

Problem 1.

For each of the following, write down a vector equivalent of the line l and convert it to parametric and Cartesian forms.

- (a) l passes through the point with position vector $-\mathbf{i} + \mathbf{k}$ and is parallel to the vector $\mathbf{i} + \mathbf{j}$.
- (b) l passes through the points $P(1, -1, 3)$ and $Q(2, 1, -2)$.
- (c) l passes through the origin and is parallel to the line $m : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$.
- (d) l is the x -axis.
- (e) l passes through the point $C(4, -1, 2)$ and is parallel to the z -axis.

Solution**Part (a)**

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda - 1 \\ y = \lambda \\ z = 1 \end{cases}$
Cartesian	$x + 1 = y, z = 1$

Part (b)

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda + 1 \\ y = 2\lambda - 1 \\ z = -5\lambda + 3 \end{cases}$
Cartesian	$x - 1 = \frac{y + 1}{2} = \frac{3 - z}{5}$

Part (c)

Form	Expression
Vector	$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda \\ y = 2\lambda \\ z = 3\lambda \end{cases}$
Cartesian	$x = \frac{y}{2} = \frac{z}{3}$

Part (d)

Form	Expression
Vector	$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda \\ y = 0 \\ z = 0 \end{cases}$
Cartesian	$x \in \mathbb{R}, y = 0, z = 0$

Part (e)

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = 4 \\ y = -1 \\ z = \lambda + 2 \end{cases}$
Cartesian	$x = 4, y = -1, z \in \mathbb{R}$

Problem 2.

For each of the following, determine if l_1 and l_2 are parallel, intersecting or skew. In the case of intersecting lines, find the position vector of the point of intersection. In addition, find the acute angle between the lines l_1 and l_2 .

(a) $l_1 : x - 1 = -y = z - 2$ and $l_2 : \frac{x-2}{2} = -\frac{y+1}{2} = \frac{z-4}{2}$

(b) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}, \alpha \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 10 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$

(c) $l_1 : \mathbf{r} = (\mathbf{i} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} - 4\mathbf{j} - \mathbf{k}), \mu \in \mathbb{R}$

Solution**Part (a)**

Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$$

Since $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, l_1 and l_2 are parallel. Since $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$ for all real μ , l_1 and l_2 are distinct.

Distinct parallel lines. $\theta = 0$.

Part (b)

Since $\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \neq \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$ for all real β , l_1 and l_2 are not parallel.

Consider $l_1 = l_2$.

$$\begin{aligned} l_1 &= l_2 \\ \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 10 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \\ \Rightarrow \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} - \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 10 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 4\alpha - 3\beta = -1 \\ -2\alpha - 8\beta = 10 \\ -3\alpha - \beta = 1 \end{cases}$$

There are no solutions to the above system. Hence, l_1 and l_2 do not intersect and are hence skew.

Let θ be the acute angle between l_1 and l_2 .

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} \right| \left| \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \right|} \\ &= \frac{7}{\sqrt{2146}} \\ \implies \theta &= 81.3^\circ \text{ (1 d.p.)}\end{aligned}$$

Skew lines. $\theta = 81.3^\circ$.

Part (c)

Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$$

Since $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \neq \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$ for all real μ , l_1 and l_2 are not parallel.

Consider $l_1 = l_2$.

$$\begin{aligned}l_1 &= l_2 \\ \implies \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \\ \implies \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}\end{aligned}$$

This gives the following system:

$$\begin{cases} -5\mu + \lambda = 0 \\ 4\mu - \lambda = -1 \\ \mu + \lambda = 6 \end{cases}$$

The above system has the unique solution $\lambda = 5$ and $\mu = 1$. Hence, l_1 and l_2 intersect at

$$\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 0 \end{pmatrix}.$$

Let θ be the acute angle between l_1 and l_2 .

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \right|} \\ &= \frac{8}{3\sqrt{14}} \\ \implies \quad \theta &= 44.5^\circ \text{ (1 d.p.)}\end{aligned}$$

Intersecting lines. $\begin{pmatrix} 6 \\ -5 \\ 0 \end{pmatrix}$. $\theta = 44.5^\circ$ (1 d.p.).
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Problem 3.

- (a) Find the shortest distance from the point $(1, 2, 3)$ to the line with equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$.
- (b) Find the length of projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\frac{x+5}{4} = \frac{y-5}{3} = 10-2z$.
- (c) Find the projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\frac{x+5}{4} = \frac{y-5}{3} = 10-2z$.

Solution**Part (a)**

Let $\vec{OP} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\vec{OA} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$. We have that A is on the line with equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

Note that $\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$.

$$\begin{aligned} \text{Shortest distance} &= \left| \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| / \left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \left| \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \right| \\ &= \frac{\sqrt{2^2 + (-3)^2 + (-4)^2}}{3} \\ &= \frac{\sqrt{29}}{3} \end{aligned}$$

The shortest distance is $\frac{\sqrt{29}}{3}$ units.

Part (b)

Note that the line has vector form

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -1/2 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\text{Length of projection} &= \left| \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| / \left| \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| \\ &= \frac{4}{\sqrt{101}}\end{aligned}$$

The length of projection is $\frac{4}{\sqrt{101}}$ units.

Part (c)

$$\begin{aligned}\text{Projection} &= \left[\begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} / \left| \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| \right] \cdot \left[\begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} / \left| \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| \right] \\ &= \frac{-4}{101} \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}\end{aligned}$$

Problem 4.

The points P and Q have coordinates $(0, -1, -1)$ and $(3, 0, 1)$ respectively, and the equations of the lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}$$

- (a) Show that P lies on l_1 but not on l_2 .
- (b) Determine if l_2 passes through Q .
- (c) Find the coordinates of the foot of the perpendicular from P to l_2 . Hence, or otherwise, find the perpendicular distance from P to l_2 .
- (d) Find the length of projection of \overrightarrow{PQ} onto l_2 .

Solution

We have that $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Part (a)

When $\lambda = -2$, we have $\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \overrightarrow{OP}$. Hence, P lies on l_1 .

Observe that all points on l_2 have a z -coordinate of 1. Since P has a z -coordinate of -1 , P does not lie on l_2 .

Part (b)

When $\mu = 3$, we have $\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \overrightarrow{OQ}$. Hence, l_2 passes through Q .

l_2 passes through Q .

Part (c)

Let the foot of the perpendicular from P to l_2 be F . Since F is on l_2 , we have that $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ for some real μ . We also have that $\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0$.

$$\begin{aligned}
& \overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & (\overrightarrow{OF} - \overrightarrow{OP}) \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & \left[\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & \left[\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & -10 + 5m = 0 \\
\Rightarrow & m = 2
\end{aligned}$$

Hence, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\boxed{F(1, 1, 1)}$$

$$\begin{aligned}
\text{Perpendicular distance} &= \left| \overrightarrow{PF} \right| \\
&= \left| \overrightarrow{OF} - \overrightarrow{OP} \right| \\
&= \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| \\
&= \sqrt{1^2 + 2^2 + 2^2} \\
&= 3
\end{aligned}$$

The perpendicular distance from P to l_2 is 3 units.

Part (d)

Note that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

$$\begin{aligned}\text{Length of projection} &= \left| \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right| / \left| \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right| \\ &= \frac{|6 - 1 + 0|}{\sqrt{2^2 + (-1)^2 + 0^2}} \\ &= \frac{5}{\sqrt{5}} \\ &= \sqrt{5}\end{aligned}$$

The length of projection of \overrightarrow{PQ} onto l_2 is $\sqrt{5}$ units.

Problem 5.

The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

respectively. Find the position vectors of the points P on l_1 and Q on l_2 such that O , P and Q are collinear, where O is the origin.

Solution

We have that $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ for some reals s and t .

For O , P and Q to be collinear, we need $\overrightarrow{OP} = \lambda \overrightarrow{OQ}$ for some real λ .

$$\begin{aligned} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} &= \lambda \left[\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \\ \Rightarrow \begin{pmatrix} s \\ 1 \\ 2 + 3s \end{pmatrix} &= \lambda \begin{pmatrix} -2 + 2t \\ 3 + t \\ 1 \end{pmatrix} \end{aligned}$$

This gives use the system:

$$\begin{cases} s = \lambda(-2 + 2t) \\ 1 = \lambda(3 + t) \\ 2 + 3s = \lambda \end{cases}$$

Substituting the third equation into the first two gives the reduced system:

$$\begin{cases} s = (2 + 3s)(-2 + 2t) \\ 1 = (2 + 3s)(3 + t) \end{cases}$$

Subtracting twice of the second equation from the first yields

$$\begin{aligned} s - 2 &= (2 + 3s)(-2 + 2t) - 2(2 + 3s)(3 + t) \\ &= (2 + 3s)(-2 + 2t) - (2 + 3s)(6 + 2t) \\ &= (2 + 3s)(-2 + 2t - (6 + 2t)) \\ &= -8(2 + 3s) \\ &= -16 - 24s \\ \Rightarrow 25s &= -14 \\ \Rightarrow s &= -\frac{14}{25} \end{aligned}$$

It quickly follows that $t = \frac{1}{8}$. Hence,

$$\begin{aligned}\overrightarrow{OP} &= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{14}{25} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 0 \\ 25 \\ 50 \end{pmatrix} - \frac{1}{25} \begin{pmatrix} 14 \\ 0 \\ 42 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix} \\ \overrightarrow{OQ} &= \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} -16 \\ 24 \\ 8 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}, \overrightarrow{OQ} = \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}}$$

Problem 6.

Relative to the origin O , the points A , B and C have position vectors $5\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$, $-4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ and $-5\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$ respectively.

- Find the Cartesian equation of the line AB .
- Find the length of projection of \overrightarrow{AC} onto the line AB . Hence, find the perpendicular distance from C to the line AB .
- Find the position vector of the foot N of the perpendicular from C to the line AB .
- The point D is such that it is a reflection of point C about the line AB . Find the position vector of D .

Solution

We have that $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} -4 \\ 4 \\ -2 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix}$.

Part (a)

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = -3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$. The line AB hence has the vector form

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

The line AB thus has the Cartesian form

$$\boxed{\frac{x-5}{3} = \frac{z-10}{4}, y=4}$$

Part (b)

Note that $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} = -5 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$.

$$\begin{aligned} \text{Length of projection} &= \left| \overrightarrow{AC} \cdot \overrightarrow{AB} \right| / \left| \overrightarrow{AB} \right| \\ &= \frac{1}{-3\sqrt{3^2 + 0^2 + 4^2}} \left| -5 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot -3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right| \\ &= 10 \end{aligned}$$

The perpendicular distance from C to the line AB is 10 units.

Part (c)

Let $\overrightarrow{AN} = \lambda \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix}$ for some real λ such that $|\overrightarrow{AN}| = 10$.

$$\begin{aligned} |\overrightarrow{AN}| &= 10 \\ \implies \lambda \cdot \sqrt{3^2 + 0^2 + 4^2} &= 10 \\ \implies \lambda &= \frac{2}{3} \end{aligned}$$

Hence, $\overrightarrow{AN} = \frac{2}{3} \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix}$. Thus, $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$.

$$\boxed{\overrightarrow{ON} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}}$$

Part (d)

Note that $\overrightarrow{NC} = \overrightarrow{OC} - \overrightarrow{ON} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}$. Since D is the reflection of C about AB , we have that $\overrightarrow{ND} = -\overrightarrow{NC}$.

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{ON} + \overrightarrow{ND} \\ &= \overrightarrow{ON} - \overrightarrow{NC} \\ &= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{OD} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}}$$

Problem 7.

The points A and B have coordinates $(0, 9, c)$ and $(d, 5, -2)$ respectively, where c and d are constants. The line l has equation $\frac{x+3}{-1} = \frac{y-1}{4} = \frac{z-5}{3}$.

- (a) Given that $d = \frac{22}{7}$ and the line AB intersects l , find the value of c . Find also the coordinates of the foot of the perpendicular from A to l .
- (b) Given instead that the lines AB and l are parallel, state the value of c and d and find the shortest distance between the lines AB and l .

Solution

We have that $\vec{OA} = \begin{pmatrix} 0 \\ 9 \\ c \end{pmatrix}$ and $\vec{OB} = \begin{pmatrix} d \\ 5 \\ -2 \end{pmatrix}$. We also have that l is given by the vector

$$\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \text{ for } \lambda \in \mathbb{R}.$$

Note that $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix}$. Hence, the line AB is given by the vector

$$\mathbf{r}_{AB} = \begin{pmatrix} d \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix} \text{ for } \mu \in \mathbb{R}.$$

Part (a)

Consider the direction vectors of AB and l . Since $\begin{pmatrix} 22/7 \\ -4 \\ -2-c \end{pmatrix} \neq \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ for all real λ and c , the lines AB and l are not parallel. Hence, AB and l intersect at only one point. Thus, there must be a unique solution to $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_{AB} \\ \Rightarrow \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= \begin{pmatrix} 22/7 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 22/7 \\ -4 \\ -2-c \end{pmatrix} \\ \Rightarrow \begin{pmatrix} -21 \\ 7 \\ 35 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= \begin{pmatrix} 22 \\ 35 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} \\ \Rightarrow \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} &= \begin{pmatrix} 43 \\ 28 \\ -49 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} -\lambda - 22\mu = 43 \\ 4\lambda + 28\mu = 28 \\ 3\lambda + (14 + 7c)\mu = -49 \end{cases}$$

Solving the first two equations gives $\lambda = \frac{91}{3}$ and $\mu = -\frac{10}{3}$. It follows from the third equation that $c = 4$.

$$\boxed{c = 4}$$

Let F be the foot of the perpendicular from A to l . We have that $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

for some real λ . We also have that $\overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0$.

$$\begin{aligned} & \overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & (\overrightarrow{OF} - \overrightarrow{OA}) \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & \left[\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & \left[\begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & \begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & -26 + 26\lambda = 0 \\ \Rightarrow & \lambda = 1 \end{aligned}$$

Hence, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix}$.

The foot of the perpendicular from A to l has coordinates $(-4, 5, 8)$.

Part (b)

Given that AB is parallel to l , one of their direction vectors must be a scalar multiple of the other. Hence, for some real λ ,

$$\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} d \\ -4 \\ -2 - c \end{pmatrix}$$

It is obvious that $\lambda = -1$, whence $c = 1$ and $d = 1$.

$$\boxed{c = 1, d = 1}$$

Note that the direction vector of l and AB is $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$. Also note that $(-3, 1, 5)$ is on l and $(1, 5, -2)$ is on AB .

$$\begin{aligned} \text{Shortest distance between } AB \text{ and } l &= \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \left(\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \right) \right| / \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{(-1)^2 + 4^2 + 3^2}} \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{26}} \left| \begin{pmatrix} -40 \\ -5 \\ -20 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{26}} \left| -5 \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} \right| \\ &= \frac{5\sqrt{8^2 + 1^2 + 4^2}}{\sqrt{26}} \\ &= \frac{45}{\sqrt{26}} \end{aligned}$$

The shortest distance between AB and l is $\frac{45}{\sqrt{26}}$ units.

Problem 8.

The equation of the line L is $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$, $t \in \mathbb{R}$. The points A and B have position vectors $\begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix}$ and $\begin{pmatrix} 13 \\ 9 \\ \alpha \end{pmatrix}$ respectively. The line L intersects the line through A and B at P .

- (a) Find α and the acute angle between line L and AB .

The point C has position vector $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ and the foot of the perpendicular from C to L is Q .

- (b) Find the position vector of Q . Hence, find the shortest distance from C to L .
 (c) Find the position vector of the point of reflection of the point C about the line L . Hence, find the reflection of the line passing through C and the point $(1, 3, 7)$ about the line L .

Solution**Part (a)**

We have that $\overrightarrow{OA} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 13 \\ 9 \\ \alpha \end{pmatrix}$. Hence, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix}$.

The line AB is thus given by $\mathbf{r}_{AB} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix}$ for $u \in \mathbb{R}$. Note that AB is not parallel to L . Hence, \overrightarrow{OP} is the only solution to the equation $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{aligned} \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix} \\ \Rightarrow t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix} &= \begin{pmatrix} 8 \\ 0 \\ 19 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 2t - 4u = 8 \\ -t - 6u = 0 \\ 5t - (\alpha - 26)u = 19 \end{cases}$$

Solving the first two equations gives $t = 3$ and $u = -\frac{1}{2}$. It follows from the third equation that $\alpha = 34$.

$$\boxed{\alpha = 34}$$

Let the acute angle between L and AB be θ .

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \right| \left| \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right|} \\ &= \frac{42}{\sqrt{30}\sqrt{116}} \\ \implies \theta &= \arccos \frac{42}{\sqrt{30}\sqrt{116}} \\ &= 44.6^\circ \text{ (1 d.p.)}\end{aligned}$$

$$\boxed{\theta = 44.6^\circ \text{ (1 d.p.)}}$$

Part (b)

Since Q is on L , we have that $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ for some real t . Further, since

$\overrightarrow{CQ} \perp L$, we have that $\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0$.

$$\begin{aligned}\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies (\overrightarrow{OQ} - \overrightarrow{OC}) \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies \left[\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies \left[\begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies \begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies 30 + 30t &= 0 \\ \implies t &= 1\end{aligned}$$

Hence, $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$.

$$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \text{Shortest distance from } C \text{ to } L &= |\overrightarrow{CQ}| \\ &= \left| \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{(-3)^2 + (-1)^2 + 1^2} \\ &= \sqrt{11} \end{aligned}$$

The shortest distance from C to L is $\sqrt{11}$ units.

Part (c)

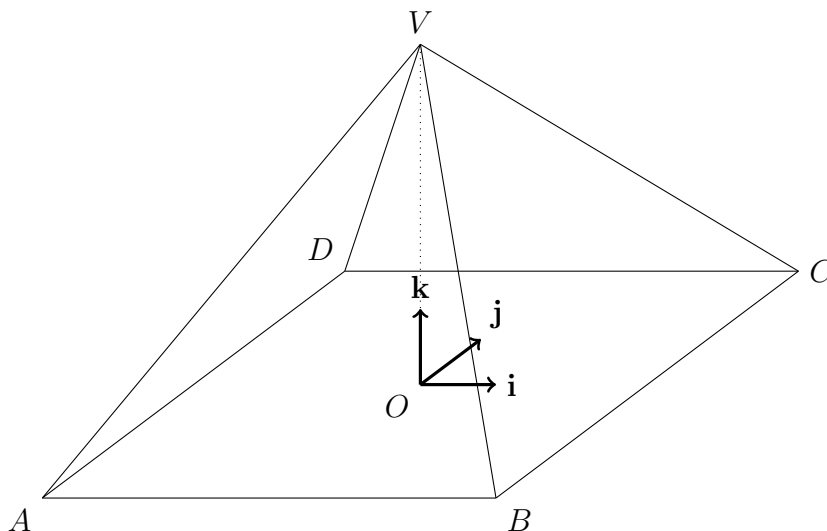
Let C' be the reflection of C about L .

$$\begin{aligned} \overrightarrow{OC'} &= \overrightarrow{OQ} - \overrightarrow{QC} \\ &= \overrightarrow{OQ} + \overrightarrow{CQ} \\ &= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} \end{aligned}$$

$$\overrightarrow{OC'} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}$$

Note that $(1, 3, 7)$ is on L and is hence invariant under a reflection about L . Let the reflection about L of the line passing through C and $(1, 3, 7)$ be L' . Since $\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -4 \end{pmatrix} = -\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$, L' has direction vector $\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$. Thus, L' is given by $\mathbf{r}' = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.

$$L' : \mathbf{r}' = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

Problem 9.

In the diagram, O is the origin of the square base $ABCD$ of a right pyramid with vertex V . The perpendicular unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} are parallel to AB , AD and OV respectively. The length of AB is 4 units and the length of OV is $2h$ units. P , Q , M and N are the mid-points of AB , BC , CV and VA respectively. The point O is taken as the origin for position vectors.

Show that the equation of the line PM may be expressed as $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$, where t is a parameter.

- Find an equation for the line QN .
- Show that the lines PM and QN intersect and that the position vector \overrightarrow{OX} of their point of intersection is $\mathbf{r} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix}$.
- Given that OX is perpendicular to VB , find the value of h and calculate the acute angle between PM and QN , giving your answer correct to the nearest 0.1° .

Solution

We are given that $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 2h \end{pmatrix}$. Hence, $\overrightarrow{CV} = \overrightarrow{OV} - \overrightarrow{OC} = \begin{pmatrix} -2 \\ -2 \\ 2h \end{pmatrix}$. Thus, $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CV} = \begin{pmatrix} -1 \\ -1 \\ h \end{pmatrix}$. Since $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}$, we have that $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$. Thus, PM is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}, t \in \mathbb{R}$$

Part (a)

Since $\overrightarrow{OM} = \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}$, by symmetry, $\overrightarrow{ON} = \begin{pmatrix} -1 \\ -1 \\ h \end{pmatrix}$. Given that $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, we have that

$\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix}$. Thus, QN is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix}, u \in \mathbb{R}$$

Part (b)

Consider $PM = QN$.

$$\begin{aligned} PM &= QN \\ \Rightarrow \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} \\ \Rightarrow t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} - u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} t + 3u = 2 \\ 3t + u = 2 \\ ht - hu = 0 \end{cases}$$

From the first two equations, we see that $t = \frac{1}{2}$ and $u = \frac{1}{2}$, which is consistent with the third equation. Hence, $\overrightarrow{OX} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix}$.

Part (c)

Note that $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}$, whence $\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \begin{pmatrix} 2 \\ -2 \\ -2h \end{pmatrix}$. Since OX is perpendicular to VB , we have that $\overrightarrow{OX} \cdot \overrightarrow{VB} = 0$.

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{VB} &= 0 \\ \implies \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} \cdot 2 \begin{pmatrix} 1 \\ -1 \\ -h \end{pmatrix} &= 0 \\ \implies 1 + 1 - h^2 &= 0 \\ \implies h^2 &= 2 \end{aligned}$$

We hence have that $h = \sqrt{2}$. Note that we reject $h = -\sqrt{2}$ since $h > 0$.

$$\boxed{h = \sqrt{2}}$$

Let the acute angle between PM and QN be θ .

$$\begin{aligned} \cos \theta &= \frac{|\overrightarrow{PM} \cdot \overrightarrow{QN}|}{|\overrightarrow{PM}| |\overrightarrow{QN}|} \\ &= \frac{1}{\sqrt{1^2 + 3^2 + \sqrt{2}^2}} \cdot \frac{1}{\sqrt{(-3)^2 + (-1)^2 + \sqrt{2}^2}} \cdot \left| \begin{pmatrix} 1 \\ 3 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ \sqrt{2} \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{12}} \cdot \frac{1}{\sqrt{12}} \cdot |-3 - 3 + 2| \\ &= \frac{1}{3} \\ \implies \theta &= \arccos \frac{1}{3} \\ &= 70.5^\circ \text{ (1 d.p.)} \end{aligned}$$

$$\boxed{\theta = 70.5^\circ \text{ (1 d.p.)}}$$