

Problem 1.

- (a) Explain why the Euler method will fail for the initial-value problem

$$\frac{dy}{dx} = y \cos \sqrt{x}, \quad y(0) = 0,$$

where $y = y(x)$ satisfies that differential equation and is not a constant.

- (b) Suppose the initial condition for the problem in part (a) is now $y(0) = 10$. Use the improved Euler method with a step size of 0.1 to find, to three decimal places, an estimate for $y(0.1)$.
- (c) Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{2}y(2 - x), \quad y > 0, \quad y(0) = 10,$$

expressing y in terms of x , and simplifying your answer as far as possible.

- (d) Explain why the solution found in part (c) will give a reasonable estimate for $y(0.1)$ in part (b).

Solution

Part (a)

By the Euler method,

$$y_1 = y_0 + \Delta x(y_0 \cos \sqrt{x_0}) = 0 + \Delta x(0 \cos 0) = 0.$$

It follows that $y_n = 0$ for all $n \in \mathbb{N}$, whence y is the zero function. However, because y is not a constant function, y cannot be the zero function, a contradiction. Hence, the Euler method fails.

Part (b)

Let $\Delta x = 0.1$, $y_0 = 10$ and $x_n = n\Delta x$.

$$\begin{aligned} \tilde{y}_1 &= y_0 + \Delta x(y_0 \cos \sqrt{x_0}) = 11 \\ y_1 &= y_0 + \frac{1}{2}\Delta x [y_0 \cos \sqrt{x_0} + \tilde{y}_1 \cos \sqrt{x_1}] = 11.023 \text{ (3 d.p.)} \end{aligned}$$

$y(0.1) \approx 11.023$

Part (c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}y(2 - x) \\ \implies \frac{1}{y} \frac{dy}{dx} &= \frac{1}{2}(2 - x) \\ \implies \int \frac{1}{y} \frac{dy}{dx} dx &= \int \frac{1}{2}(2 - x) dx \end{aligned}$$

$$\begin{aligned}\Rightarrow \quad & \int \frac{1}{y} dy = \int \frac{1}{2}(2-x) dx \\ \Rightarrow \quad & \ln y = \frac{1}{2} \left[2x - \frac{1}{2}x^2 \right] + C_1 \\ & = x - \frac{1}{4}x^2 + C_1 \\ \Rightarrow \quad & y = C \exp\left(x - \frac{1}{4}x^2\right)\end{aligned}$$

Since $y(0) = 10$, we have $C = 10$. Thus,

$$\boxed{y = 10 \exp\left(x - \frac{1}{4}x^2\right)}$$

Part (d)

For small x , we have that $\cos \sqrt{x} \approx 1 - \frac{1}{2}(\sqrt{x})^2 = \frac{1}{2}(2-x)$. Thus,

$$y \cos \sqrt{x} \approx \frac{1}{2}y(2-x),$$

whence the two differential equations and thus their solutions are approximately equal. Since $x = 0.1$ is small, the solution found in part (c) will give a reasonable estimate for $y(0.1)$ in part (b).

Problem 2.

A particle moves along a straight line which passes through a fixed point O . It is acted on by two resistive forces, one of which is proportional to its displacement x from O while the other is proportional to its speed v . As a result, the motion of the particle is governed by the differential equation

$$v \frac{dv}{dx} = -7x - 24v.$$

Given that $v = 121$ when $x = 0$, estimate the value of v when $x = 1$ using

- (a) one iteration of the Euler method,
- (b) one iteration of the improved Euler method.

Hence, explain why v is approximately a linear function of x for $0 \leq x \leq 1$.

By considering the values of $\frac{x}{v}$ for $0 \leq x \leq 1$, use the given differential equation to find an expression for this linear function.

Solution

Rewriting the given differential equation, we obtain

$$\frac{dv}{dx} = -\frac{7x}{v} - 24.$$

Let $f(x, v) = -\frac{7x}{v} - 24$, $\Delta x = 1$, $v_0 = 121$, and $x_n = n\Delta x$.

Part (a)

By the Euler method,

$$v_1 = v_0 + \Delta x f(x_0, v_0) = 97$$

$$\boxed{y(1) \approx 97}$$

Part (b)

By the improved Euler method,

$$\tilde{v}_1 = v_0 + \Delta x f(x_0, v_0) = 97$$

$$v_1 = v_0 + \frac{1}{2}\Delta x [f(x_0, v_0) + f(x_1, \tilde{v}_1)] = 96.964$$

$$\boxed{y(1) \approx 96.964}$$

The gradient of v at $x = 0$ is $f(x_0, v_0) = -24$, which is very close to the gradient of v at $x = 1$, $f(x_1, \tilde{v}_1) = -24.072$. Since the gradient of v is approximately constant for $0 \leq x \leq 1$, we have that v is approximately a linear function on that interval.

Observe that for $0 \leq x \leq 1$, $x/v \approx 0$ since $x \in [0, 1]$, while $v \geq 96$. Thus, $dv/dx \approx -24$, whence $v = -24x + C$. Since $v = 121$ when $x = 0$, we have $\boxed{v \approx -24x + 121}$.

Problem 3.

The function $y = y(x)$ satisfies

$$\frac{dy}{dx} = \frac{1}{5}(\tan x + x^3 y).$$

The value of $y(h)$ is to be found, where h is a small positive number, and $y(0) = 0$.

- (a) Use one step of the improved Euler method to find an alternative approximation to $y(h)$ in terms of h .
- (b) It can be shown that $y = y(x)$ satisfies

$$y(h) = e^{0.05h^4} \int_0^h \frac{\tan x}{5} e^{-0.05x^4} dx.$$

Assume that h is small and hence find another approximation to $y(h)$ in terms of h .

- (c) Discuss the relative merits of these two methods employed to obtain these approximations.

Solution

Part (a)

Let $f(x, y) = \frac{1}{5}(\tan x + x^3 y)$, $\Delta x = h$, $y_0 = 0$ and $x_n = n\Delta x$. By the improved Euler method,

$$\begin{aligned} \tilde{y}_1 &= y_0 + \Delta x f(x_0, y_0) = 0 \\ y_1 &= y_0 + \frac{1}{2} \Delta x [f(x_0, y_0) + f(x_1, \tilde{y}_1)] \\ &= 0 + \frac{1}{2} h \left[0 + \frac{1}{5} (\tan h + 0) \right] \\ &= \frac{h \tan h}{10} \end{aligned}$$

$$y(h) \approx \frac{h \tan h}{10}$$

Part (b)

Since h is small, we have that $e^{0.05h^4} \approx 1$. Furthermore, since we are integrating over the interval $x \in [0, h]$, the integrand $\frac{\tan x}{5} e^{-0.05x^4}$ can likewise be approximated by $\frac{x}{5}$. Our integral hence transforms to

$$\begin{aligned} y(h) &= e^{0.05h^4} \int_0^h \frac{\tan x}{5} e^{-0.05x^4} dx \\ &\approx \int_0^h \frac{x}{5} dx \\ &= \boxed{\frac{h^2}{10}} \end{aligned}$$

Part (c)

The improved Euler method involves more steps, while the approximation in (b) is more direct.