}\right)^n.$$

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Problem 3. A passcode is generated using the digits 1 to 5, with repetitions allowed. The passcodes are classified into two types. A Type A passcode has an even number of the digit 1, while a Type B passcode has an odd number of the digit 1. For example, a Type A passcode is 1231, and a Type B passcode is 1541213. Let a_n and b_n denote the number of n-digit Type A and Type B passcodes respectively.

(a) State the values of a_1 and a_2.

(b) By considering the relationship between a_n and b_n, show that

$$a_n = xa_{n-1} + y^{n-1}, \qquad n \geq 2$$

where x and y are constants to be determined.

(c) Using the substitution $c_n = za_n + y^n$, where z is a constant to be determined, find a first order linear recurrence relation for c_n. Hence, find the general term formula for a_n.

Solution.

Part (a). $a_1 = 4$, $a_2 = 17$.

Part (b). Let P be an n-digit passcode with Type T, where T is either A or B. Let Type T' be the other type.

By concatenating a digit from 1 to 5 to P, five $(n+1)$-digit passcodes can be created. Let P' denote a new passcode that is created via this process. If the digit 1 is concatenated, then P' is of Type T'. If the digit 1 is not concatenated, then P' is of Type T. There are 4 choices for such a case. This hence gives the recurrence relations

$$\begin{cases} a_n = 4a_{n-1} + b_{n-1} \\ b_n = 4b_{n-1} + a_{n-1} \end{cases}$$

Adding the two equations, we see that $a_n + b_n = 5(a_{n-1} + b_{n-1})$. Thus,

$$a_n + b_n = 5^{n-1}(a_1 + b_1) = 5^{n-1}(4+1) = 5^n.$$

Hence,

$$a_n = 4a_{n-1} + b_{n-1} = 3a_{n-1} + a_{n-1} + b_{n-1} = 3a_{n-1} + 5^{n-1},$$

whence $x = 3$ and $y = 5$.

Part (c). Observe that

$$\begin{aligned} c_n &= za_n + 5^n = z\left(3a_{n-1} + 5^{n-1}\right) + 5^n = 3\left(za_{n-1} + 5^{n-1}\right) + (2+z)5^{n-1} \\ &= 3c_{n-1} + (2+z)5^{n-1}. \end{aligned}$$

Let $z = -2$. Then,

$$c_n = 3c_{n-1} = 3^{n-1}c_1 = 3^{n-1}\left(-2a_1 + 5\right) = -3^n.$$

Note that $a_n = \frac{1}{z}\left(c_n - y^n\right)$. Thus,

$$a_n = \frac{-3^n - 5^n}{-2} = \frac{3^n + 5^n}{2}.$$

A6. Polar Coordinates

Tutorial A6

Problem 1.

(a) Find the rectangular coordinates of the following points.

(i) $(3, -\frac{\pi}{4})$

(ii) $(1, \pi)$

(iii) $(\frac{1}{2}, \frac{3}{2}\pi)$

(b) Find the polar coordinates of the following points.

(i) $(3, 3)$

(ii) $(-1, -\sqrt{3})$

(iii) $(2, 0)$

(iv) $(4, 2)$

Solution.

Part (a).

Part (a)(i). Note that $r = 3$ and $\theta = -\frac{\pi}{4}$. This gives

$$x = r\cos\theta = \frac{3}{\sqrt{2}}, \quad y = r\sin\theta = -\frac{3}{\sqrt{2}}.$$

Hence, the rectangular coordinate of the point is $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$.

Part (a)(ii). Note that $r = 1$ and $\theta = \pi$. This gives

$$x = r\cos\theta = -1, \quad y = r\sin\theta = 0.$$

Hence, the rectangular coordinate of the point is $(-1, 0)$.

Part (a)(iii). Note that $r = \frac{1}{2}$ and $\theta = \frac{3}{2}\pi$. This gives

$$x = \rho\cos\theta = 0, \quad y = r\sin\theta = -\frac{1}{2}.$$

Hence, the rectangular coordinate of the point is $(0, -\frac{1}{2})$.

Part (b).

Part (b)(i). Note that $x = 3$ and $y = -3$. This gives

$$r^2 = x^2 + y^2 \implies r = 3\sqrt{2}, \quad \tan\theta = \frac{y}{x} \implies \theta = -\frac{\pi}{4}.$$

Hence, the polar coordinate of the point is $(3\sqrt{2}, -\frac{\pi}{4})$.

Part (b)(ii). Note that $x = -1$ and $y = -\sqrt{3}$. This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan\theta = \frac{y}{x} \implies \theta = \frac{\pi}{3}.$$

Hence, the polar coordinate of the point is $(2, \frac{\pi}{3})$.

Part (b)(iii). Note that $x = 2$ and $y = 0$. This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan\theta = \frac{y}{x} \implies \theta = 0.$$

Hence, the polar coordinate of the point is $(2, 0)$.

Part (b)(iv). Note that $x = 4$ and $y = 2$. This gives

$$r^2 = x^2 + y^2 \implies r = 2\sqrt{5}, \quad \tan\theta = \frac{y}{x} \implies \theta = \arctan\frac{1}{2}.$$

Hence, the polar coordinate of the point is $(2\sqrt{5}, \arctan\frac{1}{2})$.

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Problem 2. Rewrite the following equations in polar form.

(a) $2x^2 + 3y^2 = 4$

(b) $y = 2x^2$

Solution.

Part (a).

$$2x^2 + 3y^2 = 2(r\cos\theta)^2 + 3(r\sin\theta)^2 = 4 \implies r^2 = \frac{4}{2\cos^2\theta + 3\sin^2\theta} = \frac{4}{2 + \sin^2\theta}.$$

Part (b).

$$y = 2x^2 \implies \frac{y}{x} = 2x \implies \tan\theta = 2r\cos\theta \implies r = \frac{1}{2}\tan\theta\sec\theta.$$

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Problem 3. Rewrite the following equations in rectangular form.

(a) $r = \frac{1}{1-2\cos\theta}$

(b) $r = \sin\theta$

Solution.

Part (a).

$$r = \frac{1}{1 - 2\cos\theta} \implies r - 2r\cos\theta = 1 \implies r = 2x + 1 \implies r^2 = 4x^2 + 4x + 1$$
$$\implies x^2 + y^2 = 4x^2 + 4x + 1 \implies y^2 = 3x^2 + 4x + 1.$$

Part (b).

$$r = \sin\theta \implies r^2 = r\sin\theta \implies x^2 + y^2 = y.$$

Problem 4.

(a) Show that the curve with polar equation $r = 3a\cos\theta$, where a is a positive constant, is a circle. Write down its centre and radius.

(b) By finding the Cartesian equation, sketch the curve whose polar equation is $r = a\sec\left(\theta - \frac{\pi}{4}\right)$, where a is a positive constant.

Solution.

Part (a).

$$r = 3a\cos\theta \implies r^2 = 3ar\cos\theta \implies x^2 + y^2 = 3ax \implies x^2 - 3ax + y^2 = 0.$$

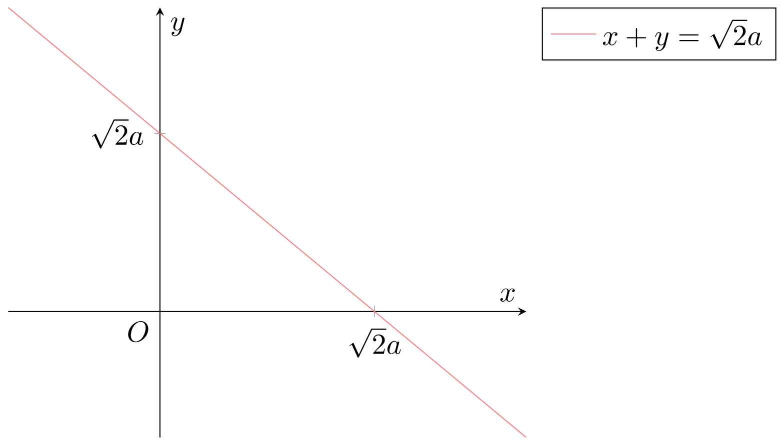
Completing the square, we get

$$\left(x - \frac{3a}{2}\right)^2 + y^2 \left(\frac{3a}{2}\right)^2.$$

Thus, the circle has centre $(\frac{3a}{2}, 0)$ and radius $\frac{3a}{2}$.

Part (b).

$$r = a\sec\left(\theta - \frac{\pi}{4}\right) \implies r\cos\left(\theta - \frac{\pi}{4}\right) = a \implies r\left(\cos\theta + \sin\theta\right) = \sqrt{2}a \implies x + y = \sqrt{2}a.$$



Problem 5. Sketch the following polar curves, where r is non-negative and $0 \leq \theta \leq 2\pi$.

(a) $r = 2$

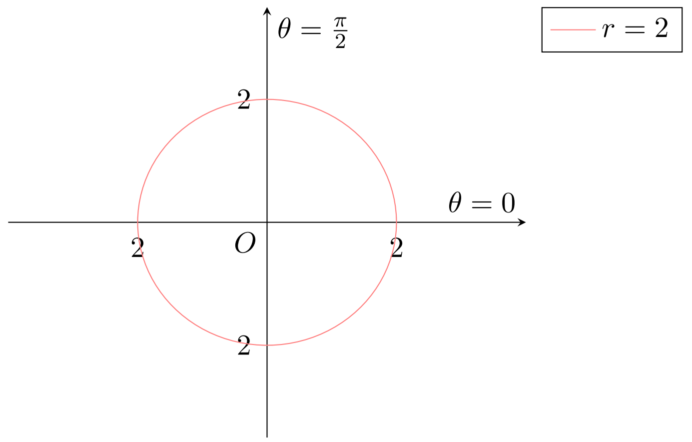
(b) $\theta = \frac{\pi}{4}$

(c) $r = \frac{1}{2}\theta$

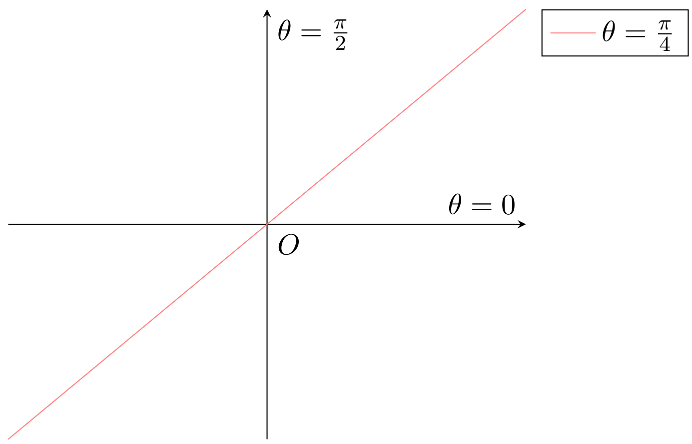
(d) $r = 2\csc\theta$

Solution.

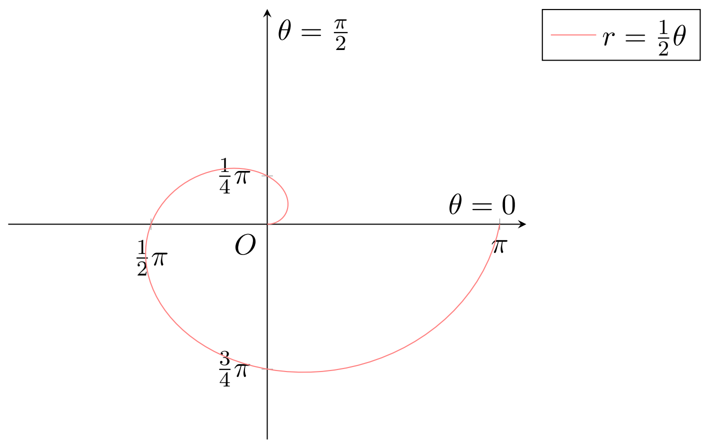
Part (a).



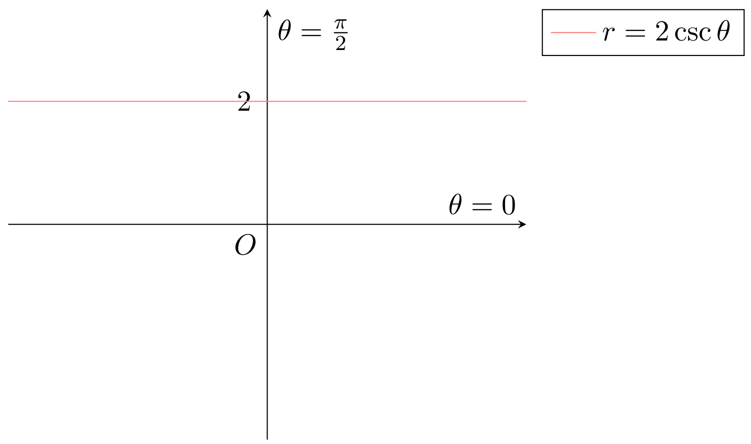
Part (b).



Part (c).



Part (d).



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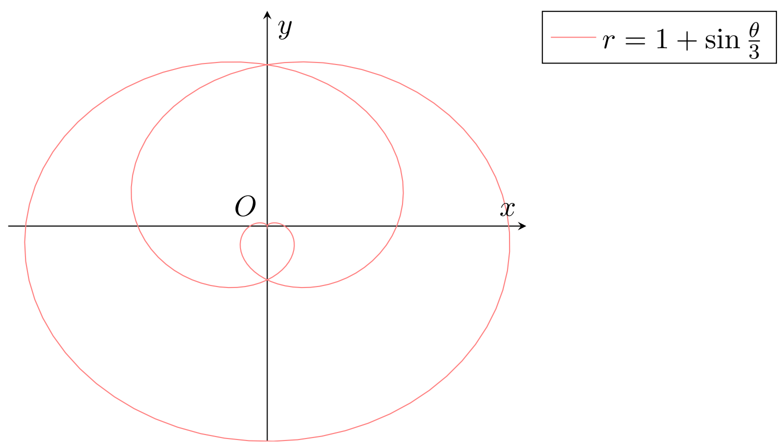
Problem 6. A sketch of the curve $r = 1 + \sin\frac{\theta}{3}$ is shown. Copy the diagram and indicate the x- and y-intercepts.



Solution. Observe that the curve is symmetric about the y-axis. Also observe that $\frac{\theta}{3} \in [0, 2\pi)$, hence we take $\theta \in [0, 6\pi)$.

For x-intercepts, $y = r\sin\theta = 0 \implies \theta = n\pi$, where $n \in \mathbb{Z}$. Due to the symmetry of the curve, we consider only $n = 0, 2, 4$.

Case 1. $n = 0 \implies r = 1 + \sin\frac{0}{3}\pi = 1$.

Case 2. $n = 2 \implies r = 1 + \sin\frac{2}{3}\pi = 1 + \frac{\sqrt{3}}{2}$.

Case 3. $n = 4 \implies r = 1 + \sin\frac{4}{3}\pi = 1 - \frac{\sqrt{3}}{2}$.

Hence, the curve intersects the x-axis at $x = 1, 1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}$. Correspondingly, the curve also intersects the x-axis at $x = -1, -1 - \frac{\sqrt{3}}{2}, -1 + \frac{\sqrt{3}}{2}$.

For y-intercepts, $x = r\cos\theta = 0 \implies \theta = (n + \frac{1}{2})\pi$, where $n \in \mathbb{Z}$. Due to the restriction on θ, we consider $n \in [0, 5)$.

Case 1. $n = 0, r = 1 + \sin\frac{1/2}{3}\pi = \frac{3}{2}$.

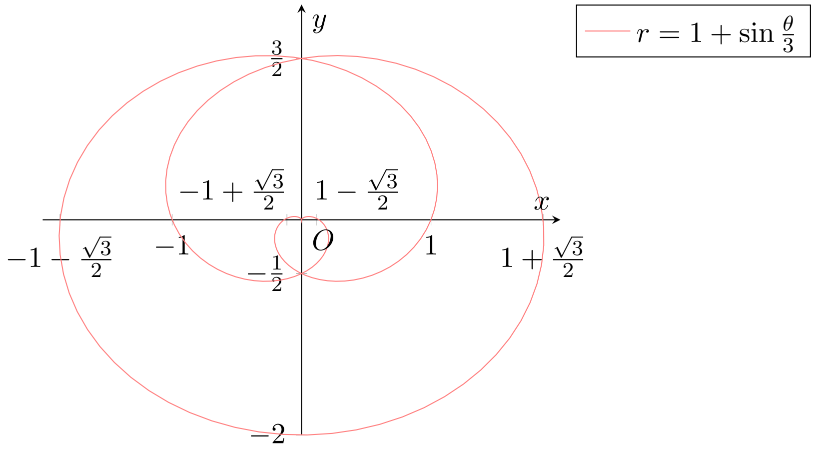
Case 2. $n = 1, r = 1 + \sin\frac{3/2}{3}\pi = 2$.

Case 3. $n = 2, r = 1 + \sin\frac{5/2}{3}\pi = \frac{3}{2}$.

Case 4. $n = 3, r = 1 + \sin\frac{7/2}{3}\pi = \frac{1}{2}$.

Case 5. $n = 4, r = 1 + \sin\frac{9/2}{3}\pi = 0$.

Hence, the curve intersects the y-axis at $y = -2, -\frac{1}{2}, \frac{3}{2}$.



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Problem 7.

(a) A graph has polar equation $r = \frac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha}$, where α is a constant. Express the equation in Cartesian form. Hence, sketch the graph in the case $\alpha = \frac{\pi}{4}$, giving the Cartesian coordinates of the intersection with the axes.

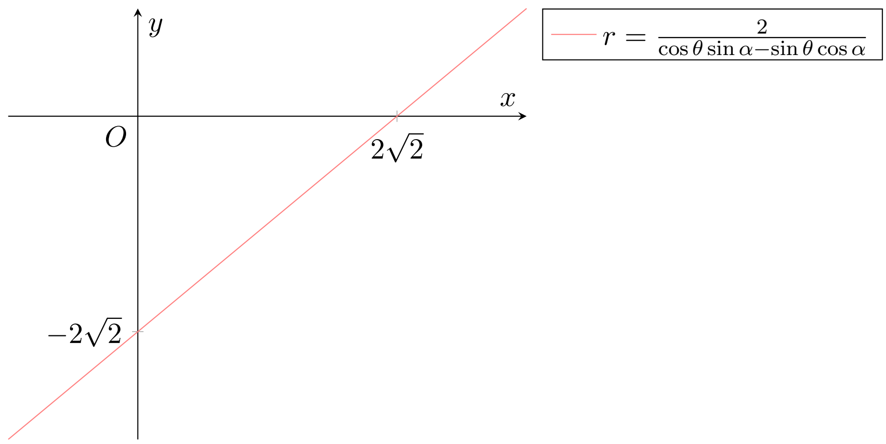
(b) A graph has Cartesian equation $(x^2 + y^2)^2 = 4x^2$. Express the equation in polar form. Hence, or otherwise, sketch the graph.

Solution.

Part (a).

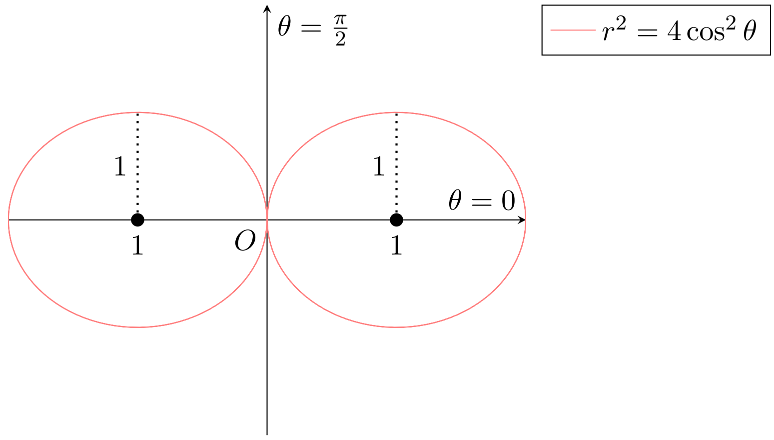
$$\begin{aligned} r = \frac{2}{\cos\theta\sin\alpha - \sin\theta\cos\alpha} &\implies r\cos\theta\sin\alpha - r\sin\theta\cos\alpha = x\sin\alpha - y\cos\alpha = 2 \\ &\implies y = x\tan\alpha - 2\sec\alpha. \end{aligned}$$

When $\alpha = \frac{\pi}{4}$, we have $y = x - 2\sqrt{2}$.



Part (b).

$$(x^2+y^2)^2 = 4x^2 \implies \left(r^2\right)^2 = 4(r\cos\theta)^2 \implies r^4 = 4r^2\cos^2\theta \implies r^2 = 4\cos^2\theta.$$



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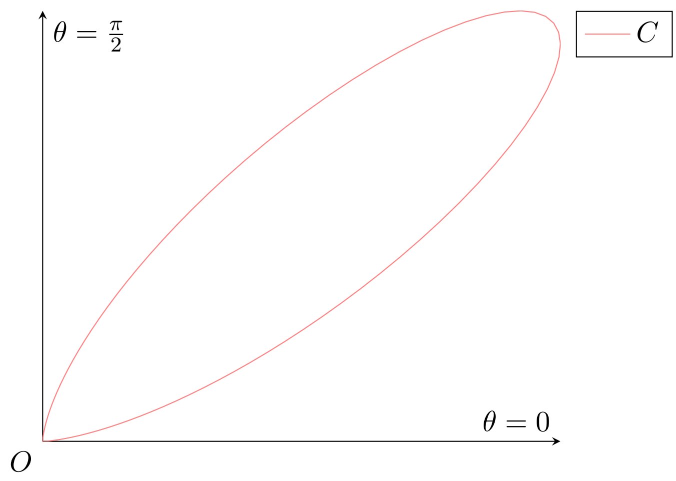
Problem 8. Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \leq \theta \leq \frac{\pi}{2}$.

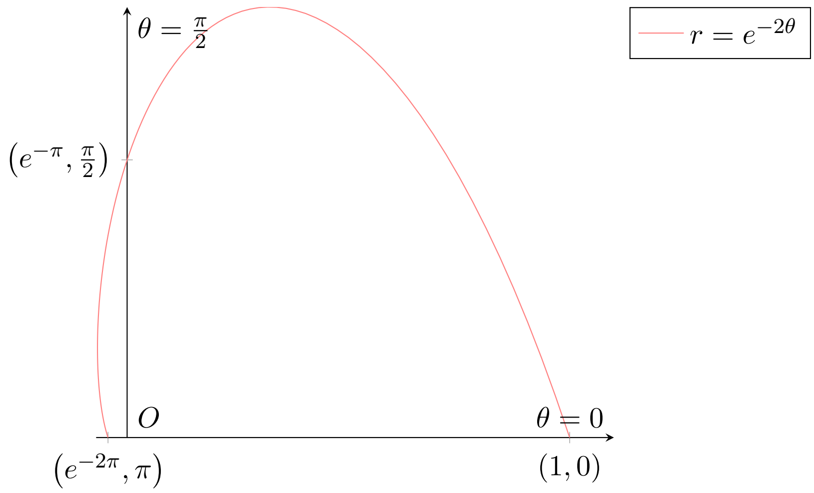
Solution.

$$\begin{aligned} x^5 + y^5 &= 5bx^2y^2 \\ \implies (r\cos\theta)^5 + (r\sin\theta)^5 &= 5b(r\cos\theta)^2(r\sin\theta)^2 \\ \implies r\left(\cos^5\theta + \sin^5\theta\right) &= 5b\cos^2\theta\sin^2\theta \\ \implies r &= \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta} \end{aligned}$$



Problem 9. The equation of a curve, in polar coordinates, is $r = e^{-2\theta}$, for $0 \le \theta \le \pi$. Sketch the curve, indicating clearly the polar coordinates of any axial intercepts.

Solution.



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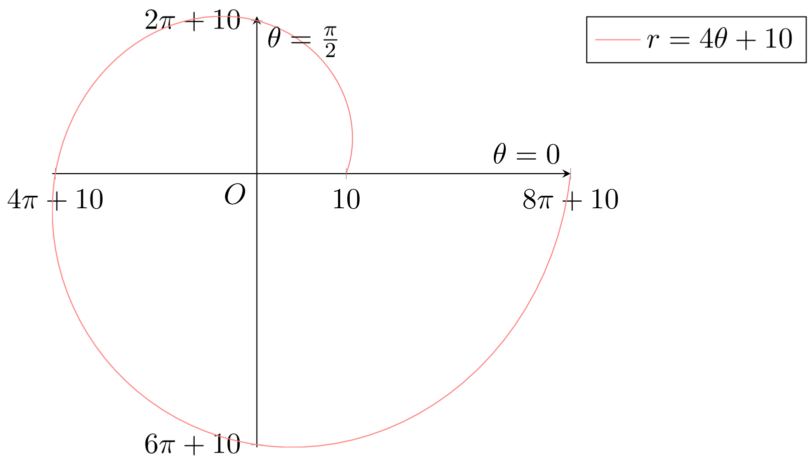
Problem 10. Suppose that a long thin rod with one end fixed at the pole of a polar coordinate system rotates counter-clockwise at the constant rate of 0.5 rad/sec. At time $t = 0$, a bug on the rod is 10 mm from the pole and is moving outward along the rod at a constant speed of 2 mm/sec. Find an equation of the form $r = f(\theta)$ for the part of motion of the bug, assuming that $\theta = 0$ when $t = 0$. Sketch the path of the bug on the polar coordinate system for $0 \le t \le 4\pi$.

Solution. Let $\theta(t)$ and $r(t)$ be functions of time, with $\theta(0) = 0$ and $r(0) = 10$. We know that $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5$ and $\frac{\mathrm{d}r}{\mathrm{d}t} = 2$. Hence,

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{\mathrm{d}r}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^{-1} = 2 \cdot (0.5)^{-1} = 4.$$

Thus, $r = 4\theta + r(0) = 4\theta + 10$.

Since $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0.5$ and $\theta(0) = 0$, we have $\theta = 0.5t$. Hence, $0 \le t \le 4\pi \implies 0 \le \theta \le 2\pi$.



Problem 11. The equation, in polar coordinates, of a curve C is $r = ae^{\frac{1}{2}\theta}$, $0 \le \theta \le 2\pi$, where a is a positive constant. Write down, in terms of θ, the Cartesian coordinates, x and y, of a general point P on the curve. Show that the gradient at P is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\tan\theta+2}{1-2\tan\theta}$.

Hence, show that the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$. Sketch the curve C.

Solution. Note that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = ae^{\frac{1}{2}\theta}\cos\theta$ and $y = ae^{\frac{1}{2}\theta}\sin\theta$. Hence, $P\left(ae^{\frac{1}{2}\theta}\cos\theta, ae^{\frac{1}{2}\theta}\sin\theta\right)$.

Observe that $\frac{\mathrm{d}r}{\mathrm{d}\theta} = \frac{1}{2}ae^{\frac{1}{2}\theta} = \frac{1}{2}r$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}r}{\mathrm{d}\theta}\sin\theta + r\cos\theta}{\frac{\mathrm{d}r}{\mathrm{d}\theta}\cos\theta - r\sin\theta} = \frac{\frac{1}{2}r\sin\theta + r\cos\theta}{\frac{1}{2}r\cos\theta - r\sin\theta} = \frac{\sin\theta + 2\cos\theta}{\cos\theta - 2\sin\theta} = \frac{\tan\theta+2}{1-2\tan\theta}.$$

Let $\mathbf{t} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ represent the direction of the tangent line. Then

$$\mathbf{t} = \begin{pmatrix} 1 \\ \mathrm{d}y/\mathrm{d}x \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\tan\theta+2}{1-2\tan\theta} \end{pmatrix} = \frac{1}{1-2\tan\theta}\begin{pmatrix} 1-2\tan\theta \\ \tan\theta+2 \end{pmatrix}$$

and

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ae^{\frac{1}{2}\theta}\cos\theta \\ ae^{\frac{1}{2}\theta}\sin\theta \end{pmatrix} = ae^{\frac{1}{2}\theta}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

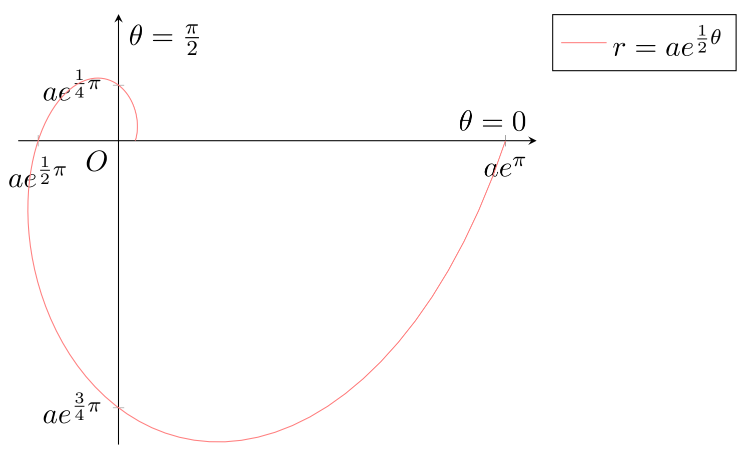
By the definition of the dot-product, we have $\mathbf{t}\cdot\overrightarrow{OP} = |\mathbf{t}||\overrightarrow{OP}|\cos\alpha$, whence

$$\begin{aligned}\cos\alpha = \frac{\mathbf{t}\cdot\overrightarrow{OP}}{|\mathbf{t}||\overrightarrow{OP}|} &= \frac{(1-2\tan\theta)\cos\theta + (\tan\theta+2)\sin\theta}{\sqrt{(1-2\tan\theta)^2 + (\tan\theta+2)^2}\cdot\sqrt{\cos^2\theta+\sin^2\theta}} \\ &= \frac{\cos\theta + \tan\theta\sin\theta}{\sqrt{5\tan^2\theta+5}} = \frac{\cos^2\theta+\sin^2\theta}{\sqrt{5\sin^2\theta+5\cos^2\theta}} = \frac{1}{\sqrt{5}}.\end{aligned}$$

Thus, $\alpha = \arccos\frac{1}{\sqrt{5}}$. Since $\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$,

$$\tan\alpha = \tan\left(\arccos\frac{1}{\sqrt{5}}\right) = \frac{\sqrt{1-\left(1/\sqrt{5}\right)^2}}{1/\sqrt{5}} = 2.$$

Hence, the tangent at P is inclined to $\overrightarrow{OP}$ at a constant angle α, where $\tan\alpha = 2$.



Problem 12. The polar equation of a curve is given by $r = e^\theta$ where $0 \le \theta \le \frac{\pi}{2}$. Cartesian axes are taken at the pole O. Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$.

Solution. Recall that $x = r\cos\theta$ and $y = r\sin\theta$, whence $x = e^\theta \cos\theta$ and $y = e^\theta \sin\theta$. Thus, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = e^\theta(\cos\theta - \sin\theta)$, and $\frac{\mathrm{d}y}{\mathrm{d}x} = e^\theta(\cos\theta + \sin\theta)$. Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{e^\theta(\cos\theta + \sin\theta)}{e^\theta(\cos\theta - \sin\theta)} = \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}.$$

At $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$, we clearly have $x = 0$ and $y = e^{\pi/2}$. Also, $\mathrm{d}y/\mathrm{d}x = -1$. By the point-slope formula, the equation of the tangent line at $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$ is given by $y = -x + e^{\frac{\pi}{2}}$.

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Problem 13. A curve C has polar equation $r = a\cot\theta$, $0 < \theta \le \pi$, where a is a positive constant.

(a) Show that $y = a$ is an asymptote of C.

(b) Find the tangent at the pole.

Hence, sketch C and find the Cartesian equation of C in the form $y^2(x^2 + y^2) = bx^2$, where b is a constant to be determined.

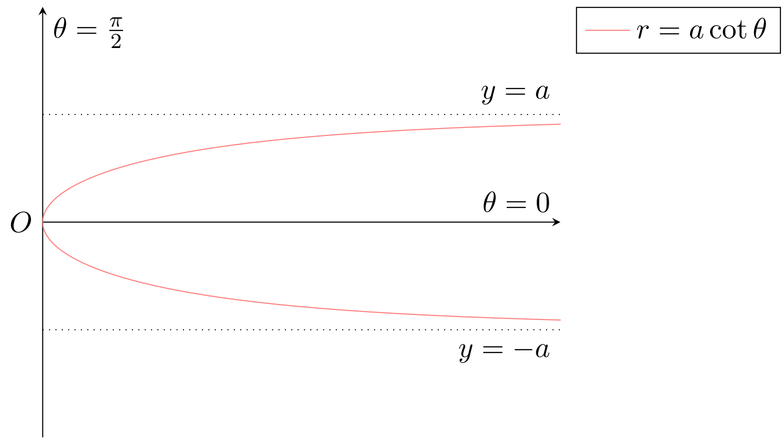
Solution.

Part (a). Note that

$$r = a\cot\theta \implies y = r\sin\theta = a\cos\theta.$$

As $\theta \to 0$, $r \to \infty$. Hence, there is an asymptote at $\theta = 0$. Since $\cos\theta = 1$ when $\theta = 0$, the line $y = a\cos\theta = a$ is an asymptote of C.

Part (b). For tangents at the pole, $r = 0 \implies \cot\theta = 0 \implies \theta = \frac{\pi}{2}$.



Note that

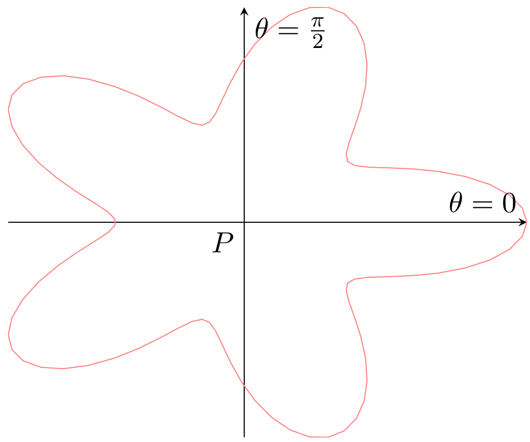
$$r = a\cot\theta = a\left(\frac{r\cos\theta}{r\sin\theta}\right) = a\left(\frac{x}{y}\right).$$

Thus,

$$x^2 + y^2 = r^2 = a^2\left(\frac{x^2}{y^2}\right) \implies y^2\left(x^2 + y^2\right) = a^2x^2,$$

whence $b = a^2$.

Problem 14.



Relative to the pole P and the initial line $\theta = 0$, the polar equation of the curve shown is either

i. $r = a + b \sin n\theta$, or

ii. $r = a + b \cos n\theta$

where a, b and n are positive constants. State, with a reason, whether the equation is (i) or (ii) and state the value of n.

The maximum value of r is $\frac{11}{2}$ and the minimum value of r is $\frac{5}{2}$. Find the values of a and b.

Solution. Since the curve is symmetrical about the horizontal half-line $\theta = 0$, the polar equation of the curve is a function of $\cos n\theta$ only. Hence, the polar equation of the curve is $r = a + b \cos n\theta$, with $n = 5$.

Observe that the maximum value of r is achieved when $\cos 5\theta = 1$, whence $r = a + b$. Thus, $a + b = \frac{11}{2}$. Also observe that the minimum value of r is achieved when $\cos 5\theta = -1$, whence $r = a - b$. Thus, $a - b = \frac{5}{2}$. Solving, we get $a = 4$ and $b = \frac{3}{2}$.

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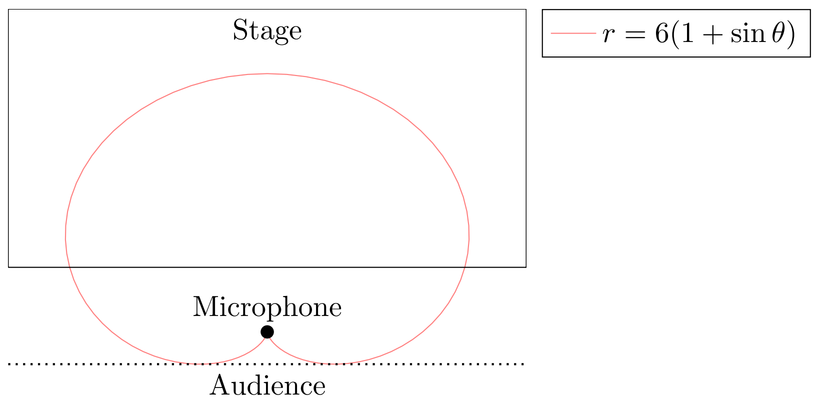
Problem 15.



Sound engineers often use a microphone with a cardioid acoustic pickup pattern to record live performances because it reduces pickup from the audience. Suppose a cardioid microphone is placed 3 metres from the front of the stage, and the boundary of the optimal pickup region is given by the cardioid with polar equation

$$r = 6(1 + \sin\theta)$$

where r is measured in metres and the microphone is at the pole.

Find the minimum distance from the front of the stage the first row of the audience can be seated such that the microphone does not pick up noise from the audience.

Solution. Note that $r = 6(1 + \sin\theta) = 6(1 + \frac{y}{r})$, whence $r^2 = 6r + 6y$. Thus,

$$r^2 - 6r - 6y = 0 \implies r = 3 \pm \sqrt{9 + 6y} \implies 9 + 6y = (r-3)^2.$$

Since $9 + 6y = (r-3)^2 \geq 0$, we have $y \geq -1.5$. Thus, the furthest distance the audience has to be from the stage is $|-1.5| + 3 = 4.5$ m.

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Problem 16. To design a flower pendant, a designer starts off with a curve C_1, given by the Cartesian equation

$$\left(x^2 + y^2\right)^2 = a^2\left(3x^2 - y^2\right)$$

where a is a positive constant.

(a) Show that a corresponding polar equation of C_1 is $r^2 = a^2(1 + 2\cos 2\theta)$.

(b) Find the equations of the tangents to C_1 at the pole.

Another curve C_2 is obtained by rotating C_1 anti-clockwise about the origin by $\frac{\pi}{3}$ radians.

(c) State a polar equation of C_2.

(d) Sketch C_1 and C_2 on the same diagram, stating clearly the exact polar coordinates of the points of intersection of the curves with the axes. Find also the exact polar coordinates of the points of intersection with C_1 and C_2.

The curve C_3 is obtained by reflecting C_2 in the line $\theta = \frac{\pi}{2}$.

(e) State a polar equation of C_3.

(f) The designer wishes to enclose the 3 curves inside a circle given by the polar equation $r = r_1$. State the minimum value of r_1 in terms of a.

Solution.

Part (a). Observe that $\left(x^2 + y^2\right)^2 = r^4$ and $3x^2 - y^2 = r^2\left(3\cos^2\theta - \sin^2\theta\right)$. Hence,

$$\left(x^2 + y^2\right)^2 = a^2\left(3x^2 - y^2\right) \implies r^2 = a^2\left(3\cos^2\theta - \sin^2\theta\right).$$

Note that

$$3\cos^2\theta - \sin^2\theta = 1 + 2\cos^2\theta - 2\sin^2\theta = 1 + 2\cos 2\theta.$$

Thus,

$$r^2 = a^2\left(1 + 2\cos 2\theta\right).$$

Part (b). For tangents at the pole,

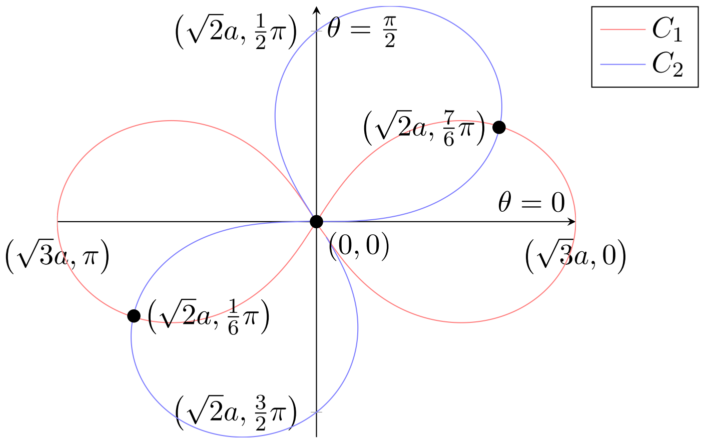
$$r = 0 \implies 1 + 2\cos 2\theta = 0 \implies \cos 2\theta = -\frac{1}{2}.$$

Note that $0 \leq 2\theta \leq 2\pi$. Hence, $2\theta = \frac{2}{3}\pi, \frac{4}{3}\pi$, whence $\theta = \frac{1}{3}\pi, \frac{2}{3}\pi$. For full lines, we also have $\theta = \frac{2}{3}\pi$ and $\theta = \frac{5}{3}\pi$.

Part (c).

$$r^2 = a^2\left[1 + 2\cos\left(2\left(\theta - \frac{\pi}{3}\right)\right)\right] = a^2\left[1 + 2\cos\left(2\theta - \frac{2}{3}\pi\right)\right].$$

Part (d).



Consider the horizontal intercepts of C_1. When $\theta = 0$, $r = \sqrt{3}a$. Hence, by symmetry, C_1 intercepts the horizontal axis at $\left(\sqrt{3}a, 0\right)$ and $\left(\sqrt{3}a, \pi\right)$.

Consider the vertical intercepts of C_2. When $\theta = \frac{\pi}{2}$, $r = \sqrt{2}a$. Hence, by symmetry, C_2 intercepts the vertical axis at $\left(\sqrt{2}a, \frac{\pi}{2}\right)$ and $\left(\sqrt{2}a, \frac{3}{2}\pi\right)$.

Now consider the intersections between C_1 and C_2. By symmetry, it is obvious that the points of intersections must lie along the half-lines $\frac{\pi}{6}$ and $\frac{7\pi}{6}$, or along the half-lines $\frac{4\pi}{6}$ and $\frac{10\pi}{6}$. By symmetry, we consider only the half-lines $\frac{\pi}{6}$ and $\frac{4\pi}{6}$.

Case 1: $\theta = \frac{\pi}{6}$. Substituting $\theta = \frac{\pi}{6}$ into the equation of C_1, we obtain $r = \sqrt{2}a$. Hence, C_1 and C_2 intersect at $\left(\sqrt{2}a, \frac{1}{6}\pi\right)$ and, by symmetry, at $\left(\sqrt{2}a, \frac{7}{6}\pi\right)$.

Case -NoValue-: *2.*$\theta = \frac{4\pi}{6}$ Substituting $\theta = \frac{4\pi}{6}$ into the equation of C_1, we obtain $r = 0$. Hence, C_1 and C_2 intersect at $(0, 0)$.

Part (e). Reflecting about the line $\theta = \frac{\pi}{2}$ is equivalent to applying the map $\theta \mapsto \theta + \frac{1}{3}\pi$ to C_1. Hence,

$$r^2 = a^2\left[1 + 2\cos\left(2\left(\theta + \frac{1}{3}\pi\right)\right)\right] = a^2\left[1 + 2\cos\left(2\theta + \frac{2}{3}\pi\right)\right].$$

Part (f). $r_1 = \sqrt{3}a$.

Assignment A6

A14. Discrete Random Variables

Tutorial A14A

Problem 1. Alfred and Bertie play a game, each starting with cash amounting to \$100. Two dice are thrown. If the total score if 5 or more, then Alfred pays $\$x$, where $0 < x \leq 8$, to Bertie. If the total score if 4 or less, then Bertie pays $\$(x+8)$ to Alfred.

(a) Show that the expectation of Alfred's cash after the first game is $\$\frac{1}{3}(304 - 2x)$.

(b) Find the expectation of Alfred's cash after six games.

(c) Find the value of x for the game to be fair.

(d) Given that $x = 3$, find the variance of Alfred's cash after the first game.

Solution.

Part (a). Note that

$$\mathrm{P}(\text{score} < 5) = \frac{3+2+1}{6^2} = \frac{1}{6} \implies \mathrm{P}(\text{score} \geq 5) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Let $\$a_n$ be the expectation of Alfred's cash after n games. Suppose Alfred and Bertie play one more game (i.e. $n+1$ total games). Then

$$a_{n+1} = \frac{5}{6}(a_n - x) + \frac{1}{6}(a_n + x + 8) = a_n + \frac{2}{3}(2 - x).$$

a_n is in AP with common difference $\frac{2}{3}(2-x)$ and is thus given by

$$a_n = a_0 + n\left[\frac{2}{3}(2-x)\right] = 100 + \frac{2n}{3}(2-x).$$

Hence, the expectation of Alfred's cash after the first game is

$$a_1 = 100 + \frac{2 \cdot 1}{3}(2-x) = \frac{1}{3}(304 - 2x).$$

Part (b). The expectation of Alfred's cash after six games is

$$a_6 = 100 + \frac{2 \cdot 6}{3}(2-x) = 108 - 4x.$$

Part (c). For the game to be fair, $a_0 = a_1 = a_2 = \cdots$, i.e. the common difference is 0. Hence, $x = 2$.

Part (d). Let the random variable X be Alfred's cash after one game. Since the payouts are unaffected by a_0, we take $a_0 = 0$. When $x = 3$, $\mathrm{E}(X) = -\frac{2}{3}$. Hence,

$$\mathrm{Var}(X) = \frac{5}{6}\left(3 - \frac{2}{3}\right)^2 + \frac{1}{6}\left(3 + 8 + \frac{2}{3}\right)^2 = \frac{245}{9}.$$

Tutorial A14B

Problem 1. In a computer game, a bug moves from left to right through a network of connected paths. The bug starts at S and, at each junction, randomly takes the left fork with probability p or the right fork with probability q, where $q = 1 - p$. The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A - I (see diagram).

(a) Show that the probability that the bug finishes its journey at D is $56p^5q^3$.

(b) Given that the probability that the bug finishes its journey at D is greater than the probability at any one of the other endpoints, find exactly the possible range of values of p.

In another version of the game, the probability that, at each junction, the bug takes the left fork is $0.9p$, the probability that the bug takes the right fork is $0.9q$ and the probability that the bug is swallowed up by a 'black hole' is 0.1.

(c) Find the probability that, in this version of the game, the bug reaches one of the endpoints A - I, without being swallowed up by a black hole.

Solution.

Part (a). Relabel each endpoint from A - I to 0 - 8. Let the random variable X be the end-point that the bug ends up at. Clearly, to reach endpoint i, the bug must take i right forks and $8 - i$ left forks. Hence, $X \sim \mathrm{B}(8, q)$ and the probability that the bug reaches endpoint 3 (i.e. endpoint D) is

$$\mathrm{P}(X = 3) = \binom{8}{3} q^3 (1-q)^{8-3} = 56p^5q^3.$$

Part (b). Since X follows a binomial distribution, it suffices to find the range of values of p that satisfy $\mathrm{P}(X = 2) < \mathrm{P}(X = 3) > \mathrm{P}(X = 4)$.

Case 1: $\mathrm{P}(X = 2) < \mathrm{P}(X = 3)$. Note that $\mathrm{P}(X = 2) = \binom{8}{2} q^2 (1-q)^{8-2} = 28p^6q^2$.

$$\mathrm{P}(X = 2) < \mathrm{P}(X = 3) \implies 28p^6q^2 < 56p^5q^3 \implies 28p < 56(1-p) \implies p < \frac{2}{3}.$$

Case 2: $\mathrm{P}(X = 3) > \mathrm{P}(X = 4)$. Note that $\mathrm{P}(X = 4) = \binom{8}{4} q^4 (1-q)^{8-4} = 70p^4q^4$.

$$\mathrm{P}(X = 3) > \mathrm{P}(X = 4) \implies 56p^5q^3 > 70p^4q^4 \implies 56p > 70(1-p) \implies p > \frac{5}{9}.$$

Hence, $\frac{5}{9} < p < \frac{2}{3}$.

Part (c). Note that the bug most take a total of 8 forks. Since the probability of not getting swallowed by a black hole at each fork is 0.9, the desired probability is clearly $0.9^8 = 0.430$ (3 s.f.).

Part II.

Group B

Part III.

Examinations