

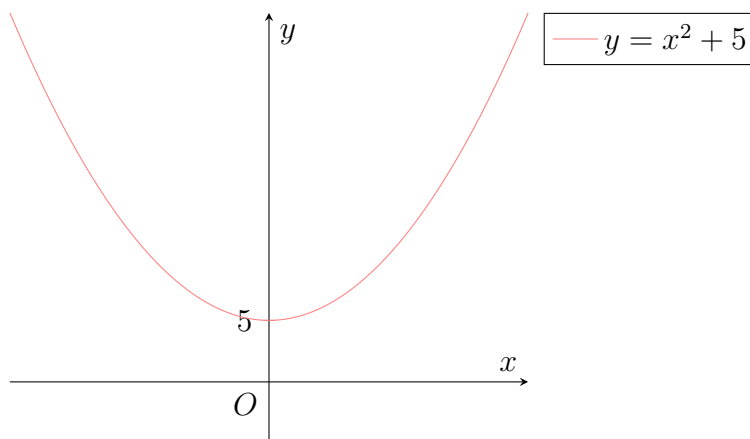
Problem 1.

Without using a calculator, sketch the following graphs and determine their symmetries.

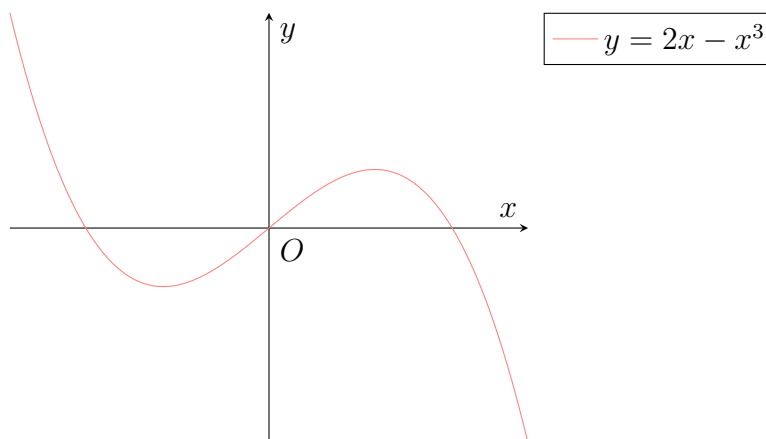
(a) $y = x^2 + 5$

(b) $y = 2x - x^3$

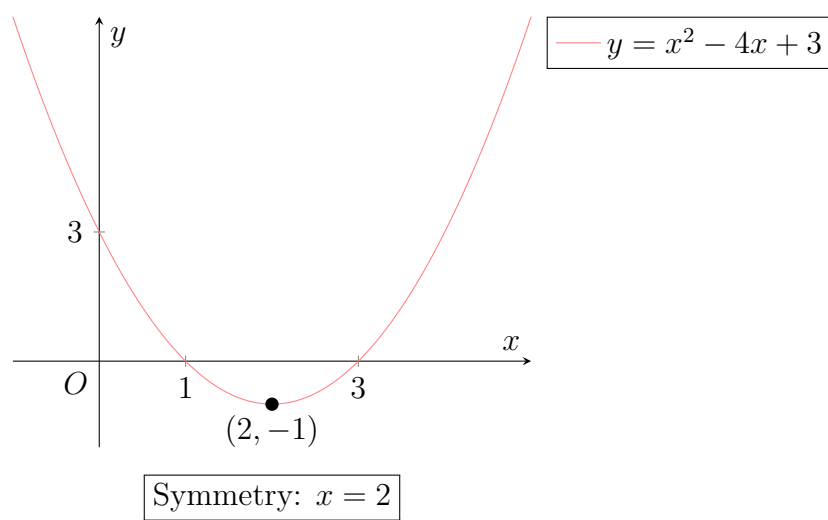
(c) $y = x^2 - 4x + 3$

Solution**Part (a)**

Symmetry: $x = 0$

Part (b)

Symmetry: $(0, 0)$

Part (c)

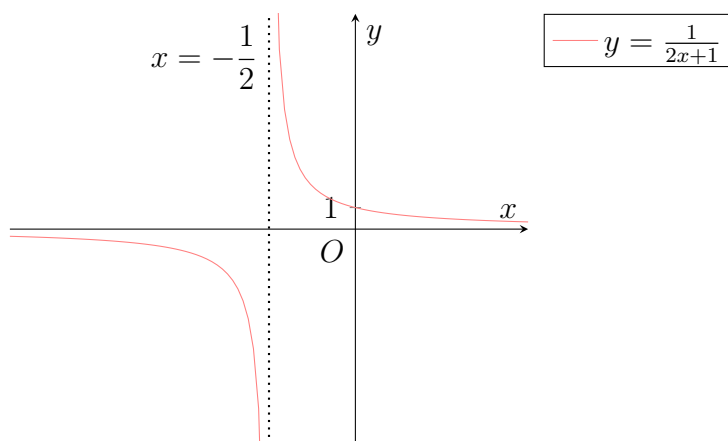
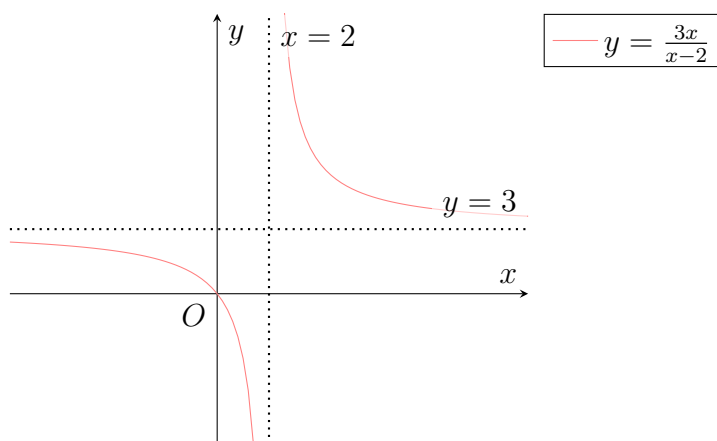
Problem 2.

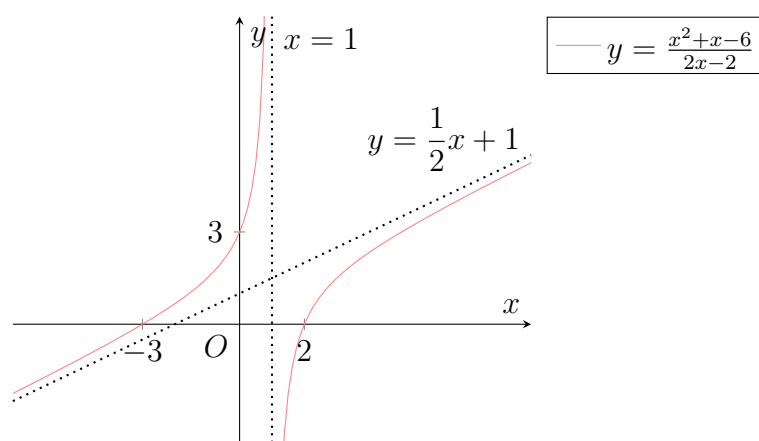
Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

(a) $y = \frac{1}{2x+1}$

(b) $y = \frac{3x}{x-2}$

(c) $y = \frac{x^2 + x - 6}{2x - 2}$

Solution**Part (a)****Part (b)**

Part (c)

Problem 3.

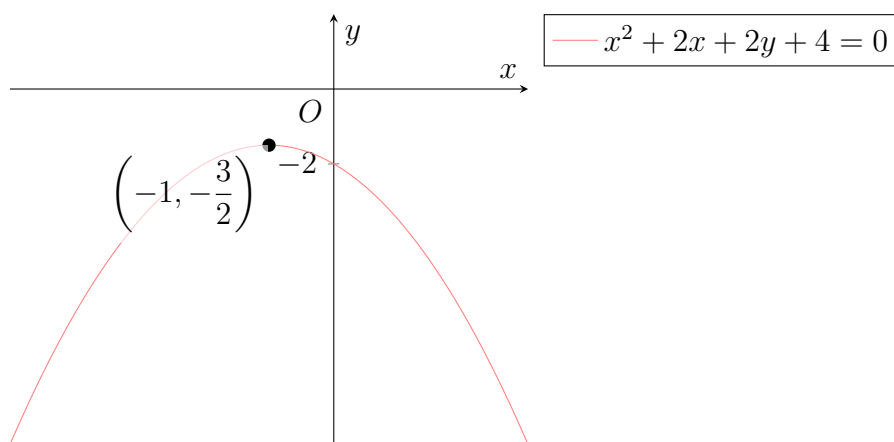
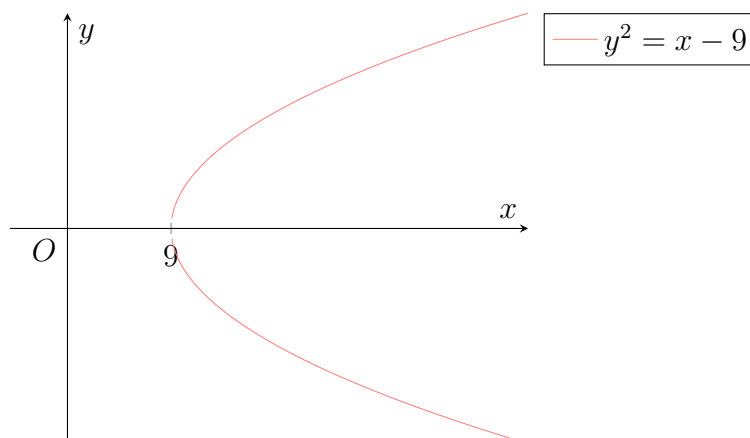
Sketch the following graphs

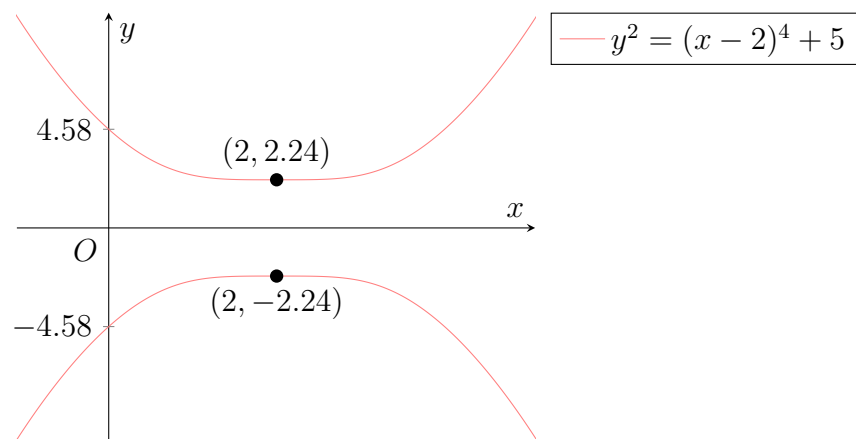
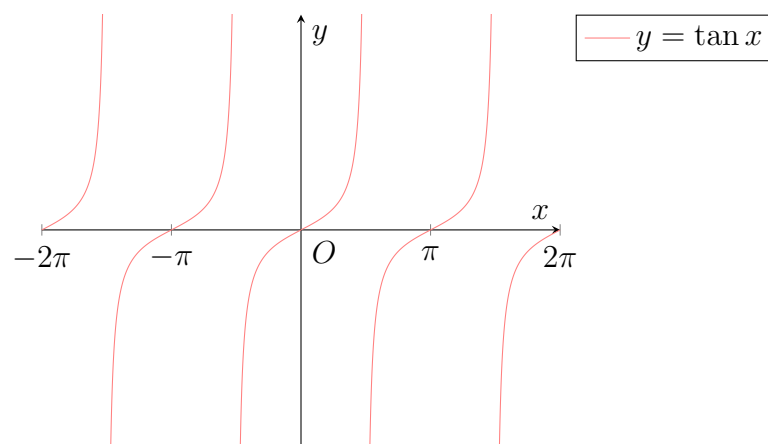
(a) $x^2 + 2x + 2y + 4 = 0$

(b) $y^2 = x - 9$

(c) $y^2 = (x - 2)^4 + 5$

(d) $y = \tan \frac{1}{2}x, -2\pi \leq x \leq 2\pi$

Solution**Part (a)****Part (b)**

Part (c)**Part (d)**

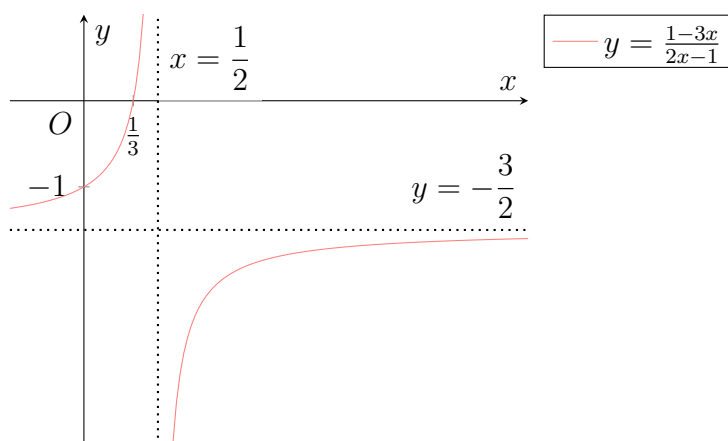
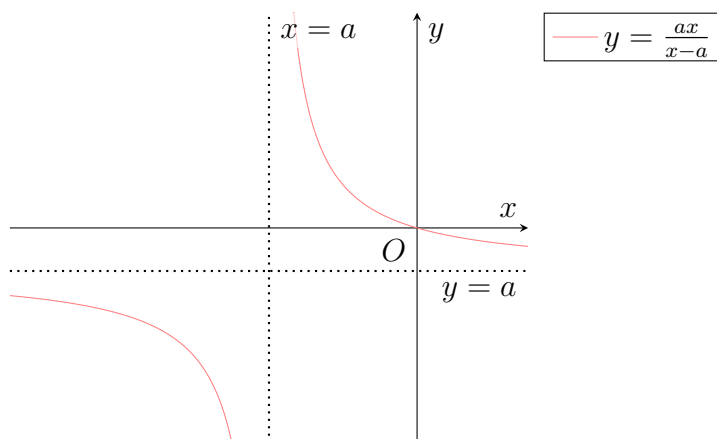
Problem 4.

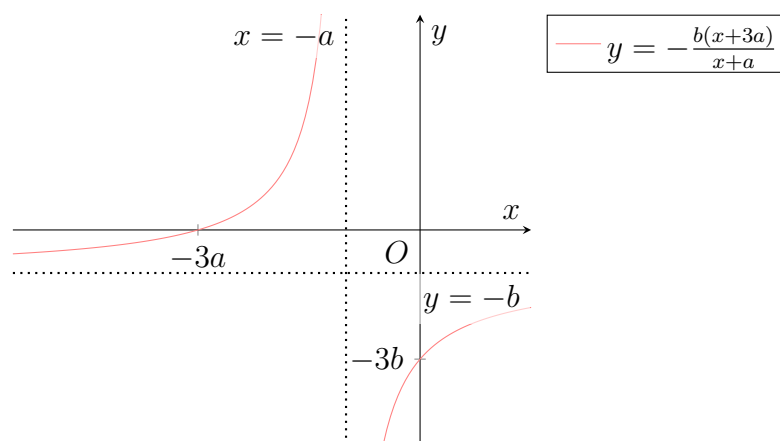
Sketch the following curves. Indicate using exact values, the equations of any asymptotes and the coordinates of any intersection with the axes.

(a) $y = \frac{1-3x}{2x-1}$

(b) $y = \frac{ax}{x-a}, a < 0$

(c) $y = -\frac{b(x+3a)}{x+a}, a, b > 0$

Solution**Part (a)****Part (b)**

Part (c)

Problem 5.

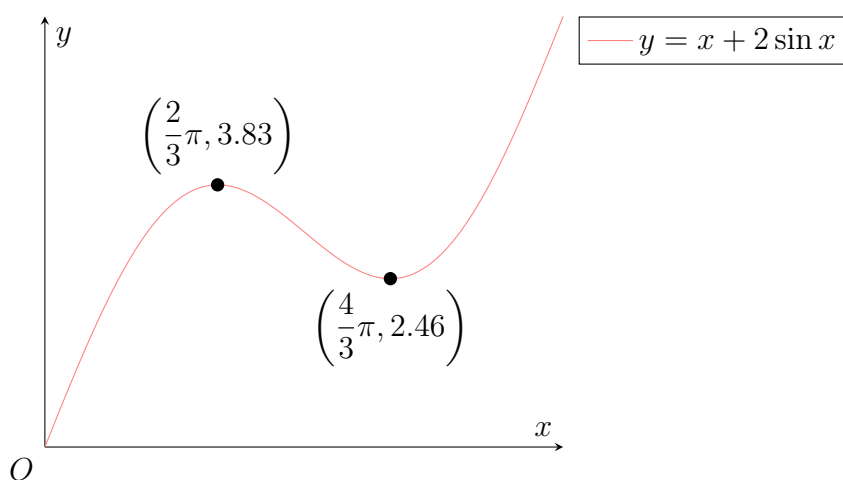
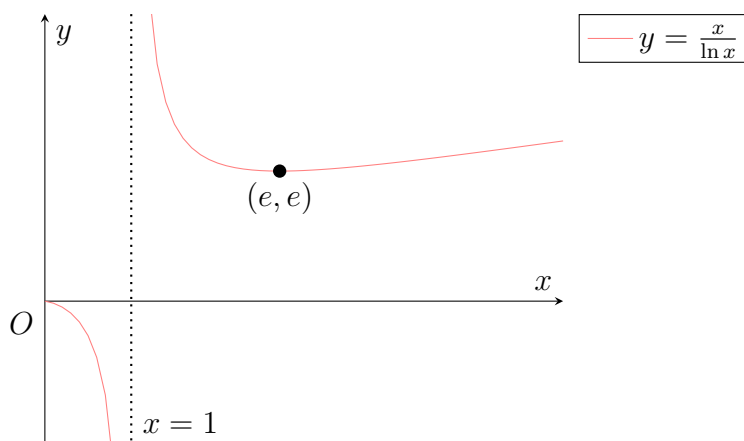
Sketch the following curves and find the coordinates of any turning points on the curves.

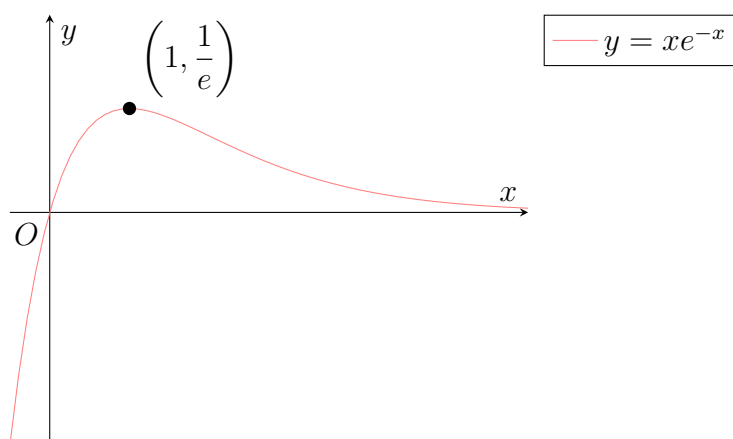
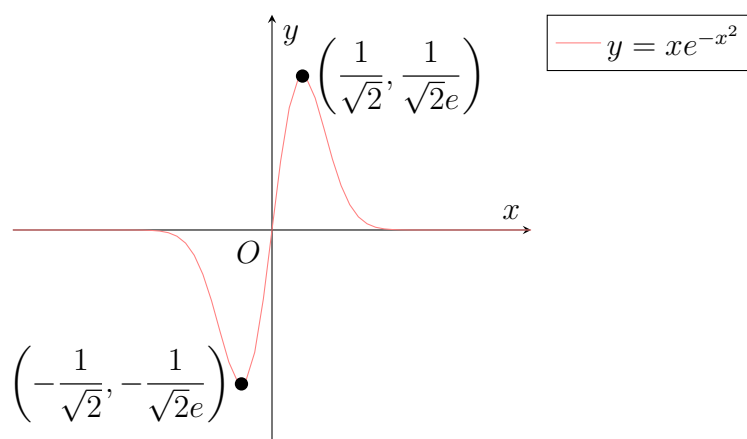
(a) $y = x + 2 \sin x$, $0 \leq x \leq 2\pi$

(b) $y = \frac{x}{\ln x}$, $x > 0$, $x \neq 1$

(c) $y = xe^{-x}$

(d) $y = xe^{-x^2}$

Solution**Part (a)****Part (b)**

Part (c)**Part (d)**

Problem 6.

The equation of a curve C is $y = 1 + \frac{6}{x-3} - \frac{24}{x+3}$.

- (a) Explain why $y = 1$ and $x = 3$ are asymptotes to the curve.
- (b) Find the coordinates of the points where C meets the axes.
- (c) Sketch C .

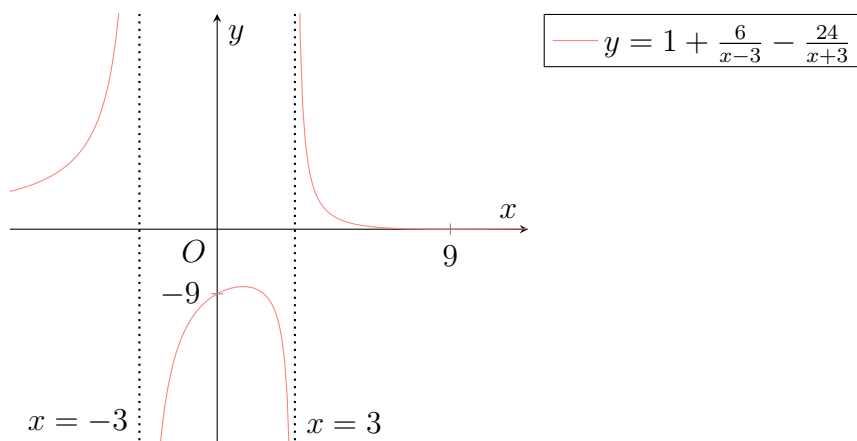
Solution**Part (a)**

As $x \rightarrow \pm\infty$, $y \rightarrow 1$. Hence, $y = 1$ is an asymptote to C . As $x \rightarrow 3^\pm$, $y \rightarrow \pm\infty$. Hence, $x = 3$ is an asymptote to C .

Part (b)

When $x = 0$, $y = -9$. When $y = 0$, $x = 9$.

C meets the axes at $(0, -9)$ and $(9, 0)$.

Part (c)

Problem 7.

The curve C has equation $y = \frac{ax^2 + bx}{x + 2}$, where $x \neq -2$. It is given that C has an asymptote $y = 1 - 2x$.

- Show (do not verify) that $a = -2$ and $b = -3$.
- Using an algebraic method, find the set of values that y can take.
- Sketch C , showing clearly the positions of any axial intercept(s), asymptote(s) and stationary point(s).
- Deduce that the equation $x^4 + 2x^3 + 2x^2 + 3x = 0$ has exactly one real non-zero root.

Solution**Part (a)**

$$\begin{aligned}
 y &= \frac{ax^2 + bx}{x + 2} \\
 &= \frac{(ax + b - 2a)(x + 2) - 2(b - 2a)}{x + 2} \\
 &= ax + b - 2a - \frac{2(b - 2a)}{x + 2}
 \end{aligned}$$

Since C has an asymptote $y = 1 - 2x$, we have $a = -2$ and $b - 2a = 1$, whence $b = -3$.

Part (b)

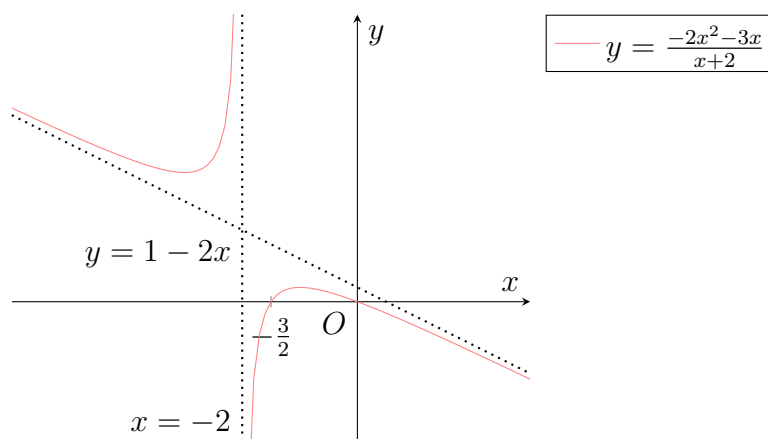
$$\begin{aligned}
 y &= \frac{-2x^2 + -3x}{x + 2} \\
 \implies y(x + 2) &= -2x^2 - 3x \\
 \implies 2x^2 + (3 + y)x + 2y &= 0
 \end{aligned}$$

For all values that y can take on, there exists a solution to $2x^2 + (3 + y)x + 2y = 0$. Hence, $\Delta \geq 0$.

$$\begin{aligned}
 (3 + y)^2 - 4(2)(2y) &\geq 0 \\
 \implies 9 + 6y + y^2 - 16y &\geq 0 \\
 \implies y^2 - 10y + 9 &\geq 0 \\
 \implies (y - 1)(y - 9) &\geq 0
 \end{aligned}$$

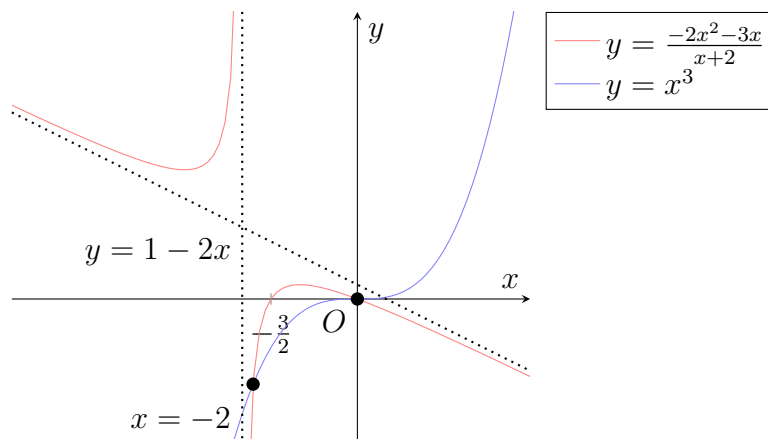


$$\boxed{\{y \in \mathbb{R} : y \leq 1 \vee y \geq 9\}}$$

Part (c)**Part (d)**

$$\begin{aligned}
 & x^4 + 2x^3 + 2x^2 + 3x = 0 \\
 \Rightarrow & \quad x^4 + 2x^3 = -2x^2 - 3x \\
 \Rightarrow & \quad x^3(x+2) = -2x^2 - 3x \\
 \Rightarrow & \quad x^3 = \frac{-2x^2 - 3x}{x+2}
 \end{aligned}$$

This motivates us to plot $y = x^3$ and $y = \frac{-2x^2 - 3x}{x+2}$ on the same graph.



We thus see that $y = x^3$ intersects $y = \frac{-2x^2 - 3x}{x+2}$ twice, with one intersection point being the origin. Thus, there is only one real non-zero root to $x^4 + 2x^3 + 2x^2 + 3x = 0$.

Problem 8.

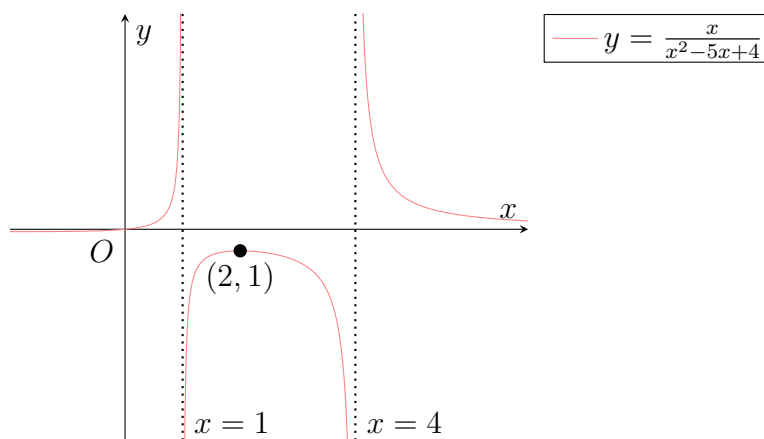
The curve C is defined by the equation $y = \frac{x}{x^2 - 5x + 4}$.

- Write down the equations of the asymptotes.
- Sketch C , indicating clearly the axial intercept(s), asymptote(s) and turning point(s).
- Find the positive value k such that the equation $\frac{x}{x^2 - 5x + 4} = kx$ has exactly 2 distinct real roots.

Solution**Part (a)**

As $x \rightarrow \pm\infty$, $y \rightarrow 0$. Hence, $y = 0$ is an asymptote. Observe that $x^2 - 5x + 4 = (x - 1)(x - 4)$. Hence, $x = 1$ and $x = 4$ are asymptotes.

Asymptotes: $y = 0$, $x = 1$, $x = 4$

Part (b)**Part (c)**

Note that $x = 0$ is always a root of $\frac{x}{x^2 - 5x + 4} = kx$. We thus aim to find the value of k such that $\frac{x}{x^2 - 5x + 4} = kx$ has only one non-zero root.

We observe that if $k > 0$, $y = kx$ will intersect with $y = \frac{x}{x^2 - 5x + 4}$ at least twice: before $x = 1$ and after $x = 4$. In order to have only one non-zero root, we must force the intersection point that comes before $x = 1$ to be at the origin $(0,0)$. Hence, k is tangential to C at $(0,0)$, thus giving $k = \left. \frac{dC}{dx} \right|_{x=0}$.

$$\begin{aligned}k &= \left. \frac{dC}{dx} \right|_{x=0} \\&= \left. \frac{d}{dx} \frac{x}{x^2 - 5x + 4} \right|_{x=0} \\&= \left. \frac{3x^2 - 10x + 4}{(x^2 - 5x + 4)^2} \right|_{x=0} \\&= \frac{1}{4}\end{aligned}$$

$$\boxed{k = \frac{1}{4}}$$