

**Problem 1.**

Given that  $y = e^{-x} \cos x$ , show that  $\frac{d^2 y}{dx^2} = -2 \left( y + \frac{dy}{dx} \right)$ . By further differentiation, find the series expansion of  $y$ , in ascending powers of  $x$ , up to and including the term in  $x^3$ . Use the series to obtain an approximate value for  $\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} dx$ , giving your answer correct to 4 decimal places.

Using the trapezium rule with 4 trapezia of equal width, find another approximation for  $\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} dx$ , giving your answer correct to 4 decimal places.

**Solution**

Note that  $y = e^{-x} \cos x = e^{-x} \operatorname{Re} e^{ix} = \operatorname{Re} e^{(i-1)x}$ . Hence, we have  $y' = \operatorname{Re} ((i-1)e^{(i-1)x})$  and  $y'' = \operatorname{Re} ((i-1)^2 e^{(i-1)x})$ . Hence,

$$\begin{aligned} y'' &= \operatorname{Re} ((i^2 - 2i + 1^2)e^{(i-1)x}) \\ &= -2 \operatorname{Re} (ie^{(i-1)x}) \\ &= -2 \operatorname{Re} ((i-1)e^{(i-1)x} + e^{(i-1)x}) \\ &= -2 (\operatorname{Re} (i-1)e^{(i-1)x} + \operatorname{Re} e^{(i-1)x}) \\ &= -2 (y' + y) \end{aligned}$$

Further differentiating, we obtain  $y^{(3)} = -2(y' + y'')$ . Evaluating  $y$  and its derivatives at 0,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= -1 \\ y''(0) &= 0 \\ y^{(3)}(0) &= 2 \end{aligned}$$

Hence, we have

$$e^{-x} \cos x = 1 - x + \frac{1}{3}x^3 + \dots$$

$$\begin{aligned} \int_0^{0.2} \frac{\cos x^2}{e^{x^2}} dx &= \int_0^{0.2} e^{-x^2} \cos x^2 dx \\ &\approx \int_0^{0.2} \left( 1 - x^2 + \frac{1}{3}(x^2)^3 \right) dx \\ &= \int_0^{0.2} \left( 1 - x^2 + \frac{1}{3}x^6 \right) dx \\ &= 0.1973 \text{ (4 d.p.)} \end{aligned}$$

$$\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} dx \approx 0.1973$$

Let  $g(x) = \frac{\cos x^2}{e^{x^2}}$ . By the trapezium rule, we have

$$\begin{aligned}\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} dx &\approx \frac{1}{2} \cdot \frac{0.2 - 0}{4} [g(0) + 2g(0.05) + 2g(0.1) + 2g(0.15) + g(0.2)] \\ &= 0.1973 \text{ (4 d.p.)}\end{aligned}$$

$\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} dx \approx 0.1973$
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**Problem 2.**

The curve  $C$  has equation  $y^2 = \frac{x}{\sqrt{1+x^2}}$ ,  $y \geq 0$ .

The finite region  $R$  is bounded by  $C$ , the  $x$ -axis and the lines  $x = 0$  and  $x = 2$ .  $R$  is rotated through  $2\pi$  radians about the  $x$ -axis.

- (a) Find the exact volume of the solid formed.

An estimate for the volume in (a) is found using the trapezium rule with 7 ordinates.

- (b) Find the percentage error resulting from using this estimate, giving your answer to 3 decimal places.

Explain, with the help of a sketch, why the estimate given by the trapezium rule is less than the actual value.

**Solution****Part (a)**

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 y^2 \, dx \\ &= \pi \int_0^2 \frac{x}{\sqrt{1+x^2}} \, dx \\ &= \frac{\pi}{2} \int_1^5 \frac{1}{\sqrt{u}} \, du \\ &= \frac{\pi}{2} [2\sqrt{u}]_1^5 \\ &= \pi(\sqrt{5} - 1) \end{aligned}$$

$$\begin{aligned} u &= 1 + x^2 \\ du &= 2x \, dx \end{aligned}$$

The volume of the solid formed is  $\pi(\sqrt{5} - 1)$  units<sup>3</sup>.

**Part (b)**

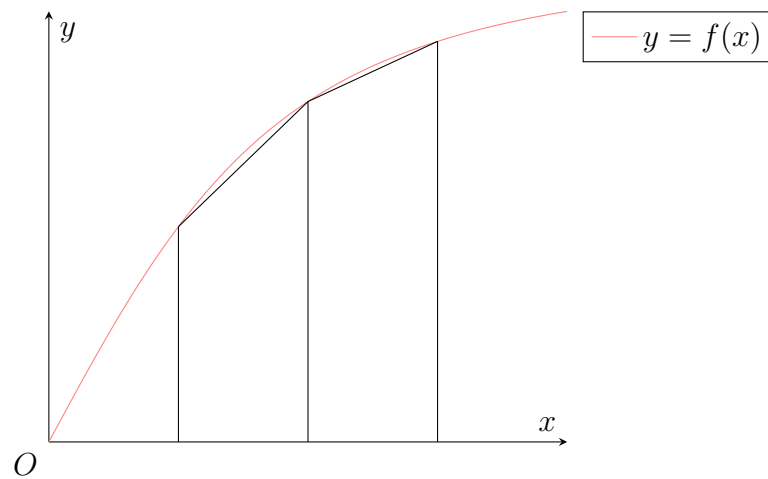
Let  $f(x) = \frac{x}{\sqrt{1+x^2}}$ . By the trapezium rule,

$$\begin{aligned} \text{Volume} &= \pi \int_0^2 f(x) \, dx \\ &\approx \pi \cdot \frac{1}{2} \cdot \frac{2-0}{6} \sum_{n=0}^5 \left[ f\left(\frac{n}{3}\right) + f\left(\frac{n+1}{3}\right) \right] \\ &= 3.8566 \text{ (5 s.f.)} \end{aligned}$$

Hence, the percentage error is  $\left| \frac{\pi(\sqrt{5} - 1) - 3.8566}{\pi(\sqrt{5} - 1)} \right| = 0.686\% \text{ (3 d.p.)}$ .

The percentage error of the estimate is 0.686%.

Consider the following graph of  $y = f(x)$ .



From the graph, the curve  $y = f(x)$  is clearly concave downwards. Hence, the approximation given by the trapezium rule is an underestimate and is thus less than the actual value.

**Problem 3.**

Prove that  $\int_{-h}^h f(x) dx = \frac{1}{3}h(y_{-1} + 4y_0 + y_1)$ , where  $y = f(x)$  is the quadratic curve passing through the points  $(-h, y_{-1})$ ,  $(0, y_0)$  and  $(h, y_1)$ .

Use Simpson's rule with 5 ordinates to find an approximation to

$$\int_{-3}^1 (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3} dx$$

Find another approximation to the same integral using the trapezium rule with 5 ordinates.

Which of these approximations would you expect to be more accurate? Justify your answer.

**Solution**

Let  $f(x) = ax^2 + bx + c$  be the quadratic such that the graph  $y = f(x)$  passes through the points  $(-h, y_{-1})$ ,  $(0, y_0)$  and  $(h, y_1)$ .

Note that we have  $y_0 = f(0) = a \cdot 0^2 + b \cdot 0 + c = c$ . We also have

$$y_{-1} + y_1 = f(-h) + f(h) = [a(-h)^2 + b(-h) + c] + [ah^2 + bh + c] = 2ah^2 + 2c$$

Hence,

$$\begin{aligned} \int_{-h}^h f(x) dx &= \int_{-h}^h (ax^2 + bx + c) dx \\ &= \left[ \frac{1}{3}x^3 + \frac{1}{2}bx^2 + cx \right]_{-h}^h \\ &= \frac{2}{3}h^3 + 2ch \\ &= \frac{1}{3}h(2h^2 + 6c) \\ &= \frac{1}{3}h(2h^2 + 2c + 4c) \\ &= \frac{1}{3}h(y_{-1} + y_1 + 4y_0) \\ &= \frac{1}{3}h(y_{-1} + 4y_0 + y_1) \end{aligned}$$

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Let  $f(x) = (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3}$ . By Simpson's rule,

$$\begin{aligned} \int_{-3}^1 (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3} dx \\ \approx \frac{1}{3} \cdot \frac{1 - (-3)}{4} [f(-3) + 4f(-2) + 2f(-1) + 4f(0) + f(1)] \\ = 11.977 \text{ (5 s.f.)} \end{aligned}$$

By Simpson's rule,  $\int_{-3}^1 (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3} dx = 11.977$

By the trapezium rule,

$$\begin{aligned} \int_{-3}^1 (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3} dx \\ \approx \frac{1}{2} \cdot \frac{1 - (-3)}{4} [f(-3) + 2f(-2) + 2f(-1) + 2f(0) + f(1)] \\ = 12.142 \text{ (5 s.f.)} \end{aligned}$$

By the trapezium rule,  $\int_{-3}^1 (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3} dx = 12.142$

The approximation given by Simpson's rule should be more accurate as Simpson's rule accounts for the concavity of the curve  $y = f(x)$ .

**Problem 4.**

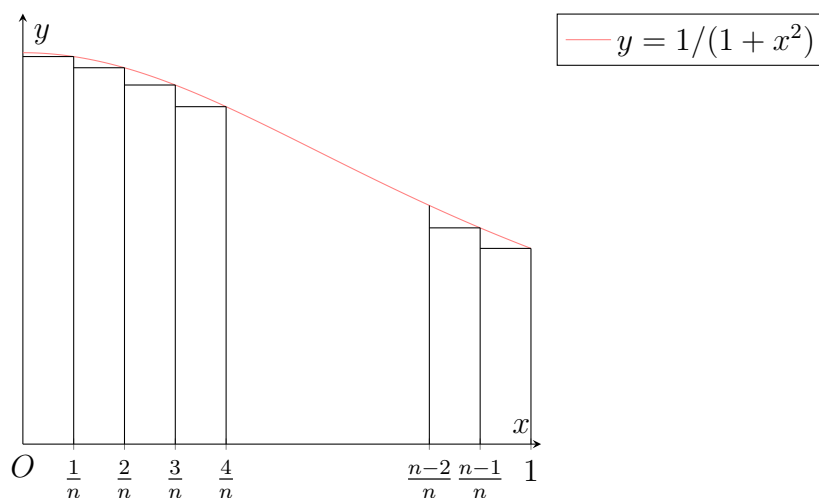
(a) Find the exact value of  $\int_0^1 \frac{1}{1+x^2} dx$ .

(b) The graph of  $y = \frac{1}{1+x^2}$  is shown in the diagram below. Rectangles, each of width  $\frac{1}{n}$ , are drawn under the curve.

Show that the total area  $A$  of all  $n$  rectangles is given by

$$A = \frac{1}{n} \left[ \frac{1}{1 + \left(\frac{1}{n}\right)^2} + \frac{1}{1 + \left(\frac{2}{n}\right)^2} + \frac{1}{1 + \left(\frac{3}{n}\right)^2} + \cdots + \frac{1}{2} \right]$$

State the limit of  $A$  as  $n \rightarrow \infty$ .



(c) It is given that

$$B = \frac{1}{n} \left[ \frac{1}{1 + \left(\frac{1}{n}\right)^4} + \frac{1}{1 + \left(\frac{2}{n}\right)^4} + \frac{1}{1 + \left(\frac{3}{n}\right)^4} + \cdots + \frac{1}{2} \right]$$

Find an approximation for the limit of  $B$  as  $n \rightarrow \infty$  by considering an appropriate graph and using the trapezium rule with 5 intervals. Given your answer correct to 2 decimal places.

**Solution**

**Part (a)**

$$\int_0^1 \frac{1}{1+x^2} dx = [\arctan x]_0^1 = \frac{\pi}{4}$$

$$\boxed{\int_0^1 \frac{1}{1+x^2} dx = \frac{\pi}{4}}$$

**Part (b)**

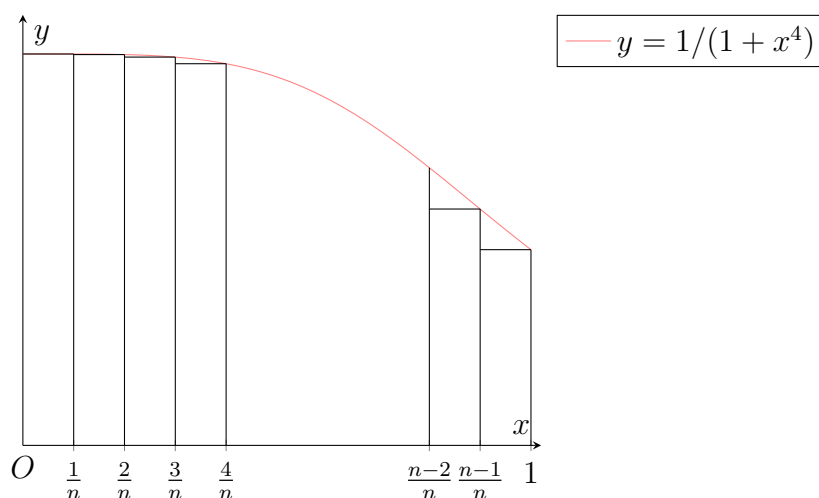
Observe that the  $k$ th rectangle has height  $\frac{1}{1 + (k/n)^2}$  and width  $1/n$ . Hence,

$$\begin{aligned}
 A &= \sum_{k=1}^n \frac{1}{n} \cdot \frac{1}{1 + (k/n)^2} \\
 &= \frac{1}{n} \sum_{k=1}^n \frac{1}{1 + (k/n)^2} \\
 &= \frac{1}{n} \left[ \frac{1}{1 + \left(\frac{1}{n}\right)^2} + \frac{1}{1 + \left(\frac{2}{n}\right)^2} + \frac{1}{1 + \left(\frac{3}{n}\right)^2} + \dots + \frac{1}{1 + \left(\frac{n}{n}\right)^2} \right] \\
 &= \frac{1}{n} \left[ \frac{1}{1 + \left(\frac{1}{n}\right)^2} + \frac{1}{1 + \left(\frac{2}{n}\right)^2} + \frac{1}{1 + \left(\frac{3}{n}\right)^2} + \dots + \frac{1}{2} \right]
 \end{aligned}$$

As  $n \rightarrow \infty$ ,  $A \rightarrow \int_0^1 \frac{1}{1 + x^2} = \frac{\pi}{4}$ .

**Part (c)**

Consider the following graph of  $y = \frac{1}{1 + x^4}$ .



Using a similar line of logic presented in part (b), we have that  $B$  is the total area of the rectangles above. Hence, as  $n \rightarrow \infty$ ,  $B \rightarrow \int_0^1 \frac{1}{1 + x^4} dx$ .

Let  $f(x) = \frac{1}{1 + x^4}$ . Using the trapezium rule,

$$\begin{aligned}
 \int_0^1 \frac{1}{1 + x^4} dx &= \frac{1}{2} \cdot \frac{1 - 0}{5} [f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \\
 &= 0.86 \text{ (2 d.p.)}
 \end{aligned}$$

As  $n \rightarrow \infty$ ,  $B \rightarrow 0.86$ .