Problem 1.

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel vectors. The point C is such that $\overrightarrow{OC} = m\overrightarrow{OA}$ where m is a constant. The point D lies on AB produced such that B divides AD in the ratio $1:2$.

(a) Express the area of triangle ADC in the form $k\,|\mathbf{a}\times\mathbf{b}|$, where k is an expression in terms of m. Show your working clearly.

(b) If $\overrightarrow{AC}$ is a unit vector, give a geometrical interpretation of the value of $\left|\mathbf{b}\times\overrightarrow{AC}\right|$ and find the possible values of m in terms of $|\mathbf{a}|$.

Solution

Part (a)

$$\begin{aligned} \overrightarrow{OC} &= m\overrightarrow{OA} \\ &= m\mathbf{a} \\ \implies \overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\ &= m\mathbf{a} - \mathbf{a} \\ &= (m-1)\mathbf{a} \end{aligned}$$

By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OB} &= \frac{1\cdot\overrightarrow{OD} + 2\cdot\overrightarrow{OA}}{1+2} \\ \implies \overrightarrow{OD} &= 3\overrightarrow{OB} - 2\overrightarrow{OA} \\ &= 3\mathbf{b} - 2\mathbf{a} \\ \implies \overrightarrow{AD} &= \overrightarrow{OD} - \overrightarrow{OA} \\ &= 3\mathbf{b} - 3\mathbf{a} \end{aligned}$$

Thus,

$$\begin{aligned} \text{Area}\,\triangle ADC &= \frac{1}{2}\left|\overrightarrow{AC}\times\overrightarrow{AD}\right| \\ &= \frac{1}{2}\left|(m-1)\mathbf{a}\times(3\mathbf{b}-3\mathbf{a})\right| \\ &= \frac{3}{2}\left|m-1\right|\left|\mathbf{a}\times(\mathbf{b}-\mathbf{a})\right| \\ &= \frac{3}{2}\left|m-1\right|\left|\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{a}\right| \\ &= \frac{3}{2}\left|m-1\right|\left|\mathbf{a}\times\mathbf{b}\right| \end{aligned}$$

whence $k = \frac{3}{2}\left|m-1\right|$.

$$\boxed{\text{Area}\,\triangle ADC = \frac{3}{2}\left|m-1\right|\left|\mathbf{a}\times\mathbf{b}\right|}$$

Part (b)

Since $\overrightarrow{AC}$ is parallel to $\mathbf{a}$, if $\overrightarrow{AC}$ is a unit vector, then $\overrightarrow{AC} = \hat{\mathbf{a}}$. Hence, $\left|\mathbf{b} \times \overrightarrow{AC}\right| = |\mathbf{b} \times \hat{\mathbf{a}}|$ is the shortest distance from B to the line OA.

$$\begin{aligned}
\left|\overrightarrow{AC}\right| &= 1 \\
\implies |(m-1)\mathbf{a}| &= 1 \\
\implies |m-1| &= \frac{1}{|\mathbf{a}|} \\
\implies m - 1 &= \pm\frac{1}{|\mathbf{a}|} \\
\implies m &= 1 \pm \frac{1}{|\mathbf{a}|}
\end{aligned}$$

$$\boxed{m = 1 \pm \frac{1}{|\mathbf{a}|}}$$

Problem 2.

Marine biologist experts calculated that when the concentration of chemical X in a sea inlet reaches 6 milligrams per litre (mg/l), the level of pollution endangers marine life. A factory wishes to release waste containing chemical X into the inlet. It claimed that the discharge will not endanger the marine life, and they provided the local authority with the following information:

- There is no presence of chemical X in the sea inlet at present.
- The plain is to discharge chemical X on a weekly basis into the sea inlet. The discharge, which will be done at the beginning of each week, will result in an increase in concentration of 2.3 mg/l of chemical X in the inlet.
- The tidal streams will remove 7% of chemical X from the inlet at the end of every day.

(a) Form a recurrence relation for the concentration level of chemical X, u_n, at the beginning of week n. Hence, find the concentration at the beginning of week n.

(b) Should the local authority allow the factory to go ahead with the discharge if they are concerned with the marine life at the sea inlet? Justify your answer.

Solution

Part (a)

$$\boxed{u_n = 0.93^7 u_{n-1} + 2.3,\ u_0 = 0}$$

Let k be the constant such that $u_n + k = 0.93^7(u_{n-1} + k)$. Then $k = \dfrac{2.3}{0.93^7 - 1}$.

$$\begin{aligned} u_n - \frac{2.3}{1-0.93^7} &= 0.93^7\left(u_{n-1} - \frac{2.3}{1-0.93^7}\right) \\ &= 0.93^{7n}\left(u_0 - \frac{2.3}{1-0.93^7}\right) \\ &= -\frac{2.3 \cdot 0.93^{7n}}{1-0.93^7} \\ \implies \qquad u_n &= \frac{2.3}{1-0.93^7} - \frac{2.3 \cdot 0.93^{7n}}{1-0.93^7} \end{aligned}$$

$$\boxed{\text{The concentration at the beginning of week } n \text{ is } \frac{2.3}{1-0.93^7} - \frac{2.3 \cdot 0.93^{7n}}{1-0.93^7} \text{ mg/l.}}$$

Part (b)

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2.3}{1-0.93^7} - \frac{2.3 \cdot 0.93^{7n}}{1-0.93^7} = \frac{2.3}{1-0.93^7} = 5.77 \text{ (3 s.f.)}$$

Since $5.77 < 6$, if the local authority's only concern is marine life, they should allow the factory to go ahead with the discharge.

$$\boxed{\text{The local authority should allow the factory to go ahead with the discharge.}}$$

Problem 3.

Referred to the origin O, the position vector of the point A is $3\mathbf{i}-2\mathbf{j}-6\mathbf{k}$ and the Cartesian equation of the line l_1 is $x-1=2-y=2z+6$.

(a) Find the position vector of the foot of perpendicular from A to l_1.

Line l_2 has the vector equation $\mathbf{r}=\begin{pmatrix}-1\\6\\-1\end{pmatrix}+\mu\begin{pmatrix}-6\\6\\-3\end{pmatrix}$, where $\mu\in\mathbb{R}$.

(b) Find the shortest distance between l_1 and l_2.

(c) Given that l_2 is the reflection of l_1 about the line l_3, find the vector equation of the line l_3.

Solution

Part (a)

Note that l_1 has vector equation

$$l_1:\mathbf{r}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix},\ \lambda\in\mathbb{R}$$

Let F be the foot of perpendicular from A to l_1. Since F is on l_1, $\overrightarrow{OF}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}$ for some $\lambda\in\mathbb{R}$. Note also that $\overrightarrow{AF}$ is perpendicular to l_1. Hence,

$$\begin{aligned}
&& \overrightarrow{AF}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}&=0\\
\implies && \left[\overrightarrow{OF}-\overrightarrow{OA}\right]\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}&=0\\
\implies && \left[\begin{pmatrix}1\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}-\begin{pmatrix}3\\-2\\-6\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}&=0\\
\implies && \left[\begin{pmatrix}-2\\4\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}&=0\\
\implies && (-4-8+3)+\lambda(4+4+1)&=0\\
\implies && -9+9\lambda&=0\\
\implies && \lambda&=1
\end{aligned}$$

Thus, $\overrightarrow{OF}=\begin{pmatrix}1\\2\\-3\end{pmatrix}+\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\0\\-2\end{pmatrix}$.

$$\boxed{\overrightarrow{OF} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}}$$

Part (b)

Note that $\begin{pmatrix} -6 \\ 6 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Hence, l_2 is parallel to l_1. Hence, the shortest distance between l_1 and l_2 is the perpendicular distance from a point on l_1 to l_2.

$$\begin{aligned}
\text{Shortest distance} &= \left| \left[\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix} \right] \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| \Big/ \left| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| \\
&= \frac{1}{\sqrt{9}} \left| \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| \\
&= \frac{2}{3} \left| \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| \\
&= \frac{2}{3} \left| \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} \right| \\
&= \frac{2}{3} \sqrt{29}
\end{aligned}$$

$$\boxed{\text{The shortest distance between } l_1 \text{ and } l_2 \text{ is } \frac{2}{3}\sqrt{29} \text{ units.}}$$

Part (c)

Observe that l_3 passes through the midpoint of $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix}$, which evaluates to $\frac{1}{2} \left[\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 6 \\ -1 \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}$. l_3 is also parallel to both l_1 and l_2. Hence,

$$\boxed{l_3 : \mathbf{r} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \nu \in \mathbb{R}}$$

Problem 4.

A first order recurrence relation is given as

$$u_{n+1}\left[u_n+\left(\frac{1}{2}\right)^n\right]+u_n\left[\left(\frac{1}{2}\right)^{n+1}-10\right]=10\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}-16$$

where $u_1=1$.

(a) Using the substitution $u_n=\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n$ where $v_{n-1}\neq 0$, show that the recurrence relation can be expressed as a second order recurrence relation of the form $v_{n+1}+av_n+16v_{n-1}=0$, where a is a constant to be found.

(b) By first solving the second order recurrence relation in (a), find an expression for u_n in terms of n.

(c) Describe what happens to the value of u_n for large values of n.

Solution

Part (a)

$$\begin{aligned}
&u_{n+1}\left[u_n+\left(\frac{1}{2}\right)^n\right]+u_n\left[\left(\frac{1}{2}\right)^{n+1}-10\right]=10\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}-16\\
\implies&\left[\frac{v_{n+1}}{v_n}-\left(\frac{1}{2}\right)^{n+1}\right]\left[\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n+\left(\frac{1}{2}\right)^n\right]+\left[\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n\right]\left[\left(\frac{1}{2}\right)^{n+1}-10\right]\\
&\qquad=10\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}-16\\
\implies&\left[\frac{v_{n+1}}{v_n}-\left(\frac{1}{2}\right)^{n+1}\right]\left(\frac{v_n}{v_{n-1}}\right)+\left[\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^n\right]\left[\left(\frac{1}{2}\right)^{n+1}-10\right]\\
&\qquad=10\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}-16\\
\implies&\frac{v_{n+1}}{v_n}\cdot\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^{n+1}\cdot\frac{v_n}{v_{n-1}}+\frac{v_n}{v_{n-1}}\cdot\left(\frac{1}{2}\right)^{n+1}-10\cdot\frac{v_n}{v_{n-1}}-\left(\frac{1}{2}\right)^{2n+1}+10\left(\frac{1}{2}\right)^n\\
&\qquad=10\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^{2n+1}-16\\
\implies&\frac{v_{n+1}}{v_{n-1}}-10\cdot\frac{v_n}{v_{n-1}}=-16\\
\implies&v_{n+1}-10v_n+16v_{n-1}=0
\end{aligned}$$

Hence, $a=-10$.

Part (b)

Consider the characteristic equation of v_n.

$$\begin{aligned}
&x^2-10x+16=0\\
\implies&(x-2)(x-8)=0
\end{aligned}$$

Hence, 2 and 8 are the roots of the characteristic equation. Thus,

$$\boxed{v_n = A \cdot 2^n + B \cdot 8^n}$$

Consider u_1.

$$\begin{aligned}
& & u_1 &= 1 \\
&\implies & \frac{v_1}{v_0} - \frac{1}{2} &= 1 \\
&\implies & \frac{2A + 8B}{A + B} &= \frac{3}{2} \\
&\implies & \frac{4A + 16B}{A + B} &= 3 \\
&\implies & 4A + 16B &= 3A + 3B \\
&\implies & A &= -13B
\end{aligned}$$

Consider u_n.

$$\begin{aligned}
u_n &= \frac{v_n}{v_{n-1}} - \left(\frac{1}{2}\right)^n \\
&= \frac{A \cdot 2^n + B \cdot 8^n}{A \cdot 2^{n-1} + B \cdot 8^{n-1}} - \left(\frac{1}{2}\right)^n \\
&= 8\left(\frac{A \cdot 2^n + B \cdot 8^n}{4A \cdot 2^n + B \cdot 8^n}\right) - \left(\frac{1}{2}\right)^n \\
&= 8\left(1 - \frac{3A \cdot 2^n}{4A \cdot 2^n + B \cdot 8^n}\right) - \left(\frac{1}{2}\right)^n \\
&= 8\left(1 - \frac{3 \cdot -13B \cdot 2^n}{4 \cdot -13B \cdot 2^n + B \cdot 8^n}\right) - \left(\frac{1}{2}\right)^n \\
&= 8\left(1 - \frac{-39 \cdot 2^n}{-52 \cdot 2^n + 8^n}\right) - \left(\frac{1}{2}\right)^n \\
&= 8 + \frac{312 \cdot 2^n}{52 \cdot 2^n - 8^n} - \left(\frac{1}{2}\right)^n
\end{aligned}$$

$$\boxed{u_n = 8 + \frac{312 \cdot 2^n}{52 \cdot 2^n - 8^n} - \left(\frac{1}{2}\right)^n}$$

Part (c)

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(8 + \frac{312 \cdot 2^n}{52 \cdot 2^n - 8^n} - \left(\frac{1}{2}\right)^n\right) = 8$$

$$\boxed{u_n \text{ converges to 8 for large values of } n.}$$