

**Problem 1.**

For each of the following, write down a vector equivalent of the line  $l$  and convert it to parametric and Cartesian forms.

- (a)  $l$  passes through the point with position vector  $-\mathbf{i} + \mathbf{k}$  and is parallel to the vector  $\mathbf{i} + \mathbf{j}$ .
- (b)  $l$  passes through the points  $P(1, -1, 3)$  and  $Q(2, 1, -2)$ .
- (c)  $l$  passes through the origin and is parallel to the line  $m : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$ .
- (d)  $l$  is the  $x$ -axis.
- (e)  $l$  passes through the point  $C(4, -1, 2)$  and is parallel to the  $z$ -axis.

**Solution****Part (a)**

Form	Expression
<b>Vector</b>	$\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$
<b>Parametric</b>	$\begin{cases} x = \lambda - 1 \\ y = \lambda \\ z = 1 \end{cases}$
<b>Cartesian</b>	$x + 1 = y, z = 1$

**Part (b)**

Form	Expression
<b>Vector</b>	$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -5 \end{pmatrix}, \lambda \in \mathbb{R}$
<b>Parametric</b>	$\begin{cases} x = \lambda + 1 \\ y = 2\lambda - 1 \\ z = -5\lambda + 3 \end{cases}$
<b>Cartesian</b>	$x - 1 = \frac{y + 1}{2} = \frac{3 - z}{5}$

**Part (c)**

Form	Expression
Vector	$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda \\ y = 2\lambda \\ z = 3\lambda \end{cases}$
Cartesian	$x = \frac{y}{2} = \frac{z}{3}$

**Part (d)**

Form	Expression
Vector	$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda \\ y = 0 \\ z = 0 \end{cases}$
Cartesian	$x \in \mathbb{R}, y = 0, z = 0$

**Part (e)**

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = 4 \\ y = -1 \\ z = \lambda + 2 \end{cases}$
Cartesian	$x = 4, y = -1, z \in \mathbb{R}$

**Problem 2.**

For each of the following, determine if  $l_1$  and  $l_2$  are parallel, intersecting or skew. In the case of intersecting lines, find the position vector of the point of intersection. In addition, find the acute angle between the lines  $l_1$  and  $l_2$ .

(a)  $l_1 : x - 1 = -y = z - 2$  and  $l_2 : \frac{x-2}{2} = -\frac{y+1}{2} = \frac{z-4}{2}$

(b)  $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}, \alpha \in \mathbb{R}$  and  $l_2 : \mathbf{r} = \begin{pmatrix} 0 \\ 10 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$

(c)  $l_1 : \mathbf{r} = (\mathbf{i} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k}), \lambda \in \mathbb{R}$  and  $l_2 : \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} - 4\mathbf{j} - \mathbf{k}), \mu \in \mathbb{R}$

**Solution****Part (a)**

Note that  $l_1$  and  $l_2$  have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}, \mu \in \mathbb{R}$$

Since  $\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ,  $l_1$  and  $l_2$  are parallel. Since  $\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}$  for all real  $\mu$ ,  $l_1$  and  $l_2$  are distinct.

Distinct parallel lines.  $\theta = 0$ .

**Part (b)**

Since  $\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} \neq \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix}$  for all real  $\beta$ ,  $l_1$  and  $l_2$  are not parallel.

Consider  $l_1 = l_2$ .

$$\begin{aligned} l_1 &= l_2 \\ \Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 10 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \\ \Rightarrow \alpha \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} - \beta \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 10 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 4\alpha - 3\beta = -1 \\ -2\alpha - 8\beta = 10 \\ -3\alpha - \beta = 1 \end{cases}$$

There are no solutions to the above system. Hence,  $l_1$  and  $l_2$  do not intersect and are hence skew.

Let  $\theta$  be the acute angle between  $l_1$  and  $l_2$ .

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix} \right| \left| \begin{pmatrix} 3 \\ 8 \\ 1 \end{pmatrix} \right|} \\ &= \frac{7}{\sqrt{2146}} \\ \implies \theta &= 81.3^\circ \text{ (1 d.p.)}\end{aligned}$$

Skew lines.  $\theta = 81.3^\circ$ .

### Part (c)

Note that  $l_1$  and  $l_2$  have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$$

Since  $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \neq \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix}$  for all real  $\mu$ ,  $l_1$  and  $l_2$  are not parallel.

Consider  $l_1 = l_2$ .

$$\begin{aligned}l_1 &= l_2 \\ \implies \begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \\ \implies \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \mu \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \\ 6 \end{pmatrix}\end{aligned}$$

This gives the following system:

$$\begin{cases} -5\mu + \lambda = 0 \\ 4\mu - \lambda = -1 \\ \mu + \lambda = 6 \end{cases}$$

The above system has the unique solution  $\lambda = 5$  and  $\mu = 1$ . Hence,  $l_1$  and  $l_2$  intersect at

$$\begin{pmatrix} 1 \\ 0 \\ -5 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ 0 \end{pmatrix}.$$

Let  $\theta$  be the acute angle between  $l_1$  and  $l_2$ .

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} 5 \\ -4 \\ -1 \end{pmatrix} \right|} \\ &= \frac{8}{3\sqrt{14}} \\ \implies \quad \theta &= 44.5^\circ \text{ (1 d.p.)}\end{aligned}$$

Intersecting lines. $\begin{pmatrix} 6 \\ -5 \\ 0 \end{pmatrix}$ . $\theta = 44.5^\circ$ (1 d.p.).
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**Problem 3.**

- (a) Find the shortest distance from the point  $(1, 2, 3)$  to the line with equation  $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$ ,  $\lambda \in \mathbb{R}$ .
- (b) Find the length of projection of  $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$  onto the line with equation  $\frac{x+5}{4} = \frac{y-5}{3} = 10-2z$ .
- (c) Find the projection of  $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$  onto the line with equation  $\frac{x+5}{4} = \frac{y-5}{3} = 10-2z$ .

**Solution****Part (a)**

Let  $\vec{OP} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$  and  $\vec{OA} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ . We have that  $A$  is on the line with equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

Note that  $\vec{AP} = \vec{OP} - \vec{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$ .

$$\begin{aligned} \text{Shortest distance} &= \left| \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| / \left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \left| \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \right| \\ &= \frac{\sqrt{2^2 + (-3)^2 + (-4)^2}}{3} \\ &= \frac{\sqrt{29}}{3} \end{aligned}$$

The shortest distance is  $\frac{\sqrt{29}}{3}$  units.

**Part (b)**

Note that the line has vector form

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 3 \\ -1/2 \end{pmatrix} \\ &= \begin{pmatrix} -5 \\ 5 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\text{Length of projection} &= \left| \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| / \left| \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| \\ &= \frac{4}{\sqrt{101}}\end{aligned}$$

The length of projection is  $\frac{4}{\sqrt{101}}$  units.

**Part (c)**

$$\begin{aligned}\text{Projection} &= \left[ \begin{pmatrix} 4 \\ -5 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} / \left| \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| \right] \cdot \left[ \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} / \left| \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix} \right| \right] \\ &= \frac{-4}{101} \begin{pmatrix} 8 \\ 6 \\ -1 \end{pmatrix}\end{aligned}$$

**Problem 4.**

The points  $P$  and  $Q$  have coordinates  $(0, -1, -1)$  and  $(3, 0, 1)$  respectively, and the equations of the lines  $l_1$  and  $l_2$  are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}$$

- (a) Show that  $P$  lies on  $l_1$  but not on  $l_2$ .
- (b) Determine if  $l_2$  passes through  $Q$ .
- (c) Find the coordinates of the foot of the perpendicular from  $P$  to  $l_2$ . Hence, or otherwise, find the perpendicular distance from  $P$  to  $l_2$ .
- (d) Find the length of projection of  $\overrightarrow{PQ}$  onto  $l_2$ .

**Solution**

We have that  $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$  and  $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ .

**Part (a)**

When  $\lambda = -2$ , we have  $\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \overrightarrow{OP}$ . Hence,  $P$  lies on  $l_1$ .

Observe that all points on  $l_2$  have a  $z$ -coordinate of 1. Since  $P$  has a  $z$ -coordinate of  $-1$ ,  $P$  does not lie on  $l_2$ .

**Part (b)**

When  $\mu = 3$ , we have  $\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \overrightarrow{OQ}$ . Hence,  $l_2$  passes through  $Q$ .

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**Part (c)**

Let the foot of the perpendicular from  $P$  to  $l_2$  be  $F$ . Since  $F$  is on  $l_2$ , we have that  $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$  for some real  $\mu$ . We also have that  $\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0$ .



$$\begin{aligned}
& \overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & (\overrightarrow{OF} - \overrightarrow{OP}) \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & \left[ \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & \left[ \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\Rightarrow & -10 + 5m = 0 \\
\Rightarrow & m = 2
\end{aligned}$$

Hence,  $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

$$\boxed{F(1, 1, 1)}$$

$$\begin{aligned}
\text{Perpendicular distance} &= \left| \overrightarrow{PF} \right| \\
&= \left| \overrightarrow{OF} - \overrightarrow{OP} \right| \\
&= \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \right| \\
&= \left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| \\
&= \sqrt{1^2 + 2^2 + 2^2} \\
&= 3
\end{aligned}$$

The perpendicular distance from  $P$  to  $l_2$  is 3 units.

**Part (d)**

Note that  $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$ .

$$\begin{aligned}\text{Length of projection} &= \left| \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right| / \left| \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \right| \\ &= \frac{|6 - 1 + 0|}{\sqrt{2^2 + (-1)^2 + 0^2}} \\ &= \frac{5}{\sqrt{5}} \\ &= \sqrt{5}\end{aligned}$$

The length of projection of  $\overrightarrow{PQ}$  onto  $l_2$  is  $\sqrt{5}$  units.

**Problem 5.**

The lines  $l_1$  and  $l_2$  have equations

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{r} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

respectively. Find the position vectors of the points  $P$  on  $l_1$  and  $Q$  on  $l_2$  such that  $O$ ,  $P$  and  $Q$  are collinear, where  $O$  is the origin.

**Solution**

We have that  $\overrightarrow{OP} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$  and  $\overrightarrow{OQ} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$  for some reals  $s$  and  $t$ .

For  $O$ ,  $P$  and  $Q$  to be collinear, we need  $\overrightarrow{OP} = \lambda \overrightarrow{OQ}$  for some real  $\lambda$ .

$$\begin{aligned} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} &= \lambda \left[ \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \\ \Rightarrow \begin{pmatrix} s \\ 1 \\ 2 + 3s \end{pmatrix} &= \lambda \begin{pmatrix} -2 + 2t \\ 3 + t \\ 1 \end{pmatrix} \end{aligned}$$

This gives us the system:

$$\begin{cases} s = \lambda(-2 + 2t) \\ 1 = \lambda(3 + t) \\ 2 + 3s = \lambda \end{cases}$$

Substituting the third equation into the first two gives the reduced system:

$$\begin{cases} s = (2 + 3s)(-2 + 2t) \\ 1 = (2 + 3s)(3 + t) \end{cases}$$

Subtracting twice of the second equation from the first yields

$$\begin{aligned} s - 2 &= (2 + 3s)(-2 + 2t) - 2(2 + 3s)(3 + t) \\ &= (2 + 3s)(-2 + 2t) - (2 + 3s)(6 + 2t) \\ &= (2 + 3s)(-2 + 2t - (6 + 2t)) \\ &= -8(2 + 3s) \\ &= -16 - 24s \\ \Rightarrow 25s &= -14 \\ \Rightarrow s &= -\frac{14}{25} \end{aligned}$$

It quickly follows that  $t = \frac{1}{8}$ . Hence,

$$\begin{aligned}\overrightarrow{OP} &= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{14}{25} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} 0 \\ 25 \\ 50 \end{pmatrix} - \frac{1}{25} \begin{pmatrix} 14 \\ 0 \\ 42 \end{pmatrix} \\ &= \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix} \\ \overrightarrow{OQ} &= \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} -16 \\ 24 \\ 8 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}, \overrightarrow{OQ} = \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}}$$

**Problem 6.**

Relative to the origin  $O$ , the points  $A$ ,  $B$  and  $C$  have position vectors  $5\mathbf{i} + 4\mathbf{j} + 10\mathbf{k}$ ,  $-4\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$  and  $-5\mathbf{i} + 9\mathbf{j} + 5\mathbf{k}$  respectively.

- Find the Cartesian equation of the line  $AB$ .
- Find the length of projection of  $\overrightarrow{AC}$  onto the line  $AB$ . Hence, find the perpendicular distance from  $C$  to the line  $AB$ .
- Find the position vector of the foot  $N$  of the perpendicular from  $C$  to the line  $AB$ .
- The point  $D$  is such that it is a reflection of point  $C$  about the line  $AB$ . Find the position vector of  $D$ .

**Solution**

We have that  $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix}$ ,  $\overrightarrow{OB} = \begin{pmatrix} -4 \\ 4 \\ -2 \end{pmatrix}$  and  $\overrightarrow{OC} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix}$ .

**Part (a)**

Note that  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -4 \\ 4 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = -3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ . The line  $AB$  hence has the vector form

$$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

The line  $AB$  thus has the Cartesian form

$$\boxed{\frac{x-5}{3} = \frac{z-10}{4}, y=4}$$

**Part (b)**

Note that  $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} = \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} = -5 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ .

$$\begin{aligned} \text{Length of projection} &= \left| \overrightarrow{AC} \cdot \overrightarrow{AB} \right| / \left| \overrightarrow{AB} \right| \\ &= \frac{1}{-3\sqrt{3^2 + 0^2 + 4^2}} \left| -5 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot -3 \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right| \\ &= 10 \end{aligned}$$

The perpendicular distance from  $C$  to the line  $AB$  is 10 units.

**Part (c)**

Let  $\overrightarrow{AN} = \lambda \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix}$  for some real  $\lambda$  such that  $|\overrightarrow{AN}| = 10$ .

$$\begin{aligned} |\overrightarrow{AN}| &= 10 \\ \implies \lambda \cdot \sqrt{3^2 + 0^2 + 4^2} &= 10 \\ \implies \lambda &= \frac{2}{3} \end{aligned}$$

Hence,  $\overrightarrow{AN} = \frac{2}{3} \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix}$ . Thus,  $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$ .

$$\boxed{\overrightarrow{ON} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}}$$

**Part (d)**

Note that  $\overrightarrow{NC} = \overrightarrow{OC} - \overrightarrow{ON} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}$ . Since  $D$  is the reflection of  $C$  about  $AB$ , we have that  $\overrightarrow{ND} = -\overrightarrow{NC}$ .

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{ON} + \overrightarrow{ND} \\ &= \overrightarrow{ON} - \overrightarrow{NC} \\ &= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{OD} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}}$$

**Problem 7.**

The points  $A$  and  $B$  have coordinates  $(0, 9, c)$  and  $(d, 5, -2)$  respectively, where  $c$  and  $d$  are constants. The line  $l$  has equation  $\frac{x+3}{-1} = \frac{y-1}{4} = \frac{z-5}{3}$ .

- (a) Given that  $d = \frac{22}{7}$  and the line  $AB$  intersects  $l$ , find the value of  $c$ . Find also the coordinates of the foot of the perpendicular from  $A$  to  $l$ .
- (b) Given instead that the lines  $AB$  and  $l$  are parallel, state the value of  $c$  and  $d$  and find the shortest distance between the lines  $AB$  and  $l$ .

**Solution**

We have that  $\vec{OA} = \begin{pmatrix} 0 \\ 9 \\ c \end{pmatrix}$  and  $\vec{OB} = \begin{pmatrix} d \\ 5 \\ -2 \end{pmatrix}$ . We also have that  $l$  is given by the vector

$$\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \text{ for } \lambda \in \mathbb{R}.$$

Note that  $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix}$ . Hence, the line  $AB$  is given by the vector

$$\mathbf{r}_{AB} = \begin{pmatrix} d \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix} \text{ for } \mu \in \mathbb{R}.$$

**Part (a)**

Consider the direction vectors of  $AB$  and  $l$ . Since  $\begin{pmatrix} 22/7 \\ -4 \\ -2-c \end{pmatrix} \neq \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$  for all real  $\lambda$  and  $c$ , the lines  $AB$  and  $l$  are not parallel. Hence,  $AB$  and  $l$  intersect at only one point. Thus, there must be a unique solution to  $\mathbf{r} = \mathbf{r}_{AB}$ .

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_{AB} \\ \Rightarrow \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= \begin{pmatrix} 22/7 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 22/7 \\ -4 \\ -2-c \end{pmatrix} \\ \Rightarrow \begin{pmatrix} -21 \\ 7 \\ 35 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= \begin{pmatrix} 22 \\ 35 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} \\ \Rightarrow \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} &= \begin{pmatrix} 43 \\ 28 \\ -49 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} -\lambda - 22\mu = 43 \\ 4\lambda + 28\mu = 28 \\ 3\lambda + (14 + 7c)\mu = -49 \end{cases}$$

Solving the first two equations gives  $\lambda = \frac{91}{3}$  and  $\mu = -\frac{10}{3}$ . It follows from the third equation that  $c = 4$ .

$$\boxed{c = 4}$$

Let  $F$  be the foot of the perpendicular from  $A$  to  $l$ . We have that  $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$

for some real  $\lambda$ . We also have that  $\overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0$ .

$$\begin{aligned} & \overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & (\overrightarrow{OF} - \overrightarrow{OA}) \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & \left[ \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & \left[ \begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & \begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0 \\ \Rightarrow & -26 + 26\lambda = 0 \\ \Rightarrow & \lambda = 1 \end{aligned}$$

Hence,  $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix}$ .

The foot of the perpendicular from  $A$  to  $l$  has coordinates  $(-4, 5, 8)$ .



**Part (b)**

Given that  $AB$  is parallel to  $l$ , one of their direction vectors must be a scalar multiple of the other. Hence, for some real  $\lambda$ ,

$$\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} d \\ -4 \\ -2 - c \end{pmatrix}$$

It is obvious that  $\lambda = -1$ , whence  $c = 1$  and  $d = 1$ .

$$\boxed{c = 1, d = 1}$$

Note that the direction vector of  $l$  and  $AB$  is  $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ . Also note that  $(-3, 1, 5)$  is on  $l$  and  $(1, 5, -2)$  is on  $AB$ .

$$\begin{aligned} \text{Shortest distance between } AB \text{ and } l &= \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \left( \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \right) \right| / \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{(-1)^2 + 4^2 + 3^2}} \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{26}} \left| \begin{pmatrix} -40 \\ -5 \\ -20 \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{26}} \left| -5 \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} \right| \\ &= \frac{5\sqrt{8^2 + 1^2 + 4^2}}{\sqrt{26}} \\ &= \frac{45}{\sqrt{26}} \end{aligned}$$

The shortest distance between  $AB$  and  $l$  is  $\frac{45}{\sqrt{26}}$  units.

**Problem 8.**

The equation of the line  $L$  is  $\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$ ,  $t \in \mathbb{R}$ . The points  $A$  and  $B$  have position vectors  $\begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix}$  and  $\begin{pmatrix} 13 \\ 9 \\ \alpha \end{pmatrix}$  respectively. The line  $L$  intersects the line through  $A$  and  $B$  at  $P$ .

- (a) Find  $\alpha$  and the acute angle between line  $L$  and  $AB$ .

The point  $C$  has position vector  $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$  and the foot of the perpendicular from  $C$  to  $L$  is  $Q$ .

- (b) Find the position vector of  $Q$ . Hence, find the shortest distance from  $C$  to  $L$ .  
 (c) Find the position vector of the point of reflection of the point  $C$  about the line  $L$ . Hence, find the reflection of the line passing through  $C$  and the point  $(1, 3, 7)$  about the line  $L$ .

**Solution****Part (a)**

We have that  $\overrightarrow{OA} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix}$  and  $\overrightarrow{OB} = \begin{pmatrix} 13 \\ 9 \\ \alpha \end{pmatrix}$ . Hence,  $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix}$ .

The line  $AB$  is thus given by  $\mathbf{r}_{AB} = \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix}$  for  $u \in \mathbb{R}$ . Note that  $AB$  is not parallel to  $L$ . Hence,  $\overrightarrow{OP}$  is the only solution to the equation  $\mathbf{r} = \mathbf{r}_{AB}$ .

$$\begin{aligned} \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= \begin{pmatrix} 9 \\ 3 \\ 26 \end{pmatrix} + u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix} \\ \Rightarrow t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - u \begin{pmatrix} 4 \\ 6 \\ \alpha - 26 \end{pmatrix} &= \begin{pmatrix} 8 \\ 0 \\ 19 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 2t - 4u = 8 \\ -t - 6u = 0 \\ 5t - (\alpha - 26)u = 19 \end{cases}$$

Solving the first two equations gives  $t = 3$  and  $u = -\frac{1}{2}$ . It follows from the third equation that  $\alpha = 34$ .

$$\boxed{\alpha = 34}$$

Let the acute angle between  $L$  and  $AB$  be  $\theta$ .

$$\begin{aligned}\cos \theta &= \frac{\left| \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \right| \left| \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix} \right|} \\ &= \frac{42}{\sqrt{30}\sqrt{116}} \\ \implies \theta &= \arccos \frac{42}{\sqrt{30}\sqrt{116}} \\ &= 44.6^\circ \text{ (1 d.p.)}\end{aligned}$$

$$\boxed{\theta = 44.6^\circ \text{ (1 d.p.)}}$$

### Part (b)

Since  $Q$  is on  $L$ , we have that  $\overrightarrow{OQ} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix}$  for some real  $t$ . Further, since

$\overrightarrow{CQ} \perp L$ , we have that  $\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = 0$ .

$$\begin{aligned}\overrightarrow{CQ} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies (\overrightarrow{OQ} - \overrightarrow{OC}) \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies \left[ \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies \left[ \begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies \begin{pmatrix} -1 \\ -2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} &= 0 \\ \implies 30 + 30t &= 0 \\ \implies t &= 1\end{aligned}$$

$$\text{Hence, } \overrightarrow{OQ} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}.$$

$$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \text{Shortest distance from } C \text{ to } L &= |\overrightarrow{CQ}| \\ &= \left| \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \right| \\ &= \sqrt{(-3)^2 + (-1)^2 + 1^2} \\ &= \sqrt{11} \end{aligned}$$

The shortest distance from  $C$  to  $L$  is  $\sqrt{11}$  units.

### Part (c)

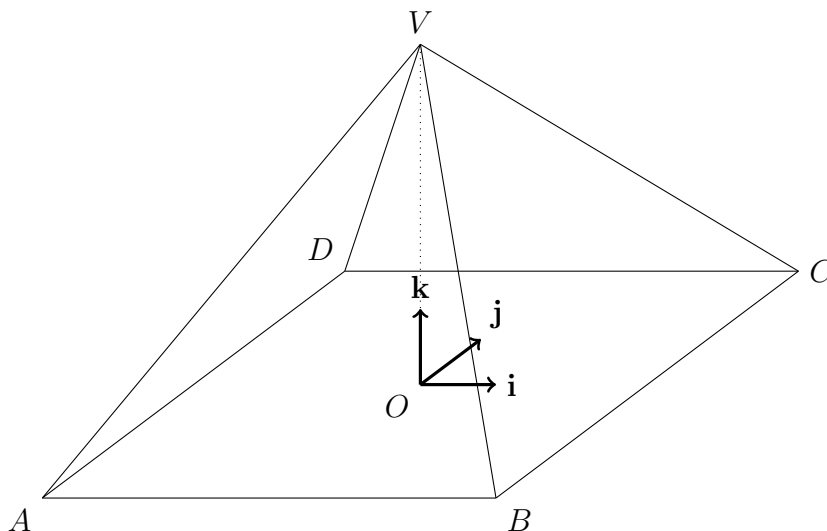
Let  $C'$  be the reflection of  $C$  about  $L$ .

$$\begin{aligned} \overrightarrow{OC'} &= \overrightarrow{OQ} - \overrightarrow{QC} \\ &= \overrightarrow{OQ} + \overrightarrow{CQ} \\ &= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} \end{aligned}$$

$$\overrightarrow{OC'} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}$$

Note that  $(1, 3, 7)$  is on  $L$  and is hence invariant under a reflection about  $L$ . Let the reflection about  $L$  of the line passing through  $C$  and  $(1, 3, 7)$  be  $L'$ . Since  $\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -4 \end{pmatrix} = -\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$ ,  $L'$  has direction vector  $\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$ . Thus,  $L'$  is given by  $\mathbf{r}' = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$  for  $\lambda \in \mathbb{R}$ .

$$L' : \mathbf{r}' = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$$

**Problem 9.**

In the diagram,  $O$  is the origin of the square base  $ABCD$  of a right pyramid with vertex  $V$ . The perpendicular unit vectors  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$  are parallel to  $AB$ ,  $AD$  and  $OV$  respectively. The length of  $AB$  is 4 units and the length of  $OV$  is  $2h$  units.  $P$ ,  $Q$ ,  $M$  and  $N$  are the mid-points of  $AB$ ,  $BC$ ,  $CV$  and  $VA$  respectively. The point  $O$  is taken as the origin for position vectors.

Show that the equation of the line  $PM$  may be expressed as  $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$ , where  $t$  is a parameter.

- Find an equation for the line  $QN$ .
- Show that the lines  $PM$  and  $QN$  intersect and that the position vector  $\overrightarrow{OX}$  of their point of intersection is  $\mathbf{r} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix}$ .
- Given that  $OX$  is perpendicular to  $VB$ , find the value of  $h$  and calculate the acute angle between  $PM$  and  $QN$ , giving your answer correct to the nearest  $0.1^\circ$ .

**Solution**

We are given that  $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$ ,  $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$  and  $\overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 2h \end{pmatrix}$ . Hence,  $\overrightarrow{CV} = \overrightarrow{OV} - \overrightarrow{OC} = \begin{pmatrix} -2 \\ -2 \\ 2h \end{pmatrix}$ . Thus,  $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CV} = \begin{pmatrix} -1 \\ -1 \\ h \end{pmatrix}$ . Since  $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}$ , we have that  $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$ . Thus,  $PM$  is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}, t \in \mathbb{R}$$

**Part (a)**

Since  $\overrightarrow{OM} = \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}$ , by symmetry,  $\overrightarrow{ON} = \begin{pmatrix} -1 \\ -1 \\ h \end{pmatrix}$ . Given that  $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ , we have that

$\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix}$ . Thus,  $QN$  is given by

$$\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix}, u \in \mathbb{R}$$

**Part (b)**

Consider  $PM = QN$ .

$$\begin{aligned} PM &= QN \\ \Rightarrow \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} &= \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} \\ \Rightarrow t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} - u \begin{pmatrix} -3 \\ -1 \\ h \end{pmatrix} &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} t + 3u = 2 \\ 3t + u = 2 \\ ht - hu = 0 \end{cases}$$

From the first two equations, we see that  $t = \frac{1}{2}$  and  $u = \frac{1}{2}$ , which is consistent with the third equation. Hence,  $\overrightarrow{OX} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix}$ .

**Part (c)**

Note that  $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}$ , whence  $\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \begin{pmatrix} 2 \\ -2 \\ -2h \end{pmatrix}$ . Since  $OX$  is perpendicular to  $VB$ , we have that  $\overrightarrow{OX} \cdot \overrightarrow{VB} = 0$ .

$$\begin{aligned} \overrightarrow{OX} \cdot \overrightarrow{VB} &= 0 \\ \implies \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} \cdot 2 \begin{pmatrix} 1 \\ -1 \\ -h \end{pmatrix} &= 0 \\ \implies 1 + 1 - h^2 &= 0 \\ \implies h^2 &= 2 \end{aligned}$$

We hence have that  $h = \sqrt{2}$ . Note that we reject  $h = -\sqrt{2}$  since  $h > 0$ .

$$\boxed{h = \sqrt{2}}$$

Let the acute angle between  $PM$  and  $QN$  be  $\theta$ .

$$\begin{aligned} \cos \theta &= \frac{|\overrightarrow{PM} \cdot \overrightarrow{QN}|}{|\overrightarrow{PM}| |\overrightarrow{QN}|} \\ &= \frac{1}{\sqrt{1^2 + 3^2 + \sqrt{2}^2}} \cdot \frac{1}{\sqrt{(-3)^2 + (-1)^2 + \sqrt{2}^2}} \cdot \left| \begin{pmatrix} 1 \\ 3 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ \sqrt{2} \end{pmatrix} \right| \\ &= \frac{1}{\sqrt{12}} \cdot \frac{1}{\sqrt{12}} \cdot |-3 - 3 + 2| \\ &= \frac{1}{3} \\ \implies \theta &= \arccos \frac{1}{3} \\ &= 70.5^\circ \text{ (1 d.p.)} \end{aligned}$$

$$\boxed{\theta = 70.5^\circ \text{ (1 d.p.)}}$$