Problem 1.

On a single Argand diagram, sketch the following loci.

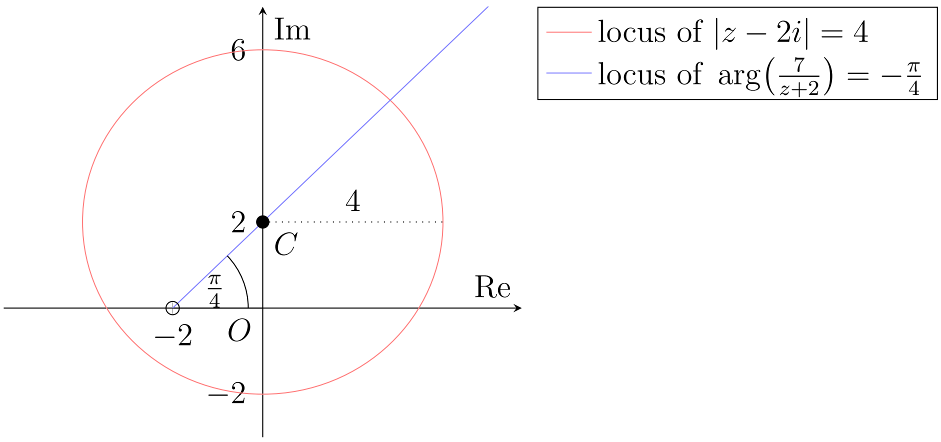
(a) $|z - 2i| = 4$.

(b) $\arg\left(\dfrac{7}{z+2}\right) = -\dfrac{\pi}{4}$.

Hence, or otherwise, find the exact value of z satisfying both equations in part (a) and (b).

Solution

Note that $\arg\left(\dfrac{7}{z+2}\right) = -\dfrac{\pi}{4} \implies \arg(z-(-2)) = \dfrac{\pi}{4}$.



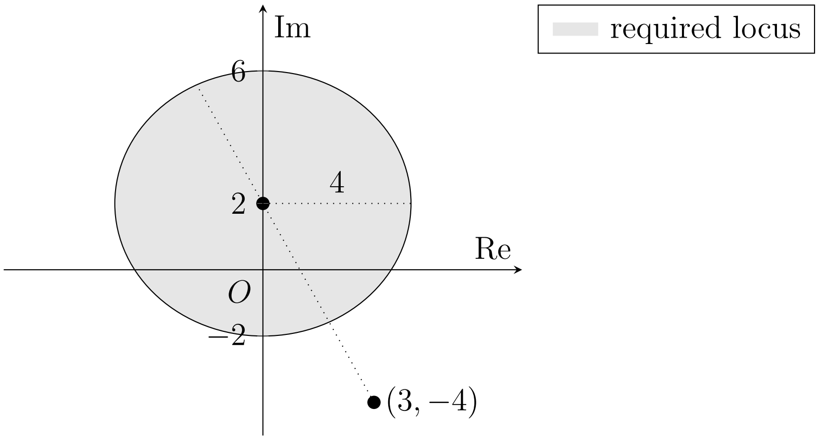
$$
\begin{aligned}
z &= 2i + \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\
&= 2i + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\
&= \boxed{\frac{\sqrt{2}}{2} + \left(2 + \frac{\sqrt{2}}{2}\right)i}
\end{aligned}
$$

Problem 2.

Given that $|z - 2i| \leq 4$, illustrate the locus of the point representing the complex number z in an Argand diagram.

Hence, find the greatest and least possible value of $|z - 3 + 4i|$, given that $|z - 2i| \leq 4$.

Solution



Note that $|z - 3 + 4i| = |z - (3 - 4i)|$ represents the distance between z and the point $(3, -4)$.

By Pythagoras' Theorem, the distance between the centre of the circle $(0, 2)$ and $(3, -4)$ is $\sqrt{(0-3)^2 + (2+4)^2} = 3\sqrt{5}$. Hence, $\max|z - 3 + 4i| = 3\sqrt{5} + 4$, while $\min|z - 3 + 4i| = 3\sqrt{5} - 4$.

$$\boxed{\max|z - 3 + 4i| = 3\sqrt{5} + 4,\ \min|z - 3 + 4i| = 3\sqrt{5} - 4}$$

Problem 3.

The point A on an Argand diagram represents the fixed complex number a, where $0 < \arg a < \frac{\pi}{2}$. The complex numbers z and w are such that $|z - 2ia| = |a|$ and $|w| = |w + ia|$.

Sketch, on a single diagram, the loci of the point representing z and w.

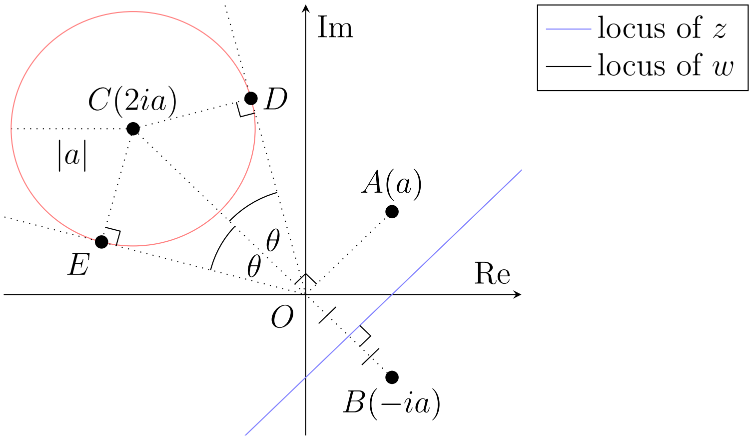
Find

(a) the minimum value of $|z - w|$ in terms of $|a|$,

(b) the range of values of $\arg \frac{1}{z}$ in terms of $\arg a$.

Solution

Note that $|w| = |w + ia| \implies |w - 0| = |w - (-ia)|$.



Part (a)

Let $B(-ia)$ and $C(2ia)$. Note that $W\left(-\frac{1}{2}ia\right)$ lies on the locus of w as well as the line passing through OC. Since CW is perpendicular to the locus of w, it follows that the minimum value of $|z - w|$ is given by

$$\begin{aligned} CW - |a| &= \left|2ia + \frac{1}{2}ia\right| - |a| \\ &= \frac{5}{2}|a|\,|i| - |a| \\ &= \frac{3}{2}|a| \end{aligned}$$

$$\boxed{\text{The minimum value of } |z - w| \text{ is } \frac{3}{2}|a|.}$$

Part (b)

Let D and E be such that OD and OE are tangent to the circle given by the locus of z. Let $\angle COD = \theta$. Observe that $\sin\theta = \frac{CD}{CO} = \frac{|a|}{|2ia|} = \frac{1}{2}$, whence $\theta = \arcsin\frac{1}{2}$. Since

$\angle COA = \arg i = \dfrac{\pi}{2}$, it follows that $\angle DOA = \dfrac{\pi}{2} - \theta = \dfrac{\pi}{2} - \arcsin\dfrac{1}{2} = \arccos\dfrac{1}{2}$. Thus, $\min\arg z = \arg a + \angle DOA = \arg a + \arccos\dfrac{1}{2}$. Meanwhile, $\angle COE = \angle COD = \theta$, whence $max \arg z = \arg a + \dfrac{\pi}{2} + \theta$. Since $\arg\dfrac{1}{z} = -\arg z$, we thus have

$$\boxed{\arg\frac{1}{z} \in \left[-\left(\arg a + \frac{\pi}{2} + \arcsin\frac{1}{2}\right), -\left(\arg a + \arccos\frac{1}{2}\right)\right]}$$

Problem 4.

(a) Solve the equation

$$z^7 - (1+i) = 0,$$

giving the roots in the form $re^{i\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$.

(b) Show the roots on an Argand diagram.

(c) The roots represented by z_1 and z_2 are such that $0 < \arg z_1 < \arg z_2 < \frac{\pi}{2}$. Explain why the locus of all points z such that $|z - z_1| = |z - z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its Cartesian equation.

(d) Describe the transformation that will map the points representing the roots of the equation $z^7 - (1+i) = 0$ to the points representing the roots of the equation $(z-2)^7 7 - (1+i) = 0$ on the Argand diagram.

Solution

Part (a)

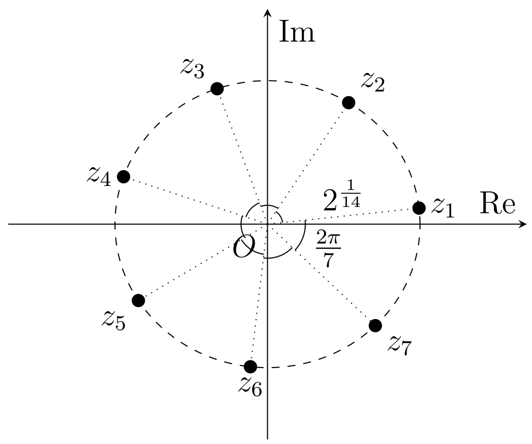
Note that $1 + i = \sqrt{2}e^{i\pi\frac{1}{4}} = 2^{\frac{1}{2}}e^{i\pi(\frac{1}{4}+2k)}$, where $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} z^7 - (1+i) &= 0 \\ \implies \quad z^7 &= 2^{\frac{1}{2}}e^{i\pi(\frac{1}{4}+2k)} \\ \implies \quad z &= 2^{\frac{1}{14}}e^{i\pi(\frac{1}{4}+2k)/7} \\ &= 2^{\frac{1}{14}}e^{i\pi(1+8k)/28} \end{aligned}$$

Taking $k \in \{-3, -2, \ldots, 2, 3\}$,

$$\boxed{z = 2^{\frac{1}{14}}e^{-i\pi\frac{23}{28}}, 2^{\frac{1}{14}}e^{-i\pi\frac{15}{28}}, 2^{\frac{1}{14}}e^{-i\pi\frac{7}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{1}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{9}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{17}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{25}{28}}}$$

Part (b)



Part (c)

Since $|z_1| = |z_2| = 2^{\frac{1}{14}}$, the distance between z_1 and the origin and the distance between z_2 and the origin are equal. Since the locus of $|z - z_1| = |z - z_2|$ represents all points equidistant from z_1 and z_2, it passes through the origin.

Observe that the midpoint of z_1 and z_2 will have argument $\frac{1}{2}\left(\frac{1}{28}\pi + \frac{9}{28}\pi\right) = \frac{5}{28}\pi$. Thus, the Cartesian equation of the locus of z is given by

$$\boxed{y = \tan\left(\frac{5}{28}\pi\right)x}$$

Part (d)

$$\boxed{\text{Translate the points 2 units in the positive real direction.}}$$