

**Problem 1.**

The points  $A$  and  $B$  have position vectors relative to the origin  $O$ , denoted by  $\mathbf{a}$  and  $\mathbf{b}$  respectively, where  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel vectors. The point  $P$  lies on  $AB$  such that  $AP : PB = \lambda : 1$ . The point  $Q$  lies on  $OP$  extended such that  $OP = 2PQ$  and  $\overrightarrow{BQ} = \overrightarrow{OA} + \mu\overrightarrow{OB}$ . Find the values of the real constants  $\lambda$  and  $\mu$ .

**Solution**

By the Ratio Theorem,

$$\begin{aligned}
 \overrightarrow{OP} &= \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} \\
 \Rightarrow \overrightarrow{OQ} &= \overrightarrow{OP} + \overrightarrow{PQ} \\
 &= \overrightarrow{OP} + \frac{1}{2}\overrightarrow{OP} \\
 &= \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} \\
 \overrightarrow{BQ} &= \overrightarrow{OA} + \mu\overrightarrow{OB} \\
 \Rightarrow \overrightarrow{OB} + \overrightarrow{BQ} &= \overrightarrow{OA} + (1 + \mu)\overrightarrow{OB} \\
 \Rightarrow \overrightarrow{OQ} &= \overrightarrow{OA} + (1 + \mu)\overrightarrow{OB} \\
 \Rightarrow \frac{3}{2} \cdot \frac{\mathbf{a} + \lambda\mathbf{b}}{1 + \lambda} &= \mathbf{a} + (1 + \mu)\mathbf{b}
 \end{aligned}$$

Since  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel, we have the following system:

$$\begin{cases} \frac{3}{2} \cdot \frac{1}{1 + \lambda} = 1 \\ \frac{3}{2} \cdot \frac{\lambda}{1 + \lambda} = 1 + \mu \end{cases}$$

which has the unique solution  $\lambda = \frac{1}{2}$  and  $\mu = -\frac{1}{2}$ .

$$\boxed{\lambda = \frac{1}{2}, \mu = -\frac{1}{2}}$$

**Problem 2.**

Given that  $\mathbf{a} = \mathbf{i} + \mathbf{j}$ ,  $\mathbf{b} = 4\mathbf{i} - 2\mathbf{j} + 6\mathbf{k}$  and  $\mathbf{p} = \lambda\mathbf{a} + (1 - \lambda)\mathbf{b}$  where  $\lambda \in \mathbb{R}$ , find the possible value(s) of  $\lambda$  for which the angle between  $\mathbf{p}$  and  $\mathbf{k}$  is  $45^\circ$ .

**Solution**

$$\begin{aligned}
 \mathbf{p} &= \lambda\mathbf{a} + (1 - \lambda)\mathbf{b} \\
 &= \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1 - \lambda) \begin{pmatrix} 4 \\ -2 \\ 6 \end{pmatrix} \\
 &= \begin{pmatrix} 4 - 3\lambda \\ -2 + 3\lambda \\ 6 - 6\lambda \end{pmatrix} \\
 \implies |\mathbf{p}|^2 &= (4 - 3\lambda)^2 + (-2 + 3\lambda)^2 + (6 - 6\lambda)^2 \\
 &= 54\lambda^2 - 108\lambda + 56
 \end{aligned}$$

Since the angle between  $\mathbf{p}$  and  $\mathbf{k}$  is  $45^\circ$ ,

$$\begin{aligned}
 \cos 45^\circ &= \frac{\mathbf{p} \cdot \mathbf{k}}{|\mathbf{p}||\mathbf{k}|} \\
 \implies \frac{1}{\sqrt{2}} &= \frac{(4 - 3\lambda) \cdot 0 + (-2 + 3\lambda) \cdot 0 + (6 - 6\lambda) \cdot 1}{|\mathbf{p}| \cdot 1} \\
 \implies \frac{|\mathbf{p}|}{\sqrt{2}} &= 6 - 6\lambda \\
 \implies \frac{|\mathbf{p}|^2}{2} &= (6 - 6\lambda)^2 \\
 \implies \frac{54\lambda^2 - 108\lambda + 56}{2} &= 36\lambda^2 - 72\lambda + 36 \\
 \implies 9\lambda^2 - 18\lambda + 8 &= 0 \\
 \implies (3\lambda - 2)(3\lambda - 4) &= 0
 \end{aligned}$$

Hence,  $\lambda = \frac{2}{3}, \frac{4}{3}$ . However, we must reject  $\lambda = \frac{4}{3}$  since  $6 - 6\lambda = \frac{|\mathbf{p}|}{\sqrt{2}} > 0 \implies \lambda < 1$ .

$$\boxed{\lambda = \frac{2}{3}}$$

**Problem 3.**

- (a)  $\mathbf{a}$  and  $\mathbf{b}$  are non-zero vectors such that  $\mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b}$ . State the relation between the directions of  $\mathbf{a}$  and  $\mathbf{b}$ , and find  $|\mathbf{b}|$ .
- (b)  $\mathbf{a}$  is a non-zero vector such that  $|\mathbf{a}| = \sqrt{3}$  and  $\mathbf{b}$  is a unit vector. Given that  $\mathbf{a}$  and  $\mathbf{b}$  are non-parallel and the angle between them is  $\frac{5}{6}\pi$ , find the exact value of the length of projection of  $\mathbf{a}$  on  $\mathbf{b}$ . By considering  $(2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b})$ , or otherwise, find the exact value of  $|2\mathbf{a} + \mathbf{b}|$ .

**Solution****Part (a)**

$\mathbf{a}$  and  $\mathbf{b}$  either have the same or opposite direction.

Let  $\mathbf{b} = \lambda\mathbf{a}$  for some  $\lambda \in \mathbb{R}$ .

$$\begin{aligned}
 & \mathbf{a} = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} \\
 \implies & \mathbf{a} = (\mathbf{a} \cdot \lambda\mathbf{a})\lambda\mathbf{a} \\
 \implies & \mathbf{a} = \lambda^2|\mathbf{a}|^2\mathbf{a} \\
 \implies & \lambda^2|\mathbf{a}|^2 = 1 \\
 \implies & \lambda|\mathbf{a}| = \pm 1 \\
 \implies & \lambda = \pm \frac{1}{|\mathbf{a}|} \\
 \implies & \mathbf{b} = \pm \frac{\mathbf{a}}{|\mathbf{a}|} \\
 \implies & \mathbf{b} = \pm \hat{\mathbf{a}} \\
 \implies & |\mathbf{b}| = 1
 \end{aligned}$$

$$\boxed{|\mathbf{b}| = 1}$$

**Part (b)**

$$\begin{aligned}
 |\mathbf{a} \cdot \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| \cos \frac{5}{6}\pi \\
 &= \sqrt{3} \cdot 1 \cdot \left(-\frac{\sqrt{3}}{2}\right) \\
 &= -\frac{3}{2} \\
 \implies \text{length of projection of } \mathbf{a} \text{ on } \mathbf{b} &= \left| \mathbf{a} \cdot \hat{\mathbf{b}} \right| \\
 &= |\mathbf{a} \cdot \mathbf{b}| \\
 &= \left| -\frac{3}{2} \right| \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\text{length of projection of } \mathbf{a} \text{ on } \mathbf{b} = \frac{3}{2}$$

$$\begin{aligned} |2\mathbf{a} + \mathbf{b}|^2 &= (2\mathbf{a} + \mathbf{b}) \cdot (2\mathbf{a} + \mathbf{b}) \\ &= 2\mathbf{a} \cdot 2\mathbf{a} + 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot 2\mathbf{a} + \mathbf{b} \cdot \mathbf{b} \\ &= 4\mathbf{a} \cdot \mathbf{a} + 4\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b} \\ &= 4 \cdot 3 + 4 \cdot \left(-\frac{3}{2}\right) + 1^2 \\ &= 7 \\ \Rightarrow |2\mathbf{a} + \mathbf{b}| &= \sqrt{7} \end{aligned}$$

$$|2\mathbf{a} + \mathbf{b}| = \sqrt{7}$$

**Problem 4.**

The points  $A, B, C, D$  have position vectors  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$  given by  $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ ,  $\mathbf{b} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ ,  $\mathbf{c} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ ,  $\mathbf{d} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$ , respectively. The point  $P$  lies on  $AB$  produced such that  $AP = 2AB$ , and the point  $Q$  is the mid-point of  $AC$ .

- (a) Show that  $PQ$  is perpendicular to  $AQ$ .
- (b) Find the area of the triangle  $APQ$ .
- (c) Find a vector perpendicular to the plane  $ABC$ .
- (d) Find the cosine of the angle between  $\overrightarrow{AD}$  and  $\overrightarrow{BD}$ .

**Solution**

We recenter the vectors such that  $\mathbf{a}$  is the origin. This gives  $\mathbf{a}' = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ,  $\mathbf{b}' = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ ,  $\mathbf{c}' = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ ,  $\mathbf{d}' = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix}$ . Hence,  $\overrightarrow{OP'}$  is clearly  $\begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$ , while  $\overrightarrow{OQ'} = \frac{1}{2}\mathbf{c}' = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

**Part (a)**

$$\begin{aligned}
 \overrightarrow{PQ} \cdot \overrightarrow{AQ} &= \overrightarrow{PQ'} \cdot \overrightarrow{OQ'} \\
 &= (\overrightarrow{OQ'} - \overrightarrow{OP'}) \cdot (\overrightarrow{OQ'}) \\
 &= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \\
 &= 1 + 0 - 1 \\
 &= 0
 \end{aligned}$$

Since  $\overrightarrow{PQ} \cdot \overrightarrow{AQ} = 0$ , the lines  $PQ$  and  $AQ$  must be perpendicular.

**Part (b)**

$$\begin{aligned}
 \text{Area } \triangle APQ &= \frac{1}{2} \left| \overrightarrow{AP} \times \overrightarrow{AQ} \right| \\
 &= \frac{1}{2} \left| \overrightarrow{OP'} \times \overrightarrow{OQ'} \right| \\
 &= \frac{1}{2} \left| \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right| \\
 &= \frac{1}{2} \left| \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \right| \\
 &= 1
 \end{aligned}$$

$$\text{Area } \triangle APQ = 1$$

**Part (c)**

$$\begin{aligned} \mathbf{b}' \times \mathbf{c}' &= \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \end{aligned}$$

$$\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \text{ is perpendicular to the plane } ABC.$$

**Part (d)**

Let the angle between  $\overrightarrow{AD}$  and  $\overrightarrow{BD}$  be  $\theta$ . Note that  $\overrightarrow{BD} = \mathbf{d}' - \mathbf{b}' = \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} =$

$$\begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix} = -3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

$$\cos \theta = \frac{\overrightarrow{AD} \cdot \overrightarrow{BD}}{|\overrightarrow{AD}| |\overrightarrow{BD}|}$$

$$\begin{aligned} &= \frac{1}{\sqrt{3^2 + (-3)^2 + (-3)^2} \cdot 3\sqrt{1^2 + (-1)^2 + (-1)^2}} \begin{pmatrix} 3 \\ -3 \\ -4 \end{pmatrix} \cdot 3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \\ &= \frac{1}{3\sqrt{102}} \cdot 3(3 \cdot 1 + (-3) \cdot (-1) + (-4) \cdot (-1)) \\ &= \frac{10}{\sqrt{102}} \end{aligned}$$

$$\cos \theta = \frac{10}{\sqrt{102}}$$