

Problem 1.

The vector \mathbf{v} is defined by $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. Find the unit vector in the direction of \mathbf{v} and hence find a vector of magnitude 25 which is parallel to \mathbf{v} .

Solution

$$\begin{aligned}\hat{\mathbf{v}} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{1}{\sqrt{3^2 + (-4)^2 + 1^2}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\end{aligned}$$

$$\boxed{\frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \frac{25}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}}$$

Problem 2.

With respect to an origin O , the position vectors of the points A , B , C and D are $4\mathbf{i} + 7\mathbf{j}$, $\mathbf{i} + 3\mathbf{j}$, $2\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + d\mathbf{j}$ respectively.

(a) Find the vectors \overrightarrow{BA} and \overrightarrow{BC} .

(b) Find the value of d if B , C and D are collinear. State the ratio $\frac{BC}{BD}$.

Solution**Part (a)**

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\boxed{\overrightarrow{BA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\overrightarrow{BC} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

Part (b)

If B , C and D are collinear, then $\overrightarrow{BC} = \lambda \overrightarrow{CD}$ for some λ .

$$\begin{aligned} \overrightarrow{BC} &= \lambda \overrightarrow{CD} \\ \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \lambda (\overrightarrow{OD} - \overrightarrow{OC}) \\ &= \lambda \left[\begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right] \\ &= \begin{pmatrix} \lambda \\ \lambda(d-4) \end{pmatrix} \end{aligned}$$

Hence, $\lambda = 1$ and $\lambda(d-4) = 1$, whence $d = 5$.

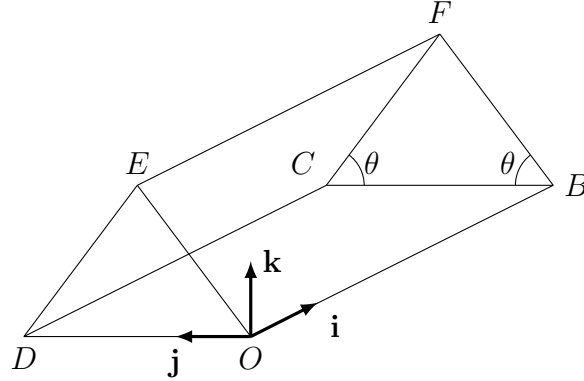
$$\boxed{d = 5}$$

$$\frac{BC}{BD} = \frac{BC}{BC + CD} = \frac{BC}{BC + BC} = \frac{1}{2}$$

$$\boxed{\frac{BC}{BD} = \frac{1}{2}}$$

Problem 3.

The diagram shows a roof, with horizontal rectangular base $OBCD$, where $OB = 10$ m and $BC = 6$ m. The triangular planes ODE and BCF are vertical and the ridge EF is horizontal to the base. The planes $OBFE$ and $DCFE$ are each inclined at an angle θ to the horizontal, where $\tan \theta = \frac{4}{3}$. The point O is taken as the origin and vectors \mathbf{i} , \mathbf{j} , \mathbf{k} , each of length 1 m, are taken along OB , OD and vertically upwards from O respectively.



Find the position vectors of the points B , C , D , E and F .

Solution

Since $OB = 10$ m, we know $\overrightarrow{OB} = 10\mathbf{i}$. Further, since $BC = 6$, we know $\overrightarrow{BC} = 6\mathbf{j}$. Thus, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = 10\mathbf{i} + 6\mathbf{j}$.

Note that $\triangle ODE \cong \triangle BCF$. Hence, $BC \cong OD \implies \overrightarrow{OD} = 6\mathbf{j}$.

We also have $\angle ODE = \angle DOE = \theta$. Thus, $\triangle ODE$ is isosceles. Let G be the mid-point of OD . Since $\tan \theta = \frac{4}{3}$, we have $\frac{EG}{DG} = \frac{4}{3}$, which implies that $EG = \frac{4}{3}DG = \frac{2}{3}OD$. Thus, $EG = \frac{2}{3} \cdot 6 = 4$ m. Hence, $\overrightarrow{OE} = \overrightarrow{OG} + \overrightarrow{GE} = \frac{1}{2}\overrightarrow{OD} + \overrightarrow{GE} = 3\mathbf{j} + 4\mathbf{k}$.

Since $BF \cong OE$, we know $\overrightarrow{BF} = \overrightarrow{OE}$. Thus, $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

$\begin{aligned}\overrightarrow{OB} &= 10\mathbf{i} \\ \overrightarrow{OC} &= 10\mathbf{i} + 6\mathbf{j} \\ \overrightarrow{OD} &= 6\mathbf{j} \\ \overrightarrow{OE} &= 3\mathbf{j} + 4\mathbf{k} \\ \overrightarrow{OF} &= 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}\end{aligned}$

Problem 4.

Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ and the angle between \mathbf{u} and \mathbf{v} given that

(a) $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

(b) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Solution**Part (a)**

We have $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$. Hence,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 1 \cdot 3 + (-1) \cdot 2 + 1 \cdot 7 \\ &= 8\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{pmatrix} -1 \cdot 7 - 2 \cdot 1 \\ 1 \cdot 3 - 7 \cdot 1 \\ 1 \cdot 2 - 3 \cdot (-1) \end{pmatrix} \\ &= \begin{pmatrix} -9 \\ -4 \\ 5 \end{pmatrix}\end{aligned}$$

Let the angle between \mathbf{u} and \mathbf{v} be θ .

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta \\ \Rightarrow 8 &= \sqrt{1^2 + (-1)^2 + 1^2} \cdot \sqrt{3^2 + 2^2 + 7^2} \cdot \cos \theta \\ \Rightarrow \theta &= \arccos\left(\frac{8}{\sqrt{3} \cdot \sqrt{62}}\right) \\ &= 54.1^\circ \text{ (1 d.p.)}\end{aligned}$$

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 8 \\ \mathbf{u} \times \mathbf{v} &= -9\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} \\ \theta &= 54.1^\circ\end{aligned}$$

Part (b)

We have $\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$. Hence,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 2 \cdot (-1) + 0 \cdot 7 + (-3) \cdot 2 \\ &= -8\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{pmatrix} 0 \cdot 2 - 7 \cdot (-3) \\ (-3) \cdot (-1) - 2 \cdot 2 \\ 2 \cdot 7 - (-1) \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 21 \\ -1 \\ 14 \end{pmatrix}\end{aligned}$$

Let the angle between \mathbf{u} and \mathbf{v} be θ .

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta \\ \Rightarrow 8 &= \sqrt{2^2 + 0^2 + (-3)^2} \cdot \sqrt{(-1)^2 + 7^2 + 2^2} \cdot \cos \theta \\ \Rightarrow \theta &= \arccos\left(\frac{-8}{\sqrt{13} \cdot \sqrt{54}}\right) \\ &= 107.6^\circ \text{ (1 d.p.)}\end{aligned}$$

$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= -8 \\ \mathbf{u} \times \mathbf{v} &= 21\mathbf{i} - \mathbf{j} + 14\mathbf{k} \\ \theta &= 107.6^\circ\end{aligned}$
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Problem 5.

Find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$ given that $\mathbf{u} = 2\mathbf{a} - \mathbf{b}$, $\mathbf{v} = -\mathbf{a} + 3\mathbf{b}$, where $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and the angle between \mathbf{a} and \mathbf{b} is 60° .

Solution

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (2\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} + 3\mathbf{b}) \\&= -2\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - 3\mathbf{b} \cdot \mathbf{b} \\&= -2|\mathbf{a}|^2 - 3|\mathbf{b}|^2 + 7|\mathbf{a}||\mathbf{b}|\cos\theta \\&= -2 \cdot 2^2 - 3 \cdot 1^2 + 7 \cdot 2 \cdot 1 \cdot \cos 60^\circ \\&= -4 \\|\mathbf{u} \times \mathbf{v}| &= |(2\mathbf{a} - \mathbf{b}) \times (-\mathbf{a} + 3\mathbf{b})| \\&= |-2\mathbf{a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - 3\mathbf{b} \times \mathbf{b}| \\&= |-2 \cdot 0 + 6\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} - 3 \cdot 0| \\&= |5\mathbf{a} \times \mathbf{b}| \\&= 5|\mathbf{a}||\mathbf{b}|\sin\theta \\&= 5 \cdot 2 \cdot 1 \cdot \frac{\sqrt{3}}{2} \\&= 5\sqrt{3}\end{aligned}$$

$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= -4 \\ \mathbf{u} \times \mathbf{v} &= 5\sqrt{3}\end{aligned}$

Problem 6.

If $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$, find

- (a) a unit vector perpendicular to both \mathbf{a} and \mathbf{b} ,
- (b) a vector perpendicular to both $(3\mathbf{b} - 5\mathbf{c})$ and $(7\mathbf{b} + \mathbf{c})$.

Solution**Part (a)**

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 4 \cdot 3 - (-1) \cdot (-1) \\ (-1) \cdot 1 - 3 \cdot 1 \\ 1 \cdot (-1) - 1 \cdot 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix} \\ \Rightarrow |\mathbf{a} \times \mathbf{b}| &= \sqrt{11^2 + (-4)^2 + (-5)^2} \\ &= \sqrt{162} \\ \Rightarrow \widehat{\mathbf{a} \times \mathbf{b}} &= \frac{1}{\sqrt{162}} \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix}\end{aligned}$$

$$\boxed{\frac{1}{\sqrt{162}} \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix}}$$

Part (b)

Observe that $(3\mathbf{b} - 5\mathbf{c}) \times (7\mathbf{b} + \mathbf{c}) = \lambda\mathbf{b} \times \mathbf{c}$ for some scalar λ . It hence suffices to find $\mathbf{b} \times \mathbf{c}$.

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{pmatrix} (-1) \cdot 0 - 1 \cdot 3 \\ 3 \cdot 2 - 0 \cdot 1 \\ 1 \cdot 1 - 2 \cdot (-1) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}\end{aligned}$$

$$\boxed{\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}}$$

Problem 7.

The position vectors of the points A , B and C are given by $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$ respectively. Find

- (a) a unit vector parallel to \overrightarrow{AB} ;
- (b) the position vector of the point D such that $ABCD$ is a parallelogram, leaving your answer in terms of λ ;
- (c) the value of λ if A , B and C are collinear;
- (d) the position vector of the point P on AB is $AP : PB = 2 : 1$.

Solution**Part (a)**

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}\end{aligned}$$

Observe that $|\overrightarrow{AB}| = \sqrt{3^2 + (-4)^2 + 6^2} = \sqrt{61}$. Hence, the required vector is

$$\boxed{\frac{1}{\sqrt{61}} \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}}$$

Part (b)

Since $ABCD$ is a parallelogram, we have that $\overrightarrow{AD} = \overrightarrow{BC}$. Thus,

$$\begin{aligned}\overrightarrow{OD} - \mathbf{a} &= \mathbf{c} - \mathbf{b} \\ \Rightarrow \overrightarrow{OD} &= \mathbf{a} - \mathbf{b} + \mathbf{c} \\ &= \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix} \\ &= \boxed{\overrightarrow{OD} = \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix}}\end{aligned}$$

Part (c)

Given that A , B and C are collinear, we have $\overrightarrow{AB} = k\overrightarrow{BC}$. Hence,

$$\begin{aligned}\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} &= k(\mathbf{c} - \mathbf{b}) \\ &= k \left[\begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right] \\ &= k \begin{pmatrix} 6 \\ \lambda + 1 \\ 12 \end{pmatrix}\end{aligned}$$

We hence see that $k = \frac{1}{2}$, which implies that $-4 = \frac{\lambda + 1}{2}$, whence $\lambda = -9$.

$$\boxed{\lambda = -9}$$

Part (d)

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{1\mathbf{a} + 2\mathbf{b}}{2 + 1} \\ &= \frac{1}{3} \left[\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2 \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} \right] \\ &= \frac{1}{3} \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{3} \begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}}$$

Problem 8.

$ABCD$ is a square, and M and N are the midpoints of BC and CD respectively. Express \overrightarrow{AC} in terms of \mathbf{p} and \mathbf{q} , where $\overrightarrow{AM} = \mathbf{p}$ and $\overrightarrow{AN} = \mathbf{q}$.

Solution

Let $ABCD$ be a square with side length $2k$ with A at the origin. Then $\mathbf{p} = \overrightarrow{AM} = k \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \overrightarrow{AN} = k \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Hence, $\mathbf{p} + \mathbf{q} = k \begin{pmatrix} 3 \\ -3 \end{pmatrix}$. Thus, $\overrightarrow{AC} = k \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{2}{3} \cdot k \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \frac{2}{3}(\mathbf{p} + \mathbf{q})$.

$$\boxed{\overrightarrow{AC} = \frac{2}{3}(\mathbf{p} + \mathbf{q})}$$

Problem 9.

The points A, B have position vectors \mathbf{a}, \mathbf{b} respectively, referred to an origin O , where \mathbf{a} and \mathbf{b} are not parallel to each other. The point C lies on AB between A and B and is such that $\frac{AC}{CB} = 2$, and D is the mid-point of OC . The line AD produced meets OB at E .

Find, in terms of \mathbf{a} and \mathbf{b} ,

- (a) the position vector of C (referred to O),
 (b) the vector \overrightarrow{AD} . Find the values of $\frac{OE}{EB}$ and $\frac{AE}{ED}$.

Solution**Part (a)**

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OC} &= \frac{1\mathbf{a} + 2\mathbf{b}}{2 + 1} \\ &= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{OC} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}}$$

Part (b)

Since D is the mid-point of OC ,

$$\begin{aligned}\overrightarrow{OD} &= \frac{1}{2}\overrightarrow{OC} \\ &= \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b} \\ \Rightarrow \overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA} \\ &= \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b} - \mathbf{a} \\ &= -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{AD} = -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}}$$

Since \mathbf{a} and \mathbf{b} are non-parallel, there exists some linear transformation \mathbf{A} such that

$\mathbf{Aa} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{Ab} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Hence,

$$\begin{aligned} \mathbf{A}\overrightarrow{AD} &= -\frac{5}{6}\mathbf{Aa} + \frac{1}{3}\mathbf{Ab} \\ &= -\frac{5}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \end{aligned}$$

Since E is on both AD and OB , we have

$$\mathbf{A}\overrightarrow{AE} = -\frac{\lambda}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \mu \\ -3 \end{pmatrix}$$

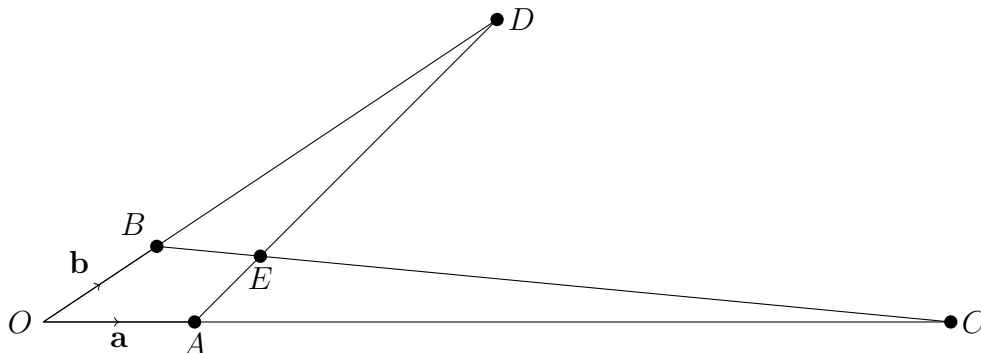
Thus, $\lambda = \frac{6}{5}$ and $\mu = -\frac{3}{5}$. Hence, $\mathbf{A}\overrightarrow{OE} = \begin{pmatrix} 2/5 \\ 0 \end{pmatrix}$, giving

$$\begin{aligned} \frac{OE}{EB} &= \frac{|\mathbf{A}\overrightarrow{OE}|}{|\mathbf{A}\overrightarrow{OB} - \mathbf{A}\overrightarrow{OE}|} \\ &= \frac{2/5}{1 - 2/5} \\ &= \frac{2}{3} \\ \frac{AE}{ED} &= \frac{|\mathbf{A}\overrightarrow{AE}|}{|\mathbf{A}\overrightarrow{AE} - \mathbf{A}\overrightarrow{AD}|} \\ &= \frac{\lambda |\mathbf{A}\overrightarrow{AD}|}{(\lambda - 1) |\mathbf{A}\overrightarrow{AD}|} \\ &= \frac{6/5}{6/5 - 1} \\ &= 6 \end{aligned}$$

$$\boxed{\frac{OE}{EB} = \frac{2}{3}, \frac{AE}{ED} = 6}$$

Problem 10.

- (a) The angle between the vectors $(3\mathbf{i} - 2\mathbf{j})$ and $(6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k})$ is $\arccos \frac{6}{13}$. Show that $2d^2 - 117d + 333 = 0$.
- (b) With reference to the origin O , the points A , B , C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AC} = 5\mathbf{a}$, $\overrightarrow{BD} = 3\mathbf{b}$. The lines AD and BC cross at E .



- (i) Find \overrightarrow{OE} in terms of \mathbf{a} and \mathbf{b} .
- (ii) The point F divides the line CD in the ratio $5 : 3$. Show that O , E and F are collinear, and find $OE : EF$.

Solution**Part (a)**

Let $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ d \\ -\sqrt{7} \end{pmatrix}$. Let θ be the angle between \mathbf{a} and \mathbf{b} .

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\Rightarrow 3 \cdot 6 + (-2) \cdot d + 0 \cdot (-\sqrt{7}) = \sqrt{3^2 + (-2)^2 + 0^2} \cdot \sqrt{6^2 + d^2 + (-\sqrt{7})^2} \cdot \cos \arccos \frac{6}{13}$$

$$\begin{aligned} \Rightarrow 18 - 2d &= \sqrt{43 + d^2} \cdot \sqrt{13} \cdot \frac{6}{13} \\ &= \sqrt{43 + d^2} \cdot \frac{6}{\sqrt{13}} \end{aligned}$$

$$\Rightarrow (18 - 2d)^2 = \frac{36}{13} (43 + d^2)$$

$$\Rightarrow (9 - d)^2 = \frac{9}{13} (43 + d^2)$$

$$\Rightarrow 13(81 - 18d + d^2) = 387 + 9d^2$$

$$\Rightarrow 1053 - 234d + 13d^2 = 387 + 9d^2$$

$$\Rightarrow 4d^2 - 234d + 666 = 0$$

$$\Rightarrow 2d^2 - 117d + 333 = 0$$

Part (b)**Subpart (i)**

Note that $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = \mathbf{a} - 4\mathbf{b}$ and $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \mathbf{b} - 6\mathbf{a}$. Since E is on both DA and CB , we have

$$\overrightarrow{OE} = \overrightarrow{OD} + \lambda \overrightarrow{DA} = \overrightarrow{OC} + \mu \overrightarrow{CB}$$

for some scalars λ and μ . Hence,

$$\begin{aligned} 4\mathbf{b} + \lambda(\mathbf{a} - 4\mathbf{b}) &= 6\mathbf{a} + \mu(\mathbf{b} - 6\mathbf{a}) \\ \implies \lambda\mathbf{a} + 4(1 - \lambda)\mathbf{b} &= 6(1 - \mu)\mathbf{a} + \mu\mathbf{b} \end{aligned}$$

Comparing the coefficients of \mathbf{a} and \mathbf{b} , we have the system

$$\begin{cases} \lambda = 6(1 - \mu) \\ \mu = 4(1 - \lambda) \end{cases}$$

which has the solution $\lambda = \frac{18}{23}$ and $\mu = \frac{20}{23}$. Hence, $\overrightarrow{OE} = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}$.

$$\boxed{\overrightarrow{OE} = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}}$$

Subpart (ii)

By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OF} &= \frac{3\mathbf{c} + 5\mathbf{d}}{5 + 3} \\ &= \frac{1}{8}(5 \cdot 4\mathbf{b} + 3 \cdot 6\mathbf{a}) \\ &= \frac{1}{8}(18\mathbf{a} + 20\mathbf{b}) \\ &= \frac{23}{8} \left(\frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b} \right) \\ &= \frac{23}{8}\overrightarrow{OE} \end{aligned}$$

$$\boxed{OE : OF = 8 : 23}$$

Problem 11.

Relative to the origin O , two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

- The point P divides the line AB in the ratio $2 : 1$. Find the coordinates of P .
- Show that AB and OP are perpendicular.
- The vector \mathbf{c} is a unit vector in the direction of \overrightarrow{OP} . Write \mathbf{c} as a column vector and give the geometrical meaning of $|\mathbf{a} \cdot \mathbf{c}|$.
- Find $\mathbf{a} \times \mathbf{p}$, where \mathbf{p} is the vector \overrightarrow{OP} , and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence, write down the area of triangle OAP .

Solution**Part (a)**

We have $\mathbf{a} = \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} = 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix}$. By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OP} &= \frac{\mathbf{a} + 2\mathbf{b}}{2 + 1} \\ &= \frac{1}{3} \left[\begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} + 2 \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} \right] \\ &= \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \end{aligned}$$

$$\boxed{P(12, -4, 6)}$$

Part (b)

Consider $\overrightarrow{AB} \cdot \overrightarrow{OP}$.

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{OP} &= \left[\begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} - \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} \right] \cdot \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \\ &= -3 \begin{pmatrix} 1 \\ 9 \\ 4 \end{pmatrix} \cdot 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\ &= -6(1 \cdot 6 + 9 \cdot (-2) + 4 \cdot 3) \\ &= 0 \end{aligned}$$

Since $\overrightarrow{AB} \cdot \overrightarrow{OP} = 0$, AB and OP must be perpendicular.

Part (c)

$$\begin{aligned}
 \mathbf{c} &= \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} \\
 &= \frac{1}{2\sqrt{6^2 + (-2)^2 + 3^2}} \cdot 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\
 &= \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}
 \end{aligned}$$

$$\boxed{\mathbf{c} = \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}}$$

$|\mathbf{a} \cdot \mathbf{c}|$ is the length of the projection of \mathbf{a} on \overrightarrow{OP} .

Part (d)

$$\begin{aligned}
 \mathbf{a} \times \mathbf{p} &= 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\
 &= 28 \begin{pmatrix} 1 \cdot 3 - (-2) \cdot 1 \\ 1 \cdot 6 - 3 \cdot 1 \\ 1 \cdot -2 - 6 \cdot 1 \end{pmatrix} \\
 &= 28 \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix}
 \end{aligned}$$

$$\boxed{\mathbf{a} \times \mathbf{p} = \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix}}$$

$|\mathbf{a} \times \mathbf{p}|$ is twice the area of $\triangle OAP$.

$$\begin{aligned}
 \text{Area } \triangle OAP &= \frac{1}{2} |\mathbf{a} \times \mathbf{p}| \\
 &= \frac{1}{2} \cdot 28 \sqrt{5^2 + 3^2 + (-8)^2} \\
 &= 14 \sqrt{98} \\
 &= 14 \cdot 7\sqrt{2} \\
 &= 98\sqrt{2}
 \end{aligned}$$

$$\boxed{\text{Area } \triangle OAP = 98\sqrt{2} \text{ units}^2}$$

Problem 12.

The points A , B and C have position vectors given by $\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{j} - \mathbf{k}$ and $2\mathbf{i} - \mathbf{j} - \mathbf{k}$ respectively.

- (a) Find the area of the triangle ABC . Hence, find the sine of the angle BAC .
- (b) Find a vector perpendicular to the plane ABC .
- (c) Find the projection vector of \overrightarrow{AC} onto \overrightarrow{AB} .
- (d) Find the distance of C to AB .

Solution

We have $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$. For simplicity, consider the translation that sends A to the origin. We thus have $\mathbf{a}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{b}' = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ and $\mathbf{c}' = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$.

Part (a)

$$\begin{aligned}
 |\mathbf{b}' \times \mathbf{c}'| &= \left| \begin{pmatrix} 2 \cdot (-2) - 0 \cdot (-2) \\ -2 \cdot 1 - (-1) \cdot (-2) \\ -1 \cdot 0 - 1 \cdot 2 \end{pmatrix} \right| \\
 &= \left| \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix} \right| \\
 &= \left| -2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right| \\
 &= 2 \cdot \sqrt{2^2 + 2^2 + 1^2} \\
 &= 6
 \end{aligned}$$

Hence, the area of $\triangle ABC = \frac{1}{2} |\mathbf{b}' \times \mathbf{c}'| = 3 \text{ units}^2$.

Area $\triangle ABC = 3 \text{ units}^2$

$$\begin{aligned}
 |\mathbf{b}' \times \mathbf{c}'| &= |\mathbf{b}'| |\mathbf{c}'| \sin BAC \\
 \implies \sin BAC &= \frac{6}{\sqrt{(-1)^2 + 2^2 + (-2)^2} \sqrt{1^2 + 0^2 + (-2)^2}} \\
 &= \frac{6}{3\sqrt{5}} \\
 &= \frac{2}{\sqrt{5}}
 \end{aligned}$$

$$\sin BAC = \frac{2}{\sqrt{5}}$$

Part (b)

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

Part (c)

Note that $\hat{\mathbf{b}}' = \frac{\mathbf{b}'}{|\mathbf{b}'|} = \frac{1}{\sqrt{(-1)^2 + 2^2 + (-2)^2}} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$. Hence,

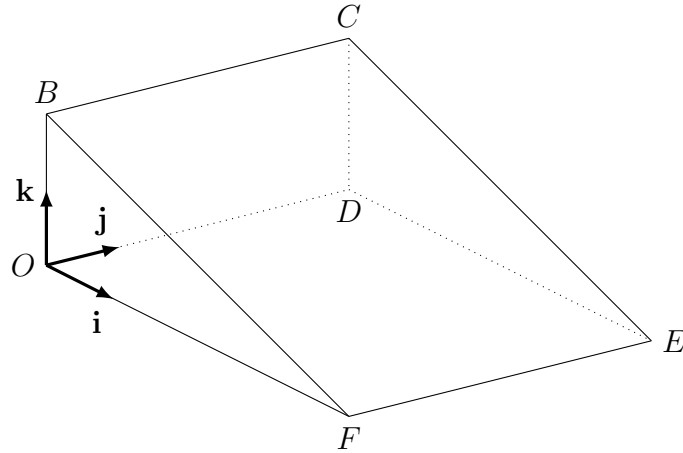
$$\begin{aligned} (\mathbf{c}' \cdot \hat{\mathbf{b}}') \hat{\mathbf{b}}' &= \left[\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right] \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{9} (1 \cdot (-1) + 0 \cdot 2 + (-2) \cdot (-2)) \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \end{aligned}$$

$$\frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

Part (d)

$$\begin{aligned} |\mathbf{c}' \times \hat{\mathbf{b}}'| &= |\hat{\mathbf{b}}' \times \mathbf{c}'| \\ &= \left| \frac{1}{3} \mathbf{b}' \times \mathbf{c}' \right| \\ &= \frac{1}{3} \cdot 6 \\ &= 2 \end{aligned}$$

The distance between C and AB is 2 units.

Problem 13.

The diagram shows a vehicle ramp $OBCDEF$ with horizontal rectangular base $ODEF$ and vertical rectangular face $OBCD$. Taking the point O as the origin, the perpendicular unit vectors \mathbf{i} , \mathbf{j} and \mathbf{k} are parallel to the edges OF , OD and OB respectively. The lengths of OF , OD and OB are $2h$ units, 3 units and h units respectively.

- (a) Show that $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$.
- (b) The point P divides the segment CF in the ratio $2 : 1$. Find \overrightarrow{OP} in terms of h .

For parts (c) and (d), let $h = 1$.

- (c) Find the length of projection of \overrightarrow{OP} onto \overrightarrow{OC} .
- (d) Using the scalar product, find the angle that the rectangular face $BCEF$ makes with the horizontal base.

Solution**Part (a)**

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OD} + \overrightarrow{DC} \\ &= \overrightarrow{OD} + \overrightarrow{OB} \\ &= 3\mathbf{j} + h\mathbf{k}\end{aligned}$$

Part (b)

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{1\overrightarrow{OC} + 2\overrightarrow{OF}}{2 + 1} \\ &= \frac{1}{3} \left[\begin{pmatrix} 0 \\ 3 \\ h \end{pmatrix} + 2 \begin{pmatrix} 2h \\ 0 \\ 0 \end{pmatrix} \right] \\ &= \frac{1}{3} \begin{pmatrix} 4h \\ 3 \\ h \end{pmatrix}\end{aligned}$$

$$\overrightarrow{OP} = \frac{1}{3} \begin{pmatrix} 4h \\ 3 \\ h \end{pmatrix}$$

Part (c)

$$\begin{aligned} |\overrightarrow{OP} \cdot \hat{\mathbf{c}}| &= \left| \frac{1}{3} \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{0^2 + 3^2 + 1^2}} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right| \\ &= \frac{1}{3\sqrt{10}} \left| \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right| \\ &= \frac{1}{3\sqrt{10}} |4 \cdot 0 + 3 \cdot 3 + 1 \cdot 1| \\ &= \frac{10}{3\sqrt{10}} \\ &= \frac{\sqrt{10}}{3} \end{aligned}$$

$$\frac{\sqrt{10}}{3} \text{ units}$$

Part (d)

Let θ be the angle the rectangular face $BCEF$ makes with the horizontal base.

$$\begin{aligned} \overrightarrow{OF} \cdot \overrightarrow{BF} &= |\overrightarrow{OF}| |\overrightarrow{BF}| \cos \theta \\ \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \left[\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] &= 2 \cdot \left| \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| \cdot \cos \theta \\ \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} &= 2 \cdot \left| \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right| \cdot \cos \theta \\ \Rightarrow 2 \cdot 2 + 0 \cdot 0 + 0 \cdot (-1) &= 2 \cdot \sqrt{2^2 + 0^2 + (-1)^2} \cdot \cos \theta \\ \Rightarrow 4 &= 2\sqrt{5} \cos \theta \\ \Rightarrow \cos \theta &= \frac{2}{\sqrt{5}} \\ \Rightarrow \theta &= \arccos\left(\frac{2}{\sqrt{5}}\right) \\ &= 26.6^\circ \text{ (1 d.p.)} \end{aligned}$$

$$26.6^\circ$$

Problem 14.

The position vectors of the points A and B relative to the origin O are $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ respectively. The point P on AB is such that $AP : PB = \lambda : 1 - \lambda$. Show that $\overrightarrow{OP} = (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}$ where λ is a real parameter.

- (a) Find the value of λ for which OP is perpendicular to AB .
 (b) Find the value of λ for which angles $\angle AOP$ and $\angle POB$ are equal.

Solution

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{\lambda \overrightarrow{OB} + (1 - \lambda) \overrightarrow{OA}}{\lambda + (1 - \lambda)} \\ &= \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2\lambda \\ -3\lambda \\ 6\lambda \end{pmatrix} + \begin{pmatrix} 1 - \lambda \\ 2 - 2\lambda \\ -2 + 2\lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \\ &= (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}\end{aligned}$$

Part (a)

For OP to be perpendicular to AB , we must have $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$.

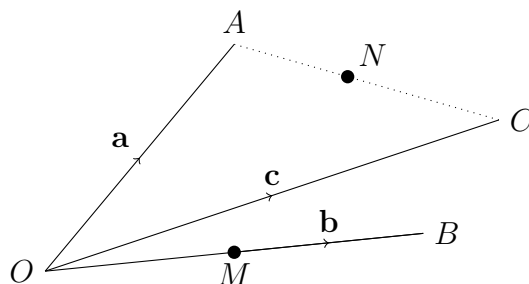
$$\begin{aligned}\overrightarrow{OP} \cdot \overrightarrow{AB} &= 0 \\ \Rightarrow \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \left[\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right] &= 0 \\ \Rightarrow \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} &= 0 \\ \Rightarrow (1 + \lambda) \cdot 1 + (2 - 5\lambda) \cdot (-5) + (-2 + 8\lambda) \cdot 8 &= 0 \\ \Rightarrow -25 + 90\lambda &= 0 \\ \Rightarrow \lambda &= \frac{5}{18}\end{aligned}$$

$$\boxed{\lambda = \frac{5}{18}}$$

Part (b)

$$\begin{aligned}
& \angle AOP = \angle POB \\
\Rightarrow & \cos \angle AOP = \cos \angle POB \\
\Rightarrow & \frac{\vec{OP} \cdot \vec{OA}}{|\vec{OP}| |\vec{OA}|} = \frac{\vec{OP} \cdot \vec{OB}}{|\vec{OP}| |\vec{OB}|} \\
\Rightarrow & |\vec{OB}| (\vec{OP} \cdot \vec{OA}) = |\vec{OA}| (\vec{OP} \cdot \vec{OB}) \\
\Rightarrow & \vec{OP} \cdot (|\vec{OB}| \vec{OA}) = \vec{OP} \cdot (|\vec{OA}| \vec{OB}) \\
\Rightarrow & \vec{OP} \cdot (|\vec{OB}| \vec{OA}) - \vec{OP} \cdot (|\vec{OA}| \vec{OB}) = 0 \\
\Rightarrow & \vec{OP} \cdot (|\vec{OB}| \vec{OA} - |\vec{OA}| \vec{OB}) = 0 \\
\Rightarrow & \vec{OP} \cdot (\sqrt{2^2 + (-3)^2 + 6^2} \vec{OA} - \sqrt{1^2 + 2^2 + (-2)^2} \vec{OB}) = 0 \\
\Rightarrow & \vec{OP} \cdot (7\vec{OA} - 3\vec{OB}) = 0 \\
\Rightarrow & \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \left[7 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right] = 0 \\
\Rightarrow & \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 23 \\ -32 \end{pmatrix} = 0 \\
\Rightarrow & (1 + \lambda) \cdot 1 + (2 - 5\lambda) \cdot 23 + (-2 + 8\lambda) \cdot (-32) = 0 \\
\Rightarrow & 111 - 370\lambda = 0 \\
\Rightarrow & \lambda = \frac{3}{10}
\end{aligned}$$

$$\boxed{\lambda = \frac{3}{10}}$$

Problem 15.

The origin O and the points A , B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$,

- (a) Explain why \mathbf{c} can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ .

The point N is on AC such that $AN : NC = 3 : 4$.

- (b) Write down the position vector of N in terms of \mathbf{a} and \mathbf{c} .
- (c) It is given that the area of triangle ONC is equal to the area of triangle OMC , where M is the mid-point of OB . By finding the areas of these triangles in terms of \mathbf{a} and \mathbf{b} , find λ in terms of μ in the case where λ and μ are both positive.

Solution**Part (a)**

Since \mathbf{a} , \mathbf{b} and \mathbf{c} are co-planar and \mathbf{a} is not parallel to \mathbf{b} , \mathbf{c} can be written as a linear combination of \mathbf{a} and \mathbf{b} .

Part (b)

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{ON} &= \frac{4\mathbf{a} + 3\mathbf{c}}{3 + 4} \\ &= \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}\end{aligned}$$

$$\boxed{\overrightarrow{ON} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}}$$

Part (c)

Since M is the mid-point of OB , we have that $M = \frac{1}{2}\mathbf{b}$. Hence,

$$\begin{aligned}\text{Area } \triangle ONC &= \text{Area } \triangle OMC \\ \Rightarrow \frac{1}{2} |\overrightarrow{ON} \times \hat{\mathbf{c}}| &= \frac{1}{2} |\overrightarrow{OM} \times \hat{\mathbf{c}}| \\ \Rightarrow \left| \left(\frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c} \right) \times \hat{\mathbf{c}} \right| &= \left| \frac{1}{2}\mathbf{b} \times \hat{\mathbf{c}} \right|\end{aligned}$$

$$\begin{aligned}\Rightarrow & \frac{4}{7} |\mathbf{a} \times \hat{\mathbf{c}}| = \frac{1}{2} |\mathbf{b} \times \hat{\mathbf{c}}| \\ \Rightarrow & \frac{4}{7} \left| \mathbf{a} \times \frac{\lambda \mathbf{a} + \mu \mathbf{b}}{|\lambda \mathbf{a} + \mu \mathbf{b}|} \right| = \frac{1}{2} \left| \mathbf{b} \times \frac{\lambda \mathbf{a} + \mu \mathbf{b}}{|\lambda \mathbf{a} + \mu \mathbf{b}|} \right| \\ \Rightarrow & \frac{4}{7} |\mathbf{a} \times (\lambda \mathbf{a} + \mu \mathbf{b})| = \frac{1}{2} |\mathbf{b} \times (\lambda \mathbf{a} + \mu \mathbf{b})| \\ \Rightarrow & \frac{4}{7} |\mathbf{a} \times \mu \mathbf{b}| = \frac{1}{2} |\mathbf{b} \times \lambda \mathbf{a}| \\ \Rightarrow & \frac{4}{7} \mu |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \lambda |\mathbf{b} \times \mathbf{a}| \\ \Rightarrow & \frac{4}{7} \mu |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \lambda |\mathbf{a} \times \mathbf{b}| \\ \Rightarrow & \left(\frac{4}{7} \mu - \frac{1}{2} \lambda \right) |\mathbf{a} \times \mathbf{b}| = 0 \\ \Rightarrow & \frac{4}{7} \mu - \frac{1}{2} \lambda = 0 \\ \Rightarrow & \lambda = \frac{8}{7} \mu\end{aligned}$$