

**Problem 1.**

The equation of a curve is  $y = 2x^3 + 3x^2 + 6x + 4$ . Find  $\frac{dy}{dx}$  and hence show that  $y$  is increasing for all real values of  $x$ .

**Solution**

$$\begin{aligned}\frac{dy}{dx} &= 2 \cdot 3x^2 + 3 \cdot 2x + 6 \\ &= 6x^2 + 6x + 6\end{aligned}$$

$$\boxed{\frac{dy}{dx} = 6x^2 + 6x + 6}$$

Observe that  $\frac{dy}{dx} = 6x^2 + 6x + 6 = 6\left(x + \frac{1}{2}\right)^2 + \frac{18}{4}$ . For all  $x \in \mathbb{R}$ , we have  $\left(x + \frac{1}{2}\right)^2 \geq 0$ . Hence,  $\frac{dy}{dx} > 0$ . Thus,  $y$  is increasing for all real values of  $x$ .

**Problem 2.**

Find, by differentiation, the  $x$ -coordinates of all the stationary points on the curve  $y = \frac{x^3}{(x+1)^2}$  stating, with reasons, the nature of each point.

**Solution**

$$\begin{aligned}
 y &= \frac{x^3}{(x+1)^2} \\
 \Rightarrow (x+1)^2 y &= x^3 \\
 \Rightarrow (x+1)^2 \cdot y' + y \cdot 2(x+1) &= 3x^2
 \end{aligned}$$

For stationary points,  $y' = 0$ .

$$\begin{aligned}
 \Rightarrow y \cdot 2(x+1) &= 3x^2 \\
 \Rightarrow \frac{x^3}{(x+1)^2} \cdot 2(x+1) &= 3x^2 \\
 \Rightarrow \frac{2x^3}{x+1} &= 3x^2 \\
 \Rightarrow 2x^3 &= 3x^2(x+1) \\
 \Rightarrow 2x^3 &= 3x^3 + 3x^2 \\
 \Rightarrow x^3 + 3x^2 &= 0 \\
 \Rightarrow x^2(x+3) &= 0
 \end{aligned}$$

Hence,  $x = 0$  or  $x = -3$ .

The  $x$ -coordinates of the stationary points are  $x = 0$  and  $x = -3$ .

$x$	$0^-$	$0$	$0^+$
$\frac{dy}{dx}$	+ve	0	+ve

Thus, there is a stationary point of inflexion at  $x = 0$ .

$x$	$(-3)^-$	$-3$	$(-3)^+$
$\frac{dy}{dx}$	+ve	0	-ve

Thus, there is a maximum point at  $x = -3$ .

At  $x = 0$ , there is a stationary point of inflexion.  
At  $x = -3$ , there is a maximum point.

**Problem 3.**

Differentiate  $f(x) = 8 \sin \frac{x}{2} - 4x$  with respect to  $x$  and deduce that  $f(x) < 0$  for  $x > 0$ .

**Solution**

$$\begin{aligned} f'(x) &= 8 \cos \frac{x}{2} \cdot \frac{1}{2} - \cos x - 4 \\ &= 4 \cos \frac{x}{2} - \cos x - 4 \end{aligned}$$

$$f'(x) = 4 \cos \frac{x}{2} - \cos x - 4$$

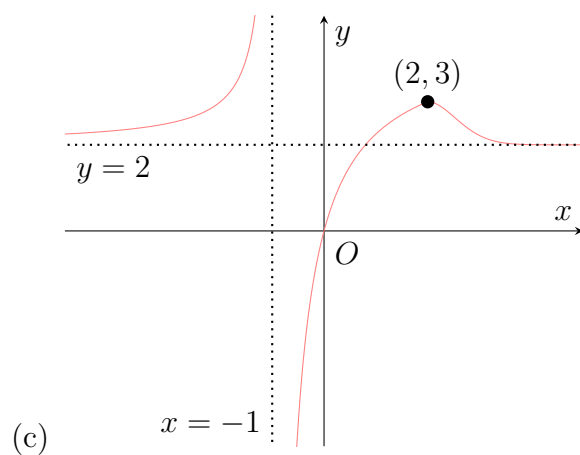
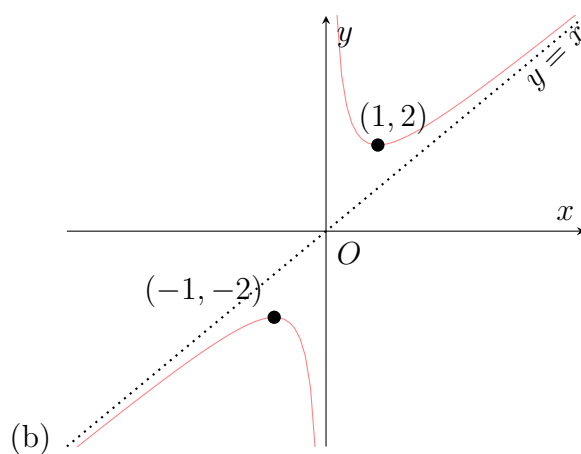
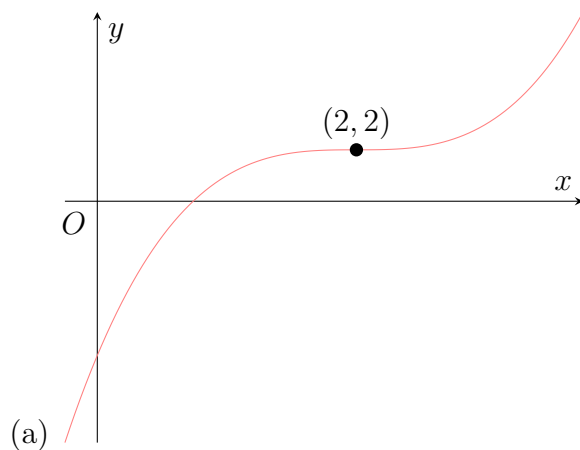
$$\begin{aligned} f'(x) &= 4 \cos \frac{x}{2} - \cos x - 4 \\ &= 4 \cos \frac{x}{2} - (2 \cos^2 \frac{x}{2} - 1) - 4 \\ &= -2 \left( \cos \frac{x}{2} - 1 \right)^2 - 1 \end{aligned}$$

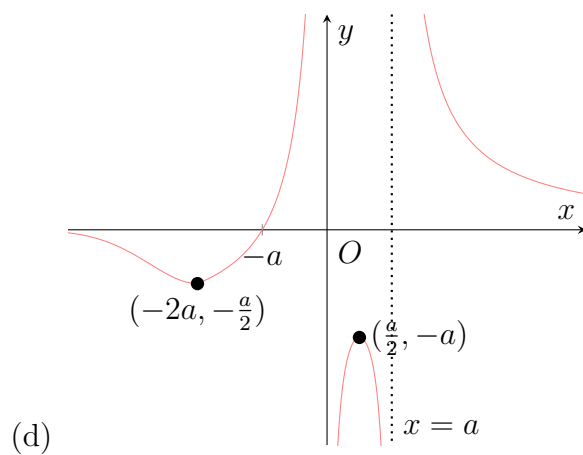
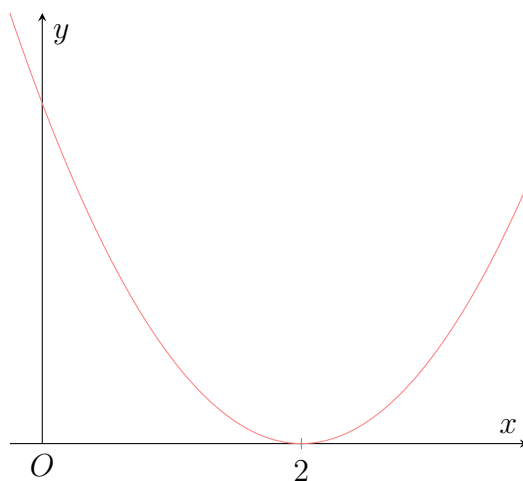
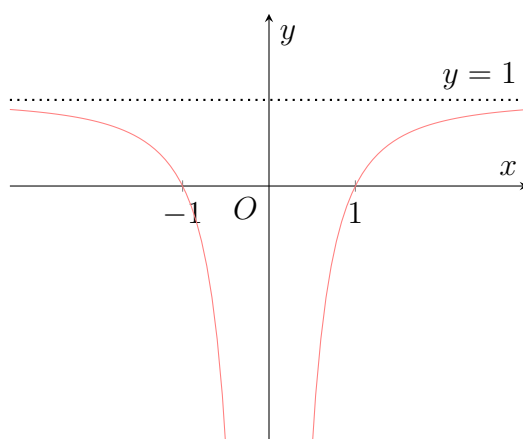
Observe that for all  $x \in \mathbb{R}$ ,  $\left( \cos \frac{x}{2} - 1 \right)^2 \geq 0$ . Hence,  $f'(x) < 0$  for all real values of  $x$ . Thus,  $f(x)$  is strictly decreasing on  $\mathbb{R}$ .

Note that  $f(0) = 8 \sin 0 - \sin 0 - 4 \cdot 0 = 0$ . Since  $f(x)$  is strictly decreasing, for all  $x > 0$ ,  $f(x) < f(0) = 0$ .

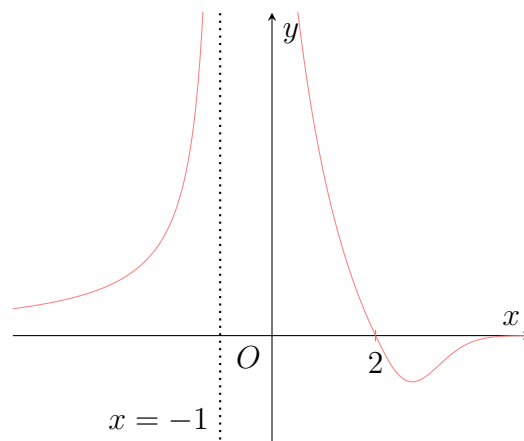
**Problem 4.**

Sketch the graphs of the derivative functions for each of the graphs of the following functions below. In each graph, the point(s) labelled in coordinate form are stationary points.

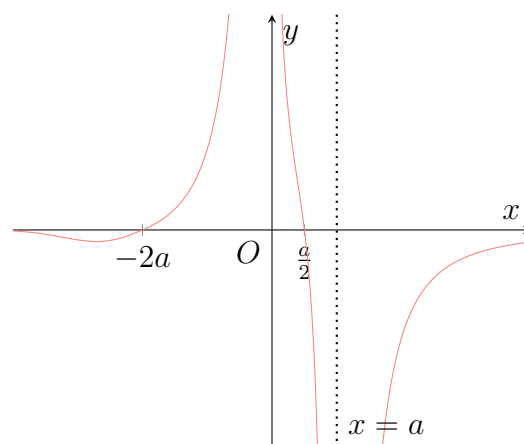


**Solution****Part (a)****Part (b)**

**Part (c)**



**Part (d)**



**Problem 5.**

- (a) Given that  $y = ax\sqrt{x+2}$  where  $a > 0$ , find  $\frac{dy}{dx}$ , expressing your answer as a single algebraic fraction. Hence show that the curve  $y = ax\sqrt{x+2}$  has only one turning point, and state its coordinates in exact form.
- (b) Sketch the graph of  $y = f'(x)$ , where  $f(x) = ax\sqrt{x+2}$ , where  $a > 0$ .

**Solution****Part (a)**

$$\begin{aligned}\frac{dy}{dx} &= a \left( x \cdot \frac{1}{2\sqrt{x+2}} + \sqrt{x+2} \right) \\ &= a \left( \frac{x}{2\sqrt{x+2}} + \frac{2(x+2)}{2\sqrt{x+2}} \right) \\ &= \frac{a(3x+4)}{2\sqrt{x+2}}\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{a(3x+4)}{2\sqrt{x+2}}}$$

Consider the stationary points of  $y = ax\sqrt{x+2}$ . For stationary points,  $\frac{dy}{dx} = 0$ .

$$\begin{aligned}\frac{dy}{dx} &= 0 \\ \implies \frac{a(3x+4)}{2\sqrt{x+2}} &= 0 \\ \implies a(3x+4) &= 0\end{aligned}$$

Since  $a > 0$ , we have  $3x+4=0$ , whence  $x = -\frac{4}{3}$ .

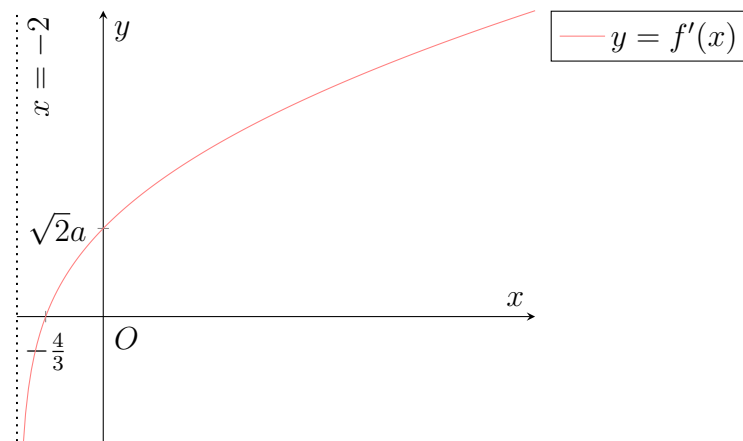
$x$	$\left(-\frac{4}{3}\right)^{-}$	$-\frac{4}{3}$	$\left(-\frac{4}{3}\right)^{+}$
$\frac{dy}{dx}$	-ve	0	+ve

Hence, at  $x = -\frac{4}{3}$ , there is a turning point (minimum point). Thus,  $y = ax\sqrt{x+2}$  has only one turning point.

Substituting  $x = -\frac{4}{3}$  into  $y = ax\sqrt{x+2}$ , we see that  $y = -\frac{4a}{3}\sqrt{\frac{2}{3}}$ . Hence, the coordinate of the turning point is  $\left(-\frac{4}{3}, -\frac{4a}{3}\sqrt{\frac{2}{3}}\right)$ .

$$\boxed{\left(-\frac{4}{3}, -\frac{4a}{3}\sqrt{\frac{2}{3}}\right)}$$

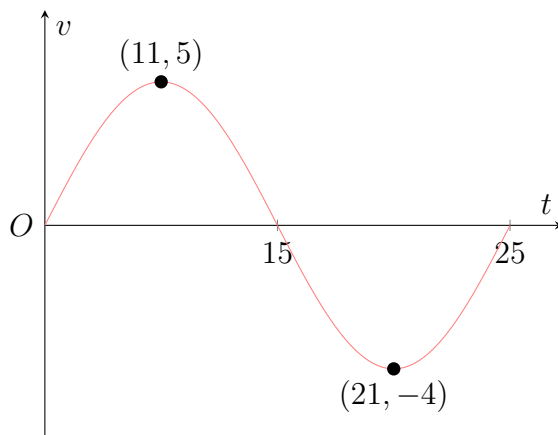
**Part (b)**





**Problem 6.**

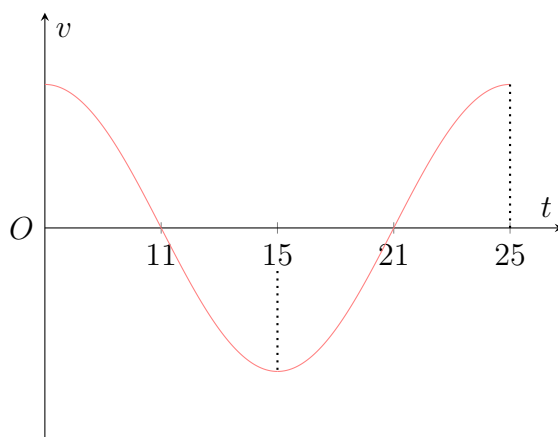
A particle  $P$  moves along the  $x$ -axis. Initially,  $P$  is at the origin  $O$ . At time  $t$  s, the velocity is  $v$   $\text{ms}^{-1}$  and the acceleration is  $a$   $\text{ms}^{-2}$ . Below is the velocity-time graph of the particle for  $0 \leq t \leq 25$ .



- (a) Describe the motion of the particle for  $0 \leq t \leq 25$ .
- (b) Sketch the acceleration-time graph of the particle  $P$ .

**Solution****Part (a)**

From  $t = 0$  to  $t = 11$ ,  $P$  speeds up and reaches a top speed of  $5 \text{ ms}^{-1}$ . From  $t = 11$  to  $t = 15$ ,  $P$  slows down. At  $t = 15$ ,  $P$  is instantaneously at rest. From  $t = 15$  to  $t = 21$ ,  $P$  speeds up and moves in the opposite direction, reaching a top speed of  $4 \text{ ms}^{-1}$ . From  $t = 21$  to  $t = 25$ ,  $P$  slows down. At  $t = 25$ ,  $P$  is instantaneously at rest.

**Part (b)**

**Problem 7.**

The function  $f$  defined by  $f(x) = \ln x - 2\left(x - \frac{1}{2}\right)$ , where  $x \in \mathbb{R}, x > 0$ . Find  $f'(x)$  and show that the function is decreasing for  $x > \frac{1}{2}$ . Hence show that for  $x > \frac{1}{2}$ ,  $2\left(x - \frac{1}{2}\right) - \ln x > \ln 2$ .

**Solution**

$$f' = \frac{1}{x} - 2$$

When  $x > \frac{1}{2}$ ,  $\frac{1}{x} < 2 \implies \frac{1}{x} - 2 < 0$ . Thus,  $f'(x) < 0$ , whence  $f(x)$  is decreasing.

Note that  $f\left(\frac{1}{2}\right) = \ln \frac{1}{2} - 2\left(\frac{1}{2} - \frac{1}{2}\right) = -\ln 2$ . Since  $f(x)$  is decreasing for all  $x > \frac{1}{2}$ ,  
 $f(x) < f\left(\frac{1}{2}\right) = -\ln 2 \implies \ln x - 2\left(x - \frac{1}{2}\right) < -\ln 2 \implies 2\left(x - \frac{1}{2}\right) - \ln x > \ln 2$ .