

Problem 1.

Evaluate the following limits.

(a) $\lim_{x \rightarrow 5} (6x + 7)$

(b) $\lim_{x \rightarrow 1} \frac{x^3 - 1}{1 - x}$

(c) $\lim_{x \rightarrow \infty} \frac{3x}{2x^2 - 5}$

Solution**Part (a)**

$$\begin{aligned}\lim_{x \rightarrow 5} (6x + 7) &= 6 \cdot 5 + 7 \\ &= 37\end{aligned}$$

$$\boxed{\lim_{x \rightarrow 5} (6x + 7) = 37}$$

Part (b)

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^3 - 1}{1 - x} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{1 - x} \\ &= \lim_{x \rightarrow 1} -(x^2 + x + 1) \\ &= -(1^2 + 1 + 1) \\ &= -3\end{aligned}$$

$$\boxed{\lim_{x \rightarrow 1} \frac{x^3 - 1}{1 - x} = -3}$$

Part (c)

$$\lim_{x \rightarrow \infty} \frac{3x}{2x^2 - 5} = \lim_{x \rightarrow \infty} \frac{3}{2x - \frac{5}{x}}$$

Note that as $x \rightarrow \infty$, $2x - \frac{5}{x} \rightarrow \infty - 0$. Hence, $\lim_{x \rightarrow \infty} \frac{1}{2x - \frac{5}{x}} = 0$.

$$\boxed{\lim_{x \rightarrow \infty} \frac{3x}{2x^2 - 5} = 0}$$

Problem 2.

Differentiate the following with respect to x from first principles.

(a) $3x + 4$

(b) x^3

Solution**Part (a)**

$$\begin{aligned}\frac{d}{dx}(3x + 4) &= \lim_{h \rightarrow 0} \frac{(3(x + h) + 4) - (3x + 4)}{h} \\ &= \lim_{h \rightarrow 0} \frac{3h}{h} \\ &= \lim_{h \rightarrow 0} 3 \\ &= 3\end{aligned}$$

$$\frac{d}{dx}(3x + 4) = 3$$

Part (b)

$$\begin{aligned}\frac{d}{dx}x^3 &= \lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{3hx^2 + 3h^2x + h^3}{h} \\ &= \lim_{h \rightarrow 0} (3x^2 + 3hx + h^2) \\ &= 3x^2\end{aligned}$$

$$\frac{d}{dx}x^3 = 3x^2$$

Problem 3.

Differentiate each of the following with respect to x , simplifying your answer.

(a) $(x^2 + 4)^2 (2x^3 - 1)$

(b) $\frac{x^2}{\sqrt{4 - x^2}}$

(c) $\sqrt{1 + \sqrt{x}}$

(d) $\left(\frac{x^3 - 1}{2x^3 + 1}\right)^4$

Solution**Part (a)**

$$\begin{aligned}\frac{d}{dx} (x^2 + 4)^2 (2x^3 - 1) &= (2x^3 - 1) \frac{d}{dx} (x^2 + 4)^2 + (x^2 + 4)^2 \cdot \frac{d}{dx} (2x^3 - 1) \\ &= (2x^3 - 1) \cdot 2 (x^2 + 4) \cdot 2x + (x^2 + 4)^2 \cdot 6x^2 \\ &= 2x (x^2 + 4) (2 (2x^3 - 1) + 3x (x^2 + 4)) \\ &= 2x (x^2 + 4) (7x^3 + 12x - 2)\end{aligned}$$

$$\frac{d}{dx} (x^2 + 4)^2 (2x^3 - 1) = 2x (x^2 + 4) (7x^3 + 12x - 2)$$

Part (b)

$$\begin{aligned}\frac{d}{dx} \frac{x^2}{\sqrt{4 - x^2}} &= \frac{\sqrt{4 - x^2} \cdot \frac{d}{dx} x^2 - x^2 \cdot \frac{d}{dx} \sqrt{4 - x^2}}{4 - x^2} \\ &= \frac{\sqrt{4 - x^2} \cdot 2x - x^2 \cdot \frac{1}{2\sqrt{4 - x^2}} \cdot -2x}{4 - x^2} \\ &= \frac{(4 - x^2) \cdot 2x - x^2 \cdot \frac{1}{2} \cdot -2x}{(4 - x^2)^{\frac{3}{2}}} \\ &= \frac{(4 - x^2) \cdot 2x + x^3}{(4 - x^2)^{\frac{3}{2}}} \\ &= \frac{x(8 - x^2)}{(4 - x^2)^{\frac{3}{2}}}\end{aligned}$$

$$\frac{d}{dx} \frac{x^2}{\sqrt{4 - x^2}} = \frac{x(8 - x^2)}{(4 - x^2)^{\frac{3}{2}}}$$

Part (c)

$$\begin{aligned}\frac{d}{dx}\sqrt{1+\sqrt{x}} &= \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} \\ &= \frac{1}{4\sqrt{x(1+\sqrt{x})}}\end{aligned}$$

$$\boxed{\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{4\sqrt{x(1+\sqrt{x})}}}$$

Part (d)

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^3-1}{2x^3+1}\right)^4 &= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{d}{dx}\frac{x^3-1}{2x^3+1} \\ &= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{(2x^3+1)\frac{d}{dx}(x^3-1) - (x^3-1)\frac{d}{dx}(2x^3+1)}{(2x^3+1)^2} \\ &= 4\left(\frac{x^3-1}{2x^3+1}\right)^3 \cdot \frac{(2x^3+1) \cdot 3x^2 - (x^3-1) \cdot 6x^2}{(2x^3+1)^2} \\ &= \frac{4(x^3-1)^3}{(2x^3+1)^3} \cdot \frac{(2x^3+1) \cdot 3x^2 - (x^3-1) \cdot 6x^2}{(2x^3+1)^2} \\ &= \frac{4(x^3-1)^3}{(2x^3+1)^5} \cdot ((2x^3+1) \cdot 3x^2 - (x^3-1) \cdot 6x^2) \\ &= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5} \cdot (2x^3+1 - (x^3-1) \cdot 2) \\ &= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5} \cdot 3 \\ &= \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}\end{aligned}$$

$$\boxed{\frac{d}{dx}\left(\frac{x^3-1}{2x^3+1}\right)^4 = \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}}$$

Problem 4.

Using a graphing calculator, find the derivative of $\frac{e^{2x}}{x^2 + 1}$ when $x = 1.5$.

Solution

$$\left. \frac{d}{dx} \frac{e^{2x}}{x^2 + 1} \right|_{x=1.5} = 6.66$$

Problem 5.

Find the derivative with respect to x of

(a) $\cos x^\circ$

(b) $\cot(1 - 2x^2)$

(c) $\tan^3(5x)$

(d) $\frac{\sec x}{1 + \tan x}$

Solution**Part (a)**

$$\begin{aligned}\frac{d}{dx} \cos x^\circ &= \frac{d}{dx} \cos\left(\frac{\pi}{180}x\right) \\ &= \frac{\pi}{180} \cdot \left(-\sin\left(\frac{\pi}{180}x\right)\right) \\ &= -\frac{\pi}{180} \sin\left(\frac{\pi}{180}x\right)\end{aligned}$$

$$\boxed{\frac{d}{dx} \cos x^\circ = -\frac{\pi}{180} \sin\left(\frac{\pi}{180}x\right)}$$

Part (b)

$$\begin{aligned}\frac{d}{dx} \cot(1 - 2x^2) &= -\csc(1 - 2x^2) \cdot (-4x) \\ &= 4x \csc(1 - 2x^2)\end{aligned}$$

$$\boxed{\frac{d}{dx} \cot(1 - 2x^2) = 4x \csc(1 - 2x^2)}$$

Part (c)

$$\begin{aligned}\frac{d}{dx} \tan^3(5x) &= 3 \tan^2(5x) \cdot \sec^2(5x) \cdot 5 \\ &= 15 \tan^2(5x) \sec^2(5x)\end{aligned}$$

$$\boxed{\frac{d}{dx} \tan^3(5x) = 15 \tan^2(5x) \sec^2(5x)}$$

Part (d)

$$\begin{aligned}\frac{d}{dx} \frac{\sec x}{1 + \tan x} &= \frac{(1 + \tan x) \cdot \frac{d}{dx} \sec x - \sec x \cdot \frac{d}{dx} (1 + \tan x)}{(1 + \tan x)^2} \\&= \frac{(1 + \tan x) \cdot \sec x \tan x - \sec x \cdot \sec^2 x}{(1 + \tan x)^2} \\&= \frac{\sec x (\tan x (1 + \tan x) - \sec^2 x)}{(1 + \tan x)^2} \\&= \frac{\sec x (\tan x + \tan^2 x) - (\tan^2 x + 1)}{(1 + \tan x)^2} \\&= \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}\end{aligned}$$

| |
|---------------------------------------------------------------------------------------|
| $\frac{d}{dx} \frac{\sec x}{1 + \tan x} = \frac{\sec x (\tan x - 1)}{(1 + \tan x)^2}$ |
|---------------------------------------------------------------------------------------|

Problem 6.

Find the derivative with respect to x of

(a) $y = e^{1+\sin 3x}$

(b) $y = x^2 e^{\frac{1}{x}}$

(c) $y = \ln \left(\frac{1-x}{\sqrt{1+x^2}} \right)$

(d) $y = \frac{\ln(2x)}{x}$

(e) $y = \log_2(3x^4 - e^x)$

(f) $y = 3^{\ln(\sin x)}$

(g) $y = a^{2 \log_a x}$

(h) $y = \sqrt[3]{\frac{e^x(x+1)}{x^2+1}}, x > 0$

Solution**Part (a)**

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} e^{1+\sin 3x} \\ &= e^{1+\sin 3x} \cdot \frac{d}{dx} (1 + \sin 3x) \\ &= e^{1+\sin 3x} \cdot \cos 3x \cdot 3 \\ &= 3e^{1+\sin 3x} \cos 3x\end{aligned}$$

| |
|------------------------------------------|
| $\frac{dy}{dx} = 3e^{1+\sin 3x} \cos 3x$ |
|------------------------------------------|

Part (b)

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} x^2 e^{\frac{1}{x}} \\ &= x^2 \cdot \frac{d}{dx} e^{\frac{1}{x}} + e^{\frac{1}{x}} \cdot \frac{d}{dx} x^2 \\ &= x^2 \cdot e^{\frac{1}{x}} \cdot \frac{d}{dx} \frac{1}{x} + e^{\frac{1}{x}} \cdot 2x \\ &= x^2 \cdot e^{\frac{1}{x}} \cdot -\frac{1}{x^2} + e^{\frac{1}{x}} \cdot 2x \\ &= -e^{\frac{1}{x}} + 2xe^{\frac{1}{x}} \\ &= e^{\frac{1}{x}}(2x - 1)\end{aligned}$$

| |
|-------------------------------------------|
| $\frac{dy}{dx} = e^{\frac{1}{x}}(2x - 1)$ |
|-------------------------------------------|

Part (c)

$$\begin{aligned}
y &= \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right) \\
&= \ln(1-x) - \ln(\sqrt{1+x^2}) \\
&= \ln(1-x) - \frac{1}{2}\ln(1+x^2) \\
\Rightarrow \frac{dy}{dx} &= \frac{1}{1-x} \cdot \frac{d}{dx}(1-x) - \frac{1}{2} \left(\frac{1}{1+x^2} \cdot \frac{d}{dx}(1+x^2) \right) \\
&= \frac{1}{1-x} \cdot -1 - \frac{1}{2} \left(\frac{1}{1+x^2} \cdot 2x \right) \\
&= \frac{1}{x-1} - \frac{x}{1+x^2} \\
&= \frac{(1+x^2) - x(x-1)}{(x-1)(1+x^2)} \\
&= \frac{1+x^2-x^2+x}{(x-1)(1+x^2)} \\
&= \frac{1+x}{(x-1)(1+x^2)}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1+x}{(x-1)(1+x^2)}}$$

Part (d)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \frac{\ln(2x)}{x} \\
&= \frac{x \cdot \frac{d}{dx} \ln(2x) - \ln(2x) \cdot \frac{d}{dx} x}{x^2} \\
&= \frac{x \cdot \frac{1}{2x} \cdot \frac{d}{dx} 2x - \ln(2x)}{x^2} \\
&= \frac{x \cdot \frac{1}{2x} \cdot 2 - \ln(2x)}{x^2} \\
&= \frac{1 - \ln(2x)}{x^2}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - \ln(2x)}{x^2}}$$

Part (e)

$$\begin{aligned}
& y = \log_2(3x^4 - e^x) \\
\Rightarrow & 2^y = 3x^4 - e^x \\
\Rightarrow & e^{\ln 2^y} = 3x^4 - e^x \\
\Rightarrow & e^{y \ln 2} = 3x^4 - e^x \\
\Rightarrow & e^{y \ln 2} \cdot \frac{dy}{dx} \cdot \ln 2 = 3 \cdot 4x^3 - e^x \\
\Rightarrow & \frac{dy}{dx} = \frac{12x^3 - e^x}{e^{y \ln 2} \ln 2} \\
& = \frac{12x^3 - e^x}{(3x^4 - e^x) \ln 2}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{12x^3 - e^x}{(3x^4 - e^x) \ln 2}}$$

Part (f)

$$\begin{aligned}
& y = 3^{\ln(\sin x)} \\
\Rightarrow & \log_3 y = \ln(\sin x) \\
\Rightarrow & \frac{\ln y}{\ln 3} = \ln(\sin x) \\
\Rightarrow & \ln y = \ln 3 \cdot \ln(\sin x) \\
\Rightarrow & \frac{1}{y} \cdot \frac{dy}{dx} = \ln 3 \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} \sin x \\
& = \ln 3 \cdot \frac{1}{\sin x} \cdot \cos x \\
\Rightarrow & \frac{dy}{dx} = \ln 3 \cdot \cot x \cdot y \\
& = \ln 3 \cdot \cot x \cdot 3^{\ln(\sin x)}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \ln 3 \cdot \cot x \cdot 3^{\ln(\sin x)}}$$

Part (g)

$$\begin{aligned}
& y = a^{2 \log_a x} \\
& = a^{\log_a x^2} \\
& = x^2 \\
\Rightarrow & \frac{dy}{dx} = \frac{d}{dx} x^2 \\
& = 2x
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = 2x}$$

Part (h)

$$\begin{aligned}
& y = \sqrt[3]{\frac{e^x(x+1)}{x^2+1}} \\
\Rightarrow & y^3 = \frac{e^x(x+1)}{x^2+1} \\
\Rightarrow & (x^2+1)y^3 = e^x(x+1) \\
\Rightarrow & (x^2+1) \cdot \frac{d}{dx}y^3 + y^3 \cdot \frac{d}{dx}(x^2+1) = e^x \cdot \frac{d}{dx}(x+1) + (x+1) \frac{d}{dx}e^x \\
\Rightarrow & (x^2+1) \cdot 3y^2 \cdot \frac{dy}{dx} + y^3 \cdot 2x = e^x \cdot 1 + (x+1) \cdot e^x \\
\Rightarrow & (x^2+1) \cdot 3y^2 \cdot \frac{dy}{dx} = e^x(x+2) - 2xy^3 \\
\Rightarrow & \frac{dy}{dx} = \frac{e^x(x+2) - 2xy^3}{(x^2+1) \cdot 3y^2} \\
& = \frac{1}{3} \left(\frac{e^x(x+2)}{(x^2+1)y^2} - \frac{2xy}{x^2+1} \right) \\
& = \frac{1}{3} \left(\frac{e^x(x+1)(x+2)}{(x^2+1)(x+1)y^2} - \frac{2xy}{x^2+1} \right) \\
& = \frac{1}{3} \left(\frac{y^3(x+2)}{(x+1)y^2} - \frac{2xy}{x^2+1} \right) \\
& = \frac{1}{3} \left(\frac{y(x+2)}{(x+1)} - \frac{2xy}{x^2+1} \right) \\
& = \frac{y}{3} \left(\frac{(x+2)}{(x+1)} - \frac{2x}{x^2+1} \right) \\
& = \frac{y}{3} \left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1} \right) \\
& = \frac{1}{3} \cdot \sqrt[3]{\frac{e^x(x+1)}{x^2+1}} \cdot \left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1} \right)
\end{aligned}$$

| |
|--------------------------------------------------------------------------------------------------------------------------------|
| $\frac{dy}{dx} = \frac{1}{3} \cdot \sqrt[3]{\frac{e^x(x+1)}{x^2+1}} \cdot \left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1} \right)$ |
|--------------------------------------------------------------------------------------------------------------------------------|

Problem 7.

Find the derivative with respect to x of

(a) $\arccos \frac{x}{10}$

(b) $\arctan \frac{1}{1-x}$

(c) $\arcsin(\tan x)$

Solution**Part (a)**

$$\begin{aligned}\frac{d}{dx} \arccos \frac{x}{10} &= -\frac{1}{\sqrt{1 - \left(\frac{x}{10}\right)^2}} \cdot \frac{1}{10} \\ &= -\frac{1}{\sqrt{100 - x^2}}\end{aligned}$$

| |
|-------------------------------------------------------------------|
| $\frac{d}{dx} \arccos \frac{x}{10} = -\frac{1}{\sqrt{100 - x^2}}$ |
|-------------------------------------------------------------------|

Part (b)

$$\begin{aligned}\frac{d}{dx} \arctan \frac{1}{1-x} &= \frac{1}{1 + \left(\frac{1}{1-x}\right)^2} \cdot -\frac{1}{(1-x)^2} \cdot -1 \\ &= \frac{1}{(1-x)^2 \left(1 + \left(\frac{1}{1-x}\right)^2\right)} \\ &= \frac{1}{(1-x)^2 + 1}\end{aligned}$$

| |
|--------------------------------------------------------------|
| $\frac{d}{dx} \arctan \frac{1}{1-x} = \frac{1}{(1-x)^2 + 1}$ |
|--------------------------------------------------------------|

Part (c)

$$\frac{d}{dx} \arcsin(\tan x) = \frac{1}{1 - \tan^2 x} \cdot \sec^2 x$$

| |
|----------------------------------------------------------------|
| $\frac{d}{dx} \arcsin(\tan x) = \frac{\sec^2 x}{1 - \tan^2 x}$ |
|----------------------------------------------------------------|

Problem 8.

Find an expression for $\frac{dy}{dx}$ in terms of x and y .

(a) $(y - x)^2 = 2a(y + x)$, where a is a constant

(b) $y^2 = e^{2x}y + xe^x$

(c) $y = x^y$

(d) $\sin x \cos y = \frac{1}{2}$

Solution**Part (a)**

$$\begin{aligned}
 & (y - x)^2 = 2a(y + x) \\
 \implies & 2(y - x) \left(\frac{dy}{dx} - 1 \right) = 2a \left(\frac{dy}{dx} + 1 \right) \\
 \implies & (y - x) \left(\frac{dy}{dx} - 1 \right) = a \left(\frac{dy}{dx} + 1 \right) \\
 \implies & (y - x) \cdot \frac{dy}{dx} - (y - x) = a \cdot \frac{dy}{dx} + a \\
 \implies & (y - x - a) \cdot \frac{dy}{dx} = a + y - x \\
 \implies & \frac{dy}{dx} = \frac{a + y - x}{y - x - a}
 \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{a + y - x}{y - x - a}}$$

Part (b)

$$\begin{aligned}
 & y^2 = e^{2x}y + xe^x \\
 \implies & 2y \cdot \frac{dy}{dx} = e^{2x} \cdot \frac{dy}{dx} + y \cdot \frac{d}{dx}e^{2x} + x \cdot \frac{d}{dx}e^x + e^x \cdot \frac{d}{dx}x \\
 \implies & 2y \cdot \frac{dy}{dx} = e^{2x} \cdot \frac{dy}{dx} + y \cdot e^{2x} \cdot 2 + xe^x + e^x \\
 \implies & (2y - e^{2x}) \cdot \frac{dy}{dx} = y \cdot e^{2x} \cdot 2 + xe^x + e^x \\
 \implies & (2y - e^{2x}) \cdot \frac{dy}{dx} = e^x (2e^x y + x + 1) \\
 \implies & \frac{dy}{dx} = \frac{e^x (2e^x y + x + 1)}{2y - e^{2x}}
 \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{e^x (2e^x y + x + 1)}{2y - e^{2x}}}$$

Part (c)

$$\begin{aligned}
& y = x^y \\
\Rightarrow & \ln y = y \ln x \\
\Rightarrow & \frac{1}{y} \cdot \frac{dy}{dx} = y \cdot \frac{d}{dx} \ln x + \ln x \cdot \frac{dy}{dx} \\
& = y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} \\
\Rightarrow & \left(\frac{1}{y} - \ln x \right) \cdot \frac{dy}{dx} = \frac{y}{x} \\
\Rightarrow & \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \ln x} \\
& = \frac{y^2}{x - xy \ln x}
\end{aligned}$$

| |
|--------------------------------------------|
| $\frac{dy}{dx} = \frac{y^2}{x - xy \ln x}$ |
|--------------------------------------------|

Part (d)

$$\begin{aligned}
& \sin x \cos y = \frac{1}{2} \\
\Rightarrow & \sin x \cdot \frac{d}{dx} \cos y + \cos y \cdot \frac{d}{dx} \sin x = 0 \\
\Rightarrow & \sin x \cdot -\sin y \cdot \frac{dy}{dx} + \cos y \cdot \cos x = 0 \\
\Rightarrow & -\sin x \sin y \cdot \frac{dy}{dx} = -\cos x \cos y \\
\Rightarrow & \frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \sin y} \\
& = \cot x \cot y
\end{aligned}$$

| |
|---------------------------------|
| $\frac{dy}{dx} = \cot x \cot y$ |
|---------------------------------|

Problem 9.

It is given that x and y satisfy the equation

$$\arctan x + \arctan y + \arctan xy = \frac{7}{12}\pi \quad (9.1)$$

- (a) Find the exact value of y when $x = 1$.
- (b) Express $\frac{d}{dx} \arctan(xy)$ in terms of x , y and y' .
- (c) Show that, when $x = 1$, $y' = -\frac{1}{3} - \frac{1}{2\sqrt{3}}$.

Solution**Part (a)**

Substituting $x = 1$ into Equation 9.1,

$$\begin{aligned} \arctan 1 + \arctan y + \arctan y &= \frac{7}{12}\pi \\ \Rightarrow \frac{1}{4}\pi + 2\arctan y &= \frac{7}{12}\pi \\ \Rightarrow \arctan y &= \frac{1}{6}\pi \\ \Rightarrow y &= \frac{1}{\sqrt{3}} \end{aligned}$$

$$\boxed{y = \frac{1}{\sqrt{3}}}$$

Part (b)

$$\begin{aligned} \frac{d}{dx} \arctan(xy) &= \frac{1}{1+(xy)^2} \cdot \frac{d}{dx}(xy) \\ &= \frac{1}{1+(xy)^2} \cdot (xy' + y) \end{aligned}$$

$$\boxed{\frac{d}{dx} \arctan(xy) = \frac{1}{1+(xy)^2} \cdot (xy' + y)}$$

Part (c)

Differentiating Equation 9.1 with respect to x ,

$$\frac{1}{1+x^2} + \frac{y'}{1+y^2} + \frac{1}{1+(xy)^2} \cdot (xy' + y) = 0$$

Substituting $x = 1$,

$$\begin{aligned}\frac{1}{2} + \frac{3}{4}y' + \frac{3}{4}(y' + y) &= 0 \\ \Rightarrow y'(\frac{3}{4} + \frac{3}{4}) &= \frac{3}{4}\left(-\frac{1}{\sqrt{3}}\right) - \frac{1}{2} \\ \Rightarrow \frac{3}{2}y' &= \frac{-3 - 2\sqrt{3}}{4\sqrt{3}} \\ \Rightarrow y' &= \frac{2}{3} \cdot \frac{-3 - 2\sqrt{3}}{4\sqrt{3}} \\ &= \frac{-1 - \frac{2}{3}\sqrt{3}}{2\sqrt{3}} \\ &= -\frac{1}{2\sqrt{3}} - \frac{1}{3}\end{aligned}$$

Problem 10.Find $\frac{dy}{dx}$ for

(a) $x = \frac{1}{1+t^2}, y = \frac{t}{1+t^2}$

(b) $x = \frac{1}{2}(e^t - e^{-t}), y = \frac{1}{2}(e^t + e^{-t})$

(c) $x = a \sec \theta, y = a \tan \theta$

(d) $x = e^{3\theta} \cos 3\theta, y = e^{3\theta} \sin 3\theta$

Solution**Part (a)**Observe that $y = xt$.

$$\begin{aligned}
\frac{dy}{dx} &= x \cdot \frac{dt}{dx} + t \\
&= x \left(\frac{dx}{dt} \right)^{-1} + t \\
&= \frac{1}{1+t^2} \left(-\frac{1}{(1+t^2)^2} \cdot 2t \right)^{-1} + t \\
&= \frac{1}{1+t^2} \cdot \left(-\frac{(1+t^2)^2}{2t} \right) + t \\
&= -\frac{1+t^2}{2t} + \frac{2t^2}{2t} \\
&= \frac{t^2 - 1}{2t}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{t^2 - 1}{2t}}$$

Part (b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\
&= \frac{dy}{dt} \cdot \left(\frac{dx}{dt} \right)^{-1} \\
&= \frac{\frac{1}{2}(e^t - e^{-t})}{\frac{1}{2}(e^t + e^{-t})} \\
&= \frac{e^t - e^{-t}}{e^t + e^{-t}}
\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{e^t - e^{-t}}{e^t + e^{-t}}}$$

Part (c)

Recall that $\tan^2 \theta + 1 = \sec^2 \theta$. Hence, $y^2 + a^2 = x^2$. Implicitly differentiating, we have

$$\begin{aligned} 2y \cdot \frac{dy}{dx} &= 2x \\ \implies \frac{dy}{dx} &= \frac{x}{y} \\ &= \frac{a \sec \theta}{a \tan \theta} \\ &= \csc \theta \end{aligned}$$

$$\boxed{\frac{dy}{dx} = \csc \theta}$$

Part (d)

Recall that $\sin^2 \theta + \cos^2 \theta = 1$. Hence, $x^2 + y^2 = e^{6\theta}$. Implicitly differentiating, we have

$$\begin{aligned} 2x + 2y \frac{dy}{dx} &= e^{6\theta} \cdot 6 \frac{d\theta}{dx} \\ \implies x + y \frac{dy}{dx} &= 3e^{6\theta} \cdot \frac{d\theta}{dx} \end{aligned} \tag{10.1}$$

Observe that $\frac{y}{x} = \tan 3\theta$. Implicitly differentiating,

$$\begin{aligned} \frac{x \frac{dy}{dx} - y}{x^2} &= \sec^2 3\theta \cdot 3 \frac{d\theta}{dx} \\ \implies x \frac{dy}{dx} - y &= 3x^2 \sec^2 3\theta \cdot \frac{d\theta}{dx} \\ &= 3(e^{3\theta} \cos 3\theta)^2 \cdot \frac{1}{\cos^2 3\theta} \cdot \frac{d\theta}{dx} \\ &= 3e^{6\theta} \cdot \frac{d\theta}{dx} \end{aligned} \tag{10.2}$$

Subtracting Equation 10.1 from Equation 10.2,

$$\begin{aligned} x \frac{dy}{dx} - y \frac{dy}{dx} - y - x &= 0 \\ \implies (x - y) \frac{dy}{dx} &= x + y \\ \implies \frac{dy}{dx} &= \frac{x + y}{x - y} \\ &= \frac{e^{3\theta}(\cos 3\theta + \sin 3\theta)}{e^{3\theta}(\cos 3\theta - \sin 3\theta)} \\ &= \frac{\cos 3\theta + \sin 3\theta}{\cos 3\theta - \sin 3\theta} \\ &= \frac{1 + \tan 3\theta}{1 - \tan 3\theta} \\ &= \tan \left(3\theta + \frac{\pi}{4} \right) \end{aligned}$$

$$\frac{dy}{dx} = \tan\left(3\theta + \frac{\pi}{4}\right)$$

Problem 11.

A curve is defined by the parametric equation

$$x = 120t - 4t^2, y = 60t - 6t^2$$

Find the value of $\frac{dy}{dx}$ at each of the points where the curve cross the x -axis.

Solution

The curve crosses the x -axis when $y = 0$.

$$\begin{aligned}y &= 0 \\ \implies 60t - 6t^2 &= 0 \\ \implies 10t - t^2 &= 0 \\ \implies t(10 - t) &= 0\end{aligned}$$

Hence, $t = 0$ or $t = 10$. Now, consider the derivative with respect to x of the curve.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \\ &= \frac{60 - 12t}{120 - 8t}\end{aligned}$$

Case 1: $t = 0$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{t=0} &= \frac{60 - 12 \cdot 0}{120 - 8 \cdot 0} \\ &= \frac{1}{2}\end{aligned}$$

Case 2: $t = 10$

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{t=10} &= \frac{60 - 12 \cdot 10}{120 - 8 \cdot 10} \\ &= -\frac{3}{2}\end{aligned}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{2} \vee -\frac{3}{2}}$$

Problem 12.

A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln t^2$, where $t > 0$. Find the value of t at the point on the curve at which the gradient is 2.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{dt} \cdot \frac{dt}{dx} \\ &= \frac{dy}{dt} \cdot \left(\frac{dx}{dt}\right)^{-1} \\ &= \frac{2t - \frac{2}{t}}{2 - \frac{1}{t}} \\ &= \frac{2t^2 - 2}{2t - 1}\end{aligned}$$

Consider $\frac{dy}{dx} = 2$.

$$\begin{aligned}\frac{dy}{dx} &= 2 \\ \implies \frac{2t^2 - 2}{2t - 1} &= 2 \\ \implies \frac{t^2 - 1}{2t - 1} &= 1 \\ \implies t^2 - 1 &= 2t - 1 \\ \implies t^2 - 2t &= 0 \\ \implies t(t - 2) &= 0\end{aligned}$$

Hence, $t = 0$ or $t = 2$. Since $t > 0$, we reject $t = 0$. Thus, $t = 2$.

$$\boxed{t = 2}$$

Problem 13.

If $y = \ln(\sin^3 2x)$, find $\frac{dy}{dx}$ and prove that $3\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 36 = 0$.

Solution

$$\begin{aligned}
 & y = \ln(\sin^3 2x) \\
 \Rightarrow & e^y = \sin^3 2x \\
 \Rightarrow & e^y \cdot \frac{dy}{dx} = 3 \sin^2 2x \cdot \cos 2x \cdot 2 \\
 & = 6 \sin^2 2x \cos 2x \\
 \Rightarrow & \frac{dy}{dx} = \frac{6 \sin^2 2x \cos 2x}{e^y} \\
 & = \frac{6 \sin^2 2x \cos 2x}{\sin^3 2x} \\
 & = \frac{6 \cos 2x}{\sin 2x} \\
 & = 6 \cot 2x
 \end{aligned}$$

$$\frac{dy}{dx} = 6 \cot 2x$$

$$\begin{aligned}
 & \frac{dy}{dx} = 6 \cot 2x \\
 \Rightarrow & \frac{d^2y}{dx^2} = 6 \cdot -\csc^2 2x \cdot 2 \\
 & = -12 \csc^2 2x \\
 & = -12 (1 + \cot^2 2x) \\
 & = -12 - 12 \cot^2 2x \\
 & = -12 - \frac{1}{3} \left(\frac{dy}{dx}\right)^2 \\
 \Rightarrow & 3\frac{d^2y}{dx^2} = -36 - \left(\frac{dy}{dx}\right)^2 \\
 \Rightarrow & 3\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 36 = 0
 \end{aligned}$$

Problem 14.

Given that $y = e^{\arcsin 2x}$, show that $(1 - 4x^2) \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} = 4y$. Further differentiate this result to obtain a differential equation for $\frac{d^3y}{dx^3}$.

Solution

$$\begin{aligned}
 & y = e^{\arcsin 2x} \\
 \Rightarrow & \ln y = \arcsin 2x \\
 \Rightarrow & \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{\sqrt{1 - (2x)^2}} \cdot 2 \\
 \Rightarrow & \frac{dy}{dx} = \frac{2y}{\sqrt{1 - 4x^2}} \\
 \Rightarrow & \frac{d^2y}{dx^2} = \frac{\sqrt{1 - 4x^2} \cdot 2 \frac{dy}{dx} - 2y \cdot \frac{1}{2\sqrt{1 - 4x^2}} \cdot -8x}{1 - 4x^2} \\
 \Rightarrow & (1 - 4x^2) \frac{d^2y}{dx^2} = 2\sqrt{1 - 4x^2} \frac{dy}{dx} + 4x \frac{2y}{\sqrt{1 - 4x^2}} \\
 & = 2\sqrt{1 - 4x^2} \cdot \frac{2y}{\sqrt{1 - 4x^2}} + 4x \frac{dy}{dx} \\
 & = 4y + 4x \frac{dy}{dx} \\
 \Rightarrow & (1 - 4x^2) \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} = 4y
 \end{aligned}$$

$$\begin{aligned}
 & (1 - 4x^2) \frac{d^2y}{dx^2} - 4x \frac{dy}{dx} = 4y \\
 \Rightarrow & (1 - 4x^2) \frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} \cdot (-8x) - 4 \left(x \frac{d^2y}{dx^2} + \frac{dy}{dx} \right) = 4 \frac{dy}{dx} \\
 \Rightarrow & (1 - 4x^2) \frac{d^3y}{dx^3} - 8x \frac{d^2y}{dx^2} - 4x \frac{d^2y}{dx^2} - 4 \frac{dy}{dx} = 4 \frac{dy}{dx} \\
 \Rightarrow & (1 - 4x^2) \frac{d^3y}{dx^3} - 12x \frac{d^2y}{dx^2} - 8 \frac{dy}{dx} = 0
 \end{aligned}$$

$$\boxed{(1 - 4x^2) \frac{d^3y}{dx^3} - 12x \frac{d^2y}{dx^2} - 8 \frac{dy}{dx} = 0}$$