Problem 1.

Evaluate the following limits.

(a) $\lim_{x \to 5} (6x + 7)$

(b) $\lim_{x \to 1} \frac{x^3 - 1}{1 - x}$

(c) $\lim_{x \to \infty} \frac{3x}{2x^2 - 5}$

Solution

Part (a)

$$\begin{aligned}\lim_{x \to 5} (6x + 7) &= 6 \cdot 5 + 7 \\ &= 37\end{aligned}$$

$$\boxed{\lim_{x \to 5} (6x + 7) = 37}$$

Part (b)

$$\begin{aligned}\lim_{x \to 1} \frac{x^3 - 1}{1 - x} &= \lim_{x \to 1} \frac{(x - 1)\left(x^2 + x + 1\right)}{1 - x} \\ &= \lim_{x \to 1} -\left(x^2 + x + 1\right) \\ &= -(1^2 + 1 + 1) \\ &= -3\end{aligned}$$

$$\boxed{\lim_{x \to 1} \frac{x^3 - 1}{1 - x} = -3}$$

Part (c)

$$\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = \lim_{x \to \infty} \frac{3}{2x - \frac{5}{x}}$$

Note that as $x \to \infty$, $2x - \frac{5}{x} \to \infty - 0$. Hence, $\lim_{x \to \infty} \frac{1}{2x - \frac{5}{x}} = 0$.

$$\boxed{\lim_{x \to \infty} \frac{3x}{2x^2 - 5} = 0}$$

Problem 2.

Differentiate the following with respect to x from first principles.

(a) $3x + 4$

(b) x^3

Solution

Part (a)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(3x+4) &= \lim_{h\to 0} \frac{(3(x+h)+4)-(3x+4)}{h} \\
&= \lim_{h\to 0} \frac{3h}{h} \\
&= \lim_{h\to 0} 3 \\
&= 3
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(3x+4) = 3}$$

Part (b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}x^3 &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{3hx^2 + 3h^2x + h^3}{h} \\
&= \lim_{h\to 0} \left(3x^2 + 3hx + h^2\right) \\
&= 3x^2
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}x^3 = 3x^2}$$

Problem 3.

Differentiate each of the following with respect to x, simplifying your answer.

(a) $\left(x^2+4\right)^2\left(2x^3-1\right)$

(b) $\dfrac{x^2}{\sqrt{4-x^2}}$

(c) $\sqrt{1+\sqrt{x}}$

(d) $\left(\dfrac{x^3-1}{2x^3+1}\right)^4$

Solution

Part (a)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+4\right)^2\left(2x^3-1\right) &= \left(2x^3-1\right)\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+4\right)^2 + \left(x^2+4\right)^2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(2x^3-1\right)\\
&= \left(2x^3-1\right)\cdot 2\left(x^2+4\right)\cdot 2x + \left(x^2+4\right)^2\cdot 6x^2\\
&= 2x\left(x^2+4\right)\left(2\left(2x^3-1\right)+3x\left(x^2+4\right)\right)\\
&= 2x\left(x^2+4\right)\left(7x^3+12x-2\right)
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+4\right)^2\left(2x^3-1\right) = 2x\left(x^2+4\right)\left(7x^3+12x-2\right)}$$

Part (b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{\sqrt{4-x^2}} &= \frac{\sqrt{4-x^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^2 - x^2\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{4-x^2}}{4-x^2}\\
&= \frac{\sqrt{4-x^2}\cdot 2x - x^2\cdot\frac{1}{2\sqrt{4-x^2}}\cdot -2x}{4-x^2}\\
&= \frac{\left(4-x^2\right)\cdot 2x - x^2\cdot\frac{1}{2}\cdot -2x}{\left(4-x^2\right)^{\frac{3}{2}}}\\
&= \frac{\left(4-x^2\right)\cdot 2x + x^3}{\left(4-x^2\right)^{\frac{3}{2}}}\\
&= \frac{x\left(8-x^2\right)}{\left(4-x^2\right)^{\frac{3}{2}}}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{\sqrt{4-x^2}} = \frac{x\left(8-x^2\right)}{\left(4-x^2\right)^{\frac{3}{2}}}}$$

Part (c)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\sqrt{x}} &= \frac{1}{2\sqrt{1+\sqrt{x}}}\cdot\frac{1}{2\sqrt{x}}\\ &= \frac{1}{4\sqrt{x(1+\sqrt{x})}}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+\sqrt{x}} = \frac{1}{4\sqrt{x(1+\sqrt{x})}}}$$

Part (d)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-1}{2x^3+1}\right)^4 &= 4\left(\frac{x^3-1}{2x^3+1}\right)^3\cdot\frac{\mathrm{d}}{\mathrm{d}x}\frac{x^3-1}{2x^3+1}\\ &= 4\left(\frac{x^3-1}{2x^3+1}\right)^3\cdot\frac{(2x^3+1)\frac{\mathrm{d}}{\mathrm{d}x}(x^3-1)-(x^3-1)\frac{\mathrm{d}}{\mathrm{d}x}(2x^3+1)}{(2x^3+1)^2}\\ &= 4\left(\frac{x^3-1}{2x^3+1}\right)^3\cdot\frac{(2x^3+1)\cdot 3x^2-(x^3-1)\cdot 6x^2}{(2x^3+1)^2}\\ &= \frac{4(x^3-1)^3}{(2x^3+1)^3}\cdot\frac{(2x^3+1)\cdot 3x^2-(x^3-1)\cdot 6x^2}{(2x^3+1)^2}\\ &= \frac{4(x^3-1)^3}{(2x^3+1)^5}\cdot\left(\left(2x^3+1\right)\cdot 3x^2-\left(x^3-1\right)\cdot 6x^2\right)\\ &= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5}\cdot\left(2x^3+1-\left(x^3-1\right)\cdot 2\right)\\ &= \frac{12x^2(x^3-1)^3}{(2x^3+1)^5}\cdot 3\\ &= \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^3-1}{2x^3+1}\right)^4 = \frac{36x^2(x^3-1)^3}{(2x^3+1)^5}}$$

Problem 4.

Using a graphing calculator, find the derivative of $\dfrac{e^{2x}}{x^2+1}$ when $x = 1.5$.

Solution

$$\boxed{\left.\frac{\mathrm{d}}{\mathrm{d}x}\frac{e^{2x}}{x^2+1}\right|_{x=1.5} = 6.66}$$

Problem 5.

Find the derivative with respect to x of

(a) $\cos x^\circ$

(b) $\cot\left(1-2x^2\right)$

(c) $\tan^3(5x)$

(d) $\dfrac{\sec x}{1+\tan x}$

Solution

Part (a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cos x^\circ &= \frac{\mathrm{d}}{\mathrm{d}x}\cos\left(\frac{\pi}{180}x\right)\\ &= \frac{\pi}{180}\cdot\left(-\sin\left(\frac{\pi}{180}x\right)\right)\\ &= -\frac{\pi}{180}\sin\left(\frac{\pi}{180}x\right)\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\cos x^\circ = -\frac{\pi}{180}\sin\left(\frac{\pi}{180}x\right)}$$

Part (b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cot\left(1-2x^2\right) &= -\csc\left(1-2x^2\right)\cdot(-4x)\\ &= 4x\csc\left(1-2x^2\right)\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\cot\left(1-2x^2\right) = 4x\csc\left(1-2x^2\right)}$$

Part (c)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan^3(5x) &= 3\tan^2(5x)\cdot\sec^2(5x)\cdot 5\\ &= 15\tan^2(5x)\sec^2(5x)\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan^3(5x) = 15\tan^2(5x)\sec^2(5x)}$$

Part (d)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sec x}{1+\tan x} &= \frac{(1+\tan x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sec x - \sec x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(1+\tan x)}{(1+\tan x)^2} \\
&= \frac{(1+\tan x)\cdot\sec x\tan x - \sec x\cdot\sec^2 x}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x(1+\tan x) - \sec^2 x)}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x + \tan^2 x) - (\tan^2 x + 1))}{(1+\tan x)^2} \\
&= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\sec x}{1+\tan x} = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}}
$$

Problem 6.

Find the derivative with respect to x of

(a) $y = e^{1+\sin 3x}$

(b) $y = x^2 e^{\frac{1}{x}}$

(c) $y = \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right)$

(d) $y = \frac{\ln(2x)}{x}$

(e) $y = \log_2(3x^4 - e^x)$

(f) $y = 3^{\ln \sin x}$

(g) $y = a^{2\log_a x}$

(h) $y = \sqrt[3]{\frac{e^x(x+1)}{x^2+1}}$, $x > 0$

Solution

Part (a)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x} e^{1+\sin 3x} \\
&= e^{1+\sin 3x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(1+\sin 3x) \\
&= e^{1+\sin 3x} \cdot \cos 3x \cdot 3 \\
&= 3e^{1+\sin 3x} \cos 3x
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 3e^{1+\sin 3x} \cos 3x}$$

Part (b)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x} x^2 e^{\frac{1}{x}} \\
&= x^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x} e^{\frac{1}{x}} + e^{\frac{1}{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}x} x^2 \\
&= x^2 \cdot e^{\frac{1}{x}} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x} + e^{\frac{1}{x}} \cdot 2x \\
&= x^2 \cdot e^{\frac{1}{x}} \cdot -\frac{1}{x^2} + e^{\frac{1}{x}} \cdot 2x \\
&= -e^{\frac{1}{x}} + 2xe^{\frac{1}{x}} \\
&= e^{\frac{1}{x}}(2x-1)
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = e^{\frac{1}{x}}(2x-1)}$$

Part (c)

$$
\begin{aligned}
y &= \ln\left(\frac{1-x}{\sqrt{1+x^2}}\right) \\
&= \ln\left(1+x^2\right) - \ln\left(\sqrt{1+x^2}\right) \\
&= \ln\left(1+x^2\right) - \frac{1}{2}\ln\left(1+x^2\right) \\
\implies \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1-x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(1-x) - \frac{1}{2}\left(\frac{1}{1+x^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(1+x^2)\right) \\
&= \frac{1}{1-x} \cdot -1 - \frac{1}{2}\left(\frac{1}{1+x^2} \cdot 2x\right) \\
&= \frac{1}{x-1} - \frac{x}{1+x^2} \\
&= \frac{(1+x^2) - x(x-1)}{(x-1)\left(1+x^2\right)} \\
&= \frac{1+x^2-x^2+x}{(x-1)\left(1+x^2\right)} \\
&= \frac{1+x}{(x-1)\left(1+x^2\right)}
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+x}{(x-1)\left(1+x^2\right)}}
$$

Part (d)

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\ln(2x))}{x} \\
&= \frac{x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\ln(2x) - \ln(2x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}x}{x^2} \\
&= \frac{x \cdot \frac{1}{2x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}2x - \ln(2x)}{x^2} \\
&= \frac{x \cdot \frac{1}{2x} \cdot 2 - \ln(2x)}{x^2} \\
&= \frac{1 - \ln(2x)}{x^2}
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1 - \ln(2x)}{x^2}}
$$

Part (e)

$$
\begin{aligned}
&& y &= \log_2(3x^4 - e^x) \\
\implies && 2^y &= 3x^4 - e^x \\
\implies && e^{\ln(2^y)} &= 3x^4 - e^x \\
\implies && e^{y\ln(2)} &= 3x^4 - e^x \\
\implies && e^{y\ln 2} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \cdot \ln 2 &= 3 \cdot 4x^3 - e^x \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12x^3 - e^x}{e^{y\ln 2}\ln 2} \\
&& &= \frac{12x^3 - e^x}{(3x^4 - e^x)\ln 2}
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12x^3 - e^x}{(3x^4 - e^x)\ln 2}}$$

Part (f)

$$
\begin{aligned}
&& y &= 3^{\ln\sin x} \\
\implies && \log_3 y &= \ln\sin x \\
\implies && \frac{\ln y}{\ln 3} &= \ln\sin x \\
\implies && \ln y &= \ln 3 \cdot \ln\sin x \\
\implies && \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 3 \cdot \frac{1}{\sin x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x \\
&& &= \ln 3 \cdot \frac{1}{\sin x} \cdot \cos x \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \ln 3 \cdot \cot x \cdot y \\
&& &= \ln 3 \cdot \cot x \cdot 3^{\ln\sin x}
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \ln 3 \cdot \cot x \cdot 3^{\ln\sin x}}$$

Part (g)

$$
\begin{aligned}
&& y &= a^{2\log_a x} \\
&& &= a^{\log_a x^2} \\
&& &= x^2 \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}x^2 \\
&& &= 2x
\end{aligned}
$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 2x}$$

Part (h)

$$
\begin{aligned}
&& y &= \sqrt[3]{\frac{e^x(x+1)}{x^2+1}} \\
\implies && y^3 &= \frac{e^x(x+1)}{x^2+1} \\
\implies && \left(x^2+1\right)y^3 &= e^x(x+1) \\
\implies && \left(x^2+1\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}y^3 + y^3\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2+1\right) &= e^x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x+1) + (x+1)\frac{\mathrm{d}}{\mathrm{d}x}e^x \\
\implies && \left(x^2+1\right)\cdot 3y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + y^3\cdot 2x &= e^x\cdot 1 + (x+1)\cdot e^x \\
\implies && \left(x^2+1\right)\cdot 3y^2\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x(x+2) - 2xy^3 \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{e^x(x+2)-2xy^3}{(x^2+1)\cdot 3y^2} \\
&& &= \frac{1}{3}\left(\frac{e^x(x+2)}{\left(x^2+1\right)y^2} - \frac{2xy}{x^2+1}\right) \\
&& &= \frac{1}{3}\left(\frac{e^x(x+1)(x+2)}{\left(x^2+1\right)(x+1)y^2} - \frac{2xy}{x^2+1}\right) \\
&& &= \frac{1}{3}\left(\frac{y^3(x+2)}{(x+1)y^2} - \frac{2xy}{x^2+1}\right) \\
&& &= \frac{1}{3}\left(\frac{y(x+2)}{(x+1)} - \frac{2xy}{x^2+1}\right) \\
&& &= \frac{y}{3}\left(\frac{(x+2)}{(x+1)} - \frac{2x}{x^2+1}\right) \\
&& &= \frac{y}{3}\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right) \\
&& &= \frac{1}{3}\cdot\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}\cdot\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{3}\cdot\sqrt[3]{\frac{e^x(x+1)}{x^2+1}}\cdot\left(1 + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)}
$$

Problem 7.

Find the derivative with respect to x of

(a) $\arccos \dfrac{x}{10}$

(b) $\arctan \dfrac{1}{1-x}$

(c) $\arcsin(\tan x)$

Solution

Part (a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \arccos \frac{x}{10} &= -\frac{1}{\sqrt{1-\left(\frac{x}{10}\right)^2}} \cdot \frac{1}{10} \\ &= -\frac{1}{\sqrt{100-x^2}}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arccos \frac{x}{10} = -\frac{1}{\sqrt{100-x^2}}}$$

Part (b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \arctan \frac{1}{1-x} &= \frac{1}{1+\left(\frac{1}{1-x}\right)^2} \cdot -\frac{1}{(1-x)^2} \cdot -1 \\ &= \frac{1}{(1-x)^2\left(1+\left(\frac{1}{1-x}\right)^2\right)} \\ &= \frac{1}{(1-x)^2+1}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arctan \frac{1}{1-x} = \frac{1}{(1-x)^2+1}}$$

Part (c)

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin(\tan x) = \frac{1}{1-\tan^2 x} \cdot \sec^2 x$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arcsin(\tan x) = \frac{\sec^2 x}{1-\tan^2 x}}$$

Problem 8.

Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

(a) $(y-x)^2 = 2a(y+x)$, where a is a constant

(b) $y^2 = e^{2x}y + xe^x$

(c) $y = x^y$

(d) $\sin x \cos y = \frac{1}{2}$

Solution

Part (a)

$$\begin{aligned}
&& (y-x)^2 &= 2a(y+x) \\
\implies && 2(y-x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) &= 2a\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right) \\
\implies && (y-x)\left(\frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) &= a\left(\frac{\mathrm{d}y}{\mathrm{d}x} + 1\right) \\
\implies && (y-x)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} - (y-x) &= a\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + a \\
\implies && (y-x-a)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= a+y-x \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{a+y-x}{y-x-a}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a+y-x}{y-x-a}}$$

Part (b)

$$\begin{aligned}
&& y^2 &= e^{2x}y + xe^x \\
\implies && 2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= e^{2x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + y\cdot\frac{\mathrm{d}}{\mathrm{d}x}e^{2x} + x\cdot\frac{\mathrm{d}}{\mathrm{d}x}e^x + e^x\cdot\frac{\mathrm{d}}{\mathrm{d}x}x \\
\implies && 2y\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= e^{2x}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + y\cdot e^{2x}\cdot 2 + xe^x + e^x \\
\implies && \left(2y - e^{2x}\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= y\cdot e^{2x}\cdot 2 + xe^x + e^x \\
\implies && \left(2y - e^{2x}\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= e^x\left(2e^xy + x + 1\right) \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{e^x\left(2e^xy + x + 1\right)}{2y - e^{2x}}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x\left(2e^xy + x + 1\right)}{2y - e^{2x}}}$$

Part (c)

$$\begin{aligned}
& y = x^y \\
\implies & \ln y = y \ln x \\
\implies & \frac{1}{y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = y \cdot \frac{\mathrm{d}}{\mathrm{d}x} \ln x + \ln x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\
& = y \cdot \frac{1}{x} + \ln x \cdot \frac{\mathrm{d}y}{\mathrm{d}x} \\
\implies & \left(\frac{1}{y} - \ln x\right) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} \\
\implies & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{y}{x}}{\frac{1}{y} - \ln x} \\
& = \frac{y^2}{x - xy \ln x}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x - xy \ln x}}$$

Part (d)

$$\begin{aligned}
& \sin x \cos y = \frac{1}{2} \\
\implies & \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \cos y + \cos y \cdot \frac{\mathrm{d}}{\mathrm{d}x} \sin x = 0 \\
\implies & \sin x \cdot - \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} + \cos y \cdot \cos x = 0 \\
\implies & - \sin x \sin y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = - \cos x \cos y \\
\implies & \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos x \cos y}{\sin x \sin y} \\
& = \cot x \cot y
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \cot x \cot y}$$

Problem 9.

It is given that x and y satisfy the equation

$$\arctan x + \arctan y + \arctan xy = \frac{7}{12}\pi \tag{9.1}$$

(a) Find the exact value of y when $x = 1$.

(b) Express $\frac{\mathrm{d}}{\mathrm{d}x} \arctan(xy)$ in terms of x, y and y'.

(c) Show that, when $x = 1$, $y' = -\frac{1}{3} - \frac{1}{2\sqrt{3}}$.

Solution

Part (a)

Substituting $x = 1$ into Equation 9.1,

$$\begin{aligned}
& \arctan 1 + \arctan y + \arctan y = \frac{7}{12}\pi \\
\implies \quad & \frac{1}{4}\pi + 2\arctan y = \frac{7}{12}\pi \\
\implies \quad & \arctan y = \frac{1}{6}\pi \\
\implies \quad & y = \frac{1}{\sqrt{3}}
\end{aligned}$$

$$\boxed{y = \frac{1}{\sqrt{3}}}$$

Part (b)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \arctan(xy) &= \frac{1}{1+(xy)^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x}(xy) \\
&= \frac{1}{1+(xy)^2} \cdot (xy' + y)
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arctan(xy) = \frac{1}{1+(xy)^2} \cdot (xy' + y)}$$

Part (c)

Differentiating Equation 9.1 with respect to x,

$$\frac{1}{1+x^2} + \frac{y'}{1+y^2} + \frac{1}{1+(xy)^2} \cdot (xy' + y) = 0$$

Substituting $x = 1$,

$$\begin{aligned}
\frac{1}{2} + \frac{3}{4}y' + \frac{3}{4}(y' + y) &= 0 \\
\implies \qquad y'(\frac{3}{4} + \frac{3}{4}) &= \frac{3}{4}\left(-\frac{1}{\sqrt{3}}\right) - \frac{1}{2} \\
\implies \qquad \frac{3}{2}y' &= \frac{-3 - 2\sqrt{3}}{4\sqrt{3}} \\
\implies \qquad y' &= \frac{2}{3} \cdot \frac{-3 - 2\sqrt{3}}{4\sqrt{3}} \\
&= \frac{-1 - \frac{2}{3}\sqrt{3}}{2\sqrt{3}} \\
&= -\frac{1}{2\sqrt{3}} - \frac{1}{3}
\end{aligned}$$

Problem 10.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for

(a) $x = \dfrac{1}{1+t^2}$, $y = \dfrac{t}{1+t^2}$

(b) $x = \dfrac{1}{2}(e^t - e^{-t})$, $y = \dfrac{1}{2}(e^t + e^{-t})$

(c) $x = a\sec\theta$, $y = a\tan\theta$

(d) $x = e^{3\theta}\cos 3\theta$, $y = e^{3\theta}\sin 3\theta$

Solution

Part (a)

Observe that $y = xt$.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= x \cdot \frac{\mathrm{d}t}{\mathrm{d}x} + t \\
&= x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} + t \\
&= \frac{1}{1+t^2}\left(-\frac{1}{(1+t^2)^2} \cdot 2t\right)^{-1} + t \\
&= \frac{1}{1+t^2} \cdot \left(-\frac{(1+t^2)^2}{2t}\right) + t \\
&= -\frac{1+t^2}{2t} + \frac{2t^2}{2t} \\
&= \frac{t^2-1}{2t}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{t^2-1}{2t}}$$

Part (b)

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \\
&= \frac{\frac{1}{2}(e^t - e^{-t})}{\frac{1}{2}(e^t + e^{-t})} \\
&= \frac{e^t - e^{-t}}{e^t + e^{-t}}
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^t - e^{-t}}{e^t + e^{-t}}}$$

Part (c)

Recall that $\tan^2\theta + 1 = \sec^2\theta$. Hence, $y^2 + a^2 = x^2$. Implicitly differentiating, we have

$$\begin{aligned} 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 2x \\ \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y} \\ &= \frac{a\sec\theta}{a\tan\theta} \\ &= \csc\theta \end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \csc\theta}$$

Part (d)

Recall that $\sin^2\theta + \cos^2\theta = 1$. Hence, $x^2 + y^2 = e^{6\theta}$. Implicitly differentiating, we have

$$\begin{aligned} 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= e^{6\theta} \cdot 6\frac{\mathrm{d}\theta}{\mathrm{d}x} \\ \implies \quad x + y\frac{\mathrm{d}y}{\mathrm{d}x} &= 3e^{6\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \end{aligned} \tag{10.1}$$

Observe that $\frac{y}{x} = \tan 3\theta$. Implicitly differentiating,

$$\begin{aligned} \frac{x\frac{\mathrm{d}y}{\mathrm{d}x} - y}{x^2} &= \sec^2 3\theta \cdot 3\frac{\mathrm{d}\theta}{\mathrm{d}x} \\ \implies x\frac{\mathrm{d}y}{\mathrm{d}x} - y &= 3x^2\sec^2 3\theta \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \\ &= 3(e^{3\theta}\cos 3\theta)^2 \cdot \frac{1}{\cos^2 3\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \\ &= 3e^{6\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \end{aligned} \tag{10.2}$$

Subtracting Equation 10.1 from Equation 10.2,

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} - y\frac{\mathrm{d}y}{\mathrm{d}x} - y - x &= 0 \\ \implies \quad (x-y)\frac{\mathrm{d}y}{\mathrm{d}x} &= x + y \\ \implies \quad \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x+y}{x-y} \\ &= \frac{e^{3\theta}(\cos 3\theta + \sin 3\theta)}{e^{3\theta}(\cos 3\theta - \sin 3\theta)} \\ &= \frac{\cos 3\theta + \sin 3\theta}{\cos 3\theta - \sin 3\theta} \\ &= \frac{1 + \tan 3\theta}{1 - \tan 3\theta} \\ &= \tan\left(3\theta + \frac{\pi}{4}\right) \end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\left(3\theta + \frac{\pi}{4}\right)}$$

Problem 11.

A curve is defined by the parametric equation

$$x = 120t - 4t^2,\ y = 60t - 6t^2$$

Find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at each of the points where the curve cross the x-axis.

Solution

The curve crosses the x-axis when $y = 0$.

$$\begin{aligned} y &= 0 \\ \implies 60t - 6t^2 &= 0 \\ \implies \quad 10t - t^2 &= 0 \\ \implies t(10 - t) &= 0 \end{aligned}$$

Hence, $t = 0$ or $t = 10$. Now, consider the derivative with respect to x of the curve.

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \\ &= \frac{60 - 12t}{120 - 8t} \end{aligned}$$

Case 1: $t = 0$

$$\begin{aligned} \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=0} &= \frac{60 - 12 \cdot 0}{120 - 8 \cdot 0} \\ &= \frac{1}{2} \end{aligned}$$

Case 2: $t = 10$

$$\begin{aligned} \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{t=10} &= \frac{60 - 12 \cdot 10}{120 - 8 \cdot 10} \\ &= -\frac{3}{2} \end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2} \vee -\frac{3}{2}}$$

Problem 12.

A curve has parametric equations $x = 2t - \ln(2t)$, $y = t^2 - \ln t^2$, where $t > 0$. Find the value of t at the point on the curve at which the gradient is 2.

Solution

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} \\
&= \frac{\mathrm{d}y}{\mathrm{d}t} \cdot \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^{-1} \\
&= \frac{2t - \frac{2}{t}}{2 - \frac{1}{t}} \\
&= \frac{2t^2 - 2}{2t - 1}
\end{aligned}$$

Consider $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.

$$\begin{aligned}
&\frac{\mathrm{d}y}{\mathrm{d}x} = 2 \\
\implies &\frac{2t^2 - 2}{2t - 1} = 2 \\
\implies &\frac{t^2 - 1}{2t - 1} = 1 \\
\implies &t^2 - 1 = 2t - 1 \\
\implies &t^2 - 2t = 0 \\
\implies &t(t - 2) = 0
\end{aligned}$$

Hence, $t = 0$ or $t = 2$. Since $t > 0$, we reject $t = 0$. Thus, $t = 2$.

$$\boxed{t = 2}$$

Problem 13.

If $y = \ln\left(\sin^3 2x\right)$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and prove that $3\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 = 0$.

Solution

$$\begin{aligned}
&& y &= \ln\left(\sin^3 2x\right) \\
\implies && e^y &= \sin^3 2x \\
\implies && e^y \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 3\sin^2 2x \cdot \cos 2x \cdot 2 \\
&& &= 6\sin^2 2x \cos 2x \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6\sin^2 2x \cos 2x}{e^y} \\
&& &= \frac{6\sin^2 2x \cos 2x}{\sin^3 2x} \\
&& &= \frac{6\cos 2x}{\sin 2x} \\
&& &= 6\cot 2x
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 6\cot 2x}$$

$$\begin{aligned}
&& \frac{\mathrm{d}y}{\mathrm{d}x} &= 6\cot 2x \\
\implies && \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 6 \cdot -\csc^2 2x \cdot 2 \\
&& &= -12\csc^2 2x \\
&& &= -12\left(1 + \cot^2 2x\right) \\
&& &= -12 - 12\cot^2 2x \\
&& &= -12 - \frac{1}{3}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\implies && 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -36 - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \\
\implies && 3\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 36 &= 0
\end{aligned}$$

Problem 14.

Given that $y = e^{\arcsin 2x}$, show that $(1-4x^2)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4y$. Differentiate this result further to obtain a differential equation for $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$.

Solution

$$\begin{aligned}
&& y &= e^{\arcsin 2x} \\
\implies && \ln y &= \arcsin 2x \\
\implies && \frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1-(2x)^2}}\cdot 2 \\
\implies && \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2y}{\sqrt{1-4x^2}} \\
\implies && \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \frac{\sqrt{1-4x^2}\cdot 2\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\cdot\frac{1}{2\sqrt{1-4x^2}}\cdot -8x}{1-4x^2} \\
\implies && \left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= 2\sqrt{1-4x^2}\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{2y}{\sqrt{1-4x^2}} \\
&& &= 2\sqrt{1-4x^2}\cdot\frac{2y}{\sqrt{1-4x^2}} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} \\
&& &= 4y + 4x\frac{\mathrm{d}y}{\mathrm{d}x} \\
\implies && \left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4y
\end{aligned}$$

$$\begin{aligned}
&& \left(1-4x^2\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}y}{\mathrm{d}x} &= 4y \\
\implies && \left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} + \frac{\mathrm{d}^2y}{\mathrm{d}x^2}\cdot(-8x) - 4\left(x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}\right) &= 4\frac{\mathrm{d}y}{\mathrm{d}x} \\
\implies && \left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 8x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} &= 4\frac{\mathrm{d}y}{\mathrm{d}x} \\
\implies && \left(1-4x^2\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 12x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

$$\boxed{(1-4x^2)\frac{\mathrm{d}^3y}{\mathrm{d}x^3} - 12x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 8\frac{\mathrm{d}y}{\mathrm{d}x} = 0}$$