Problem 1.

The vector $\mathbf{v}$ is defined by $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$. Find the unit vector in the direction of $\mathbf{v}$ and hence find a vector of magnitude 25 which is parallel to $\mathbf{v}$.

Solution

$$\begin{aligned}
\hat{\mathbf{v}} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\
&= \frac{1}{\sqrt{3^2 + (-4^2) + 1^2}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}
\end{aligned}$$

$$\boxed{\frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \frac{25}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}}$$

Problem 2.

With respect to an origin O, the position vectors of the points A, B, C and D are $4\mathbf{i}+7\mathbf{j}$, $\mathbf{i}+3\mathbf{j}$, $2\mathbf{i}+4\mathbf{j}$ and $3\mathbf{i}+d\mathbf{j}$ respectively.

(a) Find the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Find the value of d if B, C and D are collinear. State the ratio $\dfrac{BC}{BD}$.

Solution

Part (a)

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix}4\\7\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$

$$\boxed{\overrightarrow{BA} = \begin{pmatrix}3\\4\end{pmatrix}}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\boxed{\overrightarrow{BC} = \begin{pmatrix}1\\1\end{pmatrix}}$$

Part (b)

If B, C and D are collinear, then $\overrightarrow{BC} = \lambda\overrightarrow{CD}$ for some λ.

$$\begin{aligned} \overrightarrow{BC} &= \lambda\overrightarrow{CD} \\ \implies \begin{pmatrix}1\\1\end{pmatrix} &= \lambda\left(\overrightarrow{OD} - \overrightarrow{OC}\right) \\ &= \lambda\left(\begin{pmatrix}3\\d\end{pmatrix} - \begin{pmatrix}2\\4\end{pmatrix}\right) \\ &= \begin{pmatrix}\lambda\\ \lambda(d-4)\end{pmatrix} \end{aligned}$$

Hence, $\lambda = 1$ and $\lambda(d-4) = 1$, whence $d = 5$.

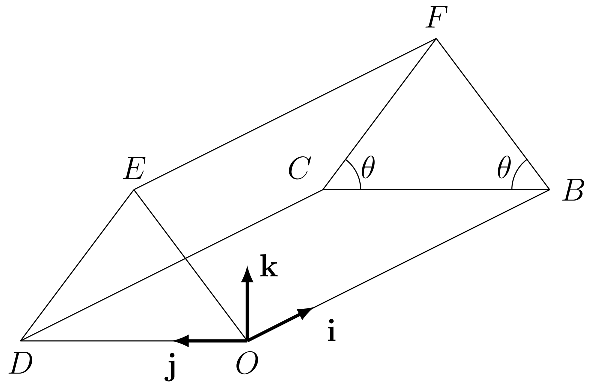
$$\boxed{d = 5}$$

$$\frac{BC}{BD} = \frac{BC}{BC + CD} = \frac{BC}{BC + BC} = \frac{1}{2}$$

$$\boxed{\frac{BC}{BD} = \frac{1}{2}}$$

Problem 3.

The diagram shows a roof, with horizontal rectangular base $OBCD$, where $OB = 10$ m and $BC = 6$ m. The triangular planes ODE and BCF are vertical and the ridge EF is horizontal to the base. The planes $OBFE$ and $DCFE$ are each inclined at an angle θ to the horizontal, where $\tan\theta = \dfrac{4}{3}$. The point O is taken as the origin and vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, each of length 1 m, are taken along OB, OD and vertically upwards from O respectively.



Find the position vectors of the points B, C, D, E and F.

Solution

Since $OB = 10$ m, we know $\overrightarrow{OB} = 10\mathbf{i}$. Further, since $BC = 6$, we know $\overrightarrow{BC} = 6\mathbf{j}$. Thus, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = 10\mathbf{i} + 6\mathbf{j}$.

Note that $\triangle ODE \cong \triangle BCF$. Hence, $BC \cong OD \implies \overrightarrow{OD} = 6\mathbf{j}$.

We also have $\angle ODE = \angle DOE = \theta$. Thus, $\triangle ODE$ is isoceles. Let G be the mid-point of OD. Since $\tan\theta = \dfrac{4}{3}$, we have $\dfrac{EG}{DG} = \dfrac{4}{3}$, which implies that $EG = \dfrac{4}{3}DG = \dfrac{2}{3}OD$. Thus, $EG = \dfrac{2}{3} \cdot 6 = 4$ m. Hence, $\overrightarrow{OE} = \overrightarrow{OG} + \overrightarrow{GE} = \dfrac{1}{2}\overrightarrow{OD} + \overrightarrow{GE} = 3\mathbf{j} + 4\mathbf{k}$.

Since $BF \cong OE$, we know $\overrightarrow{BF} = \overrightarrow{OE}$. Thus, $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$.

$$\boxed{\begin{aligned} &\overrightarrow{OB} = 10\mathbf{i} \\ &\overrightarrow{OC} = 10\mathbf{i} + 6\mathbf{j} \\ &\overrightarrow{OD} = 6\mathbf{j} \\ &\overrightarrow{OE} = 3\mathbf{j} + 4\mathbf{k} \\ &\overrightarrow{OF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \end{aligned}}$$

Problem 4.

Find $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$ and the angle between $\mathbf{u}$ and $\mathbf{v}$ given that

(a) $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

(b) $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$, $\mathbf{v} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

Solution

Part (a)

We have $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$. Hence,

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= 1 \cdot 3 + (-1) \cdot 2 + 1 \cdot 7 \\
&= 8 \\
\mathbf{u} \times \mathbf{v} &= \begin{pmatrix} -1 \cdot 7 - 2 \cdot 1 \\ 1 \cdot 3 - 7 \cdot 1 \\ 1 \cdot 2 - 3 \cdot (-1) \end{pmatrix} \\
&= \begin{pmatrix} -9 \\ -4 \\ 5 \end{pmatrix}
\end{aligned}$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ.

$$\begin{aligned}
& \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \\
\implies \quad & 8 = \sqrt{1^2 + (-1)^2 + 1^2} \cdot \sqrt{3^2 + 2^2 + 7^2} \cdot \cos\theta \\
\implies \quad & \theta = \arccos\left(\frac{8}{\sqrt{3} \cdot \sqrt{62}}\right) \\
& \quad = 54.1^\circ \text{ (1 d.p.)}
\end{aligned}$$

$$\boxed{\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 8 \\ \mathbf{u} \times \mathbf{v} &= -9\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} \\ \theta &= 54.1^\circ \end{aligned}}$$

Part (b)

We have $\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$. Hence,

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 2 \cdot (-1) + 0 \cdot 7 + (-3) \cdot 2 \\ &= -8 \\ \mathbf{u} \times \mathbf{v} &= \begin{pmatrix} 0 \cdot 2 - 7 \cdot (-3) \\ (-3) \cdot (-1) - 2 \cdot 2 \\ 2 \cdot 7 - (-1) \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 21 \\ -1 \\ 14 \end{pmatrix} \end{aligned}$$

Let the angle between $\mathbf{u}$ and $\mathbf{v}$ be θ.

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}||\mathbf{v}| \cos\theta \\ \implies \quad 8 &= \sqrt{2^2 + 0^2 + (-3)^2} \cdot \sqrt{(-1)^2 + 7^2 + 2^2} \cdot \cos\theta \\ \implies \quad \theta &= \arccos\left(\frac{-8}{\sqrt{13} \cdot \sqrt{54}}\right) \\ &= 107.6^\circ \text{ (1 d.p.)} \end{aligned}$$

$$\boxed{\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= -8 \\ \mathbf{u} \times \mathbf{v} &= 21\mathbf{i} - \mathbf{j} + 14\mathbf{k} \\ \theta &= 107.6^\circ \end{aligned}}$$

Problem 5.

Find $\mathbf{u} \cdot \mathbf{v}$ and $|\mathbf{u} \times \mathbf{v}|$ given that $\mathbf{u} = 2\mathbf{a} - \mathbf{b}$, $\mathbf{v} = -\mathbf{a} + 3\mathbf{b}$, where $|\mathbf{a}| = 2$, $|\mathbf{b}| = 1$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $60°$.

Solution

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (2\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} + 3\mathbf{b}) \\
&= -2\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - 3\mathbf{b} \cdot \mathbf{b} \\
&= -2|\mathbf{a}|^2 - 3|\mathbf{b}|^2 + 7|\mathbf{a}||\mathbf{b}| \cos\theta \\
&= -2 \cdot 2^2 - 3 \cdot 1^2 + 7 \cdot 2 \cdot 1 \cdot \cos 60° \\
&= -4 \\
|\mathbf{u} \times \mathbf{v}| &= |(2\mathbf{a} - \mathbf{b}) \times (-\mathbf{a} + 3\mathbf{b})| \\
&= |-2\mathbf{a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - 3\mathbf{b} \times \mathbf{b}| \\
&= |-2 \cdot 0 + 6\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} - 3 \cdot 0| \\
&= |5\mathbf{a} \times \mathbf{b}| \\
&= 5|\mathbf{a}||\mathbf{b}| \sin\theta \\
&= 5 \cdot 2 \cdot 1 \cdot \frac{\sqrt{3}}{2} \\
&= 5\sqrt{3}
\end{aligned}$$

$$\boxed{\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= -4 \\ |\mathbf{u} \times \mathbf{v}| &= 5\sqrt{3} \end{aligned}}$$

Problem 6.

If $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$, $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$, find

(a) a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$,

(b) a vector perpendicular to both $(3\mathbf{b} - 5\mathbf{c})$ and $(7\mathbf{b} + \mathbf{c})$.

Solution

Part (a)

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 4 \cdot 3 - (-1) \cdot (-1) \\ (-1) \cdot 1 - 3 \cdot 1 \\ 1 \cdot (-1) - 1 \cdot 4 \end{pmatrix} \\
&= \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix} \\
\implies |\mathbf{a} \times \mathbf{b}| &= \sqrt{11^2 + (-4)^2 + (-5)^2} \\
&= \sqrt{162} \\
\implies \widehat{\mathbf{a} \times \mathbf{b}} &= \frac{1}{\sqrt{162}} \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix}
\end{aligned}$$

$$\boxed{\frac{1}{\sqrt{162}} \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix}}$$

Part (b)

Observe that $(3\mathbf{b} - 5\mathbf{c}) \times (7\mathbf{b} + \mathbf{c}) = \lambda \mathbf{b} \times \mathbf{c}$ for some scalar λ. It hence suffices to find $\mathbf{b} \times \mathbf{c}$.

$$\begin{aligned}
\mathbf{b} \times \mathbf{c} &= \begin{pmatrix} (-1) \cdot 0 - 1 \cdot 3 \\ 3 \cdot 2 - 0 \cdot 1 \\ 1 \cdot 1 - 2 \cdot (-1) \end{pmatrix} \\
&= \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}
\end{aligned}$$

$$\boxed{\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}}$$

Problem 7.

The position vectors of the points A, B and C are given by $\mathbf{a} = 2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $\mathbf{b} = 5\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$ respectively. Find

(a) a unit vector parallel to $\overrightarrow{AB}$;

(b) the position vector of the point D such that $ABCD$ is a parallelogram, leaving your answer in terms of λ;

(c) the value of λ if A, B and C are collinear;

(d) the position vector of the point P on AB is $AP : PB = 2 : 1$.

Solution

Part (a)

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix}5\\-1\\2\end{pmatrix} - \begin{pmatrix}2\\3\\-4\end{pmatrix} \\ &= \begin{pmatrix}3\\-4\\6\end{pmatrix}\end{aligned}$$

Observe that $\left|\overrightarrow{AB}\right| = \sqrt{3^2 + (-4)^2 + 6^2} = \sqrt{61}$. Hence, the required vector is

$$\boxed{\frac{1}{\sqrt{61}}\begin{pmatrix}3\\-4\\6\end{pmatrix}}$$

Part (b)

Since $ABCD$ is a parallelogram, we have that $\overrightarrow{AD} = \overrightarrow{BC}$. Thus,

$$\begin{aligned} & \overrightarrow{OD} - \mathbf{a} = \mathbf{c} - \mathbf{b} \\ \implies \quad & \overrightarrow{OD} = \mathbf{a} - \mathbf{b} + \mathbf{c} \\ &= \begin{pmatrix}2\\3\\-4\end{pmatrix} - \begin{pmatrix}5\\-1\\2\end{pmatrix} + \begin{pmatrix}11\\\lambda\\14\end{pmatrix} \\ &= \begin{pmatrix}8\\\lambda+4\\8\end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OD} = \begin{pmatrix}8\\\lambda+4\\8\end{pmatrix}}$$

Part (c)

Given that A, B and C are collinear, we have $\overrightarrow{AB} = k\overrightarrow{BC}$. Hence,

$$\begin{aligned}
\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} &= k\left(\mathbf{c} - \mathbf{b}\right) \\
&= k\left(\begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right) \\
&= k\begin{pmatrix} 6 \\ \lambda + 1 \\ 12 \end{pmatrix}
\end{aligned}$$

We hence see that $k = \frac{1}{2}$, which implies that $-4 = \frac{\lambda+1}{2}$, whence $\lambda = -9$.

$$\boxed{\lambda = -9}$$

Part (d)

By the Ratio Theorem,

$$\begin{aligned}
\overrightarrow{OP} &= \frac{1\mathbf{a} + 2\mathbf{b}}{2+1} \\
&= \frac{1}{3}\left(\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right) \\
&= \frac{1}{3}\begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{3}\begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}}$$

Problem 8.

$ABCD$ is a square, and M and N are the midpoints of BC and CD respectively. Express $\overrightarrow{AC}$ in terms of $\mathbf{p}$ and $\mathbf{q}$, where $\overrightarrow{AM} = \mathbf{p}$ and $\overrightarrow{AN} = \mathbf{q}$.

Solution

Let $ABCD$ be a square with side length $2k$ with A at the origin. Then $\mathbf{p} = \overrightarrow{AM} = k\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{q} = \overrightarrow{AN} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Hence, $\mathbf{p}+\mathbf{q} = k\begin{pmatrix} 3 \\ -3 \end{pmatrix}$. Thus, $\overrightarrow{AC} = k\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{2}{3} \cdot k\begin{pmatrix} 3 \\ -3 \end{pmatrix} = \frac{2}{3}(\mathbf{p}+\mathbf{q})$.

$$\boxed{\overrightarrow{AC} = \frac{2}{3}(\mathbf{p}+\mathbf{q}).}$$

Problem 9.

The points A, B have position vectors $\mathbf{a}$, $\mathbf{b}$ respectively, referred to an origin O, where $\mathbf{a}$ and $\mathbf{b}$ are not parallel to each other. The point C lies on AB between A and B and is such that $\dfrac{AC}{CB} = 2$, and D is the mid-point of OC. The line AD produced meets OB at E.

Find, in terms of $\mathbf{a}$ and $\mathbf{b}$,

(a) the position vector of C (referred to O),

(b) the vector $\overrightarrow{AD}$. Find the values of $\dfrac{OE}{EB}$ and $\dfrac{AE}{ED}$.

Solution

Part (a)

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OC} &= \frac{1\mathbf{a} + 2\mathbf{b}}{2+1} \\ &= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{OC} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}}$$

Part (b)

Since D is the mid-point of OC,

$$\begin{aligned}\overrightarrow{OD} &= \frac{1}{2}\overrightarrow{OC} \\ &= \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b} \\ \implies \overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{A} \\ &= \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b} - \mathbf{a} \\ &= -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{AD} = -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non-parallel, there exists some linear transformation $\boldsymbol{A}$ such that $\boldsymbol{A}\mathbf{a} = \begin{pmatrix}1\\3\end{pmatrix}$ and $\boldsymbol{A}\mathbf{b} = \begin{pmatrix}1\\0\end{pmatrix}$. Hence,

$$\begin{aligned}\boldsymbol{A}\overrightarrow{AD} &= -\frac{5}{6}\boldsymbol{A}\mathbf{a} + \frac{1}{3}\boldsymbol{A}\mathbf{b} \\ &= -\frac{5}{6}\begin{pmatrix}1\\3\end{pmatrix} + \frac{1}{3}\begin{pmatrix}1\\0\end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix}1\\5\end{pmatrix}\end{aligned}$$

Since E is on both AD and OB, we have

$$\boldsymbol{A}\overrightarrow{AE} = -\frac{\lambda}{2}\begin{pmatrix}1\\5\end{pmatrix} = \begin{pmatrix}\mu\\-3\end{pmatrix}$$

Thus, $\lambda = \dfrac{6}{5}$ and $\mu = -\dfrac{3}{5}$. Hence, $\boldsymbol{A}\overrightarrow{OE} = \begin{pmatrix}\frac{2}{5}\\0\end{pmatrix}$, giving

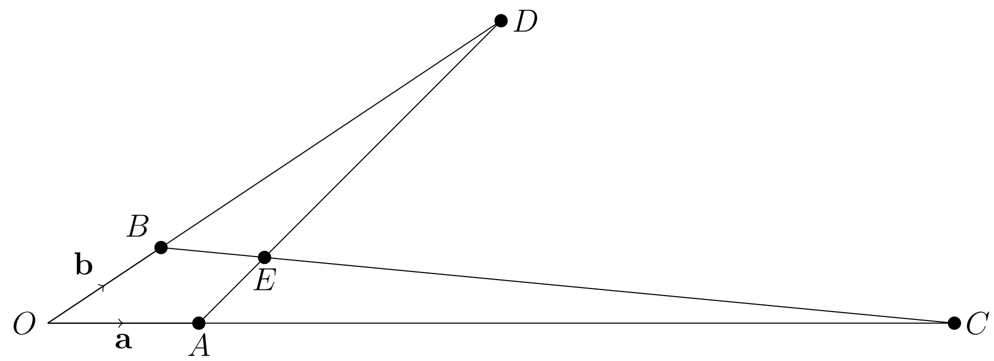
$$\begin{aligned}
\frac{OE}{EB} &= \frac{\left|\boldsymbol{A}\overrightarrow{OE}\right|}{\left|\boldsymbol{A}\overrightarrow{OB} - \boldsymbol{A}\overrightarrow{OE}\right|}\\
&= \frac{2/5}{1-2/5}\\
&= \frac{2}{3}\\
\frac{AE}{ED} &= \frac{\left|\boldsymbol{A}\overrightarrow{AE}\right|}{\left|\boldsymbol{A}\overrightarrow{AE} - \boldsymbol{A}\overrightarrow{AD}\right|}\\
&= \frac{\lambda\left|\boldsymbol{A}\overrightarrow{AD}\right|}{(\lambda-1)\left|\boldsymbol{A}\overrightarrow{AD}\right|}\\
&= \frac{6/5}{6/5-1}\\
&= 6
\end{aligned}$$

$$\boxed{\frac{OE}{EB} = \frac{2}{3},\ \frac{AE}{ED} = 6}$$

Problem 10.

(a) The angle between the vectors $(3\mathbf{i} - 2\mathbf{j})$ and $(6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k})$ is $\arccos \frac{6}{13}$. Show that $2d^2 - 117d + 333 = 0$.

(b) With reference to the origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{AC} = 5\mathbf{a}$, $\overrightarrow{BD} = 3\mathbf{b}$. The lines AD and BC cross at E.



(i) Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

(ii) The point F divides the line CD in the ratio $5 : 3$. Show that O, E and F are collinear, and find $OE : EF$.

Solution

Part (a)

Let $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 6 \\ d \\ -\sqrt{7} \end{pmatrix}$. Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\begin{aligned}
\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos\theta \\
3 \cdot 6 + (-2) \cdot d + 0 \cdot (-\sqrt{7}) &= \sqrt{3^2 + (-2)^2 + 0^2} \cdot \sqrt{6^2 + d^2 + (-\sqrt{7})^2} \cdot \cos\arccos\frac{6}{13} \\
\implies \quad 18 - 2d &= \sqrt{43 + d^2} \cdot \sqrt{13} \cdot \frac{6}{13} \\
&= \sqrt{43 + d^2} \cdot \frac{6}{\sqrt{13}} \\
\implies \quad (18 - 2d)^2 &= \frac{36}{13}\left(43 + d^2\right) \\
\implies \quad (9 - d)^2 &= \frac{9}{13}\left(43 + d^2\right) \\
\implies \quad 13\left(81 - 18d + d^2\right) &= 387 + 9d^2 \\
\implies \quad 1053 - 234d + 13d^2 &= 387 + 9d^2 \\
\implies \quad 4d^2 - 234d + 666 &= 0 \\
\implies \quad 2d^2 - 117d + 333 &= 0
\end{aligned}$$

Part (b)

Subpart (i)

Note that $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = \mathbf{a} - 4\mathbf{b}$ and $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \mathbf{b} - 6\mathbf{a}$. Since E is on both DA and CB, we have

$$\overrightarrow{OE} = \overrightarrow{OD} + \lambda\overrightarrow{DA} = \overrightarrow{OC} + \mu\overrightarrow{CB}$$

for some scalars λ and μ. Hence,

$$\begin{aligned} & 4\mathbf{b} + \lambda(\mathbf{a} - 4\mathbf{b}) = 6\mathbf{a} + \mu(\mathbf{b} - 6\mathbf{a}) \\ \implies & \lambda\mathbf{a} + 4(1-\lambda)\mathbf{b} = 6(1-\mu)\mathbf{a} + \mu\mathbf{b} \end{aligned}$$

Comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$, we have the system

$$\begin{cases} \lambda = 6(1-\mu) \\ \mu = 4(1-\lambda) \end{cases}$$

which has the solution $\lambda = \frac{18}{23}$ and $\mu = \frac{20}{23}$. Hence, $\overrightarrow{OE} = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}$.

$$\boxed{\overrightarrow{OE} = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}}$$

Subpart (ii)

By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OF} &= \frac{3\mathbf{c} + 5\mathbf{d}}{5+3} \\ &= \frac{1}{8}\left(5 \cdot 4\mathbf{b} + 3 \cdot 6\mathbf{a}\right) \\ &= \frac{1}{8}\left(18\mathbf{a} + 20\mathbf{b}\right) \\ &= \frac{23}{8}\left(\frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}\right) \\ &= \frac{23}{8}\overrightarrow{OE} \end{aligned}$$

$$\boxed{OE : OF = 8 : 23}$$

Problem 11.

Relative to the origin O, two points A and B have position vectors given by $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$ and $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$ respectively.

(a) The point P divides the line AB in the ratio $2 : 1$. Find the coordinates of P.

(b) Show that AB and OP are perpendicular.

(c) The vector $\mathbf{c}$ is a unit vector in the direction of $\overrightarrow{OP}$. Write $\mathbf{c}$ as a column vector and give the geometrical meaning of $|\mathbf{a} \cdot \mathbf{c}|$.

(d) Find $\mathbf{a} \times \mathbf{p}$, where $\mathbf{p}$ is the vector $\overrightarrow{OP}$, and give the geometrical meaning of $|\mathbf{a} \times \mathbf{p}|$. Hence write down the area of triangle OAP.

Solution

Part (a)

We have $\mathbf{a} = \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} = 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix}$. By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OP} &= \frac{\mathbf{a} + 2\mathbf{b}}{2+1} \\ &= \frac{1}{3}\left(\begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} + 2\begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix}\right) \\ &= \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \end{aligned}$$

$$\boxed{P(12, -4, 6)}$$

Part (b)

Consider $\overrightarrow{AB} \cdot \overrightarrow{OP}$.

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{OP} &= \left(\begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} - \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix}\right) \cdot \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \\ &= -3\begin{pmatrix} 1 \\ 9 \\ 4 \end{pmatrix} \cdot 2\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\ &= -6\,(1 \cdot 6 + 9 \cdot (-2) + 4 \cdot 3) \\ &= 0 \end{aligned}$$

Since $\overrightarrow{AB} \cdot \overrightarrow{OP} = 0$, AB and OP must be perpendicular.

Part (c)

$$\begin{aligned}\mathbf{c} &= \frac{\overrightarrow{OP}}{\left|\overrightarrow{OP}\right|} \\ &= \frac{1}{2\sqrt{6^2+(-2)^2+3^2}} \cdot 2\begin{pmatrix}6\\-2\\3\end{pmatrix} \\ &= \frac{1}{7}\begin{pmatrix}6\\-2\\3\end{pmatrix}\end{aligned}$$

$$\boxed{\mathbf{c} = \frac{1}{7}\begin{pmatrix}6\\-2\\3\end{pmatrix}}$$

$|\mathbf{a} \cdot \mathbf{c}|$ is the length of the projection of $\mathbf{a}$ on $\overrightarrow{OP}$.

Part (d)

$$\begin{aligned}\mathbf{a} \times \mathbf{p} &= 14\begin{pmatrix}1\\1\\1\end{pmatrix} \times 2\begin{pmatrix}6\\-2\\3\end{pmatrix} \\ &= 28\begin{pmatrix}1\cdot 3-(-2)\cdot 1\\ 1\cdot 6-3\cdot 1\\ 1\cdot -2-6\cdot 1\end{pmatrix} \\ &= 28\begin{pmatrix}5\\3\\-8\end{pmatrix}\end{aligned}$$

$$\boxed{\mathbf{a} \times \mathbf{p} = \begin{pmatrix}5\\3\\-8\end{pmatrix}}$$

$|\mathbf{a} \times \mathbf{p}|$ is twice the area of $\triangle OAP$.

$$\begin{aligned}\text{Area}\,\triangle OAP &= \frac{1}{2}|\mathbf{a} \times \mathbf{p}| \\ &= \frac{1}{2} \cdot 28\sqrt{5^2+3^2+(-8)^2} \\ &= 14\sqrt{98} \\ &= 14 \cdot 7\sqrt{2} \\ &= 98\sqrt{2}\end{aligned}$$

$$\boxed{\text{Area}\,\triangle OAP = 98\sqrt{2}\text{ units}^2}$$

Problem 12.

The points A, B and C have position vectors given by $\mathbf{i}-\mathbf{j}+\mathbf{k}$, $\mathbf{j}-\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}-\mathbf{k}$ respectively.

(a) Find the area of the triangle ABC. Hence, find the sine of the angle BAC.

(b) Find a vector perpendicular to the plane ABC.

(c) Find the projection vector of $\overrightarrow{AC}$ onto $\overrightarrow{AB}$.

(d) Find the distance of C to AB.

Solution

We have $\mathbf{a}=\begin{pmatrix}1\\-1\\1\end{pmatrix}$, $\mathbf{b}=\begin{pmatrix}0\\1\\-1\end{pmatrix}$ and $\mathbf{c}=\begin{pmatrix}2\\-1\\-1\end{pmatrix}$. For simplicity, consider the translation that sends A to the origin. We thus have $\mathbf{a}'=\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\mathbf{b}'=\begin{pmatrix}-1\\2\\-2\end{pmatrix}$ and $\mathbf{c}'=\begin{pmatrix}1\\0\\-2\end{pmatrix}$.

Part (a)

$$\begin{aligned}|\mathbf{b}'\times\mathbf{c}'| &= \left|\begin{pmatrix}2\cdot(-2)-0\cdot(-2)\\-2\cdot 1-(-1)\cdot(-2)\\-1\cdot 0-1\cdot 2\end{pmatrix}\right|\\ &= \left|\begin{pmatrix}-4\\-4\\-2\end{pmatrix}\right|\\ &= \left|-2\begin{pmatrix}2\\2\\1\end{pmatrix}\right|\\ &= 2\cdot\sqrt{2^2+2^2+1^2}\\ &= 6\end{aligned}$$

Hence, the area of $\triangle ABC=\frac{1}{2}|\mathbf{b}'\times\mathbf{c}'|=3$ units^2.

$$\boxed{\text{Area}\,\triangle ABC=3\text{ units}^2}$$

$$\begin{aligned}|\mathbf{b}'\times\mathbf{c}'| &= |\mathbf{b}'||\mathbf{c}'|\sin BAC\\ \implies \sin BAC &= \frac{6}{\sqrt{(-1)^2+2^2+(-2)^2}\sqrt{1^2+0^2+(-2)^2}}\\ &= \frac{6}{3\sqrt{5}}\\ &= \frac{2}{\sqrt{5}}\end{aligned}$$

$$\boxed{\sin BAC=\frac{2}{\sqrt{5}}}$$

Part (b)

$$\boxed{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

Part (c)

Note that $\hat{\mathbf{b}}' = \dfrac{\mathbf{b}'}{|\mathbf{b}'|} = \dfrac{1}{\sqrt{(-1)^2 + 2^2 + (-2)^2}} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \dfrac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$. Hence,

$$\begin{aligned}
\left(\mathbf{c}' \cdot \hat{\mathbf{b}}'\right) \hat{\mathbf{b}}' &= \left(\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right) \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \\
&= \frac{1}{9} \Big(1 \cdot (-1) + 0 \cdot 2 + (-2) \cdot (-2) \Big) \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}
\end{aligned}$$

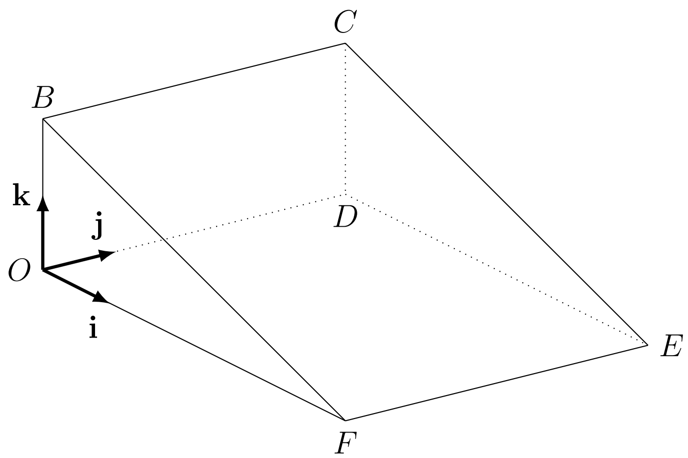
$$\boxed{\frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}}$$

Part (d)

$$\begin{aligned}
\left| \mathbf{c}' \times \hat{\mathbf{b}}' \right| &= \left| \hat{\mathbf{b}}' \times \mathbf{c}' \right| \\
&= \left| \frac{1}{3} \mathbf{b}' \times \mathbf{c}' \right| \\
&= \frac{1}{3} \cdot 6 \\
&= 2
\end{aligned}$$

$\boxed{\text{The distance between } C \text{ and } AB \text{ is 2 units.}}$

Problem 13.



The diagram shows a vehicle ramp $OBCDEF$ with horizontal rectangular base $ODEF$ and vertical rectangular face $OBCD$. Taking the point O as the origin, the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to the edges OF, OD and OB respectively. The lengths of OF, OD and OB are $2h$ units, 3 units and h units respectively.

(a) Show that $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$.

(b) The point P divides the segment CF in the ratio $2:1$. Find $\overrightarrow{OP}$ in terms of h.

For parts (c) and (d), let $h = 1$.

(c) Find the length of projection of $\overrightarrow{OP}$ onto $\overrightarrow{OC}$.

(d) Using the scalar product, find the angle that the rectangular face $BCEF$ makes with the horizontal base.

Solution

Part (a)

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OD} + \overrightarrow{DC} \\ &= \overrightarrow{OD} + \overrightarrow{OB} \\ &= 3\mathbf{j} + h\mathbf{k}\end{aligned}$$

Part (b)

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{1\overrightarrow{OC} + 2\overrightarrow{OF}}{2+1} \\ &= \frac{1}{3}\left(\begin{pmatrix}0\\3\\h\end{pmatrix} + 2\begin{pmatrix}2h\\0\\0\end{pmatrix}\right) \\ &= \frac{1}{3}\begin{pmatrix}4h\\3\\h\end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{3}\begin{pmatrix}4h\\3\\h\end{pmatrix}}$$

Part (c)

$$\begin{aligned}
\left|\overrightarrow{OP}\cdot\hat{\mathbf{c}}\right| &= \left|\frac{1}{3}\begin{pmatrix}4\\3\\1\end{pmatrix}\cdot\frac{1}{\sqrt{0^2+3^2+1^2}}\begin{pmatrix}0\\3\\1\end{pmatrix}\right| \\
&= \frac{1}{3\sqrt{10}}\left|\begin{pmatrix}4\\3\\1\end{pmatrix}\cdot\begin{pmatrix}0\\3\\1\end{pmatrix}\right| \\
&= \frac{1}{3\sqrt{10}}\left|4\cdot 0+3\cdot 3+1\cdot 1\right| \\
&= \frac{10}{3\sqrt{10}} \\
&= \frac{\sqrt{10}}{3}
\end{aligned}$$

$$\boxed{\frac{\sqrt{10}}{3}\text{ units}}$$

Part (d)

Let θ be the angle the rectangular face $BCEF$ makes with the horizontal base.

$$\begin{aligned}
&& \overrightarrow{OF}\cdot\overrightarrow{BF} &= \left|\overrightarrow{OF}\right|\left|\overrightarrow{BF}\right|\cos\theta \\
\implies && \begin{pmatrix}2\\0\\0\end{pmatrix}\cdot\left(\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right) &= 2\cdot\left|\begin{pmatrix}2\\0\\0\end{pmatrix}-\begin{pmatrix}0\\0\\1\end{pmatrix}\right|\cdot\cos\theta \\
\implies && \begin{pmatrix}2\\0\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix} &= 2\cdot\left|\begin{pmatrix}2\\0\\-1\end{pmatrix}\right|\cdot\cos\theta \\
\implies && 2\cdot 2+0\cdot 0+0\cdot(-1) &= 2\cdot\sqrt{2^2+0^2+(-1)^2}\cdot\cos\theta \\
\implies && 4 &= 2\sqrt{5}\cos\theta \\
\implies && \cos\theta &= \frac{2}{\sqrt{5}} \\
\implies && \theta &= \arccos\left(\frac{2}{\sqrt{5}}\right) \\
&& &= 26.6^\circ\text{ (1 d.p.)}
\end{aligned}$$

$$\boxed{26.6^\circ}$$

Problem 14.

The position vectors of the points A and B relative to the origin O are $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$ respectively. The point P on AB is such that $AP : PB = \lambda : 1 - \lambda$. Show that $\overrightarrow{OP} = (1+\lambda)\mathbf{i} + (2-5\lambda)\mathbf{j} + (-2+8)\mathbf{k}$ where λ is a real parameter.

(a) Find the value of λ for which OP is perpendicular to AB.

(b) Find the value of λ for which angles $\angle AOP$ and $\angle POB$ are equal.

Solution

By the Ratio Theorem,

$$\begin{aligned}
\overrightarrow{OP} &= \frac{\lambda\overrightarrow{OB} + (1-\lambda)\overrightarrow{OA}}{\lambda + (1-\lambda)} \\
&= \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix} + (1-\lambda)\begin{pmatrix}1\\2\\-2\end{pmatrix} \\
&= \begin{pmatrix}2\lambda\\-3\lambda\\6\lambda\end{pmatrix} + \begin{pmatrix}1-\lambda\\2-2\lambda\\-2+2\lambda\end{pmatrix} \\
&= \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix} \\
&= (1+\lambda)\mathbf{i} + (2-5\lambda)\mathbf{j} + (-2+8)\mathbf{k}
\end{aligned}$$

Part (a)

For OP to be perpendicular to AB, we must have $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$.

$$\begin{aligned}
& & \overrightarrow{OP} \cdot \overrightarrow{AB} &= 0 \\
&\implies & \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix} \cdot \left(\begin{pmatrix}2\\-3\\6\end{pmatrix} - \begin{pmatrix}1\\2\\-2\end{pmatrix}\right) &= 0 \\
&\implies & \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix} \cdot \begin{pmatrix}1\\-5\\8\end{pmatrix} &= 0 \\
&\implies & (1+\lambda)\cdot 1 + (2-5\lambda)\cdot(-5) + (-2+8\lambda)\cdot 8 &= 0 \\
&\implies & -25 + 90\lambda &= 0 \\
&\implies & \lambda &= \frac{5}{18}
\end{aligned}$$

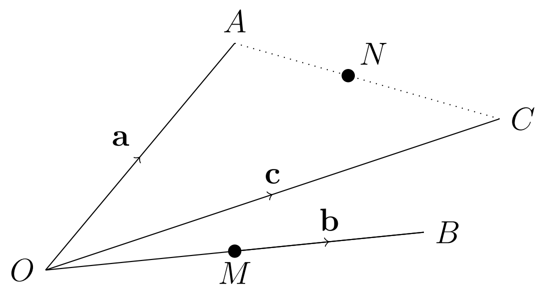
$$\boxed{\lambda = \frac{5}{18}}$$

Part (b)

$$
\begin{aligned}
& \angle AOP = \angle POB \\
\implies & \cos\angle AOP = \cos\angle POB \\
\implies & \frac{\overrightarrow{OP}\cdot\overrightarrow{OA}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OA}\right|} = \frac{\overrightarrow{OP}\cdot\overrightarrow{OB}}{\left|\overrightarrow{OP}\right|\left|\overrightarrow{OB}\right|} \\
\implies & \left|\overrightarrow{OB}\right|\left(\overrightarrow{OP}\cdot\overrightarrow{OA}\right) = \left|\overrightarrow{OA}\right|\left(\overrightarrow{OP}\cdot\overrightarrow{OB}\right) \\
\implies & \overrightarrow{OP}\cdot\left(\left|\overrightarrow{OB}\right|\overrightarrow{OA}\right) = \overrightarrow{OP}\cdot\left(\left|\overrightarrow{OA}\right|\overrightarrow{OB}\right) \\
\implies & \overrightarrow{OP}\cdot\left(\left|\overrightarrow{OB}\right|\overrightarrow{OA}\right) - \overrightarrow{OP}\cdot\left(\left|\overrightarrow{OA}\right|\overrightarrow{OB}\right) = 0 \\
\implies & \overrightarrow{OP}\cdot\left(\left|\overrightarrow{OB}\right|\overrightarrow{OA} - \left|\overrightarrow{OA}\right|\overrightarrow{OB}\right) = 0 \\
\implies & \overrightarrow{OP}\cdot\left(\sqrt{2^2+(-3)^2+6^2}\,\overrightarrow{OA} - \sqrt{1^2+2^2+(-2)^2}\,\overrightarrow{OB}\right) = 0 \\
\implies & \overrightarrow{OP}\cdot\left(7\overrightarrow{OA} - 3\overrightarrow{OB}\right) = 0 \\
\implies & \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}\cdot\left(7\begin{pmatrix}1\\2\\-2\end{pmatrix} - 3\begin{pmatrix}2\\-3\\6\end{pmatrix}\right) = 0 \\
\implies & \begin{pmatrix}1+\lambda\\2-5\lambda\\-2+8\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\23\\-32\end{pmatrix} = 0 \\
\implies & (1+\lambda)\cdot 1 + (2-5\lambda)\cdot 23 + (-2+8\lambda)\cdot(-32) = 0 \\
\implies & 111 - 370\lambda = 0 \\
\implies & \lambda = \frac{3}{10}
\end{aligned}
$$

$$\boxed{\lambda = \frac{3}{10}}$$

Problem 15.



The origin O and the points A, B and C lie in the same plane, where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$,

(a) Explain why $\mathbf{c}$ can be expressed as $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$, for constants λ and μ.

The point N is on AC such that $AN : NC = 3 : 4$.

(b) Write down the position vector of N in terms of $\mathbf{a}$ and $\mathbf{c}$.

(c) It is given that the area of triangle ONC is equal to the area of triangle OMC, where M is the mid-point of OB. By finding the areas of these triangles in terms of $\mathbf{a}$ and $\mathbf{b}$, find λ in terms of μ in the case where λ and μ are both positive.

Solution

Part (a)

Since $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are co-planar and $\mathbf{a}$ is not parallel to $\mathbf{b}$, $\mathbf{c}$ can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$.

Part (b)

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{ON} &= \frac{4\mathbf{a} + 3\mathbf{c}}{3+4} \\ &= \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}\end{aligned}$$

$$\boxed{\overrightarrow{ON} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}}$$

Part (c)

Since M is the mid-point of OB, we have that $M = \frac{1}{2}\mathbf{b}$. Hence,

$$\begin{aligned} & \text{Area}\,\triangle ONC = \text{Area}\,\triangle OMC \\ \implies \quad & \frac{1}{2}\left|\overrightarrow{ON} \times \hat{\mathbf{c}}\right| = \frac{1}{2}\left|\overrightarrow{OM} \times \hat{\mathbf{c}}\right|\end{aligned}$$

$$
\begin{aligned}
\implies \quad \left|\left(\frac{4}{7}\mathbf{a}+\frac{3}{7}\mathbf{c}\right)\times\hat{\mathbf{c}}\right| &= \left|\frac{1}{2}\mathbf{b}\times\hat{\mathbf{c}}\right| \\
\implies \quad \frac{4}{7}|\mathbf{a}\times\hat{\mathbf{c}}| &= \frac{1}{2}|\mathbf{b}\times\hat{\mathbf{c}}| \\
\implies \quad \frac{4}{7}\left|\mathbf{a}\times\frac{\lambda\mathbf{a}+\mu\mathbf{b}}{|\lambda\mathbf{a}+\mu\mathbf{b}|}\right| &= \frac{1}{2}\left|\mathbf{b}\times\frac{\lambda\mathbf{a}+\mu\mathbf{b}}{|\lambda\mathbf{a}+\mu\mathbf{b}|}\right| \\
\implies \quad \frac{4}{7}|\mathbf{a}\times(\lambda\mathbf{a}+\mu\mathbf{b})| &= \frac{1}{2}|\mathbf{b}\times(\lambda\mathbf{a}+\mu\mathbf{b})| \\
\implies \quad \frac{4}{7}|\mathbf{a}\times\mu\mathbf{b}| &= \frac{1}{2}|\mathbf{b}\times\lambda\mathbf{a}| \\
\implies \quad \frac{4}{7}\mu|\mathbf{a}\times\mathbf{b}| &= \frac{1}{2}\lambda|\mathbf{b}\times\mathbf{a}| \\
\implies \quad \frac{4}{7}\mu|\mathbf{a}\times\mathbf{b}| &= \frac{1}{2}\lambda|\mathbf{a}\times\mathbf{b}| \\
\implies \left(\frac{4}{7}\mu-\frac{1}{2}\lambda\right)|\mathbf{a}\times\mathbf{b}| &= 0 \\
\implies \quad \frac{4}{7}\mu-\frac{1}{2}\lambda &= 0 \\
\implies \quad \lambda &= \frac{8}{7}\mu
\end{aligned}
$$