Problem 1.

The curve C is defined parametrically by $x = a(2\cos\theta + \cos 2\theta)$, $y = a(2\sin\theta + \sin 2\theta)$ where $0 \leq \theta \leq \pi$ and a is a positive constant.

(a) Find the coordinates of the points at which C meets the x-axis.

(b) Sketch C.

(c) Find the exact total length of C.

(d) Find the exact area of the curve surface generated when C is rotated through 2π radians about the x-axis.

Solution

Part (a)

When C meets the x-axis, $y = 0$.

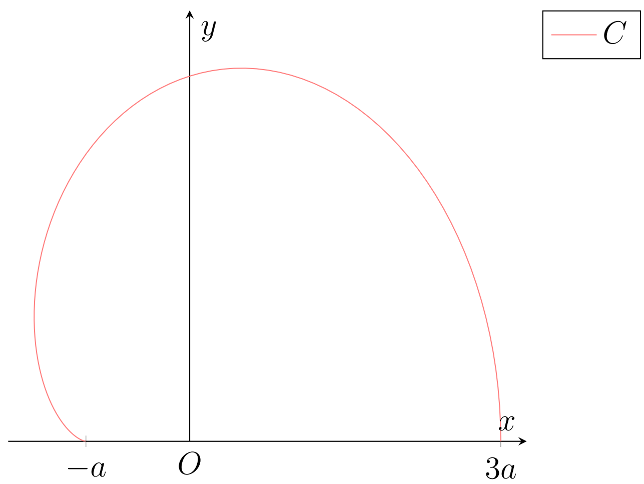
$$\begin{aligned} & y = 0 \\ \implies & a(2\sin\theta + \sin 2\theta) = 0 \\ \implies & 2\sin\theta + \sin 2\theta = 0 \\ \implies & 2\sin\theta + 2\sin\theta\cos\theta = 0 \\ \implies & \sin\theta(1 + \cos\theta) = 0 \end{aligned}$$

Note that $\sin\theta = 0 \implies \theta = 0 \vee \pi$ and $1 + \cos\theta = 0 \implies \theta = \pi$. Hence, $\theta = 0 \vee \pi$.

At $\theta = 0$, $x = a(2\cos 0 + \cos 0) = 3a$. At $\theta = \pi$, $x = a(2\cos\pi + \cos 2\pi) = -a$. Hence, C meets the x-axis at $(3a, 0)$ and $(-a, 0)$.

$$\boxed{(3a, 0),\ (-a, 0)}$$

Part (b)



Part (c)

Note that $\frac{\mathrm{d}x}{\mathrm{d}\theta} = a(-2\sin\theta - 2\sin 2\theta) = -2a(\sin\theta + \sin 2\theta)$ and $\frac{\mathrm{d}y}{\mathrm{d}\theta} = a(2\cos\theta + 2\cos 2\theta) = 2a(\cos\theta + \cos 2\theta)$. Hence,

$$\begin{aligned}
\text{Length} &= \int_0^\pi \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\
&= \int_0^\pi \sqrt{[-2a(\sin\theta + \sin 2\theta)]^2 + [2a(\cos\theta + \cos 2\theta)]^2}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a\sqrt{(\sin\theta + \sin 2\theta)^2 + (\cos\theta + \cos 2\theta)^2}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a\sqrt{\left(\sin^2\theta + 2\sin\theta\sin 2\theta + \sin^2 2\theta\right) + \left(\cos^2\theta + 2\cos\theta\cos 2\theta + \cos^2 2\theta\right)}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a\sqrt{2 + 2\sin\theta\sin 2\theta + 2\cos\theta\cos 2\theta}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a\sqrt{2 + 2\cos(2\theta - \theta)}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a\sqrt{2 + 2\cos\theta}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a\sqrt{2 + 2\left(2\cos^2(\theta/2) - 1\right)}\,\mathrm{d}\theta \\
&= \int_0^\pi 2a \cdot 2\cos(\theta/2)\,\mathrm{d}\theta \\
&= 4a\left[2\sin(\theta/2)\right]_0^\pi \\
&= 8a
\end{aligned}$$

$$\boxed{\text{The total length of } C \text{ is } 8a \text{ units.}}$$

Part (d)

$$\begin{aligned}
\text{Area} &= 2\pi\int_0^\pi a(2\sin\theta + \sin 2\theta) \cdot \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\
&= 2\pi\int_0^\pi a(2\sin\theta + \sin 2\theta) \cdot 4a\cos(\theta/2)\,\mathrm{d}\theta \\
&= 8\pi a^2\int_0^\pi (2\sin\theta + \sin 2\theta)\cos(\theta/2)\,\mathrm{d}\theta \\
&= 8\pi a^2\int_0^\pi (2\sin\theta + 2\sin\theta\cos\theta)\cos(\theta/2)\,\mathrm{d}\theta \\
&= 16\pi a^2\int_0^\pi \sin\theta(1 + \cos\theta)\cos(\theta/2)\,\mathrm{d}\theta \\
&= 16\pi a^2\int_0^\pi \sin\theta\left[1 + \left(2\cos^2(\theta/2) - 1\right)\right]\cos(\theta/2)\,\mathrm{d}\theta \\
&= 32\pi a^2\int_0^\pi \sin\theta\cos^3(\theta/2)\,\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&= 32\pi a^2 \int_0^\pi 2\sin(\theta/2)\cos(\theta/2)\cdot\ \cos^3(\theta/2)\,\mathrm{d}\theta \\
&= 64\pi a^2 \int_0^\pi \sin(\theta/2)\cos^4(\theta/2)\,\mathrm{d}\theta \\
&= 128\pi a^2 \int_0^{\pi/2} \sin u \cos^4 u\,\mathrm{d}u \\
&= 64\pi a^2 \cdot 2\int_0^{\pi/2} \sin^{2(1)-1} u \cos^{2(5/2)-1} u\,\mathrm{d}u \\
&= 64\pi a^2 \cdot B\left(1, \frac{5}{2}\right) \\
&= 64\pi a^2 \cdot \frac{\Gamma(1)\,\Gamma(5/2)}{\Gamma(1+5/2)} \\
&= 64\pi a^2 \cdot \frac{\Gamma(5/2)}{5/2\cdot\Gamma(5/2)} \\
&= 64\pi a^2 \cdot \frac{2}{5} \\
&= \frac{128}{5}\pi a^2
\end{aligned}
$$

$u = \theta/2$
$\mathrm{d}u = \mathrm{d}\theta\,/2$

$$\boxed{\text{The surface area is } \frac{128}{5}\pi a^2 \text{ units}^2.}$$

Problem 2.

The curve C is given by the equation $y = \dfrac{1}{2}(e^x + e^{-x})$.

(a) Sketch the curve C.

(b) Find the exact area bounded by C, the lines $x = 2$ and $x = -2$ and the x-axis.

(c) Points A and B are on C where $x = 2$ and $x = -2$ respectively. Find the exact length of the arc AB.

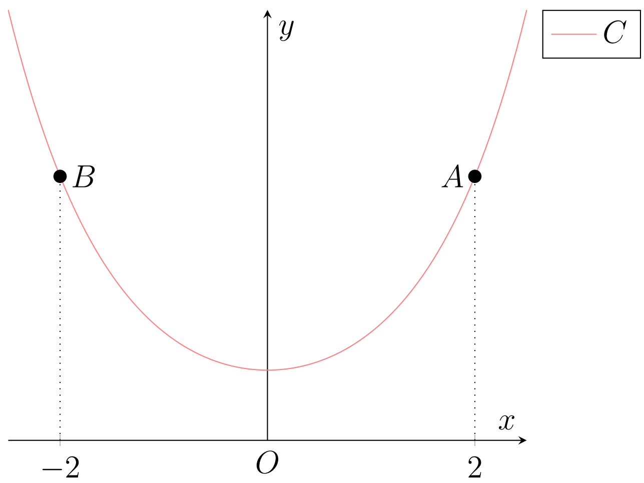
A solid, made of a certain material, is of the shape obtained by rotating the region bounded by C, the lines $x = 2$ and $x = -2$ and the x-axis about the y-axis through π radians.

(d) Find the exact amount of material required to make this solid if x is measured in cm.

(e) The solid is painted with a brush that uses 2 cm^3 of paint for every cm^2 of surface painted. Find the exact amount of paint required.

Solution

Part (a)



Part (b)

Note that $y = \dfrac{1}{2}(e^x + e^{-x}) = \cosh x$ is an even function. Hence,

$$\begin{aligned}\text{Area} &= \int_{-2}^{2} y \, \mathrm{d}x \\ &= 2\int_{0}^{2} \cosh x \, \mathrm{d}x \\ &= 2\left[\sinh x\right]_0^2 \\ &= 2\left(\sinh 2 - \sinh 0\right) \\ &= 2\left(\frac{e^2 - e^{-2}}{2} - 0\right)\end{aligned}$$

$$= e^2 - e^{-2}$$

$$\boxed{\text{The area is } e^2 - e^{-2} \text{ units}^2.}$$

Part (c)

Note that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$, whence

$$\sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2} = \sqrt{1 + \sinh^2 x} = \sqrt{\cosh^2 x} = \cosh x$$

Hence,

$$\begin{aligned}
\text{Length} &= \int_{-2}^{2} \sqrt{1 + (\mathrm{d}y/\mathrm{d}x)^2}\,\mathrm{d}x \\
&= \int_{-2}^{2} \cosh x\,\mathrm{d}x \\
&= 2\int_{0}^{2} \cosh x\,\mathrm{d}x \\
&= e^2 - e^{-2}
\end{aligned}$$

$$\boxed{\text{The length of arc } AB \text{ is } e^2 - e^{-2} \text{ units.}}$$

Part (d)

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^2 xy\,\mathrm{d}x \\
&= 2\pi \int_0^2 x\cosh x\,\mathrm{d}x
\end{aligned}$$

	D	I
$+$	x	$\cosh x$
$-$	1	$\sinh x$
$+$	0	$\cosh x$

$$\begin{aligned}
&= 2\pi \left[x\sinh x - \cosh x\right]_0^2 \\
&= 2\pi \left[(2\sinh 2 - \cosh 2) - (0\sinh 0 - \cosh 0)\right] \\
&= 2\pi \left(2 \cdot \frac{e^2 - e^2}{2} - \frac{e^2 + e^{-2}}{2} + 1\right) \\
&= \pi \left(2e^2 - 2e^{-2} - e^2 - e^{-2} + 2\right) \\
&= \pi \left(e^2 - 3e^{-2} + 2\right)
\end{aligned}$$

$$\boxed{\pi\left(e^2 - 3e^{-2} + 2\right) \text{ cm}^3 \text{ of material is required.}}$$

Part (e)

$$\begin{aligned}
\text{Area} &= \text{Area of curved surface} + \text{Area of side} + \text{Area of bottom} \\
&= 2\pi \int_0^2 x\cosh x \,\mathrm{d}x + 2^2\pi + 2^2\pi\cosh 2 \\
&= \pi\left(e^2 - 3e^{-2} + 2\right) + 4\pi + 4\pi \cdot \frac{e^2 + e^{-2}}{2} \\
&= \pi\left(e^2 - 3e^{-2} + 2\right) + 4\pi + 2\pi(e^2 + e^{-2}) \\
&= \pi\left[\left(e^2 - 3e^{-2} + 2\right) + 4 + 2(e^2 + e^{-2})\right] \\
&= \pi\left[3e^2 - e^{-2} + 6\right]
\end{aligned}$$

$$\boxed{2\pi\left[3e^2 - e^{-2} + 6\right] \text{ cm}^3 \text{ of paint is required.}}$$