# Problem 1.

The equation of a closed curve is $(x+2y)^2 + 3(x-y)^2 = 27$.

(a) Show, by differentiation, that the gradient at the point $(x, y)$ on the curve may be expressed in the form $\frac{dy}{dx} = \frac{y-4x}{7y-x}$.

(b) Find the equations of the tangents to the curve that are parallel to

(i) the $x$-axis,

(ii) the $y$-axis.

## Solution

### Part (a)

Implicitly differentiating the given equation,

$$\begin{aligned}
2(x+2y)\left(1+2y'\right) + 6(x-y)(1-y') &= 0 \\
(x+2y)\left(1+2y'\right) + 3(x-y)(1-y') &= 0 \\
\implies x + 2xy' + 2y + 4y \cdot y' + 3x - 3x \cdot y' - 3y + 3y \cdot y' &= 0 \\
\implies \qquad (-x+7y)y' + 4x - y &= 0 \\
\implies \qquad y' &= \frac{y-4x}{7y-x}
\end{aligned}$$

### Part (b)

**Subpart (i)**

When the tangent to the curve is parallel to the $x$-axis, $y' = 0$.

$$\begin{aligned}
y' &= 0 \\
\implies \frac{y-4x}{7y-x} &= 0 \\
\implies y - 4x &= 0 \\
\implies y &= 4x
\end{aligned}$$

Substituting $y = 4x$ into the given equation,

$$\begin{aligned}
(x + 2 \cdot 4x)^2 + 3(x-4x)^2 &= 27 \\
(9x)^2 + 3(-3x)^2 &= 27 \\
81x^2 + 27x^2 &= 27 \\
108x^2 &= 27 \\
x^2 &= \frac{1}{4} \\
x &= \pm\frac{1}{2}
\end{aligned}$$

Hence, $y = \pm 2$. Since the tangents to the curve is parallel to the $x$-axis, the equation of the tangents is $y = \pm 2$.

$$\boxed{y = \pm 2}$$

**Subpart (ii)**

When the tangent to the curve is parallel to the $y$-axis, $y'$ is undefined. Hence, $7y - x = 0 \implies x = 7y$. Substituting $x = 7y$ into the given equation,

$$\begin{aligned}
(7y + 2y)^2 + 3(7y - y)^2 &= 27 \\
(9y)^2 + 3(6y)^2 &= 27 \\
81y^2 + 108y^2 &= 27 \\
189y^2 &= 27 \\
y^2 &= \frac{1}{7} \\
y &= \pm\frac{1}{\sqrt{7}}
\end{aligned}$$

Hence, $x = \pm\frac{7}{\sqrt{7}} = \pm\sqrt{7}$. Since the tangents to the curve is parallel to the $y$-axis, the equation of the tangents is $x = \pm\sqrt{7}$.

$$\boxed{x = \pm\sqrt{7}}$$

# Problem 2.

A piece of wire of length 8 cm is cut into two piece, one of length $x$ cm, the other of length $(8-x)$ cm. The piece of length $x$ cm is bent to form a circle with circumference $x$ cm. The other piece is bent to form a square with perimeter $(8-x)$ cm. Show that, as $x$ varies, the sum of the areas enclosed by these two pieces of wire is a minimum when the radius of the circle is $\frac{4}{4+\pi}$ cm.

## Solution

Let the radius of the circle be $r$ cm. Then we have $x = 2\pi r \implies r = \frac{x}{2\pi}$. Let the side length of the square be $s$ cm. Then we have $8 - x = 4s \implies s = 2 - \frac{x}{4}$.

Let the total area enclosed by the circle and the square be $A(x)$.

$$\begin{aligned}
A(x) &= \pi r^2 + s^2 \\
&= \pi \left(\frac{x}{2\pi}\right)^2 + \left(2 - \frac{x}{4}\right)^2 \\
&= \frac{1}{4\pi}x^2 + \left(4 - x + \frac{1}{16}x^2\right) \\
&= \left(\frac{1}{4\pi} + \frac{1}{16}\right)x^2 - x + 4
\end{aligned}$$

Consider the stationary points of $A(x)$. For stationary points, $A'(x) = 0$.

$$\begin{aligned}
&& A'(x) &= 0 \\
\implies && \left(\frac{1}{4\pi} + \frac{1}{16}\right) \cdot 2x - 1 &= 0 \\
\implies && \left(\frac{1}{2\pi} + \frac{1}{8}\right) x - 1 &= 0 \\
\implies && x &= \frac{1}{\frac{1}{2\pi} + \frac{1}{8}} \\
&& &= \frac{16\pi}{8 + 2\pi} \\
&& &= \frac{8\pi}{4 + \pi}
\end{aligned}$$

| $x$ | $\left(\frac{8\pi}{4+\pi}\right)^-$ | $\frac{8\pi}{4+\pi}$ | $\left(\frac{8\pi}{4+\pi}\right)^+$ |
|---|---|---|---|
| $\frac{dA}{dx}$ | -ve | 0 | +ve |

Hence, the minimum value of $A(x)$ is achieved when $x = \frac{8\pi}{4+\pi}$, whence $r = \frac{1}{2\pi} \cdot \frac{8\pi}{4+\pi} = \frac{4}{4+\pi}$ cm.

# Problem 3.

A spherical balloon is being inflated in such a way that its volume is increasing at a constant rate of 150 $\text{cm}^3\text{s}^{-1}$. At time $t$ seconds, the radisu of the balloon is $r$ cm.

(a) Find $\dfrac{dr}{dt}$ when $r = 50$.

(b) Find the rate of increase of the surface area of the baloon when its radius is 50 cm.

## Solution

Let the volume of the balloon be $V(r) = \dfrac{4}{3}\pi r^3$ cm$^3$.

### Part (a)

Note that $\dfrac{dV}{dt} = 150$ and $\dfrac{dV}{dr} = 4\pi r^2$.

$$\begin{aligned}
\frac{dr}{dt} &= \frac{dr}{dV} \cdot \frac{dV}{dt} \\
&= \left(\frac{dV}{dr}\right)^{-1} \cdot \frac{dV}{dt} \\
&= \frac{1}{150} \cdot 4\pi r^2 \\
&= \frac{75}{2\pi r^2}
\end{aligned}$$

Evaluating $\dfrac{dr}{dt}$ at $r = 50$,

$$\begin{aligned}
\left.\frac{dr}{dt}\right|_{r=50} &= \frac{75}{2\pi \cdot 50^2} \\
&= \frac{3}{200\pi}
\end{aligned}$$

$$\boxed{\text{When } r = 50,\ \frac{dr}{dt} = \frac{3}{200\pi}}$$

### Part (b)

Let the surface area of the balloon be $A(r) = 4\pi r^2$. Observe that $\dfrac{dA}{dr} = 8\pi r$.

$$\begin{aligned}
\frac{dA}{dt} &= \frac{dA}{dr} \cdot \frac{dr}{dt} \\
\left.\frac{dA}{dt}\right|_{r=50} &= \left.\left(\frac{dA}{dr} \cdot \frac{dr}{dt}\right)\right|_{r=50} \\
&= 8\pi \cdot 50 \cdot \frac{3}{200\pi} \\
&= 6
\end{aligned}$$

$$\boxed{\text{The rate of increase of the surface area of the balloon when its radius is 50 cm is 6 cm/s.}}$$

# Problem 4.

A curve has parametric equations $x = 5\sec\theta$, $y = 3\tan\theta$, where $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Find the exact coordinates of the point on the curve at which the normal is parallel to the line $y = x$.

## Solution

Observe that $x^2 = 25\sec^2\theta$ and $\frac{25}{9}y^2 = 25\tan^2\theta$. Using the identity $\tan^2\theta + 1 = \sec^2\theta$,

$$\frac{25}{9}y^2 + 25 = x^2 \tag{4.1}$$

Implicitly differentiating, we get $\frac{25}{9} \cdot 2y \cdot y' = 2x \implies \frac{25}{9} \cdot y \cdot y' = x$.

Since the normal is parallel to $y = x$, the tangent is perpendicular is perpendicular to $y = x$. Hence, the tangent is parallel to $y = -x$, whence $y' = -1$.

$$\begin{aligned} \frac{25}{9} \cdot y \cdot -1 &= x \\ y &= -\frac{9}{25}x \end{aligned}$$

Substituting $y = -\frac{9}{25}x$ into Equation 4.1,

$$\begin{aligned} && \frac{25}{9}\left(-\frac{9}{25}x\right)^2 + 25 &= x^2 \\ \implies && \frac{9}{25} + 25 &= x^2 \\ \implies && \frac{16}{25}x^2 &= 25 \\ \implies && \left(\frac{4}{5}x\right)^2 &= 5^2 \\ \implies && \frac{4}{5}x &= \pm 5 \\ \implies && x &= \pm\frac{25}{4} \end{aligned}$$

Observe that for $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, $x = 5\sec\theta \geq 5$. We thus reject $x = -\frac{25}{4}$. Hence, $x = \frac{25}{4}$, whence $y = -\frac{9}{25} \cdot \frac{25}{4} = -\frac{9}{4}$. The coordinate of the required point is thus $\left(\frac{25}{4}, -\frac{9}{4}\right)$.

$$\boxed{\left(\frac{25}{4}, -\frac{9}{4}\right)}$$

# Problem 5.

The parametric equations of a curve are

$$x = t^2,\ y = \frac{2}{t} \tag{5.1}$$

(a) Find the equation of the tangent to the curve at the point $(p^2, \frac{2}{p})$, simplifying your answer.

(b) Hence find the coordinates of the points $Q$ and $R$ where this tangent meets the $x$- and $y$-axes respectively.

(c) The point $F$ is the mid-point of $QR$. Find a Cartesian equation of the curve traced by $F$ as $p$ varies.

## Solution

### Part (a)

Observe that $\frac{dx}{dt} = 2t$ and $\frac{dy}{dt} = -\frac{2}{t^2}$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^2 \\ &= -\frac{2}{t^2} \cdot \frac{1}{2t} \\ &= -\frac{1}{t^3}\end{aligned}$$

At the point $p^2, \frac{2}{p}$, $t = p$, whence $\frac{dy}{dx} = -\frac{1}{p^3}$. Using the point-slope formula, the tangent to the curve is given by the equation

$$\begin{aligned} && y - \frac{2}{p} &= -\frac{1}{p^3}\left(x - p^2\right) \\ \implies && y &= \frac{2}{p} - \frac{1}{p^3}\left(x - p^2\right) \\ && &= \frac{2}{p} - \frac{1}{p^3}x + \frac{1}{p} \\ && &= \frac{3}{p} - \frac{1}{p^3}x \end{aligned}$$

$$\boxed{y = \frac{3}{p} - \frac{1}{p^3}x}$$

**Part (b)**

**Case 1:** $y = 0$

$$\begin{aligned} 0 &= \frac{3}{p} - \frac{1}{p^3}x \\ \implies \frac{1}{p^3}x &= \frac{3}{p} \\ \implies \quad x &= 3p^2 \end{aligned}$$

Hence, $Q(3p^2, 0)$.
**Case 2:** $x = 0$

$$\begin{aligned} y &= \frac{3}{p} - \frac{1}{p^3} \cdot 0 \\ &= \frac{3}{p} \end{aligned}$$

Hence, $R\left(0, \frac{3}{p}\right)$.

$$\boxed{Q(3p^2, 0),\ R\left(0, \frac{3}{p}\right)}$$

**Part (c)**

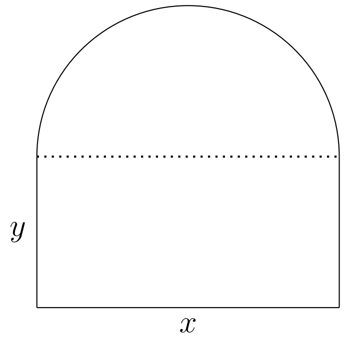
$$\begin{aligned} F &= \left(\frac{1}{2} \cdot 3p^2, \frac{1}{2} \cdot \frac{3}{p}\right) \\ &= \left(\frac{3}{2}p^2, \frac{3}{2p}\right) \end{aligned}$$

As $p$ varies, $F$ traces a curve given by the parametric equations $x = \frac{3}{2}p^2$, $y = \frac{3}{2p}$. Observe that $p = \frac{3}{2y}$.

$$\begin{aligned} x &= \frac{3}{2}\left(\frac{3}{2y}\right)^2 \\ &= \left(\frac{3}{2}\right)^3 \frac{1}{y^2} \\ \implies y^2 &= \left(\frac{3}{2}\right)^3 \frac{1}{x} \\ \implies \quad y &= \pm\sqrt{\left(\frac{3}{2}\right)^3 \frac{1}{x}} \\ \implies \quad y &= \pm\left(\frac{3}{2}\right)^{\frac{3}{2}} \frac{1}{\sqrt{x}} \end{aligned}$$

$$\boxed{y = \pm\left(\frac{3}{2}\right)^{\frac{3}{2}} \frac{1}{\sqrt{x}}}$$

# Problem 6.




A new flower-bed is being designed for a large garden. The flower-bed will occupy a rectangle $x$ m by $y$ m together with a semicircle of diameter $x$ m, as shown in the diagram. A low wall will be built around the flowerbed. The time needed to build the wall will be 3 hours per metre for the straight parts and 9 hours per metre for the semicircular part. Given that a total time of 180 hours is taken to build the wall, find, using differentiation, the values of $x$ and $y$ which give a flower-bed of maximum area.

## Solution

Observe that the length of the straight parts is $(2y + x)$ m, while the length of the semicicular part is $\frac{1}{2} \cdot 2\pi \left(\frac{x}{2}\right) = \frac{1}{2}\pi x$ m. Since a total time of 180 hours is taken to build the wall,

$$
\begin{aligned}
& 3(2y + x) + 9\left(\frac{1}{2}\pi x\right) = 180 \\
\implies \quad & (2y + x) + 3\left(\frac{1}{2}\pi x\right) = 60 \\
\implies \quad & 2y + x + \frac{3}{2}\pi x = 60 \\
\implies \quad & 4y + 2x + 3\pi x = 120 \\
\implies \quad & (2 + 3\pi)x = 120 - 4y \\
\implies \quad & x = \frac{120 - 4y}{2 + 3\pi}
\end{aligned}
$$

Let $A(y)$ be the total area enclosed by the garden, in $m^2$. Observe that $A(y) = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = xy + \frac{1}{8}\pi x^2$. Also note that $x' = -\frac{4}{2 + 3\pi}$. Now, consider the stationary points of $A(y)$. For stationary points, $A'(y) = 0$.

$$
\begin{aligned}
& A'(y) = 0 \\
\implies & (x'y + x) + \frac{1}{8}\pi(2x \cdot x') = 0 \\
\implies \quad & x'y + x + \frac{1}{4}\pi(x \cdot x') = 0
\end{aligned}
$$

Substituting $x = \dfrac{120-4y}{2+3\pi}$ and $x' = -\dfrac{4}{2+3\pi}$,

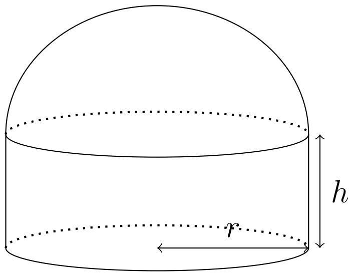
$$\begin{aligned}
& -\frac{4}{2+3\pi}\cdot y + \frac{120-4y}{2+3\pi} + \frac{1}{4}\pi\left(\frac{120-4y}{2+3\pi}\cdot -\frac{4}{2+3\pi}\right) = 0 \\
\implies\quad & -4\cdot y + (120-4y) + \frac{1}{4}\pi\left(\frac{120-4y}{2+3\pi}\cdot -4\right) = 0 \\
\implies\quad & 120 - 8y - \pi\left(\frac{120-4y}{2+3\pi}\right) = 0 \\
\implies\quad & 120(2+3\pi) - 8(2+3\pi)y - \pi(120-4y) = 0 \\
\implies\quad & 240 + 360\pi - 16y - 24\pi y - 120\pi + 4\pi y = 0 \\
\implies\quad & 240 + 240\pi - 16y - 20\pi y = 0 \\
\implies\quad & 60 + 60\pi - 4y - 5\pi y = 0 \\
\implies\quad & 4y + 5\pi y = 60 + 60\pi \\
\implies\quad & (4+5\pi)y = 60 + 60\pi \\
\implies\quad & y = \frac{60+60\pi}{4+5\pi}
\end{aligned}$$

Using the equation $x = \dfrac{120-4y}{2+3\pi}$,

$$\begin{aligned}
x &= \frac{120 - 4\cdot\frac{60+60\pi}{4+5\pi}}{2+3\pi} \\
&= \frac{120(4+5\pi) - 4(60+60\pi)}{(2+3\pi)(4+5\pi)} \\
&= \frac{480 + 600\pi - 240 - 240\pi}{(2+3\pi)(4+5\pi)} \\
&= \frac{240 + 360\pi}{(2+3\pi)(4+5\pi)} \\
&= \frac{120(2+3\pi)}{(2+3\pi)(4+5\pi)} \\
&= \frac{120}{4+5\pi}
\end{aligned}$$

$$\boxed{x = \frac{120}{4+5\pi},\ y = \frac{60+60\pi}{4+5\pi}}$$

# Problem 7.




A model of a concert hall is made up of three parts.

- The roof is modelled by the curved surface of a hemisphere of radius $r$ cm.
- The walls are modelled by the curved surface of a cylinder of radius $r$ cm and height $h$ cm.
- The floor is modelled by a circular disc of radius $r$ cm.

The three parts are joined together as shown in the diagram. The model is made of material of negligible thickness.

(a) It is given that the volume of the model is a fixed value $k$ cm$^3$, and the external surface area is a minimum. Use differentiation to find the values of $r$ and $h$ in terms of $k$. Simplify your answers.

(b) It is given instead that the volume of the model is 200 cm$^3$ and its external surface area is 180 cm$^2$. Show that there are two possible values of $r$. Given also that $r < h$, find the value of $r$ and the value of $h$.

## Solution

### Part (a)

Let the volume of the model be $V$ cm$^3$. Then

$$V = \frac{1}{2} \cdot \frac{4}{3}\pi r^3 + \pi r^2 h = k \tag{7.1}$$

$$\implies \frac{2}{3}r + h = \frac{k}{\pi r^2}$$

$$\implies h = \frac{k}{\pi r^2} - \frac{2}{3}r \tag{7.2}$$

Let the external surface area of the model be $A$ cm$^2$. Then

$$\begin{aligned} A &= \frac{1}{2} \cdot 4\pi r^2 + 2\pi r h + \pi r^2 \\ &= 3\pi r^2 + 2\pi r h \\ &= 3\pi r^2 + 2\pi r \left(\frac{k}{\pi r^2} - \frac{2}{3}r\right) \\ &= 3\pi r^2 + \frac{2k}{r} - \frac{4}{3}\pi r^2 \\ &= \frac{5}{3}\pi r^2 + \frac{2k}{r} \end{aligned} \tag{7.3}$$

Consider the stationary points of $A$. For stationary points, $\frac{dA}{dr} = 0$.

$$\begin{aligned} \frac{dA}{dr} &= 0 \\ \implies \frac{5}{3}\pi \cdot 2r - \frac{2k}{r^2} &= 0 \\ \implies \frac{5}{3}\pi r^3 - k &= 0 \\ \implies r^3 &= \frac{3k}{5\pi} \\ \implies r &= \sqrt[3]{\frac{3k}{5\pi}} \end{aligned}$$

| $r$ | $\sqrt[3]{\frac{3k}{5\pi}}^-$ | $\sqrt[3]{\frac{3k}{5\pi}}$ | $\sqrt[3]{\frac{3k}{5\pi}}^+$ |
|---|---|---|---|
| $\frac{dA}{dr}$ | -ve | 0 | +ve |

Hence, $A$ is at a minimum when $r = \sqrt[3]{\frac{3k}{5\pi}}$.

Substituting $r = \sqrt[3]{\frac{3k}{5\pi}}$ into Equation 7.1,

$$\begin{aligned} \frac{2}{3}\pi\left(\frac{3k}{5\pi}\right) + \pi r^2 h &= k \\ \implies \frac{2}{5}k + \pi r^2 h &= k \\ \implies \pi r^2 h &= \frac{3}{5}k \\ \implies r^2 h &= \frac{3k}{5\pi} \\ \implies r^2 h &= r^3 \\ \implies h &= r \\ &= \sqrt[3]{\frac{3k}{5\pi}} \end{aligned}$$

$$\boxed{r = \sqrt[3]{\frac{3k}{5\pi}},\ h = \sqrt[3]{\frac{3k}{5\pi}}}$$

**Part (b)**

From Equation 7.3, we have

$$
\begin{aligned}
\frac{5}{3}\pi r^2 + \frac{2 \cdot 200}{r} &= 180 \\
\implies \frac{5}{3}\pi r^2 + \frac{400}{r} - 180 &= 0 \\
\implies \frac{5}{3}\pi r^3 - 180r + 400 &= 0 \\
\implies \pi r^3 - 108r + 240 &= 0
\end{aligned}
$$

Let $f(r) = \pi r^3 - 108r + 240$. Consider the stationary points of $f(r)$. For stationary points, $f'(r) = 0$.

$$
\begin{aligned}
f'(r) &= 0 \\
\implies 3\pi r^2 - 108 &= 0 \\
\implies r^2 &= \frac{36}{\pi}
\end{aligned}
$$

Hence, $r = \pm\sqrt{\frac{36}{\pi}} = \pm\frac{6}{\sqrt{\pi}}$. Consider the case when $r = \frac{6}{\sqrt{\pi}}$.

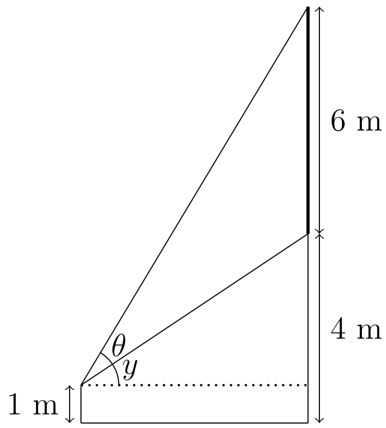
| $r$ | $\frac{6}{\sqrt{\pi}}^{-}$ | $\frac{6}{\sqrt{\pi}}$ | $\frac{6}{\sqrt{\pi}}^{+}$ |
|---|---|---|---|
| $f'(r)$ | -ve | 0 | +ve |

Hence, $f(r)$ attains a minimum when $r = \frac{6}{\sqrt{\pi}}$. Since $f\left(\frac{6}{\sqrt{\pi}}\right) < 0$ and $f(0) > 0$, there exist positive $r_1$ and $r_2$ such that $r_1 < r_2$ and $f(r_1) = f(r_2) = 0$. There are hence two possible values of $r$, which are 3.04 and 3.72 respectively.

From Equation 7.2, we know that $h = \frac{200}{\pi r^2} - \frac{2}{3}r$. When $r = 3.04$, $h = 4.88 > r$. When $r = 3.72$, $h = 2.12 < r$. Thus, given that $r < h$, $r = 3.04$ and $h = 4.88$.

$$\boxed{r = 3.04,\ h = 4.88}$$

# Problem 8.




A movie screen on a vertical wall is 6 m high and 4 m above the horizontal floor. A boy who is standing at $x$ m away from the wall has eye level at 1 m above the floor as shown in the diagram.

The viewing angle of the boy at that position is $\theta$ and the angle of elevation of the bottom of the screen is $y$.

(a) Express $y$ in terms of $x$.

(b) By expressing $\theta$ in terms of $x$ or otherwise, find the stationary value of $\theta$, giving your answers in exact form. Determine if the value is a maximum or minimum value, showing your working clearly.

## Solution

### Part (a)

Observe that $\tan y = \frac{3}{x}$, whence $y = \arctan \frac{3}{x}$.

$$\boxed{y = \arctan \frac{3}{x}}$$

### Part (b)

Observe that $\tan(y+\theta) = \frac{9}{x}$.

$$\begin{aligned} \tan(y+\theta) &= \frac{9}{x} \\ \implies \frac{\tan y + \tan\theta}{1 - \tan y \tan\theta} &= \frac{9}{x} \\ \implies \frac{\frac{3}{x} + \tan\theta}{1 - \frac{3}{x}\tan\theta} &= \frac{9}{x} \\ \implies \frac{3 + x\tan\theta}{x - 3\tan\theta} &= \frac{9}{x} \\ \implies x(3 + x\tan\theta) &= 9(x - 3\tan\theta) \end{aligned}$$

$$\begin{aligned}
\implies \quad 3x + x^2 \tan\theta &= 9x - 27\tan\theta \\
\implies x^2 \tan\theta + 27\tan\theta &= 6x \\
\implies \quad (x^2 + 27)\tan\theta &= 6x \\
\implies \quad \tan\theta &= \frac{6x}{x^2 + 27}
\end{aligned}$$

Implicitly differentiating,

$$\begin{aligned}
\sec^2(\theta) \cdot \theta' &= \frac{(x^2 + 27) \cdot 6 - 6x \cdot 2x}{(x^2 + 27)^2} \\
\implies \quad \theta' &= \frac{-6x^2 + 162}{(x^2 + 27)^2 \sec^2\theta}
\end{aligned}$$

Using the identity $\tan^2\theta + 1 = \sec^2\theta$, we have $\sec^2\theta = \left(\frac{6x}{x^2 + 27}\right)^2 + 1$.

$$\begin{aligned}
\theta' &= \frac{-6x^2 + 162}{(x^2 + 27)^2 \left(\left(\frac{6x}{x^2 + 27}\right)^2 + 1\right)} \\
&= \frac{-6x^2 + 162}{(6x)^2 + (x^2 + 27)^2} \\
&= \frac{-6x^2 + 162}{36x^2 + (x^2 + 27)^2}
\end{aligned}$$

For stationary points, $\theta' = 0$. Hence,

$$\begin{aligned}
-6x^2 + 162 &= 0 \\
\implies \quad x^2 &= 27 \\
\implies \quad x &= \pm\sqrt{27}
\end{aligned}$$

Since $x > 0$, we only take $x = \sqrt{27} = 3\sqrt{3}$. Thus, $\tan\theta = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$, whence $\theta = \frac{\pi}{6}$.

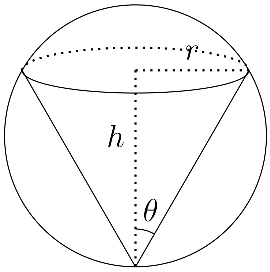
$$\boxed{\theta = \frac{\pi}{6}}$$

| $x$ | $\sqrt{27}^-$ | $\sqrt{27}$ | $\sqrt{27}^+$ |
|---|---|---|---|
| $\theta'$ | +ve | 0 | -ve |

Thus, by the First Derivative Test, $\theta = \frac{\pi}{6}$ is a maximum value.

$$\boxed{\theta = \frac{\pi}{6} \text{ is a maximum value.}}$$

# Problem 9.




The diagram shows a right inverted cone of radius $r$, height $h$ and semi-vertical angle $\theta$, which is inscribed in a sphere of radius 1 unit.

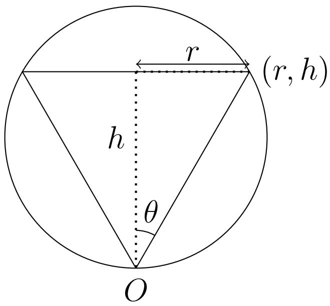
Prove that $r^2 = 2h - h^2$.

(a) As $r$ and $h$ varies, determine the exact maximum volume of the cone.

(b) Show that $h = 2\cos^2\theta$. The volume of the cone is increasing at a rate of 6 unit$^3$/s when $h = \dfrac{3}{2}$. Determine the rate of change of $\theta$ at this instant, leaving your answer in an exact form.

## Solution

Consider the following diagram of the cone and sphere.




Let the origin be the tip of the cone. Since the sphere has radius 1 unit, the circle is given by the equation $x^2 + (y-1)^2 = 1$. Since the point $(r, h)$ lies on the circle,

$$\begin{aligned} & r^2 + (h-1)^2 = 1 \\ \implies & r^2 + h^2 - 2h + 1 = 1 \\ \implies & r^2 = 2h - h^2 \end{aligned} \tag{9.1}$$

### Part (a)

Implicitly differentiating Equation 9.1,

$$\begin{aligned} & 2r = 2h' - 2h \cdot h' \\ \implies & r = h' - h \cdot h' \\ & \phantom{r} = h'(1-h) \\ \implies & h' = \frac{r}{1-h} \end{aligned}$$

Let the volume of the cone be $V(r) = \frac{1}{3}\pi r^2 h$. Differentiating $V(r)$,

$$\begin{aligned} V'(r) &= \frac{1}{3}\pi(r^2h' + h \cdot 2r) \\ &= \frac{1}{3}\pi\left(\left(2h - h^2\right)\left(\frac{r}{1-h}\right) + 2hr\right) \\ &= \frac{1}{3}\pi\left(hr \cdot \frac{2-h}{1-h} + 2hr\right) \\ &= \frac{1}{3}\pi rh\left(\frac{2-h}{1-h} + 2\right) \\ &= \frac{1}{3} \cdot \frac{\pi rh}{1-h}(2 - h + 2(1-h)) \\ &= \frac{1}{3} \cdot \frac{\pi rh}{1-h}(2 - h + 2 - 2h) \\ &= \frac{1}{3} \cdot \frac{\pi rh}{1-h}(4 - 3h) \end{aligned}$$

Consider the stationary values of $V(r)$. For stationary values, $V'(r) = 0$.

$$\begin{aligned} && V'(r) &= 0 \\ \implies && \frac{1}{3} \cdot \frac{\pi rh}{1-h}(4-3h) &= 0 \\ \implies && 4 - 3h &= 0 \\ \implies && h &= \frac{4}{3} \end{aligned}$$

Substituting $h = \frac{4}{3}$ into Equation 9.1, we obtain $r^2 = 2 \cdot \frac{4}{3} - \left(\frac{4}{3}\right)^2 = \frac{8}{9}$, whence $r = \frac{2\sqrt{2}}{3}$ (we reject $r = -\frac{2\sqrt{2}}{3}$ as $r > 0$).

| $r$ | $\frac{2\sqrt{2}}{3}^-$ | $\frac{2\sqrt{2}}{3}$ | $\frac{2\sqrt{2}}{3}^+$ |
|---|---|---|---|
| $V'(r)$ | +ve | 0 | -ve |

Hence, the maximum volume is achieved when $r = \frac{2\sqrt{2}}{3}$.

$$\begin{aligned} V\left(\frac{2\sqrt{2}}{3}\right) &= \frac{1}{3}\pi \cdot \frac{8}{9} \cdot \frac{4}{3} \\ &= \frac{32}{81}\pi \end{aligned}$$

$$\boxed{\text{The maximum volume of the cone is } \frac{32}{81}\pi \text{ units}^3.}$$

**Part (b)**

From the diagram, we see that $\cos\theta = \dfrac{h}{\sqrt{r^2+h^2}}$.

$$\begin{aligned}
\cos\theta &= \frac{h}{\sqrt{r^2+h^2}} \\
\implies \quad \cos^2\theta &= \frac{h^2}{r^2+h^2} \\
\implies 2\cos^2\theta &= \frac{2h^2}{r^2+h^2} \\
&= \frac{2h^2}{2h-h^2+h^2} \\
&= \frac{2h^2}{2h} \\
&= h
\end{aligned}$$

$$\begin{aligned}
2\cos^2\theta - 1 &= h-1 \\
\implies \qquad \cos 2\theta &= h-1 \\
\implies \qquad \cos^2 2\theta &= (h-1)^2 \\
\implies \qquad \sin^2 2\theta &= 1-(h-1)^2 \\
\implies \qquad \sin 2\theta &= \pm\sqrt{1-(h-1)^2}
\end{aligned}$$

Since $0 < \theta < \dfrac{\pi}{2}$, we know $\sin 2\theta > 0$. We thus take $\sin 2\theta = \sqrt{1-(h-1)^2}$.
Implicitly differentiating $2\cos^2\theta = h$ with respect to $h$,

$$\begin{aligned}
2\cdot 2\cos\theta\cdot -\sin\theta\cdot\frac{d\theta}{dh} &= 1 \\
\implies \qquad \frac{d\theta}{dh} &= \frac{1}{-4\sin\theta\cos\theta} \\
&= \frac{1}{-2\sin 2\theta} \\
&= \frac{1}{-2\sqrt{1-(h-1)^2}}
\end{aligned}$$

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{d\theta}{dh}\cdot\frac{dh}{dr}\cdot\frac{dr}{dV}\cdot\frac{dV}{dt} \\
&= \frac{d\theta}{dh}\cdot\frac{dh}{dr}\cdot\left(\frac{dV}{dr}\right)^{-1}\cdot\frac{dV}{dt} \\
&= \frac{1}{-2\sqrt{1-(h-1)^2}}\cdot\frac{r}{1-h}\cdot\left(\frac{1}{3}\cdot\frac{\pi r h}{1-h}(4-3h)\right)^{-1}\cdot\frac{dV}{dt} \\
&= \frac{1}{-2\sqrt{1-(h-1)^2}}\cdot\frac{r}{1-h}\cdot 3\cdot\frac{1-h}{\pi r h(4-3h)}\cdot\frac{dV}{dt} \\
&= \frac{3}{\pi}\cdot\frac{1}{-2\sqrt{1-(h-1)^2}}\cdot\frac{1}{h(4-3h)}\cdot\frac{dV}{dt}
\end{aligned}$$

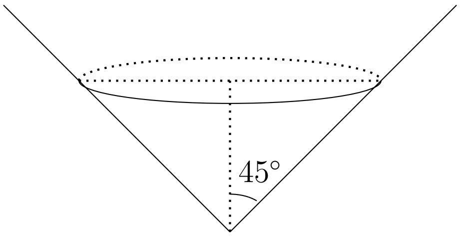
Evaluating $\dfrac{d\theta}{dt}$ at $h = \dfrac{3}{2}$,

$$\begin{aligned}
\left.\frac{d\theta}{dt}\right|_{h=\frac{3}{2}} &= \frac{3}{\pi} \cdot \frac{1}{-2\sqrt{1-(\frac{3}{2}-1)^2}} \cdot \frac{1}{\frac{3}{2}(4-3\cdot\frac{3}{2})} \cdot 6 \\
&= \frac{18}{\pi} \cdot -\frac{1}{\sqrt{3}} \cdot -\frac{4}{3} \\
&= \frac{24}{\pi} \cdot \frac{\sqrt{3}}{3} \\
&= \frac{8\sqrt{3}}{\pi}
\end{aligned}$$

Hence, $\theta$ is increasing at a rate of $\dfrac{8\sqrt{3}}{\pi}$ radians per second when $h = \dfrac{3}{2}$.

$$\boxed{\frac{8\sqrt{3}}{\pi}\text{ rad/s}}$$

# Problem 10.

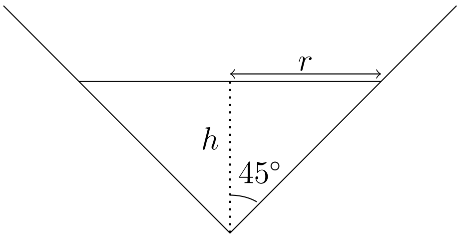



A hollow cone of semi-vertical angle 45° is held with its axis vertical and vertex downwards. At the beginning of an experiment, it is follows with 390 cm$^3$ of liquid. The liquid runs out thorugh a small hole at the vertex at a constant rate of 2 cm$^3$/s.

Find the rate at which the depth of the liquid is decreasing 3 minutes after the start of the experiment.

## Solution

Consider the following diagram.




Let the volume of liquid be $V = \dfrac{1}{3}\pi r^2 h$ cm$^3$. From the diagram, we have $r = h$. Thus,

$$V = \frac{1}{3}\pi h^3$$

Differentiating $V$ with respect to $h$,

$$\begin{aligned}\frac{dV}{dh} &= \frac{1}{3}\pi \cdot 3h^2 \\ &= \pi h^2\end{aligned}$$

Let $t$ be the time since the start of the experiment in seconds. Consider $\dfrac{dh}{dt}$.

$$\begin{aligned}\frac{dh}{dt} &= \frac{dh}{dV} \cdot \frac{dV}{dt} \\ &= \left(\frac{dh}{dV}\right)^{-1} \cdot \frac{dV}{dt} \\ &= \frac{1}{\pi h^2} \cdot -2 \\ &= \frac{-2}{\pi h^2}\end{aligned}$$

When $t = 180$, there is $390 - 180 \cdot 2 = 30$ cm$^3$ of liquid left in the cone.

$$\begin{aligned} V &= 30 \\ \implies \frac{1}{3}\pi h^3 &= 30 \\ \implies h^3 &= \frac{90}{\pi} \\ \implies h &= \sqrt[3]{\frac{90}{\pi}} \end{aligned}$$

Evaluating $\frac{dh}{dt}$ at $t = 180$,

$$\begin{aligned} \left.\frac{dh}{dt}\right|_{t=180} &= \left.\frac{dh}{dt}\right|_{h=\sqrt[3]{\frac{90}{\pi}}} \\ &= \frac{-2}{\pi\left(\sqrt[3]{\frac{90}{\pi}}\right)^2} \\ &= -0.0680 \text{ (3 s.f.)} \end{aligned}$$

$\boxed{\text{The depth of the liquid is decreasing at a rate of 0.0680 cm/s 3 minutes after the start of the experiment.}}$

# Problem 11.

A particle is projected from the origin $O$ and it moes freely under gravity in the $x$-$y$ plane. At time $t$ s after projection, the particle is at the point $(x, y)$ where $x = 30t$ and $y = 20t - 5t^2$, with $x$ and $y$ measured in meteres.

(a) Given that the particle passes through two points $A$ and $B$ which are at a distance 15 m above the $x$-axis, find the time taken for the particle to travel from $A$ to $B$. Find also the distance $AB$.

(b) It is known that the particle always travels in a direction tangential to its path. Show that, when $x = 10$, the particle is travelling at an angle of $\arctan \frac{5}{9}$ above the horizontal.

The speed of the particle is given by $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$. Find the speed of the particle when $x = 10$.

(c) Show that the equation of trajectory is $y = \frac{2}{3}x - \frac{1}{180}x^2$.

## Solution

### Part (a)

Consider $y = 15$.

$$\begin{aligned} y &= 15 \\ 20t - 5t^2 &= 15 \\ t^2 - 4t + 3 &= 0 \\ (t-1)(t-3) &= 0 \end{aligned}$$

Hence, $t = 1$ or $t = 3$. Thus, the particle takes $3 - 1 = 2$ seconds to travel from $A$ to $B$.

$$\boxed{2 \text{ seconds}}$$

**Case 1:** $t = 1$ When $t = 1$, $x = 30 \cdot 1 = 30$. Thus, $A(30, 15)$.
**Case 2:** $t = 3$ When $t = 3$, $x = 30 \cdot 3 = 90$. Thus, $B(90, 15)$.

The distance $AB$ is thus $90 - 30 = 60$ m.

$$\boxed{AB = 60 \text{ m}}$$

### Part (b)

Note that $\frac{dx}{dt} = 30$ and $\frac{dy}{dt} = 20 - 10t$. Thus

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \\ &= (20 - 10t) \cdot \frac{1}{30} \\ &= \frac{2-t}{3}\end{aligned}$$

When $x = 10$, $t = \frac{1}{3}$. Evaluating $\frac{dy}{dx}$ at $t = \frac{1}{3}$,

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{t=\frac{1}{3}} &= \frac{2 - \frac{1}{3}}{3} \\ &= \frac{5}{9}\end{aligned}$$

Hence, the line tangent to the curve at $x = 10$ has gradient $\frac{5}{9}$. Thus, the particle is travelling at an angle of $\arctan \frac{5}{9}$ above the horizontal when $x = 10$.

$$\begin{aligned}\left.\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\right|_{t=\frac{1}{3}} &= \sqrt{30^2 + (20 - \tfrac{10}{3})^2} \\ &= 34.3 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The particle is travelling at a speed of 34.3 m/s when } x = 10.}$$

**Part (c)**

Note that $t = \frac{x}{30}$. Hence,

$$\begin{aligned}y &= 20t - 5t^2 \\ &= 20 \cdot \frac{x}{30} - 5\left(\frac{x}{30}\right)^2 \\ &= \frac{2}{3}x - \frac{1}{180}x^2\end{aligned}$$