

# Double Math Solutions

<https://asdia.dev/projects/doublemath>

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**Part I.**

**Group A**



# A1. Equations and Inequalities

## Tutorial A1

**Problem 1.** Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

$$(a) \begin{cases} a + 2b - 3c = -5 \\ -2a - 4b - 6c = 10 \\ 3a + 7b - 2c = -13 \end{cases}$$

$$(b) \begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$$

$$(c) \begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$$

$$(d) \begin{cases} 1/p + 1/q + 1/r = 5 \\ 2/p - 3/q - 4/r = -11 \\ 3/p + 2/q - 1/r = -6 \end{cases}$$

$$(e) \begin{cases} 2 \sin \alpha - \cos \beta + 3 \tan \gamma = 3 \\ 4 \sin \alpha + 2 \cos \beta - 2 \tan \gamma = 2, \text{ where } 0 \leq \alpha \leq 2\pi, 0 \leq \beta \leq 2\pi, \text{ and } 0 \leq \gamma < \pi. \\ 6 \sin \alpha - 3 \cos \beta + \tan \gamma = 9 \end{cases}$$

**Solution.**

**Part (a).** Unique solution:  $a = -9$ ,  $b = 2$ ,  $c = 0$ .

**Part (b).** No solution.

**Part (c).** Infinitely many solutions:  $x_1 = 8 - t$ ,  $x_2 = t - 3$ ,  $x_3 = t$ .

**Part (d).** Solving, we obtain

$$\frac{1}{p} = 2, \quad \frac{1}{q} = -3, \quad \frac{1}{r} = 6.$$

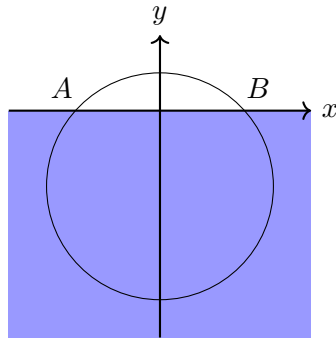
There is hence a unique solution:  $p = \frac{1}{2}$ ,  $q = -\frac{1}{3}$ ,  $r = \frac{1}{6}$ .

**Part (e).** Solving, we obtain

$$\sin \alpha = 1, \quad \cos \beta = -1, \quad \tan \gamma = 0.$$

There is hence a unique solution:  $\alpha = \frac{\pi}{2}$ ,  $\beta = \pi$ ,  $\gamma = 0$ .

**Problem 2.** The following figure shows the circular cross-section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0.$$

$A$  and  $B$  are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when  $AB$  is 40 cm apart horizontally, determine the values of  $a$ ,  $b$  and  $c$  by forming a system of linear equations.

**Solution.** Since  $AB = 40$ , we have  $A(-20, 0)$  and  $B(20, 0)$ . We also know  $(0, 10)$  lies on the circle. Substituting these points into the given equation, we have the following system of equations:

$$\begin{cases} -20a & + c = -400 \\ 20a & + c = -400 \\ & 10b + c = -100 \end{cases}$$

Solving, we obtain  $a = 0$ ,  $b = 30$ ,  $c = -400$ .

\* \* \* \* \*

**Problem 3.** Find the exact solution set of the following inequalities.

- (a)  $x^2 - 2 \geq 0$
- (b)  $4x^2 - 12x + 10 > 0$
- (c)  $x^2 + 4x + 13 < 0$
- (d)  $x^3 < 6x - x^2$
- (e)  $x^2(x - 1)(x + 3) \geq 0$

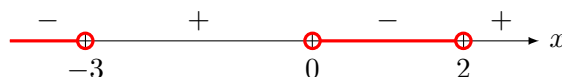
**Solution.**

**Part (a).** Note that  $x^2 - 2 \geq 0 \implies x \leq -\sqrt{2}$  or  $x \geq \sqrt{2}$ . The solution set is thus  $\{x \in \mathbb{R} : x \leq -\sqrt{2} \text{ or } x \geq \sqrt{2}\}$ .

**Part (b).** Completing the square, we see that  $4x^2 - 12x + 10 > 0 \implies (x - \frac{3}{2})^2 + \frac{19}{4} > 0$ . Since  $(x - \frac{3}{2})^2 \geq 0$ , all  $x \in \mathbb{R}$  satisfy the inequality, whence the solution set is  $\mathbb{R}$ .

**Part (c).** Completing the square, we have  $x^2 + 4x + 13 < 0 \implies (x + 2)^2 + 9 < 0$ . Since  $(x + 2)^2 \geq 0$ , there is no solution to the inequality, whence the solution set is  $\emptyset$ .

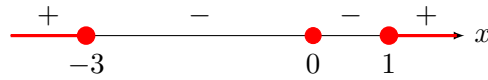
**Part (d).** Note that  $x^3 < 6x - x^2 \implies x(x + 3)(x - 2) < 0$ .





The solution set is thus  $\{x \in \mathbb{R}: x < -3 \text{ or } 0 < x < 2\}$ .

**Part (e).**



The solution set is thus  $\{x \in \mathbb{R}: x \leq -3 \text{ or } x = 0 \text{ or } x \geq 1\}$ .

\* \* \* \* \*

**Problem 4.** Find the exact solution set of the following inequalities.

(a)  $|3x + 5| < 4$

(b)  $|x - 2| < 2x$

**Solution.**

**Part (a).** If  $3x + 5 < 4$ , then  $x < -\frac{1}{3}$ . If  $-(3x + 5) < 4$ , then  $x > -3$ . Combining both inequalities, we have  $-3 < x < -\frac{1}{3}$ . Thus, the solution set is  $\{x \in \mathbb{R}: -3 < x < -\frac{1}{3}\}$ .

**Part (b).** If  $x - 2 < 2x$ , then  $x > -2$ . If  $-(x - 2) < 2x$ , then  $x > \frac{2}{3}$ . Combining both inequalities, we have  $x > \frac{2}{3}$ . Thus, the solution set is  $\{x \in \mathbb{R}: x > \frac{2}{3}\}$ .

\* \* \* \* \*

**Problem 5.** It is given that  $p(x) = x^4 + ax^3 + bx^2 + cx + d$ , where  $a$ ,  $b$ ,  $c$  and  $d$  are constants. Given that the curve with equation  $y = p(x)$  is symmetrical about the  $y$ -axis, and that it passes through the points with coordinates  $(1, 2)$  and  $(2, 11)$ , find the values of  $a$ ,  $b$ ,  $c$  and  $d$ .

**Solution.** We know that  $(1, 2)$  and  $(2, 11)$  lie on the curve. Since  $y = p(x)$  is symmetrical about the  $y$ -axis, we have that  $(-1, 2)$  and  $(-2, 11)$  also lie on the curve. Substituting these points into  $y = p(x)$ , we obtain the following system of equations:

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving, we obtain  $a = 0$ ,  $b = -2$ ,  $c = 0$ ,  $d = 3$ .

\* \* \* \* \*

**Problem 6.** Mr Mok invested \$50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of \$3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

**Solution.** Let  $a$ ,  $b$  and  $c$  be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a + b + c = 50000 \\ \frac{6}{100}a + \frac{8}{100}b + \frac{10}{100}c = 3700 \\ a = 2c \end{cases}$$

Solving, we have  $a = 30000$ ,  $b = 5000$  and  $c = 15000$ . Thus, Mr Mok invested \$30,000, \$5,000 and \$15,000 in Funds A, B and C respectively.

\* \* \* \* \*

**Problem 7.** Solve the following inequalities with exact answers.

(a)  $2x - 1 \geq \frac{6}{x}$

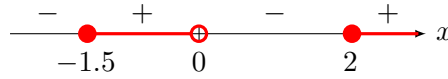
(b)  $x - \frac{1}{x} < 1$

(c)  $-1 < \frac{2x+3}{x-1} < 1$

**Solution.**

**Part (a).** Note that  $x \neq 0$ .

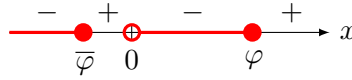
$$2x - 1 \geq \frac{6}{x} \implies x^2(2x - 1) \geq 6x \implies x(2x^2 - x - 6) \geq 0 \implies x(2x + 3)(x - 2) \geq 0.$$



Thus,  $-\frac{3}{2} \leq x < 0$  or  $x \geq 2$ .

**Part (b).** Note that  $x \neq 0$ .

$$x - \frac{1}{x} < 1 \implies x^3 - x < x^2 \implies x(x^2 - x - 1) < 0 \implies x(x - \varphi)(x - \bar{\varphi}) < 0.$$



Thus,  $x \leq \bar{\varphi}$  or  $0 < x \leq \varphi$ .

**Part (c).**

$$-1 < \frac{2x+3}{x-1} < 1 \implies -3 < \frac{5}{x-1} < -1 \implies -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5} \implies -4 < x < -\frac{2}{3}.$$

\* \* \* \* \*

**Problem 8.** Without using a calculator, solve the inequality  $\frac{x^2+x+1}{x^2+x-2} < 0$ .

**Solution.** Observe that  $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4} > 0$ . The inequality thus reduces to  $\frac{1}{x^2+x-2} < 0$ .

$$\frac{1}{x^2+x-2} < 0 \implies x^2+x-2 < 0 \implies (x-1)(x+2) < 0.$$



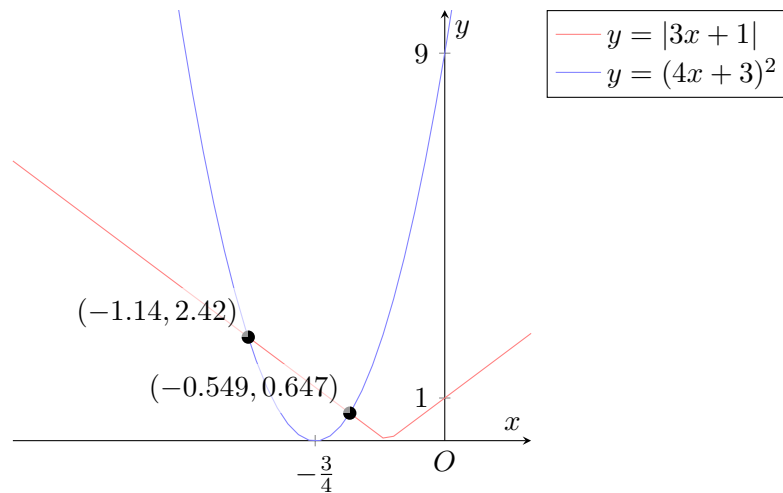
Hence,  $-2 < x < 1$ .

**Problem 9.** Solve the following inequalities using a graphical method.

- (a)  $|3x + 1| < (4x + 3)^2$
- (b)  $|3x + 1| \geq |2x + 7|$
- (c)  $|x - 2| \geq x + |x|$
- (d)  $5x^2 + 4x - 3 > \ln(x + 1)$

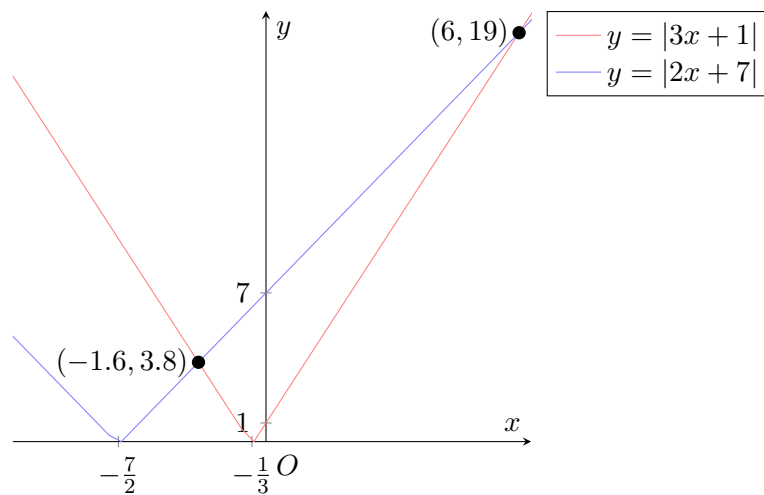
**Solution.**

**Part (a).**



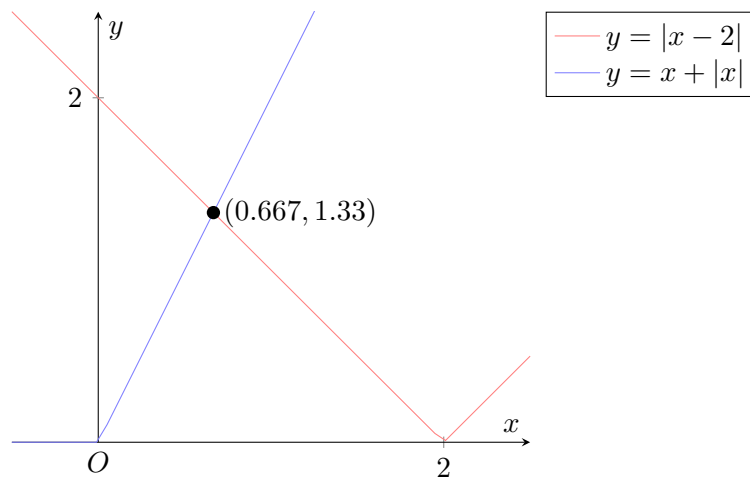
Thus,  $x < -1.14$  or  $x > -0.549$ .

**Part (b).**



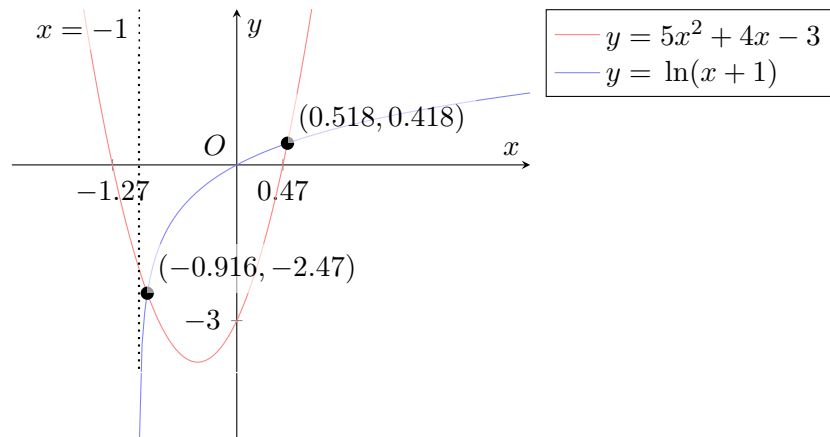
Thus,  $x \leq -1.6$  or  $x \geq 6$ .

**Part (c).**



Thus,  $x \leq 0.667$ .

**Part (d).**

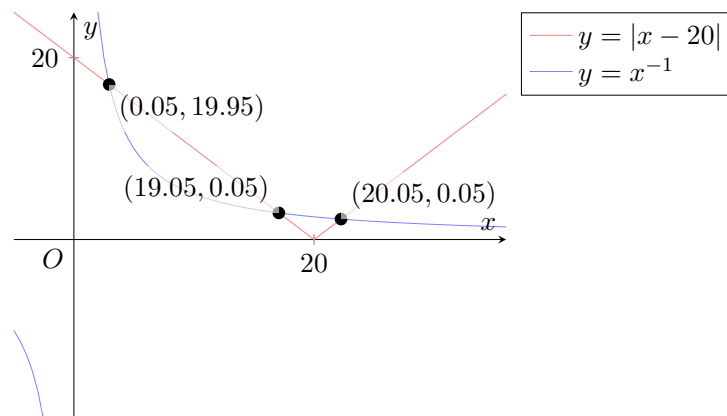


Thus,  $-1 < x < -0.916$  or  $x > 0.518$ .

\* \* \* \* \*

**Problem 10.** Sketch the graphs of  $y = |x - 20|$  and  $y = \frac{1}{x}$  on the same diagram. Hence or otherwise, solve the inequality  $|x - 20| < \frac{1}{x}$ , leaving your answers correct to 2 decimal places.

**Solution.**



Thus,  $0 < x < 0.05$  or  $19.95 < x < 20.05$ .

\* \* \* \* \*

**Problem 11.** Solve the inequality  $\frac{x-9}{x^2-9} \leq 1$ . Hence, solve the inequalities

(a)  $\frac{|x|-9}{x^2-9} \leq 1$

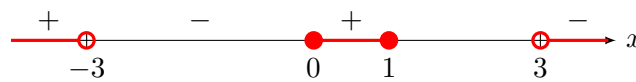
(b)  $\frac{x+9}{x^2-9} \geq -1$

**Solution.** Note that  $x^2 - 9 \neq 0 \implies x \neq \pm 3$ .

$$\frac{x-9}{x^2-9} \leq 1 \implies (x-9)(x^2-9) \leq (x^2-9)^2.$$

Expanding and factoring, we get

$$x^4 - x^3 - 9x^2 + 9x = x(x+3)(x-1)(x-3) \geq 0.$$



Thus,  $x < -3$  or  $0 \leq x \leq 1$  or  $x > 3$ .

**Part (a).** Consider the substitution  $x \mapsto |x|$ . Then

$$|x| < -3 \text{ or } 0 \leq |x| \leq 1 \text{ or } |x| > 3.$$

This immediately gives us  $x < -3$  or  $-1 \leq x \leq 1$  or  $x > 3$ .

**Part (b).** Consider the substitution  $x \mapsto -x$ . Then

$$-x < -3 \text{ or } 0 \leq -x \leq 1 \text{ or } -x > 3.$$

This immediately gives us  $x < -3$  or  $-1 \leq x \leq 0$  or  $x > 3$ .

\* \* \* \* \*

**Problem 12.** Solve the inequality  $\frac{x-5}{1-x} \geq 1$ . Hence, solve  $0 < \frac{1-\ln x}{\ln x-5} \leq 1$ .

**Solution.** Note that  $x \neq 1$ .

$$\frac{x-5}{1-x} \geq 1 \implies (x-5)(1-x) \geq (1-x)^2 \implies 2x^2 - 8x + 6 \leq 0 \implies 2(x-1)(x-3) \leq 0.$$



Thus,  $1 < x \leq 3$ .

Consider the substitution  $x \mapsto \ln x$ . Taking reciprocals, we have our desired inequality  $0 < \frac{1-\ln x}{\ln x-5} \leq 1$ . Hence,

$$1 < \ln x \leq 3 \implies e < x \leq e^3.$$

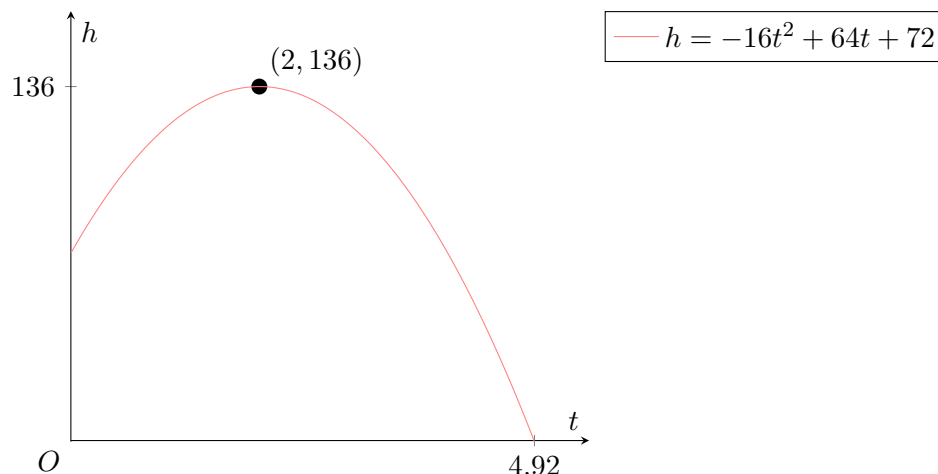
\* \* \* \* \*

**Problem 13.** A small rocket is launched from a height of 72 m from the ground. The height of the rocket in metres,  $h$ , is represented by the equation  $h = -16t^2 + 64t + 72$ , where  $t$  is the time in seconds after the launch.

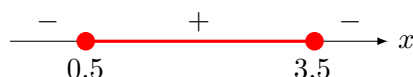
- (a) Sketch the graph of  $h$  against  $t$ .
- (b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

**Solution.**

**Part (a).**



**Part (b).** Note that  $-16t^2 + 64t + 72 \geq 100 \implies -4(2t - 1)(2t - 7) \geq 0$ .

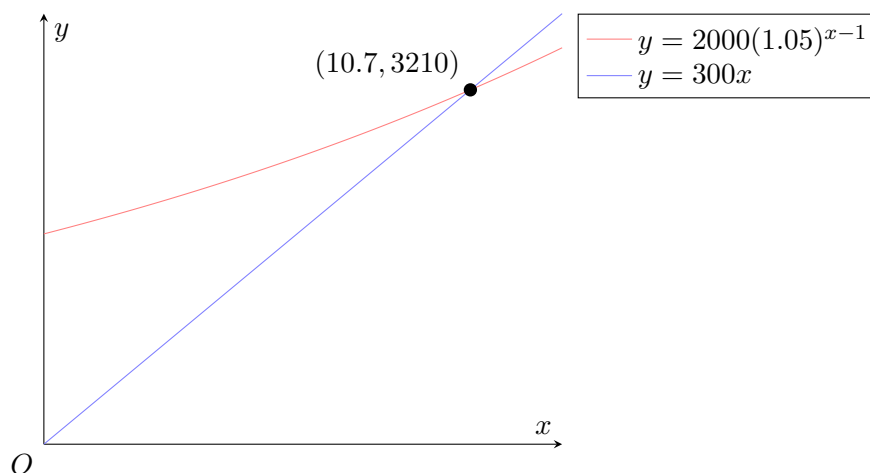


Thus, the rocket will remain at or above 100 m from the ground for 3 seconds.

\* \* \* \* \*

**Problem 14.** Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of  $2000(1.05)^{n-1}$  dollars in the  $n$ th quarter, where  $n$  is a positive integer. She also gives a regular donation of  $\$300n$  in the  $n$ th quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

**Solution.** Consider the curves  $y = 2000(1.05)^{x-1}$  and  $y = 300x$ .



Hence, Xinxin will stop donating in the 11th quarter.

## Assignment A1

**Problem 1.** A traveller just returned from Germany, France and Spain. The amount (in dollars) that he spent each day on housing, food and incidental expenses in each country are shown in the table below.

| Country | Housing | Food | Incidental Expenses |
|---------|---------|------|---------------------|
| Germany | 28      | 30   | 14                  |
| France  | 23      | 25   | 8                   |
| Spain   | 19      | 22   | 12                  |

The traveller's records of the trip indicate a total of \$191 spent for housing, \$430 for food and \$180 for incidental expenses. Calculate the number of days the traveller spent in each country.

He did his account again and the amount spent on food is \$337. Is this record correct? Why?

**Solution.** Let  $g$ ,  $f$  and  $s$  represent the number of days the traveller spent in Germany, France and Spain respectively. From the table, we obtain the following system of equations:

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 430 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution  $g = 4$ ,  $f = 8$  and  $s = 5$ . The traveller thus spent 4 days in Germany, 8 days in France and 5 days in Spain.

Consider the scenario where the amount spent on food is \$337.

$$\begin{cases} 23f + 28g + 19s = 391 \\ 25f + 30g + 22s = 337 \\ 8f + 14g + 12s = 180 \end{cases}$$

This gives the unique solution  $g = 66$ ,  $f = -27$  and  $s = -44$ . The record is hence incorrect as  $f$  and  $s$  must be positive.

\* \* \* \* \*

### Problem 2.

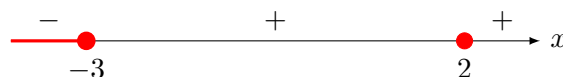
(a) Solve algebraically  $x^2 - 9 \geq (x + 3)(x^2 - 3x + 1)$ .

(b) Solve algebraically  $\frac{7-2x}{3-x^2} \leq 1$ .

**Solution.**

**Part (a).**

$$\begin{aligned} x^2 - 9 &\geq (x + 3)(x^2 - 3x + 1) \\ \implies (x + 3)(x - 3) &\geq (x + 3)(x^2 - 3x + 1) \\ \implies (x + 3)(x^2 - 4x + 4) &\leq 0 \\ \implies (x + 3)(x - 2)^2 &\leq 0 \end{aligned}$$



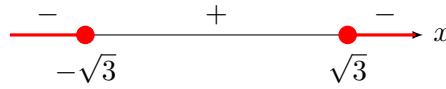
Thus,  $x \leq -3$  or  $x = 2$ .

**Part (b).** Note that  $3 - x^2 \neq 0 \implies x \neq \pm\sqrt{3}$ .

$$\begin{aligned} & \frac{7-2x}{3-x^2} \leq 1 \\ \implies & \frac{7-2x}{3-x^2} - \frac{3-x^2}{3-x^2} \leq 0 \\ \implies & \frac{x^2-2x+4}{3-x^2} \leq 0 \end{aligned}$$

Observe that  $x^2 - 2x + 4 = (x-1)^2 + 3 > 0$ . Dividing through by  $x^2 - 2x + 4$ , we obtain

$$\begin{aligned} & \frac{1}{3-x^2} \leq 0 \\ \implies & 3-x^2 \leq 0 \end{aligned}$$



Thus,  $x < -\sqrt{3}$  or  $x > \sqrt{3}$ .

\* \* \* \* \*

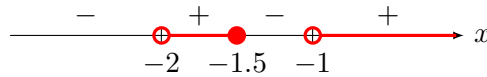
### Problem 3.

- (a) Without using a calculator, solve the inequality  $\frac{3x+4}{x^2+3x+2} \geq \frac{1}{x+2}$ .
- (b) Hence, deduce the set of values of  $x$  that satisfies  $\frac{3|x|+4}{x^2+3|x|+2} \geq \frac{1}{|x|+2}$ .

**Solution.**

**Part (a).** Note that  $x^2 + 3x + 2 \neq 0$  and  $x + 2 \neq 0$ , whence  $x \neq -1, -2$ .

$$\begin{aligned} & \frac{3x+4}{x^2+3x+2} \geq \frac{1}{x+2} \\ \implies & \frac{3x+4}{(x+2)(x+1)} \geq \frac{1}{x+2} \\ \implies & (3x+4)(x+2)(x+1) \geq (x+2)(x+1)^2 \\ \implies & (x+2)(x+1)(2x+3) \geq 0 \end{aligned}$$



Thus,  $-2 < x \leq -\frac{3}{2}$  or  $x > -1$ .

**Part (b).** Observe that  $|x|^2 = x^2$ . Hence, with the map  $x \mapsto |x|$ , we obtain

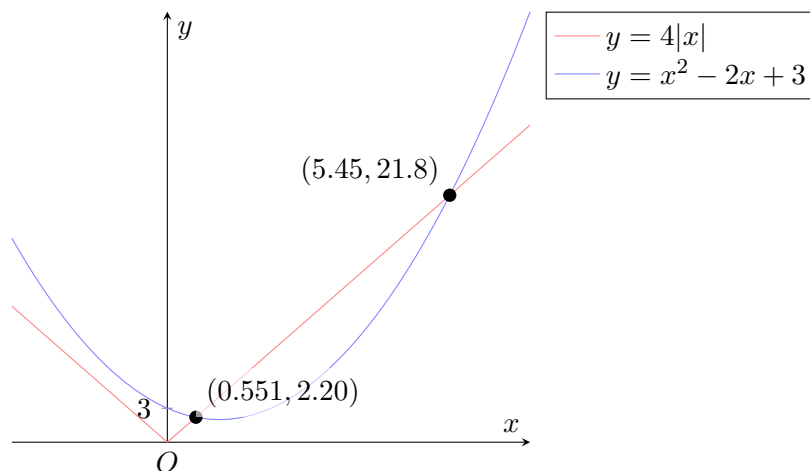
$$-2 < |x| \leq -\frac{3}{2} \text{ or } |x| > -1.$$

Since  $|x| \geq 0$ , we have that  $|x| > -1$  is satisfied for all real  $x$ . Hence, the solution set is  $\mathbb{R}$ .



**Problem 4.** On the same diagram, sketch the graphs of  $y = 4|x|$  and  $y = x^2 - 2x + 3$ . Hence or otherwise, solve the inequality  $4|x| \geq x^2 - 2x + 3$ .

**Solution.**



From the graph, we see that  $0.551 \leq x \leq 5.45$ .

## A2. Numerical Methods of Finding Roots

### Tutorial A2

**Problem 1.** Without using a graphing calculator, show that the equation  $x^3 + 2x^2 - 2 = 0$  has exactly one positive root.

This root is denoted by  $\alpha$  and is to be found using two different iterative methods, starting with the same initial approximation in each case.

- (a) Show that  $\alpha$  is a root of the equation  $x = \sqrt{\frac{2}{x+2}}$ , and use the iterative formula  $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$ , with  $x_1 = 1$ , to find  $\alpha$  correct to 2 significant figures.
- (b) Use the Newton-Raphson method, with  $x_1 = 1$ , to find  $\alpha$  correct to 3 significant figures.

**Solution.** Let  $f(x) = x^3 + 2x^2 - 2$ . Observe that for all  $x > 0$ , we have  $f'(x) = 3x^2 + 4x > 0$ . Hence,  $f(x)$  is strictly increasing on  $(0, \infty)$ . Since  $f(0)f(1) = (-2)(1) < 0$ , it follows that  $f(x)$  has exactly one positive root.

**Part (a).** We know  $f(\alpha) = 0$ . Hence,

$$\alpha^3 + 2\alpha^2 - 2 = 0 \implies \alpha^2(\alpha + 2) = 2 \implies \alpha^2 = \frac{2}{\alpha + 2} \implies \alpha = \sqrt{\frac{2}{\alpha + 2}}.$$

Note that we reject the negative branch since  $\alpha > 0$ . We hence see that  $\alpha$  is a root of the equation  $x = \sqrt{\frac{2}{x+2}}$ . Using the iterative formula  $x_{n+1} = \sqrt{\frac{2}{x_n+2}}$  with  $x_1 = 1$ , we have

| $n$ | $x_n$   |
|-----|---------|
| 1   | 1       |
| 2   | 0.81650 |
| 3   | 0.84268 |
| 4   | 0.83879 |

Hence,  $\alpha = 0.84$  (2 s.f.).

**Part (b).** Using the Newton-Raphson method ( $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ) with  $x_1 = 1$ , we have

| $n$ | $x_n$    |
|-----|----------|
| 1   | 1        |
| 2   | 0.857143 |
| 3   | 0.839545 |
| 4   | 0.839287 |
| 5   | 0.839287 |

Hence,  $\alpha = 0.839$  (3 s.f.).

**Problem 2.**

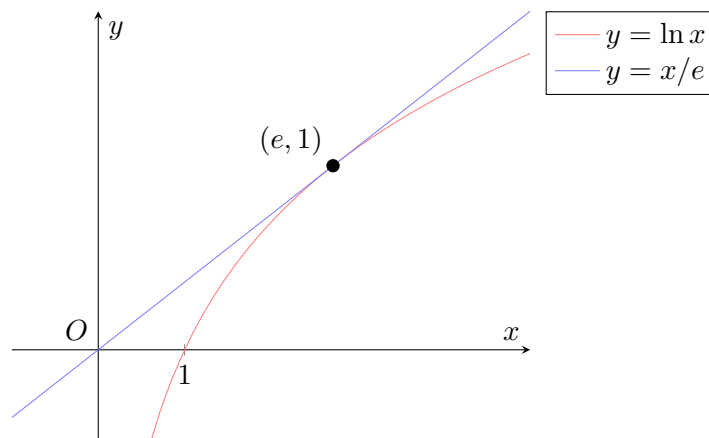
- (a) Show that the tangent at the point  $(e, 1)$  to the graph  $y = \ln x$  passes through the origin, and deduce that the line  $y = mx$  cuts the graph  $y = \ln x$  in two points provided that  $0 < m < \frac{1}{e}$ .
- (b) For each root of the equation  $\ln x = \frac{1}{3}x$ , find an integer  $n$  such that the interval  $n < x < n + 1$  contains the root. Using linear interpolation, based on  $x = n$  and  $x = n + 1$ , find a first approximation to the smaller root, giving your answer to 1 decimal place. Using your first approximation, obtain, by the Newton-Raphson method, a second approximation to the smaller root, giving your answer to 2 decimal places.

**Solution.**

**Part (a).** Note that the derivative of  $y = \ln x$  at  $x = e$  is  $\frac{1}{e}$ . Using the point slope formula, we see that the equation of the tangent at the point  $(e, 1)$  is given by

$$y - 1 = \frac{x - e}{e} \implies y = \frac{x}{e}.$$

Since  $x = 0, y = 0$  is clearly a solution, the tangent passes through the origin. From the graph below, it is clear that for  $y = mx$  to intersect  $y = \ln x$  twice, we must have  $0 < m < \frac{1}{e}$ .



**Part (b).** Consider  $f(x) = \frac{1}{3}x - \ln x$ . Let  $\alpha$  and  $\beta$  be the smaller and larger root to  $f(x) = 0$  respectively. Observe that  $f(1)f(2) = (1)(-0.03) < 0$  and  $f(4)f(5) = (-0.05)(0.06) < 0$ . Thus, for the smaller root  $\alpha$ ,  $n = 1$ , while for the larger root  $\beta$ ,  $n = 4$ .

Let  $x_1$  be the first approximation to  $\alpha$ . Using linear interpolation, we have

$$x_1 = \frac{1f(2) - 2f(1)}{f(2) - f(1)} = 1.9 \text{ (1 d.p.)}$$

Note that  $f'(x) = \frac{1}{3} - \frac{1}{x}$ . Using the Newton-Raphson method ( $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ), we have

| $n$ | $x_n$   |
|-----|---------|
| 1   | 1.9     |
| 2   | 1.85585 |
| 3   | 1.85718 |

Hence,  $\alpha = 1.86$  (2 d.p.).

**Problem 3.** Find the exact coordinates of the turning points on the graph of  $y = f(x)$  where  $f(x) = x^3 - x^2 - x - 1$ . Deduce that the equation  $f(x) = 0$  has only one real root  $\alpha$ , and prove that  $\alpha$  lies between 1 and 2. Use the Newton-Raphson method applied to the equation  $f(x) = 0$  to find a second approximation  $x_2$  to  $\alpha$ , taking  $x_1$ , the first approximation, to be 2. With reference to a graph of  $y = f(x)$ , explain why all further approximations to  $\alpha$  by this process are always larger than  $\alpha$ .

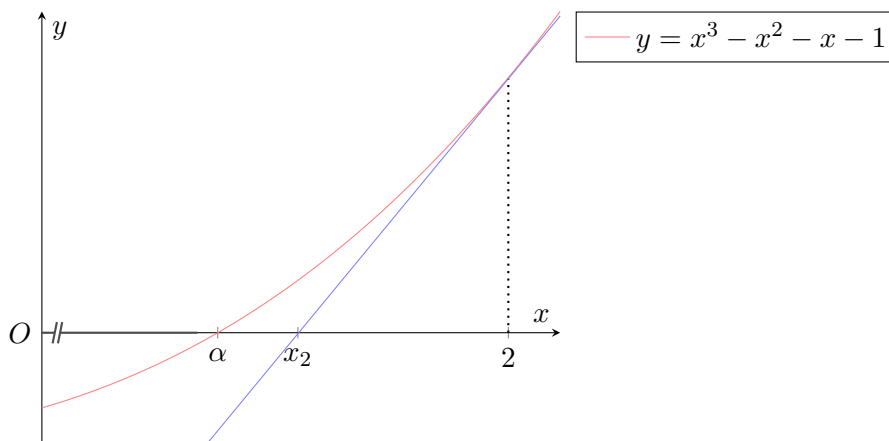
**Solution.** For turning points,  $f'(x) = 0$ .

$$f'(x) = 0 \implies 3x^2 - 2x - 1 = 0 \implies (3x + 1)(x - 1) = 0.$$

Hence,  $x = -\frac{1}{3}$  or  $x = 1$ . When  $x = -\frac{1}{3}$ , we have  $y = -0.815$ , giving the coordinate  $(-\frac{1}{3}, -0.815)$ . When  $x = 1$ , we have  $y = -2$ , giving the coordinate  $(1, -2)$ .

Observe that  $f(x)$  is strictly increasing for all  $x > 1$ . Further, since both turning points have a negative  $y$ -coordinate, it follows that  $y < 0$  for all  $x \leq 1$ . Since  $f(1)f(2) = (-2)(1) < 0$ , the equation  $f(x) = 0$  has only one real root.

Using the Newton-Raphson method with  $x_1 = 2$ , we have  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{13}{7}$ .



Since  $x_2$  lies on the right of  $\alpha$ , the Newton-Raphson method gives an over-estimation given an initial approximation of 2. Thus, all further approximations to  $\alpha$  will also be larger than  $\alpha$ .

\* \* \* \* \*

**Problem 4.** A curve  $C$  has equation  $y = x^5 + 50x$ . Find the least value of  $\frac{dy}{dx}$  and hence give a reason why the equation  $x^5 + 50x = 10^5$  has exactly one real root. Use the Newton-Raphson method, with a suitable first approximation, to find, correct to 4 decimal places, the root of the equation  $x^5 + 50x = 10^5$ . You should demonstrate that your answer has the required accuracy.

**Solution.** Since  $y = x^5 + 50x$ , we have  $\frac{dy}{dx} = 5x^4 + 50$ . Since  $x^4 \geq 0$  for all real  $x$ , the minimum value of  $\frac{dy}{dx}$  is 50.

Let  $f(x) = x^5 + 50x$ . Since  $\min \frac{df}{dx} = 50 > 0$ , it follows that  $f(x)$  is strictly increasing. Hence,  $f(x)$  will intersect only once with the line  $y = 10^5$ , whence the equation  $x^5 + 50x = 10^5$  has exactly one real root.

Observe that  $f(9)f(10) = (-40901)(50) < 0$ . Thus, there must be a root in the interval  $(9, 10)$ . We now use the Newton-Raphson method ( $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ) with  $x_1 = 9$  as the first approximation.

| $n$ | $x_n$      |
|-----|------------|
| 1   | 9          |
| 2   | 10.2178921 |
| 3   | 10.0017491 |
| 4   | 9.9901221  |
| 5   | 9.9899912  |
| 6   | 9.9899900  |

Thus, the root is 9.9900 (4 d.p.).

Observe that  $f(9.98995)f(9.99005) = (-2.00)(3.00) < 0$ . Hence, the root lies in the interval  $(9.98995, 9.99005)$  whence the calculated root has the required accuracy.

\* \* \* \* \*

### Problem 5.

- (a) A function  $f$  is such that  $f(4) = 1.158$  and  $f(5) = -3.381$ , correct to 3 decimal places in each case. Assuming that there is a value of  $x$  between 4 and 5 for which  $f(x) = 0$ , use linear interpolation to estimate this value.

For the case when  $f(x) = \tan x$ , and  $x$  is measured in radians, the value of  $f(4)$  and  $f(5)$  are as given above. Explain, with the aid of a sketch, why linear interpolation using these values does not give an approximation to a solution of the equation  $\tan x = 0$ .

- (b) Show, by means of a graphical argument or otherwise, that the equation  $\ln(x-1) = -2x$  has exactly one real root, and show that this root lies between 1 and 2.

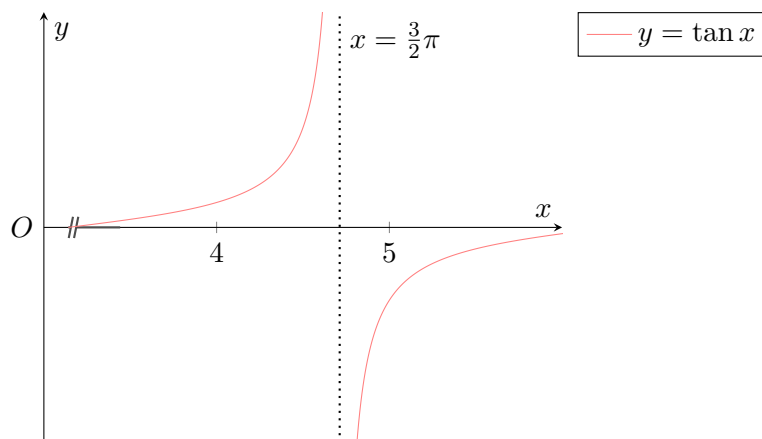
The equation may be written in the form  $\ln(x-1) + 2x = 0$ . Show that neither  $x = 1$  nor  $x = 2$  is a suitable initial value for the Newton-Raphson method in this case.

The equation may also be written in the form  $x - 1 - e^{-2x} = 0$ . For this form, use two applications of the Newton-Raphson method, starting with  $x = 1$ , to obtain an approximation to the root, giving 3 decimal places in your answer.

### Solution.

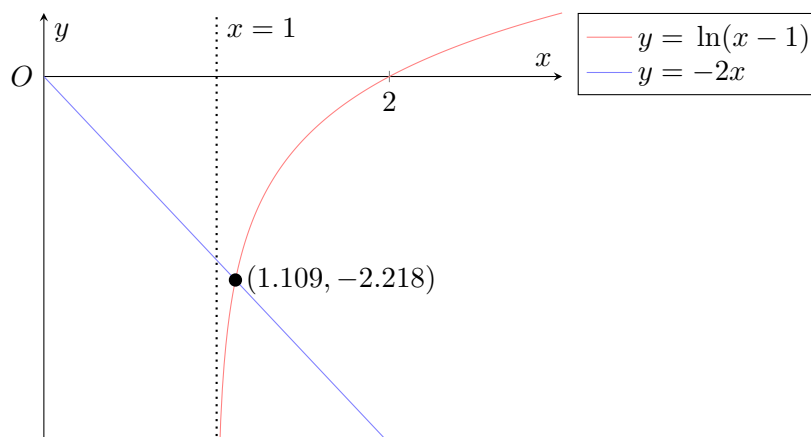
**Part (a).** Let the root of  $f(x) = 0$  be  $\alpha$ . Using linear interpolation on the interval  $[4, 5]$ , we have

$$\alpha = \frac{4f(5) - 5f(4)}{f(5) - f(4)} = 4.255 \text{ (3 d.p.)}.$$



Since  $\tan x$  has a vertical asymptote at  $x = \frac{3}{2}\pi$ , it is not continuous on  $[4, 5]$ . Thus, linear interpolation diverges when applied to the equation  $\tan x = 0$ .

**Part (b).**



Since there is only one intersection between the graphs  $y = \ln(x-1)$  and  $y = -2x$ , there is only one real root to the equation  $\ln(x-1) = -2x$ . Furthermore, since  $y = -2x$  is negative for all  $x > 0$  and  $y = \ln(x-1)$  is negative only when  $1 < x < 2$ , it follows that the root must lie between 1 and 2.

Let  $f(x) = \ln(x-1) + 2x$ . Then  $f'(x) = \frac{1}{x-1} + 2$ . Note that the Newton-Raphson method is given by  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ .

Since  $f'(1)$  is undefined, an initial approximation of  $x_1 = 1$  cannot be used for the Newton-Raphson method, which requires a division by  $f'(1)$ .

Using the Newton-Raphson method with the initial approximation  $x_2 = 2$ , we see that  $x_2 = 1$ . Once again, because  $f'(1)$  is undefined,  $x_1 = 2$  is also not a suitable initial value.

Let  $g(x) = x - 1 - e^{-2x}$ . Then  $g'(x) = 1 + 2e^{-2x}$ . Using the Newton-Raphson method with the initial approximation  $x_1 = 1$ , we have

| $n$ | $x_n$    |
|-----|----------|
| 1   | 1        |
| 2   | 1.106507 |
| 3   | 1.108857 |

Hence,  $x = 1.109$  (3 d.p.).

\* \* \* \* \*

**Problem 6.** The equation  $x = 3 \ln x$  has two roots  $\alpha$  and  $\beta$ , where  $1 < \alpha < 2$  and  $4 < \beta < 5$ . Using the iterative formula  $x_{n+1} = F(x_n)$ , where  $F(x) = 3 \ln x$ , and starting with  $x_0 = 4.5$ , find the value of  $\beta$  correct to 3 significant figures. Find a suitable  $F(x)$  for computing  $\alpha$ .

**Solution.** Using the iterative formula  $x_{n+1} = F(x_n)$ , we have

| $n$ | $x_n$   | $n$ | $x_n$   |
|-----|---------|-----|---------|
| 0   | 4.5     | 5   | 4.53175 |
| 1   | 4.51223 | 6   | 4.53333 |
| 2   | 4.52038 | 7   | 4.53437 |
| 3   | 4.52579 | 8   | 4.53506 |
| 4   | 4.52937 | 9   | 4.53551 |

Hence,  $\beta = 4.54$  (3 s.f.).

Note that  $x = 3 \ln x \implies x = e^{x/3}$ . Observe that  $\frac{d}{dx}(e^{x/3}) = \frac{1}{3}e^{x/3}$ , which is between  $-1$  and  $1$  for all  $1 < x < 2$ . Thus, the iterative formula  $x_{n+1} = F(x_n)$  will converge, whence  $F(x) = e^{x/3}$  is suitable for computing  $\alpha$ .

\* \* \* \* \*

**Problem 7.** Show that the cubic equation  $x^3 + 3x - 15 = 0$  has only one real root. This root is near  $x = 2$ . The cubic equation can be written in any one of the forms below:

(a)  $x = \frac{1}{3}(15 - x^3)$

(b)  $x = \frac{15}{x^2 + 3}$

(c)  $x = (15 - 3x)^{1/3}$

Determine which of these forms would be suitable for the use of the iterative formula  $x_{r+1} = F(x_r)$ , where  $r = 1, 2, 3, \dots$

Hence, find the root correct to 3 decimal places.

**Solution.** Let  $f(x) = x^3 + 3x - 15$ . Then  $f'(x) = 3x^2 + 3 > 0$  for all real  $x$ . Hence,  $f$  is strictly increasing. Since  $f$  is continuous,  $f(x) = 0$  has only one real root.

**Part (a).** Let  $g_1(x) = \frac{1}{3}(15 - x^3)$ . Then  $g'_1(x) = -x^2$ . For values of  $x$  near 2,  $|g'_1(x)| > 1$ . Hence, the iterative formula  $x_{n+1} = g_1(x_n)$  will diverge and  $g_1(x)$  is unsuitable.

**Part (b).** Let  $g_2(x) = \frac{15}{x^2 + 3}$ . Then  $g'_2(x) = \frac{-30x}{(x^2 + 3)^2}$ . For values of  $x$  near 2,  $|g'_2(x)| > 1$ . Hence, the iterative formula  $x_{n+1} = g_2(x_n)$  will diverge and  $g_2(x)$  is unsuitable.

**Part (c).** Let  $g_3(x) = (15 - 3x)^{1/3}$ . Then  $g'_3(x) = -(15 - 3x)^{-2/3}$ . For values of  $x$  near 2,  $|g'_3(x)| < 1$ . Hence, the iterative formula  $x_{n+1} = g_3(x_n)$  will converge and  $g_3(x)$  is suitable.

Using the iterative formula  $x_{r+1} = g_3(x_r)$ , we get

| $r$ | $x_r$    |
|-----|----------|
| 1   | 2        |
| 2   | 2.080084 |
| 3   | 2.061408 |
| 4   | 2.065793 |
| 5   | 2.064765 |

Hence,  $x = 2.065$  (3 d.p.).

\* \* \* \* \*

**Problem 8.** The equation of a curve is  $y = f(x)$ . The curve passes through the points  $(a, f(a))$  and  $(b, f(b))$ , where  $0 < a < b$ ,  $f(a) > 0$  and  $f(b) < 0$ . The equation  $f(x) = 0$  has precisely one root  $\alpha$  such that  $a < \alpha < b$ . Derive an expression, in terms of  $a$ ,  $b$ ,  $f(a)$  and  $f(b)$ , for the estimated value of  $\alpha$  based on linear interpolation.

Let  $f(x) = 3e^{-x} - x$ . Show that  $f(x) = 0$  has a root  $\alpha$  such that  $1 < \alpha < 2$ , and that for all  $x$ ,  $f'(x) < 0$  and  $f''(x) > 0$ . Obtain an estimate of  $\alpha$  using linear interpolation to 2 decimal places, and explain by means of a sketch whether the value obtained is an over-estimate or an under-estimate.

Use one application of the Newton-Raphson method to obtain a better estimate of  $\alpha$ , giving your answer to 2 decimal places.

**Solution.** Using the point-slope formula, the equation of the line that passes through both  $(a, f(a))$  and  $(b, f(b))$  is

$$y - f(a) = \frac{f(a) - f(b)}{a - b}(x - a).$$

Note that  $(\alpha, 0)$  is approximately the solution to the above equation. Thus,

$$0 - f(a) \approx \frac{f(a) - f(b)}{a - b}(\alpha - a) \implies \alpha \approx \frac{bf(a) - af(b)}{f(a) - f(b)}.$$

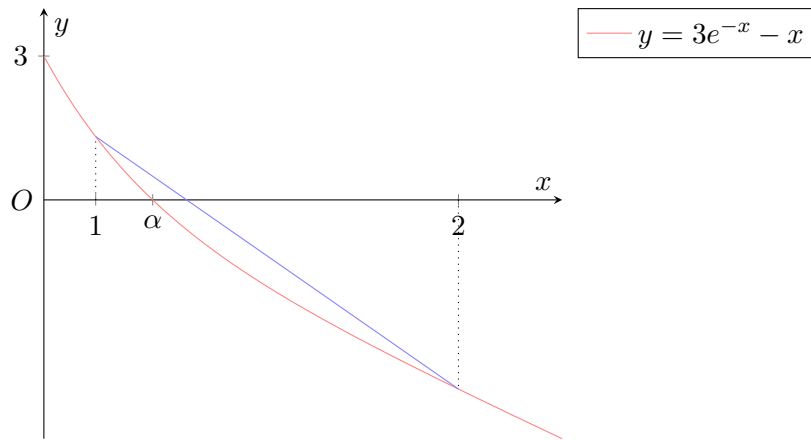
Since  $f(x)$  is continuous, and  $f(1)f(2) = (0.10)(-1.6) < 0$ , there exists a root  $\alpha \in (1, 2)$ .

Note that  $f'(x) = -3e^{-x} - 1$  and  $f''(x) = 3e^{-x}$ . Since  $e^{-x} > 0$  for all  $x$ , we have that  $f'(x) < 0$  and  $f''(x) > 0$  for all  $x$ .

Using linear interpolation on the interval  $(1, 2)$ , we have

$$\alpha = \frac{2 \cdot f(1) - 1 \cdot f(2)}{f(1) - f(2)} = 1.06 \text{ (2 d.p.)}.$$

Since  $f'(x) < 0$  and  $f''(x) > 0$ , we know that  $f(x)$  is strictly decreasing and is concave upwards.  $f(x)$  hence has the following shape:



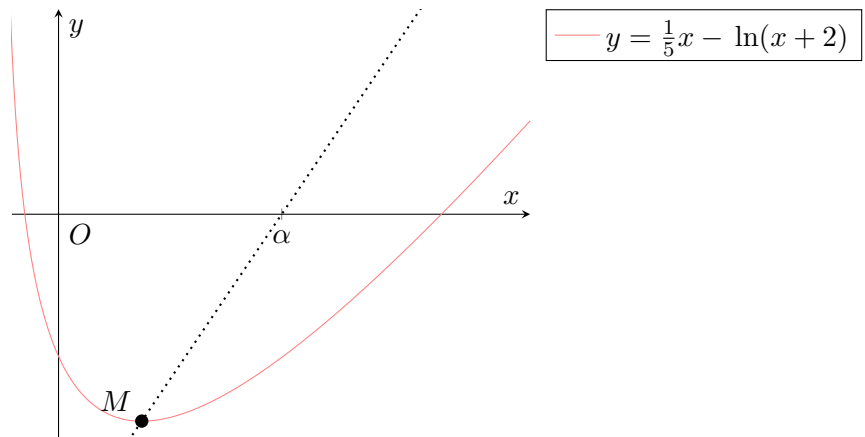
From the graph, we see that the value obtained is an over-estimate.

Using the Newton-Raphson method with the initial approximation  $x_1 = 1.06$ , we get

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.05 \text{ (2 d.p.)}.$$

\* \* \* \* \*

### Problem 9.



The diagram shows a sketch of the graph  $y = \frac{1}{5}x - \ln(x+2)$ . Find the  $x$ -coordinate of the minimum point  $M$  on the graph, and verify that  $y$  is positive when  $x = 20$ .



Show that the gradient of the curve is always less than  $\frac{1}{5}$ . Hence, by considering the line through  $M$  having gradient  $\frac{1}{5}$ , show that the positive root of the equation  $\frac{1}{5}x - \ln(x+2) = 0$  is greater than 8.

Use linear interpolation, once only, on the interval  $[8, 20]$ , to find an approximate value  $a$  for this positive root, giving your answer to 1 decimal place.

Using  $a$  as an initial value, carry out one application of the Newton-Raphson method to obtain another approximation to the positive root, giving your answer to 2 decimal places.

**Solution.** For stationary points,  $y' = 0$ .

$$y' = 0 \implies \frac{1}{5} - \frac{1}{x+2} \implies x = 3.$$

By the second derivative test, we see that  $y''(x) = \frac{1}{(x+2)^2} > 0$ . Hence, the  $x$ -coordinate of  $M$  is 3. Substituting  $x = 20$  into the equation of the curve gives  $y = 4 - \ln 22 = 0.909 > 0$ .

We know that  $y' = \frac{1}{5} - \frac{1}{x+2}$ , hence  $y' < \frac{1}{5}$  for all  $x > -2$ . Since the domain of the curve is  $x > -2$ ,  $y'$  is always less than  $\frac{1}{5}$ .

Let  $(\alpha, 0)$  be the coordinates of the root of the line through  $M$  having gradient  $\frac{1}{5}$ . We know that the coordinates of  $M$  are  $(3, \frac{3}{5} - \ln 5)$ . Taking the gradient of the line segment joining  $M$  and  $(\alpha, 0)$ , we get

$$\frac{(\frac{3}{5} - \ln 5) - 0}{3 - \alpha} = \frac{1}{5} \implies \alpha = 5 \ln 5 = 8.05 > 8.$$

Since the gradient of the curve is always less than  $\frac{1}{5}$ ,  $\alpha$  represents the lowest possible value of the positive root of the curve. Hence, the positive root of the equation  $\frac{1}{5}x - \ln(x+2) = 0$  is greater than 8.

Let  $f(x) = \frac{1}{5}x - \ln(x+2)$ . Using linear interpolation on the interval  $[8, 20]$ , we have

$$\alpha = \frac{8f(20) - 20f(8)}{f(20) - f(8)} = 13.2 \text{ (1 d.p.)}.$$

Using the Newton-Raphson method with the initial approximation  $x_1 = 13.2$ , we have

$$\alpha = x_1 - \frac{f(x_1)}{f'(x_1)} = 13.81 \text{ (2 d.p.)}.$$

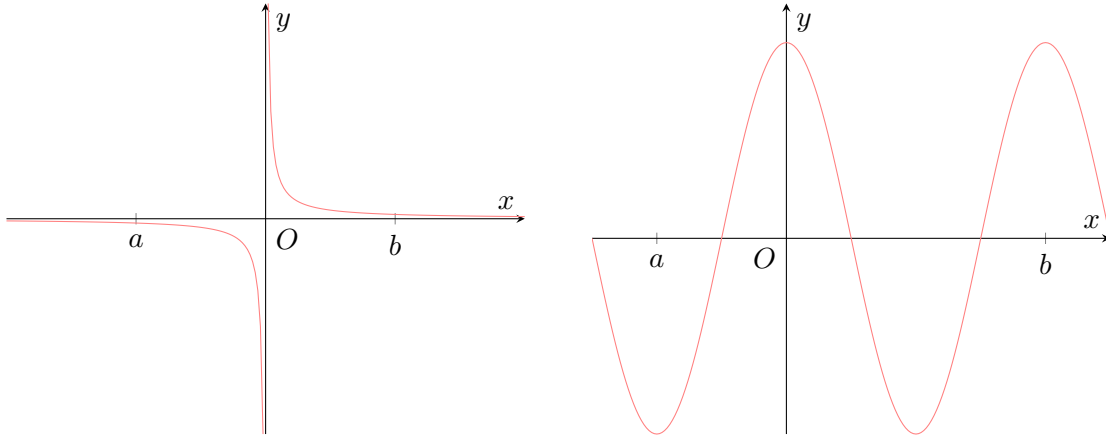
\* \* \* \* \*

### Problem 10.

- The function  $f$  is such that  $f(a)f(b) < 0$ , where  $a < b$ . A student concludes that the equation  $f(x) = 0$  has exactly one root in the interval  $(a, b)$ . Draw sketches to illustrate two distinct ways in which the student could be wrong.
- The equation  $\sec^2 x - e^2 = 0$  has a root  $\alpha$  in the interval  $[1.5, 2.5]$ . A student uses linear interpolation once on this interval to find an approximation to  $\alpha$ . Find the approximation to  $\alpha$  given by this method and comment on the suitability of the method in this case.
- The equation  $\sec^2 x - e^x = 0$  also has a root  $\beta$  in the interval  $(0.1, 0.9)$ . Use the Newton-Raphson method, with  $f(x) = \sec^2 x - e^x$  and initial approximation 0.5, to find a sequence of approximations  $\{x_1, x_2, x_3, \dots\}$  to  $\beta$ . Describe what is happening to  $x_n$  for large  $n$ , and use a graph of the function to explain why the sequence is not converging to  $\beta$ .

**Solution.**

**Part (a).**



**Part (b).** Let  $f(x) = \sec^2 x - e^x$ . Using linear interpolation on the interval  $[1.5, 2.5]$ ,

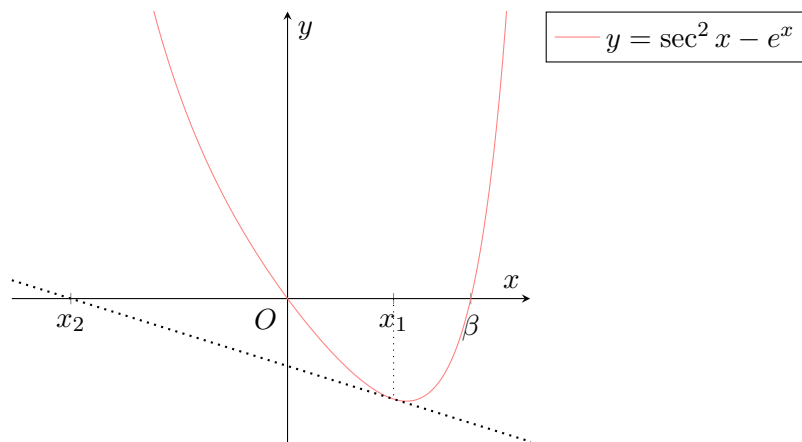
$$a = \frac{1.5f(2.5) - 2.5f(1.5)}{f(2.5) - f(1.5)} = 1.06 \text{ (2 d.p.)}.$$

$\sec^2 x$  is not continuous on the interval  $[1.5, 2.5]$  due to the presence of an asymptote at  $x = \frac{\pi}{2}$ . Hence, linear interpolation is not suitable in this case.

**Part (c).** We know  $f'(x) = 2 \sec^2 x \tan x - e^x$ . Using the Newton-Raphson method with the initial approximation  $x_1 = 0.5$ ,

| $r$ | $x_r$    |
|-----|----------|
| 1   | 0.5      |
| 2   | -1.02272 |
| 3   | -0.75526 |
| 4   | -0.40306 |
| 5   | -0.09667 |
| 6   | -0.00466 |
| 7   | -0.00000 |

As  $n \rightarrow \infty$ ,  $x_n \rightarrow 0^-$ .



From the above graph, we see that the initial approximation of  $x_1 = 0.5$  is past the turning point. Hence, all subsequent approximations will converge to the root at 0 instead of the root at  $\beta$ . Thus, the sequence does not converge to  $\beta$ .

**Problem 11.** The function  $f$  is given by  $f(x) = \sqrt{1-x^2} + \cos x - 1$  for  $0 \leq x \leq 1$ . It is known, from graphical work, that the equation  $f(x) = 0$  has a single root  $x = \alpha$ .

- (a) Express  $g(x)$  in terms of  $x$ , where  $g(x) = x - \frac{f(x)}{f'(x)}$ .

A student attempts to use the Newton-Raphson method, based on the form  $x_{n+1} = g(x_n)$ , to calculate the value of  $\alpha$  correct to 3 decimal places.

- (b) (i) The student first uses an initial approximation to  $\alpha$  of  $x_1 = 0$ . Explain why this will be unsuccessful in finding a value for  $\alpha$ .
- (ii) The student next uses an initial approximation to  $\alpha$  of  $x_1 = 1$ . Explain why this will also be unsuccessful in finding a value for  $\alpha$ .
- (iii) The student then uses an initial approximate to  $\alpha$  of  $x_1 = 0.5$ . Investigate what happens in this case.
- (iv) By choosing a suitable value for  $x_1$ , use the Newton-Raphson method, based on the form  $x_{n+1} = g(x_n)$ , to determine  $\alpha$  correct to 3 decimal places.

**Solution.**

**Part (a).** We know  $f'(x) = \frac{-x}{\sqrt{1-x^2}} - \sin x$ . Hence,

$$g(x) = x - \frac{\sqrt{1-x^2} + \cos x - 1}{\frac{-x}{\sqrt{1-x^2}} - \sin x}.$$

**Part (b).**

**Part (b)(i).** Observe that  $f'(0) = 0$ . Hence,  $g(0)$  is undefined. Thus, starting with an initial approximation of  $x_1 = 0$  will be unsuccessful in finding a value for  $\alpha$ .

**Part (b)(ii).** Observe that  $\sqrt{1-x^2}$  is 0 when  $x = 1$ . Hence,  $f'(1)$  is undefined. Thus,  $g(1)$  is also undefined. Hence, starting with an initial approximation of  $x_1 = 1$  will also be unsuccessful in finding a value for  $\alpha$ .

**Part (b)(iii).** When  $x_1 = 0.5$ , we have  $x_2 = g(x_1) = 1.20$ . Since  $g(x)$  is only defined for  $0 \leq x \leq 1$ ,  $x_3 = g(x_2)$  is undefined. Hence, an initial approximation of  $x_1 = 0.5$  will also be unsuccessful in finding a value for  $\alpha$ .

**Part (b)(iv).** Using the Newton-Raphson method with  $x_1 = 0.9$ , we have

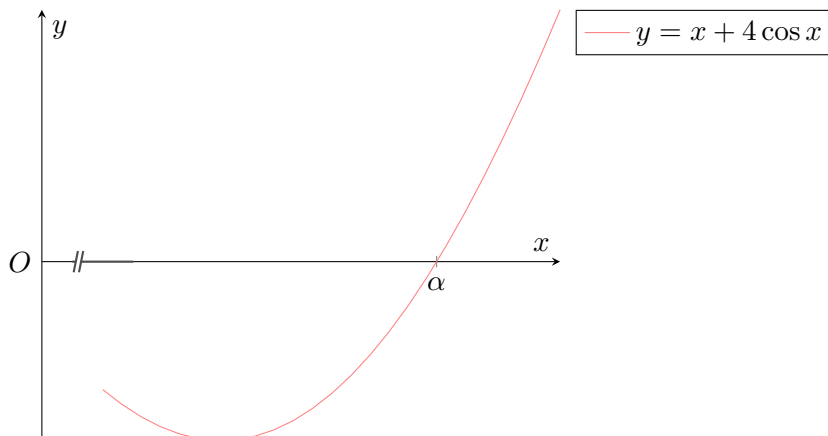
| $r$ | $x_r$   |
|-----|---------|
| 1   | 0.9     |
| 2   | 0.92019 |
| 3   | 0.91928 |
| 4   | 0.91928 |

Thus,  $\alpha = 0.919$  (3 d.p.).

## Assignment A2

**Problem 1.** By considering the graphs of  $y = \cos x$  and  $y = -\frac{1}{4}x$ , or otherwise, show that the equation  $x + 4 \cos x = 0$  has one negative root and two positive roots.

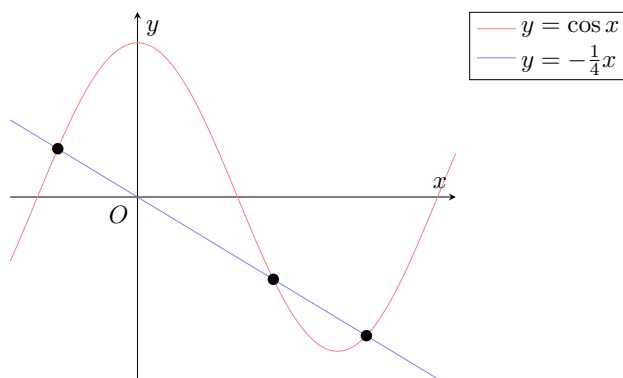
Use linear interpolation, once only, on the interval  $[-1.5, 1]$  to find an approximation to the negative root of the equation  $x + 4 \cos x = 0$  correct to 2 decimal places.



The diagram shows part of the graph of  $y = x + 4 \cos x$  near the larger positive root,  $\alpha$ , of the equation  $x + 4 \cos x = 0$ . Explain why, when using the Newton-Raphson method to find  $\alpha$ , an initial approximation which is smaller than  $\alpha$  may not be satisfactory.

Use the Newton-Raphson method to find  $\alpha$  correct to 2 significant figures. You should demonstrate that your answer has the required accuracy.

**Solution.**



Note that  $x + 4 \cos x = 0 \implies \cos x = -\frac{1}{4}x$ . Plotting the graphs of  $y = \cos x$  and  $y = -\frac{1}{4}x$ , we see that there is one negative root and two positive roots. Hence, the equation  $x + 4 \cos x = 0$  has one negative root and two positive roots.

Let  $f(x) = x + 4 \cos x$ . Let  $\beta$  be the negative root of the equation  $f(x) = 0$ . Using linear interpolation on the interval  $[-1.5, -1]$ ,

$$\beta = \frac{-1.5f(-1) - (-1)f(-1.5)}{f(-1) - f(-1.5)} = -1.24 \text{ (2 d.p.)}.$$

There is a minimum at  $x = m$  such that  $m$  is between the two positive roots. Hence, when using the Newton-Raphson method, an initial approximation which is smaller than  $m$  would result in subsequent approximations being further away from the desired root  $\alpha$ . Hence, an initial approximation that is smaller than  $\alpha$  may not be satisfactory.

We know from the above graph that  $\alpha \in (\pi, \frac{3}{2}\pi)$ . We hence pick  $\frac{3}{2}\pi$  as our initial approximation. Using the Newton-Raphson method  $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$  with  $x_1 = \frac{3}{2}\pi$ , we have

| $r$ | $x_r$            |
|-----|------------------|
| 1   | $\frac{3}{2}\pi$ |
| 2   | 3.7699           |
| 3   | 3.6106           |
| 4   | 3.5955           |
| 5   | 3.5953           |

Since  $f(3.55)f(3.65) = (-0.1)(0.2) < 0$ , we have  $\alpha \in (3.55, 3.65)$ . Hence,  $\alpha = 3.6$  (2 s.f.).

\* \* \* \* \*

**Problem 2.** Find the coordinates of the stationary points on the graph  $y = x^3 + x^2$ . Sketch the graph and hence write down the set of values of the constant  $k$  for which the equation  $x^3 + x^2 = k$  has three distinct real roots.

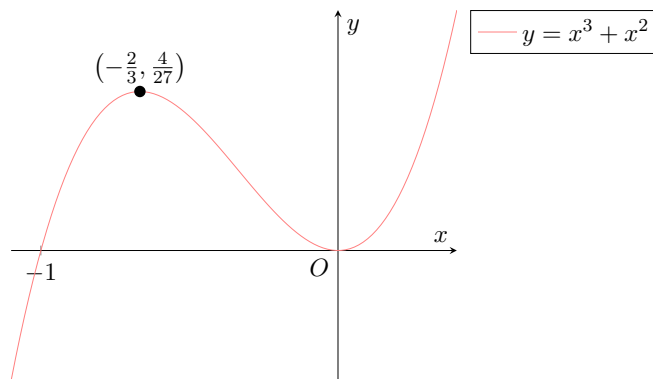
The positive root of the equation  $x^3 + x^2 = 0.1$  is denoted by  $\alpha$ .

- Find a first approximation to  $\alpha$  by linear interpolation on the interval  $0 \leq x \leq 1$ .
- With the aid of a suitable figure, indicate why, in this case, linear interpolation does not give a good approximation to  $\alpha$ .
- Find an alternative first approximation to  $\alpha$  by using the fact that if  $x$  is small then  $x^3$  is negligible when compared to  $x^2$ .

**Solution.** For stationary points,  $y' = 0$ .

$$y' = 0 \implies 3x^2 + 2x = 0 \implies x(3x + 2) = 0.$$

Hence,  $x = 0$  or  $x = -\frac{2}{3}$ . When  $x = 0$ ,  $y = 0$ . When  $x = -\frac{2}{3}$ ,  $y = \frac{4}{27}$ . Thus, the coordinates of the stationary points of  $y = x^3 + x^2$  are  $(0, 0)$  and  $(-\frac{2}{3}, \frac{4}{27})$ .

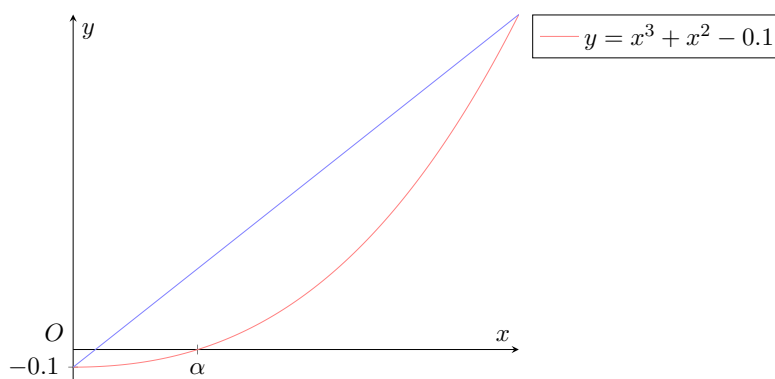


Therefore,  $k \in (0, \frac{4}{27})$ . The solution set of  $k$  is thus  $\{k \in \mathbb{R} : 0 < k < \frac{4}{27}\}$ .

**Part (a).** Let  $f(x) = x^2 + x^2 - 0.1$ . Using linear interpolation on the interval  $[0, 1]$ ,

$$\alpha = \frac{0f(1) - 1f(0)}{f(1) - f(0)} = \frac{1}{20}.$$

**Part (b).**



On the interval  $[0, 1]$ , the gradient of  $y = x^3 + x^2 - 0.1$  changes considerably. Hence, linear interpolation gives an approximation much less than the actual value.

**Part (c).** For small  $x$ ,  $x^3$  is negligible when compared to  $x^2$ . Consider  $g(x) = x^2 - 0.1$ . Then the positive root of  $g(x) = 0$  is approximately  $\alpha$ . Hence, an alternative approximation to  $\alpha$  is  $\sqrt{0.1} = 0.316$  (3 s.f.).

\* \* \* \* \*

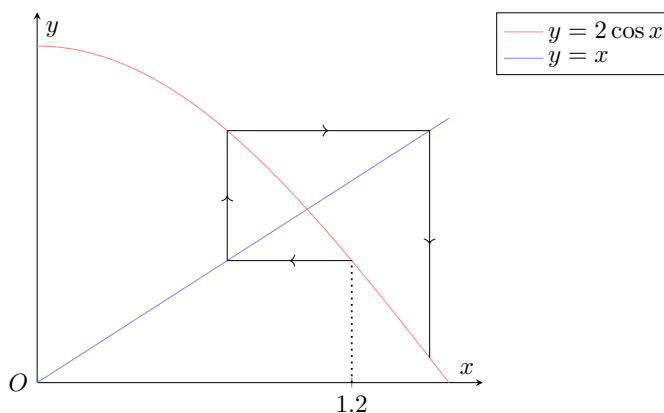
**Problem 3.** The equation  $2 \cos x - x = 0$  has a root  $\alpha$  in the interval  $[1, 1.2]$ . Iterations of the form  $x_{n+1} = F(x_n)$  are based on each of the following rearrangements of the equation:

- (a)  $x = 2 \cos x$
- (b)  $x = \cos x + \frac{1}{2}x$
- (c)  $x = \frac{2}{3}(\cos x + x)$

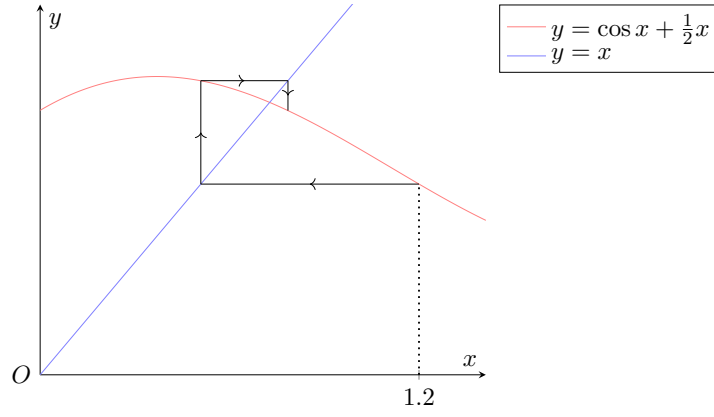
Determine which iteration will converge to  $\alpha$  and illustrate your answer by a ‘staircase’ or ‘cobweb’ diagram. Use the most appropriate iteration with  $x_1 = 1$ , to find  $\alpha$  to 4 significant figures. You should demonstrate that your answer has the required accuracy.

**Solution.**

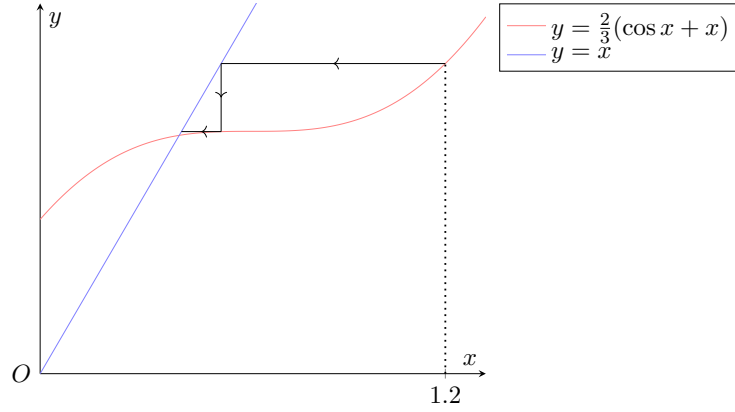
**Part (a).** Consider  $f(x) = 2 \cos x$ . Then  $f'(x) = -2 \sin x$ . Observe that  $\sin x$  is increasing on  $[1, 1.2]$ . Since  $\sin 1 > \frac{1}{2}$ ,  $|f'(x)| > 1$  for all  $x \in [1, 1.2]$ . Thus, fixed-point iteration fails and will not converge to  $\alpha$ .



**Part (b).** Consider  $f(x) = \cos x + \frac{1}{2}x$ . Then  $f'(x) = -\sin x + \frac{1}{2} - (\sin x - \frac{1}{2})$ . Since  $0 \leq \sin x \leq 1$  for  $x \in [0, \frac{\pi}{2}]$ , and  $[1, 1.2] \subset [0, \frac{\pi}{2}]$ , we know  $-\frac{1}{2} \leq \sin x - \frac{1}{2} \leq \frac{1}{2}$  for  $x \in [1, 1.2]$ . Thus,  $0 \leq |\sin x - \frac{1}{2}| \leq \frac{1}{2}$  for  $x \in [1, 1.2]$ . Hence, fixed-point iteration will work and converge to  $\alpha$ .



**Part (c).** Consider  $f(x) = \frac{2}{3}(\cos x + x)$ . Then  $f'(x) = \frac{2}{3}(-\sin x + 1)$ . For fixed-point iteration to converge to  $\alpha$ , we need  $|f'(x)| < 1$  for  $x$  near  $\alpha$ . It thus suffices to show that  $|-\sin x + 1| < \frac{3}{2}$  for all  $x \in [1, 1.2]$ . Observe that  $1 - \sin x$  is strictly decreasing and positive for  $x \in [0, \frac{\pi}{2}]$ . Since  $1 - \sin 1 < \frac{3}{2}$ , and  $[1, 1.2] \subset [0, \frac{\pi}{2}]$ , we have that  $|-\sin x + 1| < \frac{3}{2}$  for all  $x \in [1, 1.2]$ . Thus,  $|f'(x)| < 1$  for  $x$  near  $\alpha$ . Hence, fixed-point iteration will work and converge to  $\alpha$ .



For  $x \in [1, 1.2]$ ,  $|\frac{2}{3}(-\sin x + 1)| < |-\sin x + \frac{1}{2}| < 1$ . Thus,  $x_{n+1} = \frac{2}{3}(\cos x_n + x_n)$  is the most suitable iteration as it will converge to  $\alpha$  the quickest. Using  $F(x_{n+1}) = \frac{2}{3}(\cos x_n + x_n)$  with  $x_1 = 1$ ,

| $r$ | $x_r$   |
|-----|---------|
| 1   | 1       |
| 2   | 1.02687 |
| 3   | 1.02958 |
| 4   | 1.02984 |
| 5   | 1.02986 |

Since  $F(1.0295) > 1.0295$  and  $F(1.0305) < 1.0305$ , we have  $\alpha \in (1.0295, 1.0305)$ . Hence,  $\alpha = 1.030$  (4 s.f.).

## A3. Sequences and Series I

### Tutorial A3

**Problem 1.** Determine the behaviour of the following sequences.

(a)  $u_n = 3\left(\frac{1}{2}\right)^{n-1}$

(b)  $v_n = 2 - n$

(c)  $t_n = (-1)^n$

(d)  $w_n = 4$

**Solution.**

**Part (a).** Decreasing, converges to 0.

**Part (b).** Decreasing, diverges.

**Part (c).** Alternating, diverges.

**Part (d).** Constant, converges to 4.

\* \* \* \* \*

**Problem 2.** Find the sum of all even numbers from 20 to 100 inclusive.

**Solution.** The even numbers from 20 to 100 inclusive form an AP with common difference 2, first term 20 and last term 100. Since we are adding a total of  $\frac{100-20}{2} + 1 = 41$  terms, we get a sum of  $41\left(\frac{20+100}{2}\right) = 2460$ .

\* \* \* \* \*

**Problem 3.** A geometric series has first term 3, last term 384 and sum 765. Find the common ratio.

**Solution.** Let the  $n$ th term of the geometric series be  $ar^{n-1}$ , where  $1 \leq n \leq k$ . We hence have  $3r^{k-1} = 384$ , which gives  $r^k = 128r$ . Thus,

$$\frac{3(1-r^k)}{1-r} = 765 \implies \frac{3(1-128r)}{1-r} = 765 \implies r = 2.$$

\* \* \* \* \*

**Problem 4.**

(a) Find the first four terms of the following sequence  $u_{n+1} = \frac{u_n+1}{u_n+2}$ ,  $u_1 = 0$ ,  $n \geq 1$ .

(b) Write down the recurrence relation between the terms of these sequences.

(i)  $-1, 2, -4, 8, -16, \dots$

(ii)  $1, 3, 7, 15, 31, \dots$

**Solution.**

**Part (a).** Using G.C., the first four terms of  $u_n$  are 0,  $\frac{1}{2}$ ,  $\frac{3}{5}$  and  $\frac{8}{13}$ .



**Part (b).**

**Part (b)(i).**  $u_{n+1} = -2u_n$ ,  $u_1 = -1$ ,  $n \geq 1$ .

**Part (b)(ii).**  $u_{n+1} = 2u_n + 1$ ,  $u_1 = 1$ ,  $n \geq 1$ .

\* \* \* \* \*

**Problem 5.** The sum of the first  $n$  terms of a series,  $S_n$ , is given by  $S_n = 2n(n + 5)$ . Find the  $n$ th term and show that the terms are in arithmetic progression.

**Solution.** We have

$$u_n = S_n - S_{n-1} = 2n(n + 5) - 2(n - 1)(n + 4) = 4n + 8.$$

Observe that  $u_n - u_{n-1} = [4n + 8] - [4(n - 1) + 8] = 4$  is a constant. Hence,  $u_n$  is in AP.

\* \* \* \* \*

**Problem 6.** The sum of the first  $n$  terms,  $S_n$ , is given by

$$S_n = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}.$$

- (a) Find an expression for the  $n$ th term of the series.
- (b) Hence or otherwise, show that it is a geometric series.
- (c) State the values of the first term and the common ratio.
- (d) Give a reason why the sum of the series converges as  $n$  approaches infinity and write down its value.

**Solution.**

**Part (a).** Note that

$$u_n = S_n - S_{n-1} = \left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n+1}\right] - \left[\frac{1}{2} - \left(\frac{1}{2}\right)^n\right] = \left(\frac{1}{2}\right)^{n+1}.$$

**Part (b).** Since  $\frac{u_{n+1}}{u_n} = \frac{(1/2)^{n+2}}{(1/2)^{n+1}} = \frac{1}{2}$  is constant,  $u_n$  is in GP.

**Part (c).** The first term is  $\frac{1}{4}$  and the common ratio is  $\frac{1}{2}$ .

**Part (d).** As  $n \rightarrow \infty$ , we clearly have  $\left(\frac{1}{2}\right)^{n+1} \rightarrow 0$ . Hence,  $S_\infty = \frac{1}{2}$ .

\* \* \* \* \*

**Problem 7.** The first term of an arithmetic series is  $\ln x$  and the  $r$ th term is  $\ln(xk^{r-1})$ , where  $k$  is a real constant. Show that the sum of the first  $n$  terms of the series is  $S_n = \frac{n}{2} \ln(x^2 k^{n-1})$ . If  $k = 1$  and  $x \neq 1$ , find the sum of the series  $e^{S_1} + e^{S_2} + e^{S_3} + \dots + e^{S_n}$ .

**Solution.** Let  $u_n$  be the  $n$ th term in the arithmetic series. Then

$$u_n = \ln(xk^{r-1}) = \ln x + (r - 1) \ln k.$$

We thus see that the arithmetic series has first term  $\ln x$  and common difference of  $\ln k$ . Thus,

$$S_n = n \left( \frac{\ln x + (\ln x + (n - 1) \ln k)}{2} \right) = \frac{n}{2} \ln(x^2 k^{n-1}).$$

When  $k = 1$ , we have  $S_n = \ln(x^n)$ , whence  $e^{S_n} = x^n$ . Thus,

$$e^{S_1} + e^{S_2} + e^{S_3} + \dots + e^{S_n} = x + x^2 + x^3 + \dots + x^n = \frac{x(1 - x^{n+1})}{1 - x}.$$

\* \* \* \* \*

**Problem 8.** A baker wants to bake a 1-metre tall birthday cake. It comprises 10 cylindrical cakes each of equal height 10 cm. The diameter of the cake at the lowest layer is 30 cm. The diameter of each subsequent layer is 4% less than the diameter of the cake below. Find the volume of this cake in  $\text{cm}^3$ , giving your answer to the nearest integer.

**Solution.** Let the diameter of the  $n$ th layer be  $d_n$  cm. We have  $d_{n+1} = 0.96d_n$  and  $d_1 = 30$ , whence  $d_n = 30 \cdot 0.96^{n-1}$ . Let the  $n$ th layer have volume  $v_n \text{ cm}^3$ . Then

$$v_n = 10\pi \left( \frac{d_n}{2} \right)^2 = 10\pi \left( \frac{900 \cdot 0.9216^{n-1}}{4} \right) = 2250\pi \cdot 0.9216^{n-1}.$$

The volume of the cake in  $\text{cm}^3$  is thus given by

$$2250\pi \left( \frac{1 - 0.9216^{10}}{1 - 0.9216} \right) = 50309.$$

\* \* \* \* \*

**Problem 9.** The sum to infinity of a geometric progression is 5 and the sum to infinity of another series is formed by taking the first, fourth, seventh, tenth, ... terms is 4. Find the exact common ratio of the series.

**Solution.** Let the  $n$ th term of the geometric progression be given by  $ar^{n-1}$ . Then, we have

$$\frac{a}{1 - r} = 5 \implies a = 5(1 - r). \quad (1)$$

Note that the first, fourth, seventh, tenth, ... terms forms a new geometric series with common ratio  $r^3$ :  $a, ar^3, ar^6, ar^9, \dots$ . Thus,

$$\frac{a}{1 - r^3} = 4 \implies a = 4(1 - r^3). \quad (2)$$

Equating (1) and (2), we have

$$5(1 - r) = 4(1 - r^3) \implies 4r^3 + 5r + 1 = 0 \implies (r - 1)(4r^2 + 4r - 1) = 0.$$

Since  $|r| < 1$ , we only have  $4r^2 + 4r - 1 = 0$ , which has solutions  $r = \frac{-1+\sqrt{2}}{2}$  or  $r = \frac{-1-\sqrt{2}}{2}$ . Once again, since  $|r| < 1$ , we reject  $r = \frac{-1-\sqrt{2}}{2}$ . Hence,  $r = \frac{-1+\sqrt{2}}{2}$ .

\* \* \* \* \*

**Problem 10.** A geometric series has common ratio  $r$ , and an arithmetic series has first term  $a$  and common difference  $d$ , where  $a$  and  $d$  are non-zero. The first three terms of the geometric series are equal to the first, fourth and sixth terms respectively of the arithmetic series.

(a) Show that  $3r^2 - 5r + 2 = 0$

(b) Deduce that the geometric series is convergent and find, in terms of  $a$ , the sum of infinity.

- (c) The sum of the first  $n$  terms of the arithmetic series is denoted by  $S$ . Given that  $a > 0$ , find the set of possible values of  $n$  for which  $S$  exceeds  $4a$ .

**Solution.**

**Part (a).** Let the  $n$ th term of the geometric series be  $G_n = G_1 r^{n-1}$ . Let the  $n$ th term of the arithmetic series be  $A_n = a + (n-1)d$ .

Since  $G_1 = A_1$ , we have  $G_1 = a$ . We can thus write  $G_n = ar^{n-1}$ . From  $G_2 = A_4$ , we have  $ar = a + 3d$ , which gives  $a = \frac{3d}{r-1}$ . From  $G_3 = A_6$ , we have  $ar^2 = a + 5d$ . Thus,

$$\frac{3d}{r-1} \cdot r^2 = \frac{3d}{r-1} + 5d \implies \frac{3r^2}{r-1} = \frac{3}{r-1} + 5 \implies 3r^2 - 5r + 2 = 0.$$

**Part (b).** Note that the roots to  $3r^2 - 5r + 2 = 0$  are  $r = 1$  and  $r = \frac{2}{3}$ . Clearly,  $r \neq 1$  since  $a = \frac{3d}{r-1}$  would be undefined. Hence,  $r = \frac{2}{3}$ , whence the geometric series is convergent.

Let  $S_\infty$  be the sum to infinity of  $G_n$ . Then  $S_\infty = \frac{a}{1-r} = 3a$ .

**Part (c).** Note that  $d = \frac{1}{3}a(r-1) = -\frac{a}{9}$ . Hence,

$$S = n \left( \frac{a + [a + (n-1)d]}{2} \right) = n \left( \frac{2a + (n-1)(-\frac{a}{9})}{2} \right) = \frac{an}{18}(19-n).$$

Consider  $S > 4a$ .

$$S > 4a \implies \frac{n}{18}(19-n) > 4 \implies -n^2 + 19n - 72 > 0.$$

Using G.C., we see that  $5.23 < n < 13.8$ . Since  $n$  is an integer, the set of values that  $n$  can take on is  $\{n \in \mathbb{Z} : 6 \leq n \leq 13\}$ .

\* \* \* \* \*

**Problem 11.** Two musical instruments,  $A$  and  $B$ , consist of metal bars of decreasing lengths.

- (a) The first bar of instrument  $A$  has length 20 cm and the lengths of the bars form a geometric progression. The 25th bar has length 5 cm. Show that the total length of all the bars must be less than 357 cm, no matter how many bars there are.

Instrument  $B$  consists of only 25 bars which are identical to the first 25 bars of instrument  $A$ .

- (b) Find the total length,  $L$  cm, of all the bars of instrument  $B$  and the length of the 13th bar.
- (c) Unfortunately, the manufacturer misunderstands the instructions and constructs instrument  $B$  wrongly, so that the lengths of the bars are in arithmetic progression with a common difference  $d$  cm. If the total length of the 25 bars is still  $L$  cm and the length of the 25th bar is still 5 cm, find the value of  $d$  and the length of the longest bar.

**Solution.**

**Part (a).** Let  $u_n = u_1 r^{n-1}$  be the length of the  $n$ th bar. Since  $u_1 = 20$ , we have  $u_n = 20r^{n-1}$ . Since  $u_{25} = 5$ , we have  $r = 4^{-\frac{1}{24}}$ . Hence,  $u_n = 20 \cdot 4^{-\frac{n-1}{24}}$ . Now, consider the sum to infinity of  $u_n$ :

$$S_\infty = \frac{u_1}{1-r} = \frac{20}{1-4^{-1/24}} = 356.3 < 357.$$

Hence, no matter how many bars there are, the total length of the bars will never exceed 357 cm.

**Part (b).** We have

$$L = u_1 \left( \frac{1 - r^{25}}{1 - r} \right) = 20 \left( \frac{1 - 4^{-25/24}}{1 - 4^{-1/24}} \right) = 272.26 = 272 \text{ (3 s.f.)}.$$

Note that

$$u_{13} = 20 \cdot \left( 4^{-1/24} \right)^{13-1} = 10.$$

The 13th bar is hence 10 cm long.

**Part (c).** Let  $v_n = a + (n - 1)d$  be the length of the wrongly-manufactured bars. Since the length of the 25th bar is still 5 cm, we know  $v_{25} = a + 24d = 5$ . Now, consider the total lengths of the bars, which is still  $L$  cm.

$$L = 25 \left( \frac{a + 5}{2} \right) = 272.26.$$

Solving, we see that  $a = 16.781$ . Hence,  $d = \frac{5-a}{24} = -0.491$ , and the longest bar is  $16.8 =$  cm long.

\* \* \* \* \*

**Problem 12.** A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man a woman both invest money.

- (a) The man decides to invest  $\$x$  at the beginning of one year and then a further  $\$x$  at the beginning of the second and each subsequent year. He also decides that he will not draw any money out of the account, but just leave it, and any interest, to build up.
  - (i) How much will there be in the account at the end of 1 year, including the interest?
  - (ii) Show that, at the end of  $n$  years, when the interest for the last year has been added, he will have a total of  $\$21(1.05^n - 1)x$  in his account.
  - (iii) After how many complete years will he have, for the first time, at least  $\$12x$  in his account?
- (b) The woman decides that, to assist her in her everyday expenses, she will withdraw the interest as soon as it has been added. She invests  $\$y$  at the beginning of each year. Show that, at the end of  $n$  years, she will have received a total of  $\$ \frac{1}{40} n(n+1)y$  in interest.

**Solution.**

**Part (a).**

**Part (a)(i).** There will be  $\$1.05x$  in the account at the end of 1 year.

**Part (a)(ii).** Let  $\$u_n x$  be the amount of money in the account at the end of  $n$  years. Then,  $u_n$  satisfies the recurrence relation  $u_{n+1} = 1.05(1 + u_n)$ , with  $u_1 = 1.05$ . Observe that

$$u_1 = 1.05 \implies u_2 = 1.05 + 1.05^2 \implies u_3 = 1.05 + 1.05^2 + 1.05^3 \implies \dots$$

We thus have

$$u_n = 1.05 + 1.05^2 + \dots + 1.05^n = 1.05 \left( \frac{1 - 1.05^n}{1 - 1.05} \right) = 21(1.05^n - 1).$$

Hence, there will be  $\$21(1.05^n - 1)x$  in the account after  $n$  years.

**Part (a)(iii).** Consider the inequality  $u_n \geq 12x$ .

$$u_n \geq 12x \implies 21(1.05^n - 1) \geq 12 \implies n \geq 9.26.$$

Since  $n$  is an integer, the smallest value of  $n$  is 10. Hence, after 10 years, he will have at least \$12x in his account for the first time.

**Part (b).** After  $n$  years, the woman will have \$ny in her account. Hence, the interest she gains at the end of the  $n$ th year is  $\frac{1}{20}ny$ . Thus, the total interest she will gain after  $n$  years is

$$\frac{y}{20} + \frac{2y}{20} + \cdots + \frac{ny}{20} = \frac{y}{20} (1 + 2 + \cdots + n) = \frac{y}{20} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)y}{40}.$$

\* \* \* \* \*

**Problem 13.** The sum,  $S_n$ , of the first  $n$  terms of a sequence  $U_1, U_2, U_3, \dots$  is given by

$$S_n = \frac{n}{2}(c - 7n),$$

where  $c$  is a constant.

- (a) Find  $U_n$  in terms of  $c$  and  $n$ .
- (b) Find a recurrence relation of the form  $U_{n+1} = f(U_n)$ .

**Solution.**

**Part (a).** Observe that

$$U_n = S_n - S_{n-1} = \frac{n}{2}(c - 7n) - \frac{n-1}{2}(c - 7(n-1)) = -7n + \frac{7+c}{2}.$$

**Part (b).** Observe that  $U_{n+1} - U_n = -7$ . Thus,

$$U_{n+1} = U_n - 7, \quad U_1 = \frac{7+c}{2}, \quad n \geq 1.$$

\* \* \* \* \*

**Problem 14.** The positive numbers  $x_n$  satisfy the relation

$$x_{n+1} = \sqrt{\frac{9}{2} + \frac{1}{x_n}}$$

for  $n = 1, 2, 3, \dots$

- (a) Given that  $n \rightarrow \infty$ ,  $x_n \rightarrow \theta$ , find the exact value of  $\theta$ .
- (b) By considering  $x_{n+1}^2 - \theta^2$ , or otherwise, show that if  $x_n > \theta$ , then  $0 < x_{n+1} < \theta$ .

**Solution.**

**Part (a).** Observe that

$$\theta = \lim_{n \rightarrow \infty} \sqrt{\frac{9}{2} + \frac{1}{x_n}} = \sqrt{\frac{9}{2} + \frac{1}{\theta}} \implies 2\theta^3 - 9\theta - 2 = 0 \implies (\theta + 2)(2\theta^2 - 4\theta - 1) = 0.$$

We reject  $\theta = -2$  since  $\theta > 0$ . We thus consider  $2\theta^2 - 4\theta - 1 = 0$ , which has roots  $\theta = 1 + \sqrt{\frac{3}{2}}$  and  $\theta = 1 - \sqrt{\frac{3}{2}}$ . Once again, we reject  $\theta = 1 - \sqrt{\frac{3}{2}}$  since  $\theta > 0$ . Thus,  $\theta = 1 + \sqrt{\frac{3}{2}}$ .

**Part (b).** Suppose  $x_n > \theta$ . Then

$$x_{n+1}^2 = \frac{9}{2} + \frac{1}{x_n} < \frac{9}{2} + \frac{1}{\theta} = \theta^2 \implies 0 < x_{n+1} < \theta.$$

## Assignment A3

**Problem 1.** A university student has a goal of saving at least \$1 000 000 (in Singapore dollars). He begins working at the start of the year 2019. In order to achieve his goal, he saves 40% of his annual salary at the end of each year. If his annual salary in the year 2019 is \$40800, and it increases by 5% (of his previous year's salary) every year, find

- (a) his annual savings in 2027 (to the nearest dollar),
- (b) his total savings at the end of  $n$  years.

What is the minimum number of complete years for which he has to work in order to achieve his goal?

**Solution.** Let  $\$u_n$  be his annual salary in the  $n$ th year after 2019, with  $n \in \mathbb{N}$ . Then  $u_{n+1} = 1.05 \cdot u_n$ , with  $u_0 = 40800$ . Hence,  $u_n = 40800 \cdot 1.05^n$ . Let  $\$v_n$  be the amount saved in the  $n$ th year after 2019. Then  $v_n = 0.40 \cdot u_n = 16320 \cdot 1.05^n$ .

**Part (a).** In 2027,  $n = 8$ . Hence, his annual savings in 2027, in dollars, is given by

$$v_8 = 16320 \cdot 1.05^8 = 24112 \text{ (to the nearest integer).}$$

**Part (b).** His total savings at the end of  $n$  years, in dollars, is given by

$$16320 (1.05^0 + 1.05^1 + \cdots + 1.05^n) = 16320 \left( \frac{1 - 1.05^{n+1}}{1 - 1.05} \right) = 326400 (1.05^{n+1} - 1).$$

Consider  $326400 (1.05^n - 1) \geq 1000000$ . Using G.C., we see that  $n \geq 28.7$ . Thus, he needs to work for a minimum of 29 complete years to reach his goal.

\* \* \* \* \*

## Problem 2.

- (a) A rope of length  $200\pi$  cm is cut into pieces to form as many circles as possible, whose radii follow an arithmetic progression with common difference 0.25 cm. Given that the smallest circle has an area of  $\pi$  cm<sup>2</sup>, find the area of the largest circle in terms of  $\pi$ .
- (b) The sum of the first  $n$  terms of a sequence is given by  $S_n = \alpha^{-n} - 1$ , where  $\alpha$  is a non-zero constant,  $\alpha \neq 1$ .
  - (i) Show that the sequence is a geometric progression and state its common ratio in terms of  $\alpha$ .
  - (ii) Find the set of values of  $\alpha$  for which the sum to infinity of the sequence exists.
  - (iii) Find the value of the sum to infinity.

## Solution.

**Part (a).** Let the sequence  $r_n$  be the radius of the  $n$ th smallest circle, in centimetres. Hence,  $r_n = \frac{1}{4} + r_{n-1}$ . Since the smallest circle has area  $\pi$  cm<sup>2</sup>,  $r_1 = 1$ . Thus,  $r_n = 1 + \frac{1}{4}(n-1)$ .

Consider the  $n$ th partial sum of the circumferences:

$$2\pi r_1 + 2\pi r_2 + \cdots + 2\pi r_n = 2\pi \cdot n \left( \frac{1 + [1 + \frac{1}{4}(n-1)]}{2} \right) = \frac{\pi(n^2 + 7n)}{4}.$$

Since the rope has length  $200\pi$  cm, we have the inequality

$$\frac{\pi(n^2 + 7n)}{4} \leq 200\pi \implies n^2 - 7n - 800 \leq 0 \implies (n + 32)(n - 25) \leq 0.$$

Hence,  $n \leq 25$ . Since the rope is cut to form as many circles as possible,  $n = 25$ . Thus, the largest circle has area  $\pi \cdot r_{25}^2 = 49\pi$  cm<sup>2</sup>.

**Part (b).** Let the sequence being summed by  $u_1, u_2, \dots$ . Observe that

$$u_n = S_n - S_{n-1} = (\alpha^{-n} - 1) - (\alpha^{-(n-1)} - 1) = \alpha^{-n}(1 - \alpha).$$

**Part (b)(i).** Observe that

$$\frac{u_{n+1}}{u_n} = \frac{\alpha^{-(n+1)}(1 - \alpha)}{\alpha^{-n}(1 - \alpha)} = \alpha^{-1},$$

which is a constant. Thus,  $u_n$  is in GP with common ratio  $\alpha^{-1}$ .

**Part (b)(ii).** Consider  $S_\infty = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (\alpha^{-n} - 1)$ . For  $S_\infty$  to exist, we need  $\lim_{n \rightarrow \infty} \alpha^{-n}$  to exist. Hence,  $|\alpha^{-1}| < 1$ , whence  $|\alpha| > 1$ . Thus,  $\alpha < -1$  or  $\alpha > 1$ . The solution set of  $\alpha$  is thus  $\{x \in \mathbb{R} : x < -1 \text{ or } x > 1\}$ .

**Part (b)(iii).** Since  $|\alpha^{-1}| < 1$ , we know  $\lim_{n \rightarrow \infty} \alpha^{-n} = 0$ . Hence,  $S_\infty = -1$ .

\* \* \* \* \*

**Problem 3.** A sequence  $u_1, u_2, u_3, \dots$  is such that  $u_{n+1} = 2u_n + An$ , where  $A$  is a constant and  $n \geq 1$ .

(a) Given that  $u_1 = 5$  and  $u_2 = 15$ , find  $A$  and  $u_3$ .

It is known that the  $n$ th term of this sequence is given by

$$u_n = a(2^n) + bn + c,$$

where  $a$ ,  $b$  and  $c$  are constants.

(b) Find  $a$ ,  $b$  and  $c$ .

**Solution.**

**Part (a).** Substituting  $n = 1$  into the recurrence relation yields  $u_2 = 2u_1 + A$ . Thus,  $A = u_2 - 2u_1 = 5$ . Substituting  $n = 2$  into the recurrence relation yields  $u_3 = 2u_2 + 2A = 40$ .

**Part (b).** Since  $u_1 = 5$ ,  $u_2 = 15$  and  $u_3 = 40$ , we have the following system

$$\begin{cases} 2a + b + c = 5 \\ 4a + 2b + c = 15 \\ 8a + 3b + c = 40 \end{cases}$$

which has the unique solution  $a = \frac{15}{2}$ ,  $b = -5$  and  $c = -5$



**Problem 4.** The graphs of  $y = 2^x/3$  and  $y = x$  intersect at  $x = \alpha$  and  $x = \beta$  where  $\alpha < \beta$ . A sequence of real numbers  $x_1, x_2, x_3, \dots$  satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3} \cdot 2^{x_n}, \quad n \geq 1.$$

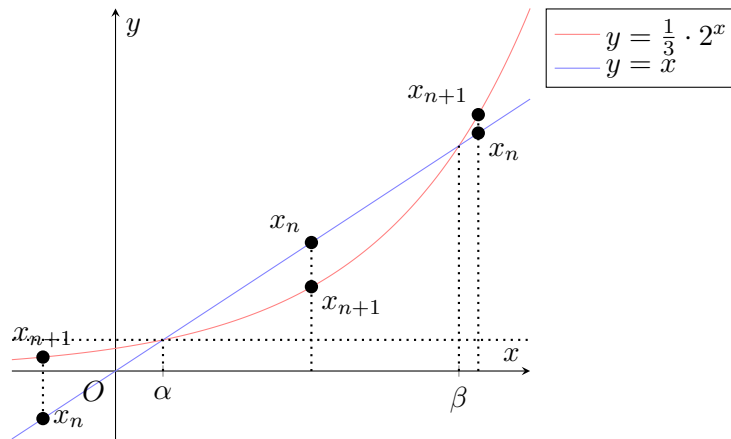
- (a) Prove algebraically that, if the sequence converges, then it converges to either  $\alpha$  or  $\beta$ .
- (b) By using the graphs of  $y = \frac{1}{3} \cdot 2^x$  and  $y = x$ , prove that
- if  $\alpha < x_n < \beta$ , then  $\alpha < x_{n+1} < x_n$
  - if  $x_n < \alpha$ , then  $x_n < x_{n+1} < \alpha$
  - if  $x_n > \beta$ , then  $x_n < x_{n+1}$

Describe the behaviour of the sequence for the three cases.

**Solution.**

**Part (a).** Let  $L = \lim_{n \rightarrow \infty} x_n$ . Then  $L = \frac{1}{3} \cdot 2^L$ . Since  $y = x$  and  $y = \frac{1}{3} \cdot 2^x$  intersect only at  $x = \alpha$  and  $x = \beta$ , then  $\alpha$  and  $\beta$  are the only roots of  $x = \frac{1}{3} \cdot 2^x$ . Since  $L$  is also a root of  $x = \frac{1}{3} \cdot 2^x$ ,  $L$  must be either  $\alpha$  or  $\beta$ .

**Part (b).**



If  $\alpha < x_n < \beta$ , then  $x_n$  is decreasing and converges to  $\alpha$ . If  $x_n < \alpha$ , then  $x_n$  is increasing and converges to  $\alpha$ . If  $x_n > \beta$ , then  $x_n$  is increasing and diverges.

## A4. Sequences and Series II

### Tutorial A4

**Problem 1.** True or False? Explain your answers briefly.

(a)  $\sum_{r=1}^n (2r + 3) = \sum_{k=1}^n (2k + 3)$

(b)  $\sum_{r=1}^n \left(\frac{1}{r} + 5\right) = \sum_{r=1}^n \frac{1}{r} + 5$

(c)  $\sum_{r=1}^n \frac{1}{r} = 1/\sum_{r=1}^n r$

(d)  $\sum_{r=1}^n c = \sum_{r=0}^{n-1} (c + 1)$

**Solution.**

**Part (a).** True: A change in index does not affect the sum.

**Part (b).** False: In general,  $\sum_{r=1}^n 5$  is not equal to 5.

**Part (c).** False: In general,  $\sum \frac{a}{b} \neq \sum a / \sum b$ .

**Part (d).** False: Since  $c$  is a constant,  $\sum_{r=1}^n c = nc \neq n(c + 1) = \sum_{r=0}^{n-1} (c + 1)$ .

\* \* \* \* \*

**Problem 2.** Write the following series in sigma notation twice, with  $r = 1$  as the lower limit in the first and  $r = 0$  as the lower limit in the second.

(a)  $-2 + 1 + 4 + \dots + 40$

(b)  $a^2 + a^4 + a^6 + \dots + a^{50}$

(c)  $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + n\text{th term}$

(d)  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$  to  $n$  terms

(e)  $\frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{28 \cdot 30}$

**Solution.**

**Part (a).**

$$-2 + 1 + 4 + \dots + 40 = \sum_{r=1}^{15} (3r - 5) = \sum_{r=0}^{14} (3r - 2).$$

**Part (b).**

$$a^2 + a^4 + a^6 + \dots + a^{50} = \sum_{r=1}^{25} a^{2r} = \sum_{r=0}^{24} a^{2r+2}.$$

**Part (c).**

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + n\text{th term} = \sum_{r=1}^n \frac{1}{2r+1} = \sum_{r=0}^{n-1} \frac{1}{2r+3}.$$

**Part (d).**

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots \text{ to } n \text{ terms} = \sum_{r=1}^n \left(-\frac{1}{2}\right)^{r-1} = \sum_{r=0}^{n-1} \left(-\frac{1}{2}\right)^r.$$

**Part (e).**

$$\frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{28 \cdot 30} = \sum_{r=1}^{27} \frac{1}{(r+1)(r+3)} = \sum_{r=0}^{26} \frac{1}{(r+2)(r+4)}.$$

\* \* \* \* \*

**Problem 3.** Without using the G.C., evaluate the following sums.

- (a)  $\sum_{r=1}^{50} (2r - 7)$
- (b)  $\sum_{r=1}^a (1 - a - r)$
- (c)  $\sum_{r=2}^n (\ln r + 3^r)$
- (d)  $\sum_{r=1}^{\infty} \left(\frac{2^r - 1}{3^r}\right)$

**Solution.**

**Part (a).**

$$\sum_{r=1}^{50} (2r - 7) = 2 \sum_{r=1}^{50} r - 7 \sum_{r=1}^{50} 1 = 2 \left(\frac{50 \cdot 51}{2}\right) - 7(50) = 2200.$$

**Part (b).**

$$\sum_{r=1}^a (1 - a - r) = (1 - a) \sum_{r=1}^a 1 - \sum_{r=1}^a r = (1 - a)a - \frac{a(a+1)}{2} = \frac{a}{2}(1 - 3a).$$

**Part (c).**

$$\sum_{r=2}^n (\ln r + 3^r) = \sum_{r=2}^n \ln r + \sum_{r=2}^n 3^r = \ln n! + 3^2 \left(\frac{1 - 3^{n-2+1}}{1 - 3}\right) = \ln n! + \frac{9}{2}(3^{n-1} - 1).$$

**Part (d).**

$$\sum_{r=1}^{\infty} \left(\frac{2^r - 1}{3^r}\right) = \sum_{r=1}^{\infty} \left(\frac{2}{3}\right)^r - \sum_{r=1}^{\infty} \left(\frac{1}{3}\right)^r = \frac{2/3}{1 - 2/3} - \frac{1/3}{1 - 1/3} = \frac{3}{2}.$$

\* \* \* \* \*

**Problem 4.** The  $n$ th term of a series is  $2^{n-2} + 3n$ . Find the sum of the first  $N$  terms.

**Solution.**

$$\begin{aligned} \sum_{n=1}^N (2^{n-2} + 3n) &= \sum_{n=1}^N 2^{n-2} + 3 \sum_{n=1}^N n \\ &= 2^{1-2} \left(\frac{(2^N - 1)}{2 - 1}\right) + 3 \left(\frac{N(N+1)}{2}\right) \\ &= \frac{1}{2} (2^N + 3N^2 + 3N - 1). \end{aligned}$$

**Problem 5.** The  $r$ th term,  $u_r$ , of a series is given by  $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$ . Express  $\sum_{r=1}^n u_r$  in the form  $A \left(1 - \frac{B}{27^n}\right)$ , where  $A$  and  $B$  are constants. Deduce the sum to infinity of the series.

**Solution.** Observe that

$$u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1} = 12 \left(\frac{1}{3}\right)^{3r} = 12 \left(\frac{1}{27}\right)^r.$$

Hence,

$$\sum_{r=1}^n u_r = 12 \cdot \frac{1}{27} \left( \frac{1 - 1/27^n}{1 - 1/27} \right) = \frac{6}{13} \left( 1 - \frac{1}{27^n} \right),$$

whence  $A = \frac{6}{13}$  and  $B = 1$ . In the limit as  $n \rightarrow \infty$ ,  $\frac{1}{27^n} \rightarrow 0$ . Hence, the sum to infinity is  $\frac{6}{13}$ .

\* \* \* \* \*

**Problem 6.** The  $r$ th term,  $u_r$ , of a series is given by  $u_r = \ln \frac{r}{r+1}$ . Find  $\sum_{r=1}^n u_r$  in terms of  $n$ . Comment on whether the series converges.

**Solution.** Observe that  $u_r = \ln \frac{r}{r+1} = \ln r - \ln(r+1)$ . Hence,

$$\begin{aligned} \sum_{r=1}^n u_r &= \sum_{r=1}^n (\ln r - \ln(r+1)) \\ &= [\ln 1 - \ln 2] + [\ln 2 - \ln 3] + \cdots + [\ln n - \ln(n+1)] \\ &= \ln 1 - \ln(n+1) = \ln \frac{1}{n+1}. \end{aligned}$$

As  $n \rightarrow \infty$ ,  $\ln \frac{1}{n+1} \rightarrow \ln 0$ . Hence, the series diverges to negative infinity.

\* \* \* \* \*

**Problem 7.** Given that  $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ , without using the G.C., find the following sums.

(a)  $\sum_{r=0}^n [r(r+4) + n]$

(b)  $\sum_{r=n+1}^{2n} (2r-1)^2$

(c)  $\sum_{r=-15}^{20} r(r-2)$

**Solution.**

**Part (a).**

$$\begin{aligned} \sum_{r=0}^n [r(r+4) + n] &= \sum_{r=0}^n (r^2 + 4r + n) \\ &= \frac{n}{6}(n+1)(2n+1) + 4 \left[ \frac{n(n+1)}{2} \right] + n(n+1) \\ &= \frac{n}{6}(n+1)(2n+19). \end{aligned}$$

**Part (b).**

$$\begin{aligned}
 \sum_{r=n+1}^{2n} (2r-1)^2 &= \sum_{r=1}^n (2(r+n)-1)^2 = \sum_{r=1}^n (4r^2 + 4(2n-1)r + (2n-1)^2) \\
 &= 4 \left[ \frac{n}{6}(n+1)(2n+1) \right] + 4(2n-1) \left[ \frac{n(n+1)}{2} \right] + (2n-1)^2 n \\
 &= \frac{1}{3}n(28n^2 - 1)
 \end{aligned}$$

**Part (c).**

$$\begin{aligned}
 \sum_{r=-15}^{20} r(r-2) &= \sum_{r=1}^{36} (r-16)[(r-16)-2] = \sum_{r=1}^{36} (r^2 - 34r + 288) \\
 &= \frac{36}{6} [(36+1)(2 \cdot 36 + 1)] - 34 \left[ \frac{36 \cdot 37}{2} \right] + 288(36) \\
 &= 3930
 \end{aligned}$$

\* \* \* \* \*

**Problem 8.** Let  $S = \sum_{r=0}^{\infty} \frac{(x-2)^r}{3^r}$  where  $x \neq 2$ . Find the range of values of  $x$  such that the series  $S$  converges. Given that  $x = 1$ , find

- (a) the value of  $S$
- (b)  $S_n$ , in terms of  $n$ , where  $S_n = \sum_{r=0}^{n-1} \frac{(x-2)^r}{3^r}$
- (c) the least value of  $n$  for which  $|S_n - S|$  is less than 0.001% of  $S$

**Solution.** Note that

$$S = \sum_{r=0}^{\infty} \frac{(x-2)^r}{3^r} = \sum_{r=0}^{\infty} \left( \frac{x-2}{3} \right)^r.$$

Hence, for  $S$  to converge, we must have  $\left| \frac{x-2}{3} \right| < 1$ , which gives  $-1 < x < 5$ ,  $x \neq 2$ .

**Part (a).** When  $x = 1$ , we get

$$S = \sum_{r=0}^{\infty} \left( -\frac{1}{3} \right)^r = \frac{1}{1 - (-\frac{1}{3})} = \frac{3}{4}.$$

**Part (b).** We have

$$S_n = \sum_{r=0}^{n-1} \left( -\frac{1}{3} \right)^r = \frac{1 - (-\frac{1}{3})^n}{1 - (-\frac{1}{3})} = \frac{3}{4} \left[ 1 - \left( -\frac{1}{3} \right)^n \right].$$

**Part (c).** Observe that

$$|S_n - S| < 0.001\% S \implies \left| \frac{S_n - S}{S} \right| < \frac{1}{100000} \implies \left| \frac{\frac{3}{4}(1 - (-\frac{1}{3})^n)}{\frac{3}{4}} - 1 \right| < \frac{1}{100000}.$$

Using G.C., the least value of  $n$  that satisfies the above inequality is 11.

**Problem 9.** Given that  $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ ,

(a) write down  $\sum_{r=1}^{2k} r^2$  in terms of  $k$

(b) find  $2^2 + 4^2 + 6^2 + \dots + (2k)^2$ .

Hence, show that  $\sum_{r=1}^k (2r-1)^2 = \frac{k}{3}(2k+1)(2k-1)$ .

**Solution.**

**Part (a).**

$$\sum_{r=1}^{2k} r^2 = \frac{2k}{6}(2k+1)(2(2k)+1) = \frac{k}{3}(2k+1)(4k+1).$$

**Part (b).**

$$2^2 + 4^2 + 6^2 + \dots + (2k)^2 = \sum_{r=1}^k (2r)^2 = \sum_{r=1}^k 4r^2 = \frac{2k}{3}(k+1)(2k+1).$$

From parts (a) and (b), we clearly have

$$\sum_{r=1}^k (2r-1)^2 = \sum_{r=1}^{2k} r^2 - \sum_{r=1}^k (2r)^2 = \frac{k}{3}(2k+1)(4k+1) - \frac{2k}{3}(k+1)(2k+1) = \frac{k}{3}(2k+1)(2k-1).$$

\* \* \* \* \*

**Problem 10.** Given that  $u_n = e^{nx} - e^{(n+1)x}$ , find  $\sum_{n=1}^N u_n$  in terms of  $N$  and  $x$ . Hence, determine the set of values of  $x$  for which the infinite series  $u_1 + u_2 + u_3 + \dots$  is convergent and give the sum to infinity for cases where this exists.

**Solution.**

$$\sum_{n=1}^N u_n = (e^x - e^{2x}) + (e^{2x} - e^{3x}) + \dots + (e^{Nx} - e^{(N+1)x}) = e^x - e^{(N+1)x}.$$

For the infinite series to converge, we require  $|e^x| < 1$ . Hence,  $x \in \mathbb{R}_0^-$ .

We now consider the sum to infinity.

*Case 1.* Suppose  $x = 0$ . Then  $e^x = 1$ , whence the sum to infinity is clearly 0.

*Case 2.* Suppose  $x < 0$ . Then  $\lim_{N \rightarrow \infty} e^{(N+1)x} \rightarrow 0$ . Thus, the sum to infinity is  $e^x$ .

\* \* \* \* \*

**Problem 11.** Given that  $r$  is a positive integer and  $f(r) = \frac{1}{r^2}$ , express  $f(r) - f(r+1)$  as a single fraction. Hence, prove that  $\sum_{r=1}^{4n} \left( \frac{2r+1}{r^2(r+1)^2} \right) = 1 - \frac{1}{(4n+1)^2}$ . Give a reason why the series is convergent and state the sum to infinity. Find  $\sum_{r=2}^{4n} \left( \frac{2r-1}{r^2(r-1)^2} \right)$ .

**Solution.**

$$f(r) - f(r+1) = \frac{1}{r^2} - \frac{1}{(r+1)^2} = \frac{(r+1)^2 - r^2}{r^2(r+1)^2} = \frac{2r+1}{r^2(r+1)^2}.$$

$$\begin{aligned} \sum_{r=1}^{4n} \left( \frac{2r+1}{r^2(r+1)^2} \right) &= \sum_{r=1}^{4n} [f(r) - f(r+1)] \\ &= [f(1) - f(2)] + [f(2) - f(3)] + \dots + [f(4n) - f(4n+1)] \\ &= f(1) - f(4n+1) = 1 - \frac{1}{(4n+1)^2} \end{aligned}$$

As  $n \rightarrow \infty$ ,  $\frac{1}{(4n+1)^2} \rightarrow 0$ . Hence, the series converges to 1.

$$\begin{aligned} \sum_{r=2}^{4n} \left( \frac{2r-1}{r^2(r-1)^2} \right) &= \sum_{r=1}^{4n-1} \left( \frac{2r+1}{r^2(r+1)^2} \right) = \sum_{r=1}^{4n-1} [f(r) - f(r+1)] \\ &= [f(1) - f(2)] + [f(2) - f(3)] + \cdots + [f(4n-1) - f(4n)] \\ &= 1 - f(4n) = 1 - \frac{1}{16n^2} \end{aligned}$$

\* \* \* \* \*

### Problem 12.

- (a) Express  $\frac{1}{(2x+1)(2x+3)(2x+5)}$  in partial fractions.  
 (b) Hence, show that  $\sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$ .  
 (c) Deduce the sum of  $\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{41 \cdot 43 \cdot 45}$ .

### Solution.

**Part (a).** Using the cover-up rule, we obtain

$$\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}.$$

**Part (b).**

$$\begin{aligned} \sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} &= \sum_{r=1}^n \left( \frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)} \right) \\ &= \frac{1}{8} \left[ \left( \sum_{r=1}^n \frac{1}{2r+1} - \sum_{r=1}^n \frac{1}{2r+3} \right) - \left( \sum_{r=1}^n \frac{1}{2r+3} - \sum_{r=1}^n \frac{1}{2r+5} \right) \right] \end{aligned}$$

Observe that the two terms in brackets clearly telescope, leaving us with

$$\sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{8} \left[ \left( \frac{1}{3} - \frac{1}{2n+3} \right) - \left( \frac{1}{5} - \frac{1}{2n+5} \right) \right],$$

which simplifies to

$$\sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$$

as desired.

**Part (c).**

$$\begin{aligned} &\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{41 \cdot 43 \cdot 45} \\ &= \frac{1}{1 \cdot 3 \cdot 5} + \sum_{r=1}^{20} \frac{1}{(2r+1)(2r+3)(2r+5)} \\ &= \frac{1}{15} + \left( \frac{1}{60} - \frac{1}{4(2 \cdot 20 + 3)(2 \cdot 20 + 5)} \right) \\ &= \frac{161}{1935}. \end{aligned}$$

## A5. Recurrence Relations

### Tutorial A5

**Problem 1.** Solve these recurrence relations together with the initial conditions.

(a)  $u_n = 2u_{n-1}$ , for  $n \geq 1$ ,  $u_0 = 3$

(b)  $u_n = 3u_{n-1} + 7$ , for  $n \geq 1$ ,  $u_0 = 5$

**Solution.**

**Part (a).**  $u_n = 2^n \cdot u_0 = 3 \cdot 2^n$ .

**Part (b).** Let  $k$  be a constant such that  $u_n + k = 3(u_{n-1} + k)$ . Then  $k = \frac{7}{2}$ . Hence,

$$u_n + \frac{7}{2} = 3 \left( u_{n-1} + \frac{7}{2} \right) \implies u_n + \frac{7}{2} = 3^n \left( u_0 + \frac{7}{2} \right) \implies u_n = \frac{17}{2} \cdot 3^n - \frac{7}{2}.$$

\* \* \* \* \*

**Problem 2.** Solve these recurrence relations together with the initial conditions.

(a)  $u_n = 5u_{n-1} - 6u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = 1$ ,  $u_1 = 0$

(b)  $u_n = 4u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = 0$ ,  $u_1 = 4$

(c)  $u_n = 4u_{n-1} - 4u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = 6$ ,  $u_1 = 8$

(d)  $u_n = -6u_{n-1} - 9u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = 3$ ,  $u_1 = -3$

(e)  $u_n = 2u_{n-1} - 2u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = 2$ ,  $u_1 = 6$

**Solution.**

**Part (a).** Note that the characteristic equation of  $u_n$ ,  $x^2 - 5x + 6 = 0$ , has roots 2 and 3. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n.$$

From  $u_0 = 1$  and  $u_1 = 0$ , we have the equations  $A + B = 1$  and  $2A + 3B = 0$ . Solving, we see that  $A = 3$  and  $B = 2$ , whence

$$u_n = 3 \cdot 2^n + 2 \cdot 3^n.$$

**Part (b).** Note that the characteristic equation of  $u_n$ ,  $x^2 - 4 = 0$ , has roots  $-2$  and  $2$ . Thus,

$$u_n = A(-2)^n + B \cdot 2^n.$$

From  $u_0 = 0$  and  $u_1 = 4$ , we get  $A + B = 0$  and  $-2A + 2B = 4$ . Solving, we see that  $A = -1$  and  $B = 1$ , whence

$$u_n = -(-2)^n + 2^n.$$



**Part (c).** Note that the characteristic equation of  $u_n$ ,  $x^2 - 4x + 4 = 0$ , has only one root, 2. Thus,

$$u_n = (A + Bn)2^n.$$

From  $u_0 = 6$  and  $u_1 = 8$ , we obtain  $A = 6$  and  $A + B = 4$ , whence  $B = -2$ . Thus,

$$u_n = (6 - 2n)2^n.$$

**Part (d).** Note that the characteristic equation of  $u_n$ ,  $x^2 + 6x + 9 = 0$ , has only one root,  $-3$ . Thus,

$$u_n = (A + Bn)(-3)^n.$$

From  $u_0 = 3$  and  $u_1 = -3$ , we get  $A = 3$  and  $A + B = 1$ , whence  $B = -2$ . Thus,

$$u_n = (3 - 2n)2^n.$$

**Part (e).** Consider the characteristic equation of  $u_n$ ,  $x^2 - 2x + 2 = 0$ . By the quadratic formula, this has roots  $x = 1 \pm i = \sqrt{2} \exp(\pm \frac{i\pi}{4})$ . Hence,

$$u_n = A \cdot 2^{\frac{1}{2}n} \cos\left(\frac{n\pi}{4}\right) + B \cdot 2^{\frac{1}{2}n} \sin\left(\frac{n\pi}{4}\right).$$

From  $u_0 = 2$ , we obtain  $A = 2$ . From  $u_1 = 6$ , we obtain  $A + B = 6$ , whence  $B = 4$ . Thus,

$$u_n = 2^{\frac{1}{2}n+1} \cos\left(\frac{n\pi}{4}\right) + 2^{\frac{1}{2}n+2} \sin\left(\frac{n\pi}{4}\right).$$

\* \* \* \* \*

### Problem 3.

- (a) A sequence is defined by the formula  $b_n = \frac{n!n!}{(2n)!} \cdot 2^n$ , where  $n \in \mathbb{Z}^+$ . Show that the sequence satisfies the recurrence relation  $b_{n+1} = \frac{n+1}{2n+1} b_n$ .
- (b) A sequence is defined recursively by the formula

$$u_{n+1} = 2u_n + 3, \quad n \in \mathbb{Z}_0^+, u_0 = a$$

Show that  $u_n = 2^n a + 3(2^n - 1)$ .

### Solution.

**Part (a).**

$$b_{n+1} = \frac{(n+1)!(n+1)!}{(2n+2)!} \cdot 2^{n+1} = \frac{2(n+1)^2}{(2n+1)(2n+2)} \cdot \left[ \frac{n!n!}{(2n)!} \cdot 2^n \right] = \frac{n+1}{2n+1} b_n.$$

**Part (b).** Let  $k$  be a constant such that  $u_{n+1} + k = 2(u_n + k)$ . Then  $k = 3$ . Hence,

$$u_{n+1} + 3 = 2(u_n + 3) \implies u_n + 3 = 2^n(u_0 + 3) \implies u_n = 2^n(a + 3) - 3 = 2^n a + 3(2^n - 1).$$

**Problem 4.** The volume of water, in litres, in a storage tank decreases by 10% by the end of each day. However, 90 litres of water is also pumped into the tank at the end of each day. The volume of water in the tank at the end of  $n$  days is denoted by  $x_n$  and  $x_0$  is the initial volume of water in the tank.

- (a) Write down a recurrence relation to represent the above situation.
- (b) Show that  $x_n = 0.9^n(x_0 - 900) + 900$ .
- (c) Deduce the amount of water in the tank when  $n$  becomes very large.

**Solution.**

**Part (a).**  $x_{n+1} = 0.9x_n + 90$ ,  $n \in \mathbb{N}$

**Part (b).** Let  $k$  be a constant such that  $x_{n+1} + k = 0.9(x_n + k)$ . Then  $k = -900$ . Hence,

$$x_{n+1} - 900 = 0.9(x_n - 900) \implies x_n - 900 = 0.9^n(x_0 - 900) \implies x_n = 0.9^n(x_0 - 900) + 900.$$

**Part (c).** As  $n \rightarrow \infty$ ,  $0.9^n \rightarrow 0$ . Hence, the amount of water in the tank will converge to 900 litres.

\* \* \* \* \*

**Problem 5.** A deposit of \$100,000 is made to an investment fund at the beginning of a year. On the last day of each year, two dividends are awarded and reinvested into the fund. The first dividend is 20% of the amount in the account during that year. The second dividend is 45% of the amount in the account in the previous year.

- (a) Find a recurrence relation  $\{P_n\}$  where  $P_n$  is the amount at the start of the  $n$ th year if no money is ever withdrawn.
- (b) How much is in the account after  $n$  years if no money is ever withdrawn?

**Solution.**

**Part (a).**

$$P_{n+2} = P_{n+1} + 0.2P_{n+1} + 0.45P_n = 1.2P_{n+1} + 0.45P_n.$$

**Part (b).** Note that the characteristic equation of  $P_n$ ,  $x^2 - 1.2x - 0.45 = 0$ , has roots  $-\frac{3}{10}$  and  $\frac{3}{2}$ . Thus,

$$P_n = A \left( -\frac{3}{10} \right)^n + B \left( \frac{3}{2} \right)^n.$$

From  $P_0 = 0$  and  $P_1 = 100000$ , we have  $A + B = 0$  and  $-\frac{3}{10}A + \frac{3}{2}B = 100000$ . Solving, we have  $A = -\frac{500000}{9}$  and  $B = \frac{500000}{9}$ . Thus,

$$P_n = \frac{500000}{9} \left[ \left( \frac{3}{2} \right)^n - \left( -\frac{3}{10} \right)^n \right].$$

Hence, there will be  $\$ \left\{ \frac{500000}{9} \left[ \left( \frac{3}{2} \right)^n - \left( -\frac{3}{10} \right)^n \right] \right\}$  in the account after  $n$  years if no money is ever withdrawn

**Problem 6.** A pair of rabbits does not breed until they are two months old. After they are two months old, each pair of rabbit produces another pair each month.

- Find a recurrence relation  $\{f_n\}$  where  $f_n$  is the total number of pairs of rabbits, assuming that no rabbits ever die.
- What is the number of pairs of rabbits at the end of the  $n$ th month, assuming that no rabbits ever die?

**Solution.**

**Part (a).**  $f_{n+2} = f_{n+1} + f_n$ ,  $n \geq 2$ ,  $f_0 = 0$ ,  $f_1 = 1$

**Part (b).** Consider the characteristic equation of  $f_n$ ,  $x^2 - x - 1 = 0$ . By the quadratic formula, the roots of the characteristic equation are  $\frac{1+\sqrt{5}}{2}$  and  $\frac{1-\sqrt{5}}{2}$ . Hence

$$f_n = A \left( \frac{1+\sqrt{5}}{2} \right)^n + B \left( \frac{1-\sqrt{5}}{2} \right)^n.$$

From  $f_0 = 0$ , we get  $A + B = 0$ . From  $f_1 = 1$ , we get  $A \left( \frac{1+\sqrt{5}}{2} \right) + B \left( \frac{1-\sqrt{5}}{2} \right) = 1$ . Solving, we get  $A = \frac{1}{\sqrt{5}}$  and  $B = -\frac{1}{\sqrt{5}}$ . Hence,

$$f_n = \frac{1}{\sqrt{5}} \left( \frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1-\sqrt{5}}{2} \right)^n.$$

\* \* \* \* \*

**Problem 7.** For  $n \in \{2^j : j \in \mathbb{Z}, j \geq 1\}$ , it is given that  $T_n = 3T_{n/2} + 17$ , where  $T_1 = 4$ . By considering the substitution  $n = 2^i$  and another suitable substitution, show that the recurrence relation can be expressed in the form

$$t_i = 3t_{i-1} + 17, \quad i \in \mathbb{Z}^+$$

Hence, find an expression for  $T_n$  in terms of  $n$ .

**Solution.** Let  $n = 2^i \iff i = \log_2 n$ . The given recurrence relation transforms to

$$T_{2^i} = 3T_{2^{i-1}} + 17, T_{2^0} = 4.$$

Let  $t_i = T_{2^i}$ . Then

$$t_i = 3t_{i-1} + 17, t_0 = 4.$$

Let  $k$  be a constant such that  $t_i + k = 3(t_{i-1} + k)$ . Then  $k = \frac{17}{2}$ . We thus obtain a formula for  $t_i$ :

$$t_i + \frac{17}{2} = 3 \left( t_{i-1} + \frac{17}{2} \right) \implies t_i + \frac{17}{2} = 3^i \left( t_0 + \frac{17}{2} \right) \implies t_i = \frac{25}{2} \cdot 3^i - \frac{17}{2}.$$

Thus,

$$T_{2^i} = \frac{25}{2} \cdot 3^i - \frac{17}{2} \implies T_n = \frac{25}{2} \cdot 3^{\log_2 n} - \frac{17}{2}.$$

**Problem 8.** Consider the sequence  $\{a_n\}$  given by the recurrence relation

$$a_{n+1} = 2a_n + 5^n, \quad n \geq 1$$

- (a) Given that  $a_n = k(5^n)$  satisfies the recurrent relation, find the value of the constant  $k$ .
- (b) Hence, by considering the sequence  $\{b_n\}$  where  $b_n = a_n - k(5^n)$ , find the particular solution to the recurrence relation for which  $a_1 = 2$ .

**Solution.**

**Part (a).**

$$a_{n+1} = 2a_n + 5^n \implies k(5^{n+1}) = 2 \cdot k(5^n) + 5^n \implies 5k = 2k + 1 \implies k = \frac{1}{3}.$$

**Part (b).**

$$b_n = a_n - \frac{5^n}{3} = (2a_{n-1} - 5^{n-1}) - \frac{5^n}{3} = 2a_{n-1} - \frac{2}{3} \cdot 5^{n-1} = 2 \left( a_{n-1} - \frac{5^{n-1}}{3} \right) = 2b_{n-1}.$$

Hence,  $b_n = b_1 \cdot 2^{n-1}$ . Note that  $b_1 = a_1 - \frac{5}{3} = \frac{1}{3}$ . Thus,  $b_n = \frac{2^{n-1}}{3}$ , which gives

$$b_n = a_n - \frac{5^n}{3} = \frac{2^{n-1}}{3} \implies a_n = \frac{2^n + 2 \cdot 5^n}{6}.$$

\* \* \* \* \*

**Problem 9.** The sequence  $\{X_n\}$  is given by

$$\sqrt{X_{n+2}} = \frac{X_{n+1}}{X_n^2}, \quad n \geq 1.$$

By applying the natural logarithm to the recurrence relation, use a suitable substitution to find the general solution of the sequence, expressing your answer in trigonometric form.

**Solution.** Taking the natural logarithm of the recurrence relation and simplifying, we get

$$\ln X_{n+2} = 2 \ln X_{n+1} - 4 \ln X_n.$$

Let  $L_n = \ln X_n \iff X_n = \exp(L_n)$ . Then,

$$L_{n+2} = 2L_{n+1} - 4L_n.$$

Consider the characteristic equation of  $L_n$ ,  $x^2 - 2x + 4 = 0$ . By the quadratic formula, this has roots  $1 \pm \sqrt{3}i = 2 \exp(\pm \frac{i\pi}{3})$ . Thus, we can express  $L_n$  as

$$L_n = A \cdot 2^n \cos \frac{n\pi}{3} + B \cdot 2^n \sin \frac{n\pi}{3} = 2^n \left( A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3} \right).$$

Thus,  $X_n$  has the general solution

$$X_n = \exp \left( 2^n \left( A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3} \right) \right).$$

**Problem 10.** The sequence  $\{X_n\}$  is given by  $X_1 = 2$ ,  $X_2 = 15$  and

$$X_{n+2} = 5 \left( 1 + \frac{1}{n+2} \right) X_{n+1} - 6 \left( 1 + \frac{2}{n+1} \right) X_n, \quad n \geq 1.$$

By dividing the recurrence relation throughout by  $n+3$ , use a suitable substitution to determine  $X_n$  as a function of  $n$ .

**Solution.** Dividing the recurrence relation by  $n+3$ , we obtain

$$\frac{X_{n+2}}{n+3} = 5 \left( \frac{1}{n+3} + \frac{1}{(n+2)(n+3)} \right) X_{n+1} - 6 \left( \frac{1}{n+3} + \frac{2}{(n+1)(n+3)} \right) X_n.$$

Note that  $\frac{1}{(n+2)(n+3)} = \frac{1}{n+2} - \frac{1}{n+3}$  and  $\frac{2}{(n+1)(n+3)} = \frac{1}{n+1} - \frac{1}{n+3}$ . Thus,

$$\frac{X_{n+2}}{n+3} = 5 \left( \frac{X_{n+1}}{n+2} \right) - 6 \left( \frac{X_n}{n+1} \right).$$

Let  $Y_n = \frac{n+1}{X_n} \iff X_n = (n+1)Y_n$ . Then,

$$Y_{n+2} = 5Y_{n+1} - 6Y_n.$$

Note that the characteristic equation of  $Y_n$ ,  $x^2 - 5x + 6 = 0$ , has roots 2 and 3. Hence,

$$Y_n = A \cdot 2^n + B \cdot 3^n \implies X_n = (n+1)(A \cdot 2^n + B \cdot 3^n).$$

From  $X_1 = 2$  and  $X_2 = 15$ , we have  $2A + 3B = 1$  and  $4A + 9B = 5$ . Solving, we obtain  $A = -1$  and  $B = 1$ . Thus,

$$X_n = (n+1)(3^n - 2^n).$$

\* \* \* \* \*

**Problem 11.** A logistics company set up an online platform providing delivery services to users on a monthly paid subscription basis. The company's sales manager models the number of subscribers that the company has at the end of each month. She notes that approximately 10% of the existing subscribers leave each month, and that there will be a constant number  $k$  of new subscribers in each subsequent month after the first.

Let  $T_n$ ,  $n \geq 1$ , denote the number of subscribers the company has at the end of the  $n$ th month after the online platform was set up.

(a) Express  $T_{n+1}$  in terms of  $T_n$ .

The company has 250 subscribers at the end of the first month.

(b) Find  $T_n$  in terms of  $n$  and  $k$ .

(c) Find the least number of subscribers the company needs to attract in each subsequent month after the first if it aims to have at least 350 subscribers by the end of the 12th month.

Let  $k = 50$  for the rest of the question.

The monthly running cost of the company is assumed to be fixed at \$4,000. The monthly subscription fee is \$10 per user which is charged at the end of each month.

(d) Given that the  $m$ th month is the first month in which the company's revenue up to and including that month is able to cover its cost up to and including that month, find the value of  $m$ .

- (e) Using your answer to part (b), determine the long-term behaviour of the number of subscribers that the company has. Hence, explain whether this behaviour is appropriate in terms of long-term prospects for the company's success.

**Solution.**

**Part (a).**  $T_{n+1} = 0.9T_n + k$

**Part (b).** Let  $m$  be a constant such that  $T_{n+1} + m = 0.9(T_n + m)$ . Then  $m = -10k$ . Hence,

$$T_{n+1} - 10k = 0.9(T_n - 10k) \implies T_n - 10k = 0.9^{n-1}(T_0 - 10k).$$

Since  $T_0 = 250$ , we get

$$T_n = 0.9^{n-1}(250 - 10k) + 10k.$$

**Part (c).** Consider  $T_{12} \geq 350$ .

$$T_{12} \geq 350 \implies 0.9^{12-1}(250 - 10k) + 10k \geq 350.$$

Using G.C.,  $k \geq 39.6$ . Hence, the company needs to attract at least 40 subscribers in each subsequent month.

**Part (d).** Since  $k = 50$ ,  $T_n = -250 \cdot 0.9^{n-1} + 500$ . Let  $\$S_m$  be the total revenue for the first  $m$  months.

$$\begin{aligned} S_m &= 10 \sum_{n=1}^m T_n = 10 \sum_{n=1}^m (-250 \cdot 0.9^{n-1} + 500) \\ &= 10 \left[ -250 \left( \frac{1 - 0.9^m}{1 - 0.9} \right) + 500m \right] = 25000(0.9^m - 1) + 5000m. \end{aligned}$$

Note that the total cost for the first  $m$  months is  $\$4000m$ . Hence, the total profit for the first  $m$  months is given by  $\$(S_m - 4000m)$ . Hence, we consider  $S_m - 4000m \geq 0$ :

$$S_m - 4000m \geq 0 \implies 25000(0.9^m - 1) + 1000m \geq 0.$$

Using G.C., we obtain  $m \geq 22.7$ , whence the least value of  $m$  is 23.

**Part (e).** As  $n \rightarrow \infty$ ,  $0.9^{n-1} \rightarrow 0$ . Hence,  $T_n \rightarrow 500$ . Hence, as  $n$  becomes very large, the profit per month approaches  $500 \cdot 10 - 4000 = 1000$  dollars. Thus, this behaviour is appropriate as the business will remain profitable in the long run.

## Assignment A5

**Problem 1.** In an auction at a charity gala dinner, a group of wealthy businessmen are competing with each other to be the highest bidder. Each time one of them makes a bid amount, another counter-bids by 50% more, less a service charge of ten dollars (e.g. If A bids \$1000, then B will bid \$1490). Let  $u_n$  be the amount at the  $n$ th bid and  $u_1$  be the initial amount.

- (a) Write down a recurrence relation that describes the bidding process.
- (b) Show that  $u_n = \$(1.5^{n-1}(u_1 - 20) + 20)$ .
- (c) The target amount to be raised is \$1 234 567 and the bidding stops when the bid amount meets or crosses this target amount. Given that  $u_1 = 111$ ,
  - (i) state the least number of bids required to meet this amount.
  - (ii) find the winning bid amount, correct to the nearest thousand dollars.

**Solution.**

**Part (a).**  $u_{n+1} = 1.5u_n - 10$ .

**Part (b).** Let  $k$  be the constant such that  $u_{n+1} + k = 1.5(u_n + k)$ . Then  $k = -20$ . Hence,  $u_{n+1} - 20 = 1.5(u_n - 20)$ .

$$u_{n+1} - 20 = 1.5(u_n - 20) \implies u_n - 20 = 1.5^{n-1}(u_1 - 20) \implies u_n = 1.5^{n-1}(u_1 - 20) + 20.$$

**Part (c).**

**Part (c)(i).** Let  $m$  be the least integer such that  $u_m \geq 1234567$ . Consider  $u_m \geq 1234567$ :

$$u_m \geq 1234567 \implies 1.5^{m-1}(111 - 20) + 20 \geq 1234567.$$

Using G.C.,  $m \geq 24.5$ . Hence, it takes at least 25 bids to meet this amount.

**Part (c)(ii).** Since  $u_{25} = 1.5^{25-1}(111 - 20) + 20 = 1532000$  (to the nearest thousand), the winning bid is \$1 532 000.

\* \* \* \* \*

**Problem 2.** Solve these recurrence relations together with the initial conditions.

- (a)  $u_{n+2} = -u_n + 2u_{n+1}$ , for  $n \geq 0$ ,  $u_0 = 5$ ,  $u_1 = -1$ .
- (b)  $4u_n = 4u_{n-1} + u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = a$ ,  $u_1 = b$ ,  $a, b \in \mathbb{R}$ .

**Solution.**

**Part (a).** Observe that the characteristic equation of  $u_n$ ,  $x^2 - 2x + 1 = 0$ , has only one root, namely  $x = 1$ . Thus,

$$u_n = (A + Bn) \cdot 1^n = A + Bn.$$

Thus,  $u_n$  is in AP. Since  $u_0 = 5$  and  $u_1 = -1$ , it follows that

$$u_n = 5 - 6n.$$

**Part (b).** Rewriting the given recurrence relation, we have  $u_n = u_{n-1} + \frac{1}{4}u_{n-2}$ . Thus, the characteristic equation is  $x^2 - x - \frac{1}{4} = 0$ , which has roots  $\frac{1}{2}(1 \pm \sqrt{2})$ . Thus,

$$u_n = A \left( \frac{1 + \sqrt{2}}{2} \right)^n + B \left( \frac{1 - \sqrt{2}}{2} \right)^n.$$

Since  $u_0 = a$ , we obviously have  $A+B = a$ . Since  $u_1 = b$ , we get  $A\left(\frac{1+\sqrt{2}}{2}\right) + B\left(\frac{1-\sqrt{2}}{2}\right) = b$ . Solving, we get

$$A = \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b, \quad B = \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b.$$

Thus,

$$u_n = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b\right)\left(\frac{1+\sqrt{2}}{2}\right)^n + \left(\frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b\right)\left(\frac{1-\sqrt{2}}{2}\right)^n.$$

\* \* \* \* \*

**Problem 3.** A passcode is generated using the digits 1 to 5, with repetitions allowed. The passcodes are classified into two types. A Type  $A$  passcode has an even number of the digit 1, while a Type  $B$  passcode has an odd number of the digit 1. For example, a Type  $A$  passcode is 1231, and a Type  $B$  passcode is 1541213. Let  $a_n$  and  $b_n$  denote the number of  $n$ -digit Type  $A$  and Type  $B$  passcodes respectively.

- (a) State the values of  $a_1$  and  $a_2$ .  
 (b) By considering the relationship between  $a_n$  and  $b_n$ , show that

$$a_n = xa_{n-1} + y^{n-1}, \quad n \geq 2$$

where  $x$  and  $y$  are constants to be determined.

- (c) Using the substitution  $c_n = za_n + y^n$ , where  $z$  is a constant to be determined, find a first order linear recurrence relation for  $c_n$ . Hence, find the general term formula for  $a_n$ .

**Solution.**

**Part (a).**  $a_1 = 4$ ,  $a_2 = 17$ .

**Part (b).** Let  $P$  be an  $n$ -digit passcode with Type  $T$ , where  $T$  is either  $A$  or  $B$ . Let Type  $T'$  be the other type.

By concatenating a digit from 1 to 5 to  $P$ , five  $(n+1)$ -digit passcodes can be created. Let  $P'$  denote a new passcode that is created via this process. If the digit 1 is concatenated, then  $P'$  is of Type  $T'$ . If the digit 1 is not concatenated, then  $P'$  is of Type  $T$ . There are 4 choices for such a case. This hence gives the recurrence relations

$$\begin{cases} a_n = 4a_{n-1} + b_{n-1} \\ b_n = 4b_{n-1} + a_{n-1} \end{cases}$$

Adding the two equations, we see that  $a_n + b_n = 5(a_{n-1} + b_{n-1})$ . Thus,

$$a_n + b_n = 5^{n-1}(a_1 + b_1) = 5^{n-1}(4 + 1) = 5^n.$$

Hence,

$$a_n = 4a_{n-1} + b_{n-1} = 3a_{n-1} + a_{n-1} + b_{n-1} = 3a_{n-1} + 5^{n-1},$$

whence  $x = 3$  and  $y = 5$ .



**Part (c).** Observe that

$$\begin{aligned}c_n &= za_n + 5^n = z(3a_{n-1} + 5^{n-1}) + 5^n = 3(za_{n-1} + 5^{n-1}) + (2+z)5^{n-1} \\&= 3c_{n-1} + (2+z)5^{n-1}.\end{aligned}$$

Let  $z = -2$ . Then,

$$c_n = 3c_{n-1} = 3^{n-1}c_1 = 3^{n-1}(-2a_1 + 5) = -3^n.$$

Note that  $a_n = \frac{1}{z}(c_n - y^n)$ . Thus,

$$a_n = \frac{-3^n - 5^n}{-2} = \frac{3^n + 5^n}{2}.$$

## A6. Polar Coordinates

### Tutorial A6

#### Problem 1.

(a) Find the rectangular coordinates of the following points.

(i)  $(3, -\frac{\pi}{4})$

(ii)  $(1, \pi)$

(iii)  $(\frac{1}{2}, \frac{3}{2}\pi)$

(b) Find the polar coordinates of the following points.

(i)  $(3, 3)$

(ii)  $(-1, -\sqrt{3})$

(iii)  $(2, 0)$

(iv)  $(4, 2)$

#### Solution.

##### Part (a).

**Part (a)(i).** Note that  $r = 3$  and  $\theta = -\frac{\pi}{4}$ . This gives

$$x = r \cos \theta = \frac{3}{\sqrt{2}}, \quad y = r \sin \theta = -\frac{3}{\sqrt{2}}.$$

Hence, the rectangular coordinate of the point is  $(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}})$ .

**Part (a)(ii).** Note that  $r = 1$  and  $\theta = \pi$ . This gives

$$x = r \cos \theta = -1, \quad y = r \sin \theta = 0.$$

Hence, the rectangular coordinate of the point is  $(-1, 0)$ .

**Part (a)(iii).** Note that  $r = \frac{1}{2}$  and  $\theta = \frac{3}{2}\pi$ . This gives

$$x = r \cos \theta = 0, \quad y = r \sin \theta = -\frac{1}{2}.$$

Hence, the rectangular coordinate of the point is  $(0, -\frac{1}{2})$ .

##### Part (b).

**Part (b)(i).** Note that  $x = 3$  and  $y = -3$ . This gives

$$r^2 = x^2 + y^2 \implies r = 3\sqrt{2}, \quad \tan \theta = \frac{y}{x} \implies \theta = -\frac{\pi}{4}.$$

Hence, the polar coordinate of the point is  $(3\sqrt{2}, -\frac{\pi}{4})$ .

**Part (b)(ii).** Note that  $x = -1$  and  $y = -\sqrt{3}$ . This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan \theta = \frac{y}{x} \implies \theta = \frac{\pi}{3}.$$

Hence, the polar coordinate of the point is  $(2, \frac{\pi}{3})$ .

**Part (b)(iii).** Note that  $x = 2$  and  $y = 0$ . This gives

$$r^2 = x^2 + y^2 \implies r = 2, \quad \tan \theta = \frac{y}{x} \implies \theta = 0.$$

Hence, the polar coordinate of the point is  $(2, 0)$ .

**Part (b)(iv).** Note that  $x = 4$  and  $y = 2$ . This gives

$$r^2 = x^2 + y^2 \implies r = 2\sqrt{5}, \quad \tan \theta = \frac{y}{x} \implies \theta = \arctan \frac{1}{2}.$$

Hence, the polar coordinate of the point is  $(2\sqrt{5}, \arctan \frac{1}{2})$ .

\* \* \* \* \*

**Problem 2.** Rewrite the following equations in polar form.

(a)  $2x^2 + 3y^2 = 4$

(b)  $y = 2x^2$

**Solution.**

**Part (a).**

$$2x^2 + 3y^2 = 2(r \cos \theta)^2 + 3(r \sin \theta)^2 = 4 \implies r^2 = \frac{4}{2 \cos^2 \theta + 3 \sin^2 \theta} = \frac{4}{2 + \sin^2 \theta}.$$

**Part (b).**

$$y = 2x^2 \implies \frac{y}{x} = 2x \implies \tan \theta = 2r \cos \theta \implies r = \frac{1}{2} \tan \theta \sec \theta.$$

\* \* \* \* \*

**Problem 3.** Rewrite the following equations in rectangular form.

(a)  $r = \frac{1}{1-2 \cos \theta}$

(b)  $r = \sin \theta$

**Solution.**

**Part (a).**

$$\begin{aligned} r = \frac{1}{1-2 \cos \theta} &\implies r - 2r \cos \theta = 1 \implies r = 2x + 1 \implies r^2 = 4x^2 + 4x + 1 \\ &\implies x^2 + y^2 = 4x^2 + 4x + 1 \implies y^2 = 3x^2 + 4x + 1. \end{aligned}$$

**Part (b).**

$$r = \sin \theta \implies r^2 = r \sin \theta \implies x^2 + y^2 = y.$$

**Problem 4.**

- (a) Show that the curve with polar equation  $r = 3a \cos \theta$ , where  $a$  is a positive constant, is a circle. Write down its centre and radius.
- (b) By finding the Cartesian equation, sketch the curve whose polar equation is  $r = a \sec\left(\theta - \frac{\pi}{4}\right)$ , where  $a$  is a positive constant.

**Solution.****Part (a).**

$$r = 3a \cos \theta \implies r^2 = 3ar \cos \theta \implies x^2 + y^2 = 3ax \implies x^2 - 3ax + y^2 = 0.$$

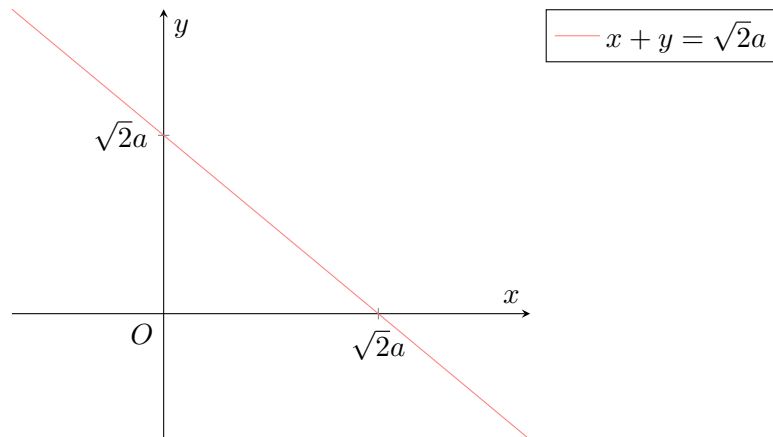
Completing the square, we get

$$\left(x - \frac{3a}{2}\right)^2 + y^2 = \left(\frac{3a}{2}\right)^2.$$

Thus, the circle has centre  $\left(\frac{3a}{2}, 0\right)$  and radius  $\frac{3a}{2}$ .

**Part (b).**

$$r = a \sec\left(\theta - \frac{\pi}{4}\right) \implies r \cos\left(\theta - \frac{\pi}{4}\right) = a \implies r(\cos \theta + \sin \theta) = \sqrt{2}a \implies x + y = \sqrt{2}a.$$



**Problem 5.** Sketch the following polar curves, where  $r$  is non-negative and  $0 \leq \theta \leq 2\pi$ .

(a)  $r = 2$

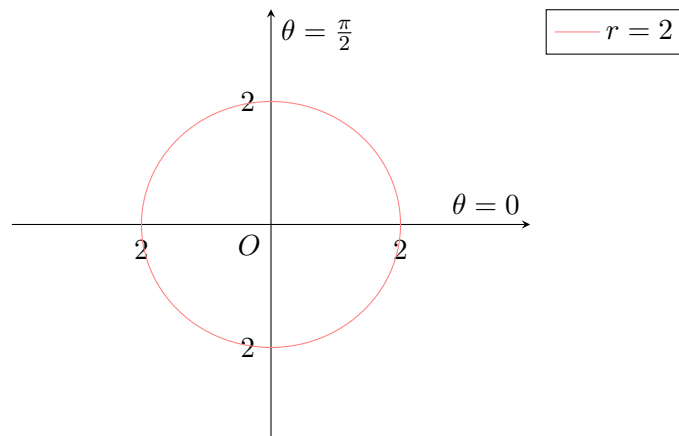
(b)  $\theta = \frac{\pi}{4}$

(c)  $r = \frac{1}{2}\theta$

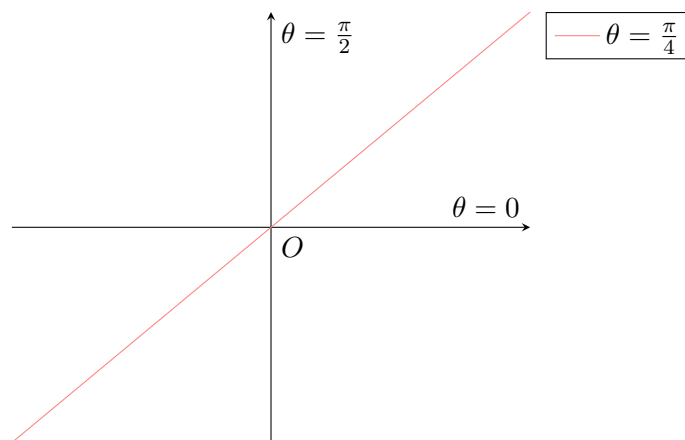
(d)  $r = 2 \csc \theta$

**Solution.**

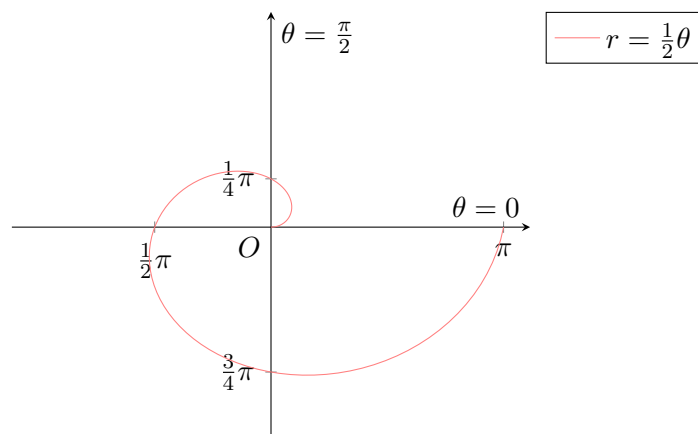
**Part (a).**



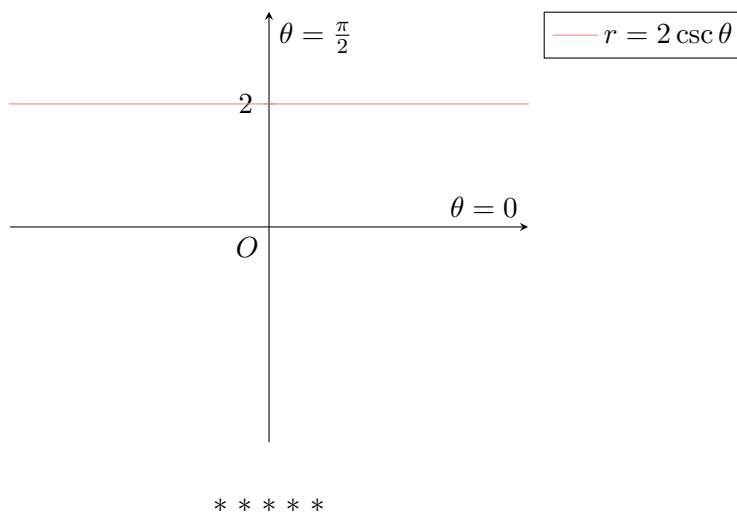
**Part (b).**



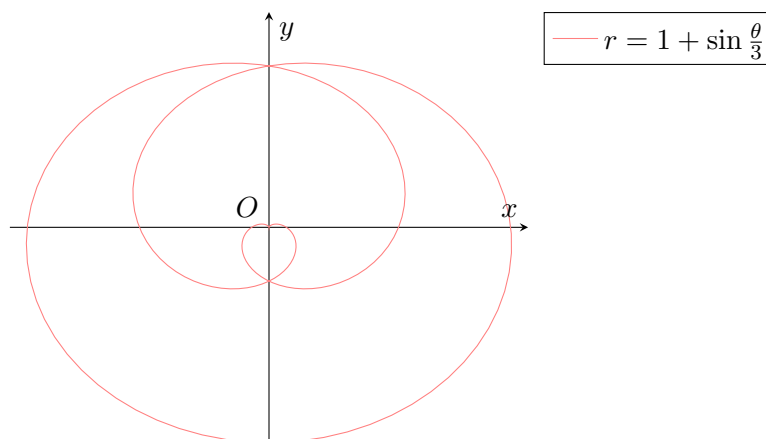
**Part (c).**



**Part (d).**



**Problem 6.** A sketch of the curve  $r = 1 + \sin \frac{\theta}{3}$  is shown. Copy the diagram and indicate the  $x$ - and  $y$ -intercepts.



**Solution.** Observe that the curve is symmetric about the  $y$ -axis. Also observe that  $\frac{\theta}{3} \in [0, 2\pi)$ , hence we take  $\theta \in [0, 6\pi)$ .

For  $x$ -intercepts,  $y = r \sin \theta = 0 \implies \theta = n\pi$ , where  $n \in \mathbb{Z}$ . Due to the symmetry of the curve, we consider only  $n = 0, 2, 4$ .

*Case 1.*  $n = 0 \implies r = 1 + \sin \frac{0}{3}\pi = 1$ .

*Case 2.*  $n = 2 \implies r = 1 + \sin \frac{2}{3}\pi = 1 + \frac{\sqrt{3}}{2}$ .

*Case 3.*  $n = 4 \implies r = 1 + \sin \frac{4}{3}\pi = 1 - \frac{\sqrt{3}}{2}$ .

Hence, the curve intersects the  $x$ -axis at  $x = 1, 1 + \frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}$ . Correspondingly, the curve also intersects the  $x$ -axis at  $x = -1, -1 - \frac{\sqrt{3}}{2}, -1 + \frac{\sqrt{3}}{2}$ .

For  $y$ -intercepts,  $x = r \cos \theta = 0 \implies \theta = (n + \frac{1}{2})\pi$ , where  $n \in \mathbb{Z}$ . Due to the restriction on  $\theta$ , we consider  $n \in [0, 5)$ .

*Case 1.*  $n = 0, r = 1 + \sin \frac{1/2}{3}\pi = \frac{3}{2}$ .

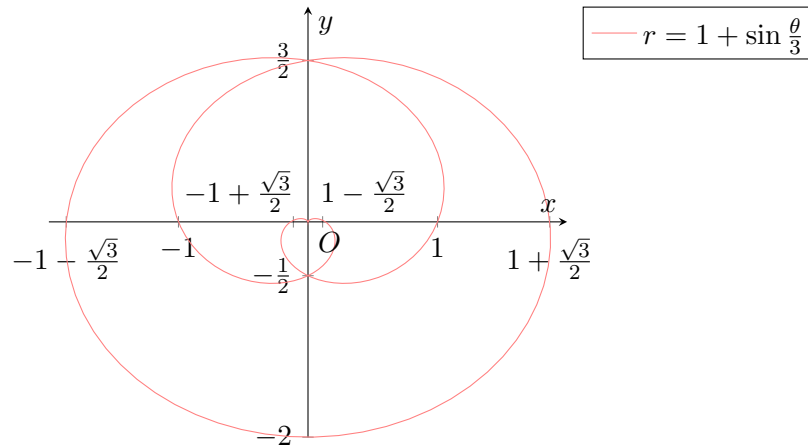
*Case 2.*  $n = 1, r = 1 + \sin \frac{3/2}{3}\pi = 2$ .

*Case 3.*  $n = 2, r = 1 + \sin \frac{5/2}{3}\pi = \frac{3}{2}$ .

*Case 4.*  $n = 3, r = 1 + \sin \frac{7/2}{3}\pi = \frac{1}{2}$ .

*Case 5.*  $n = 4, r = 1 + \sin \frac{9/2}{3}\pi = 0$ .

Hence, the curve intersects the  $y$ -axis at  $y = -2, -\frac{1}{2}, \frac{3}{2}$ .



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**Problem 7.**

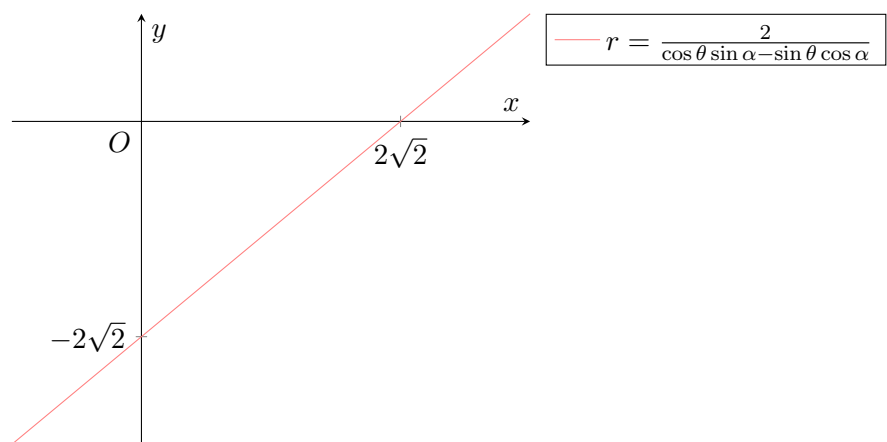
- (a) A graph has polar equation  $r = \frac{2}{\cos \theta \sin \alpha - \sin \theta \cos \alpha}$ , where  $\alpha$  is a constant. Express the equation in Cartesian form. Hence, sketch the graph in the case  $\alpha = \frac{\pi}{4}$ , giving the Cartesian coordinates of the intersection with the axes.
- (b) A graph has Cartesian equation  $(x^2 + y^2)^2 = 4x^2$ . Express the equation in polar form. Hence, or otherwise, sketch the graph.

**Solution.****Part (a).**

$$r = \frac{2}{\cos \theta \sin \alpha - \sin \theta \cos \alpha} \implies r \cos \theta \sin \alpha - r \sin \theta \cos \alpha = x \sin \alpha - y \cos \alpha = 2$$

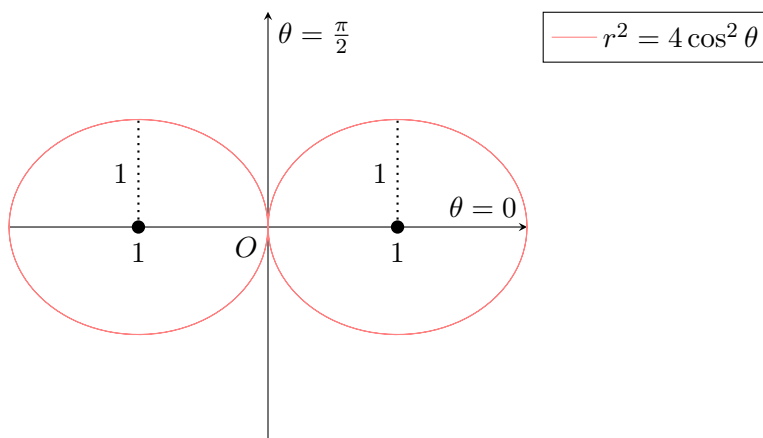
$$\implies y = x \tan \alpha - 2 \sec \alpha.$$

When  $\alpha = \frac{\pi}{4}$ , we have  $y = x - 2\sqrt{2}$ .



**Part (b).**

$$(x^2 + y^2)^2 = 4x^2 \implies (r^2)^2 = 4(r \cos \theta)^2 \implies r^4 = 4r^2 \cos^2 \theta \implies r^2 = 4 \cos^2 \theta.$$

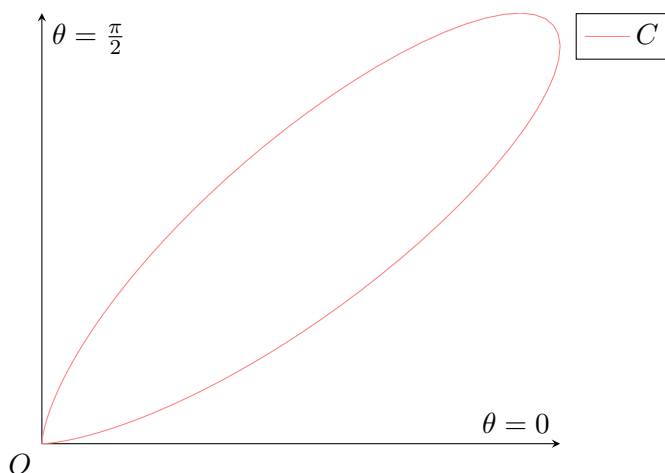


\* \* \* \* \*

**Problem 8.** Find the polar equation of the curve  $C$  with equation  $x^5 + y^5 = 5bx^2y^2$ , where  $b$  is a positive constant. Sketch the part of the curve  $C$  where  $0 \leq \theta \leq \frac{\pi}{2}$ .

**Solution.**

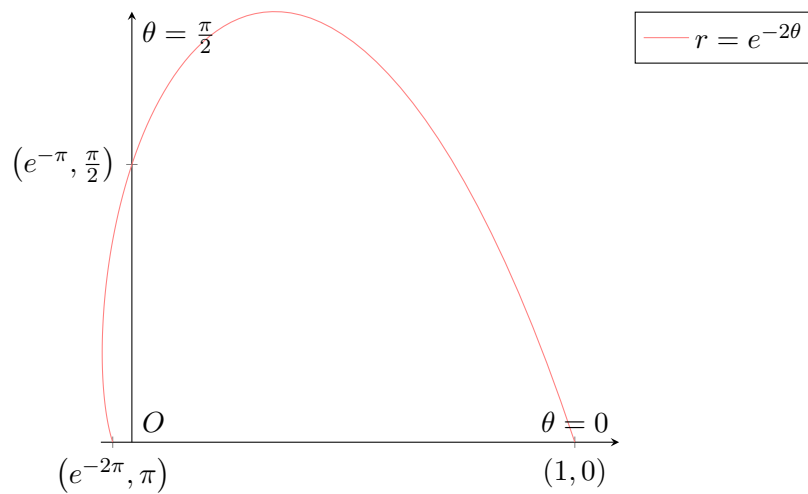
$$\begin{aligned} x^5 + y^5 &= 5bx^2y^2 \\ \implies (r \cos \theta)^5 + (r \sin \theta)^5 &= 5b(r \cos \theta)^2(r \sin \theta)^2 \\ \implies r(\cos^5 \theta + \sin^5 \theta) &= 5b \cos^2 \theta \sin^2 \theta \\ \implies r &= \frac{5b \cos^2 \theta \sin^2 \theta}{\cos^5 \theta + \sin^5 \theta} \end{aligned}$$





**Problem 9.** The equation of a curve, in polar coordinates, is  $r = e^{-2\theta}$ , for  $0 \leq \theta \leq \pi$ . Sketch the curve, indicating clearly the polar coordinates of any axial intercepts.

**Solution.**



\* \* \* \* \*

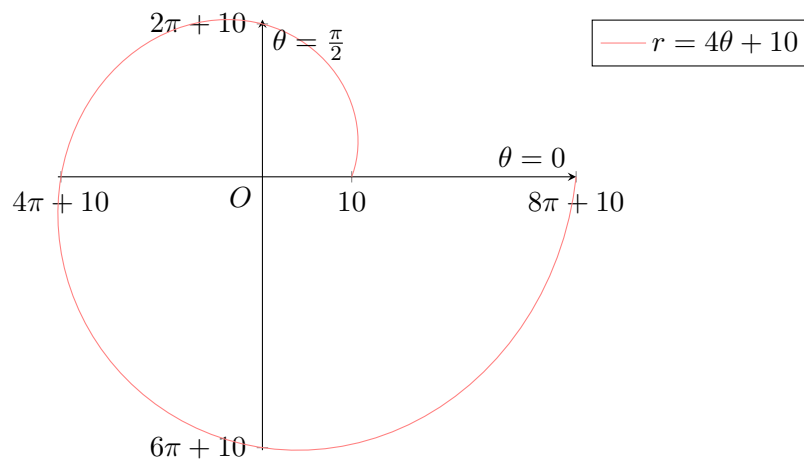
**Problem 10.** Suppose that a long thin rod with one end fixed at the pole of a polar coordinate system rotates counter-clockwise at the constant rate of 0.5 rad/sec. At time  $t = 0$ , a bug on the rod is 10 mm from the pole and is moving outward along the rod at a constant speed of 2 mm/sec. Find an equation of the form  $r = f(\theta)$  for the part of motion of the bug, assuming that  $\theta = 0$  when  $t = 0$ . Sketch the path of the bug on the polar coordinate system for  $0 \leq t \leq 4\pi$ .

**Solution.** Let  $\theta(t)$  and  $r(t)$  be functions of time, with  $\theta(0) = 0$  and  $r(0) = 10$ . We know that  $\frac{d\theta}{dt} = 0.5$  and  $\frac{dr}{dt} = 2$ . Hence,

$$\frac{dr}{d\theta} = \frac{dr}{dt} \cdot \frac{dt}{d\theta} = \frac{dr}{dt} \cdot \left(\frac{d\theta}{dt}\right)^{-1} = 2 \cdot (0.5)^{-1} = 4.$$

Thus,  $r = 4\theta + r(0) = 4\theta + 10$ .

Since  $\frac{d\theta}{dt} = 0.5$  and  $\theta(0) = 0$ , we have  $\theta = 0.5t$ . Hence,  $0 \leq t \leq 4\pi \implies 0 \leq \theta \leq 2\pi$ .



**Problem 11.** The equation, in polar coordinates, of a curve  $C$  is  $r = ae^{\frac{1}{2}\theta}$ ,  $0 \leq \theta \leq 2\pi$ , where  $a$  is a positive constant. Write down, in terms of  $\theta$ , the Cartesian coordinates,  $x$  and  $y$ , of a general point  $P$  on the curve. Show that the gradient at  $P$  is given by  $\frac{dy}{dx} = \frac{\tan \theta + 2}{1 - 2 \tan \theta}$ .

Hence, show that the tangent at  $P$  is inclined to  $\overrightarrow{OP}$  at a constant angle  $\alpha$ , where  $\tan \alpha = 2$ . Sketch the curve  $C$ .

**Solution.** Note that  $x = r \cos \theta$  and  $y = r \sin \theta$ , whence  $x = ae^{\frac{1}{2}\theta} \cos \theta$  and  $y = ae^{\frac{1}{2}\theta} \sin \theta$ . Hence,  $P \left( ae^{\frac{1}{2}\theta} \cos \theta, ae^{\frac{1}{2}\theta} \sin \theta \right)$ .

Observe that  $\frac{dr}{d\theta} = \frac{1}{2}ae^{\frac{1}{2}\theta} = \frac{1}{2}r$ . Hence,

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} = \frac{\frac{1}{2}r \sin \theta + r \cos \theta}{\frac{1}{2}r \cos \theta - r \sin \theta} = \frac{\sin \theta + 2 \cos \theta}{\cos \theta - 2 \sin \theta} = \frac{\tan \theta + 2}{1 - 2 \tan \theta}.$$

Let  $\mathbf{t} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$  represent the direction of the tangent line. Then

$$\mathbf{t} = \begin{pmatrix} 1 \\ dy/dx \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{\tan \theta + 2}{1 - 2 \tan \theta} \end{pmatrix} = \frac{1}{1 - 2 \tan \theta} \begin{pmatrix} 1 - 2 \tan \theta \\ \tan \theta + 2 \end{pmatrix}$$

and

$$\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ae^{\frac{1}{2}\theta} \cos \theta \\ ae^{\frac{1}{2}\theta} \sin \theta \end{pmatrix} = ae^{\frac{1}{2}\theta} \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}.$$

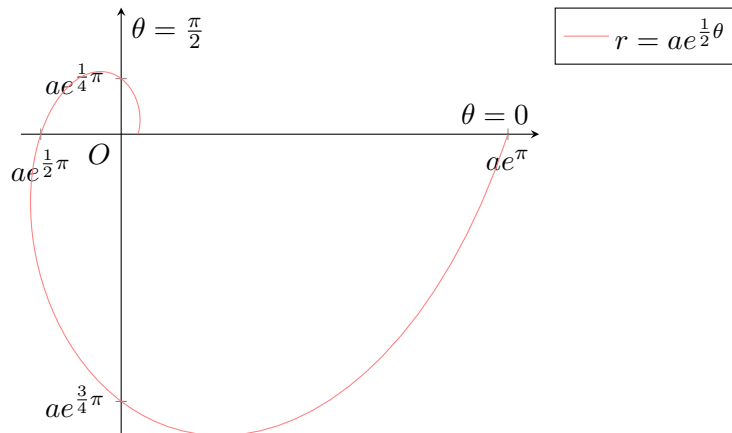
By the definition of the dot-product, we have  $\mathbf{t} \cdot \overrightarrow{OP} = |\mathbf{t}| |\overrightarrow{OP}| \cos \alpha$ , whence

$$\begin{aligned} \cos \alpha &= \frac{\mathbf{t} \cdot \overrightarrow{OP}}{|\mathbf{t}| |\overrightarrow{OP}|} = \frac{(1 - 2 \tan \theta) \cos \theta + (\tan \theta + 2) \sin \theta}{\sqrt{(1 - 2 \tan \theta)^2 + (\tan \theta + 2)^2} \cdot \sqrt{\cos^2 \theta + \sin^2 \theta}} \\ &= \frac{\cos \theta + \tan \theta \sin \theta}{\sqrt{5 \tan^2 \theta + 5}} = \frac{\cos^2 \theta + \sin^2 \theta}{\sqrt{5 \sin^2 \theta + 5 \cos^2 \theta}} = \frac{1}{\sqrt{5}}. \end{aligned}$$

Thus,  $\alpha = \arccos \frac{1}{\sqrt{5}}$ . Since  $\tan(\arccos x) = \frac{\sqrt{1-x^2}}{x}$ ,

$$\tan \alpha = \tan \left( \arccos \frac{1}{\sqrt{5}} \right) = \frac{\sqrt{1 - (1/\sqrt{5})^2}}{1/\sqrt{5}} = 2.$$

Hence, the tangent at  $P$  is inclined to  $\overrightarrow{OP}$  at a constant angle  $\alpha$ , where  $\tan \alpha = 2$ .



**Problem 12.** The polar equation of a curve is given by  $r = e^\theta$  where  $0 \leq \theta \leq \frac{\pi}{2}$ . Cartesian axes are taken at the pole  $O$ . Express  $x$  and  $y$  in terms of  $\theta$  and hence find the Cartesian equation of the tangent at  $(e^{\frac{\pi}{2}}, \frac{\pi}{2})$ .

**Solution.** Recall that  $x = r \cos \theta$  and  $y = r \sin \theta$ , whence  $x = e^\theta \cos \theta$  and  $y = e^\theta \sin \theta$ . Thus,  $\frac{dx}{d\theta} = e^\theta(\cos \theta - \sin \theta)$ , and  $\frac{dy}{d\theta} = e^\theta(\cos \theta + \sin \theta)$ . Hence,

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{e^\theta(\cos \theta + \sin \theta)}{e^\theta(\cos \theta - \sin \theta)} = \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta}.$$

At  $(e^{\frac{\pi}{2}}, \frac{\pi}{2})$ , we clearly have  $x = 0$  and  $y = e^{\pi/2}$ . Also,  $dy/dx = -1$ . By the point-slope formula, the equation of the tangent line at  $(e^{\frac{\pi}{2}}, \frac{\pi}{2})$  is given by  $y = -x + e^{\frac{\pi}{2}}$ .

\* \* \* \* \*

**Problem 13.** A curve  $C$  has polar equation  $r = a \cot \theta$ ,  $0 < \theta \leq \pi$ , where  $a$  is a positive constant.

(a) Show that  $y = a$  is an asymptote of  $C$ .

(b) Find the tangent at the pole.

Hence, sketch  $C$  and find the Cartesian equation of  $C$  in the form  $y^2(x^2 + y^2) = bx^2$ , where  $b$  is a constant to be determined.

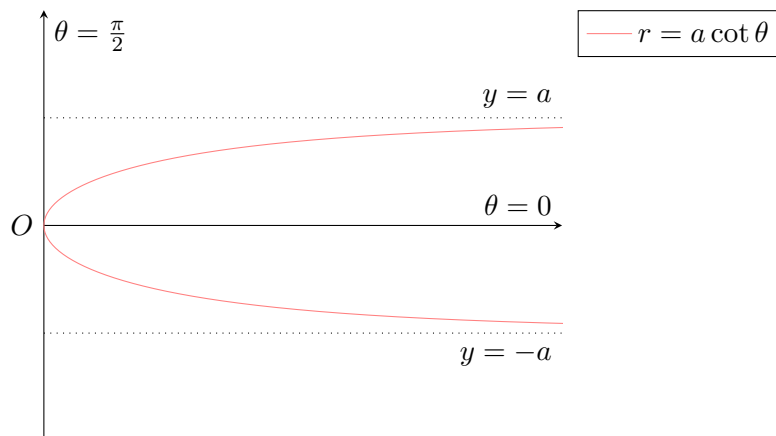
**Solution.**

**Part (a).** Note that

$$r = a \cot \theta \implies y = r \sin \theta = a \cos \theta.$$

As  $\theta \rightarrow 0$ ,  $r \rightarrow \infty$ . Hence, there is an asymptote at  $\theta = 0$ . Since  $\cos \theta = 1$  when  $\theta = 0$ , the line  $y = a \cos \theta = a$  is an asymptote of  $C$ .

**Part (b).** For tangents at the pole,  $r = 0 \implies \cot \theta = 0 \implies \theta = \frac{\pi}{2}$ .



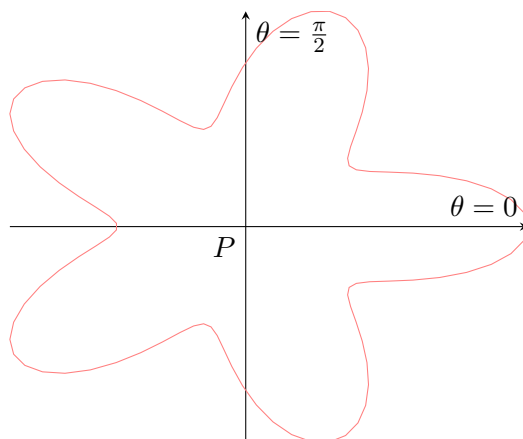
Note that

$$r = a \cot \theta = a \left( \frac{r \cos \theta}{r \sin \theta} \right) = a \left( \frac{x}{y} \right).$$

Thus,

$$x^2 + y^2 = r^2 = a^2 \left( \frac{x^2}{y^2} \right) \implies y^2 (x^2 + y^2) = a^2 x^2,$$

whence  $b = a^2$ .

**Problem 14.**

Relative to the pole  $P$  and the initial line  $\theta = 0$ , the polar equation of the curve shown is either

- i.  $r = a + b \sin n\theta$ , or
- ii.  $r = a + b \cos n\theta$

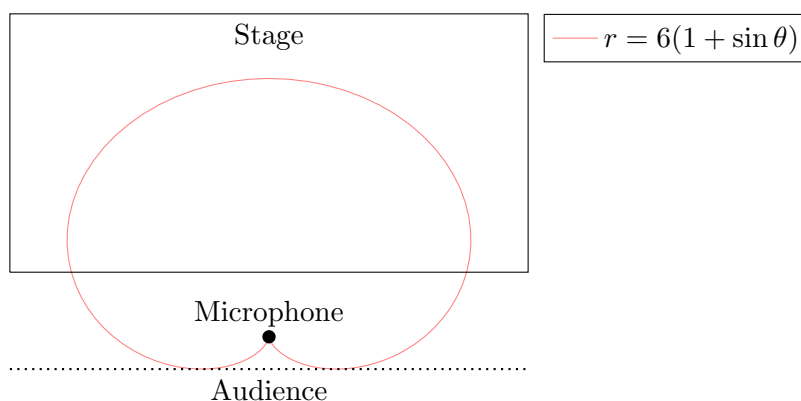
where  $a$ ,  $b$  and  $n$  are positive constants. State, with a reason, whether the equation is (i) or (ii) and state the value of  $n$ .

The maximum value of  $r$  is  $\frac{11}{2}$  and the minimum value of  $r$  is  $\frac{5}{2}$ . Find the values of  $a$  and  $b$ .

**Solution.** Since the curve is symmetrical about the horizontal half-line  $\theta = 0$ , the polar equation of the curve is a function of  $\cos n\theta$  only. Hence, the polar equation of the curve is  $r = a + b \cos n\theta$ , with  $n = 5$ .

Observe that the maximum value of  $r$  is achieved when  $\cos 5\theta = 1$ , whence  $r = a + b$ . Thus,  $a + b = \frac{11}{2}$ . Also observe that the minimum value of  $r$  is achieved when  $\cos 5\theta = -1$ , whence  $r = a - b$ . Thus,  $a - b = \frac{5}{2}$ . Solving, we get  $a = 4$  and  $b = \frac{3}{2}$ .

\* \* \* \* \*

**Problem 15.**

Sound engineers often use a microphone with a cardioid acoustic pickup pattern to record live performances because it reduces pickup from the audience. Suppose a cardioid microphone is placed 3 metres from the front of the stage, and the boundary of the optimal pickup region is given by the cardioid with polar equation

$$r = 6(1 + \sin \theta)$$

where  $r$  is measured in metres and the microphone is at the pole.

Find the minimum distance from the front of the stage the first row of the audience can be seated such that the microphone does not pick up noise from the audience.

**Solution.** Note that  $r = 6(1 + \sin \theta) = 6(1 + \frac{y}{r})$ , whence  $r^2 = 6r + 6y$ . Thus,

$$r^2 - 6r - 6y = 0 \implies r = 3 \pm \sqrt{9 + 6y} \implies 9 + 6y = (r - 3)^2.$$

Since  $9 + 6y = (r - 3)^2 \geq 0$ , we have  $y \geq -1.5$ . Thus, the furthest distance the audience has to be from the stage is  $|-1.5| + 3 = 4.5$  m.

\* \* \* \* \*

**Problem 16.** To design a flower pendant, a designer starts off with a curve  $C_1$ , given by the Cartesian equation

$$(x^2 + y^2)^2 = a^2 (3x^2 - y^2)$$

where  $a$  is a positive constant.

- (a) Show that a corresponding polar equation of  $C_1$  is  $r^2 = a^2(1 + 2 \cos 2\theta)$ .
- (b) Find the equations of the tangents to  $C_1$  at the pole.

Another curve  $C_2$  is obtained by rotating  $C_1$  anti-clockwise about the origin by  $\frac{\pi}{3}$  radians.

- (c) State a polar equation of  $C_2$ .
- (d) Sketch  $C_1$  and  $C_2$  on the same diagram, stating clearly the exact polar coordinates of the points of intersection of the curves with the axes. Find also the exact polar coordinates of the points of intersection with  $C_1$  and  $C_2$ .

The curve  $C_3$  is obtained by reflecting  $C_2$  in the line  $\theta = \frac{\pi}{2}$ .

- (e) State a polar equation of  $C_3$ .
- (f) The designer wishes to enclose the 3 curves inside a circle given by the polar equation  $r = r_1$ . State the minimum value of  $r_1$  in terms of  $a$ .

**Solution.**

**Part (a).** Observe that  $(x^2 + y^2)^2 = r^4$  and  $3x^2 - y^2 = r^2 (3 \cos^2 \theta - \sin^2 \theta)$ . Hence,

$$(x^2 + y^2)^2 = a^2 (3x^2 - y^2) \implies r^2 = a^2 (3 \cos^2 \theta - \sin^2 \theta).$$

Note that

$$3 \cos^2 \theta - \sin^2 \theta = 1 + 2 \cos^2 \theta - 2 \sin^2 \theta = 1 + 2 \cos 2\theta.$$

Thus,

$$r^2 = a^2 (1 + 2 \cos 2\theta).$$

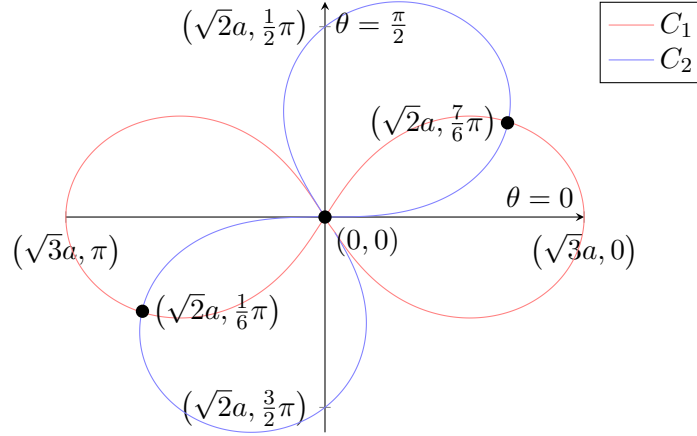
**Part (b).** For tangents at the pole,

$$r = 0 \implies 1 + 2 \cos 2\theta = 0 \implies \cos 2\theta = -\frac{1}{2}.$$

Since  $0 \leq 2\theta \leq 2\pi$ , we have  $\theta = \pi/3, 2\pi/3$ . For full lines, we also have  $\theta = 4\pi/3$  and  $\theta = 5\pi/3$ .

**Part (c).**

$$r^2 = a^2 \left[ 1 + 2 \cos \left( 2 \left( \theta - \frac{\pi}{3} \right) \right) \right] = a^2 \left[ 1 + 2 \cos \left( 2\theta - \frac{2}{3}\pi \right) \right].$$

**Part (d).**

Consider the horizontal intercepts of  $C_1$ . When  $\theta = 0$ ,  $r = \sqrt{3}a$ . Hence, by symmetry,  $C_1$  intercepts the horizontal axis at  $(\sqrt{3}a, 0)$  and  $(\sqrt{3}a, \pi)$ .

Consider the vertical intercepts of  $C_2$ . When  $\theta = \pi/2$ ,  $r = \sqrt{2}a$ . Hence, by symmetry,  $C_2$  intercepts the vertical axis at  $(\sqrt{2}a, \pi/2)$  and  $(\sqrt{2}a, 3\pi/2)$ .

Now consider the intersections between  $C_1$  and  $C_2$ . By symmetry, it is obvious that the points of intersections must lie along the half-lines  $\pi/6$  and  $7\pi/6$ , or along the half-lines  $4\pi/6$  and  $10\pi/6$ . By symmetry, we consider only the half-lines  $\pi/6$  and  $4\pi/6$ .

*Case 1:*  $\theta = \pi/6$ . Substituting  $\theta = \pi/6$  into the equation of  $C_1$ , we obtain  $r = \sqrt{2}a$ . Hence,  $C_1$  and  $C_2$  intersect at  $(\sqrt{2}a, \pi/6)$  and, by symmetry, at  $(\sqrt{2}a, 7\pi/6)$ .

*Case 2:*  $\theta = 4\pi/6$ . Substituting  $\theta = 4\pi/6$  into the equation of  $C_1$ , we obtain  $r = 0$ . Hence,  $C_1$  and  $C_2$  intersect at  $(0, 0)$ .

**Part (e).** Reflecting about the line  $\theta = \pi/2$  is equivalent to applying the map  $\theta \mapsto \theta + \pi/3$  to  $C_1$ . Hence,

$$r^2 = a^2 \left[ 1 + 2 \cos \left( 2 \left( \theta + \frac{1}{3}\pi \right) \right) \right] = a^2 \left[ 1 + 2 \cos \left( 2\theta + \frac{2}{3}\pi \right) \right].$$

**Part (f).**  $r_1 = \sqrt{3}a$ .

## Assignment A6

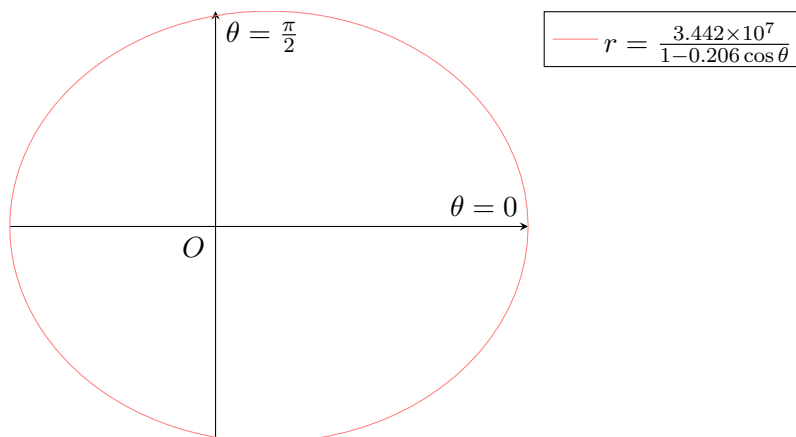
**Problem 1.** The planet Mercury travels around the sun in an elliptical orbit given approximately by

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cos \theta},$$

where  $r$  is measured in miles and the sun is at the pole.

Sketch the orbit and find the distance from Mercury to the sun at the aphelion (the greatest distance from the sun) and at the perihelion (the shortest distance from the sun).

**Solution.**



Observe that  $r$  attains a maximum when  $\cos \theta$  is also at its maximum. Since the maximum value of  $\cos \theta$  is 1,

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cdot 1} = 4.34 \times 10^7 \text{ (3 s.f.)}.$$

Hence, the distance from Mercury to the sun at the aphelion is  $4.34 \times 10^7$  miles.

Observe that  $r$  attains a minimum when  $\cos \theta$  is also at its minimum. Since the minimum value of  $\cos \theta$  is  $-1$ ,

$$r = \frac{3.442 \times 10^7}{1 - 0.206 \cdot -1} = 2.85 \times 10^7 \text{ (3 s.f.)}.$$

Hence, the distance from Mercury to the sun at the perihelion is  $2.85 \times 10^7$  miles.

\* \* \* \* \*

**Problem 2.** A variable point  $P$  has polar coordinates  $(r, \theta)$ , and fixed points  $A$  and  $B$  have polar coordinates  $(1, 0)$  and  $(1, \pi)$  respectively. Given that  $P$  moves so that the product  $PA \cdot PB = 2$ , show that

$$r^2 = \cos 2\theta + \sqrt{3 + \cos^2 2\theta}.$$

- Given that  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$ , find the maximum and minimum values of  $r$ , and the values of  $\theta$  at which they occur.
- Verify that the path taken by  $P$  is symmetric about the lines  $\theta = 0$  and  $\theta = \frac{\pi}{2}$ , giving your reasons.

**Solution.** Note that  $A$  and  $B$  have Cartesian coordinates  $(1, 0)$  and  $(-1, 0)$  respectively. Let  $P(x, y)$ . Then

$$PA^2 = (x - 1)^2 + y^2, \quad PB^2 = (x + 1)^2 + y^2.$$

Hence,

$$PA \cdot PB = ((x-1)^2 + y^2)((x+1)^2 + y^2) = (x^2 + y^2)^2 - 2(x^2 - y^2) + 1.$$

Since  $x^2 - y^2 = r^2(\cos^2 \theta - \sin^2 \theta) = r^2 \cos 2\theta$ , the polar equation of the locus of  $P$  is

$$r^4 - 2r^2 \cos 2\theta + 1 = (PA \cdot PB)^2 = 4 \implies r^4 - 2r^2 \cos 2\theta - 3 = 0.$$

By the quadratic formula, we have

$$r^2 = \frac{2 \cos 2\theta \pm \sqrt{4 \cos^2 2\theta + 12}}{2} = \cos 2\theta \pm \sqrt{\cos^2 2\theta + 3}.$$

Since  $\sqrt{\cos^2 2\theta + 3} > \cos 2\theta$  and  $r^2 \geq 0$ , we reject the negative case. Thus,

$$r^2 = \cos 2\theta + \sqrt{3 + \cos^2 2\theta}.$$

**Part (a).** Differentiating with respect to  $\theta$ , we obtain

$$2r \frac{dr}{d\theta} = -2 \sin 2\theta \left( 1 + \frac{1}{2\sqrt{3 + \cos^2 2\theta}} \right).$$

For stationary points,  $dr/d\theta = 0$ . Since  $1 + 1/(2\sqrt{3 + \cos^2 2\theta}) > 0$ , we must have  $\sin 2\theta = 0$ , whence  $\theta = 0, \pi/2, \pi, 3\pi/2$ . By symmetry, we only consider  $\theta = 0$  and  $\theta = \pi/2$ .

*Case 1.* When  $\theta = 0$ , we have  $r^2 = 3$ , whence  $r = \sqrt{3}$ .

*Case 2.* When  $\theta = \pi/2$ , we have  $r^2 = 1$ , whence  $r = 1$ .

Thus,  $\max r = \sqrt{3}$  and occurs when  $\theta = 0, \pi$ , while  $\min r = 1$  and occurs when  $\theta = \pi/2, 3\pi/2$ .

**Part (b).** Recall that the path taken by  $P$  is given by

$$((x-1)^2 + y^2)((x+1)^2 + y^2) = 4.$$

Observe that the above equation is invariant under the transformations  $x \mapsto -x$  and  $y \mapsto -y$ . Hence, the path is symmetric about both the  $x$ - and  $y$ -axes, i.e. the lines  $\theta = 0$  and  $\theta = \pi/2$ .

\* \* \* \* \*

### Problem 3.

(a) Explain why the curve with equation  $x^3 + 2xy^2 - a^2y = 0$  where  $a$  is a positive constant lies entirely in the region  $|x| \leq 2^{-3/4}a$ .

(b) Show that the polar equation of this curve is  $r^2 = \frac{a^2 \tan \theta}{2 - \cos^2 \theta}$ .

(c) Sketch the curve.

### Solution.

**Part (a).** Consider the discriminant  $\Delta$  of  $x^3 + 2xy^2 - a^2y = 0$  with respect to  $y$ :

$$\Delta = (-a^2)^2 - 4(2x) = a^4 - 8x^4.$$

For points on the curve, we clearly have  $\Delta \geq 0$ . Thus,

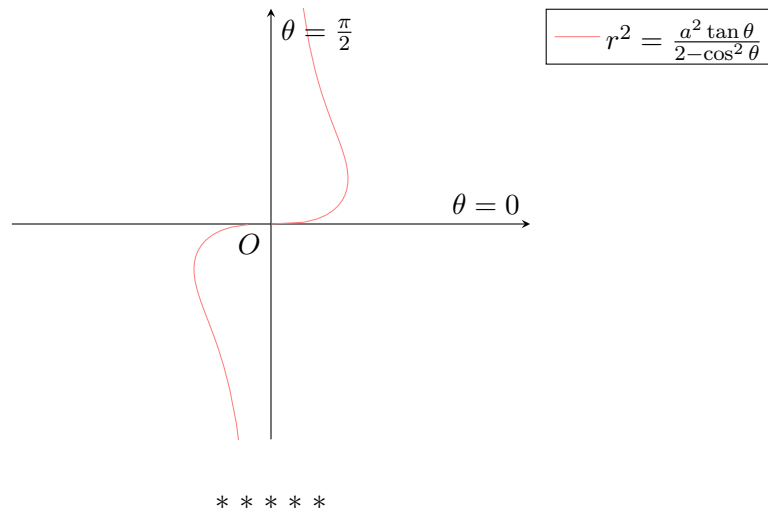
$$a^4 - 8x^4 \geq 0 \implies x^4 \leq 2^{-3}a^4 \implies |x| \leq 2^{-3/4}a.$$



**Part (b).**

$$\begin{aligned} x^3 + 2xy^2 - a^2y &= 0 \implies 2(x^2 + y^2) - x^2 - a^2 \frac{y}{x} = 0 \implies 2r^2 - r^2 \cos^2 \theta - a^2 \tan \theta = 0 \\ \implies r^2 &= \frac{a^2 \tan \theta}{2 - \cos^2 \theta}. \end{aligned}$$

**Part (c).**

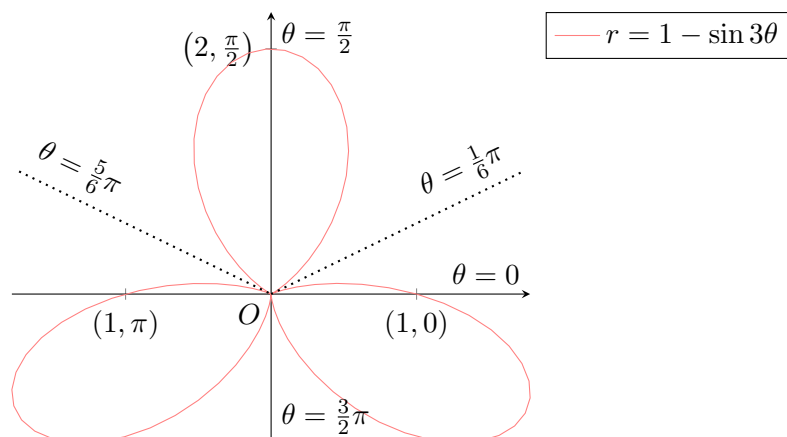


**Problem 4.** The curve  $C$  has polar equation  $r = 1 - \sin 3\theta$ , where  $0 \leq \theta \leq 2\pi$ .

- Sketch the curve  $C$ , showing the tangents at the pole and the intersections with the axes.
- Find the gradient of the curve at the point where  $\theta = \frac{\pi}{3}$ , giving your answer in the form  $a + b\sqrt{3}$ , where  $a$  and  $b$  are constants to be determined.

**Solution.**

**Part (a).**



When  $\theta = 0$  or  $\theta = \pi$ , we have  $r = 1$ . Thus,  $C$  intersects the horizontal axis at  $(1, 0)$  and  $(1, \pi)$ . When  $\theta = \pi/2$ , we have  $r = 2$ . Thus,  $C$  intersects the vertical axis at  $(2, \pi/2)$ . When  $\theta = 3\pi/2$ , we have  $r = 0$ . Thus,  $C$  passes through the pole.

For tangents at the pole,  $r = 0 \implies \sin 3\theta = 1 \implies \theta = \pi/6, 5\pi/6, 3\pi/2$ .

**Part (b).** Note that  $dr/d\theta = -3\cos 3\theta$  evaluates to 3 when  $\theta = \pi/3$ . Thus,

$$\left. \frac{dy}{dx} \right|_{\theta=\frac{\pi}{3}} = \frac{\left. \frac{dr}{d\theta} \sin \theta + r \cos \theta \right|_{\theta=\frac{\pi}{3}}}{\left. \frac{dr}{d\theta} \cos \theta - r \sin \theta \right|_{\theta=\frac{\pi}{3}}} = \frac{3\sqrt{3} + 1}{3 - \sqrt{3}} = \frac{12 + 10\sqrt{3}}{6} = 2 + \frac{5}{3}\sqrt{3}.$$

Hence, when  $\theta = \pi/3$ , the gradient of the curve is  $2 + 5\sqrt{3}/2$ .

## A7. Vectors I

### Tutorial A7

**Problem 1.** The vector  $\mathbf{v}$  is defined by  $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ . Find the unit vector in the direction of  $\mathbf{v}$  and hence find a vector of magnitude 25 which is parallel to  $\mathbf{v}$ .

**Solution.**

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{3^2 + (-4)^2 + 1^2}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \quad 25\hat{\mathbf{v}} = \frac{25}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}.$$

\* \* \* \* \*

**Problem 2.** With respect to an origin  $O$ , the position vectors of the points  $A$ ,  $B$ ,  $C$  and  $D$  are  $4\mathbf{i} + 7\mathbf{j}$ ,  $\mathbf{i} + 3\mathbf{j}$ ,  $2\mathbf{i} + 4\mathbf{j}$  and  $3\mathbf{i} + d\mathbf{j}$  respectively.

(a) Find the vectors  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ .

(b) Find the value of  $d$  if  $B$ ,  $C$  and  $D$  are collinear. State the ratio  $\frac{BC}{BD}$ .

**Solution.**

**Part (a).** Note that

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad \overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

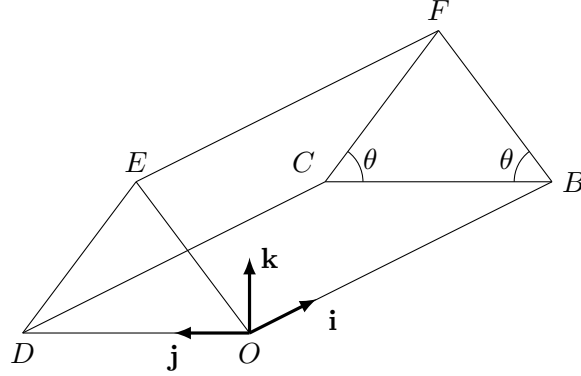
**Part (b).** If  $B$ ,  $C$  and  $D$  are collinear, then  $\overrightarrow{BC} = \lambda \overrightarrow{CD}$  for some  $\lambda \in \mathbb{R}$ .

$$\overrightarrow{BC} = \lambda \overrightarrow{CD} \implies \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda \left( \overrightarrow{OD} - \overrightarrow{OC} \right) = \lambda \left[ \begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right] = \begin{pmatrix} \lambda \\ \lambda(d-4) \end{pmatrix}.$$

Hence,  $\lambda = 1$  and  $\lambda(d-4) = 1$ , whence  $d = 5$ . Also,  $\overrightarrow{BC} = \overrightarrow{CD}$ . Thus,

$$\frac{BC}{BD} = \frac{BC}{BC + CD} = \frac{BC}{BC + BC} = \frac{1}{2}.$$

**Problem 3.** The diagram shows a roof, with horizontal rectangular base  $OBCD$ , where  $OB = 10$  m and  $BC = 6$  m. The triangular planes  $ODE$  and  $BCF$  are vertical and the ridge  $EF$  is horizontal to the base. The planes  $OBFE$  and  $DCFE$  are each inclined at an angle  $\theta$  to the horizontal, where  $\tan \theta = 4/3$ . The point  $O$  is taken as the origin and vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ , each of length 1 m, are taken along  $OB$ ,  $OD$  and vertically upwards from  $O$  respectively.



Find the position vectors of the points  $B$ ,  $C$ ,  $D$ ,  $E$  and  $F$ .

**Solution.** Note that  $\overrightarrow{OB} = 10\mathbf{i}$  and  $\overrightarrow{BC} = 6\mathbf{j}$ . Thus,  $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = 10\mathbf{i} + 6\mathbf{j}$ . Also, note that  $\triangle ODE \cong \triangle BCF$ . Hence,  $\overrightarrow{OD} = \overrightarrow{BC} = 6\mathbf{j}$ . Note that  $\triangle ODE$  is isosceles. Let  $G$  be the mid-point of  $OD$ . Since  $\tan \theta = 4/3$ , we have

$$\frac{EG}{DG} = \frac{4}{3} \implies EG = \frac{4}{3}DG = \frac{2}{3}OD = \frac{2}{3} \cdot 6 = 4 \implies \overrightarrow{GE} = 4\mathbf{k}.$$

Hence,

$$\overrightarrow{OE} = \overrightarrow{OG} + \overrightarrow{GE} = \frac{1}{2}\overrightarrow{OD} + \overrightarrow{GE} = 3\mathbf{j} + 4\mathbf{k}.$$

Hence,

$$\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \overrightarrow{OB} + \overrightarrow{OE} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

Thus,

$$\overrightarrow{OB} = 10\mathbf{i}, \quad \overrightarrow{OC} = 10\mathbf{i} + 6\mathbf{j}, \quad \overrightarrow{OD} = 6\mathbf{j}, \quad \overrightarrow{OE} = 3\mathbf{j} + 4\mathbf{k}, \quad \overrightarrow{OF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}.$$

\* \* \* \* \*

**Problem 4.** Find  $\mathbf{u} \cdot \mathbf{v}$ ,  $\mathbf{u} \times \mathbf{v}$  and the angle between  $\mathbf{u}$  and  $\mathbf{v}$  given that

(a)  $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

(b)  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$ ,  $\mathbf{v} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

**Solution.**

**Part (a).** We have  $\mathbf{u} = \langle 1, -1, 1 \rangle$  and  $\mathbf{v} = \langle 3, 2, 7 \rangle$ . Hence,

$$\mathbf{u} \cdot \mathbf{v} = (1)(3) + (-1)(2) + (1)(7) = 8, \quad \mathbf{u} \times \mathbf{v} = \begin{pmatrix} (-1)(7) - (2)(1) \\ (1)(3) - (7)(1) \\ (1)(2) - (3)(-1) \end{pmatrix} = \begin{pmatrix} -9 \\ -4 \\ 5 \end{pmatrix}.$$

Let the angle between  $\mathbf{u}$  and  $\mathbf{v}$  be  $\theta$ .

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{8}{\sqrt{3}\sqrt{62}} \implies \theta = 54.1^\circ \text{ (1 d.p.)}.$$

**Part (b).** We have  $\mathbf{u} = \langle 2, 0, -3 \rangle$  and  $\mathbf{v} = \langle -1, 7, 2 \rangle$ . Hence,

$$\mathbf{u} \cdot \mathbf{v} = (2)(-1) + (0)(7) + (-3)(2) = -8, \quad \mathbf{u} \cdot \mathbf{v} = \begin{pmatrix} (0)(2) - (7)(-3) \\ (-3)(-1) - (2)(2) \\ (2)(7) - (-1)(0) \end{pmatrix} = \begin{pmatrix} 21 \\ -1 \\ 14 \end{pmatrix}.$$

Let the angle between  $\mathbf{u}$  and  $\mathbf{v}$  be  $\theta$ .

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}| |\mathbf{v}|} = \frac{-8}{\sqrt{13}\sqrt{54}} \implies \theta = 107.6^\circ \text{ (1 d.p.)}.$$

\* \* \* \* \*

**Problem 5.** Find  $\mathbf{u} \cdot \mathbf{v}$  and  $|\mathbf{u} \times \mathbf{v}|$  given that  $\mathbf{u} = 2\mathbf{a} - \mathbf{b}$ ,  $\mathbf{v} = -\mathbf{a} + 3\mathbf{b}$ , where  $|\mathbf{a}| = 2$ ,  $|\mathbf{b}| = 1$  and the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $60^\circ$ .

**Solution.**

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (2\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} + 3\mathbf{b}) = -2\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - 3\mathbf{b} \cdot \mathbf{b} \\ &= -2|\mathbf{a}|^2 - 3|\mathbf{b}|^2 + 7|\mathbf{a}||\mathbf{b}|\cos \theta = -2(2)^2 - 3(1)^2 + 7(2)(1)\cos 60^\circ = -4. \\ |\mathbf{u} \times \mathbf{v}| &= |(2\mathbf{a} - \mathbf{b}) \times (-\mathbf{a} + 3\mathbf{b})| = |-2\mathbf{a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - 3\mathbf{b} \times \mathbf{b}| \\ &= |5\mathbf{a} \times \mathbf{b}| = 5|\mathbf{a}||\mathbf{b}|\sin \theta = 5(2)(1)\sin 60^\circ = 5\sqrt{3}. \end{aligned}$$

\* \* \* \* \*

**Problem 6.** If  $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$ , find

- (a) a unit vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ ,
- (b) a vector perpendicular to both  $(3\mathbf{b} - 5\mathbf{c})$  and  $(7\mathbf{b} + \mathbf{c})$ .

**Solution.**

**Part (a).** Note that  $\mathbf{a} \times \mathbf{b} = \langle 11, -4, -5 \rangle$ . Hence,  $\widehat{\mathbf{a} \times \mathbf{b}} = \frac{1}{\sqrt{162}} \langle 11, -4, -5 \rangle$ .

**Part (b).** Observe that  $(3\mathbf{b} - 5\mathbf{c}) \times (7\mathbf{b} + \mathbf{c}) = \lambda \mathbf{b} \times \mathbf{c}$  for some  $\lambda \in \mathbb{R}$ . It hence suffices to find  $\mathbf{b} \times \mathbf{c}$ , which works out to be  $\langle -3, 6, 3 \rangle$ .

\* \* \* \* \*

**Problem 7.** The position vectors of the points  $A$ ,  $B$  and  $C$  are given by  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ ,  $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$  respectively. Find

- (a) a unit vector parallel to  $\overrightarrow{AB}$ ;
- (b) the position vector of the point  $D$  such that  $ABCD$  is a parallelogram, leaving your answer in terms of  $\lambda$ ;
- (c) the value of  $\lambda$  if  $A$ ,  $B$  and  $C$  are collinear;
- (d) the position vector of the point  $P$  on  $AB$  is  $AP : PB = 2 : 1$ .

**Solution.**

**Part (a).**

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}.$$

Note that  $|\overrightarrow{AB}| = \sqrt{61}$ . Hence, the required vector is  $\frac{1}{\sqrt{61}} \langle 3, -4, 6 \rangle$ .

**Part (b).** Since  $ABCD$  is a parallelogram, we have that  $\overrightarrow{AD} = \overrightarrow{BC}$ . Thus,

$$\overrightarrow{OD} - \mathbf{a} = \mathbf{c} - \mathbf{b} \implies \overrightarrow{OD} = \mathbf{a} - \mathbf{b} + \mathbf{c} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} = \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix}.$$

**Part (c).** Given that  $A$ ,  $B$  and  $C$  are collinear, we have  $\overrightarrow{AB} = k\overrightarrow{BC}$  for some  $k \in \mathbb{R}$ . Hence,

$$\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} = k(\mathbf{c} - \mathbf{b}) = k \left[ \begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right] = k \begin{pmatrix} 6 \\ \lambda + 1 \\ 12 \end{pmatrix}.$$

We hence see that  $k = 1/2$ , whence  $\lambda = -9$ .

**Part (d).** By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + 2\mathbf{b}}{2 + 1} = \frac{1}{3} \left[ \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right] = \frac{1}{3} \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}.$$

\* \* \* \* \*

**Problem 8.**  $ABCD$  is a square, and  $M$  and  $N$  are the midpoints of  $BC$  and  $CD$  respectively. Express  $\overrightarrow{AC}$  in terms of  $\mathbf{p}$  and  $\mathbf{q}$ , where  $\overrightarrow{AM} = \mathbf{p}$  and  $\overrightarrow{AN} = \mathbf{q}$ .

**Solution.** Let  $ABCD$  be a square with side length  $2k$  with  $A$  at the origin. Then  $\mathbf{p} = \overrightarrow{AM} = \langle 2k, -k \rangle$  and  $\mathbf{q} = \overrightarrow{AN} = \langle k, -2k \rangle$ . Hence,  $\mathbf{p} + \mathbf{q} = \langle 3k, -3k \rangle$ . Thus,  $\overrightarrow{AC} = \langle 2k, -2k \rangle = \frac{2}{3} \langle 3k, -3k \rangle = \frac{2}{3} (\mathbf{p} + \mathbf{q})$ .

\* \* \* \* \*

**Problem 9.** The points  $A$ ,  $B$  have position vectors  $\mathbf{a}$ ,  $\mathbf{b}$  respectively, referred to an origin  $O$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are not parallel to each other. The point  $C$  lies on  $AB$  between  $A$  and  $B$  and is such that  $\frac{AC}{CB} = 2$ , and  $D$  is the mid-point of  $OC$ . The line  $AD$  produced meets  $OB$  at  $E$ .

Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ ,

(a) the position vector of  $C$  (referred to  $O$ ),

(b) the vector  $\overrightarrow{AD}$ . Find the values of  $\frac{OE}{EB}$  and  $\frac{AE}{ED}$ .

**Solution.**

**Part (a).** By the ratio theorem,

$$\overrightarrow{OC} = \frac{\mathbf{a} + 2\mathbf{b}}{2 + 1} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}.$$

**Part (b).** Using Menalaus' theorem on  $\triangle BCO$ ,

$$\frac{BA}{AC} \frac{CD}{DO} \frac{OE}{EB} = 1 \implies \frac{OE}{EB} = \frac{2}{3}.$$

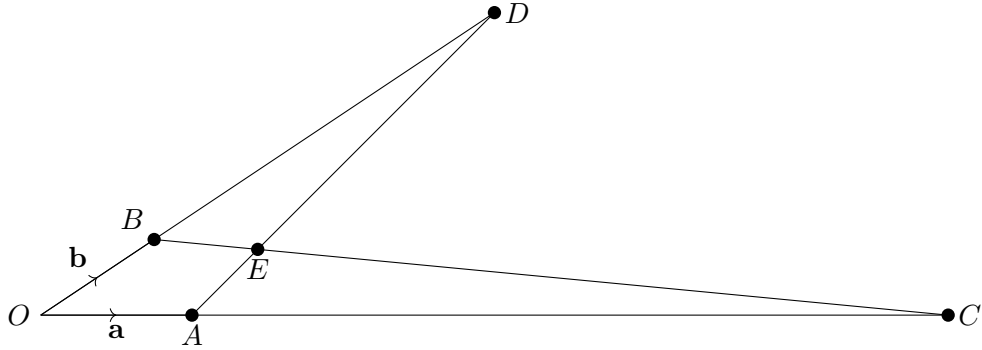
Using Menalaus' theorem on  $\triangle BEA$ ,

$$\frac{BO}{OE} \frac{ED}{DA} \frac{AC}{CB} = 1 \implies \frac{ED}{AD} = \frac{1}{5} \implies \frac{AE}{ED} = \frac{AD + DE}{ED} = 6.$$

\* \* \* \* \*

**Problem 10.**

- (a) The angle between the vectors  $(3\mathbf{i} - 2\mathbf{j})$  and  $(6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k})$  is  $\arccos \frac{6}{13}$ . Show that  $2d^2 - 117d + 333 = 0$ .
- (b) With reference to the origin  $O$ , the points  $A, B, C$  and  $D$  are such that  $\overrightarrow{OA} = \mathbf{a}$ ,  $\overrightarrow{OB} = \mathbf{b}$ ,  $\overrightarrow{AC} = 5\mathbf{a}$ ,  $\overrightarrow{BD} = 3\mathbf{b}$ . The lines  $AD$  and  $BC$  cross at  $E$ .



- (i) Find  $\overrightarrow{OE}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .
- (ii) The point  $F$  divides the line  $CD$  in the ratio  $5 : 3$ . Show that  $O, E$  and  $F$  are collinear, and find  $OE : EF$ .

**Solution.**

**Part (a).** Let  $\mathbf{a} = \langle 3, -2, 0 \rangle$  and  $\mathbf{b} = \langle 6, d, -\sqrt{7} \rangle$ . Note that  $\mathbf{a} \cdot \mathbf{b} = 18 - 2d$ . Let  $\theta$  be the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\begin{aligned} \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \implies \frac{6}{13} = \frac{18 - 2d}{\sqrt{43 + d^2} \sqrt{13}} \implies \frac{9}{13} = \frac{(9 - d)^2}{43 + d^2} \\ &\implies 9(43 + d^2) = 13(d^2 - 18d + 81) \implies 2d^2 - 117d + 333 = 0. \end{aligned}$$

**Part (b).**

**Part (b)(i).** By Menalaus' theorem,

$$\frac{OC}{CA} \frac{AE}{ED} \frac{DB}{BO} = 1 \implies \frac{AE}{ED} = \frac{5}{18} \implies \overrightarrow{AE} = \frac{5}{23} \overrightarrow{AD} \implies \overrightarrow{OE} = \overrightarrow{OA} + \frac{5}{23} \overrightarrow{AD}.$$

Since  $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = 4\mathbf{b} - \mathbf{a}$ . Thus,

$$\overrightarrow{OE} = \mathbf{a} + \frac{5}{23} (4\mathbf{b} - \mathbf{a}) = \frac{18}{23} \mathbf{a} + \frac{20}{23} \mathbf{b}.$$

**Part (b)(ii).** By the ratio theorem,

$$\overrightarrow{OF} = \frac{3\mathbf{c} + 5\mathbf{d}}{5 + 3} = \frac{23}{8} \left( \frac{18}{23} \mathbf{a} + \frac{20}{23} \mathbf{b} \right) = \frac{23}{8} \overrightarrow{OE}.$$

Thus,  $OE : OF = 8 : 23$ .

\* \* \* \* \*

**Problem 11.** Relative to the origin  $O$ , two points  $A$  and  $B$  have position vectors given by  $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$  and  $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$  respectively.

- (a) The point  $P$  divides the line  $AB$  in the ratio  $2 : 1$ . Find the coordinates of  $P$ .
- (b) Show that  $AB$  and  $OP$  are perpendicular.
- (c) The vector  $\mathbf{c}$  is a unit vector in the direction of  $\overrightarrow{OP}$ . Write  $\mathbf{c}$  as a column vector and give the geometrical meaning of  $|\mathbf{a} \cdot \mathbf{c}|$ .
- (d) Find  $\mathbf{a} \times \mathbf{p}$ , where  $\mathbf{p}$  is the vector  $\overrightarrow{OP}$ , and give the geometrical meaning of  $|\mathbf{a} \times \mathbf{p}|$ . Hence, write down the area of triangle  $OAP$ .

**Solution.**

**Part (a).** We have  $\mathbf{a} = \langle 14, 14, 14 \rangle = 14 \langle 1, 1, 1 \rangle$  and  $\mathbf{b} = \langle 11, -13, 2 \rangle$ . By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + 2\mathbf{b}}{2 + 1} = \frac{1}{3} \left[ \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} + 2 \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} \right] = \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}.$$

Hence,  $P(12, -4, 6)$

**Part (b).** Consider  $\overrightarrow{AB} \cdot \overrightarrow{OP}$ .

$$\overrightarrow{AB} \cdot \overrightarrow{OP} = \left[ \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} - \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} \right] \cdot \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ 9 \\ 4 \end{pmatrix} \cdot 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = 0.$$

Since  $\overrightarrow{AB} \cdot \overrightarrow{OP} = 0$ ,  $AB$  and  $OP$  must be perpendicular.

**Part (c).** We have

$$\mathbf{c} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} = \frac{1}{\sqrt{6^2 + (-2)^2 + 3^2}} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}.$$

$|\mathbf{a} \cdot \mathbf{c}|$  is the length of the projection of  $\mathbf{a}$  on  $\overrightarrow{OP}$ .

**Part (d).** We have

$$\mathbf{a} \times \mathbf{p} = 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} = 28 \begin{pmatrix} 1 \cdot 3 - (-2) \cdot 1 \\ 1 \cdot 6 - 3 \cdot 1 \\ 1 \cdot -2 - 6 \cdot 1 \end{pmatrix} = 28 \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix}.$$

$|\mathbf{a} \times \mathbf{p}|$  is twice the area of  $\triangle OAP$ .

$$[\triangle OAP] = \frac{1}{2} |\mathbf{a} \times \mathbf{p}| = 14\sqrt{98} = 98\sqrt{2} \text{ units}^2.$$



**Problem 12.** The points  $A$ ,  $B$  and  $C$  have position vectors given by  $\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{j} - \mathbf{k}$  and  $2\mathbf{i} - \mathbf{j} - \mathbf{k}$  respectively.

- Find the area of the triangle  $ABC$ . Hence, find the sine of the angle  $BAC$ .
- Find a vector perpendicular to the plane  $ABC$ .
- Find the projection vector of  $\overrightarrow{AC}$  onto  $\overrightarrow{AB}$ .
- Find the distance of  $C$  to  $AB$ .

**Solution.**

**Part (a).** We have  $\overrightarrow{OA} = \langle 1, -1, 1 \rangle$ ,  $\overrightarrow{OB} = \langle 0, 1, -1 \rangle$  and  $\overrightarrow{OC} = \langle 2, -1, -1 \rangle$ . Note that  $\overrightarrow{AB} = \langle -1, 2, -2 \rangle$  and  $\overrightarrow{AC} = \langle 1, 0, -2 \rangle$ . Thus,

$$[\triangle ABC] = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \left| \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix} \right| = \frac{1}{2} \cdot 6 = 3 \text{ units}^2.$$

We have

$$\sin BAC = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{|\overrightarrow{AB}| |\overrightarrow{AC}|} = \frac{6}{3\sqrt{5}} = \frac{2\sqrt{5}}{5}.$$

**Part (b).**  $\langle 2, 2, 1 \rangle$  is parallel to  $\overrightarrow{AB} \times \overrightarrow{AC}$  and is hence perpendicular to the plane  $ABC$ .

**Part (c).** The projection vector of  $\overrightarrow{AC}$  onto  $\overrightarrow{AB}$  is given by

$$\left( \overrightarrow{AC} \cdot \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \right) \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}.$$

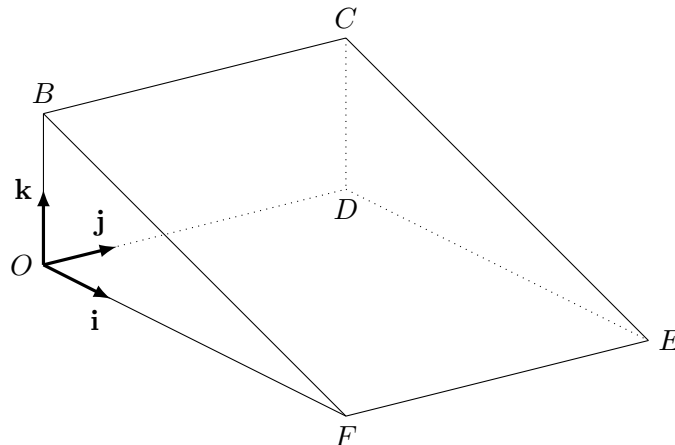
**Part (d).** Observe that

$$\left| \overrightarrow{AC} \times \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} \right| = \frac{1}{3} |\overrightarrow{AB} \times \overrightarrow{AC}| = 2.$$

Hence, the perpendicular distance between  $C$  and  $AB$  is 2 units.

\* \* \* \* \*

**Problem 13.**



The diagram shows a vehicle ramp  $OBCDEF$  with horizontal rectangular base  $ODEF$  and vertical rectangular face  $OBCD$ . Taking the point  $O$  as the origin, the perpendicular unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are parallel to the edges  $OF$ ,  $OD$  and  $OB$  respectively. The lengths of  $OF$ ,  $OD$  and  $OB$  are  $2h$  units,  $3$  units and  $h$  units respectively.

- (a) Show that  $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$ .
- (b) The point  $P$  divides the segment  $CF$  in the ratio  $2 : 1$ . Find  $\overrightarrow{OP}$  in terms of  $h$ .
- For parts (c) and (d), let  $h = 1$ .
- (c) Find the length of projection of  $\overrightarrow{OP}$  onto  $\overrightarrow{OC}$ .
- (d) Using the scalar product, find the angle that the rectangular face  $BCEF$  makes with the horizontal base.

**Solution.**

**Part (a).** We have

$$\overrightarrow{OC} = \overrightarrow{OD} + \overrightarrow{DC} = \overrightarrow{OD} + \overrightarrow{OB} = 3\mathbf{j} + h\mathbf{k}.$$

**Part (b).** By the ratio theorem,

$$\overrightarrow{OP} = \frac{\overrightarrow{OC} + 2\overrightarrow{OF}}{2 + 1} = \frac{1}{3} \left[ \begin{pmatrix} 0 \\ 3 \\ h \end{pmatrix} + 2 \begin{pmatrix} 2h \\ 0 \\ 0 \end{pmatrix} \right] = \frac{1}{3} \begin{pmatrix} 4h \\ 3 \\ h \end{pmatrix}.$$

**Part (c).** The length of projection of  $\overrightarrow{OP}$  onto  $\overrightarrow{OC}$  is given by

$$\left| \overrightarrow{OP} \cdot \frac{\overrightarrow{OC}}{|\overrightarrow{OC}|} \right| = \frac{1}{3\sqrt{10}} \left| \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right| = \frac{\sqrt{10}}{3} \text{ units.}$$

**Part (d).** Note that  $\overrightarrow{OF} = \langle 2, 0, 0 \rangle$  and  $\overrightarrow{BF} = \overrightarrow{OF} - \overrightarrow{OB} = \langle 2, 0, -1 \rangle$ . Let  $\theta$  be the angle the rectangular face  $BCEF$  makes with the horizontal base.

$$\cos \theta = \frac{\overrightarrow{OF} \cdot \overrightarrow{BF}}{|\overrightarrow{OF}| |\overrightarrow{BF}|} = \frac{4}{2\sqrt{5}} \implies \theta = 26.6^\circ \text{ (1 d.p.)}.$$

\* \* \* \* \*

**Problem 14.** The position vectors of the points  $A$  and  $B$  relative to the origin  $O$  are  $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$  and  $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$  respectively. The point  $P$  on  $AB$  is such that  $AP : PB = \lambda : 1 - \lambda$ . Show that  $\overrightarrow{OP} = (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}$  where  $\lambda$  is a real parameter.

- (a) Find the value of  $\lambda$  for which  $OP$  is perpendicular to  $AB$ .
- (b) Find the value of  $\lambda$  for which angles  $\angle AOP$  and  $\angle POB$  are equal.

**Solution.** By the ratio theorem,

$$\overrightarrow{OP} = \frac{\lambda \overrightarrow{OB} + (1 - \lambda) \overrightarrow{OA}}{\lambda + (1 - \lambda)} = \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix}.$$

**Part (a).** Note that  $\vec{AB} = \vec{OB} - \vec{OA} = \langle 1, -5, 8 \rangle$ . For  $OP$  to be perpendicular to  $AB$ , we must have  $\vec{OP} \cdot \vec{AB} = 0$ .

$$\vec{OP} \cdot \vec{AB} = 0 \implies \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} = 0 \implies -25 + 90\lambda = 0 \implies \lambda = \frac{5}{18}.$$

**Part (b).** Suppose  $\angle AOP = \angle POB$ . Then  $\cos \angle AOP = \cos \angle POB$ . Thus,

$$\frac{\vec{OP} \cdot \vec{OA}}{|\vec{OP}| |\vec{OA}|} = \frac{\vec{OP} \cdot \vec{OB}}{|\vec{OP}| |\vec{OB}|} \implies \vec{OP} \cdot \left( \frac{1}{3} \vec{OA} - \frac{1}{7} \vec{OB} \right) = 0 \implies \vec{OP} \cdot (7\vec{OA} - 3\vec{OB}) = 0.$$

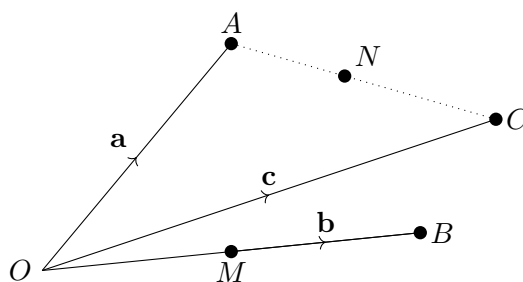
This gives

$$\begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \left[ 7 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right] = \begin{pmatrix} 1+\lambda \\ 2-5\lambda \\ -2+8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 23 \\ -32 \end{pmatrix} = 0.$$

Taking the dot product and simplifying, we see that  $111 - 370\lambda = 0$ , whence  $\lambda = \frac{3}{10}$ .

\* \* \* \* \*

### Problem 15.



The origin  $O$  and the points  $A$ ,  $B$  and  $C$  lie in the same plane, where  $\vec{OA} = \mathbf{a}$ ,  $\vec{OB} = \mathbf{b}$  and  $\vec{OC} = \mathbf{c}$ ,

(a) Explain why  $\mathbf{c}$  can be expressed as  $\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}$ , for constants  $\lambda$  and  $\mu$ .

The point  $N$  is on  $AC$  such that  $AN : NC = 3 : 4$ .

(b) Write down the position vector of  $N$  in terms of  $\mathbf{a}$  and  $\mathbf{c}$ .

(c) It is given that the area of triangle  $ONC$  is equal to the area of triangle  $OMC$ , where  $M$  is the mid-point of  $OB$ . By finding the areas of these triangles in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , find  $\lambda$  in terms of  $\mu$  in the case where  $\lambda$  and  $\mu$  are both positive.

### Solution.

**Part (a).** Since  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are co-planar and  $\mathbf{a}$  is not parallel to  $\mathbf{b}$ ,  $\mathbf{c}$  can be written as a linear combination of  $\mathbf{a}$  and  $\mathbf{b}$ .

**Part (b).** By the ratio theorem,

$$\vec{ON} = \frac{4\mathbf{a} + 3\mathbf{c}}{3 + 4} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}.$$

**Part (c).** Let  $\mathbf{c} = \lambda \mathbf{a} + \mu \mathbf{b}$ . The area of  $\triangle ONC$  is given by

$$[\triangle ONC] = \frac{1}{2} |\vec{ON} \times \mathbf{c}| = \frac{1}{2} \left| \left[ \frac{4}{7}\mathbf{a} + \frac{3}{7}(\lambda \mathbf{a} + \mu \mathbf{b}) \right] \times \frac{(\lambda \mathbf{a} + \mu \mathbf{b})}{|\mathbf{c}|} \right| = \frac{2}{7|\mathbf{c}|} |\mathbf{a} \times \mathbf{b}|.$$

Meanwhile, the area of  $\triangle OMC$  is given by

$$[\triangle OMC] = \frac{1}{2} \left| \overrightarrow{OM} \times \hat{\mathbf{c}} \right| = \frac{1}{2} \left| \frac{1}{2} \mathbf{b} \times \frac{(\lambda \mathbf{a} + \mu \mathbf{b})}{|\mathbf{c}|} \right| = \frac{\lambda}{4|\mathbf{c}|} |\mathbf{a} \times \mathbf{b}|.$$

Since the two areas are equal,

$$[\triangle ONC] = [\triangle OMC] \implies \frac{2}{7|\mathbf{c}|} |\mathbf{a} \times \mathbf{b}| = \frac{\lambda}{4|\mathbf{c}|} |\mathbf{a} \times \mathbf{b}| \implies \lambda = \frac{8}{7}\mu.$$

## Assignment A7

**Problem 1.** The points  $A$  and  $B$  have position vectors relative to the origin  $O$ , denoted by  $\mathbf{a}$  and  $\mathbf{b}$  respectively, where  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel vectors. The point  $P$  lies on  $AB$  such that  $AP : PB = \lambda : 1$ . The point  $Q$  lies on  $OP$  extended such that  $OP = 2PQ$  and  $\overrightarrow{BQ} = \overrightarrow{OA} + \mu\overrightarrow{OB}$ . Find the values of the real constants  $\lambda$  and  $\mu$ .

**Solution.** By the ratio theorem,

$$\overrightarrow{OP} = \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} \implies \overrightarrow{OQ} = \frac{3}{2}\overrightarrow{OP} = \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda}.$$

However, we also have

$$\overrightarrow{OQ} = \overrightarrow{OB} + \overrightarrow{BQ} = \mathbf{b} + (1 + \mu)\mathbf{a}.$$

This gives the equality

$$\frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} = \mathbf{a} + (1 + \mu)\mathbf{b}.$$

Since  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel, we can compare the  $\mathbf{a}$ - and  $\mathbf{b}$ -components of both vectors separately. This gives us

$$\frac{3}{2} \cdot \frac{1}{1 + \lambda} = 1, \quad \frac{3}{2} \cdot \frac{\lambda}{1 + \lambda} = 1 + \mu,$$

which has the unique solution  $\lambda = 1/2$  and  $\mu = -1/2$ .

\* \* \* \* \*

**Problem 2.** Given that  $\mathbf{a} = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$  and  $\mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$  where  $\lambda \in \mathbb{R}$ , find the possible value(s) of  $\lambda$  for which the angle between  $\mathbf{p}$  and  $\mathbf{k}$  is  $45^\circ$ .

**Solution.** Observe that

$$\mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b} = \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 - 3\lambda \\ -2 + 3\lambda \\ 6 - 6\lambda \end{pmatrix}.$$

Thus,

$$|\mathbf{p}|^2 = (4 - 3\lambda)^2 + (-2 + 3\lambda)^2 + (6 - 6\lambda)^2 = 54\lambda^2 - 108\lambda + 56.$$

Since the angle between  $\mathbf{p}$  and  $\mathbf{k}$  is  $45^\circ$ ,

$$\cos 45^\circ = \frac{\mathbf{p} \cdot \mathbf{k}}{|\mathbf{p}| |\mathbf{k}|} \implies \frac{1}{\sqrt{2}} = \frac{6 - 6\lambda}{|\mathbf{p}|} \implies \frac{|\mathbf{p}|^2}{2} = (6 - 6\lambda)^2.$$

We thus obtain the quadratic equation

$$\frac{54\lambda^2 - 108\lambda + 56}{2} = 36\lambda^2 - 72\lambda + 36 \implies 9\lambda^2 - 18\lambda + 8 = 0,$$

which has solutions  $\lambda = 2/3$  and  $\lambda = 4/3$ . However, we must reject  $\lambda = 4/3$  since  $6 - 6\lambda = |\mathbf{p}|/\sqrt{2} > 0 \implies \lambda < 1$ . Thus,  $\lambda = 2/3$ .

**Problem 3.**

- (a)  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero vectors such that  $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$ . State the relation between the directions of  $\mathbf{a}$  and  $\mathbf{b}$ , and find  $|\mathbf{b}|$ .
- (b)  $\mathbf{a}$  is a non-zero vector such that  $|\mathbf{a}| = \sqrt{3}$  and  $\mathbf{b}$  is a unit vector. Given that  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel and the angle between them is  $5\pi/6$ , find the exact value of the length of projection of  $\mathbf{a}$  on  $\mathbf{b}$ . By considering  $(2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b})$ , or otherwise, find the exact value of  $|2\mathbf{a} + \mathbf{b}|$ .

**Solution.**

**Part (a).**  $\mathbf{a}$  and  $\mathbf{b}$  either have the same or opposite direction. Let  $\mathbf{b} = \lambda\mathbf{a}$  for some  $\lambda \in \mathbb{R}$ .

$$\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} = (\mathbf{a} \cdot \lambda\mathbf{a})\lambda\mathbf{a} = \lambda^2 |\mathbf{a}|^2 \mathbf{a} \implies \lambda^2 |\mathbf{a}|^2 = 1 \implies |\mathbf{b}| = |\lambda| |\mathbf{a}| = 1.$$

**Part (b).** Note that  $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}| |\mathbf{b}| \cos(5\pi/6) = -3/2$ . Hence, the length of projection of  $\mathbf{a}$  on  $\mathbf{b}$  is  $|\mathbf{a} \cdot \hat{\mathbf{b}}| = 3/2$  units.

Observe that

$$|2\mathbf{a} + \mathbf{b}|^2 = (2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) = 4|\mathbf{a}|^2 + 4(\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 = 7.$$

Thus,  $|2\mathbf{a} + \mathbf{b}| = \sqrt{7}$ .

\* \* \* \* \*

**Problem 4.** The points  $A, B, C, D$  have position vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  given by  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{d} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$ , respectively. The point  $P$  lies on  $AB$  produced such that  $AP = 2AB$ , and the point  $Q$  is the mid-point of  $AC$ .

- (a) Show that  $PQ$  is perpendicular to  $AQ$ .
- (b) Find the area of the triangle  $APQ$ .
- (c) Find a vector perpendicular to the plane  $ABC$ .
- (d) Find the cosine of the angle between  $\overrightarrow{AD}$  and  $\overrightarrow{BD}$ .

**Solution.** Note that  $\overrightarrow{AB} = \langle 0, 0, -1 \rangle$ ,  $\overrightarrow{AC} = \langle 2, 0, -2 \rangle$  and  $\overrightarrow{AD} = \langle 3, -3, -4 \rangle$ .

**Part (a).** Note that

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + 2\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

and

$$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

Thus,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Since  $\overrightarrow{PQ} \cdot \overrightarrow{AQ} = 0$ , the two vectors are perpendicular, whence  $PQ \perp AQ$ .

**Part (b).** Note that  $\overrightarrow{AP} = \langle 0, 0, -2 \rangle$ . Hence,

$$[\triangle APQ] = \frac{1}{2} \left| \overrightarrow{AP} \times \overrightarrow{AQ} \right| = \frac{1}{2} \left| \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right| = 1 \text{ units}^2.$$

**Part (c).** The vector  $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0, -2, 0 \rangle$  is perpendicular to the plane  $ABC$ .

**Part (d).** Let the angle between  $\overrightarrow{AD}$  and  $\overrightarrow{BD}$  be  $\theta$ . Note that  $\overrightarrow{BD} = -3 \langle -1, 1, 1 \rangle$ . Hence,

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \overrightarrow{BD}}{\left| \overrightarrow{AD} \right| \left| \overrightarrow{BD} \right|} = \frac{30}{\sqrt{34} \cdot 3\sqrt{3}} = \frac{10}{\sqrt{102}}.$$

## A8. Vectors II

### Tutorial A8

**Problem 1.** For each of the following, write down a vector equivalent of the line  $l$  and convert it to parametric and Cartesian forms.

- (a)  $l$  passes through the point with position vector  $-\mathbf{i} + \mathbf{k}$  and is parallel to the vector  $\mathbf{i} + \mathbf{j}$ .
- (b)  $l$  passes through the points  $P(1, -1, 3)$  and  $Q(2, 1, -2)$ .
- (c)  $l$  passes through the origin and is parallel to the line  $m : \mathbf{r} = \langle 1, -1, 3 \rangle + \lambda \langle 1, 2, 3 \rangle$ , where  $\lambda \in \mathbb{R}$ .
- (d)  $l$  is the  $x$ -axis.
- (e)  $l$  passes through the point  $C(4, -1, 2)$  and is parallel to the  $z$ -axis.

**Solution.**

**Part (a).**

| Form       | Expression  |
|------------|---|
| Vector     | $\mathbf{r} = \langle -1, 0, 1 \rangle + \lambda \langle 1, 1, 0 \rangle, \lambda \in \mathbb{R}$ |
| Parametric | $x = \lambda - 1, y = \lambda, z = 1$   |
| Cartesian  | $x + 1 = y, z = 1$  |

**Part (b).**

| Form       | Expression   |
|------------|--|
| Vector     | $\mathbf{r} = \langle 1, -1, 3 \rangle + \lambda \langle 1, 2, -5 \rangle, \lambda \in \mathbb{R}$ |
| Parametric | $x = \lambda + 1, y = 2\lambda - 1, z = -5\lambda + 3$   |
| Cartesian  | $x - 1 = \frac{y+1}{2} = \frac{3-z}{5}$  |

**Part (c).**

| Form       | Expression   |
|------------|--|
| Vector     | $\mathbf{r} = \lambda \langle 1, 2, 3 \rangle, \lambda \in \mathbb{R}$ |
| Parametric | $x = \lambda, y = 2\lambda, z = 3\lambda$                              |
| Cartesian  | $x = \frac{y}{2} = \frac{z}{3}$  |

**Part (d).**

| Form       | Expression   |
|------------|--|
| Vector     | $\mathbf{r} = \lambda \langle 1, 0, 0 \rangle, \lambda \in \mathbb{R}$ |
| Parametric | $x = \lambda, y = 0, z = 0$  |
| Cartesian  | $x \in \mathbb{R}, y = 0, z = 0$                                       |



**Part (e).**

| Form              | Expression  |
|-------------------|---|
| <b>Vector</b>     | $\mathbf{r} = \langle 4, -1, 2 \rangle + \lambda \langle 0, 0, 1 \rangle, \lambda \in \mathbb{R}$ |
| <b>Parametric</b> | $x = 4, y = -1, z = \lambda + 2$  |
| <b>Cartesian</b>  | $x = 4, y = -1, z \in \mathbb{R}$   |

\* \* \* \* \*

**Problem 2.** For each of the following, determine if  $l_1$  and  $l_2$  are parallel, intersecting or skew. In the case of intersecting lines, find the position vector of the point of intersection. In addition, find the acute angle between the lines  $l_1$  and  $l_2$ .

(a)  $l_1 : x - 1 = -y = z - 2$  and  $l_2 : \frac{x-2}{2} = -\frac{y+1}{2} = \frac{z-4}{2}$

(b)  $l_1 : \mathbf{r} = \langle 1, 0, 0 \rangle + \alpha \langle 4, -2, -3 \rangle, \alpha \in \mathbb{R}$  and  $l_2 : \mathbf{r} = \langle 0, 10, 1 \rangle + \beta \langle 3, 8, 1 \rangle$

(c)  $l_1 : \mathbf{r} = (\mathbf{i} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$  and  $l_2 : \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} - 4\mathbf{j} - \mathbf{k}), \mu \in \mathbb{R}$

**Solution.**

**Part (a).** Note that  $l_1$  and  $l_2$  have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}.$$

Since  $\langle 2, -2, 2 \rangle = 2\langle 1, -1, 1 \rangle$ ,  $l_1$  and  $l_2$  are parallel ( $\theta = 0$ ). Since  $\langle 1, 0, 2 \rangle \neq \langle 2, 1, 4 \rangle + \mu \langle 2, -2, 2 \rangle$  for all real  $\mu$ , we have that  $l_1$  and  $l_2$  are distinct.

**Part (b).** Since  $\langle 4, -2, 3 \rangle \neq \beta \langle 3, 8, 1 \rangle$  for all real  $\beta$ , it follows that  $l_1$  and  $l_2$  are not parallel.

Consider  $l_1 = l_2$ .

$$l_1 = l_2 \implies \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \implies \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} - \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 10 \\ 1 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 4\alpha - 3\beta = -1 \\ -2\alpha - 8\beta = 10 \\ -3\alpha - \beta = 1 \end{cases}$$

There are no solutions to the above system. Hence,  $l_1$  and  $l_2$  do not intersect and are thus skew.

Let  $\theta$  be the acute angle between  $l_1$  and  $l_2$ .

$$\cos \theta = \frac{|\langle 4, -2, -3 \rangle \cdot \langle 3, 8, 1 \rangle|}{|\langle 4, -2, -3 \rangle| |\langle 3, 8, 1 \rangle|} = \frac{7}{\sqrt{2146}} \implies \theta = 81.3^\circ \text{ (1 d.p.)}.$$

**Part (c).** Note that  $l_1$  and  $l_2$  have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}, \lambda, \mu \in \mathbb{R}.$$

Since  $\langle 1, -1, 1 \rangle \neq \mu \langle 5, -4, -1 \rangle$  for all real  $\mu$ , it follows that  $l_1$  and  $l_2$  are not parallel. Consider  $l_1 = l_2$ .

$$l_1 = l_2 \implies \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \implies \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} -5\mu + \lambda = 0 \\ 4\mu - \lambda = -1 \\ \mu + \lambda = 6 \end{cases}$$

The above system has the unique solution  $\lambda = 5$  and  $\mu = 1$ . Hence,  $l_1$  and  $l_2$  intersect at  $\langle 1, 0, -5 \rangle + 5 \langle 1, -1, 1 \rangle = \langle 6, -5, 0 \rangle$ .

Let  $\theta$  be the acute angle between  $l_1$  and  $l_2$ .

$$\cos \theta = \frac{|\langle 1, -1, 1 \rangle \cdot \langle 5, -4, -1 \rangle|}{|\langle 1, -1, 1 \rangle| |\langle 5, -4, -1 \rangle|} = \frac{8}{3\sqrt{14}} \implies \theta = 44.5^\circ \text{ (1 d.p.)}.$$

\* \* \* \* \*

### Problem 3.

- Find the shortest distance from the point  $(1, 2, 3)$  to the line with equation  $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ ,  $\lambda \in \mathbb{R}$ .
- Find the length of projection of  $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$  onto the line with equation  $\frac{x+5}{4} = \frac{y-5}{3} = 10 - 2z$ .
- Find the projection of  $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$  onto the line with equation  $\frac{x+5}{4} = \frac{y-5}{3} = 10 - 2z$ .

### Solution.

**Part (a).** Let  $\overrightarrow{OP} = \langle 1, 2, 3 \rangle$  and  $\overrightarrow{OA} = \langle 3, 2, 4 \rangle$ . Note that  $\overrightarrow{AP} = \langle -2, 0, -1 \rangle$ . The shortest distance between  $P$  and the line is thus

$$\text{Shortest distance} = \frac{|\langle -2, 0, -1 \rangle \times \langle 1, 2, 2 \rangle|}{|\langle 1, 2, 2 \rangle|} = \frac{|\langle 2, -3, -4 \rangle|}{3} = \frac{\sqrt{29}}{3} \text{ units.}$$

**Part (b).** Note that the line has vector form

$$\mathbf{r} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda' \begin{pmatrix} 4 \\ 3 \\ -1/2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}.$$

The length of projection of  $\langle 4, -5, 6 \rangle$  onto the line is thus given by

$$\text{Length of projection} = \frac{|\langle 4, -5, 6 \rangle \cdot \langle 8, 6, -1 \rangle|}{|\langle 8, 6, -1 \rangle|} = \frac{4}{\sqrt{101}} \text{ units.}$$

**Part (c).**

$$\text{Projection} = \left[ \frac{\langle 4, -5, 6 \rangle \cdot \langle 8, 6, -1 \rangle}{|\langle 8, 6, -1 \rangle|} \right] \cdot \frac{\langle 8, 6, -1 \rangle}{|\langle 8, 6, -1 \rangle|} = \frac{-4}{101} \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}$$

**Problem 4.** The points  $P$  and  $Q$  have coordinates  $(0, -1, -1)$  and  $(3, 0, 1)$  respectively, and the equations of the lines  $l_1$  and  $l_2$  are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}.$$

- (a) Show that  $P$  lies on  $l_1$  but not on  $l_2$ .
- (b) Determine if  $l_2$  passes through  $Q$ .
- (c) Find the coordinates of the foot of the perpendicular from  $P$  to  $l_2$ . Hence, or otherwise, find the perpendicular distance from  $P$  to  $l_2$ .
- (d) Find the length of projection of  $\overrightarrow{PQ}$  onto  $l_2$ .

**Solution.** We have that  $\overrightarrow{OP} = \langle 0, -1, -1 \rangle$  and  $\overrightarrow{OQ} = \langle 3, 0, 1 \rangle$ .

**Part (a).** When  $\lambda = -2$ , we have  $\langle 0, 1, -3 \rangle - 2\langle 0, 1, -1 \rangle = \langle 0, -1, -1 \rangle = \overrightarrow{OP}$ . Hence,  $P$  lies on  $l_1$ .

Observe that all points on  $l_2$  have a  $z$ -coordinate of 1. Since  $P$  has a  $z$ -coordinate of  $-1$ ,  $P$  does not lie on  $l_2$ .

**Part (b).** When  $\mu = 3$ , we have  $\langle -3, 3, 1 \rangle + 3\langle 2, -1, 0 \rangle = \langle 3, 0, 1 \rangle = \overrightarrow{OQ}$ . Hence,  $l_2$  passes through  $Q$ .

**Part (c).** Let the foot of the perpendicular from  $P$  to  $l_2$  be  $F$ . Since  $F$  is on  $l_2$ , we have that  $\overrightarrow{OF} = \langle -3, 3, 1 \rangle + \mu \langle 2, -1, 0 \rangle$  for some real  $\mu$ . We also have that  $\overrightarrow{PF} \cdot \langle 2, -1, 0 \rangle = 0$ . Note that

$$\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 + 2\mu \\ 4 - \mu \\ 2 \end{pmatrix}.$$

Hence,

$$\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \implies \begin{pmatrix} -3 + 2\mu \\ 4 - \mu \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \implies -10 + 5\mu = 0 \implies \mu = 2.$$

Hence,  $\overrightarrow{OF} = \langle -3, 3, 1 \rangle + 2\langle 2, -1, 0 \rangle = \langle 1, 1, 1 \rangle$ . Thus,  $F(1, 1, 1)$ . The perpendicular distance from  $P$  to  $l_2$  is thus  $|\overrightarrow{PF}| = |\langle 1, 2, 2 \rangle| = 3$  units.

**Part (d).** Note that  $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ . The length of projection of  $\overrightarrow{PQ}$  onto  $l_2$  is thus given by

$$\text{Length of projection} = \frac{|\langle 3, 1, 2 \rangle \cdot \langle 2, -1, 0 \rangle|}{|\langle 2, -1, 0 \rangle|} = \frac{5}{\sqrt{5}} = \sqrt{5} \text{ units.}$$

\* \* \* \* \*

**Problem 5.** The lines  $l_1$  and  $l_2$  have equations

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

respectively. Find the position vectors of the points  $P$  on  $l_1$  and  $Q$  on  $l_2$  such that  $O$ ,  $P$  and  $Q$  are collinear, where  $O$  is the origin.

**Solution.** We have that  $\overrightarrow{OP} = \langle 0, 1, 2 \rangle + s \langle 1, 0, 3 \rangle$  and  $\overrightarrow{OQ} = \langle -2, 3, 1 \rangle + t \langle 2, 1, 0 \rangle$  for some  $s, t \in \mathbb{R}$ . For  $O, P$  and  $Q$  to be collinear, we need  $\overrightarrow{OP} = \lambda \overrightarrow{OQ}$  for some  $\lambda \in \mathbb{R}$ :

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \lambda \left[ \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \implies \begin{pmatrix} s \\ 1 \\ 2+3s \end{pmatrix} = \lambda \begin{pmatrix} -2+2t \\ 3+t \\ 1 \end{pmatrix}.$$

This gives us the system:

$$\begin{cases} s = \lambda(-2+2t) \\ 1 = \lambda(3+t) \\ 2+3s = \lambda \end{cases}$$

Substituting the third equation into the first two gives the reduced system:

$$\begin{cases} s = (2+3s)(-2+2t) \\ 1 = (2+3s)(3+t) \end{cases}$$

Subtracting twice of the second equation from the first yields  $s - 2 = -8(2+3s)$ , whence  $s = -14/25$ . It quickly follows that  $t = 1/8$ . Hence,

$$\overrightarrow{OP} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{14}{25} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}, \quad \overrightarrow{OQ} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}.$$

\* \* \* \* \*

**Problem 6.** Relative to the origin  $O$ , the points  $A, B$  and  $C$  have position vectors  $5\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$ ,  $-4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$  and  $-5\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$  respectively.

- Find the Cartesian equation of the line  $AB$ .
- Find the length of projection of  $\overrightarrow{AC}$  onto the line  $AB$ . Hence, find the perpendicular distance from  $C$  to the line  $AB$ .
- Find the position vector of the foot  $N$  of the perpendicular from  $C$  to the line  $AB$ .
- The point  $D$  is such that it is a reflection of point  $C$  about the line  $AB$ . Find the position vector of  $D$ .

**Solution.** We have that  $\overrightarrow{OA} = \langle 5, 4, 10 \rangle$ ,  $\overrightarrow{OB} = \langle -4, 4, -2 \rangle$  and  $\overrightarrow{OC} = \langle -5, 9, 5 \rangle$ .

**Part (a).** Note that  $\overrightarrow{AB} = \langle -9, 0, -12 \rangle = -3 \langle 3, 0, 4 \rangle$ . The line  $AB$  hence has the vector form

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

and Cartesian form  $\frac{x-5}{3} = \frac{z-10}{4}$ ,  $y = 4$ .

**Part (b).** Note that  $\overrightarrow{AC} = \langle -10, 5, -5 \rangle = -5 \langle 2, -1, 1 \rangle$ . Hence, the length of projection of  $\overrightarrow{AC}$  onto the line  $AB$  is given by

$$\text{Length of projection} = \frac{|\overrightarrow{AC} \cdot \overrightarrow{AB}|}{|\overrightarrow{AB}|} = \frac{1}{15} \left| 5 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot 3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right| = 10 \text{ units.}$$

Since  $|\overrightarrow{AC}| = 5\sqrt{6}$ , the perpendicular distance from  $C$  to the line  $AB$  is  $\sqrt{(5\sqrt{6})^2 - 10^2} = 5\sqrt{2}$  units.

**Part (c).** Let  $\overrightarrow{AN} = \lambda \langle -9, 0, -12 \rangle$  for some  $\lambda \in \mathbb{R}$  such that  $|\overrightarrow{AN}| = 10$ .

$$|\overrightarrow{AN}| = 10 \implies 15\lambda = 10 \implies \lambda = \frac{2}{3}.$$

Hence,  $\overrightarrow{AN} = \frac{2}{3} \langle -9, 0, -12 \rangle = \langle -6, 0, -8 \rangle$ . Thus,  $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \langle -1, 4, 2 \rangle$ .

**Part (d).** Note that  $\overrightarrow{NC} = \overrightarrow{OC} - \overrightarrow{ON} = \langle -4, 5, 3 \rangle$ . Since  $D$  is the reflection of  $C$  about  $AB$ , we have that  $\overrightarrow{ND} = -\overrightarrow{NC}$ . Thus,

$$\overrightarrow{OD} = \overrightarrow{ON} + \overrightarrow{ND} = \overrightarrow{ON} - \overrightarrow{NC} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}.$$

\* \* \* \* \*

**Problem 7.** The points  $A$  and  $B$  have coordinates  $(0, 9, c)$  and  $(d, 5, -2)$  respectively, where  $c$  and  $d$  are constants. The line  $l$  has equation  $\frac{x+3}{-1} = \frac{y-1}{4} = \frac{z-5}{3}$ .

- (a) Given that  $d = \frac{22}{7}$  and the line  $AB$  intersects  $l$ , find the value of  $c$ . Find also the coordinates of the foot of the perpendicular from  $A$  to  $l$ .
- (b) Given instead that the lines  $AB$  and  $l$  are parallel, state the value of  $c$  and  $d$  and find the shortest distance between the lines  $AB$  and  $l$ .

**Solution.** We have that  $\overrightarrow{OA} = \langle 0, 9, c \rangle$  and  $\overrightarrow{OB} = \langle d, 5, -2 \rangle$ . We also have that the line  $l$  is given by the vector  $\mathbf{r} = \langle -3, 1, 5 \rangle + \lambda \langle -1, 4, 3 \rangle$  for  $\lambda \in \mathbb{R}$ .

Note that  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \langle d, -4, -2 - c \rangle$ . Hence, the line  $AB$  is given by the vector  $\mathbf{r}_{AB} = \langle d, 5, -2 \rangle + \mu \langle d, -4, -2 - c \rangle$  for  $\mu \in \mathbb{R}$ .

**Part (a).** Consider the direction vectors of  $AB$  and  $l$ . Since  $\langle 22/7, -4, -2 - c \rangle \neq \lambda \langle -1, 4, 3 \rangle$  for all real  $\lambda$  and  $c$ , the lines  $AB$  and  $l$  are not parallel. Hence,  $AB$  and  $l$  intersect at only one point. Thus, there must be a unique solution to  $\mathbf{r} = \mathbf{r}_{AB}$ .

$$\begin{aligned} \mathbf{r} = \mathbf{r}_{AB} &\implies \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 22/7 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 22/7 \\ -4 \\ -2 - c \end{pmatrix} \\ &\implies \lambda \begin{pmatrix} -7 \\ 28 \\ 21 \end{pmatrix} - \mu \begin{pmatrix} 22 \\ -28 \\ -14 - 7c \end{pmatrix} = \begin{pmatrix} 43 \\ 28 \\ -49 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} -\lambda - 22\mu = 43 \\ 4\lambda + 28\mu = 28 \\ 3\lambda + (14 + 7c)\mu = -49 \end{cases}$$

Solving the first two equations gives  $\lambda = 91/3$  and  $\mu = -10/3$ . It follows from the third equation that  $c = 4$ .

Let  $F$  be the foot of the perpendicular from  $A$  to  $l$ . We have that  $\overrightarrow{OF} = \langle -3, 1, 5 \rangle + \lambda \langle -1, 4, 3 \rangle$  for some  $\lambda \in \mathbb{R}$ . We also have that  $\overrightarrow{AF} \cdot \langle -1, 4, 3 \rangle = 0$ . Note that

$$\overrightarrow{AF} = \overrightarrow{OF} - \overrightarrow{OA} = \begin{pmatrix} -3 - \lambda \\ -8 + 4\lambda \\ 1 + 3\lambda \end{pmatrix}.$$

Hence,

$$\overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \implies \begin{pmatrix} -3 - \lambda \\ -8 + 4\lambda \\ 1 + 3\lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \implies -26 + 26\lambda = 0 \implies \lambda = 1.$$

Hence,  $\overrightarrow{OF} = \langle -3, 1, 5 \rangle + \langle -1, 4, 3 \rangle = \langle -4, 5, 8 \rangle$ . The foot of the perpendicular from  $A$  to  $l$  hence has coordinates  $(-4, 5, 8)$ .

**Part (b).** Given that  $AB$  is parallel to  $l$ , one of their direction vectors must be a scalar multiple of the other. Hence, for some real  $\lambda$ ,  $\langle -1, 4, 3 \rangle = \lambda \langle d, -4, -2 - c \rangle$ . It is obvious that  $\lambda = -1$ , whence  $c = 1$  and  $d = 1$ .

Note that the direction vector of  $l$  and  $AB$  is  $\langle -1, 4, 3 \rangle$ . Also note that  $l$  passes through  $(-3, 1, 5)$  and  $AB$  passes through  $(1, 5, -2)$ . Since  $\langle 1, 5, -2 \rangle - \langle -3, 1, 5 \rangle = \langle 4, 4, -7 \rangle$ , the shortest distance between  $AB$  and  $l$  is

$$\frac{|\langle -1, 4, 3 \rangle \times \langle 4, 4, -7 \rangle|}{|\langle -1, 4, 3 \rangle|} = \frac{1}{\sqrt{26}} \left| \begin{pmatrix} -40 \\ -5 \\ -20 \end{pmatrix} \right| = \frac{45}{\sqrt{26}} \text{ units.}$$

\* \* \* \* \*

**Problem 8.** The equation of the line  $L$  is  $\mathbf{r} = \langle 1, 3, 7 \rangle + t \langle 2, -1, 5 \rangle$ ,  $t \in \mathbb{R}$ . The points  $A$  and  $B$  have position vectors  $\langle 9, 3, 26 \rangle$  and  $\langle 13, 9, \alpha \rangle$  respectively. The line  $L$  intersects the line through  $A$  and  $B$  at  $P$ .

- (a) Find  $\alpha$  and the acute angle between line  $L$  and  $AB$ .

The point  $C$  has position vector  $\langle 2, 5, 1 \rangle$  and the foot of the perpendicular from  $C$  to  $L$  is  $Q$ .

- (b) Find the position vector of  $Q$ . Hence, find the shortest distance from  $C$  to  $L$ .
- (c) Find the position vector of the point of reflection of the point  $C$  about the line  $L$ . Hence, find the reflection of the line passing through  $C$  and the point  $(1, 3, 7)$  about the line  $L$ .

**Solution.**

**Part (a).** We have that  $\overrightarrow{OA} = \langle 9, 3, 26 \rangle$  and  $\overrightarrow{OB} = \langle 13, 9, \alpha \rangle$ . Hence,  $\overrightarrow{AB} = \langle 4, 6, \alpha - 26 \rangle$ . The line  $AB$  is thus given by  $\mathbf{r}_{AB} = \langle 9, 3, 26 \rangle + u \langle 4, 6, \alpha - 26 \rangle$  for  $u \in \mathbb{R}$ . Note that  $AB$  is not parallel to  $L$ . Hence,  $\overrightarrow{OP}$  is the only solution to the equation  $\mathbf{r} = \mathbf{r}_{AB}$ .

$$\begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix} \implies t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \\ 19 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 2t - 4u = 8 \\ -t - 6u = 0 \\ 5t - (\alpha - 26)u = 19 \end{cases}$$

Solving the first two equations gives  $t = 3$  and  $u = -\frac{1}{2}$ . It follows from the third equation that  $\alpha = 34$ .

Let the acute angle between  $L$  and  $AB$  be  $\theta$ .

$$\cos \theta = \frac{|\langle 2, -1, 5 \rangle \cdot \langle 4, 6, 8 \rangle|}{|\langle 2, -1, 5 \rangle| |\langle 4, 6, 8 \rangle|} = \frac{42}{\sqrt{30}\sqrt{116}} \implies \theta = 44.6^\circ \text{ (1 d.p.)}.$$

**Part (b).** Since  $Q$  is on  $L$ , we have that  $\overrightarrow{OQ} = \langle 1, 3, 7 \rangle + t \langle 2, -1, 5 \rangle$  for some real  $t$ . Further, since  $\overrightarrow{CQ} \perp L$ , we have that  $\overrightarrow{CQ} \cdot \langle 2, -1, 5 \rangle = 0$ . Note that

$$\overrightarrow{CQ} = \overrightarrow{OQ} - \overrightarrow{OC} = \begin{pmatrix} -1+2t \\ -2-t \\ 6+5t \end{pmatrix}.$$

Thus,

$$\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0 \implies \begin{pmatrix} -1+2t \\ -2-t \\ 6+5t \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0 \implies 30 + 30t = 0 \implies t = -1.$$

Hence,  $\overrightarrow{OQ} = \langle 1, 3, 7 \rangle + \langle 2, -1, 5 \rangle = \langle -1, 4, 2 \rangle$ . The shortest distance from  $C$  to  $L$  is thus

$$|\overrightarrow{CQ}| = \left| \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right| = \left| \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right| = \sqrt{11} \text{ units.}$$

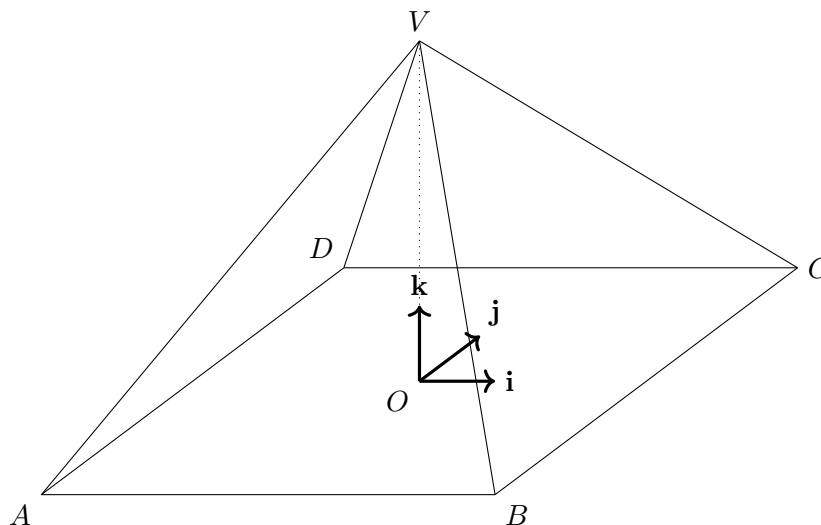
**Part (c).** Let  $C'$  be the reflection of  $C$  about  $L$ . Note that

$$\overrightarrow{OC'} = \overrightarrow{OQ} - \overrightarrow{QC} = \overrightarrow{OQ} + \overrightarrow{CQ} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}.$$

Note that  $(1, 3, 7)$  is on  $L$  and is hence invariant under a reflection about  $L$ . Let the reflection about  $L$  of the line passing through  $C$  and  $(1, 3, 7)$  be  $L'$ . Since  $\langle -4, 3, 3 \rangle - \langle 1, 3, 7 \rangle = \langle -5, 0, -4 \rangle \parallel \langle 5, 0, 4 \rangle$ ,  $L'$  hence has direction vector  $\langle 5, 0, 4 \rangle$ . Thus,  $L'$  is given by  $\mathbf{r}' = \langle 1, 3, 7 \rangle + \lambda \langle 5, 0, 4 \rangle$  for  $\lambda \in \mathbb{R}$ .

\* \* \* \* \*

### Problem 9.



In the diagram,  $O$  is the origin of the square base  $ABCD$  of a right pyramid with vertex  $V$ . The perpendicular unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are parallel to  $AB$ ,  $AD$  and  $OV$  respectively. The length of  $AB$  is 4 units and the length of  $OV$  is  $2h$  units.  $P$ ,  $Q$ ,  $M$  and  $N$  are the mid-points of  $AB$ ,  $BC$ ,  $CV$  and  $VA$  respectively. The point  $O$  is taken as the origin for position vectors.

Show that the equation of the line  $PM$  may be expressed as  $\mathbf{r} = \langle 0, -2, 0 \rangle + t \langle 1, 3, h \rangle$ , where  $t$  is a parameter.

- Find an equation for the line  $QN$ .
- Show that the lines  $PM$  and  $QN$  intersect and that the position vector  $\overrightarrow{OX}$  of their point of intersection is  $\mathbf{r} = \frac{1}{2} \langle 1, -1, h \rangle$ .
- Given that  $OX$  is perpendicular to  $VB$ , find the value of  $h$  and calculate the acute angle between  $PM$  and  $QN$ , giving your answer correct to the nearest  $0.1^\circ$ .

**Solution.** We are given that  $\overrightarrow{OP} = \langle 0, -2, 0 \rangle$ ,  $\overrightarrow{OC} = \langle 2, 2, 0 \rangle$  and  $\overrightarrow{OV} = \langle 0, 0, 2h \rangle$ . Hence,  $\overrightarrow{CV} = \overrightarrow{OV} - \overrightarrow{OC} = \langle -2, -2, 2h \rangle$ . Thus,  $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CV} = \langle -1, -1, h \rangle$ . Since  $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \langle 1, 1, h \rangle$ , we have that  $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \langle 1, 3, h \rangle$ . Thus,  $PM$  is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}, t \in \mathbb{R}.$$

**Part (a).** Since  $\overrightarrow{OM} = \langle 1, 1, h \rangle$ , by symmetry,  $\overrightarrow{ON} = \langle -1, -1, h \rangle$ . Given that  $\overrightarrow{OQ} = \langle 2, 0, 0 \rangle$ , we have that  $\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \langle -3, -1, h \rangle$ . Thus,  $QN$  is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix}, u \in \mathbb{R}.$$

**Part (b).** Consider  $PM = QN$ .

$$PM = QN \implies \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} \implies t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} - u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} t + 3u = 2 \\ 3t - u = 2 \\ ht - hu = 0 \end{cases}$$

From the first two equations, we see that  $t = \frac{1}{2}$  and  $u = \frac{1}{2}$ , which is consistent with the third equation. Hence,  $\overrightarrow{OX} = \langle 0, -2, 0 \rangle + \frac{1}{2} \langle 1, 3, h \rangle = \frac{1}{2} \langle 1, -1, h \rangle$ .

**Part (c).** Note that  $\overrightarrow{OB} = \langle 2, -2, 6 \rangle$ , whence  $\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \langle 2, -2, -2h \rangle$ . Since  $OX$  is perpendicular to  $VB$ , we have that  $\overrightarrow{OX} \cdot \overrightarrow{VB} = 0$ .

$$\overrightarrow{OX} \cdot \overrightarrow{VB} = 0 \implies \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} \cdot 2 \begin{pmatrix} 1 \\ -1 \\ -h \end{pmatrix} = 0 \implies h^2 = 2.$$

We hence have that  $h = \sqrt{2}$ . Note that we reject  $h = -\sqrt{2}$  since  $h > 0$ .

Let the acute angle between  $PM$  and  $QN$  be  $\theta$ .

$$\cos \theta = \frac{|\overrightarrow{PM} \cdot \overrightarrow{QN}|}{|\overrightarrow{PM}| |\overrightarrow{QN}|} = \frac{1}{\sqrt{12}\sqrt{12}} \left| \begin{pmatrix} 1 \\ 3 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ \sqrt{2} \end{pmatrix} \right| = \frac{1}{3} \implies \theta = 70.5^\circ \text{ (1 d.p.)}.$$



## Assignment A8

**Problem 1.** Find the position vector of the foot of the perpendicular from the point with position vector  $\mathbf{c}$  to the line with equation  $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ ,  $\lambda \in \mathbb{R}$ . Leave your answers in terms of  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$ .

**Solution.** Let the foot of the perpendicular be  $F$ . We have that  $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b}$  for some real  $\lambda$ , and  $\overrightarrow{CF} \cdot \mathbf{b} = 0$ . Note that  $\overrightarrow{CF} = \overrightarrow{OF} - \overrightarrow{OC} = \mathbf{a} + \lambda\mathbf{b} - \mathbf{c}$ . Thus,

$$\overrightarrow{CF} \cdot \mathbf{b} = 0 \implies (\mathbf{a} + \lambda\mathbf{b} - \mathbf{c}) \cdot \mathbf{b} = 0 \implies \lambda|\mathbf{b}|^2 + (\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 0 \implies \lambda = \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2}.$$

Thus,

$$\overrightarrow{OF} = \mathbf{a} + \left( \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b}.$$

\* \* \* \* \*

**Problem 2.** The point  $O$  is the origin, and points  $A$ ,  $B$ ,  $C$  have position vectors given by  $\overrightarrow{OA} = 6\mathbf{i}$ ,  $\overrightarrow{OB} = 3\mathbf{j}$ ,  $\overrightarrow{OC} = 4\mathbf{k}$ . The point  $P$  is on the line  $AB$  between  $A$  and  $B$ , and is such that  $AP = 2PB$ . The point  $Q$  has position vector given by  $\overrightarrow{OQ} = q\mathbf{i}$ , where  $q$  is a scalar.

- Express, in terms of  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ , the vector  $\overrightarrow{CP}$ .
- Show that the line  $BQ$  has equation  $\mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j})$ , where  $t$  is a parameter. Give an equation of the line  $CP$  in a similar form.
- Find the value of  $q$  for which the lines  $CP$  and  $BQ$  are perpendicular.
- Find the sine of the acute angle between the lines  $CP$  and  $BQ$  in terms of  $q$ .

**Solution.** We have that  $\overrightarrow{OA} = \langle 6, 0, 0 \rangle$ ,  $\overrightarrow{OB} = \langle 0, 3, 0 \rangle$  and  $\overrightarrow{OC} = \langle 0, 0, 4 \rangle$ .

**Part (a).** By the ratio theorem,

$$\overrightarrow{OP} = \frac{2\overrightarrow{OB} + \overrightarrow{OA}}{1+2} = \frac{1}{3} \left[ 2 \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \right] = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \implies \overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}.$$

Hence,  $\overrightarrow{CP} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$ .

**Part (b).** Note that  $\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB} = \langle q, -3, 0 \rangle$ . Thus,  $BQ$  is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix}, t \in \mathbb{R} \iff \mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j}), t \in \mathbb{R}.$$

Note that  $\overrightarrow{CP} = \langle 2, 2, -4 \rangle = 2\langle 1, 1, -2 \rangle$ . Hence,  $CP$  is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, u \in \mathbb{R} \iff \mathbf{r} = 4\mathbf{k} + u(\mathbf{i} + \mathbf{j} - 2\mathbf{k}), u \in \mathbb{R}.$$

**Part (c).** Since  $CP$  is perpendicular to  $BQ$ , we have  $\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0$ . Thus,

$$\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0 \implies 2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} = 0 \implies q - 3 + 0 = 0 \implies q = 3.$$

**Part (d).** Let  $\theta$  be the acute angle between  $CP$  and  $BQ$ .

$$\sin \theta = \frac{|\langle 1, 1, -2 \rangle \times \langle q, -3, 0 \rangle|}{|\langle 1, 1, -2 \rangle| |\langle q, -3, 0 \rangle|} = \frac{|\langle -6, 2q, 3-q \rangle|}{\sqrt{6}\sqrt{q^2+9}} = \sqrt{\frac{5q^2-6q+45}{6q^2+54}}.$$

\* \* \* \* \*

**Problem 3.** Line  $l_1$  passes through the point  $A$  with position vector  $3\mathbf{i} - 2\mathbf{k}$  and is parallel to  $-2\mathbf{i} + 4\mathbf{j} - \mathbf{j}$ . Line  $l_2$  has Cartesian equation given by  $\frac{x-1}{2} = y = z + 3$ .

- Show that the two lines intersect and find the coordinates of their point of intersection.
- Find the acute angle between the two lines  $l_1$  and  $l_2$ . Hence, or otherwise, find the shortest distance from point  $A$  to line  $l_2$ .
- Find the position vector of the foot  $N$  of the perpendicular from  $A$  to the line  $l_2$ . The point  $B$  lies on the line  $AN$  produced and is such that  $N$  is the mid-point of  $AB$ . Find the position vector of  $B$ .

**Solution.** We have

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}, \quad l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}.$$

**Part (a).** Consider  $l_1 = l_2$ .

$$l_1 = l_2 \implies \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \implies \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}.$$

This gives the following system:

$$\begin{cases} 2\lambda + 2\mu = 2 \\ -4\lambda + \mu = 0 \\ \lambda + \mu = 1 \end{cases}$$

which has the unique solution  $\mu = 4/5$  and  $\lambda = 1/5$ . Thus, the intersection point  $P$  has position vector  $\langle 3, 0, -2 \rangle + \frac{1}{5} \langle -2, 4, -1 \rangle = \frac{1}{5} \langle 13, 4, -11 \rangle$  and thus has coordinates  $(13/5, 4/5, -11/5)$ .

**Part (b).** Let  $\theta$  be the acute angle between  $l_1$  and  $l_2$ .

$$\cos \theta = \frac{|\langle -2, 4, -1 \rangle \cdot \langle 2, 1, 1 \rangle|}{|\langle -2, 4, -1 \rangle| |\langle 2, 1, 1 \rangle|} = \frac{1}{\sqrt{126}} \implies \theta = 84.9^\circ \text{ (1 d.p.)}.$$

Note that

$$AP = \sqrt{\left(\frac{17}{5} - 3\right)^2 + \left(-\frac{4}{5} - 0\right)^2 + \left(-\frac{9}{5} - (-2)\right)^2} = \sqrt{\frac{21}{25}} = \frac{\sqrt{21}}{5}.$$

Since  $\sin \theta = \frac{AN}{AP}$ , we have that  $AN = AP \sin \theta$ . Note that

$$\sin \theta = \sin \arccos \frac{1}{\sqrt{126}} = \frac{\sqrt{(\sqrt{126})^2 - 1}}{\sqrt{126}} = \frac{\sqrt{125}}{\sqrt{126}} = \frac{5\sqrt{5}}{\sqrt{6}\sqrt{21}}.$$

Thus,

$$AN = \frac{\sqrt{21}}{5} \cdot \frac{5\sqrt{5}}{\sqrt{6}\sqrt{21}} = \sqrt{\frac{5}{6}}.$$

The shortest distance between  $A$  and  $l_2$  is hence  $\sqrt{\frac{5}{6}}$  units.

**Part (c).** Since  $N$  is on  $l_2$ , we have that  $\overrightarrow{ON} = \langle 1, 0, -3 \rangle + \mu \langle 2, 1, 1 \rangle$  for some real  $\mu$ . Additionally, since  $\overrightarrow{AN} \perp l_2$ , we have  $\overrightarrow{AN} \cdot \langle 2, 1, 1 \rangle = 0$ . Note that

$$\overrightarrow{AN} = \overrightarrow{ON} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix}.$$

Thus,

$$\overrightarrow{AN} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \implies \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = 0 \implies -5 + 6\mu = 0 \implies \mu = \frac{5}{6}.$$

Hence,

$$\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}.$$

Note that  $\overrightarrow{ON} = \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$ . Hence,

$$\overrightarrow{OB} = 2\overrightarrow{ON} - \overrightarrow{OA} = \frac{2}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}.$$

## A14. Discrete Random Variables

### Tutorial A14A

**Problem 1.** Alfred and Bertie play a game, each starting with cash amounting to \$100. Two dice are thrown. If the total score is 5 or more, then Alfred pays \$ $x$ , where  $0 < x \leq 8$ , to Bertie. If the total score is 4 or less, then Bertie pays \$( $x + 8$ ) to Alfred.

- (a) Show that the expectation of Alfred's cash after the first game is  $\$ \frac{1}{3}(304 - 2x)$ .
- (b) Find the expectation of Alfred's cash after six games.
- (c) Find the value of  $x$  for the game to be fair.
- (d) Given that  $x = 3$ , find the variance of Alfred's cash after the first game.

**Solution.**

**Part (a).** Note that

$$P(\text{score} < 5) = \frac{3 + 2 + 1}{6^2} = \frac{1}{6} \implies P(\text{score} \geq 5) = 1 - \frac{1}{6} = \frac{5}{6}.$$

Let  $a_n$  be the expectation of Alfred's cash after  $n$  games. Suppose Alfred and Bertie play one more game (i.e.  $n + 1$  total games). Then

$$a_{n+1} = \frac{5}{6}(a_n - x) + \frac{1}{6}(a_n + x + 8) = a_n + \frac{2}{3}(2 - x).$$

$a_n$  is in AP with common difference  $\frac{2}{3}(2 - x)$  and is thus given by

$$a_n = a_0 + n \left[ \frac{2}{3}(2 - x) \right] = 100 + \frac{2n}{3}(2 - x).$$

Hence, the expectation of Alfred's cash after the first game is

$$a_1 = 100 + \frac{2 \cdot 1}{3}(2 - x) = \frac{1}{3}(304 - 2x).$$

**Part (b).** The expectation of Alfred's cash after six games is

$$a_6 = 100 + \frac{2 \cdot 6}{3}(2 - x) = 108 - 4x.$$

**Part (c).** For the game to be fair,  $a_0 = a_1 = a_2 = \dots$ , i.e. the common difference is 0. Hence,  $x = 2$ .

**Part (d).** Let the random variable  $X$  be Alfred's cash after one game. Since the payouts are unaffected by  $a_0$ , we take  $a_0 = 0$ . When  $x = 3$ ,  $E(X) = -\frac{2}{3}$ . Hence,

$$\text{Var}(X) = \frac{5}{6} \left( 3 - \frac{2}{3} \right)^2 + \frac{1}{6} \left( 3 + 8 + \frac{2}{3} \right)^2 = \frac{245}{9}.$$

## Tutorial A14B

**Problem 1.** In a computer game, a bug moves from left to right through a network of connected paths. The bug starts at  $S$  and, at each junction, randomly takes the left fork with probability  $p$  or the right fork with probability  $q$ , where  $q = 1 - p$ . The forks taken at each junction are independent. The bug finishes its journey at one of the 9 endpoints labelled A - I (see diagram).

- (a) Show that the probability that the bug finishes its journey at D is  $56p^5q^3$ .
- (b) Given that the probability that the bug finishes its journey at D is greater than the probability at any one of the other endpoints, find exactly the possible range of values of  $p$ .

In another version of the game, the probability that, at each junction, the bug takes the left fork is  $0.9p$ , the probability that the bug takes the right fork is  $0.9q$  and the probability that the bug is swallowed up by a 'black hole' is 0.1.

- (c) Find the probability that, in this version of the game, the bug reaches one of the endpoints A - I, without being swallowed up by a black hole.

### Solution.

**Part (a).** Relabel each endpoint from A - I to 0 - 8. Let the random variable  $X$  be the end-point that the bug ends up at. Clearly, to reach endpoint  $i$ , the bug must take  $i$  right forks and  $8 - i$  left forks. Hence,  $X \sim B(8, q)$  and the probability that the bug reaches endpoint 3 (i.e. endpoint D) is

$$P(X = 3) = \binom{8}{3} q^3 (1 - q)^{8-3} = 56p^5q^3.$$

**Part (b).** Since  $X$  follows a binomial distribution, it suffices to find the range of values of  $p$  that satisfy  $P(X = 2) < P(X = 3) > P(X = 4)$ .

*Case 1:*  $P(X = 2) < P(X = 3)$ . Note that  $P(X = 2) = \binom{8}{2} q^2 (1 - q)^{8-2} = 28p^6q^2$ .

$$P(X = 2) < P(X = 3) \implies 28p^6q^2 < 56p^5q^3 \implies 28p < 56(1 - p) \implies p < \frac{2}{3}.$$

*Case 2:*  $P(X = 3) > P(X = 4)$ . Note that  $P(X = 4) = \binom{8}{4} q^4 (1 - q)^{8-4} = 70p^4q^4$ .

$$P(X = 3) > P(X = 4) \implies 56p^5q^3 > 70p^4q^4 \implies 56p > 70(1 - p) \implies p > \frac{5}{9}.$$

Hence,  $\frac{5}{9} < p < \frac{2}{3}$ .

**Part (c).** Note that the bug must take a total of 8 forks. Since the probability of not getting swallowed by a black hole at each fork is 0.9, the desired probability is clearly  $0.9^8 = 0.430$  (3 s.f.).



**Part II.**

**Group B**





**Part III.**

**Examinations**

