

Problem 1.

The points A and B have position vectors relative to the origin O , denoted by \mathbf{a} and \mathbf{b} respectively, where \mathbf{a} and \mathbf{b} are non-parallel vectors. The point P lies on AB such that $AP : PB = \lambda : 1$. The point Q lies on OP extended such that $OP = 2PQ$ and $\overrightarrow{BQ} = \overrightarrow{OA} + \mu\overrightarrow{OB}$. Find the values of the real constants λ and μ .

Solution

By the Ratio Theorem,

$$\begin{aligned}
 \overrightarrow{OP} &= \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} \\
 \Rightarrow \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\
 &= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{OP} \\
 &= \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} \\
 \overrightarrow{BQ} &= \overrightarrow{OA} + \mu\overrightarrow{OB} \\
 \Rightarrow \overrightarrow{OB} + \overrightarrow{BQ} &= \overrightarrow{OA} + (1 + \mu)\overrightarrow{OB} \\
 \Rightarrow \overrightarrow{OQ} &= \overrightarrow{OA} + (1 + \mu)\overrightarrow{OB} \\
 \Rightarrow \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} &= \mathbf{a} + (1 + \mu)\mathbf{b}
 \end{aligned}$$

Since \mathbf{a} and \mathbf{b} are non-parallel, we have the following system:

$$\left\{ \begin{array}{l} \frac{3}{2} \cdot \frac{1}{1 + \lambda} = 1 \\ \frac{3}{2} \cdot \frac{\lambda}{1 + \lambda} = 1 + \mu \end{array} \right.$$

which has the unique solution $\lambda = \frac{1}{2}$ and $\mu = -\frac{1}{2}$.

$$\boxed{\lambda = \frac{1}{2}, \mu = -\frac{1}{2}}$$

Problem 2.

Given that $\mathbf{a} = \mathbf{i} + \mathbf{j}$, $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$ and $\mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$ where $\lambda \in \mathbb{R}$, find the possible value(s) of λ for which the angle between \mathbf{p} and \mathbf{k} is 45° .

Solution

$$\begin{aligned}
 \mathbf{p} &= \lambda\mathbf{a} + (1 - \lambda)\mathbf{b} \\
 &= \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} \\
 &= \begin{pmatrix} 4 - 3\lambda \\ -2 + 3\lambda \\ 6 - 6\lambda \end{pmatrix} \\
 \implies |\mathbf{p}|^2 &= (4 - 3\lambda)^2 + (-2 + 3\lambda)^2 + (6 - 6\lambda)^2 \\
 &= 54\lambda^2 - 108\lambda + 56
 \end{aligned}$$

Since the angle between \mathbf{p} and \mathbf{k} is 45° ,

$$\begin{aligned}
 \cos 45^\circ &= \frac{\mathbf{p} \cdot \mathbf{k}}{|\mathbf{p}| |\mathbf{k}|} \\
 \implies \frac{1}{\sqrt{2}} &= \frac{(4 - 3\lambda) \cdot 0 + (-2 + 3\lambda) \cdot 0 + (6 - 6\lambda) \cdot 1}{|\mathbf{p}| \cdot 1} \\
 \implies \frac{|\mathbf{p}|}{\sqrt{2}} &= 6 - 6\lambda \\
 \implies \frac{|\mathbf{p}|^2}{2} &= (6 - 6\lambda)^2 \\
 \implies \frac{54\lambda^2 - 108\lambda + 56}{2} &= 36\lambda^2 - 72\lambda + 36 \\
 \implies 9\lambda^2 - 18\lambda + 8 &= 0 \\
 \implies (3\lambda - 2)(3\lambda - 4) &= 0
 \end{aligned}$$

Hence, $\lambda = \frac{2}{3}, \frac{4}{3}$. However, we must reject $\lambda = \frac{4}{3}$ since $6 - 6\lambda = \frac{|\mathbf{p}|}{\sqrt{2}} > 0 \implies \lambda < 1$.

$$\boxed{\lambda = \frac{2}{3}}$$

Problem 3.

- (a) \mathbf{a} and \mathbf{b} are non-zero vectors such that $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$. State the relation between the directions of \mathbf{a} and \mathbf{b} , and find $|\mathbf{b}|$.
- (b) \mathbf{a} is a non-zero vector such that $|\mathbf{a}| = \sqrt{3}$ and \mathbf{b} is a unit vector. Given that \mathbf{a} and \mathbf{b} are non-parallel and the angle between them is $\frac{5}{6}\pi$, find the exact value of the length of projection of \mathbf{a} on \mathbf{b} . By considering $(2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b})$, or otherwise, find the exact value of $|2\mathbf{a} + \mathbf{b}|$.

Solution**Part (a)**

\mathbf{a} and \mathbf{b} either have the same or opposite direction.

Let $\mathbf{b} = \lambda \mathbf{a}$ for some $\lambda \in \mathbb{R}$.

$$\begin{aligned}
 & \mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} \\
 \implies & \mathbf{a} = (\mathbf{a} \cdot \lambda \mathbf{a})\lambda \mathbf{a} \\
 \implies & \mathbf{a} = \lambda^2 |\mathbf{a}|^2 \mathbf{a} \\
 \implies & \lambda^2 |\mathbf{a}|^2 = 1 \\
 \implies & \lambda |\mathbf{a}| = \pm 1 \\
 \implies & \lambda = \pm \frac{1}{|\mathbf{a}|} \\
 \implies & \mathbf{b} = \pm \frac{\mathbf{a}}{|\mathbf{a}|} \\
 \implies & \mathbf{b} = \pm \hat{\mathbf{a}} \\
 \implies & |\mathbf{b}| = 1
 \end{aligned}$$

$$\boxed{|\mathbf{b}| = 1}$$

Part (b)

$$\begin{aligned}
 |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}| |\mathbf{b}| \cos \frac{5}{6}\pi \\
 &= \sqrt{3} \cdot 1 \cdot \left(-\frac{\sqrt{3}}{2}\right) \\
 &= -\frac{3}{2} \\
 \implies \text{Length of projection of } \mathbf{a} \text{ on } \mathbf{b} &= \left| \mathbf{a} \cdot \hat{\mathbf{b}} \right| \\
 &= |\mathbf{a} \cdot \mathbf{b}| \\
 &= \left| -\frac{3}{2} \right| \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\boxed{\text{Length of projection of } \mathbf{a} \text{ on } \mathbf{b} = \frac{3}{2}}$$

$$\begin{aligned} |2\mathbf{a} + \mathbf{b}|^2 &= (2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) \\ &= 2\mathbf{a} \cdot 2\mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot 2\mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= 4\mathbf{a} \cdot \mathbf{a} + 4\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ &= 4 \cdot 3 + 4 \cdot \left(-\frac{3}{2}\right) + 1^2 \\ &= 7 \\ \Rightarrow |2\mathbf{a} + \mathbf{b}| &= \sqrt{7} \end{aligned}$$

$$\boxed{|2\mathbf{a} + \mathbf{b}| = \sqrt{7}}$$

Problem 4.

The points A, B, C, D have position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ given by $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{d} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$, respectively. The point P lies on AB produced such that $AP = 2AB$, and the point Q is the mid-point of AC .

- (a) Show that PQ is perpendicular to AQ .
- (b) Find the area of the triangle APQ .
- (c) Find a vector perpendicular to the plane ABC .
- (d) Find the cosine of the angle between \overrightarrow{AD} and \overrightarrow{BD} .

Solution

We re-centre the vectors such that \mathbf{a} is the origin. This gives $\mathbf{a}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{b}' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$, $\mathbf{c}' = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$, $\mathbf{d}' = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$. Hence, $\overrightarrow{OP'}$ is clearly $\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$, while $\overrightarrow{OQ'} = \frac{1}{2}\mathbf{c}' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

Part (a)

$$\begin{aligned} \overrightarrow{PQ} \cdot \overrightarrow{AQ} &= \overrightarrow{PQ'} \cdot \overrightarrow{OQ'} \\ &= (\overrightarrow{OQ'} - \overrightarrow{OP'}) \cdot (\overrightarrow{OQ'}) \\ &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\ &= 1 + 0 - 1 \\ &= 0 \end{aligned}$$

Since $\overrightarrow{PQ} \cdot \overrightarrow{AQ} = 0$, the lines PQ and AQ must be perpendicular.

Part (b)

$$\begin{aligned} \text{Area } \triangle APQ &= \frac{1}{2} \left| \overrightarrow{AP} \times \overrightarrow{AQ} \right| \\ &= \frac{1}{2} \left| \overrightarrow{OP'} \times \overrightarrow{OQ'} \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right| \\ &= \frac{1}{2} \left| \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \right| \\ &= 1 \end{aligned}$$

$$\boxed{\text{Area } \triangle APQ = 1}$$

Part (c)

$$\begin{aligned} \mathbf{b}' \times \mathbf{c}' &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \end{aligned}$$

$$\boxed{\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \text{ is perpendicular to the plane } ABC.}$$

Part (d)

Let the angle between \overrightarrow{AD} and \overrightarrow{BD} be θ . Note that $\overrightarrow{BD} = \mathbf{d}' - \mathbf{b}' = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$

$$\begin{aligned} \cos \theta &= \frac{\overrightarrow{AD} \cdot \overrightarrow{BD}}{|\overrightarrow{AD}| |\overrightarrow{BD}|} \\ &= \frac{1}{\sqrt{3^2 + (-3)^2 + (-3)^2} \cdot 3\sqrt{1^2 + (-1)^2 + (-1)^2}} \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} \cdot 3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\ &= \frac{1}{3\sqrt{102}} \cdot 3(3 \cdot 1 + (-3) \cdot (-1) + (-4) \cdot (-1)) \\ &= \frac{10}{\sqrt{102}} \end{aligned}$$

$$\boxed{\cos \theta = \frac{10}{\sqrt{102}}}$$