

**Problem 1.**

The vector  $\mathbf{v}$  is defined by  $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ . Find the unit vector in the direction of  $\mathbf{v}$  and hence find a vector of magnitude 25 which is parallel to  $\mathbf{v}$ .

**Solution**

$$\begin{aligned}\hat{\mathbf{v}} &= \frac{\mathbf{v}}{|\mathbf{v}|} \\ &= \frac{1}{\sqrt{3^2 + (-4)^2 + 1^2}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \\ &= \frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}\end{aligned}$$

$$\boxed{\frac{1}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}, \frac{25}{\sqrt{26}} \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix}}$$

**Problem 2.**

With respect to an origin  $O$ , the position vectors of the points  $A$ ,  $B$ ,  $C$  and  $D$  are  $4\mathbf{i} + 7\mathbf{j}$ ,  $\mathbf{i} + 3\mathbf{j}$ ,  $2\mathbf{i} + 4\mathbf{j}$  and  $3\mathbf{i} + d\mathbf{j}$  respectively.

(a) Find the vectors  $\overrightarrow{BA}$  and  $\overrightarrow{BC}$ .

(b) Find the value of  $d$  if  $B$ ,  $C$  and  $D$  are collinear. State the ratio  $\frac{BC}{BD}$ .

**Solution****Part (a)**

$$\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\boxed{\overrightarrow{BA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\boxed{\overrightarrow{BC} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

**Part (b)**

If  $B$ ,  $C$  and  $D$  are collinear, then  $\overrightarrow{BC} = \lambda \overrightarrow{CD}$  for some  $\lambda$ .

$$\begin{aligned} \overrightarrow{BC} &= \lambda \overrightarrow{CD} \\ \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \lambda (\overrightarrow{OD} - \overrightarrow{OC}) \\ &= \lambda \left( \begin{pmatrix} 3 \\ d \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right) \\ &= \begin{pmatrix} \lambda \\ \lambda(d-4) \end{pmatrix} \end{aligned}$$

Hence,  $\lambda = 1$  and  $\lambda(d-4) = 1$ , whence  $d = 5$ .

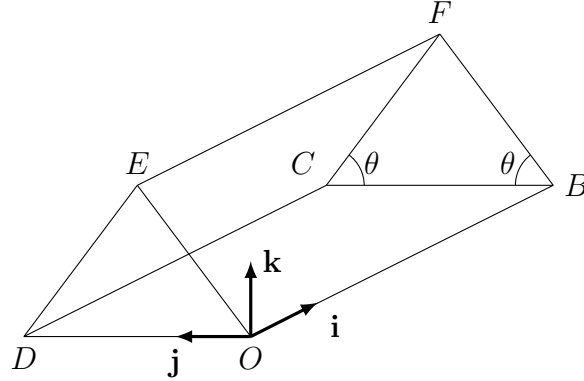
$$\boxed{d = 5}$$

$$\frac{BC}{BD} = \frac{BC}{BC + CD} = \frac{BC}{BC + BC} = \frac{1}{2}$$

$$\boxed{\frac{BC}{BD} = \frac{1}{2}}$$

**Problem 3.**

The diagram shows a roof, with horizontal rectangular base  $OBCD$ , where  $OB = 10$  m and  $BC = 6$  m. The triangular planes  $ODE$  and  $BCF$  are vertical and the ridge  $EF$  is horizontal to the base. The planes  $OBFE$  and  $DCFE$  are each inclined at an angle  $\theta$  to the horizontal, where  $\tan \theta = \frac{4}{3}$ . The point  $O$  is taken as the origin and vectors  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ , each of length 1 m, are taken along  $OB$ ,  $OD$  and vertically upwards from  $O$  respectively.



Find the position vectors of the points  $B$ ,  $C$ ,  $D$ ,  $E$  and  $F$ .

**Solution**

Since  $OB = 10$  m, we know  $\vec{OB} = 10\mathbf{i}$ . Further, since  $BC = 6$ , we know  $\vec{BC} = 6\mathbf{j}$ . Thus,  $\vec{OC} = \vec{OB} + \vec{BC} = 10\mathbf{i} + 6\mathbf{j}$ .

Note that  $\triangle ODE \cong \triangle BCF$ . Hence,  $BC \cong OD \implies \vec{OD} = 6\mathbf{j}$ .

We also have  $\angle ODE = \angle DOE = \theta$ . Thus,  $\triangle ODE$  is isosceles. Let  $G$  be the mid-point of  $OD$ . Since  $\tan \theta = \frac{4}{3}$ , we have  $\frac{EG}{DG} = \frac{4}{3}$ , which implies that  $EG = \frac{4}{3}DG = \frac{2}{3}OD$ . Thus,  $EG = \frac{2}{3} \cdot 6 = 4$  m. Hence,  $\vec{OE} = \vec{OG} + \vec{GE} = \frac{1}{2}\vec{OD} + \vec{GE} = 3\mathbf{j} + 4\mathbf{k}$ .

Since  $BF \cong OE$ , we know  $\vec{BF} = \vec{OE}$ . Thus,  $\vec{OF} = \vec{OB} + \vec{BF} = 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ .

$\begin{aligned}\vec{OB} &= 10\mathbf{i} \\ \vec{OC} &= 10\mathbf{i} + 6\mathbf{j} \\ \vec{OD} &= 6\mathbf{j} \\ \vec{OE} &= 3\mathbf{j} + 4\mathbf{k} \\ \vec{OF} &= 10\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}\end{aligned}$
--

**Problem 4.**

Find  $\mathbf{u} \cdot \mathbf{v}$ ,  $\mathbf{u} \times \mathbf{v}$  and the angle between  $\mathbf{u}$  and  $\mathbf{v}$  given that

(a)  $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}$

(b)  $\mathbf{u} = 2\mathbf{i} - 3\mathbf{k}$ ,  $\mathbf{v} = -\mathbf{i} + 7\mathbf{j} + 2\mathbf{k}$

**Solution****Part (a)**

We have  $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}$ . Hence,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 1 \cdot 3 + (-1) \cdot 2 + 1 \cdot 7 \\ &= 8\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{pmatrix} -1 \cdot 7 - 2 \cdot 1 \\ 1 \cdot 3 - 7 \cdot 1 \\ 1 \cdot 2 - 3 \cdot (-1) \end{pmatrix} \\ &= \begin{pmatrix} -9 \\ -4 \\ 5 \end{pmatrix}\end{aligned}$$

Let the angle between  $\mathbf{u}$  and  $\mathbf{v}$  be  $\theta$ .

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta \\ \Rightarrow 8 &= \sqrt{1^2 + (-1)^2 + 1^2} \cdot \sqrt{3^2 + 2^2 + 7^2} \cdot \cos \theta \\ \Rightarrow \theta &= \arccos\left(\frac{8}{\sqrt{3} \cdot \sqrt{62}}\right) \\ &= 54.1^\circ \text{ (1 d.p.)}\end{aligned}$$

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 8 \\ \mathbf{u} \times \mathbf{v} &= -9\mathbf{i} - 4\mathbf{j} + 5\mathbf{k} \\ \theta &= 54.1^\circ\end{aligned}$$

**Part (b)**

We have  $\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$ . Hence,

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= 2 \cdot (-1) + 0 \cdot 7 + (-3) \cdot 2 \\ &= -8\end{aligned}$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{pmatrix} 0 \cdot 2 - 7 \cdot (-3) \\ (-3) \cdot (-1) - 2 \cdot 2 \\ 2 \cdot 7 - (-1) \cdot 0 \end{pmatrix} \\ &= \begin{pmatrix} 21 \\ -1 \\ 14 \end{pmatrix}\end{aligned}$$

Let the angle between  $\mathbf{u}$  and  $\mathbf{v}$  be  $\theta$ .

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= |\mathbf{u}| |\mathbf{v}| \cos \theta \\ \Rightarrow 8 &= \sqrt{2^2 + 0^2 + (-3)^2} \cdot \sqrt{(-1)^2 + 7^2 + 2^2} \cdot \cos \theta \\ \Rightarrow \theta &= \arccos\left(\frac{-8}{\sqrt{13} \cdot \sqrt{54}}\right) \\ &= 107.6^\circ \text{ (1 d.p.)}\end{aligned}$$

$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= -8 \\ \mathbf{u} \times \mathbf{v} &= 21\mathbf{i} - \mathbf{j} + 14\mathbf{k} \\ \theta &= 107.6^\circ\end{aligned}$
--

**Problem 5.**

Find  $\mathbf{u} \cdot \mathbf{v}$  and  $|\mathbf{u} \times \mathbf{v}|$  given that  $\mathbf{u} = 2\mathbf{a} - \mathbf{b}$ ,  $\mathbf{v} = -\mathbf{a} + 3\mathbf{b}$ , where  $|\mathbf{a}| = 2$ ,  $|\mathbf{b}| = 1$  and the angle between  $\mathbf{a}$  and  $\mathbf{b}$  is  $60^\circ$ .

**Solution**

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= (2\mathbf{a} - \mathbf{b}) \cdot (-\mathbf{a} + 3\mathbf{b}) \\&= -2\mathbf{a} \cdot \mathbf{a} + 6\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{a} - 3\mathbf{b} \cdot \mathbf{b} \\&= -2|\mathbf{a}|^2 - 3|\mathbf{b}|^2 + 7|\mathbf{a}||\mathbf{b}|\cos\theta \\&= -2 \cdot 2^2 - 3 \cdot 1^2 + 7 \cdot 2 \cdot 1 \cdot \cos 60^\circ \\&= -4 \\|\mathbf{u} \times \mathbf{v}| &= |(2\mathbf{a} - \mathbf{b}) \times (-\mathbf{a} + 3\mathbf{b})| \\&= |-2\mathbf{a} \times \mathbf{a} + 6\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{a} - 3\mathbf{b} \times \mathbf{b}| \\&= |-2 \cdot 0 + 6\mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{b} - 3 \cdot 0| \\&= |5\mathbf{a} \times \mathbf{b}| \\&= 5|\mathbf{a}||\mathbf{b}|\sin\theta \\&= 5 \cdot 2 \cdot 1 \cdot \frac{\sqrt{3}}{2} \\&= 5\sqrt{3}\end{aligned}$$

$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= -4 \\ \mathbf{u} \times \mathbf{v}  &= 5\sqrt{3}\end{aligned}$
---

**Problem 6.**

If  $\mathbf{a} = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$  and  $\mathbf{c} = 2\mathbf{i} + \mathbf{j}$ , find

- (a) a unit vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ ,
- (b) a vector perpendicular to both  $(3\mathbf{b} - 5\mathbf{c})$  and  $(7\mathbf{b} + \mathbf{c})$ .

**Solution****Part (a)**

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= \begin{pmatrix} 4 \cdot 3 - (-1) \cdot (-1) \\ (-1) \cdot 1 - 3 \cdot 1 \\ 1 \cdot (-1) - 1 \cdot 4 \end{pmatrix} \\ &= \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix} \\ \Rightarrow |\mathbf{a} \times \mathbf{b}| &= \sqrt{11^2 + (-4)^2 + (-5)^2} \\ &= \sqrt{162} \\ \Rightarrow \widehat{\mathbf{a} \times \mathbf{b}} &= \frac{1}{\sqrt{162}} \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix}\end{aligned}$$

$$\boxed{\frac{1}{\sqrt{162}} \begin{pmatrix} 11 \\ -4 \\ -5 \end{pmatrix}}$$

**Part (b)**

Observe that  $(3\mathbf{b} - 5\mathbf{c}) \times (7\mathbf{b} + \mathbf{c}) = \lambda\mathbf{b} \times \mathbf{c}$  for some scalar  $\lambda$ . It hence suffices to find  $\mathbf{b} \times \mathbf{c}$ .

$$\begin{aligned}\mathbf{b} \times \mathbf{c} &= \begin{pmatrix} (-1) \cdot 0 - 1 \cdot 3 \\ 3 \cdot 2 - 0 \cdot 1 \\ 1 \cdot 1 - 2 \cdot (-1) \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}\end{aligned}$$

$$\boxed{\begin{pmatrix} -3 \\ 6 \\ 3 \end{pmatrix}}$$

**Problem 7.**

The position vectors of the points  $A$ ,  $B$  and  $C$  are given by  $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ ,  $\mathbf{b} = 5\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{c} = 11\mathbf{i} + \lambda\mathbf{j} + 14\mathbf{k}$  respectively. Find

- (a) a unit vector parallel to  $\overrightarrow{AB}$ ;
- (b) the position vector of the point  $D$  such that  $ABCD$  is a parallelogram, leaving your answer in terms of  $\lambda$ ;
- (c) the value of  $\lambda$  if  $A$ ,  $B$  and  $C$  are collinear;
- (d) the position vector of the point  $P$  on  $AB$  is  $AP : PB = 2 : 1$ .

**Solution****Part (a)**

$$\begin{aligned}\overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}\end{aligned}$$

Observe that  $|\overrightarrow{AB}| = \sqrt{3^2 + (-4)^2 + 6^2} = \sqrt{61}$ . Hence, the required vector is

$$\boxed{\frac{1}{\sqrt{61}} \begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix}}$$

**Part (b)**

Since  $ABCD$  is a parallelogram, we have that  $\overrightarrow{AD} = \overrightarrow{BC}$ . Thus,

$$\begin{aligned}\overrightarrow{OD} - \mathbf{a} &= \mathbf{c} - \mathbf{b} \\ \Rightarrow \overrightarrow{OD} &= \mathbf{a} - \mathbf{b} + \mathbf{c} \\ &= \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OD} = \begin{pmatrix} 8 \\ \lambda + 4 \\ 8 \end{pmatrix}}$$



**Part (c)**

Given that  $A$ ,  $B$  and  $C$  are collinear, we have  $\overrightarrow{AB} = k\overrightarrow{BC}$ . Hence,

$$\begin{aligned}\begin{pmatrix} 3 \\ -4 \\ 6 \end{pmatrix} &= k(\mathbf{c} - \mathbf{b}) \\ &= k\left(\begin{pmatrix} 11 \\ \lambda \\ 14 \end{pmatrix} - \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right) \\ &= k\begin{pmatrix} 6 \\ \lambda + 1 \\ 12 \end{pmatrix}\end{aligned}$$

We hence see that  $k = \frac{1}{2}$ , which implies that  $-4 = \frac{\lambda + 1}{2}$ , whence  $\lambda = -9$ .

$$\boxed{\lambda = -9}$$

**Part (d)**

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{1\mathbf{a} + 2\mathbf{b}}{2 + 1} \\ &= \frac{1}{3}\left(\begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} + 2\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}\right) \\ &= \frac{1}{3}\begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{3}\begin{pmatrix} 12 \\ 1 \\ 0 \end{pmatrix}}$$

**Problem 8.**

$ABCD$  is a square, and  $M$  and  $N$  are the midpoints of  $BC$  and  $CD$  respectively. Express  $\overrightarrow{AC}$  in terms of  $\mathbf{p}$  and  $\mathbf{q}$ , where  $\overrightarrow{AM} = \mathbf{p}$  and  $\overrightarrow{AN} = \mathbf{q}$ .

**Solution**

Let  $ABCD$  be a square with side length  $2k$  with  $A$  at the origin. Then  $\mathbf{p} = \overrightarrow{AM} = k \begin{pmatrix} 2 \\ -1 \end{pmatrix}$  and  $\mathbf{q} = \overrightarrow{AN} = k \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ . Hence,  $\mathbf{p} + \mathbf{q} = k \begin{pmatrix} 3 \\ -3 \end{pmatrix}$ . Thus,  $\overrightarrow{AC} = k \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \frac{2}{3} \cdot k \begin{pmatrix} 3 \\ -3 \end{pmatrix} = \frac{2}{3}(\mathbf{p} + \mathbf{q})$ .

$$\boxed{\overrightarrow{AC} = \frac{2}{3}(\mathbf{p} + \mathbf{q})}$$

**Problem 9.**

The points  $A, B$  have position vectors  $\mathbf{a}, \mathbf{b}$  respectively, referred to an origin  $O$ , where  $\mathbf{a}$  and  $\mathbf{b}$  are not parallel to each other. The point  $C$  lies on  $AB$  between  $A$  and  $B$  and is such that  $\frac{AC}{CB} = 2$ , and  $D$  is the mid-point of  $OC$ . The line  $AD$  produced meets  $OB$  at  $E$ .

Find, in terms of  $\mathbf{a}$  and  $\mathbf{b}$ ,

- (a) the position vector of  $C$  (referred to  $O$ ),
- (b) the vector  $\overrightarrow{AD}$ . Find the values of  $\frac{OE}{EB}$  and  $\frac{AE}{ED}$ .

**Solution****Part (a)**

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OC} &= \frac{1\mathbf{a} + 2\mathbf{b}}{2 + 1} \\ &= \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{OC} = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}}$$

**Part (b)**

Since  $D$  is the mid-point of  $OC$ ,

$$\begin{aligned}\overrightarrow{OD} &= \frac{1}{2}\overrightarrow{OC} \\ &= \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b} \\ \Rightarrow \overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA} \\ &= \frac{1}{6}\mathbf{a} + \frac{1}{3}\mathbf{b} - \mathbf{a} \\ &= -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}\end{aligned}$$

$$\boxed{\overrightarrow{AD} = -\frac{5}{6}\mathbf{a} + \frac{1}{3}\mathbf{b}}$$

Since  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel, there exists some linear transformation  $\mathbf{A}$  such that

$\mathbf{Aa} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$  and  $\mathbf{Ab} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . Hence,

$$\begin{aligned} \mathbf{A}\overrightarrow{AD} &= -\frac{5}{6}\mathbf{Aa} + \frac{1}{3}\mathbf{Ab} \\ &= -\frac{5}{6}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} \end{aligned}$$

Since  $E$  is on both  $AD$  and  $OB$ , we have

$$\mathbf{A}\overrightarrow{AE} = -\frac{\lambda}{2}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} \mu \\ -3 \end{pmatrix}$$

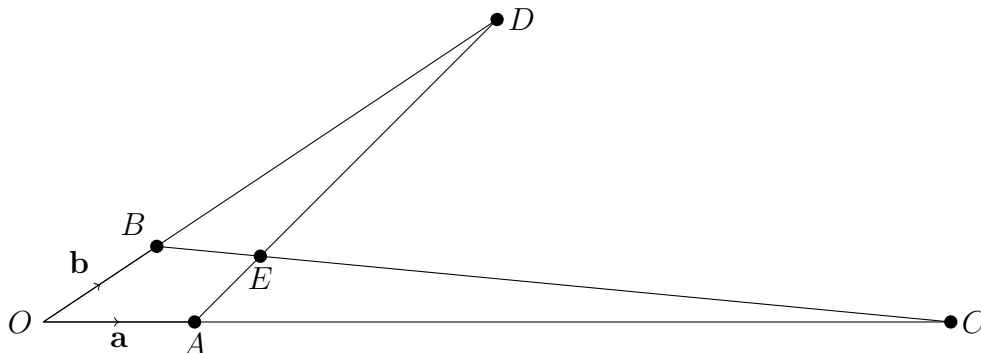
Thus,  $\lambda = \frac{6}{5}$  and  $\mu = -\frac{3}{5}$ . Hence,  $\mathbf{A}\overrightarrow{OE} = \begin{pmatrix} 2/5 \\ 0 \end{pmatrix}$ , giving

$$\begin{aligned} \frac{OE}{EB} &= \frac{|\mathbf{A}\overrightarrow{OE}|}{|\mathbf{A}\overrightarrow{OB} - \mathbf{A}\overrightarrow{OE}|} \\ &= \frac{2/5}{1 - 2/5} \\ &= \frac{2}{3} \\ \frac{AE}{ED} &= \frac{|\mathbf{A}\overrightarrow{AE}|}{|\mathbf{A}\overrightarrow{AE} - \mathbf{A}\overrightarrow{AD}|} \\ &= \frac{\lambda |\mathbf{A}\overrightarrow{AD}|}{(\lambda - 1) |\mathbf{A}\overrightarrow{AD}|} \\ &= \frac{6/5}{6/5 - 1} \\ &= 6 \end{aligned}$$

$$\boxed{\frac{OE}{EB} = \frac{2}{3}, \frac{AE}{ED} = 6}$$

**Problem 10.**

- (a) The angle between the vectors  $(3\mathbf{i} - 2\mathbf{j})$  and  $(6\mathbf{i} + d\mathbf{j} - \sqrt{7}\mathbf{k})$  is  $\arccos \frac{6}{13}$ . Show that  $2d^2 - 117d + 333 = 0$ .
- (b) With reference to the origin  $O$ , the points  $A$ ,  $B$ ,  $C$  and  $D$  are such that  $\overrightarrow{OA} = \mathbf{a}$ ,  $\overrightarrow{OB} = \mathbf{b}$ ,  $\overrightarrow{AC} = 5\mathbf{a}$ ,  $\overrightarrow{BD} = 3\mathbf{b}$ . The lines  $AD$  and  $BC$  cross at  $E$ .



- (i) Find  $\overrightarrow{OE}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .
- (ii) The point  $F$  divides the line  $CD$  in the ratio  $5 : 3$ . Show that  $O$ ,  $E$  and  $F$  are collinear, and find  $OE : EF$ .

**Solution****Part (a)**

Let  $\mathbf{a} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 6 \\ d \\ -\sqrt{7} \end{pmatrix}$ . Let  $\theta$  be the angle between  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

$$\Rightarrow 3 \cdot 6 + (-2) \cdot d + 0 \cdot (-\sqrt{7}) = \sqrt{3^2 + (-2)^2 + 0^2} \cdot \sqrt{6^2 + d^2 + (-\sqrt{7})^2} \cdot \cos \arccos \frac{6}{13}$$

$$\begin{aligned} \Rightarrow 18 - 2d &= \sqrt{43 + d^2} \cdot \sqrt{13} \cdot \frac{6}{13} \\ &= \sqrt{43 + d^2} \cdot \frac{6}{\sqrt{13}} \end{aligned}$$

$$\Rightarrow (18 - 2d)^2 = \frac{36}{13} (43 + d^2)$$

$$\Rightarrow (9 - d)^2 = \frac{9}{13} (43 + d^2)$$

$$\Rightarrow 13(81 - 18d + d^2) = 387 + 9d^2$$

$$\Rightarrow 1053 - 234d + 13d^2 = 387 + 9d^2$$

$$\Rightarrow 4d^2 - 234d + 666 = 0$$

$$\Rightarrow 2d^2 - 117d + 333 = 0$$

**Part (b)****Subpart (i)**

Note that  $\overrightarrow{DA} = \overrightarrow{OA} - \overrightarrow{OD} = \mathbf{a} - 4\mathbf{b}$  and  $\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = \mathbf{b} - 6\mathbf{a}$ . Since  $E$  is on both  $DA$  and  $CB$ , we have

$$\overrightarrow{OE} = \overrightarrow{OD} + \lambda \overrightarrow{DA} = \overrightarrow{OC} + \mu \overrightarrow{CB}$$

for some scalars  $\lambda$  and  $\mu$ . Hence,

$$\begin{aligned} 4\mathbf{b} + \lambda(\mathbf{a} - 4\mathbf{b}) &= 6\mathbf{a} + \mu(\mathbf{b} - 6\mathbf{a}) \\ \implies \lambda\mathbf{a} + 4(1 - \lambda)\mathbf{b} &= 6(1 - \mu)\mathbf{a} + \mu\mathbf{b} \end{aligned}$$

Comparing the coefficients of  $\mathbf{a}$  and  $\mathbf{b}$ , we have the system

$$\begin{cases} \lambda = 6(1 - \mu) \\ \mu = 4(1 - \lambda) \end{cases}$$

which has the solution  $\lambda = \frac{18}{23}$  and  $\mu = \frac{20}{23}$ . Hence,  $\overrightarrow{OE} = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}$ .

$$\boxed{\overrightarrow{OE} = \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b}}$$

**Subpart (ii)**

By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OF} &= \frac{3\mathbf{c} + 5\mathbf{d}}{5 + 3} \\ &= \frac{1}{8}(5 \cdot 4\mathbf{b} + 3 \cdot 6\mathbf{a}) \\ &= \frac{1}{8}(18\mathbf{a} + 20\mathbf{b}) \\ &= \frac{23}{8} \left( \frac{18}{23}\mathbf{a} + \frac{20}{23}\mathbf{b} \right) \\ &= \frac{23}{8}\overrightarrow{OE} \end{aligned}$$

$$\boxed{OE : OF = 8 : 23}$$

**Problem 11.**

Relative to the origin  $O$ , two points  $A$  and  $B$  have position vectors given by  $\mathbf{a} = 14\mathbf{i} + 14\mathbf{j} + 14\mathbf{k}$  and  $\mathbf{b} = 11\mathbf{i} - 13\mathbf{j} + 2\mathbf{k}$  respectively.

- The point  $P$  divides the line  $AB$  in the ratio  $2 : 1$ . Find the coordinates of  $P$ .
- Show that  $AB$  and  $OP$  are perpendicular.
- The vector  $\mathbf{c}$  is a unit vector in the direction of  $\overrightarrow{OP}$ . Write  $\mathbf{c}$  as a column vector and give the geometrical meaning of  $|\mathbf{a} \cdot \mathbf{c}|$ .
- Find  $\mathbf{a} \times \mathbf{p}$ , where  $\mathbf{p}$  is the vector  $\overrightarrow{OP}$ , and give the geometrical meaning of  $|\mathbf{a} \times \mathbf{p}|$ . Hence, write down the area of triangle  $OAP$ .

**Solution****Part (a)**

We have  $\mathbf{a} = \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} = 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix}$ . By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OP} &= \frac{\mathbf{a} + 2\mathbf{b}}{2 + 1} \\ &= \frac{1}{3} \left[ \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} + 2 \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} \right] \\ &= \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \end{aligned}$$

$$\boxed{P(12, -4, 6)}$$

**Part (b)**

Consider  $\overrightarrow{AB} \cdot \overrightarrow{OP}$ .

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{OP} &= \left[ \begin{pmatrix} 11 \\ -13 \\ 2 \end{pmatrix} - \begin{pmatrix} 14 \\ 14 \\ 14 \end{pmatrix} \right] \cdot \begin{pmatrix} 12 \\ -4 \\ 6 \end{pmatrix} \\ &= -3 \begin{pmatrix} 1 \\ 9 \\ 4 \end{pmatrix} \cdot 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\ &= -6(1 \cdot 6 + 9 \cdot (-2) + 4 \cdot 3) \\ &= 0 \end{aligned}$$

Since  $\overrightarrow{AB} \cdot \overrightarrow{OP} = 0$ ,  $AB$  and  $OP$  must be perpendicular.

**Part (c)**

$$\begin{aligned}
\mathbf{c} &= \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|} \\
&= \frac{1}{2\sqrt{6^2 + (-2)^2 + 3^2}} \cdot 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\
&= \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{c} = \frac{1}{7} \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}}$$

$|\mathbf{a} \cdot \mathbf{c}|$  is the length of the projection of  $\mathbf{a}$  on  $\overrightarrow{OP}$ .

**Part (d)**

$$\begin{aligned}
\mathbf{a} \times \mathbf{p} &= 14 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times 2 \begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix} \\
&= 28 \begin{pmatrix} 1 \cdot 3 - (-2) \cdot 1 \\ 1 \cdot 6 - 3 \cdot 1 \\ 1 \cdot -2 - 6 \cdot 1 \end{pmatrix} \\
&= 28 \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix}
\end{aligned}$$

$$\boxed{\mathbf{a} \times \mathbf{p} = \begin{pmatrix} 5 \\ 3 \\ -8 \end{pmatrix}}$$

$|\mathbf{a} \times \mathbf{p}|$  is twice the area of  $\triangle OAP$ .

$$\begin{aligned}
\text{Area } \triangle OAP &= \frac{1}{2} |\mathbf{a} \times \mathbf{p}| \\
&= \frac{1}{2} \cdot 28 \sqrt{5^2 + 3^2 + (-8)^2} \\
&= 14 \sqrt{98} \\
&= 14 \cdot 7\sqrt{2} \\
&= 98\sqrt{2}
\end{aligned}$$

$$\boxed{\text{Area } \triangle OAP = 98\sqrt{2} \text{ units}^2}$$



**Problem 12.**

The points  $A$ ,  $B$  and  $C$  have position vectors given by  $\mathbf{i} - \mathbf{j} + \mathbf{k}$ ,  $\mathbf{j} - \mathbf{k}$  and  $2\mathbf{i} - \mathbf{j} - \mathbf{k}$  respectively.

- Find the area of the triangle  $ABC$ . Hence, find the sine of the angle  $BAC$ .
- Find a vector perpendicular to the plane  $ABC$ .
- Find the projection vector of  $\overrightarrow{AC}$  onto  $\overrightarrow{AB}$ .
- Find the distance of  $C$  to  $AB$ .

**Solution**

We have  $\mathbf{a} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ,  $\mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  and  $\mathbf{c} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$ . For simplicity, consider the translation that sends  $A$  to the origin. We thus have  $\mathbf{a}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{b}' = \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$  and  $\mathbf{c}' = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ .

**Part (a)**

$$\begin{aligned}
 |\mathbf{b}' \times \mathbf{c}'| &= \left| \begin{pmatrix} 2 \cdot (-2) - 0 \cdot (-2) \\ -2 \cdot 1 - (-1) \cdot (-2) \\ -1 \cdot 0 - 1 \cdot 2 \end{pmatrix} \right| \\
 &= \left| \begin{pmatrix} -4 \\ -4 \\ -2 \end{pmatrix} \right| \\
 &= \left| -2 \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right| \\
 &= 2 \cdot \sqrt{2^2 + 2^2 + 1^2} \\
 &= 6
 \end{aligned}$$

Hence, the area of  $\triangle ABC = \frac{1}{2} |\mathbf{b}' \times \mathbf{c}'| = 3 \text{ units}^2$ .

Area  $\triangle ABC = 3 \text{ units}^2$

$$\begin{aligned}
 |\mathbf{b}' \times \mathbf{c}'| &= |\mathbf{b}'| |\mathbf{c}'| \sin BAC \\
 \implies \sin BAC &= \frac{6}{\sqrt{(-1)^2 + 2^2 + (-2)^2} \sqrt{1^2 + 0^2 + (-2)^2}} \\
 &= \frac{6}{3\sqrt{5}} \\
 &= \frac{2}{\sqrt{5}}
 \end{aligned}$$

$$\sin BAC = \frac{2}{\sqrt{5}}$$

**Part (b)**

$$\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

**Part (c)**

Note that  $\hat{\mathbf{b}}' = \frac{\mathbf{b}'}{|\mathbf{b}'|} = \frac{1}{\sqrt{(-1)^2 + 2^2 + (-2)^2}} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ . Hence,

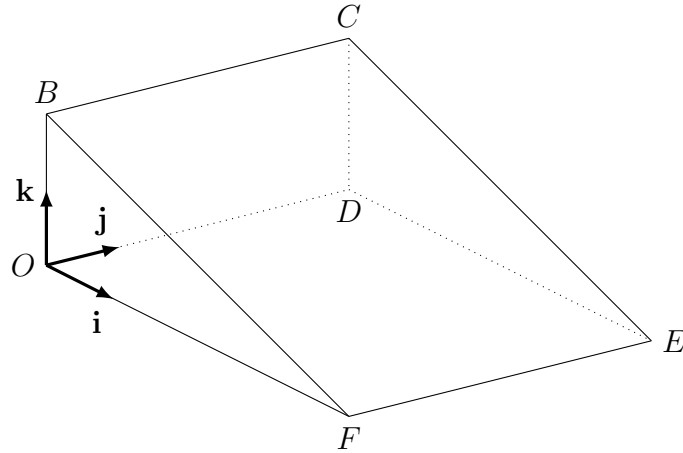
$$\begin{aligned} (\mathbf{c}' \cdot \hat{\mathbf{b}}') \hat{\mathbf{b}}' &= \left[ \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \right] \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{9} (1 \cdot (-1) + 0 \cdot 2 + (-2) \cdot (-2)) \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \end{aligned}$$

$$\frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$$

**Part (d)**

$$\begin{aligned} |\mathbf{c}' \times \hat{\mathbf{b}}'| &= |\hat{\mathbf{b}}' \times \mathbf{c}'| \\ &= \left| \frac{1}{3} \mathbf{b}' \times \mathbf{c}' \right| \\ &= \frac{1}{3} \cdot 6 \\ &= 2 \end{aligned}$$

The distance between  $C$  and  $AB$  is 2 units.

**Problem 13.**

The diagram shows a vehicle ramp  $OBCDEF$  with horizontal rectangular base  $ODEF$  and vertical rectangular face  $OBCD$ . Taking the point  $O$  as the origin, the perpendicular unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are parallel to the edges  $OF$ ,  $OD$  and  $OB$  respectively. The lengths of  $OF$ ,  $OD$  and  $OB$  are  $2h$  units, 3 units and  $h$  units respectively.

- (a) Show that  $\overrightarrow{OC} = 3\mathbf{j} + h\mathbf{k}$ .
- (b) The point  $P$  divides the segment  $CF$  in the ratio  $2 : 1$ . Find  $\overrightarrow{OP}$  in terms of  $h$ .

For parts (c) and (d), let  $h = 1$ .

- (c) Find the length of projection of  $\overrightarrow{OP}$  onto  $\overrightarrow{OC}$ .
- (d) Using the scalar product, find the angle that the rectangular face  $BCEF$  makes with the horizontal base.

**Solution****Part (a)**

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{OD} + \overrightarrow{DC} \\ &= \overrightarrow{OD} + \overrightarrow{OB} \\ &= 3\mathbf{j} + h\mathbf{k}\end{aligned}$$

**Part (b)**

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{1\overrightarrow{OC} + 2\overrightarrow{OF}}{2 + 1} \\ &= \frac{1}{3} \left[ \begin{pmatrix} 0 \\ 3 \\ h \end{pmatrix} + 2 \begin{pmatrix} 2h \\ 0 \\ 0 \end{pmatrix} \right] \\ &= \frac{1}{3} \begin{pmatrix} 4h \\ 3 \\ h \end{pmatrix}\end{aligned}$$

$$\overrightarrow{OP} = \frac{1}{3} \begin{pmatrix} 4h \\ 3 \\ h \end{pmatrix}$$

**Part (c)**

$$\begin{aligned} |\overrightarrow{OP} \cdot \hat{\mathbf{c}}| &= \left| \frac{1}{3} \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \frac{1}{\sqrt{0^2 + 3^2 + 1^2}} \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right| \\ &= \frac{1}{3\sqrt{10}} \left| \begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right| \\ &= \frac{1}{3\sqrt{10}} |4 \cdot 0 + 3 \cdot 3 + 1 \cdot 1| \\ &= \frac{10}{3\sqrt{10}} \\ &= \frac{\sqrt{10}}{3} \end{aligned}$$

$$\frac{\sqrt{10}}{3} \text{ units}$$

**Part (d)**

Let  $\theta$  be the angle the rectangular face  $BCEF$  makes with the horizontal base.

$$\begin{aligned} \overrightarrow{OF} \cdot \overrightarrow{BF} &= |\overrightarrow{OF}| |\overrightarrow{BF}| \cos \theta \\ \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \left[ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] &= 2 \cdot \left| \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right| \cdot \cos \theta \\ \Rightarrow \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} &= 2 \cdot \left| \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \right| \cdot \cos \theta \\ \Rightarrow 2 \cdot 2 + 0 \cdot 0 + 0 \cdot (-1) &= 2 \cdot \sqrt{2^2 + 0^2 + (-1)^2} \cdot \cos \theta \\ \Rightarrow 4 &= 2\sqrt{5} \cos \theta \\ \Rightarrow \cos \theta &= \frac{2}{\sqrt{5}} \\ \Rightarrow \theta &= \arccos\left(\frac{2}{\sqrt{5}}\right) \\ &= 26.6^\circ \text{ (1 d.p.)} \end{aligned}$$

$$26.6^\circ$$

**Problem 14.**

The position vectors of the points  $A$  and  $B$  relative to the origin  $O$  are  $\overrightarrow{OA} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$  and  $\overrightarrow{OB} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$  respectively. The point  $P$  on  $AB$  is such that  $AP : PB = \lambda : 1 - \lambda$ . Show that  $\overrightarrow{OP} = (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}$  where  $\lambda$  is a real parameter.

- (a) Find the value of  $\lambda$  for which  $OP$  is perpendicular to  $AB$ .  
 (b) Find the value of  $\lambda$  for which angles  $\angle AOP$  and  $\angle POB$  are equal.

**Solution**

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{OP} &= \frac{\lambda \overrightarrow{OB} + (1 - \lambda) \overrightarrow{OA}}{\lambda + (1 - \lambda)} \\ &= \lambda \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2\lambda \\ -3\lambda \\ 6\lambda \end{pmatrix} + \begin{pmatrix} 1 - \lambda \\ 2 - 2\lambda \\ -2 + 2\lambda \end{pmatrix} \\ &= \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \\ &= (1 + \lambda)\mathbf{i} + (2 - 5\lambda)\mathbf{j} + (-2 + 8\lambda)\mathbf{k}\end{aligned}$$

**Part (a)**

For  $OP$  to be perpendicular to  $AB$ , we must have  $\overrightarrow{OP} \cdot \overrightarrow{AB} = 0$ .

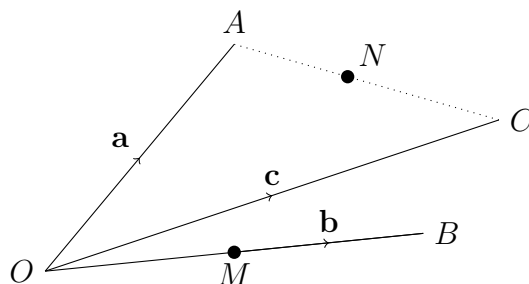
$$\begin{aligned}\overrightarrow{OP} \cdot \overrightarrow{AB} &= 0 \\ \Rightarrow \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \left[ \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \right] &= 0 \\ \Rightarrow \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -5 \\ 8 \end{pmatrix} &= 0 \\ \Rightarrow (1 + \lambda) \cdot 1 + (2 - 5\lambda) \cdot (-5) + (-2 + 8\lambda) \cdot 8 &= 0 \\ \Rightarrow -25 + 90\lambda &= 0 \\ \Rightarrow \lambda &= \frac{5}{18}\end{aligned}$$

$$\boxed{\lambda = \frac{5}{18}}$$

**Part (b)**

$$\begin{aligned}
& \angle AOP = \angle POB \\
\Rightarrow & \cos \angle AOP = \cos \angle POB \\
\Rightarrow & \frac{\vec{OP} \cdot \vec{OA}}{|\vec{OP}| |\vec{OA}|} = \frac{\vec{OP} \cdot \vec{OB}}{|\vec{OP}| |\vec{OB}|} \\
\Rightarrow & |\vec{OB}| (\vec{OP} \cdot \vec{OA}) = |\vec{OA}| (\vec{OP} \cdot \vec{OB}) \\
\Rightarrow & \vec{OP} \cdot (|\vec{OB}| \vec{OA}) = \vec{OP} \cdot (|\vec{OA}| \vec{OB}) \\
\Rightarrow & \vec{OP} \cdot (|\vec{OB}| \vec{OA}) - \vec{OP} \cdot (|\vec{OA}| \vec{OB}) = 0 \\
\Rightarrow & \vec{OP} \cdot (|\vec{OB}| \vec{OA} - |\vec{OA}| \vec{OB}) = 0 \\
\Rightarrow & \vec{OP} \cdot (\sqrt{2^2 + (-3)^2 + 6^2} \vec{OA} - \sqrt{1^2 + 2^2 + (-2)^2} \vec{OB}) = 0 \\
\Rightarrow & \vec{OP} \cdot (7\vec{OA} - 3\vec{OB}) = 0 \\
\Rightarrow & \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \left[ 7 \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} \right] = 0 \\
\Rightarrow & \begin{pmatrix} 1 + \lambda \\ 2 - 5\lambda \\ -2 + 8\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 23 \\ -32 \end{pmatrix} = 0 \\
\Rightarrow & (1 + \lambda) \cdot 1 + (2 - 5\lambda) \cdot 23 + (-2 + 8\lambda) \cdot (-32) = 0 \\
\Rightarrow & 111 - 370\lambda = 0 \\
\Rightarrow & \lambda = \frac{3}{10}
\end{aligned}$$

$$\boxed{\lambda = \frac{3}{10}}$$

**Problem 15.**

The origin  $O$  and the points  $A$ ,  $B$  and  $C$  lie in the same plane, where  $\overrightarrow{OA} = \mathbf{a}$ ,  $\overrightarrow{OB} = \mathbf{b}$  and  $\overrightarrow{OC} = \mathbf{c}$ ,

- (a) Explain why  $\mathbf{c}$  can be expressed as  $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ , for constants  $\lambda$  and  $\mu$ .

The point  $N$  is on  $AC$  such that  $AN : NC = 3 : 4$ .

- (b) Write down the position vector of  $N$  in terms of  $\mathbf{a}$  and  $\mathbf{c}$ .
- (c) It is given that the area of triangle  $ONC$  is equal to the area of triangle  $OMC$ , where  $M$  is the mid-point of  $OB$ . By finding the areas of these triangles in terms of  $\mathbf{a}$  and  $\mathbf{b}$ , find  $\lambda$  in terms of  $\mu$  in the case where  $\lambda$  and  $\mu$  are both positive.

**Solution****Part (a)**

Since  $\mathbf{a}$ ,  $\mathbf{b}$  and  $\mathbf{c}$  are co-planar and  $\mathbf{a}$  is not parallel to  $\mathbf{b}$ ,  $\mathbf{c}$  can be written as a linear combination of  $\mathbf{a}$  and  $\mathbf{b}$ .

**Part (b)**

By the Ratio Theorem,

$$\begin{aligned}\overrightarrow{ON} &= \frac{4\mathbf{a} + 3\mathbf{c}}{3 + 4} \\ &= \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}\end{aligned}$$

$$\boxed{\overrightarrow{ON} = \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c}}$$

**Part (c)**

Since  $M$  is the mid-point of  $OB$ , we have that  $M = \frac{1}{2}\mathbf{b}$ . Hence,

$$\begin{aligned}\text{Area } \triangle ONC &= \text{Area } \triangle OMC \\ \Rightarrow \frac{1}{2} |\overrightarrow{ON} \times \hat{\mathbf{c}}| &= \frac{1}{2} |\overrightarrow{OM} \times \hat{\mathbf{c}}| \\ \Rightarrow \left| \left( \frac{4}{7}\mathbf{a} + \frac{3}{7}\mathbf{c} \right) \times \hat{\mathbf{c}} \right| &= \left| \frac{1}{2}\mathbf{b} \times \hat{\mathbf{c}} \right|\end{aligned}$$

$$\begin{aligned}\Rightarrow & \frac{4}{7} |\mathbf{a} \times \hat{\mathbf{c}}| = \frac{1}{2} |\mathbf{b} \times \hat{\mathbf{c}}| \\ \Rightarrow & \frac{4}{7} \left| \mathbf{a} \times \frac{\lambda \mathbf{a} + \mu \mathbf{b}}{|\lambda \mathbf{a} + \mu \mathbf{b}|} \right| = \frac{1}{2} \left| \mathbf{b} \times \frac{\lambda \mathbf{a} + \mu \mathbf{b}}{|\lambda \mathbf{a} + \mu \mathbf{b}|} \right| \\ \Rightarrow & \frac{4}{7} |\mathbf{a} \times (\lambda \mathbf{a} + \mu \mathbf{b})| = \frac{1}{2} |\mathbf{b} \times (\lambda \mathbf{a} + \mu \mathbf{b})| \\ \Rightarrow & \frac{4}{7} |\mathbf{a} \times \mu \mathbf{b}| = \frac{1}{2} |\mathbf{b} \times \lambda \mathbf{a}| \\ \Rightarrow & \frac{4}{7} \mu |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \lambda |\mathbf{b} \times \mathbf{a}| \\ \Rightarrow & \frac{4}{7} \mu |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} \lambda |\mathbf{a} \times \mathbf{b}| \\ \Rightarrow & \left( \frac{4}{7} \mu - \frac{1}{2} \lambda \right) |\mathbf{a} \times \mathbf{b}| = 0 \\ \Rightarrow & \frac{4}{7} \mu - \frac{1}{2} \lambda = 0 \\ \Rightarrow & \lambda = \frac{8}{7} \mu\end{aligned}$$