Problem 1.

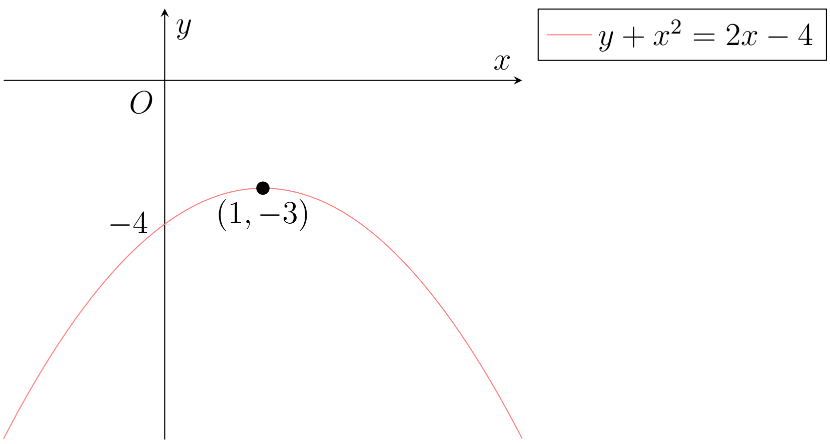
Sketch clearly labelled diagrams of each of the following curves, giving exact values of axial intercepts, stationary points and equations of asymptotes, if any.

(a) $y + x^2 = 2x - 4$

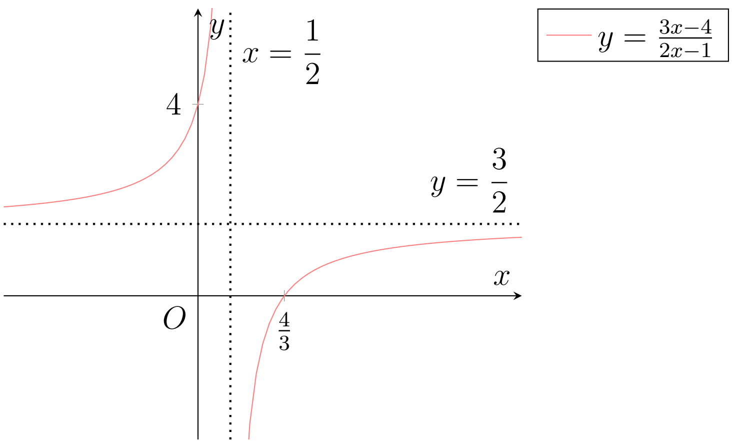
(b) $y = \dfrac{3x-4}{2x-1}$

Solution

Part (a)



Part (b)



Problem 2.

On separate diagrams, sketch the graphs of

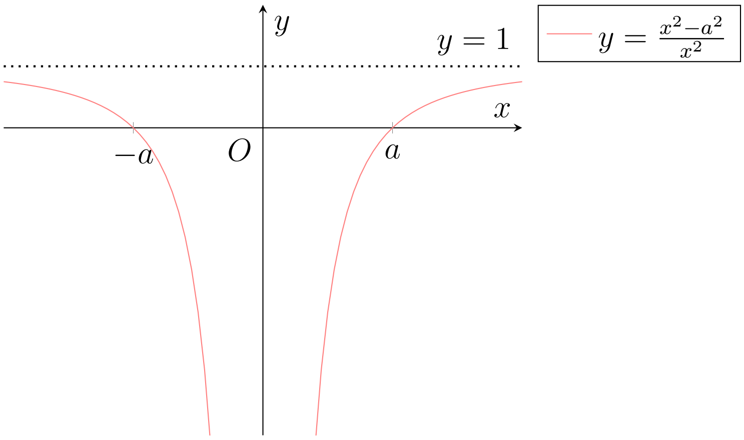
(a) $y = \dfrac{x^2 - a^2}{x^2}$, $a > 0$

(b) $y = \dfrac{x-1}{2x(x+3)}$

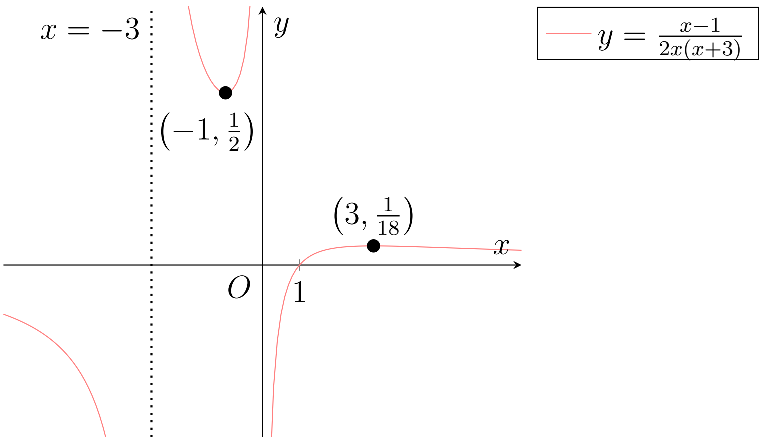
Indicate clearly the coordinates of axial intercepts, stationary points and equations of asymptotes, if any.

Solution

Part (a)



Part (b)



Problem 3.

The curve C has equation $y = \dfrac{ax^2 + bx - 2}{x + 4}$, where a and b are constants. It is given that $y = 2x - 5$ is an asymptote of C.

(a) Find the values of a and b.

(b) Sketch C.

(c) Using an algebraic method, find the set of values that y cannot take.

(d) By drawing a sketch of another suitable curve in the same diagram as your sketch of C in part (b), deduce the number of distinct real roots of the equation $x^3 + 6x^2 + 3x - 2 = 0$.

Solution

Part (a)

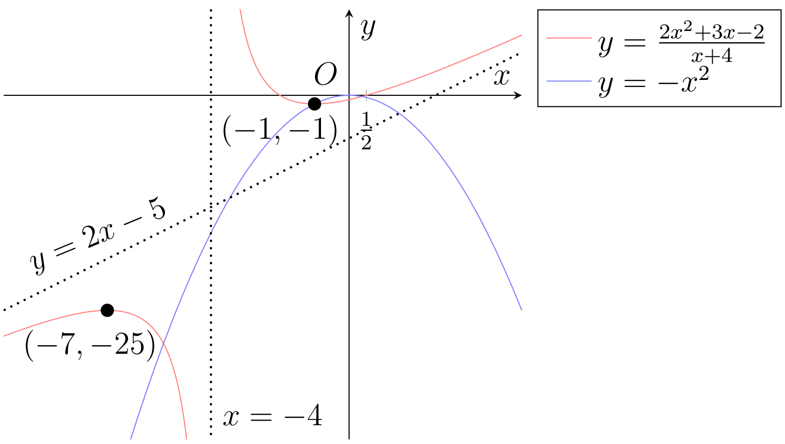
Since $y = 2x - 5$ is an asymptote of C, $\dfrac{ax^2 + bx - 2}{x + 4}$ can be expressed in the form $2x - 5 + \dfrac{k}{x+4}$, where k is a constant.

$$\begin{aligned} \frac{ax^2 + bx - 2}{x + 4} &= 2x - 5 + \frac{k}{x+4} \\ \implies ax^2 + bx - 2 &= (2x - 5)(x + 4) + k \\ \implies ax^2 + bx - 2 &= 2x^2 + 3x - 20 + k \end{aligned}$$

Comparing coefficients of x^2, x and constant terms, we have $a = 2$, $b = 3$ and $k = 18$.

$$\boxed{a = 2,\ b = 3}$$

Part (b)

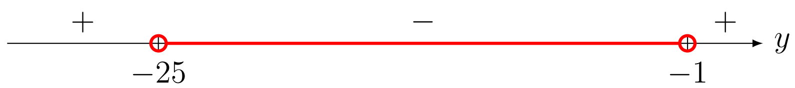


Part (c)

$$\begin{aligned} y &= \frac{2x^2+3x-2}{x+4} \\ \implies \quad (x+4)y &= 2x^2+3x-2 \\ \implies 2x^2+(3-y)x-(2+4y) &= 0 \end{aligned}$$

For values that y cannot take on, there exist no solutions to $2x^2+(3-y)x-(2+4y)=0$. Hence, $\Delta < 0$.

$$\begin{aligned} \Delta &< 0 \\ \implies (3-y)^2 - 4(2)\left(-(2+4y)\right) &< 0 \\ \implies \quad y^2+26y+25 &< 0 \\ \implies \quad (y+25)(y+1) &< 0 \end{aligned}$$



We thus see that y cannot take on a value between -25 and -1.

$$\boxed{\{y \in \mathbb{R} \colon -25 < y < -1\}}$$

Part (d)

$$\begin{aligned} x^3+6x^2+3x-2 &= 0 \\ \implies \quad \frac{x^3+6x^2+3x-2}{x+4} &= 0 \\ \implies \frac{x^3+4x^2}{x+4} + \frac{2x^2+3x-2}{x+4} &= 0 \\ \implies \quad x^2 + \frac{2x^2+3x-2}{x+4} &= 0 \\ \implies \quad C &= -x^2 \end{aligned}$$

Plotting $y=-x^2$ on the same digram, we see that there are 3 intersections between $y=x^2$ and C. Hence, there are 3 distinct real roots to $x^3+6x^2+3x-2=0$.