Problem 1.

Evaluate the following limits.

(a) $\lim_{x \to 5}(6x + 7)$

(b) $\lim_{x \to 1} \dfrac{x^3 - 1}{1 - x}$

(c) $\lim_{x \to \infty} \dfrac{3x}{2x^2 - 5}$

Solution

Part (a)

$$\lim_{x \to 5}(6x + 7) = 6 \cdot 5 + 7$$
$$= 37$$

$$\boxed{\lim_{x \to 5}(6x + 7) = 37}$$

Part (b)

$$\lim_{x \to 1} \frac{x^3 - 1}{1 - x} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{1 - x}$$
$$= \lim_{x \to 1} -(x^2 + x + 1)$$
$$= -(1^2 + 1 + 1)$$
$$= -3$$

$$\boxed{\lim_{x \to 1} \frac{x^3 - 1}{1 - x} = -3}$$

Part (c)

$$\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = \lim_{x \to \infty} \frac{3}{2x - \frac{5}{x}}$$

Note that as $x \to \infty$, $2x - \dfrac{5}{x} \to \infty - 0$. Hence, $\lim_{x \to \infty} \dfrac{1}{2x - \frac{5}{x}} = 0$.

$$\boxed{\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = 0}$$

Problem 2.

Differentiate the following with respect to x from first principles.

(a) $3x + 4$

(b) x^3

Solution

Part (a)

$$\frac{d}{dx}(3x + 4) = \lim_{h \to 0} \frac{(3(x + h) + 4) - (3x + 4)}{h}$$
$$= \lim_{h \to 0} \frac{3h}{h}$$
$$= \lim_{h \to 0} 3$$
$$= 3$$

$$\boxed{\frac{d}{dx}(3x + 4) = 3}$$

Part (b)

$$\frac{d}{dx}x^3 = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h}$$
$$= \lim_{h \to 0} \frac{3hx^2 + 3h^2x + h^3}{h}$$
$$= \lim_{h \to 0} \left(3x^2 + 3hx + h^2\right)$$
$$= 3x^2$$

$$\boxed{\frac{d}{dx}x^3 = 3x^2}$$

Problem 3.

Differentiate each of the following with respect to x, simplifying your answer.

(a) $\left(x^2 + 4\right)^2 \left(2x^3 - 1\right)$

(b) $\dfrac{x^2}{\sqrt{4 - x^2}}$

(c) $\sqrt{1 + \sqrt{x}}$

(d) $\left(\dfrac{x^3 - 1}{2x^3 + 1}\right)^4$

Solution

Part (a)

$$\begin{aligned}
\frac{d}{dx}\left(x^2 + 4\right)^2\left(2x^3 - 1\right) &= \left(2x^3 - 1\right)\frac{d}{dx}\left(x^2 + 4\right)^2 + \left(x^2 + 4\right)^2 \cdot \frac{d}{dx}\left(2x^3 - 1\right) \\
&= \left(2x^3 - 1\right) \cdot 2\left(x^2 + 4\right) \cdot 2x + \left(x^2 + 4\right)^2 \cdot 6x^2 \\
&= 2x\left(x^2 + 4\right)\left(2\left(2x^3 - 1\right) + 3x\left(x^2 + 4\right)\right) \\
&= 2x\left(x^2 + 4\right)\left(7x^3 + 12x - 2\right)
\end{aligned}$$

$$\boxed{\frac{d}{dx}\left(x^2 + 4\right)^2\left(2x^3 - 1\right) = 2x\left(x^2 + 4\right)\left(7x^3 + 12x - 2\right)}$$

Part (b)

$$\begin{aligned}
\frac{d}{dx}\frac{x^2}{\sqrt{4 - x^2}} &= \frac{\sqrt{4 - x^2} \cdot \dfrac{d}{dx}x^2 - x^2 \cdot \dfrac{d}{dx}\sqrt{4 - x^2}}{4 - x^2} \\
&= \frac{\sqrt{4 - x^2} \cdot 2x - x^2 \cdot \dfrac{1}{2\sqrt{4 - x^2}} \cdot -2x}{4 - x^2} \\
&= \frac{\left(4 - x^2\right) \cdot 2x - x^2 \cdot \dfrac{1}{2} \cdot -2x}{\left(4 - x^2\right)^{\frac{3}{2}}} \\
&= \frac{\left(4 - x^2\right) \cdot 2x + x^3}{\left(4 - x^2\right)^{\frac{3}{2}}} \\
&= \frac{x\left(8 - x^2\right)}{\left(4 - x^2\right)^{\frac{3}{2}}}
\end{aligned}$$

$$\boxed{\frac{d}{dx}\frac{x^2}{\sqrt{4 - x^2}} = \frac{x\left(8 - x^2\right)}{\left(4 - x^2\right)^{\frac{3}{2}}}}$$

Part (c)

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{4\sqrt{x(1+\sqrt{x})}}$$

$$\boxed{\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{4\sqrt{x(1+\sqrt{x})}}}$$

Part (d)

$$\frac{d}{dx}\left(\frac{x^3-1}{2x^3+1}\right)^4 = 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{d}{dx}\frac{x^3-1}{2x^3+1}$$

$$= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{(2x^3+1)\frac{d}{dx}(x^3-1) - (x^3-1)\frac{d}{dx}(2x^3+1)}{(2x^3+1)^2}$$

$$= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{(2x^3+1)\cdot 3x^2 - (x^3-1)\cdot 6x^2}{(2x^3+1)^2}$$

$$= \frac{4(x^3-1)^3}{(2x^3+1)^3} \cdot \frac{(2x^3+1)\cdot 3x^2 - (x^3-1)\cdot 6x^2}{(2x^3+1)^2}$$

$$= \frac{4(x^3-1)^3}{(2x^3+1)^5} \cdot \left((2x^3+1)\cdot 3x^2 - (x^3-1)\cdot 6x^2\right)$$

$$= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5} \cdot \left(2x^3+1 - (x^3-1)\cdot 2\right)$$

$$= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5} \cdot 3$$

$$= \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}$$

$$\boxed{\frac{d}{dx}\left(\frac{x^3-1}{2x^3+1}\right)^4 = \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}}$$

Problem 4.

Using a graphing calculator, find the derivative of $\dfrac{e^{2x}}{x^2 + 1}$ when $x = 1.5$.

Solution

$$\boxed{\left. \frac{d}{dx} \frac{e^{2x}}{x^2 + 1} \right|_{x=1.5} = 6.66}$$

Problem 5.

Find the derivative with respect to x of

(a) $\cos x°$

(b) $\cot\left(1 - 2x^2\right)$

(c) $\tan^3(5x)$

(d) $\dfrac{\sec x}{1 + \tan x}$

Solution

Part (a)

$$\frac{d}{dx}\cos x° = \frac{d}{dx}\cos\left(\frac{\pi}{180}x\right)$$
$$= \frac{\pi}{180} \cdot \left(-\sin\left(\frac{\pi}{180}x\right)\right)$$
$$= -\frac{\pi}{180}\sin\left(\frac{\pi}{180}x\right)$$

$$\boxed{\frac{d}{dx}\cos x° = -\frac{\pi}{180}\sin\left(\frac{\pi}{180}x\right)}$$

Part (b)

$$\frac{d}{dx}\cot\left(1 - 2x^2\right) = -\csc\left(1 - 2x^2\right)\cdot(-4x)$$
$$= 4x\csc\left(1 - 2x^2\right)$$

$$\boxed{\frac{d}{dx}\cot\left(1 - 2x^2\right) = 4x\csc\left(1 - 2x^2\right)}$$

Part (c)

$$\frac{d}{dx}\tan^3(5x) = 3\tan^2(5x)\cdot\sec^2(5x)\cdot 5$$
$$= 15\tan^2(5x)\sec^2(5x)$$

$$\boxed{\frac{d}{dx}\tan^3(5x) = 15\tan^2(5x)\sec^2(5x)}$$

Part (d)

$$\frac{d}{dx}\frac{\sec x}{1+\tan x} = \frac{(1+\tan x)\cdot\frac{d}{dx}\sec x - \sec x\cdot\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\cdot\sec x\tan x - \sec x\cdot\sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x(1+\tan x) - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x) - (\tan^2 x + 1))}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

$$\boxed{\frac{d}{dx}\frac{\sec x}{1+\tan x} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}}$$

Problem 6.

Find the derivative with respect to x of

(a) $y = e^{1+\sin 3x}$

(b) $y = x^2 e^{\frac{1}{x}}$

(c) $y = \ln\left(\dfrac{1-x}{\sqrt{1+x^2}}\right)$

(d) $y = \dfrac{\ln(2x)}{x}$

(e) $y = \log_2(3x^4 - e^x)$

(f) $y = 3^{\ln(\sin x)}$

(g) $y = a^{2\log_a x}$

(h) $y = \sqrt[3]{\dfrac{e^x(x+1)}{x^2+1}}$, $x > 0$

Solution

Part (a)

$$\frac{dy}{dx} = \frac{d}{dx}e^{1+\sin 3x}$$
$$= e^{1+\sin 3x} \cdot \frac{d}{dx}(1 + \sin 3x)$$
$$= e^{1+\sin 3x} \cdot \cos 3x \cdot 3$$
$$= 3e^{1+\sin 3x}\cos 3x$$

$$\boxed{\frac{dy}{dx} = 3e^{1+\sin 3x}\cos 3x}$$

Part (b)

$$\frac{dy}{dx} = \frac{d}{dx}x^2 e^{\frac{1}{x}}$$
$$= x^2 \cdot \frac{d}{dx}e^{\frac{1}{x}} + e^{\frac{1}{x}} \cdot \frac{d}{dx}x^2$$
$$= x^2 \cdot e^{\frac{1}{x}} \cdot \frac{d}{dx}\frac{1}{x} + e^{\frac{1}{x}} \cdot 2x$$
$$= x^2 \cdot e^{\frac{1}{x}} \cdot -\frac{1}{x^2} + e^{\frac{1}{x}} \cdot 2x$$
$$= -e^{\frac{1}{x}} + 2xe^{\frac{1}{x}}$$
$$= e^{\frac{1}{x}}(2x - 1)$$

$$\boxed{\frac{dy}{dx} = e^{\frac{1}{x}}(2x - 1)}$$

Part (c)

$$y = \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right)$$
$$= \ln(1-x) - \ln(\sqrt{1+x^2})$$
$$= \ln(1-x) - \frac{1}{2}\ln(1+x^2)$$
$$\implies \frac{dy}{dx} = \frac{1}{1-x} \cdot \frac{d}{dx}(1-x) - \frac{1}{2}\left(\frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2)\right)$$
$$= \frac{1}{1-x} \cdot -1 - \frac{1}{2}\left(\frac{1}{1+x^2} \cdot 2x\right)$$
$$= \frac{1}{x-1} - \frac{x}{1+x^2}$$
$$= \frac{(1+x^2) - x(x-1)}{(x-1)(1+x^2)}$$
$$= \frac{1+x^2 - x^2 + x}{(x-1)(1+x^2)}$$
$$= \frac{1+x}{(x-1)(1+x^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{1+x}{(x-1)(1+x^2)}}$$

Part (d)

$$\frac{dy}{dx} = \frac{d}{dx}\frac{\ln(2x)}{x}$$
$$= \frac{x \cdot \dfrac{d}{dx}\ln(2x) - \ln(2x) \cdot \dfrac{d}{dx}x}{x^2}$$
$$= \frac{x \cdot \dfrac{1}{2x} \cdot \dfrac{d}{dx}2x - \ln(2x)}{x^2}$$
$$= \frac{x \cdot \dfrac{1}{2x} \cdot 2 - \ln(2x)}{x^2}$$
$$= \frac{1 - \ln(2x)}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - \ln(2x)}{x^2}}$$

Part (e)

$$y = \log_2(3x^4 - e^x)$$
$$\implies \quad 2^y = 3x^4 - e^x$$
$$\implies \quad e^{\ln 2^y} = 3x^4 - e^x$$
$$\implies \quad e^{y\ln 2} = 3x^4 - e^x$$
$$\implies e^{y\ln 2} \cdot \frac{dy}{dx} \cdot \ln 2 = 3 \cdot 4x^3 - e^x$$
$$\implies \quad \frac{dy}{dx} = \frac{12x^3 - e^x}{e^{y\ln 2}\ln 2}$$
$$= \frac{12x^3 - e^x}{(3x^4 - e^x)\ln 2}$$

$$\boxed{\frac{dy}{dx} = \frac{12x^3 - e^x}{(3x^4 - e^x)\ln 2}}$$

Part (f)

$$y = 3^{\ln(\sin x)}$$
$$\implies \quad \log_3 y = \ln(\sin x)$$
$$\implies \quad \frac{\ln y}{\ln 3} = \ln(\sin x)$$
$$\implies \quad \ln y = \ln 3 \cdot \ln(\sin x)$$
$$\implies \frac{1}{y} \cdot \frac{dy}{dx} = \ln 3 \cdot \frac{1}{\sin x} \cdot \frac{d}{dx}\sin x$$
$$= \ln 3 \cdot \frac{1}{\sin x} \cdot \cos x$$
$$\implies \quad \frac{dy}{dx} = \ln 3 \cdot \cot x \cdot y$$
$$= \ln 3 \cdot \cot x \cdot 3^{\ln(\sin x)}$$

$$\boxed{\frac{dy}{dx} = \ln 3 \cdot \cot x \cdot 3^{\ln(\sin x)}}$$

Part (g)

$$y = a^{2\log_a x}$$
$$= a^{\log_a x^2}$$
$$= x^2$$
$$\implies \frac{dy}{dx} = \frac{d}{dx}x^2$$
$$= 2x$$

$$\boxed{\frac{dy}{dx} = 2x}$$

Part (h)

$$y = \sqrt[3]{\frac{e^x(x+1)}{x^2+1}}$$

$$\implies \qquad y^3 = \frac{e^x(x+1)}{x^2+1}$$

$$\implies \qquad \left(x^2+1\right)y^3 = e^x(x+1)$$

$$\implies \left(x^2+1\right)\cdot\frac{d}{dx}y^3 + y^3\cdot\frac{d}{dx}\left(x^2+1\right) = e^x\cdot\frac{d}{dx}(x+1)+(x+1)\frac{d}{dx}e^x$$

$$\implies \qquad \left(x^2+1\right)\cdot 3y^2\cdot\frac{dy}{dx} + y^3\cdot 2x = e^x\cdot 1 + (x+1)\cdot e^x$$

$$\implies \qquad \left(x^2+1\right)\cdot 3y^2\cdot\frac{dy}{dx} = e^x(x+2) - 2xy^3$$

$$\implies \qquad \frac{dy}{dx} = \frac{e^x(x+2) - 2xy^3}{\left(x^2+1\right)\cdot 3y^2}$$

$$= \frac{1}{3}\left(\frac{e^x(x+2)}{\left(x^2+1\right)y^2} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{1}{3}\left(\frac{e^x(x+1)(x+2)}{\left(x^2+1\right)(x+1)y^2} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{1}{3}\left(\frac{y^3(x+2)}{(x+1)y^2} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{1}{3}\left(\frac{y(x+2)}{(x+1)} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{y}{3}\left(\frac{(x+2)}{(x+1)} - \frac{2x}{x^2+1}\right)$$

$$= \frac{y}{3}\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)$$

$$= \frac{1}{3}\cdot\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}\cdot\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)$$

$$\boxed{\frac{dy}{dx} = \frac{1}{3}\cdot\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}\cdot\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)}$$

Problem 7.

Find the derivative with respect to x of

(a) $\arccos \dfrac{x}{10}$

(b) $\arctan \dfrac{1}{1-x}$

(c) $\arcsin(\tan x)$

Solution

Part (a)

$$\frac{d}{dx} \arccos \frac{x}{10} = -\frac{1}{\sqrt{1 - \left(\frac{x}{10}\right)^2}} \cdot \frac{1}{10}$$

$$= -\frac{1}{\sqrt{100 - x^2}}$$

$$\boxed{\frac{d}{dx} \arccos \frac{x}{10} = -\frac{1}{\sqrt{100 - x^2}}}$$

Part (b)

$$\frac{d}{dx} \arctan \frac{1}{1-x} = \frac{1}{1 + \left(\frac{1}{1-x}\right)^2} \cdot -\frac{1}{(1-x)^2} \cdot -1$$

$$= \frac{1}{(1-x)^2 \left(1 + \left(\frac{1}{1-x}\right)^2\right)}$$

$$= \frac{1}{(1-x)^2 + 1}$$

$$\boxed{\frac{d}{dx} \arctan \frac{1}{1-x} = \frac{1}{(1-x)^2 + 1}}$$

Part (c)

$$\frac{d}{dx} \arcsin(\tan x) = \frac{1}{1 - \tan^2 x} \cdot \sec^2 x$$

$$\boxed{\frac{d}{dx} \arcsin(\tan x) = \frac{\sec^2 x}{1 - \tan^2 x}}$$

Eytan Chong

Problem 8.

Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

(a) $(y - x)^2 = 2a(y + x)$, where a is a constant

(b) $y^2 = e^{2x}y + xe^x$

(c) $y = x^y$

(d) $\sin x \cos y = \dfrac{1}{2}$

Solution

Part (a)

$$(y - x)^2 = 2a(y + x)$$

$$\implies \quad 2(y - x)\left(\frac{dy}{dx} - 1\right) = 2a\left(\frac{dy}{dx} + 1\right)$$

$$\implies \quad (y - x)\left(\frac{dy}{dx} - 1\right) = a\left(\frac{dy}{dx} + 1\right)$$

$$\implies (y - x) \cdot \frac{dy}{dx} - (y - x) = a \cdot \frac{dy}{dx} + a$$

$$\implies \quad (y - x - a) \cdot \frac{dy}{dx} = a + y - x$$

$$\implies \quad \frac{dy}{dx} = \frac{a + y - x}{y - x - a}$$

$$\boxed{\frac{dy}{dx} = \frac{a + y - x}{y - x - a}}$$

Part (b)

$$y^2 = e^{2x}y + xe^x$$

$$\implies \quad 2y \cdot \frac{dy}{dx} = e^{2x} \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}e^{2x} + x \cdot \frac{d}{dx}e^x + e^x \cdot \frac{d}{dx}x$$

$$\implies \quad 2y \cdot \frac{dy}{dx} = e^{2x} \cdot \frac{dy}{dx} + y \cdot e^{2x} \cdot 2 + xe^x + e^x$$

$$\implies \left(2y - e^{2x}\right) \cdot \frac{dy}{dx} = y \cdot e^{2x} \cdot 2 + xe^x + e^x$$

$$\implies \left(2y - e^{2x}\right) \cdot \frac{dy}{dx} = e^x\left(2e^xy + x + 1\right)$$

$$\implies \quad \frac{dy}{dx} = \frac{e^x\left(2e^xy + x + 1\right)}{2y - e^{2x}}$$

$$\boxed{\frac{dy}{dx} = \frac{e^x\left(2e^xy + x + 1\right)}{2y - e^{2x}}}$$

Part (c)

$$y = x^y$$
$$\implies \ln y = y \ln x$$
$$\implies \frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{d}{dx} \ln x + \ln x \cdot \frac{dy}{dx}$$
$$= y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx}$$
$$\implies \left(\frac{1}{y} - \ln x \right) \cdot \frac{dy}{dx} = \frac{y}{x}$$
$$\implies \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \ln x}$$
$$= \frac{y^2}{x - xy \ln x}$$

$$\boxed{\frac{dy}{dx} = \frac{y^2}{x - xy \ln x}}$$

Part (d)

$$\sin x \cos y = \frac{1}{2}$$
$$\implies \sin x \cdot \frac{d}{dx} \cos y + \cos y \cdot \frac{d}{dx} \sin x = 0$$
$$\implies \sin x \cdot - \sin y \cdot \frac{dy}{dx} + \cos y \cdot \cos x = 0$$
$$\implies - \sin x \sin y \cdot \frac{dy}{dx} = - \cos x \cos y$$
$$\implies \frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \sin y}$$
$$= \cot x \cot y$$

$$\boxed{\frac{dy}{dx} = \cot x \cot y}$$

Problem 9.

It is given that x and y satisfy the equation

$$\arctan x + \arctan y + \arctan xy = \frac{7}{12}\pi \tag{9.1}$$

(a) Find the exact value of y when $x = 1$.

(b) Express $\dfrac{d}{dx}\arctan(xy)$ in terms of x, y and y'.

(c) Show that, when $x = 1$, $y' = -\dfrac{1}{3} - \dfrac{1}{2\sqrt{3}}$.

Solution

Part (a)

Substituting $x = 1$ into Equation 9.1,

$$\arctan 1 + \arctan y + \arctan y = \frac{7}{12}\pi$$

$$\implies \frac{1}{4}\pi + 2\arctan y = \frac{7}{12}\pi$$

$$\implies \arctan y = \frac{1}{6}\pi$$

$$\implies y = \frac{1}{\sqrt{3}}$$

$$\boxed{y = \frac{1}{\sqrt{3}}}$$

Part (b)

$$\frac{d}{dx}\arctan(xy) = \frac{1}{1+(xy)^2} \cdot \frac{d}{dx}(xy)$$

$$= \frac{1}{1+(xy)^2} \cdot (xy' + y)$$

$$\boxed{\frac{d}{dx}\arctan(xy) = \frac{1}{1+(xy)^2} \cdot (xy' + y)}$$

Part (c)

Differentiating Equation 9.1 with respect to x,

$$\frac{1}{1+x^2} + \frac{y'}{1+y^2} + \frac{1}{1+(xy)^2} \cdot (xy' + y) = 0$$

Substituting $x = 1$,

$$\frac{1}{2} + \frac{3}{4}y' + \frac{3}{4}(y' + y) = 0$$

$$\implies \quad y'(\frac{3}{4} + \frac{3}{4}) = \frac{3}{4}\left(-\frac{1}{\sqrt{3}}\right) - \frac{1}{2}$$

$$\implies \quad \frac{3}{2}y' = \frac{-3 - 2\sqrt{3}}{4\sqrt{3}}$$

$$\implies \quad y' = \frac{2}{3} \cdot \frac{-3 - 2\sqrt{3}}{4\sqrt{3}}$$

$$= \frac{-1 - \frac{2}{3}\sqrt{3}}{2\sqrt{3}}$$

$$= -\frac{1}{2\sqrt{3}} - \frac{1}{3}$$

Problem 10.

Find $\dfrac{dy}{dx}$ for

(a) $x = \dfrac{1}{1+t^2}$, $y = \dfrac{t}{1+t^2}$

(b) $x = \dfrac{1}{2}(e^t - e^{-t})$, $y = \dfrac{1}{2}(e^t + e^{-t})$

(c) $x = a\sec\theta$, $y = a\tan\theta$

(d) $x = e^{3\theta}\cos 3\theta$, $y = e^{3\theta}\sin 3\theta$

Solution

Part (a)

Observe that $y = xt$.

$$
\begin{aligned}
\frac{dy}{dx} &= x \cdot \frac{dt}{dx} + t \\
&= x\left(\frac{dx}{dt}\right)^{-1} + t \\
&= \frac{1}{1+t^2}\left(-\frac{1}{(1+t^2)^2}\cdot 2t\right)^{-1} + t \\
&= \frac{1}{1+t^2}\cdot\left(-\frac{(1+t^2)^2}{2t}\right) + t \\
&= -\frac{1+t^2}{2t} + \frac{2t^2}{2t} \\
&= \frac{t^2-1}{2t}
\end{aligned}
$$

$$\boxed{\frac{dy}{dx} = \frac{t^2-1}{2t}}$$

Part (b)

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt}\cdot\frac{dt}{dx} \\
&= \frac{dy}{dt}\cdot\left(\frac{dx}{dt}\right)^{-1} \\
&= \frac{\frac{1}{2}(e^t - e^{-t})}{\frac{1}{2}(e^t + e^{-t})} \\
&= \frac{e^t - e^{-t}}{e^t + e^{-t}}
\end{aligned}
$$

$$\boxed{\frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t + e^{-t}}}$$

Part (c)

Recall that $\tan^2\theta + 1 = \sec^2\theta$. Hence, $y^2 + a^2 = x^2$. Implicitly differentiating, we have

$$2y \cdot \frac{dy}{dx} = 2x$$

$$\implies \quad \frac{dy}{dx} = \frac{x}{y}$$

$$= \frac{a\sec\theta}{a\tan\theta}$$

$$= \csc\theta$$

$$\boxed{\frac{dy}{dx} = \csc\theta}$$

Part (d)

Recall that $\sin^2\theta + \cos^2\theta = 1$. Hence, $x^2 + y^2 = e^{6\theta}$. Implicitly differentiating, we have

$$2x + 2y\frac{dy}{dx} = e^{6\theta} \cdot 6\frac{d\theta}{dx}$$

$$\implies \quad x + y\frac{dy}{dx} = 3e^{6\theta} \cdot \frac{d\theta}{dx} \tag{10.1}$$

Observe that $\frac{y}{x} = \tan 3\theta$. Implicitly differentiating,

$$\frac{x\frac{dy}{dx} - y}{x^2} = \sec^2 3\theta \cdot 3\frac{d\theta}{dx}$$

$$\implies x\frac{dy}{dx} - y = 3x^2 \sec^2 3\theta \cdot \frac{d\theta}{dx}$$

$$= 3(e^{3\theta}\cos 3\theta)^2 \cdot \frac{1}{\cos^2 3\theta} \cdot \frac{d\theta}{dx}$$

$$= 3e^{6\theta} \cdot \frac{d\theta}{dx} \tag{10.2}$$

Subtracting Equation 10.1 from Equation 10.2,

$$x\frac{dy}{dx} - y\frac{dy}{dx} - y - x = 0$$

$$\implies \quad (x - y)\frac{dy}{dx} = x + y$$

$$\implies \quad \frac{dy}{dx} = \frac{x + y}{x - y}$$

$$= \frac{e^{3\theta}(\cos 3\theta + \sin 3\theta)}{e^{3\theta}(\cos 3\theta - \sin 3\theta)}$$

$$= \frac{\cos 3\theta + \sin 3\theta}{\cos 3\theta - \sin 3\theta}$$

$$= \frac{1 + \tan 3\theta}{1 - \tan 3\theta}$$

$$= \tan\left(3\theta + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \tan\left(3\theta + \frac{\pi}{4}\right)$$

$$\frac{dy}{dx} = \tan\left(3\theta + \frac{\pi}{4}\right)$$

Problem 11.

A curve is defined by the parametric equation

$$x = 120t - 4t^2,\ y = 60t - 6t^2$$

Find the value of $\dfrac{dy}{dx}$ at each of the points where the curve cross the x-axis.

Solution

The curve crosses the x-axis when $y = 0$.

$$\begin{aligned} y &= 0 \\ \implies 60t - 6t^2 &= 0 \\ \implies 10t - t^2 &= 0 \\ \implies t(10 - t) &= 0 \end{aligned}$$

Hence, $t = 0$ or $t = 10$. Now, consider the derivative with respect to x of the curve.

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \\ &= \frac{60 - 12t}{120 - 8t} \end{aligned}$$

Case 1: $t = 0$

$$\begin{aligned} \frac{dy}{dx}\bigg|_{t=0} &= \frac{60 - 12 \cdot 0}{120 - 8 \cdot 0} \\ &= \frac{1}{2} \end{aligned}$$

Case 2: $t = 10$

$$\begin{aligned} \frac{dy}{dx}\bigg|_{t=10} &= \frac{60 - 12 \cdot 10}{120 - 8 \cdot 10} \\ &= -\frac{3}{2} \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2} \vee -\frac{3}{2}}$$

Problem 12.

A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln t^2$, where $t > 0$. Find the value of t at the point on the curve at which the gradient is 2.

Solution

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\
&= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \\
&= \frac{2t - \frac{2}{t}}{2 - \frac{1}{t}} \\
&= \frac{2t^2 - 2}{2t - 1}
\end{aligned}
$$

Consider $\dfrac{dy}{dx} = 2$.

$$
\begin{aligned}
\frac{dy}{dx} &= 2 \\
\implies \frac{2t^2 - 2}{2t - 1} &= 2 \\
\implies \frac{t^2 - 1}{2t - 1} &= 1 \\
\implies t^2 - 1 &= 2t - 1 \\
\implies t^2 - 2t &= 0 \\
\implies t(t - 2) &= 0
\end{aligned}
$$

Hence, $t = 0$ or $t = 2$. Since $t > 0$, we reject $t = 0$. Thus, $t = 2$.

$$\boxed{t = 2}$$

Problem 13.

If $y = \ln\left(\sin^3 2x\right)$, find $\dfrac{dy}{dx}$ and prove that $3\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 + 36 = 0$.

Solution

$$
\begin{aligned}
y &= \ln\left(\sin^3 2x\right) \\
\implies \quad e^y &= \sin^3 2x \\
\implies e^y \cdot \frac{dy}{dx} &= 3\sin^2 2x \cdot \cos 2x \cdot 2 \\
&= 6\sin^2 2x \cos 2x \\
\implies \quad \frac{dy}{dx} &= \frac{6\sin^2 2x \cos 2x}{e^y} \\
&= \frac{6\sin^2 2x \cos 2x}{\sin^3 2x} \\
&= \frac{6\cos 2x}{\sin 2x} \\
&= 6\cot 2x
\end{aligned}
$$

$$
\boxed{\frac{dy}{dx} = 6\cot 2x}
$$

$$
\begin{aligned}
\frac{dy}{dx} &= 6\cot 2x \\
\implies \quad \frac{d^2y}{dx^2} &= 6\cdot -\csc^2 2x \cdot 2 \\
&= -12\csc^2 2x \\
&= -12\left(1 + \cot^2 2x\right) \\
&= -12 - 12\cot^2 2x \\
&= -12 - \frac{1}{3}\left(\frac{dy}{dx}\right)^2 \\
\implies \quad 3\frac{d^2y}{dx^2} &= -36 - \left(\frac{dy}{dx}\right)^2 \\
\implies 3\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 &+ 36 = 0
\end{aligned}
$$

Problem 14.

Given that $y = e^{\arcsin 2x}$, show that $(1 - 4x^2)\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} = 4y$. Further differentiate this result to obtain a differential equation for $\dfrac{d^3y}{dx^3}$.

Solution

$$y = e^{\arcsin 2x}$$

$$\implies \ln y = \arcsin 2x$$

$$\implies \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x)^2}} \cdot 2$$

$$\implies \frac{dy}{dx} = \frac{2y}{\sqrt{1 - 4x^2}}$$

$$\implies \frac{d^2y}{dx^2} = \frac{\sqrt{1 - 4x^2} \cdot 2\dfrac{dy}{dx} - 2y \cdot \dfrac{1}{2\sqrt{1 - 4x^2}} \cdot -8x}{1 - 4x^2}$$

$$\implies \left(1 - 4x^2\right)\frac{d^2y}{dx^2} = 2\sqrt{1 - 4x^2}\frac{dy}{dx} + 4x\frac{2y}{\sqrt{1 - 4x^2}}$$

$$= 2\sqrt{1 - 4x^2} \cdot \frac{2y}{\sqrt{1 - 4x^2}} + 4x\frac{dy}{dx}$$

$$= 4y + 4x\frac{dy}{dx}$$

$$\implies \left(1 - 4x^2\right)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 4y$$

$$\left(1 - 4x^2\right)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 4y$$

$$\implies \left(1 - 4x^2\right)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \cdot (-8x) - 4\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = 4\frac{dy}{dx}$$

$$\implies \left(1 - 4x^2\right)\frac{d^3y}{dx^3} - 8x\frac{d^2y}{dx^2} - 4x\frac{d^2y}{dx^2} - 4\frac{dy}{dx} = 4\frac{dy}{dx}$$

$$\implies \left(1 - 4x^2\right)\frac{d^3y}{dx^3} - 12x\frac{d^2y}{dx^2} - 8\frac{dy}{dx} = 0$$

$$\boxed{\left(1 - 4x^2\right)\frac{d^3y}{dx^3} - 12x\frac{d^2y}{dx^2} - 8\frac{dy}{dx} = 0}$$