

**Problem 1.**

True or False? Explain your answers briefly.

(a)  $\sum_{r=1}^n (2r + 3) = \sum_{k=1}^n (2k + 3)$

(b)  $\sum_{r=1}^n \left( \frac{1}{r} + 5 \right) = \sum_{r=1}^n \frac{1}{r} + 5$

(c)  $\sum_{r=1}^n \frac{1}{r} = \frac{1}{\sum_{r=1}^n r}$

(d)  $\sum_{r=1}^n c = \sum_{r=0}^{n-1} (c + 1)$

**Solution****Part (a)**

Since both sums differ only by dummy variables, they are equal.

True

**Part (b)**

Summation is distributive. Since  $\sum_{r=1}^n 5$  is not equal to 5 in general, the equality does not hold.

False

**Part (c)**

In general,  $\sum \frac{a}{b} \neq \frac{\sum a}{\sum b}$ .

False

**Part (d)**

Since  $c$  is a constant with respect to  $r$ ,  $\sum_{r=1}^n c = nc \neq n(c + 1) = \sum_{r=0}^{n-1} (c + 1)$ .

False

**Problem 2.**

Write the following series in  $\Sigma$  notation twice, with  $r = 1$  as the lower limit in the first and  $r = 0$  as the lower limit in the second.

- (a)  $-2 + 1 + 4 + \dots + 40$
- (b)  $a^2 + a^4 + a^6 + \dots + a^{50}$
- (c)  $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + n\text{th term}$
- (d)  $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots$  to  $n$  terms
- (e)  $\frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{28 \cdot 30}$

**Solution****Part (a)**

Observe that  $-2 + 1 + 4 + \dots + 40$  is in arithmetic progression with a common difference of 3.

$$\begin{aligned} -2 + 1 + 4 + \dots + 40 &= \sum_{r=1}^{15} (3r - 5) \\ &= \sum_{r=0}^{14} (3(r+1) - 5) \\ &= \sum_{r=0}^{14} (3r - 2) \end{aligned}$$

$-2 + 1 + 4 + \dots + 40 = \sum_{r=1}^{15} (3r - 5) = \sum_{r=0}^{14} (3r - 2)$
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**Part (b)**

Observe that  $a^2 + a^4 + a^6 + \dots + a^{50}$  is in geometric progression with a common ratio of  $a^2$ .

$$\begin{aligned} a^2 + a^4 + a^6 + \dots + a^{50} &= \sum_{r=1}^{25} a^{2r} \\ &= \sum_{r=0}^{24} a^{2(r+1)} \\ &= \sum_{r=0}^{24} a^{2r+2} \end{aligned}$$

$a^2 + a^4 + a^6 + \dots + a^{50} = \sum_{r=1}^{25} a^{2r} = \sum_{r=0}^{24} a^{2r+2}$
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**Part (c)**

$$\begin{aligned}
\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + n\text{th term} &= \sum_{r=1}^n \frac{1}{2r+1} \\
&= \sum_{r=0}^{n-1} \frac{1}{2(r+1)+1} \\
&= \sum_{r=0}^{n-1} \frac{1}{2r+3}
\end{aligned}$$

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + n\text{th term} = \sum_{r=1}^n \frac{1}{2r+1} = \sum_{r=0}^{n-1} \frac{1}{2r+3}$$

**Part (d)**

$$\begin{aligned}
1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots \text{ to } n \text{ terms} &= \sum_{r=1}^n \left(-\frac{1}{2}\right)^{r-1} \\
&= \sum_{r=0}^{n-1} \left(-\frac{1}{2}\right)^{(r+1)-1} \\
&= \sum_{r=0}^{n-1} \left(-\frac{1}{2}\right)^r
\end{aligned}$$

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \dots \text{ to } n \text{ terms} = \sum_{r=1}^n \left(-\frac{1}{2}\right)^{r-1} = \sum_{r=0}^{n-1} \left(-\frac{1}{2}\right)^r$$

**Part (e)**

$$\begin{aligned}
\frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{28 \cdot 30} &= \sum_{r=1}^{27} \frac{1}{(r+1)(r+3)} \\
&= \sum_{r=0}^{26} \frac{1}{((r+1)+1)((r+1)+3)} \\
&= \sum_{r=0}^{26} \frac{1}{(r+2)(r+4)}
\end{aligned}$$

$$\frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \frac{1}{4 \cdot 6} + \dots + \frac{1}{28 \cdot 30} = \sum_{r=1}^{27} \frac{1}{(r+1)(r+3)} = \sum_{r=0}^{26} \frac{1}{(r+2)(r+4)}$$

**Problem 3.**

Without using the GC, evaluate the following sums.

$$(a) \sum_{r=1}^{50} (2r - 7)$$

$$(b) \sum_{r=1}^a (1 - a - r)$$

$$(c) \sum_{r=2}^n (\ln r + 3^r)$$

$$(d) \sum_{r=1}^{\infty} \left( \frac{2^r - 1}{3^r} \right)$$

**Solution****Part (a)**

$$\begin{aligned} \sum_{r=1}^{50} (2r - 7) &= 2 \sum_{r=1}^{50} r - 7 \sum_{r=1}^{50} 1 \\ &= 2 \cdot \frac{50 \cdot 51}{2} - 7 \cdot 50 \\ &= 2200 \end{aligned}$$

$$\sum_{r=1}^{50} (2r - 7) = 2200$$

**Part (b)**

$$\begin{aligned} \sum_{r=1}^a (1 - a - r) &= (1 - a) \sum_{r=1}^a 1 - \sum_{r=1}^a r \\ &= (1 - a) \cdot a - \frac{a(a+1)}{2} \\ &= \frac{a}{2} \cdot 2(1 - a) - \frac{a}{2} \cdot (a + 1) \\ &= \frac{a}{2} (2 - 2a - (a + 1)) \\ &= \frac{a}{2} (1 - 3a) \end{aligned}$$

$$\sum_{r=1}^a (1 - a - r) = \frac{a}{2} (1 - 3a)$$

**Part (c)**

$$\begin{aligned}
\sum_{r=2}^n (\ln r + 3^r) &= \sum_{r=2}^n \ln r + \sum_{r=2}^n 3^r \\
&= \ln n! + \sum_{r=1}^{n-1} 3^{r+1} \\
&= \ln n! + 3 \sum_{r=1}^{n-1} 3^r \\
&= \ln n! + 3 \cdot \frac{3(3^{n-1} - 1)}{3 - 1} \\
&= \ln n! + \frac{9}{2} (3^{n-1} - 1)
\end{aligned}$$

$$\boxed{\sum_{r=2}^n (\ln r + 3^r) = \ln n! + \frac{9}{2} (3^{n-1} - 1)}$$

**Part (d)**

$$\begin{aligned}
\sum_{r=1}^{\infty} \left( \frac{2^r - 1}{3^r} \right) &= \sum_{r=1}^{\infty} \frac{2^r}{3^r} - \sum_{r=1}^{\infty} \frac{1}{3^r} \\
&= \sum_{r=1}^{\infty} \left( \frac{2}{3} \right)^r - \sum_{r=1}^{\infty} \left( \frac{1}{3} \right)^r \\
&= \frac{\frac{2}{3}}{1 - \frac{2}{3}} - \frac{\frac{1}{3}}{1 - \frac{1}{3}} \\
&= \frac{3}{2}
\end{aligned}$$

$$\boxed{\sum_{r=1}^{\infty} \left( \frac{2^r - 1}{3^r} \right) = \frac{3}{2}}$$

**Problem 4.**

The  $n$ th term of a series is  $2^{n-2} + 3n$ . Find the sum of the first  $N$  terms.

**Solution**

$$\begin{aligned}\sum_{r=1}^N (2^{r-2} + 3r) &= \sum_{r=1}^N 2^{r-2} + 3 \sum_{r=1}^N r \\ &= \frac{2^{1-2} (2^N - 1)}{2 - 1} + \frac{3N(N+1)}{2} \\ &= \frac{2^N - 1}{2} + \frac{3N(N+1)}{2} \\ &= \frac{1}{2} (2^N + 3N^2 + 3N - 1)\end{aligned}$$

The sum of the first  $N$  terms is  $\frac{1}{2} (2^N + 3N^2 + 3N - 1)$

**Problem 5.**

The  $r$ th term,  $u_r$ , of a series is given by  $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$ . Express  $\sum_{r=1}^n u_r$  in the form  $A \left(1 - \frac{B}{27^n}\right)$ , where  $A$  and  $B$  are constants. Deduce the sum to infinity of the series.

**Solution**

$$\begin{aligned}
 \sum_{r=1}^n u_r &= \sum_{r=1}^n \left( \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1} \right) \\
 &= \sum_{r=1}^n \left(\frac{1}{3}\right)^{3r-2} + \sum_{r=1}^n \left(\frac{1}{3}\right)^{3r-1} \\
 &= 9 \sum_{r=1}^n \left(\frac{1}{3}\right)^{3r} + 3 \sum_{r=1}^n \left(\frac{1}{3}\right)^{3r} \\
 &= 12 \sum_{r=1}^n \left(\frac{1}{3}\right)^{3r} \\
 &= 12 \sum_{r=1}^n \left(\frac{1}{27}\right)^r \\
 &= 12 \cdot \frac{\frac{1}{27}(1 - (\frac{1}{27})^n)}{1 - \frac{1}{27}} \\
 &= 12 \cdot \frac{1 - (\frac{1}{27})^n}{27 - 1} \\
 &= \frac{6}{13} \left(1 - \frac{1}{27^n}\right)
 \end{aligned}$$

$$\boxed{\sum_{r=1}^n u_r = \frac{6}{13} \left(1 - \frac{1}{27^n}\right)}$$

$$\begin{aligned}
 \sum_{r=1}^{\infty} u_r &= \lim_{n \rightarrow \infty} \sum_{r=1}^n u_r \\
 &= \lim_{n \rightarrow \infty} \frac{6}{13} \left(1 - \frac{1}{27^n}\right) \\
 &= \frac{6}{13} (1 - 0) \\
 &= \frac{6}{13}
 \end{aligned}$$

$$\boxed{\sum_{r=1}^{\infty} u_r = \frac{6}{13}}$$

**Problem 6.**

The  $r$ th term,  $u_r$ , of a series is given by  $u_r = \ln \frac{r}{r+1}$ . Find  $\sum_{r=1}^n u_r$  in terms of  $n$ . Comment on whether the series converges.

**Solution**

$$\begin{aligned}
 \sum_{r=1}^n u_r &= \sum_{r=1}^n \ln \frac{r}{r+1} \\
 &= \sum_{r=1}^n (\ln r - \ln(r+1)) \\
 &= \sum_{r=1}^n \ln r - \sum_{r=1}^n \ln(r+1) \\
 &= \sum_{r=1}^n \ln r - \sum_{r=2}^{n+1} \ln r \\
 &= \ln 1 + \sum_{r=2}^{n+1} \ln r - \ln(n+1) - \sum_{r=2}^{n+1} \ln r \\
 &= \ln 1 - \ln(n+1) \\
 &= \ln \frac{1}{n+1}
 \end{aligned}$$

$$\boxed{\sum_{r=1}^n u_r = \ln \frac{1}{n+1}}$$

Observe that as  $n \rightarrow \infty$ ,  $\ln \frac{1}{n+1} \rightarrow \ln 0$ . Hence,  $\sum_{r=1}^n u_r$  diverges to infinity. Thus,  $u_n$  does not converge.



**Problem 7.**

Given that  $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ , without using the GC, find the following sums.

(a)  $\sum_{r=0}^n (r(r+4) + n)$

(b)  $\sum_{r=n+1}^{2n} (2r-1)^2$

(c)  $\sum_{r=-15}^{20} r(r-2)$

**Solution****Part (a)**

$$\begin{aligned}
 \sum_{r=0}^n (r(r+4) + n) &= \sum_{r=0}^n (r^2 + 4r + n) \\
 &= \sum_{r=0}^n r^2 + 4 \sum_{r=0}^n r + n \sum_{r=0}^n 1 \\
 &= \sum_{r=1}^n r^2 + 4 \sum_{r=1}^n r + n \sum_{r=0}^n 1 \\
 &= \frac{n}{6}(n+1)(2n+1) + 4 \cdot \frac{n(n+1)}{2} + n(n+1) \\
 &= \frac{n}{6}(n+1)(2n+1) + 2n(n+1) + n(n+1) \\
 &= (n+1) \left( \frac{n}{6}(2n+1) + 2n + n \right) \\
 &= \frac{n}{6}(n+1)(2n+1+12+6) \\
 &= \frac{n}{6}(n+1)(2n+19)
 \end{aligned}$$

$\sum_{r=0}^n (r(r+4) + n) = \frac{n}{6}(n+1)(2n+19)$
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**Part (b)**

$$\begin{aligned}
\sum_{r=n+1}^{2n} (2r-1)^2 &= \sum_{r=1}^n (2(r+n)-1)^2 \\
&= \sum_{r=1}^n (2r+2n-1)^2 \\
&= \sum_{r=1}^n (4r^2 + 2(2r)(2n-1) + (2n-1)^2) \\
&= \sum_{r=1}^n (4r^2 + 4(2n-1)r + (2n-1)^2) \\
&= 4 \sum_{r=1}^n r^2 + 4(2n-1) \sum_{r=1}^n r + (2n-1)^2 \sum_{r=1}^n 1 \\
&= 4 \cdot \frac{n}{6} (n+1)(2n+1) + 4(2n-1) \frac{n(n+1)}{2} + n(2n-1)^2 \\
&= \frac{2}{3} \cdot n(n+1)(2n+1) + 2n(2n-1)(n+1) + n(2n-1)^2 \\
&= \frac{1}{3} n (2(n+1)(2n+1) + 6(2n-1)(n+1) + 3(2n-1)^2) \\
&= \frac{1}{3} n (4n^2 + 4n + 2n + 2 + 12n^2 - 6n + 12n - 6 + 12n^2 - 12n + 3) \\
&= \frac{1}{3} n (28n^2 - 1)
\end{aligned}$$

$$\boxed{\sum_{r=n+1}^{2n} (2r-1)^2 = \frac{1}{3} n (28n^2 - 1)}$$

**Part (c)**

$$\begin{aligned}
\sum_{r=-15}^{20} r(r-2) &= \sum_{r=1}^{36} (r-16)((r-16)-2) \\
&= \sum_{r=1}^{36} (r-16)(r-18) \\
&= \sum_{r=1}^{36} (r^2 - 34r + 288) \\
&= \sum_{r=1}^{36} r^2 - 34 \sum_{r=1}^{36} r + 288 \sum_{r=1}^{36} 1 \\
&= \frac{36}{6} \cdot (36+1)(2 \cdot 36 + 1) - 34 \cdot \frac{36 \cdot 37}{2} + 288 \cdot 36 \\
&= 3930
\end{aligned}$$

$$\boxed{\sum_{r=-15}^{20} r(r-2) = 3930}$$

**Problem 8.**

Let  $S = \sum_{r=0}^{\infty} \frac{(x-2)^r}{3^r}$  where  $x \neq 2$ . Find the range of values of  $x$  such that the series  $S$  converges. Given that  $x = 1$ , find

(a) the value of  $S$

(b)  $S_n$ , in terms of  $n$ , where  $S_n = \sum_{r=0}^{n-1} \frac{(x-2)^r}{3^r}$

(c) the least value of  $n$  for which  $|S_n - S|$  is less than 0.001% of  $S$

**Solution**

$$\begin{aligned} S &= \sum_{r=0}^{\infty} \frac{(x-2)^r}{3^r} \\ &= \sum_{r=0}^{\infty} \left( \frac{x-2}{3} \right)^r \end{aligned}$$

For  $S$  to converge, we must have  $\left| \frac{x-2}{3} \right| < 1$ .

**Case 1:**  $\frac{x-2}{3} < 1 \implies x-2 < 3 \implies x < 5$

**Case 2:**  $-\frac{x-2}{3} < 1 \implies \frac{x-2}{3} > -1 \implies x-2 > -3 \implies x > -1$

Putting both inequalities together, we see that  $-1 < x < 5$ .

For  $S$  to converge, we must have  $-1 < x < 5$ ,  $x \neq 2$ .

**Part (a)**

When  $x = 1$ ,

$$\begin{aligned} S &= \sum_{r=0}^{\infty} \left( -\frac{1}{3} \right)^r \\ &= \frac{1}{1 - (-\frac{1}{3})} \\ &= \frac{3}{4} \end{aligned}$$

$$S = \frac{3}{4}$$

**Part (b)**

$$\begin{aligned}
S_n &= \sum_{r=0}^{n-1} \frac{(-1)^r}{3^r} \\
&= \sum_{r=0}^{n-1} \left(-\frac{1}{3}\right)^r \\
&= \sum_{r=1}^n \left(-\frac{1}{3}\right)^{r-1} \\
&= -3 \sum_{r=1}^n \left(-\frac{1}{3}\right)^r \\
&= -3 \cdot \frac{-\frac{1}{3}(1 - (-\frac{1}{3})^n)}{1 - (-\frac{1}{3})} \\
&= \frac{3}{4} \left(1 - \left(-\frac{1}{3}\right)^n\right)
\end{aligned}$$

$$S_n = \frac{3}{4} \left(1 - \left(-\frac{1}{3}\right)^n\right)$$

**Part (c)**

$$\begin{aligned}
&|S_n - S| < 0.001\%S \\
\Rightarrow &\left|\frac{S_n - S}{S}\right| < \frac{0.001}{100} \\
\Rightarrow &\left|\frac{\frac{3}{4}(1 - (-\frac{1}{3})^n)}{\frac{3}{4}} - 1\right| < \frac{1}{100000} \\
\Rightarrow &\left|-\left(-\frac{1}{3}\right)^n\right| < \frac{1}{100000} \\
\Rightarrow &\left(\frac{1}{3}\right)^n < \frac{1}{100000} \\
\Rightarrow &n > \log_{\frac{1}{3}} \frac{1}{100000} \\
&= 10.5
\end{aligned}$$

Since  $n \in \mathbb{N}$ , the least value of  $n$  is 11.

The least value of  $n$  is 11.

**Problem 9.**

Given that  $\sum_{r=1}^n r^2 = \frac{n}{6}(n+1)(2n+1)$ ,

(a) write down  $\sum_{r=1}^{2k} r^2$  in terms of  $k$

(b) find  $2^2 + 4^2 + 6^2 + \dots + (2k)^2$ .

Hence, show that  $\sum_{r=1}^k (2r-1)^2 = \frac{k}{3}(2k+1)(2k-1)$ .

**Solution****Part (a)**

$$\begin{aligned}\sum_{r=1}^{2k} r^2 &= \frac{2k}{6}(2k+1)(2(2k)+1) \\ &= \frac{k}{3}(2k+1)(4k+1)\end{aligned}$$

$$\sum_{r=1}^{2k} r^2 = \frac{k}{3}(2k+1)(4k+1)$$

**Part (b)**

$$\begin{aligned}2^2 + 4^2 + 6^2 + \dots + (2k)^2 &= \sum_{r=1}^k (2r)^2 \\ &= \sum_{r=1}^k 4r^2 \\ &= 4 \sum_{r=1}^k r^2 \\ &= 4 \cdot \frac{k}{6}(k+1)(2k+1) \\ &= \frac{2k}{3}(k+1)(2k+1)\end{aligned}$$

$$2^2 + 4^2 + 6^2 + \dots + (2k)^2 = \frac{2k}{3}(k+1)(2k+1)$$

$$\begin{aligned}\sum_{r=1}^k (2r-1)^2 &= \sum_{r=1}^{2k} r^2 - \sum_{r=1}^k (2r)^2 \\&= \frac{k}{3}(2k+1)(4k+1) - \frac{2k}{3}(k+1)(2k+1) \\&= \frac{k}{3}(2k+1)((4k+1) - 2(k+1)) \\&= \frac{k}{3}(2k+1)(2k-1)\end{aligned}$$

**Problem 10.**

Given that  $u_n = e^{nx} - e^{(n+1)x}$ , find  $\sum_{n=1}^N u_n$  in terms of  $N$  and  $x$ . Hence determine the set of values of  $x$  for which the infinite series  $u_1 + u_2 + u_3 + \dots$  is convergent and give the sum to infinity for cases where this exists.

**Solution**

$$\begin{aligned}
 \sum_{n=1}^N u_n &= \sum_{n=1}^N (e^{nx} - e^{(n+1)x}) \\
 &= \sum_{n=1}^N e^{nx} - \sum_{n=1}^N e^{(n+1)x} \\
 &= \sum_{n=1}^N e^{nx} - \sum_{n=2}^{N+1} e^{nx} \\
 &= \left( e^x + \sum_{n=2}^N e^{nx} \right) - \left( \sum_{n=2}^N e^{nx} + e^{(N+1)x} \right) \\
 &= e^x - e^{(N+1)x} \\
 &= e^x(1 - e^{Nx})
 \end{aligned}$$

$$\boxed{\sum_{n=1}^N u_n = e^x(1 - e^{Nx})}$$

For the infinite series  $u_1 + u_2 + u_3 + \dots$  to converge, we require  $e^x$  to converge. Hence,  $x \leq 0$ . Equivalently,  $x \in \mathbb{R}_0^-$ .

$$\boxed{\mathbb{R}_0^-}$$

**Case 1:**  $x = 0$

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n &= \lim_{N \rightarrow \infty} e^x(1 - e^{Nx}) \\
 &= \lim_{N \rightarrow \infty} e^0(1 - e^{N \cdot 0}) \\
 &= \lim_{N \rightarrow \infty} 1(1 - 1) \\
 &= 0
 \end{aligned}$$

When  $x = 0$ , the sum to infinity is 0.

**Case 2:**  $x < 0$

$$\begin{aligned}
 \lim_{N \rightarrow \infty} \sum_{n=1}^N u_n &= \lim_{N \rightarrow \infty} e^x(1 - e^{Nx}) \\
 &= e^x(1 - 0) \\
 &= e^x
 \end{aligned}$$

When  $x < 0$ , the sum to infinity is  $e^x$ .

**Problem 11.**

Given that  $r$  is a positive integer and  $f(r) = \frac{1}{r^2}$ , express  $f(r) - f(r+1)$  as a single fraction. Hence prove that  $\sum_{r=1}^{4n} \left( \frac{2r+1}{r^2(r+1)^2} \right) = 1 - \frac{1}{(4n+1)^2}$ . Give a reason why the series is convergent and state the sum to infinity. Find  $\sum_{r=2}^{4n} \left( \frac{2r-1}{r^2(r-1)^2} \right)$ .

**Solution**

$$\begin{aligned} f(r) - f(r+1) &= \frac{1}{r^2} - \frac{1}{(r+1)^2} \\ &= \frac{(r+1)^2 - r^2}{r^2(r+1)^2} \\ &= \frac{2r+1}{r^2(r+1)^2} \end{aligned}$$

$$f(r) - f(r+1) = \frac{2r+1}{r^2(r+1)^2}$$

$$\begin{aligned} \sum_{r=1}^{4n} \left( \frac{2r+1}{r^2(r+1)^2} \right) &= \sum_{r=1}^{4n} (f(r) - f(r+1)) \\ &= \sum_{r=1}^{4n} f(r) - \sum_{r=1}^{4n} f(r+1) \\ &= \sum_{r=1}^{4n} f(r) - \sum_{r=2}^{4n+1} f(r) \\ &= \left( f(1) + \sum_{r=2}^{4n} f(r) \right) - \left( \sum_{r=2}^{4n} f(r) + f(4n+1) \right) \\ &= f(1) - f(4n+1) \\ &= 1 - \frac{1}{(4n+1)^2} \end{aligned}$$

As  $r \rightarrow \infty$ ,  $\sum_{r=1}^{4n} \left( \frac{2r+1}{r^2(r+1)^2} \right) = 1 - \frac{1}{(4n+1)^2} \rightarrow 1$ . Hence, the series is convergent and converges to 1.

The sum to infinity is 1.



$$\begin{aligned}
\sum_{r=2}^{4n} \left( \frac{2r-1}{r^2(r-1)^2} \right) &= \sum_{r=1}^{4n-1} \left( \frac{2(r+1)-1}{(r+1)^2 r^2} \right) \\
&= \sum_{r=1}^{4n-1} \left( \frac{2r+1}{r^2(r+1)^2} \right) \\
&= \sum_{r=1}^{4n-1} (f(r) - f(r+1)) \\
&= \sum_{r=1}^{4n} (f(r) - f(r+1)) - (f(4n) - f(4n+1)) \\
&= 1 - f(4n+1) - (f(4n) - f(4n+1)) \\
&= 1 - f(4n) \\
&= 1 - \frac{1}{(4n)^2} \\
&= 1 - \frac{1}{16n^2}
\end{aligned}$$

$\sum_{r=2}^{4n} \left( \frac{2r-1}{r^2(r-1)^2} \right) = 1 - \frac{1}{16n^2}$
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**Problem 12.**

- (a) Express  $\frac{1}{(2x+1)(2x+3)(2x+5)}$  in partial fractions.
- (b) Hence show that  $\sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} = \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}$ .
- (c) Deduce the sum of  $\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{41 \cdot 43 \cdot 45}$ .

**Solution****Part (a)**

Let  $u = 2x + 3$ . Then  $\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{(u-2)u(u+2)}$ .

$$\begin{aligned} \frac{1}{(u-2)u(u+2)} &= \frac{A}{u-2} + \frac{B}{u} + \frac{C}{u+2} \\ \Rightarrow 1 &= A(u)(u+2) + B(u-2)(u+2) + C(u-2)u \\ &= Au^2 + 2Au + Bu^2 - 4B + Cu^2 - 2Cu \\ &= (A+B+C)u^2 + (A-C)u - 4B \end{aligned}$$

Comparing the coefficients of  $u^2$ ,  $u$  and constant terms, we have the following system of equations.

$$\begin{cases} A+B+C = 0 \\ A-C = 0 \\ -4B = 1 \end{cases}$$

Solving, we obtain  $A = \frac{1}{8}$ ,  $B = -\frac{1}{4}$  and  $C = \frac{1}{8}$ . Hence,

$$\begin{aligned} \frac{1}{(u-2)u(u+2)} &= \frac{1}{8(u-2)} - \frac{1}{4u} + \frac{1}{8(u+2)} \\ \Rightarrow \frac{1}{(2x+1)(2x+3)(2x+5)} &= \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)} \end{aligned}$$

$\frac{1}{(2x+1)(2x+3)(2x+5)} = \frac{1}{8(2x+1)} - \frac{1}{4(2x+3)} + \frac{1}{8(2x+5)}$
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**Part (b)**

$$\begin{aligned}
& \sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} \\
&= \sum_{r=1}^n \left( \frac{1}{8(2r+1)} - \frac{1}{4(2r+3)} + \frac{1}{8(2r+5)} \right) \\
&= \frac{1}{8} \sum_{r=1}^n \frac{1}{2r+1} - \frac{1}{4} \sum_{r=1}^n \frac{1}{2r+3} + \frac{1}{8} \sum_{r=1}^n \frac{1}{2r+5} \\
&= \left( \frac{1}{8} \sum_{r=1}^n \frac{1}{2r+1} - \frac{1}{8} \sum_{r=1}^n \frac{1}{2r+3} \right) + \left( \frac{1}{8} \sum_{r=1}^n \frac{1}{2r+5} - \frac{1}{8} \sum_{r=1}^n \frac{1}{2r+3} \right) \\
&= \frac{1}{8} \left( \left( \sum_{r=1}^n \frac{1}{2r+1} - \sum_{r=1}^n \frac{1}{2r+3} \right) + \left( \sum_{r=1}^n \frac{1}{2r+5} - \sum_{r=1}^n \frac{1}{2r+3} \right) \right) \quad (12.1)
\end{aligned}$$

Consider  $\sum_{r=1}^n \frac{1}{2r+1} - \sum_{r=1}^n \frac{1}{2r+3}$ .

$$\begin{aligned}
\sum_{r=1}^n \frac{1}{2r+1} - \sum_{r=1}^n \frac{1}{2r+3} &= \sum_{r=1}^n \frac{1}{2r+1} - \sum_{r=2}^{n+1} \frac{1}{2r+1} \\
&= \left( \frac{1}{3} + \sum_{r=2}^n \frac{1}{2r+1} \right) - \left( \sum_{r=2}^n \frac{1}{2r+1} + \frac{1}{2(n+1)+1} \right) \\
&= \frac{1}{3} - \frac{1}{2n+3} \quad (12.2)
\end{aligned}$$

Consider  $\sum_{r=1}^n \frac{1}{2r+5} - \sum_{r=1}^n \frac{1}{2r+3}$ .

$$\begin{aligned}
\sum_{r=1}^n \frac{1}{2r+5} - \sum_{r=1}^n \frac{1}{2r+3} &= \sum_{r=1}^n \frac{1}{2r+5} - \sum_{r=0}^{n-1} \frac{1}{2r+5} \\
&= \left( \sum_{r=0}^n \frac{1}{2r+5} - \frac{1}{5} \right) - \left( \sum_{r=0}^n \frac{1}{2r+5} - \frac{1}{2n+5} \right) \\
&= \frac{1}{2n+5} - \frac{1}{5} \quad (12.3)
\end{aligned}$$

Substituting Equations 12.2 and 12.3 into Equation 12.1, we have

$$\begin{aligned}
\sum_{r=1}^n \frac{1}{(2r+1)(2r+3)(2r+5)} &= \frac{1}{8} \left( \frac{1}{3} - \frac{1}{2n+3} + \frac{1}{2n+5} - \frac{1}{5} \right) \\
&= \frac{1}{8} \left( \frac{2}{15} - \frac{1}{2n+3} + \frac{1}{2n+5} \right) \\
&= \frac{1}{8} \left( \frac{2}{15} - \frac{2}{(2n+3)(2n+5)} \right) \\
&= \frac{1}{60} - \frac{1}{4(2n+3)(2n+5)}
\end{aligned}$$

**Part (c)**

$$\begin{aligned} & \frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{41 \cdot 43 \cdot 45} \\ &= \sum_{r=0}^{20} \frac{1}{(2r+1)(2r+3)(2r+5)} \\ &= \frac{1}{1 \cdot 3 \cdot 5} + \sum_{r=1}^{20} \frac{1}{(2r+1)(2r+3)(2r+5)} \\ &= \frac{1}{15} + \frac{1}{60} - \frac{1}{4(2 \cdot 20 + 3)(2 \cdot 20 + 5)} \\ &= \frac{161}{1935} \end{aligned}$$

$\frac{1}{1 \cdot 3 \cdot 5} + \frac{1}{3 \cdot 5 \cdot 7} + \frac{1}{5 \cdot 7 \cdot 9} + \cdots + \frac{1}{41 \cdot 43 \cdot 45} = \frac{161}{1935}$
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