Problem 1.

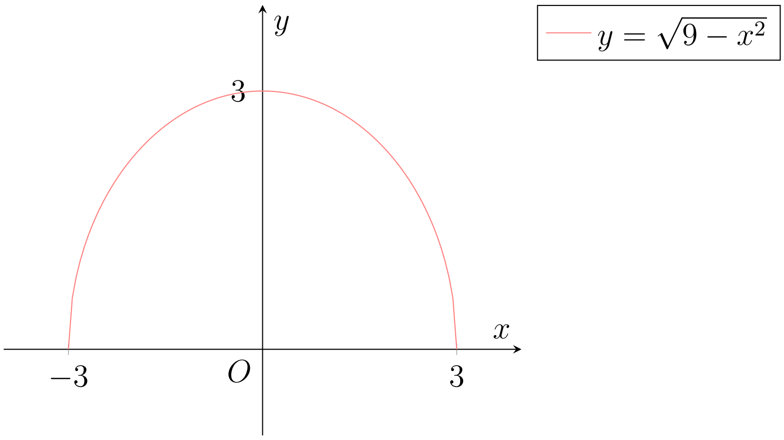
Sketch the following graphs and determine whether each graph represents a function for the given domain.

(a) $y = \sqrt{9 - x^2}$, $-3 \leq x \leq 3$

(b) $x = (y - 4)^2$, $y \in \mathbb{R}$

Solution

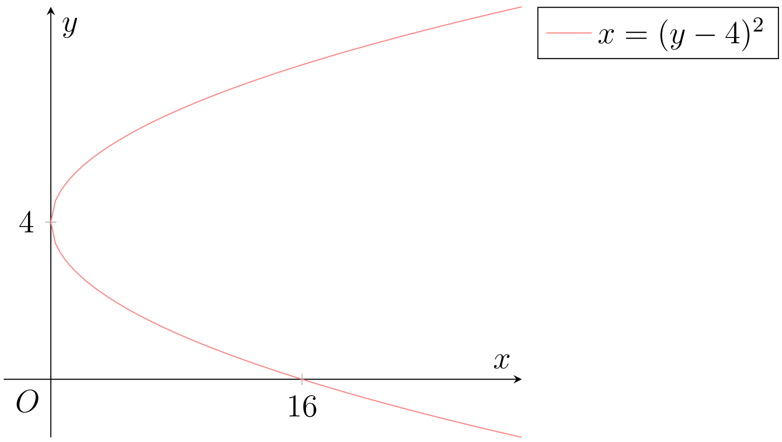
Part (a)



$y = \sqrt{9 - x^2}$ passes the vertical line test for $-3 \leq x \leq 3$ and is hence a function.

$$\boxed{y = \sqrt{9 - x^2},\ -3 \leq x \leq 3 \text{ is a function.}}$$

Part (b)



$x = (y - 4)^2$ does not pass the vertical line test for $y \in \mathbb{R}$ and is hence not a function.

$$\boxed{x = (y - 4)^2,\ y \in \mathbb{R} \text{ is not a function.}}$$

Problem 2.

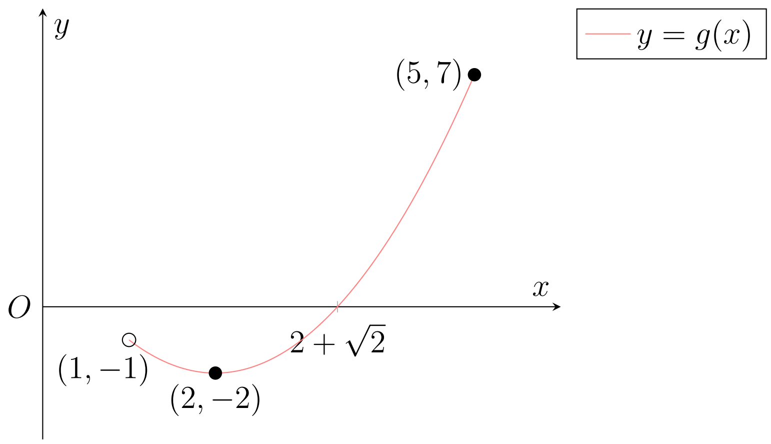
Sketch the graph and find the range for each the following functions.

(a) $g\colon x \mapsto x^2 - 4x + 2,\ 1 < x \leq 5$

(b) $h\colon x \mapsto |2x-3|\,,\ x < 3$

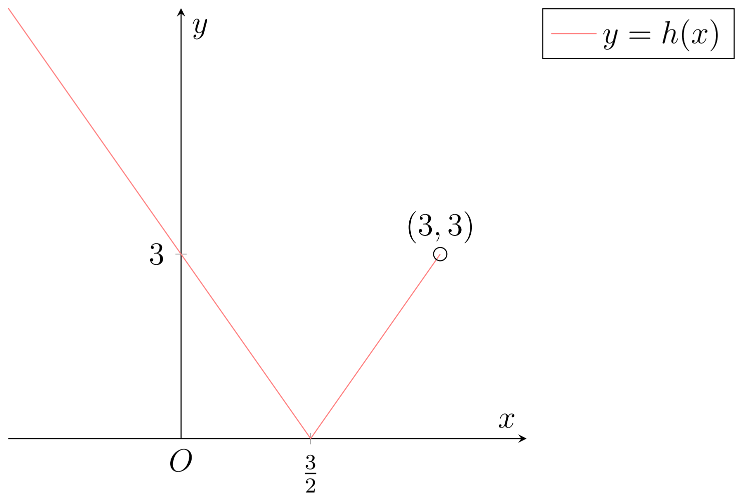
Solution

Part (a)



$$\boxed{R_g = [-2, 7)}$$

Part (b)



$$\boxed{R_h = [0, \infty)}$$

Problem 3.

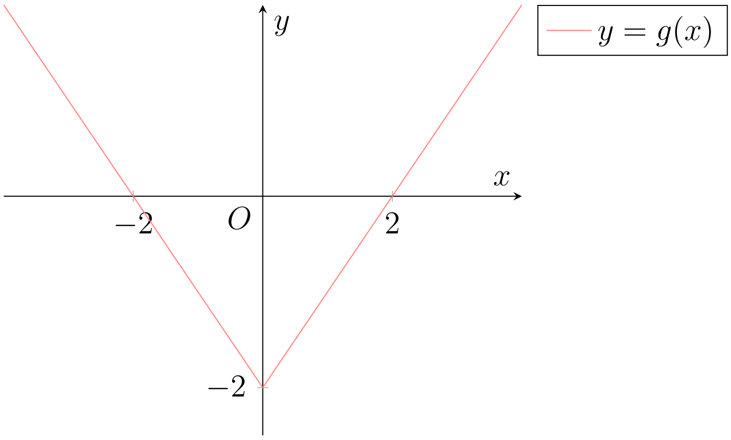
For each of the following functions, sketch its graph and determine if the function is one-one. If it is, find its inverse in a similar form.

(a) $g \colon x \mapsto |x| - 2,\ x \in \mathbb{R}$

(b) $h \colon x \mapsto x^2 + 2x + 5,\ x \leq -2$

Solution

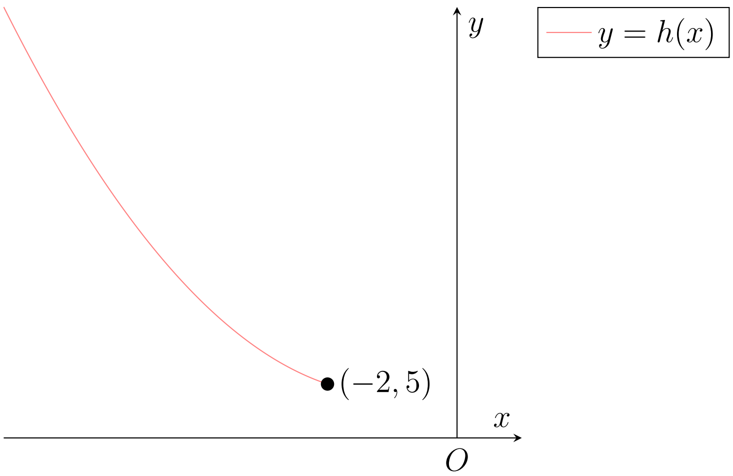
Part (a)



$y = g(x)$ does not pass the horizontal line test. Hence, g is not one-one.

$$\boxed{g \text{ is not one-one.}}$$

Part (b)



$y = h(x)$ passes the horizontal line test. Hence, h is one-one.

$$\boxed{h \text{ is not one-one.}}$$

Note that $y = h(x) \implies x = h^{-1}(y)$. Now consider $y = h(x)$.

$$\begin{aligned} && y &= h(x) \\ \implies && y &= x^2 + 2x + 5 \\ \implies && y &= x^2 + 2x + 1 + 4 \\ \implies && y &= (x+1)^2 + 4 \\ \implies && (x+1)^2 &= y - 4 \\ \implies && x + 1 &= \pm\sqrt{y-4} \end{aligned}$$

Now, since $x \leq -2$, we have $x + 1 \leq -1$. Hence, we reject $x + 1 = \sqrt{y-4}$ since $\sqrt{y-4} \geq 0$.

$$\begin{aligned} \implies && x + 1 &= -\sqrt{y-4} \\ \implies && x &= -1 - \sqrt{y-4} \end{aligned}$$

Hence, $h^{-1}(x) = -1 - \sqrt{x-4}$. Note that $D_{h^{-1}} = R_h = [5, \infty)$. Hence,

$$\boxed{h^{-1} \colon x \mapsto -1 - \sqrt{x-4},\ x \geq 5}$$

Problem 4.

The function f is defined by

$$f\colon x \mapsto x + \frac{1}{x},\ x \neq 0$$

(a) Sketch the graph of f and explain why f^{-1} does not exist.

(b) The function h is defined by $h\colon x \mapsto f(x)$, $x \in \mathbb{R}$, $x \geq \alpha$, where $\alpha \in \mathbb{R}^+$. Find the smallest value of α such that the inverse function of h exists.

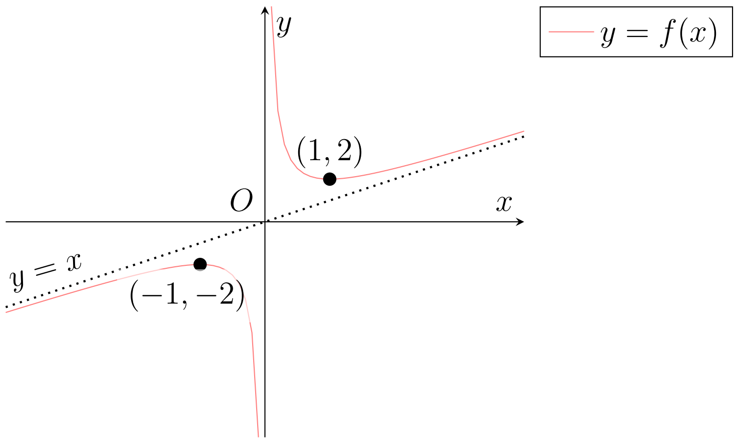
Using this value of α,

(c) State the range of h.

(d) Express h^{-1} in a similar form and sketch on a single diagram, the graphs of h and h^{-1}, showing clearly their geometrical relationship.

Solution

Part (a)



$y = f(x)$ does not pass the horizontal line test. Hence, f is not one-one. Hence, f^{-1} does not exist.

Part (b)

Consider $f'(x) = 0$ for $x > 0$.

$$\begin{aligned} & f'(x) = 0 \\ \implies & 1 - \frac{1}{x^2} = 0 \\ \implies & x^2 = 1 \\ \implies & x = 1 \qquad (\text{rej. } x = -1 \because x > 0) \end{aligned}$$

Looking at the graph of $y = f(x)$, we see that $f(x)$ achieves a minimum at $x = 1$. Hence, f is increasing for all $x \geq 1$. Thus, the smallest value of α is 1.

$$\boxed{\min \alpha = 1}$$

Part (c)

Note $f(1) = 2$. Hence, from the graph,

$$\boxed{R_h = [2, \infty)}$$

Part (d)

Note that $y = h(x) \implies x = h^{-1}(y)$. Now consider $y = h(x)$.

$$\begin{aligned}
& y = h(x) \\
\implies \quad & y = x + \frac{1}{x} \\
\implies \quad & xy = x^2 + 1 \\
\implies \quad & x^2 - yx + 1 = 0 \\
\implies \quad & x = \frac{1}{2}(y \pm \sqrt{y^2 - 4})
\end{aligned}$$

Note that $f(2) = \frac{5}{2}$. Since $2 = \frac{1}{2}\left(\frac{5}{2} + \sqrt{\left(\frac{5}{2}\right)^2 - 4}\right)$ and $2 \neq \frac{1}{2}\left(\frac{5}{2} - \sqrt{\left(\frac{5}{2}\right)^2 - 4}\right)$, we reject $x = \frac{1}{2}(y - \sqrt{y^2 - 4})$. Hence, $h^{-1}(x) = \frac{1}{2}(x + \sqrt{x^2 - 4})$. Note that $D_{f^{-1}} = R_f = [2, \infty)$. Thus,

$$\boxed{h^{-1} \colon x \mapsto \frac{1}{2}\left(x + \sqrt{x^2 - 4}\right), \ x \geq 2}$$

Problem 5.

The function f is defined as follows:

$$f : x \mapsto x^3 + x - 7, \, x \in \mathbb{R}$$

(a) By using a graphical method or otherwise, show that the inverse of f exists.

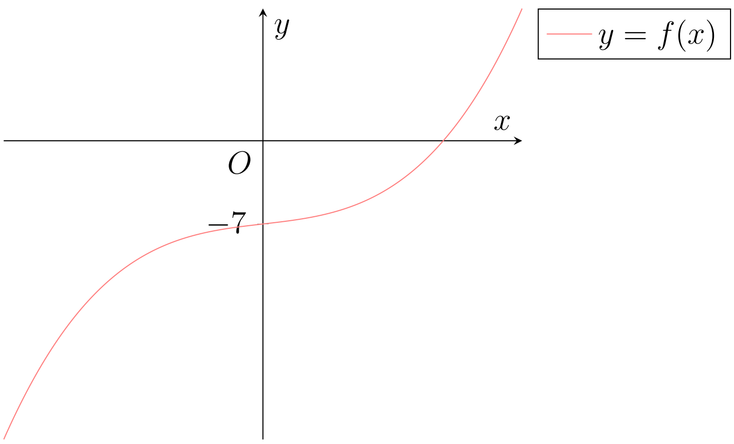
(b) Solve exactly the equation $f^{-1}(x) = 0$. Sketch the graph of f^{-1} together with the graph of f on the same diagram.

(c) Find, in exact form, the coordinates of the intersection point(s) of the graphs of f and f^{-1}.

(d) Given that the gradient of the tangent to the curve with equation $y = f^{-1}(x)$ is $\frac{1}{4}$ at the point with $x = p$, find the possible values of p.

Solution

Part (a)

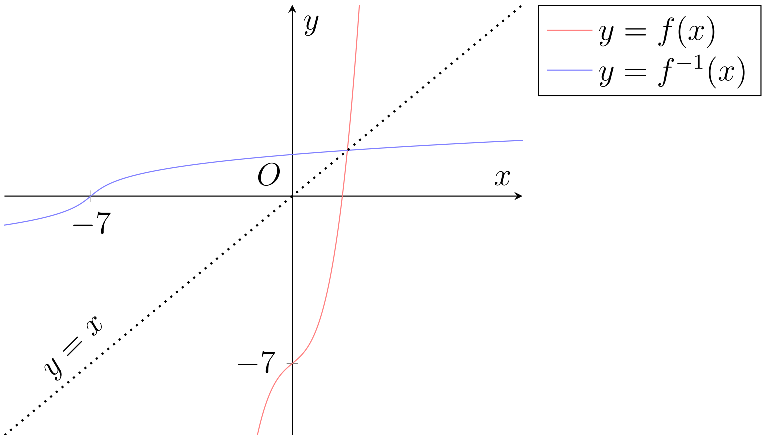


$y = f(x)$ passes the horizontal line test. Hence, f is one-one. Thus, f^{-1} exists.

Part (b)

$$\begin{aligned} & f^{-1}(x) = 0 \\ \implies & \qquad x = f(0) \\ \implies & \qquad x = 0^3 + 0 - 7 \\ & \qquad\;\; = -7 \end{aligned}$$

$$\boxed{x = -7}$$



Part (c)

Let (α, β) be the coordinates of the intersection between $f(x)$ and f^{-1}. From the graph, we see that $\alpha = \beta$, hence $f(\alpha) = \alpha$.

$$\begin{aligned} & f(\alpha) = \alpha \\ \implies & \alpha^3 + \alpha - 7 = \alpha \\ \implies & \alpha^3 = 7 \\ \implies & \alpha = \sqrt[3]{7} \end{aligned}$$

$$\boxed{(\sqrt[3]{7}, \sqrt[3]{7})}$$

Part (d)

$$\begin{aligned} & [f^{-1}(x)]' = \frac{1}{f'(f^{-1}(x))} \\ \implies & [f^{-1}(x)]'\Big|_{x=p} = \frac{1}{f'(f^{-1}(x))}\Bigg|_{x=p} \\ \implies & \frac{1}{4} = \frac{1}{f'(f^{-1}(x))}\Bigg|_{x=p} \\ \implies & f'(f^{-1}(x))\Big|_{x=p} = 4 \end{aligned}$$

Note that $f'(x) = 3x^2 + 1$.

$$\begin{aligned} \implies & \left(3 \cdot f^{-1}(x)^2 + 1\right)\Big|_{x=p} = 4 \\ \implies & 3 \cdot f^{-1}(p)^2 + 1 = 4 \\ \implies & f^{-1}(p)^2 = 1 \\ \implies & f^{-1}(p) = \pm 1 \end{aligned}$$

Case 1: $f^{-1}(p) = 1$

$$\begin{aligned} f^{-1}(p) &= 1 \\ \implies \quad p &= f(1) \\ \implies \quad p &= 1^3 + 1 - 7 \\ &= -5 \end{aligned}$$

Case 2: $f^{-1}(p) = -1$

$$\begin{aligned} f^{-1}(p) &= -1 \\ \implies \quad p &= f(-1) \\ \implies \quad p &= (-1)^3 - 1 - 7 \\ &= -9 \end{aligned}$$

$$\boxed{p = -9 \lor -5}$$

Problem 6.

The functions g and h are defined as follows:

$$\begin{aligned} g &: x \mapsto \ln(x+2), & x &\in (-1,1) \\ h &: x \mapsto x^2 - 2x - 1, & x &\in \mathbb{R}^+ \end{aligned}$$

(a) Sketch, on separate diagrams, the graphs of g and h.

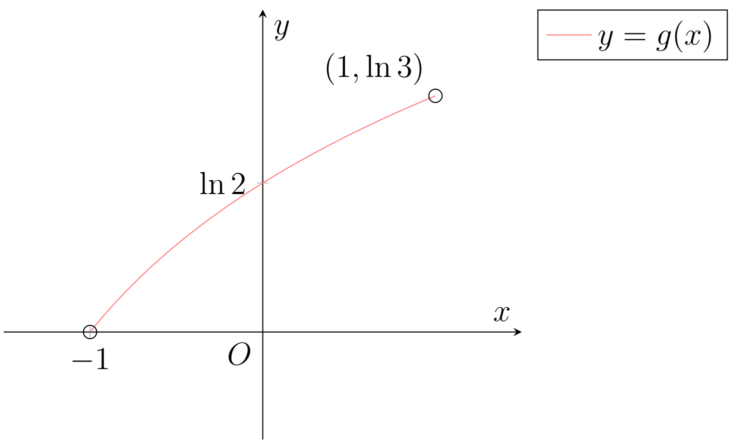
(b) Determine whether the composite function gh exists.

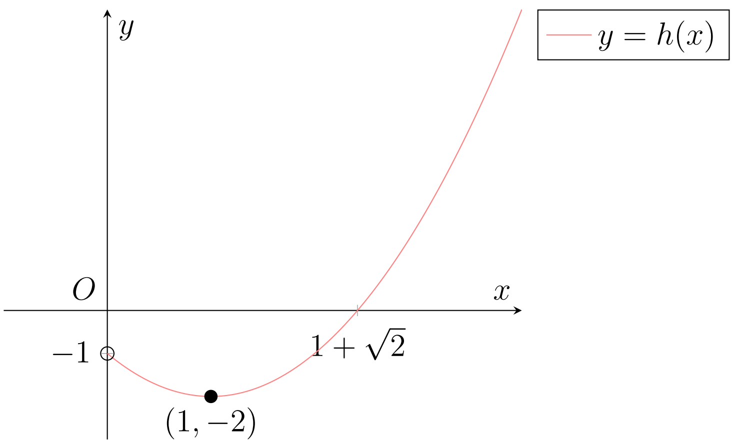
(c) Give the rule and domain of the composite function hg and find its range.

(d) The image of a under the composite function hg is -1.5. Find the value of a.

Solution

Part (a)





Part (b)

Observe that $R_h = [-2, \infty)$ and $D_g = (-1, 1)$. Hence, $R_h \not\subseteq D_g$. Thus, gh does not exist.

$$\boxed{gh \text{ does not exist.}}$$

Part (c)

$$\begin{aligned} hg(x) &= h(\,\ln(x+2)) \\ &= \,\ln(x+2)^2 - 2\ln(x+2) - 1 \end{aligned}$$

Note that $D_{hg} = D_g = (-1, 1)$.

$$\boxed{hg\colon x \mapsto \,\ln(x+2)^2 - 2\ln(x+2) - 1,\; x \in (-1, 1)}$$

Observe that h is decreasing on the interval $(0, 1]$ and increasing on the interval $[1, \infty)$. Note that $R_g = (0, \ln 3)$. Hence,

$$\begin{aligned} R_{hg} &= [-2, \max\{h(0), h(\ln 3)\}) \\ &= [-2, -1) \end{aligned}$$

Part (d)

Note that $h(x) = (x-1)^2 - 2$. Hence, $h^{-1}(x) = 1 + \sqrt{x+2}$ (we reject $h^{-1}(x) = 1 - \sqrt{x+2}$ since $R_{h^{-1}} = D_h = \mathbb{R}^+$). Further note that $g^{-1} = e^x - 2$.

$$\begin{aligned} hg(a) &= -1.5 \\ \implies \quad g(a) &= h^{-1}(-1.5) \\ &= 1 + \sqrt{-1.5+2} \\ &= 1 + \frac{1}{\sqrt{2}} \\ \implies \quad a &= g^{-1}\left(1 + \frac{1}{\sqrt{2}}\right) \\ &= e^{1+\frac{1}{\sqrt{2}}} - 2 \end{aligned}$$

$$\boxed{a = e^{1+\frac{1}{\sqrt{2}}} - 2}$$

Problem 7.

The functions f and g are defined as follows:

$$\begin{aligned} f\colon x &\mapsto 3 - x, && x \in \mathbb{R} \\ g\colon x &\mapsto \frac{4}{x}, && x \in \mathbb{R},\ x \neq 0 \end{aligned}$$

(a) Show that the composite function fg exists and express the definition of fg in a similar form. Find the range of fg.

(b) Find, in similar form, g^2 and g^3, and deduce g^{2017}.

(c) Find the set of values of x for which $g(x) = g^{-1}(x)$.

Solution

Part (a)

Note that $R_g = \mathbb{R} \setminus \{0\}$ and $D_g = \mathbb{R}$. Hence, $R_g \subseteq D_g$. Thus, fg exists.

$$\begin{aligned} fg(x) &= f\left(\frac{4}{x}\right) \\ &= 3 - \frac{4}{x} \end{aligned}$$

Observe that $D_{fg} = D_g = \mathbb{R} \setminus \{0\}$.

$$\boxed{fg\colon x \mapsto 3 - \frac{4}{x},\ x \in \mathbb{R} \setminus \{0\}}$$

Since $\frac{4}{x}$ can take on any value except 0, then $fg(x) = 3 - \frac{4}{x}$ can take on any value except 3.

$$\boxed{R_{fg} = \mathbb{R} \setminus \{3\}}$$

Part (b)

$$\begin{aligned} g^2(x) &= g\left(\frac{4}{x}\right) \\ &= \frac{4}{\frac{4}{x}} \\ &= x \end{aligned}$$

$$\boxed{g^2\colon x \mapsto x,\ x \in \mathbb{R} \setminus \{0\}}$$

$$\begin{aligned} g^3(x) &= g(g^2(x)) \\ &= g(x) \\ &= \frac{4}{x} \end{aligned}$$

$$\boxed{g^3 \colon x \mapsto \frac{4}{x},\, x \in \mathbb{R} \setminus \{0\}}$$

$$\begin{aligned} g^{2017} &= g^{2016}(g(x)) \\ &= \left(g^2\right)^{1008}(g(x)) \\ &= g(x) \\ &= \frac{4}{x} \end{aligned}$$

$$\boxed{g^{2017} \colon x \mapsto \frac{4}{x},\, x \in \mathbb{R} \setminus \{0\}}$$

Part (c)

$$\begin{aligned} & g(x) = g^{-1}(x) \\ \implies & g^2(x) = x \end{aligned}$$

From the definition of $g^2(x)$, we know that $g^2(x) = x$ for all x in D_{g^2}.

$$\boxed{\mathbb{R} \setminus \{0\}}$$

Problem 8.

The function f is defined by

$$f(x) = \begin{cases} 2x+1, & 0 \leq x < 2 \\ (x-4)^2+1, & 2 \leq x < 4 \end{cases}$$

It is further given that $f(x) = f(x+4)$ for all real values of x.

(a) Find the values of $f(1)$ and $f(5)$ and hence explain why f is not one-one.

(b) Sketch the graph of $y = f(x)$ for $-4 \leq x < 8$.

(c) Find the range of f for $-4 \leq x < 8$.

Solution

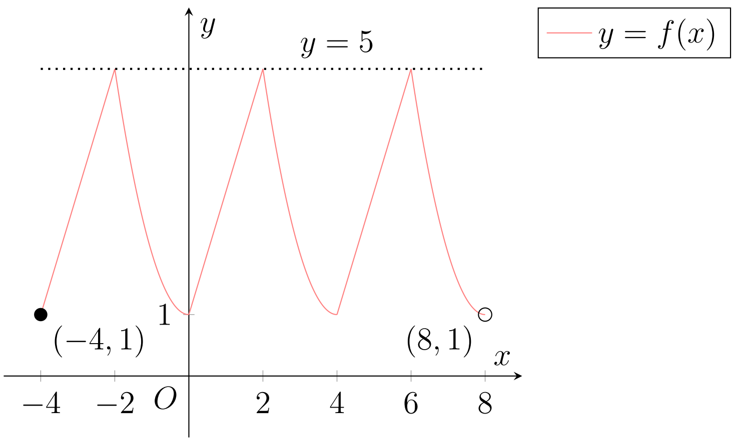
Part (a)

$$\begin{aligned} f(1) &= 2(1)+1 \\ &= 3 \\ f(5) &= f(1+4) \\ &= f(1) \\ &= 3 \end{aligned}$$

$$\boxed{f(1) = 3,\ f(5) = 3}$$

Since $f(1) = f(5)$, but $1 \neq 5$, f is not one-one.

Part (b)



Part (c)

$$\boxed{R_f = [1, 5]}$$

Problem 9.

(a) The function f is given by $f\colon x \mapsto 1 + \sqrt{x}$ for $x \in \mathbb{R}^+$.

(i) Find $f^{-1}(x)$ and state the domain of f^{-1}.

(ii) Find $f^2(x)$ and the range of f^2.

(iii) Show that if $f^2(x) = x$ then $x^3 - 4x^2 + 4x - 1 = 0$. Hence, find the value of x for which $f^2(x) = x$. Explain why this value of x satisfies the equation $f(x) = f^{-1}(x)$.

(b) The function g, with domain the set of non-negative integers, is given by

$$g(n) = \begin{cases} 1, & n = 0 \\ 2 + g\left(\frac{1}{2}n\right), & n \text{ even} \\ 1 + g(n-1), & n \text{ odd} \end{cases}$$

(i) Find $g(4)$, $g(7)$ and $g(12)$.

(ii) Does g have an inverse? Justify your answer.

Solution

Part (a)

Subpart (i)

Let $y = f(x)$. Then $x = f^{-1}(y)$.

$$\begin{aligned} & y = f(x) \\ \implies & y = 1 + \sqrt{x} \\ \implies & \sqrt{x} = y - 1 \\ \implies & x = (y-1)^2 \end{aligned}$$

$$\boxed{f^{-1}(x) = (x-1)^2}$$

Observe that $D_{f^{-1}} = R_f = (1, \infty)$.

$$\boxed{D_{f^{-1}} = (1, \infty)}$$

Subpart (ii)

$$\begin{aligned} f^2(x) &= f(1 + \sqrt{x}) \\ &= 1 + \sqrt{1 + \sqrt{x}} \end{aligned}$$

$$\boxed{f^2(x) = 1 + \sqrt{1 + \sqrt{x}}}$$

Observe that $\sqrt{1 + \sqrt{x}} > 1$. Hence, $1 + \sqrt{1 + \sqrt{x}} > 1 + 1 = 2$.

$$\boxed{R_{f^2} = (2, \infty)}$$

Subpart (iii)

$$\begin{aligned}
& & f^2(x) &= x \\
&\implies & 1 + \sqrt{1 + \sqrt{x}} &= x \\
&\implies & \sqrt{1 + \sqrt{x}} &= x - 1 \\
&\implies & 1 + \sqrt{x} &= (x-1)^2 \\
&\implies & \sqrt{x} &= (x-1)^2 - 1 \\
&\implies & &= x(x-2) \\
&\implies & x &= (x(x-2))^2 \\
&\implies & x(x-2)^2 &= 1 \qquad (\because x \neq 0) \\
&\implies & x\left(x^2 - 4x + 4\right) &= 1 \\
&\implies & x^3 - 4x^2 + 4x &= 1 \\
&\implies & x^3 - 4x^2 + 4x - 1 &= 0
\end{aligned}$$

Hence, if $f^2(x) = x$, then $x^3 - 4x^2 + 4x - 1 = 0$.

$$\begin{aligned}
& & f^2(x) &= x \\
&\implies & x^3 - 4x^2 + 4x - 1 &= 0 \\
&\implies & (x-1)\left(x^2 - 3x + 1\right) &= 0
\end{aligned}$$

Hence, $x = 1$ or $(x^2 - 3x + 1) = 0$. However, since $x \geq 2$, x cannot be 1. We thus consider $(x^2 - 3x + 1) = 0$.

$$\begin{aligned}
& & \left(x^2 - 3x + 1\right) &= 0 \\
&\implies & x &= \frac{3 \pm \sqrt{5}}{2}
\end{aligned}$$

Observe that $\frac{3-\sqrt{5}}{2} < 2$ and $\frac{3+\sqrt{5}}{2} > 2$. Thus, we reject $x = \frac{3-\sqrt{5}}{2}$ and take $x = \frac{3+\sqrt{5}}{2}$.

$$\boxed{x = \frac{3+\sqrt{5}}{2}}$$

Consider $f(x) = f^{-1}(x)$. Applying f on both sides of the equation, we have $f^2(x) = f(x)$. Since $x = \frac{3+\sqrt{5}}{2}$ satisfies $f^2(x) = f(x)$, it also satisfies $f(x) = f^{-1}(x)$.

Part (b)

Subpart (i)

$$
\begin{aligned}
g(4) &= 2 + g(2) \\
&= 2 + 2 + g(1) \\
&= 2 + 2 + 1 + g(0) \\
&= 2 + 2 + 1 + 1 \\
&= 6 \\
g(7) &= 1 + g(6) \\
&= 1 + 2 + g(3) \\
&= 1 + 2 + 1 + g(2) \\
&= 1 + 2 + 1 + (g(4) - 2) \\
&= 1 + 2 + 1 + 6 - 2 \\
&= 8 \\
g(12) &= 2 + g(6) \\
&= 2 + (g(7) - 1) \\
&= 2 + 8 - 1 \\
&= 9
\end{aligned}
$$

$$\boxed{g(4) = 6,\ g(7) = 8,\ g(12) = 9}$$

Subpart (ii)
Consider $g(5)$ and $g(6)$.

$$
\begin{aligned}
g(5) &= 1 + g(4) \\
&= 1 + 6 \\
&= 7 \\
g(6) &= g(7) - 1 \\
&= 8 - 1 \\
&= 7
\end{aligned}
$$

Since $g(5) = g(6)$, but $5 \neq 6$, g is not one-one. Hence, g^{-1} does not exist.

$$\boxed{g \text{ does not have an inverse.}}$$