

Problem 1.

Given that $y = 1$ when $x = 1$, find the particular solution of the differential equation $\frac{dy}{dx} = \frac{y^2}{x}$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^2}{x} \\ \Rightarrow \frac{1}{y^2} \frac{dy}{dx} &= \frac{1}{x} \\ \Rightarrow \int \frac{1}{y^2} \frac{dy}{dx} dx &= \int \frac{1}{x} dx \\ \Rightarrow \int \frac{1}{y^2} dy &= \int \frac{1}{x} dx \\ \Rightarrow -\frac{1}{y} &= \ln|x| + C_1 \\ \Rightarrow y &= -\frac{1}{\ln|x| + C_1} \\ &= \frac{1}{C - \ln|x|}\end{aligned}$$

$$\text{At } x = 1, y = 1, 1 = \frac{1}{C - \ln|1|} \Rightarrow C = 1.$$

$$\boxed{y = \frac{1}{1 - \ln|x|}}$$

Problem 2.

Two variables x and t are connected by the differential equation $\frac{dx}{dt} = \frac{kx}{10-x}$, where $0 < x < 10$ and where k is a constant. It is given that $x = 1$ when $t = 0$ and that $x = 2$ when $t = 1$. Find the value of t when $x = 5$, given your answer to three s.f.

Solution

$$\begin{aligned}\frac{dx}{dt} &= \frac{kx}{10-x} \\ \implies \frac{10-x}{x} \frac{dx}{dt} &= k \\ \implies \int \frac{10-x}{x} \frac{dx}{dt} dt &= k \int dt \\ \implies \int \frac{10-x}{x} dx &= k \int dt \\ \implies \int \left(\frac{10}{x} - 1 \right) dx &= kt + C \\ \implies 10 \ln x - x &= kt + C\end{aligned}$$

At $t = 0$, $-x = 1$, $C = -1$.

At $t = 1$, $x = 2$, $10 \ln 2 - 2 = k - 1 \implies k = \ln 2 - 1$.

When $x = 5$, $10 \ln 5 - 5 = (\ln 2 - 1)t - 1 \implies t = \frac{10 \ln 5 - 4}{10 \ln 2 - 1} = 2.04$ (3 s.f.).

When $x = 5$, $t = 2.04$

Problem 3.

Use the substitution $y = u - 2x$ to find the general solution of the differential equation $\frac{dy}{dx} = -\frac{8x + 4y + 1}{4x + 2y + 1}$.

Solution

Note that $y = u - 2x \implies u = y + 2x$. Also, $\frac{dy}{dx} = \frac{du}{dx} - 2$.

$$\begin{aligned}
 & \frac{dy}{dx} = -\frac{8x + 4y + 1}{4x + 2y + 1} \\
 \implies & \frac{du}{dx} - 2 = -\frac{8x + 4(u - 2x) + 1}{4x + 2(u - 2x) + 1} \\
 & = -\frac{4u + 1}{2u + 1} \\
 \implies & \frac{du}{dx} = 2 - \frac{4u + 1}{2u + 1} \\
 & = \frac{4u + 2}{2u + 1} - \frac{4u + 1}{2u + 1} \\
 & = \frac{1}{2u + 1} \\
 \implies & (2u + 1) \frac{du}{dx} = 1 \\
 \implies & \int (2u + 1) \frac{du}{dx} dx = \int dx \\
 \implies & \int (2u + 1) du = \int dx \\
 \implies & u^2 + u = x + C \\
 \implies & (y + 2x)^2 + (y + 2x) = x + C \\
 \implies & (y + 2x)^2 + y + x = C
 \end{aligned}$$

$$\boxed{(y + 2x)^2 + y + x = C}$$

Problem 4.

By using the substitution $z = ye^{2x}$, find the general solution of the differential equation $\frac{dy}{dx} + 2y = xe^{-2x}$.

Find the particular solution of the differential equation given that $\frac{dy}{dx} = 1$ when $x = 0$.

Solution

Note that $z = ye^{2x} \implies \frac{dz}{dx} = \frac{dy}{dx}e^{2x} + 2ye^{2x} = \frac{dy}{dx}e^{2x} + 2z$. Hence, $\frac{dy}{dx} = \frac{dz}{dx}e^{-2x} - 2y$.

$$\begin{aligned} \frac{dy}{dx} + 2y &= xe^{-2x} \\ \implies \frac{dz}{dx}e^{-2x} - 2y + 2y &= xe^{-2x} \\ \implies \frac{dz}{dx} &= x \\ \implies z &= \frac{x^2}{2} + C \\ \implies ye^{2x} &= \frac{x^2}{2} + C \\ \implies y &= \frac{x^2}{2e^{2x}} + \frac{C}{e^{2x}} \end{aligned}$$

$$\boxed{y = \frac{x^2}{2e^{2x}} + \frac{C}{e^{2x}}}$$

At $x = 0$, $\frac{dy}{dx} = 1$. Hence, $1 + 2y = 0 \implies y = -\frac{1}{2}$. Thus, $C = -\frac{1}{2}$.

$$\boxed{y = \frac{x^2 - 1}{2e^{2x}}}$$

Problem 5.

Find the general solution of the differential equation $\frac{dy}{dx} = 6xy^3$.

Find its particular solution given that $y = 0.5$ when $x = 0$.

Determine the interval of validity for the particular solution.

Solution

$$\begin{aligned}\frac{dy}{dx} &= 6xy^3 \\ \Rightarrow \frac{1}{y^3} \frac{dy}{dx} &= 6x \\ \Rightarrow \int \frac{1}{y^3} \frac{dy}{dx} dx &= 6 \int x dx \\ \Rightarrow \int \frac{1}{y^3} dy &= 6 \int x dx \\ \Rightarrow -\frac{1}{2} \frac{1}{y^2} &= 3x^2 + C_1 \\ \Rightarrow \frac{1}{y^2} &= C - 6x^2 \\ \Rightarrow y^2 &= \frac{1}{C - 6x^2}\end{aligned}$$

$$y^2 = \frac{1}{C - 6x^2}$$

At $x = 0$, $y = 0.5$, $\frac{1}{4} = \frac{1}{C} \Rightarrow C = 4$.

$$y^2 = \frac{1}{4 - 6x^2}$$

Observe that we require $4 - 6x^2 > 0$, whence $-\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}$.

The interval of validity is $\left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$.

Problem 6.

- (a) Find the general solution of the differential equation $\frac{dy}{dx} = \frac{3x}{x^2 + 1}$.
- (b) What can you say about the gradient of every solution as $x \rightarrow \pm\infty$?
- (c) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$. Hence sketch the graph of this solution.

Solution**Part (a)**

$$\begin{aligned}\frac{dy}{dx} &= \frac{3x}{x^2 + 1} \\ &= \frac{3}{2} \frac{2x}{x^2 + 1} \\ \Rightarrow y &= \frac{3}{2} \ln(x^2 + 1) + C\end{aligned}$$

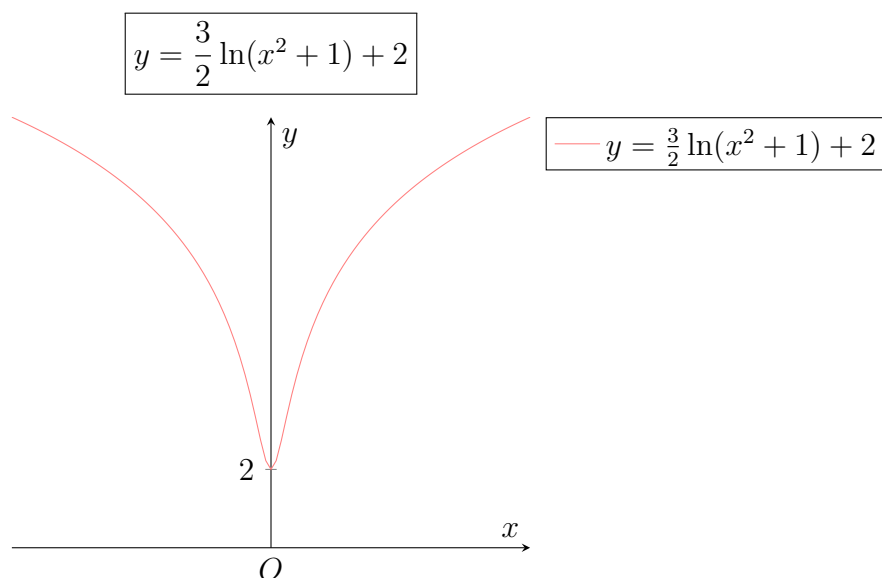
$$y = \frac{3}{2} \ln(x^2 + 1) + C$$

Part (b)

As $x \rightarrow \pm\infty$, $\frac{3x}{x^2 + 1} \rightarrow 0^+$. Hence, the gradient of every solution approaches 0 from above.

Part (c)

When $x = 0$ and $y = 2$, $C = 2$.



Problem 7.

The variables x , y and z are connected by the following differential equations.

$$\frac{dz}{dx} = 3 - 2z \quad (7.1)$$

$$\frac{dy}{dx} = z \quad (7.2)$$

(a) Given that $z < \frac{3}{2}$, solve equation 7.1 to find z in terms of x .

(b) Hence find y in terms of x .

(c) Use the result in part (b) to show that

$$\frac{d^2y}{dx^2} = a \frac{dy}{dx} + b$$

for constants a and b to be determined.

(d) The curve of the solution in part (b) passes through the points $(0, 1)$ and $(2, 3 + e^{-4})$. Sketch this curve, indicating its axial intercept and asymptote (if any).

Solution**Part (a)**

$$\begin{aligned} & \frac{dz}{dx} = 3 - 2z \\ \Rightarrow & \frac{1}{3 - 2z} \frac{dz}{dx} = 1 \\ \Rightarrow & \int \frac{1}{3 - 2z} \frac{dz}{dx} dx = \int dx \\ \Rightarrow & \int \frac{1}{3 - 2z} dz = \int dx \\ \Rightarrow & -\frac{1}{2} \ln(3 - 2z) = x + C_1 \\ \Rightarrow & \ln(3 - 2z) = C_2 - 2x \\ \Rightarrow & 3 - 2z = C_3 e^{-2x} \\ \Rightarrow & z = \frac{3}{2} - C_4 e^{-2x} \end{aligned}$$

$$\boxed{z = \frac{3}{2} - A e^{-2x}, A > 0}$$

Part (b)

$$\begin{aligned} & \frac{dy}{dx} = \frac{3}{2} - C_4 e^{-2x} \\ \Rightarrow & y = \int \left(\frac{3}{2} - C_4 e^{-2x} \right) dx \\ & = \frac{3}{2}x - \frac{C_4}{2} e^{-2x} + C_6 \end{aligned}$$

$$y = \frac{3}{2}x - \frac{A}{2}e^{-2x} + B$$

Part (c)

$$\begin{aligned}\frac{dy}{dx} &= \frac{3}{2} + C_4 e^{-2x} \\ \Rightarrow \frac{d^2y}{dx^2} &= -2C_4 e^{-2x} \\ &= -2\left(\frac{dy}{dx} - \frac{3}{2}\right) \\ &= -2\frac{dy}{dx} + 3\end{aligned}$$

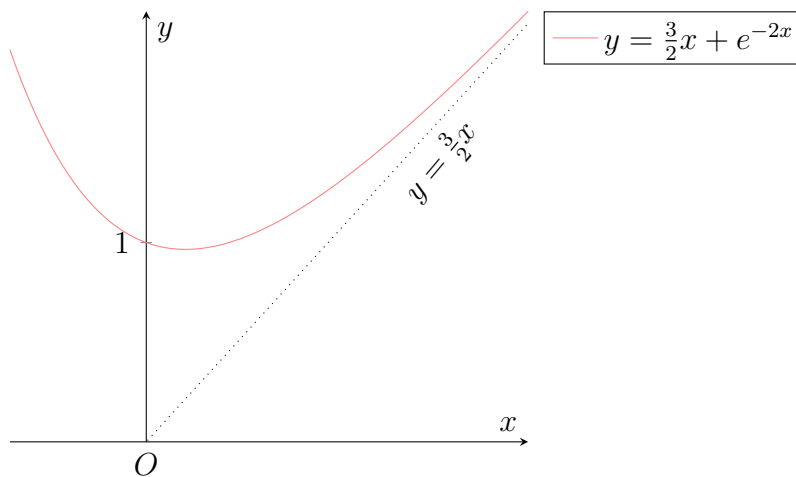
$$a = -2, b = 3$$

Part (d)

At $(0, 1)$, we obtain $1 = -\frac{A}{2} + B$.

At $(2, 3 + e^{-4})$, we obtain $3 + e^{-4} = 3 - \frac{A}{2}e^{-4} + B \Rightarrow 1 = -\frac{A}{2} + Be^4$.

Hence, $B = Be^4$, whence $B = 0$ and $A = -2$. The curve thus has equation $y = \frac{3}{2}x + e^{-2x}$.



Problem 8.

A bottle containing liquid is taken from a refrigerator and placed in a room where the temperature is a constant 20°C . As the liquid warms up, the rate of increase of its temperature $\theta^\circ\text{C}$ after time t minutes is proportional to the temperature difference $(20 - \theta)^\circ\text{C}$. Initially the temperature of the liquid is 10°C and the rate of increase of the temperature is 1°C per minute. By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-t/10}$.

Find the time it takes the liquid to reach a temperature of 15°C , and state what happens to θ for large values of t . Sketch a graph of θ against t .

Solution

Since $\frac{d\theta}{dt} \propto (20 - \theta)$, we have $\frac{d\theta}{dt} = k(20 - \theta)$, where k is a constant. We now solve for θ .

$$\begin{aligned} \frac{d\theta}{dt} &= k(20 - \theta) \\ \Rightarrow \frac{1}{20 - \theta} \frac{d\theta}{dt} &= k \\ \Rightarrow \int \frac{1}{20 - \theta} \frac{d\theta}{dt} dt &= k \int dt \\ \Rightarrow \int \frac{1}{20 - \theta} d\theta &= k \int dt \\ \Rightarrow -\ln(20 - \theta) &= kt + C_1 \\ \Rightarrow \ln(20 - \theta) &= C_2 - kt \\ \Rightarrow 20 - \theta &= Ce^{-kt} \\ \Rightarrow \theta &= 20 - Ce^{-kt} \end{aligned}$$

At $t = 0$, $\theta = 10$. Hence, $10 = 20 - C \Rightarrow C = 10$. We also have $\left. \frac{d\theta}{dt} \right|_0 = 1$. Hence,

$$1 = k[20 - (20 - 10e^0)] = 10k \Rightarrow k = \frac{1}{10}. \text{ Thus,}$$

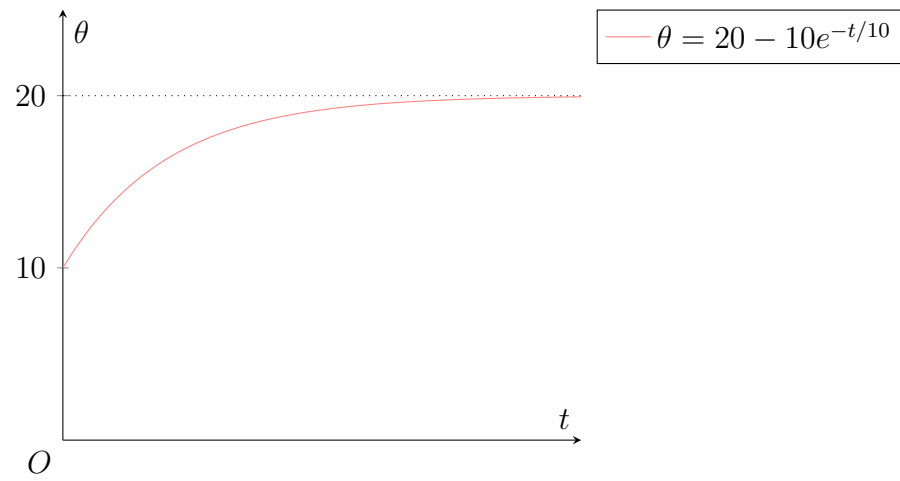
$$\theta = 20 - 10e^{-t/10}.$$

Consider $\theta = 15$.

$$\begin{aligned} 15 &= 20 - 10e^{-t/10} \\ \Rightarrow e^{-t/10} &= \frac{1}{2} \\ \Rightarrow -\frac{t}{10} &= -\ln 2 \\ \Rightarrow t &= 10 \ln 2 \end{aligned}$$

It takes $10 \ln 2$ minutes for the liquid to reach a temperature of 15°C .

As $t \rightarrow \infty$, $\theta \rightarrow 20$.



Problem 9.

(a) Find $\int \frac{1}{100 - v^2} dv$.

- (b) A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and
- t
- seconds later its speed
- v
- metres per second satisfies the differential equation

$$\frac{dv}{dt} = 10 - 0.1v^2.$$

- (i) Find t in terms of v . Hence find the exact time the stone takes to reach a speed of 5 metres per second.
- (ii) Find the speed of the stone after 1 second.
- (iii) What happens to the speed of the stone for large values of t ?

Solution**Part (a)**

$$\begin{aligned} \int \frac{1}{100 - v^2} dv &= \frac{1}{2(10)} \ln \left(\frac{10 + v}{10 - v} \right) + C \\ &= \frac{1}{20} \ln \left(\frac{10 + v}{10 - v} \right) + C \end{aligned}$$

$$\boxed{\int \frac{1}{100 - v^2} dv = \frac{1}{20} \ln \left(\frac{10 + v}{10 - v} \right) + C}$$

Part (b)**Subpart (i)**

$$\begin{aligned} \frac{dv}{dt} &= 10 - 0.1v^2 \\ &= \frac{1}{10}(100 - v^2) \\ \Rightarrow \frac{1}{100 - v^2} \frac{dv}{dt} &= \frac{1}{10} \\ \Rightarrow \int \frac{1}{100 - v^2} \frac{dv}{dt} dt &= \frac{1}{10} \int dt \\ \Rightarrow \int \frac{1}{100 - v^2} dv &= \frac{1}{10} \int dt \\ \Rightarrow \frac{1}{20} \ln \left(\frac{10 + v}{10 - v} \right) + C &= \frac{1}{10} t \\ \Rightarrow t &= \frac{1}{2} \ln \left(\frac{10 + v}{10 - v} \right) + C \end{aligned}$$

At $t = 0$, $v = 0$. Hence, $C = 0$.

$$t = \frac{1}{2} \ln \left(\frac{10+v}{10-v} \right)$$

Subpart (ii)

Consider $t = 1$.

$$\begin{aligned} \frac{1}{2} \ln \left(\frac{10+v}{10-v} \right) &= 1 \\ \implies \ln \left(\frac{10+v}{10-v} \right) &= 2 \\ \implies \frac{10+v}{10-v} &= e^2 \\ \implies 10+v &= e^2(10-v) \\ \implies v(1+e^2) &= 10(e^2-1) \\ \implies v &= \frac{10(e^2-1)}{e^2+1} \end{aligned}$$

After 1 second, the speed of the stone is $\frac{10(e^2-1)}{e^2+1}$ m/s.

Subpart (iii)

As $t \rightarrow \infty$, we have $\ln \left(\frac{10+v}{10-v} \right) \rightarrow \infty \implies \frac{10+v}{10-v} \rightarrow \infty$. Thus, $v \rightarrow 10^-$.

For large values of t , the speed of the stone approaches 10 m/s.

Problem 10.

Two scientists are investigating the change of a certain population of an animal species of size n thousand at time t years. It is known that due to its inability to reproduce effectively, the species is unable to replace itself in the long run.

- (a) One scientist suggests that n and t are related by the differential equation $\frac{d^2n}{dt^2} = 10 - 6t$. Given that $n = 100$ when $t = 0$, show that the general solution of this differential equation is $n = 5t^2 - t^3 + Ct + 100$, where C is a constant. Sketch the solution curve of the particular solution when $C = 0$, stating the axial intercepts clearly.
- (b) The other scientist suggests that n and t are related by the differential equation $\frac{dn}{dt} = 3 - 0.02n$. Find n in terms of t , given again that $n = 100$ when $t = 0$. Explain in simple terms what will eventually happen to the population using this model.

Which is a more appropriate model in modeling the population of the animal species?

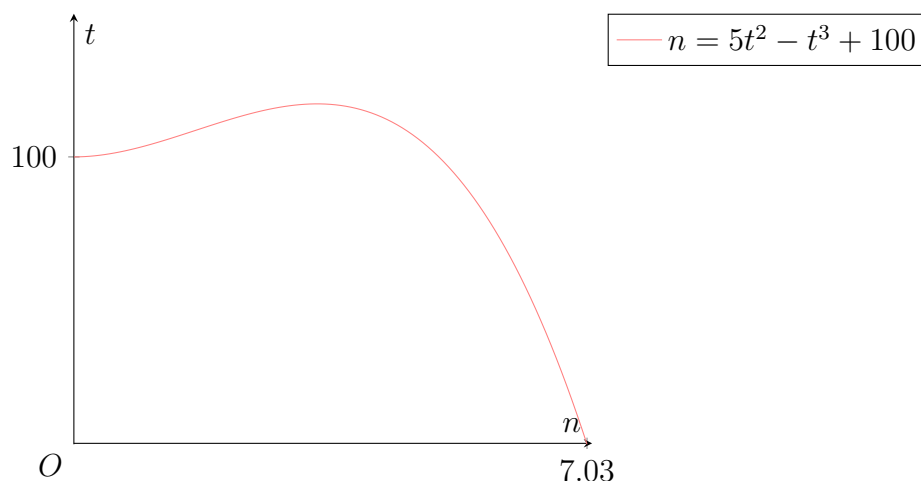
Solution**Part (a)**

$$\begin{aligned}\frac{d^2n}{dt^2} &= 10 - 6t \\ \Rightarrow \frac{dn}{dt} &= \int (10 - 6t) dt \\ &= 10t - 3t^2 + C \\ \Rightarrow n &= \int (10t - 3t^2 + C) dt \\ &= 5t^2 - t^3 + Ct + C'\end{aligned}$$

When $t = 0$ and $n = 100$, we have $C' = 100$. Thus,

$$n = 5t^2 - t^3 + Ct + 100.$$

When $C = 0$, $n = 5t^2 - t^3 + 100$.



Part (b)

$$\begin{aligned}
\frac{dn}{dt} &= 3 - 0.02n \\
&= \frac{150 - n}{50} \\
\Rightarrow \frac{1}{150 - n} \frac{dn}{dt} &= \frac{1}{50} \\
\Rightarrow \int \frac{1}{150 - n} \frac{dn}{dt} dt &= \frac{1}{50} \int dt \\
\Rightarrow \int \frac{1}{150 - n} dn &= \frac{1}{50} \int dt \\
\Rightarrow -\ln(150 - n) &= \frac{1}{50}t + C_1 \\
\Rightarrow \ln(150 - n) &= C_2 - \frac{1}{50}t \\
\Rightarrow 150 - n &= Ce^{-t/50} \\
\Rightarrow n &= 150 - Ce^{-t/50}
\end{aligned}$$

When $t = 0$ and $n = 100$, we have $C = 50$.

$$\boxed{n = 150 - 50e^{-t/50}}$$

As $t \rightarrow \infty$, $n \rightarrow 150$. Hence, the population will decrease before plateauing at 150 thousand.

The first model is more appropriate, as it account for the fact that the species will eventually go extinct ($n = 0$) due to the fact that they cannot replace itself in the long run.

Problem 11.

A rectangular tank has a horizontal base. Water is flowing into the tank at a constant rate, and flows out at a rate which is proportional to the depth of water in the tank. At time t seconds, the depth of water in the tank is x metres. If the depth is 0.5 m, it remains at this constant value. Show that $\frac{dx}{dt} = -k(2x - 1)$, where k is a positive constant. When $t = 0$, the depth of water in the tank is 0.75 m and is decreasing at a rate of 0.01 m s^{-1} . Find the time at which the depth of water is 0.55 m.

Solution

Let $V_i \text{ m}^3/\text{s}$ be the rate at which water is flowing into the tank. Note that $V_i \geq 0$. Let the rate at which water is flowing out of the tank be $V_o x \text{ m}^3/\text{s}$. Let the base of the container be $A \text{ m}^2$. Then $dx/dt = 1/A \cdot (V_i - V_o x)$. At $x = 0.5$, the volume of water in the tank is constant, i.e. $dx/dt|_{0.5} = 0$. This gives $V_i - 0.5V_o = 0$, whence $V_o = 2V_i$. Thus, $dx/dt = 1/A \cdot (V_i - 2V_i x) = -V_i/A \cdot (2x - 1)$. Letting $k = V_i/A$, we have

$$\frac{dx}{dt} = -k(2x - 1)$$

as desired.

We now solve for t .

$$\begin{aligned} \frac{dx}{dt} &= -k(2x - 1) \\ \Rightarrow \frac{1}{2x - 1} \frac{dx}{dt} &= -k \\ \Rightarrow \int \frac{1}{2x - 1} \frac{dx}{dt} dt &= -k \int dt \\ \Rightarrow \int \frac{1}{2x - 1} dx &= -k \int dt \\ \Rightarrow \frac{1}{2} \ln(2x - 1) + C_1 &= -kt \\ \Rightarrow \ln(2x - 1) + C_2 &= -2kt \\ \Rightarrow t &= -\frac{1}{2k}(\ln(2x - 1) + C_2) \end{aligned}$$

At $t = 0$, we have $x = 0.75$. This gives $0 = \ln(2 \cdot 0.75 - 1) + C_2$, whence $C_2 = \ln 2$. We also have $dx/dt|_0 = -0.01$. We thus obtain $-0.01 = -k(2 \cdot 0.75 - 1)$, whence $k = 0.02$. Thus,

$$t = -\frac{1}{0.04}(\ln(2x - 1) + \ln 2) = -25 \ln(4x - 2).$$

Hence, when $x = 0.55$, we have $t = -25 \ln 0.2 = 25 \ln 5$.

The depth of the water is 0.55 m when $t = 25 \ln 5 \text{ s}$.

Problem 12.

In a model of mortgage repayment, the sum of money owed to the Building Society is denoted by x and the time is denoted by t . Both x and t are taken to be continuous variables. The sum of money owed to the Building Society increases, due to interest, at a rate proportional to the sum of money owed. Money is also repaid at a constant rate r .

When $x = a$, interest and repayment balance. Show that, for $x > 0$, $\frac{dx}{dt} = \frac{r}{a}(x - a)$.

Given that, when $t = 0$, $x = A$, find x in terms of t , r , a and A .

On a single, clearly labelled sketch, show the graph of x against t in the two cases:

- (a) $A > a$.
- (b) $A < a$.

State the circumstances under which the loan is repaid in a finite time T and show that, in this case, $T = \frac{a}{r} \ln \frac{a}{a - A}$.

Solution

Let the rate at which money is owed to the Building Society be kx . Then $\frac{dx}{dt} = kx - r$. At $x = a$, interest and repayment balance, i.e. $dx/dt|_a = 0$. This gives $ka - r = 0 \implies k = r/a$. Thus,

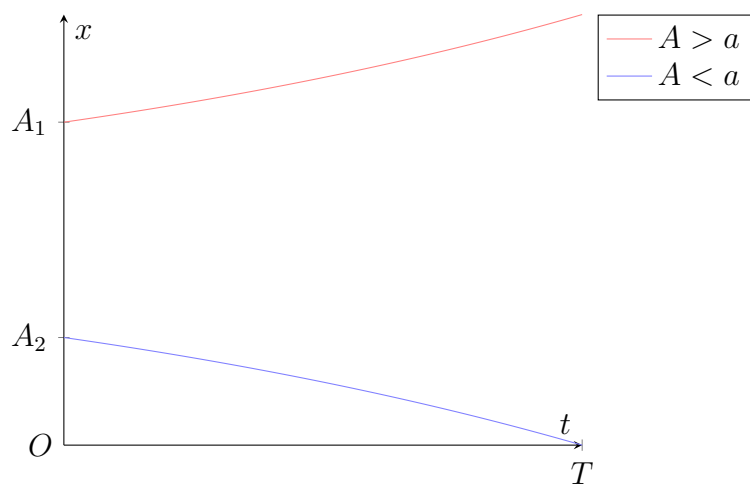
$$\frac{dx}{dt} = \frac{r}{a}x - r = \frac{r}{a}(x - a).$$

We now solve for x .

$$\begin{aligned} \frac{dx}{dt} &= \frac{r}{a}(x - a) \\ \implies \frac{1}{x - a} \frac{dx}{dt} &= \frac{r}{a} \\ \implies \int \frac{1}{x - a} \frac{dx}{dt} dt &= \frac{r}{a} \int dt \\ \implies \int \frac{1}{x - a} dx &= \frac{r}{a} \int dt \\ \implies \ln |x - a| &= \frac{r}{a}t + C_1 \\ \implies x - a &= Ce^{rt/a} \\ \implies x &= Ce^{rt/a} + a \end{aligned}$$

When $t = 0$, we have $x = A$. This gives $A = C + a$, whence $C = A - a$. Thus,

$$\boxed{x = (A - a)e^{rt/a} + a}$$



For the loan to be repaid in finite time, $A < a$. At time T , the loan has been repaid, i.e. $x = 0$. Note that $C_1 = \ln C = \ln(A - a)$. Hence,

$$\begin{aligned}
 & \frac{r}{a}T + \ln(A - a) = \ln(0 - a) \\
 \Rightarrow & \quad \frac{r}{a}T = \ln a - \ln(A - a) \\
 & \quad \quad \quad = \ln \frac{a}{A - a} \\
 \Rightarrow & \quad T = \frac{a}{r} \ln \frac{a}{A - a}
 \end{aligned}$$