Problem 1.

Evaluate the following limits.

(a) $\lim\limits_{x \to 5}(6x + 7)$

(b) $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{1 - x}$

(c) $\lim\limits_{x \to \infty} \dfrac{3x}{2x^2 - 5}$

Solution

Part (a)

$$\lim_{x \to 5}(6x + 7) = 6 \cdot 5 + 7$$
$$= 37$$

$$\boxed{\lim_{x \to 5}(6x + 7) = 37}$$

Part (b)

$$\lim_{x \to 1} \frac{x^3 - 1}{1 - x} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{1 - x}$$
$$= \lim_{x \to 1} -\left(x^2 + x + 1\right)$$
$$= -(1^2 + 1 + 1)$$
$$= -3$$

$$\boxed{\lim_{x \to 1} \frac{x^3 - 1}{1 - x} = -3}$$

Part (c)

$$\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = \lim_{x \to \infty} \frac{3}{2x - \frac{5}{x}}$$

Note that as $x \to \infty$, $2x - \dfrac{5}{x} \to \infty - 0$. Hence, $\lim\limits_{x \to \infty} \dfrac{1}{2x - \frac{5}{x}} = 0$.

$$\boxed{\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = 0}$$

Problem 2.

Differentiate the following with respect to x from first principles.

(a) $3x + 4$

(b) x^3

Solution

Part (a)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(3x + 4) &= \lim_{h \to 0} \frac{(3(x + h) + 4) - (3x + 4)}{h} \\
&= \lim_{h \to 0} \frac{3h}{h} \\
&= \lim_{h \to 0} 3 \\
&= 3
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(3x + 4) = 3}
$$

Part (b)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}x^3 &= \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\
&= \lim_{h \to 0} \frac{3hx^2 + 3h^2x + h^3}{h} \\
&= \lim_{h \to 0} \left(3x^2 + 3hx + h^2\right) \\
&= 3x^2
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2}
$$

Problem 3.

Differentiate each of the following with respect to x, simplifying your answer.

(a) $(x^2 + 4)^2 (2x^3 - 1)$

(b) $\dfrac{x^2}{\sqrt{4 - x^2}}$

(c) $\sqrt{1 + \sqrt{x}}$

(d) $\left(\dfrac{x^3 - 1}{2x^3 + 1}\right)^4$

Solution

Part (a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 4\right)^2\left(2x^3 - 1\right) = \left(2x^3 - 1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 4\right)^2 + \left(x^2 + 4\right)^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3 - 1\right)$$
$$= \left(2x^3 - 1\right) \cdot 2\left(x^2 + 4\right) \cdot 2x + \left(x^2 + 4\right)^2 \cdot 6x^2$$
$$= 2x\left(x^2 + 4\right)\left(2\left(2x^3 - 1\right) + 3x\left(x^2 + 4\right)\right)$$
$$= 2x\left(x^2 + 4\right)\left(7x^3 + 12x - 2\right)$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 4\right)^2\left(2x^3 - 1\right) = 2x\left(x^2 + 4\right)\left(7x^3 + 12x - 2\right)}$$

Part (b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{\sqrt{4 - x^2}} = \frac{\sqrt{4 - x^2} \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}x^2 - x^2 \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{4 - x^2}}{4 - x^2}$$
$$= \frac{\sqrt{4 - x^2} \cdot 2x - x^2 \cdot \dfrac{1}{2\sqrt{4-x^2}} \cdot -2x}{4 - x^2}$$
$$= \frac{\left(4 - x^2\right) \cdot 2x - x^2 \cdot \frac{1}{2} \cdot -2x}{\left(4 - x^2\right)^{\frac{3}{2}}}$$
$$= \frac{\left(4 - x^2\right) \cdot 2x + x^3}{\left(4 - x^2\right)^{\frac{3}{2}}}$$
$$= \frac{x\left(8 - x^2\right)}{\left(4 - x^2\right)^{\frac{3}{2}}}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{\sqrt{4 - x^2}} = \frac{x\left(8 - x^2\right)}{\left(4 - x^2\right)^{\frac{3}{2}}}}$$

Part (c)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{4\sqrt{x(1+\sqrt{x})}}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\sqrt{x}} = \frac{1}{4\sqrt{x(1+\sqrt{x})}}}$$

Part (d)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-1}{2x^3+1}\right)^4 = 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3-1}{2x^3+1}$$

$$= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{(2x^3+1)\dfrac{\mathrm{d}}{\mathrm{d}x}(x^3-1) - (x^3-1)\dfrac{\mathrm{d}}{\mathrm{d}x}(2x^3+1)}{(2x^3+1)^2}$$

$$= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{(2x^3+1)\cdot 3x^2 - (x^3-1)\cdot 6x^2}{(2x^3+1)^2}$$

$$= \frac{4(x^3-1)^3}{(2x^3+1)^3} \cdot \frac{(2x^3+1)\cdot 3x^2 - (x^3-1)\cdot 6x^2}{(2x^3+1)^2}$$

$$= \frac{4(x^3-1)^3}{(2x^3+1)^5} \cdot \left((2x^3+1)\cdot 3x^2 - (x^3-1)\cdot 6x^2\right)$$

$$= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5} \cdot \left(2x^3+1 - (x^3-1)\cdot 2\right)$$

$$= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5} \cdot 3$$

$$= \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-1}{2x^3+1}\right)^4 = \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}}$$

Problem 4.

Using a graphing calculator, find the derivative of $\dfrac{e^{2x}}{x^2 + 1}$ when $x = 1.5$.

Solution

$$\boxed{\left.\frac{\mathrm{d}}{\mathrm{d}x} \frac{e^{2x}}{x^2 + 1}\right|_{x=1.5} = 6.66}$$

Problem 5.

Find the derivative with respect to x of

(a) $\cos x°$

(b) $\cot(1 - 2x^2)$

(c) $\tan^3(5x)$

(d) $\dfrac{\sec x}{1 + \tan x}$

Solution

Part (a)

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x° = \frac{\mathrm{d}}{\mathrm{d}x} \cos\left(\frac{\pi}{180}x\right)$$
$$= \frac{\pi}{180} \cdot \left(-\sin\left(\frac{\pi}{180}x\right)\right)$$
$$= -\frac{\pi}{180} \sin\left(\frac{\pi}{180}x\right)$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \cos x° = -\frac{\pi}{180} \sin\left(\frac{\pi}{180}x\right)}$$

Part (b)

$$\frac{\mathrm{d}}{\mathrm{d}x} \cot\left(1 - 2x^2\right) = -\csc\left(1 - 2x^2\right) \cdot (-4x)$$
$$= 4x \csc\left(1 - 2x^2\right)$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \cot\left(1 - 2x^2\right) = 4x \csc\left(1 - 2x^2\right)}$$

Part (c)

$$\frac{\mathrm{d}}{\mathrm{d}x} \tan^3(5x) = 3 \tan^2(5x) \cdot \sec^2(5x) \cdot 5$$
$$= 15 \tan^2(5x) \sec^2(5x)$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \tan^3(5x) = 15 \tan^2(5x) \sec^2(5x)}$$

Part (d)

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sec x}{1+\tan x} = \frac{(1+\tan x)\cdot \dfrac{\mathrm{d}}{\mathrm{d}x}\sec x - \sec x \cdot \dfrac{\mathrm{d}}{\mathrm{d}x}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)\cdot \sec x \tan x - \sec x \cdot \sec^2 x}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x(1+\tan x) - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x + \tan^2 x) - (\tan^2 x + 1))}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sec x}{1+\tan x} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}}$$

Problem 6.

Find the derivative with respect to x of

(a) $y = e^{1+\sin 3x}$

(b) $y = x^2 e^{\frac{1}{x}}$

(c) $y = \ln\left(\dfrac{1-x}{\sqrt{1+x^2}}\right)$

(d) $y = \dfrac{\ln(2x)}{x}$

(e) $y = \log_2(3x^4 - e^x)$

(f) $y = 3^{\ln \sin x}$

(g) $y = a^{2\log_a x}$

(h) $y = \sqrt[3]{\dfrac{e^x(x+1)}{x^2+1}}$, $x > 0$

Solution

Part (a)

$$\frac{dy}{dx} = \frac{d}{dx} e^{1+\sin 3x}$$
$$= e^{1+\sin 3x} \cdot \frac{d}{dx}(1 + \sin 3x)$$
$$= e^{1+\sin 3x} \cdot \cos(3x) \cdot 3$$
$$= 3e^{1+\sin 3x} \cos(3x)$$

$$\boxed{\frac{dy}{dx} = 3e^{1+\sin 3x} \cos(3x)}$$

Part (b)

$$\frac{dy}{dx} = \frac{d}{dx} x^2 e^{\frac{1}{x}}$$
$$= x^2 \cdot \frac{d}{dx} e^{\frac{1}{x}} + e^{\frac{1}{x}} \cdot \frac{d}{dx} x^2$$
$$= x^2 \cdot e^{\frac{1}{x}} \cdot \frac{d}{dx}\frac{1}{x} + e^{\frac{1}{x}} \cdot 2x$$
$$= x^2 \cdot e^{\frac{1}{x}} \cdot -\frac{1}{x^2} + e^{\frac{1}{x}} \cdot 2x$$
$$= -e^{\frac{1}{x}} + 2xe^{\frac{1}{x}}$$
$$= e^{\frac{1}{x}}(2x - 1)$$

$$\boxed{\frac{dy}{dx} = e^{\frac{1}{x}}(2x - 1)}$$

Part (c)

$$y = \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right)$$

$$= \ln\left(1+x^2\right) - \ln\left(\sqrt{1+x^2}\right)$$

$$= \ln\left(1+x^2\right) - \frac{1}{2}\ln\left(1+x^2\right)$$

$$\implies \frac{dy}{dx} = \frac{1}{1-x} \cdot \frac{d}{dx}(1-x) - \frac{1}{2}\left(\frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2)\right)$$

$$= \frac{1}{1-x} \cdot -1 - \frac{1}{2}\left(\frac{1}{1+x^2} \cdot 2x\right)$$

$$= \frac{1}{x-1} - \frac{x}{1+x^2}$$

$$= \frac{(1+x^2) - x(x-1)}{(x-1)(1+x^2)}$$

$$= \frac{1+x^2 - x^2 + x}{(x-1)(1+x^2)}$$

$$= \frac{1+x}{(x-1)(1+x^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{1+x}{(x-1)(1+x^2)}}$$

Part (d)

$$\frac{dy}{dx} = \frac{d}{dx}\frac{\ln(2x))}{x}$$

$$= \frac{x \cdot \dfrac{d}{dx}\ln(2x) - \ln(2x) \cdot \dfrac{d}{dx}x}{x^2}$$

$$= \frac{x \cdot \dfrac{1}{2x} \cdot \dfrac{d}{dx}2x - \ln(2x)}{x^2}$$

$$= \frac{x \cdot \dfrac{1}{2x} \cdot 2 - \ln(2x)}{x^2}$$

$$= \frac{1 - \ln(2x)}{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - \ln(2x)}{x^2}}$$

Part (e)

$$y = \log_2(3x^4 - e^x)$$
$$\implies \quad 2^y = 3x^4 - e^x$$
$$\implies \quad e^{\ln(2^y)} = 3x^4 - e^x$$
$$\implies \quad e^{y\ln(2)} = 3x^4 - e^x$$
$$\implies e^{y\ln 2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \ln 2 = 3 \cdot 4x^3 - e^x$$
$$\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12x^3 - e^x}{e^{y\ln 2}\ln 2}$$
$$= \frac{12x^3 - e^x}{(3x^4 - e^x)\ln 2}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12x^3 - e^x}{(3x^4 - e^x)\ln 2}}$$

Part (f)

$$y = 3^{\ln \sin x}$$
$$\implies \log_3 y = \ln \sin x$$
$$\implies \frac{\ln y}{\ln 3} = \ln \sin x$$
$$\implies \ln y = \ln 3 \cdot \ln \sin x$$
$$\implies \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3 \cdot \frac{1}{\sin(x)} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x$$
$$= \ln 3 \cdot \frac{1}{\sin(x)} \cdot \cos x$$
$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3 \cdot \cot(x) \cdot y$$
$$= \ln 3 \cdot \cot(x) \cdot 3^{\ln \sin x}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3 \cdot \cot(x) \cdot 3^{\ln \sin x}}$$

Part (g)

$$y = a^{2\log_a x}$$
$$= a^{\log_a x^2}$$
$$= x^2$$
$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}x^2$$
$$= 2x$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 2x}$$

Part (h)

$$y = \sqrt[3]{\frac{e^x(x+1)}{x^2+1}}$$

$$\implies \qquad y^3 = \frac{e^x(x+1)}{x^2+1}$$

$$\implies \qquad \left(x^2+1\right)y^3 = e^x(x+1)$$

$$\implies \left(x^2+1\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}y^3 + y^3\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right) = e^x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x+1) + (x+1)\frac{\mathrm{d}}{\mathrm{d}x}e^x$$

$$\implies \qquad \left(x^2+1\right)\cdot 3y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + y^3\cdot 2x = e^x\cdot 1 + (x+1)\cdot e^x$$

$$\implies \qquad \left(x^2+1\right)\cdot 3y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = e^x(x+2) - 2xy^3$$

$$\implies \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x(x+2) - 2xy^3}{\left(x^2+1\right)\cdot 3y^2}$$

$$= \frac{1}{3}\left(\frac{e^x(x+2)}{\left(x^2+1\right)y^2} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{1}{3}\left(\frac{e^x(x+1)(x+2)}{\left(x^2+1\right)(x+1)y^2} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{1}{3}\left(\frac{y^3(x+2)}{(x+1)y^2} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{1}{3}\left(\frac{y(x+2)}{(x+1)} - \frac{2xy}{x^2+1}\right)$$

$$= \frac{y}{3}\left(\frac{(x+2)}{(x+1)} - \frac{2x}{x^2+1}\right)$$

$$= \frac{y}{3}\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)$$

$$= \frac{1}{3}\cdot\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}\cdot\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\cdot\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}\cdot\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)}$$

Problem 7.

Find the derivative with respect to x of

(a) $\arccos \dfrac{x}{10}$

(b) $\arctan \dfrac{1}{1-x}$

(c) $\arcsin(\tan x)$

Solution

Part (a)

$$\frac{\mathrm{d}}{\mathrm{d}x} \arccos \frac{x}{10} = -\frac{1}{\sqrt{1 - \left(\frac{x}{10}\right)^2}} \cdot \frac{1}{10}$$

$$= -\frac{1}{\sqrt{100 - x^2}}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arccos \frac{x}{10} = -\frac{1}{\sqrt{100 - x^2}}}$$

Part (b)

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan \frac{1}{1-x} = \frac{1}{1 + \left(\frac{1}{1-x}\right)^2} \cdot -\frac{1}{(1-x)^2} \cdot -1$$

$$= \frac{1}{(1-x)^2 \left(1 + \left(\frac{1}{1-x}\right)^2\right)}$$

$$= \frac{1}{(1-x)^2 + 1}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arctan \frac{1}{1-x} = \frac{1}{(1-x)^2 + 1}}$$

Part (c)

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin(\tan x) = \frac{1}{1 - \tan^2 x} \cdot \sec^2 x$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arcsin(\tan x) = \frac{\sec^2 x}{1 - \tan^2 x}}$$

Problem 8.

Find an expression for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(a) $(y - x)^2 = 2a(y + x)$, where a is a constant

(b) $y^2 = e^{2x}y + xe^x$

(c) $y = x^y$

(d) $\sin x \cos y = \dfrac{1}{2}$

Solution

Part (a)

$$(y - x)^2 = 2a(y + x)$$
$$\implies \quad 2(y - x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = 2a\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right)$$
$$\implies \quad (y - x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = a\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right)$$
$$\implies (y - x) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} - (y - x) = a \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + a$$
$$\implies \quad (y - x - a) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = a + y - x$$
$$\implies \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a + y - x}{y - x - a}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a + y - x}{y - x - a}}$$

Part (b)

$$y^2 = e^{2x}y + xe^x$$
$$\implies \qquad 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + y \cdot \frac{\mathrm{d}}{\mathrm{d}x}e^{2x} + x \cdot \frac{\mathrm{d}}{\mathrm{d}x}e^x + e^x \cdot \frac{\mathrm{d}}{\mathrm{d}x}x$$
$$\implies \qquad 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = e^{2x} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + y \cdot e^{2x} \cdot 2 + xe^x + e^x$$
$$\implies \left(2y - e^{2x}\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot e^{2x} \cdot 2 + xe^x + e^x$$
$$\implies \left(2y - e^{2x}\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = e^x\left(2e^x y + x + 1\right)$$
$$\implies \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x\left(2e^x y + x + 1\right)}{2y - e^{2x}}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x\left(2e^x y + x + 1\right)}{2y - e^{2x}}}$$

Part (c)

$$y = x^y$$

$$\implies \qquad \ln y = y \ln x$$

$$\implies \qquad \frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{d}{dx} \ln x + \ln x \cdot \frac{dy}{dx}$$

$$= y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx}$$

$$\implies \left(\frac{1}{y} - \ln x \right) \cdot \frac{dy}{dx} = \frac{y}{x}$$

$$\implies \qquad \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \ln x}$$

$$= \frac{y^2}{x - xy \ln x}$$

$$\boxed{\frac{dy}{dx} = \frac{y^2}{x - xy \ln x}}$$

Part (d)

$$\sin x \cos y = \frac{1}{2}$$

$$\implies \sin x \cdot \frac{d}{dx} \cos y + \cos y \cdot \frac{d}{dx} \sin x = 0$$

$$\implies \sin x \cdot - \sin y \cdot \frac{dy}{dx} + \cos y \cdot \cos x = 0$$

$$\implies \qquad - \sin x \sin y \cdot \frac{dy}{dx} = - \cos x \cos y$$

$$\implies \qquad \frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \sin y}$$

$$= \cot x \cot y$$

$$\boxed{\frac{dy}{dx} = \cot x \cot y}$$

Problem 9.

It is given that x and y satisfy the equation

$$\arctan(x) + \arctan(y) + \arctan(xy) = \frac{7}{12}\pi \qquad (9.1)$$

(a) Find the exact value of y when $x = 1$.

(b) Express $\dfrac{d}{dx}\arctan((xy))$ in terms of x, y and y'.

(c) Show that, when $x = 1$, $y' = -\dfrac{1}{3} - \dfrac{1}{2\sqrt{3}}$.

Solution

Part (a)

Substituting $x = 1$ into Equation 9.1,

$$\arctan 1 + \arctan(y) + \arctan(y) = \frac{7}{12}\pi$$
$$\implies \frac{1}{4}\pi + 2\arctan(y) = \frac{7}{12}\pi$$
$$\implies \arctan(y) = \frac{1}{6}\pi$$
$$\implies y = \frac{1}{\sqrt{3}}$$

$$\boxed{y = \frac{1}{\sqrt{3}}}$$

Part (b)

$$\frac{d}{dx}\arctan((xy)) = \frac{1}{1 + (xy)^2} \cdot \frac{d}{dx}(xy)$$
$$= \frac{1}{1 + (xy)^2} \cdot (xy' + y)$$

$$\boxed{\frac{d}{dx}\arctan((xy)) = \frac{1}{1 + (xy)^2} \cdot (xy' + y)}$$

Part (c)

Differentiating Equation 9.1 with respect to x,

$$\frac{1}{1 + x^2} + \frac{y'}{1 + y^2} + \frac{1}{1 + (xy)^2} \cdot (xy' + y) = 0$$

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Substituting $x = 1$,

$$\frac{1}{2} + \frac{3}{4}y' + \frac{3}{4}(y' + y) = 0$$

$$\implies \quad y'\left(\frac{3}{4} + \frac{3}{4}\right) = \frac{3}{4}\left(-\frac{1}{\sqrt{3}}\right) - \frac{1}{2}$$

$$\implies \quad \frac{3}{2}y' = \frac{-3 - 2\sqrt{3}}{4\sqrt{3}}$$

$$\implies \quad y' = \frac{2}{3} \cdot \frac{-3 - 2\sqrt{3}}{4\sqrt{3}}$$

$$= \frac{-1 - \frac{2}{3}\sqrt{3}}{2\sqrt{3}}$$

$$= -\frac{1}{2\sqrt{3}} - \frac{1}{3}$$

Problem 10.

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for

  (a) $x = \dfrac{1}{1 + t^2}$, $y = \dfrac{t}{1 + t^2}$

  (b) $x = \dfrac{1}{2}(e^t - e^{-t})$, $y = \dfrac{1}{2}(e^t + e^{-t})$

  (c) $x = a \sec \theta$, $y = a \tan \theta$

  (d) $x = e^{3\theta} \cos(3\theta)$, $y = e^{3\theta} \sin(3\theta)$

Solution

Part (a)

Observe that $y = xt$.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x \cdot \frac{\mathrm{d}t}{\mathrm{d}x} + t \\
&= x \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^{-1} + t \\
&= \frac{1}{1 + t^2} \left(-\frac{1}{(1 + t^2)^2} \cdot 2t \right)^{-1} + t \\
&= \frac{1}{1 + t^2} \cdot \left(-\frac{(1 + t^2)^2}{2t} \right) + t \\
&= -\frac{1 + t^2}{2t} + \frac{2t^2}{2t} \\
&= \frac{t^2 - 1}{2t}
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2 - 1}{2t}}$$

Part (b)

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^{-1} \\
&= \frac{\frac{1}{2}(e^t - e^{-t})}{\frac{1}{2}(e^t + e^{-t})} \\
&= \frac{e^t - e^{-t}}{e^t + e^{-t}}
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^t - e^{-t}}{e^t + e^{-t}}}$$

Part (c)

Recall that $\tan^2 \theta + 1 = \sec^2 \theta$. Hence, $y^2 + a^2 = x^2$. Implicitly differentiating, we have

$$2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}$$

$$= \frac{a \sec \theta}{a \tan \theta}$$

$$= \csc \theta$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \csc \theta}$$

Part (d)

Recall that $\sin^2 \theta + \cos^2 \theta = 1$. Hence, $x^2 + y^2 = e^{6\theta}$. Implicitly differentiating, we have

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = e^{6\theta} \cdot 6\frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$\implies \quad x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 3e^{6\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{10.1}$$

Observe that $\dfrac{y}{x} = \tan(3\theta)$. Implicitly differentiating,

$$\frac{x\dfrac{\mathrm{d}y}{\mathrm{d}x} - y}{x^2} = \sec^2 3\theta \cdot 3\frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$\implies x\frac{\mathrm{d}y}{\mathrm{d}x} - y = 3x^2 \sec^2 3\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$= 3(e^{3\theta}\cos(3\theta))^2 \cdot \frac{1}{\cos^2 3\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x}$$

$$= 3e^{6\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{10.2}$$

Subtracting Equation 10.1 from Equation 10.2,

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}y}{\mathrm{d}x} - y - x = 0$$

$$\implies \qquad (x - y)\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$$

$$\implies \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{x - y}$$

$$= \frac{e^{3\theta}(\cos 3\theta + \sin 3\theta)}{e^{3\theta}(\cos 3\theta - \sin 3\theta)}$$

$$= \frac{\cos 3\theta + \sin 3\theta}{\cos 3\theta - \sin 3\theta}$$

$$= \frac{1 + \tan 3\theta}{1 - \tan 3\theta}$$

$$= \tan\left(3\theta + \frac{\pi}{4}\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(3\theta + \frac{\pi}{4}\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(3\theta + \frac{\pi}{4}\right)$$

Problem 11.

A curve is defined by the parametric equation

$$x = 120t - 4t^2, \; y = 60t - 6t^2$$

Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at each of the points where the curve cross the x-axis.

Solution

The curve crosses the x-axis when $y = 0$.

$$
\begin{aligned}
y &= 0 \\
\implies 60t - 6t^2 &= 0 \\
\implies 10t - t^2 &= 0 \\
\implies t(10 - t) &= 0
\end{aligned}
$$

Hence, $t = 0$ or $t = 10$. Now, consider the derivative with respect to x of the curve.

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t} \right)^{-1} \\
&= \frac{60 - 12t}{120 - 8t}
\end{aligned}
$$

Case 1: $t = 0$

$$
\begin{aligned}
\left. \frac{\mathrm{d}y}{\mathrm{d}x} \right|_{t=0} &= \frac{60 - 12 \cdot 0}{120 - 8 \cdot 0} \\
&= \frac{1}{2}
\end{aligned}
$$

Case 2: $t = 10$

$$
\begin{aligned}
\left. \frac{\mathrm{d}y}{\mathrm{d}x} \right|_{t=10} &= \frac{60 - 12 \cdot 10}{120 - 8 \cdot 10} \\
&= -\frac{3}{2}
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \lor -\frac{3}{2}}$$

Problem 12.

A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln t^2$, where $t > 0$. Find the value of t at the point on the curve at which the gradient is 2.

Solution

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \\
&= \frac{2t - \frac{2}{t}}{2 - \frac{1}{t}} \\
&= \frac{2t^2 - 2}{2t - 1}
\end{aligned}$$

Consider $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 2 \\
\implies \frac{2t^2 - 2}{2t - 1} &= 2 \\
\implies \frac{t^2 - 1}{2t - 1} &= 1 \\
\implies t^2 - 1 &= 2t - 1 \\
\implies t^2 - 2t &= 0 \\
\implies t(t - 2) &= 0
\end{aligned}$$

Hence, $t = 0$ or $t = 2$. Since $t > 0$, we reject $t = 0$. Thus, $t = 2$.

$$\boxed{t = 2}$$

Problem 13.

If $y = \ln\left(\sin^3 2x\right)$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and prove that $3\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 = 0$.

Solution

$$
\begin{aligned}
y &= \ln\left(\sin^3 2x\right) \\
\implies \quad e^y &= \sin^3 2x \\
\implies e^y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 3\sin^2 2x \cdot \cos 2x \cdot 2 \\
&= 6\sin^2 2x \cos 2x \\
\implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6\sin^2 2x \cos 2x}{e^y} \\
&= \frac{6\sin^2 2x \cos 2x}{\sin^3 2x} \\
&= \frac{6\cos 2x}{\sin 2x} \\
&= 6\cot 2x
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 6\cot 2x}
$$

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= 6\cot 2x \\
\implies \quad \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= 6 \cdot -\csc^2 2x \cdot 2 \\
&= -12\csc^2 2x \\
&= -12\left(1 + \cot^2 2x\right) \\
&= -12 - 12\cot^2 2x \\
&= -12 - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\implies \quad 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} &= -36 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\implies 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 &= 0
\end{aligned}
$$

Problem 14.

Given that $y = e^{\arcsin(2x)}$, show that $(1 - 4x^2)\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4y$. Differentiate this result further to obtain a differential equation for $\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}$.

Solution

$$y = e^{\arcsin(2x)}$$

$$\implies \ln y = \arcsin(2x)$$

$$\implies \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1 - (2x)^2}} \cdot 2$$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{\sqrt{1 - 4x^2}}$$

$$\implies \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\sqrt{1 - 4x^2} \cdot 2\dfrac{\mathrm{d}y}{\mathrm{d}x} - 2y \cdot \dfrac{1}{2\sqrt{1 - 4x^2}} \cdot -8x}{1 - 4x^2}$$

$$\implies (1 - 4x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2\sqrt{1 - 4x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{2y}{\sqrt{1 - 4x^2}}$$

$$= 2\sqrt{1 - 4x^2} \cdot \frac{2y}{\sqrt{1 - 4x^2}} + 4x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$= 4y + 4x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\implies (1 - 4x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 4y$$

$$(1 - 4x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} = 4y$$

$$\implies (1 - 4x^2)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \cdot (-8x) - 4\left(x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) = 4\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\implies (1 - 4x^2)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 8x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} = 4\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\implies (1 - 4x^2)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 12x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\boxed{(1 - 4x^2)\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 12x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} = 0}$$