Problem 1.

Omitted.

Problem 2.

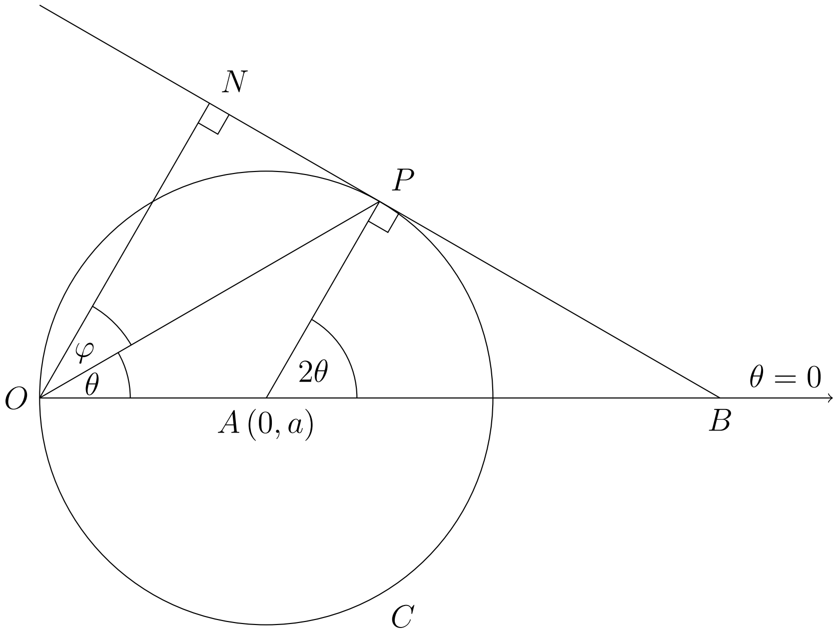
A point P lies on the curve C with polar equation $r = 2a\cos\theta$, $-\dfrac{\pi}{2} < \theta \leq \dfrac{\pi}{2}$, where a is a positive constant. The point N is the foot of the perpendicular from the pole to the tangent of C at P.

(a) Sketch C, P and N on the same diagram.

(b) By considering the polar coordinates of N, show that, as P varies, the locus of N is given by the polar equation $r = a(1+\cos\theta)$, $-\pi < \theta \leq \pi$.

Solution

Part (a)



Part (b)

Consider the diagram above. Let $N(\theta+\varphi, r)$. Let B be the intersection between the tangent at P and the half-line $\theta = 0$. Note that C is a circle centred at A with radius a.

Since angle at centre is twice angle at circumference, $\angle PAB = 2\angle POA = 2\theta$. Since tangent to circle is parallel to radius, $AP \perp PB$. Hence, $ON \parallel AP \implies \angle NOA = \angle PAB \implies \varphi = \theta$.

Since $\cos\angle NOP = \dfrac{ON}{OP}$, we have

$$\begin{aligned} ON &= OP\cos\theta \\ &= (2a\cos\theta)\cos\theta \\ &= 2a\cos^2\theta \\ &= a\left(2\cos^2\theta - 1 + 1\right) \\ &= a(\cos 2\theta + 1) \end{aligned}$$

Hence, $N(\theta + \varphi, r) = \Big(2\theta, a(\cos 2\theta + 1)\Big) = \Big(\theta, a(\cos\theta + 1)\Big)$. Thus, the locus of N is given by $r = a(1 + \cos\theta)$.

Problem 3.

The sequence $\{u_n\}$ is given by the recurrence relation

$$u_{n+2} = 5u_{n+1} - 6u_n,\ n \in \mathbb{Z}^+$$

together with terms $u_1 = a$ and $u_2 = b$.

(a) Find the expression of u_n in terms of a and b.

(b) Find algebraically the possible limits of $\dfrac{u_n}{u_{n-1}}$.

Solution

Part (a)

Consider the characteristic equation of the recurrence relation.

$$\begin{aligned} x^2 - 5x + 6 &= 0 \\ \implies (x-2)(x-3) &= 0 \end{aligned}$$

Hence, the roots of the characteristic equation are 2 and 3. Thus,

$$u_n = A \cdot 2^n + B \cdot 3^n$$

Substituting $n = 1$ and $n = 2$, we have

$$\begin{cases} 2A + 3B = a \\ 4A + 9B = b \end{cases}$$

which has solution $A = \dfrac{3a-b}{2}$ and $B = \dfrac{b-2a}{3}$. Thus,

$$\boxed{u_n = \frac{3a-b}{2} \cdot 2^n + \frac{b-2a}{3} \cdot 3^n}$$

Part (b)

Let $L = \lim_{n\to\infty} \dfrac{u_n}{u_{n-1}}$.

$$\begin{aligned} && u_{n+2} &= 5u_{n+1} - 6u_n \\ \implies && \frac{u_{n+2}}{u_{n+1}} &= 5 - 6 \cdot \frac{u_n}{u_{n+1}} \\ \implies && \lim_{n\to\infty} \frac{u_{n+2}}{u_{n+1}} &= \lim_{n\to\infty} \left(5 - 6 \cdot \frac{u_n}{u_{n+1}}\right) \\ \implies && L &= 5 - \frac{6}{L} \\ \implies && L^2 - 5L + 6 &= 0 \\ \implies && (L-2)(L-3) &= 0 \\ \implies && L &= 2 \vee 3 \end{aligned}$$

$$\boxed{\text{The possible limits are 2 and 3.}}$$

Problem 4.

Omitted.

Problem 5.

The points A and B have Cartesian coordinates $(a, 0)$ and $(-a, 0)$ respectively, where a is a positive constant. The point P is such that $AP \cdot BP = a^2$. The curve C describes the locus of P.

(a) Show that C has polar equation $r^2 = 2a^2 \cos 2\theta$, $0 \leq \theta \leq 2\pi$.

(b) Sketch the graph of C, indicating all key features and symmetries of the curve.

(c) Find the exact area of the region enclosed by the curve C.

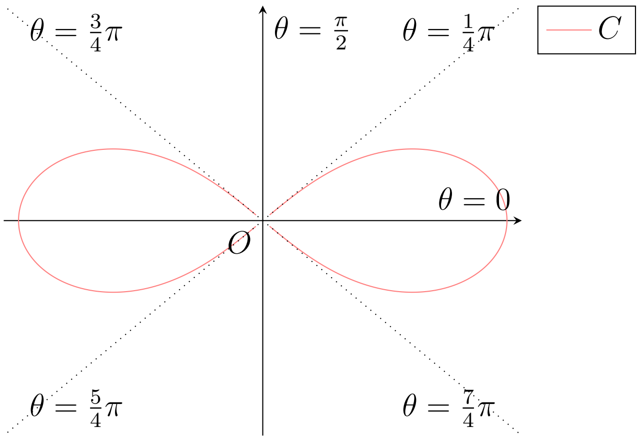
Solution

Part (a)

Let $P(x, y)$.

$$\begin{aligned}
& & AP \cdot BP &= a^2 \\
&\implies & AP^2 \cdot BP^2 &= a^4 \\
&\implies & \Big((x-a)^2 + y^2\Big)\Big((x+a)^2 + y^2\Big) &= a^4 \\
&\implies & (x-a)^2(x+a)^2 + y^2(x+a)^2 + y^2(x-a)^2 + y^4 &= a^4 \\
&\implies & \left(x^2 - a^2\right)^2 + y^2\Big((x+a)^2 + (x-a)^2\Big) + y^4 &= a^4 \\
&\implies & \left(x^4 - 2a^2x^2 + a^4\right) + y^2\Big(\left(x^2 + 2ax + a^2\right) + \left(x^2 - 2ax + a^2\right)\Big) + y^4 &= a^4 \\
&\implies & x^4 - 2a^2x^2 + 2y^2\Big(x^2 + a^2\Big) + y^4 &= 0 \\
&\implies & x^4 - 2a^2x^2 + 2x^2y^2 + 2a^2y^2 + y^4 &= 0 \\
&\implies & \left(x^4 + 2x^2y^2 + y^4\right) + \left(2a^2y^2 - 2a^2x^2\right) &= 0 \\
&\implies & \left(x^2 + y^2\right)^2 + 2a^2\left(y^2 - x^2\right) &= 0 \\
&\implies & \left(r^2\right)^2 + 2a^2\left((r\sin\theta)^2 - (r\cos\theta)^2\right) &= 0 \\
&\implies & r^4 + 2a^2r^2\left(\sin^2\theta - \cos^2\theta\right) &= 0 \\
&\implies & r^4 - 2a^2r^2\cos 2\theta &= 0 \\
&\implies & r^2 - 2a^2\cos 2\theta &= 0 \\
&\implies & r^2 &= 2a^2\cos 2\theta
\end{aligned}$$

Part (b)



Part (c)

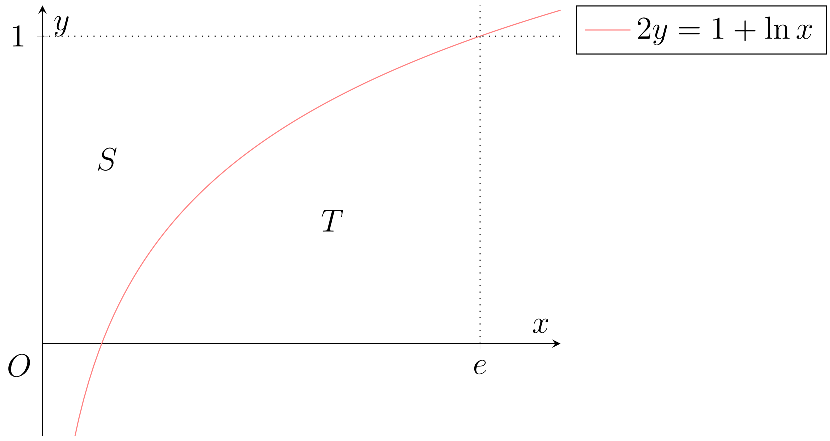
$$\begin{aligned}\text{Area} &= 2 \cdot \frac{1}{2} \int_{-\pi/4}^{\pi/4} r^2 \, \mathrm{d}\theta \\ &= \int_{-\pi/4}^{\pi/4} 2a^2 \cos 2\theta \, \mathrm{d}\theta \\ &= a^2 \int_{-\pi/2}^{\pi/2} \cos u \, \mathrm{d}u \\ &= a^2 \left[\sin u\right]_{-\pi/2}^{\pi/2} \\ &= 2a^2\end{aligned}$$

$$\begin{aligned} u &= 2\theta \\ du &= 2\,\mathrm{d}\theta \end{aligned}$$

$\boxed{\text{The area of the region enclosed by } C \text{ is } 2a^2 \text{ units}^2.}$

Problem 6.

In the diagram, the curve with equation $2y = 1 + \ln x$ for $x > 0$ divides the rectangle bounded by the axes, the lines $y = 1$ and $x = e$ into two regions, S and T.



(a) Show that the volume of the solid generated when S is rotated completely about the x-axis is given by $2\pi \int_0^1 F(y) \, dy$, where $F(y)$ is a function to be determined.

(b) Find the exact value of $\int_0^1 F(y) \, dy$.

(c) By using the result in part (b), find the exact value of $\int_{e^{-1}}^{e} (\ln x + 1)^2 \, dx$.

(d) The arc of the curve between the x-intercept and $x = e$ is rotated through 2π radians about the x-axis. Find the area of the surface generated.

Without further calculation, deduce the area of the surface generated when the arc of a curve with equation $y = e^{2x-1}$ between the y-intercept and $y = e$ is rotated through 2π radians about the y-axis, justifying your answer.

Solution

Part (a)

Note that $2y = 1 + \ln x \implies x = e^{2y-1}$. Using the shell method,

$$\begin{aligned} \text{Volume}_S &= 2\pi \int_0^1 xy \, dy \\ &= 2\pi \int_0^1 ye^{2y-1} \, dy \end{aligned}$$

Part (b)

$$\begin{aligned} \int_0^1 ye^{2y-1} \, dy &= \frac{1}{e} \int_0^1 ye^{2y} \, dy \\ &= \frac{1}{4e} \int_0^2 ue^u \, du \end{aligned}$$

$u = 2y$
$du = 2 \, dy$

	D	I
$+$	u	e^u
$-$	1	e^u
$+$	0	e^u

$$\begin{aligned}\int_0^1 ye^{2y-1}\,\mathrm{d}y &= \frac{1}{4e}\left[ue^u - e^u\right]_0^2 \\ &= \frac{1}{4e}\left(e^2+1\right) \\ &= \frac{1}{4}\left(e+e^{-1}\right)\end{aligned}$$

$$\boxed{\int_0^1 ye^{2y-1}\,\mathrm{d}y = \frac{1}{4}\left(e+e^{-1}\right)}$$

Part (c)

Let $I = \displaystyle\int_{e^{-1}}^{e} (\ln x + 1)^2\,\mathrm{d}x$. Observe that the volume of the solid generated when T is rotated completely about the x-axis is given by

$$\begin{aligned}\text{Volume}_T &= \int_{e^{-1}}^{e} \left(\frac{1+\ln x}{2}\right)^2 \mathrm{d}x \\ &= \frac{\pi}{4}\int_{e^{-1}}^{e} (\ln x + 1)^2\,\mathrm{d}x \\ &= \frac{\pi}{4} I\end{aligned}$$

Note that when the entire rectangle is rotated completely about the x-axis, its volume is given by πe. Hence,

$$\begin{aligned} & \text{Volume}_S + \text{Volume}_T = \pi e \\ \implies & 2\pi \cdot \frac{1}{4}\left(e - e^{-1}\right) + \frac{\pi}{4} I = \pi e \\ \implies & 2\left(e+e^{-1}\right) + I = 4e \\ \implies & I = 4e - 2\left(e+e^{-1}\right) \\ & = 2\left(e - e^{-1}\right)\end{aligned}$$

$$\boxed{\int_{e^{-1}}^{e} (\ln x + 1)^2\,\mathrm{d}x = 2\left(e-e^{-1}\right)}$$

Part (d)

Note that $2y = 1 + \ln x \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{2x}$. Hence,

$$\text{Surface area} = 2\pi \int_{e^{-1}}^{e} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

$$
\begin{aligned}
&= 2\pi \int_{e^{-1}}^{e} \left(\frac{1+\ln x}{2}\right) \sqrt{1+\left(\frac{1}{2x}\right)^2}\,\mathrm{d}x \\
&= 10.3 \text{ (3 s.f.)}
\end{aligned}
$$

$$\boxed{\text{The area of the surface generated is } 10.3 \text{ units}^2.}$$

The curve described is the inverse of $2y = 1 + \ln x$. Hence, the surface area generated by the new curve is the same as that of $2y = 1 + \ln x$.

$$\boxed{\text{The surface area generated by the new curve is also } 10.3 \text{ units}^2.}$$

Problem 7.

Omitted.

Problem 8.

The function g is given by $g(x) = x^{3/2} - 2\sqrt{x} - 1$ for $x \geq 0$. Show that the equation $g(x) = 0$ has only one real root, $x = \alpha$. State an integer n such that $n < \alpha < n + 1$.

(a) To find an approximate value for α, the following rearrangements of $g(x) = 0$ are suggested as a basis for the iteration method of the form $x_{n+1} = f(x_n)$.

- $x = \frac{1}{4}\left(x^{3/2} - 1\right)^2$
- $x = 2 + \frac{1}{\sqrt{x}}$
- $x = x^{5/2} - x^{3/2}$

(i) By considering $f'(x)$, identify the iteration method which converges to α. Use a graph to explain why the chosen iteration method converges to α.

(ii) Using the appropriate iteration method found in part(a)(i), and $x_0 = n$, find the value of α, correct to 4 decimal places, and demonstrate how to verify its correctness.

(b) (i) Show that a Newton-Raphson iteration method for the root α is given by

$$x_{n+1} = \frac{x_n^{3/2} + 2x_n^{1/2} + 2}{3x_n^{1/2} - 2x_n^{-1/2}}$$

(ii) Explain why the Newton-Raphson iteration method fails when the initial value x_0 is less than or equal to $\frac{2}{3}$.

Solution

Let $y = \sqrt{x}$. Hence, $g(x)$ is equivalent to $y^3 - 2y - 1$. Let this function be $h(y)$. Observe that $h'(y) = 3y^2 - 2$. Thus, the sole stationary point of $h(y)$ occurs at $y = \sqrt{\frac{2}{3}}$. Note that we reject $y = -\sqrt{\frac{2}{3}}$ since $y = \sqrt{x} \geq 0$.

y	$\left(\sqrt{\frac{2}{3}}\right)^-$	$\sqrt{\frac{2}{3}}$	$\left(\sqrt{\frac{2}{3}}\right)^+$
$h'(y)$	-ve	0	+ve

From the First Derivative Test, we see that $y = \sqrt{\frac{2}{3}}$ is a minimum point. Hence, $h(y)$ is strictly decreasing on the interval $\left[0, \sqrt{\frac{2}{3}}\right)$ and strictly increasing on the interval $\left(\sqrt{\frac{2}{3}}, \infty\right)$. Since $h(0) = -1 < 0$, we have that $h(y) < 0$ for all $y \leq \sqrt{\frac{2}{3}}$. Since $h(y)$

is strictly increasing for $y > \sqrt{\frac{2}{3}}$, and $h\left(\sqrt{\frac{2}{3}}\right) < 0$, $h(y)$ has only one real root. Thus, $g(x)$ has only one real root.

Observe that $g(2)\,g(3) = (-1)(0.732) < 0$. Hence, $\alpha \in (2, 3)$. Thus,

$$\boxed{n = 2}$$

Part (a)

Subpart (i)

Case 1: $f(x) = \frac{1}{4}\left(x^{3/2} - 1\right)^2$

$$\begin{aligned} f'(x) &= \frac{1}{4} \cdot 2\left(x^{3/2} - 1\right) \cdot \frac{3}{2} x^{1/2} \\ &= \frac{3}{4} x^{1/2}\left(x^{3/2} - 1\right) \end{aligned}$$

Note that for all $x \in (2, 3)$, $|f'(x)| > 1$. Hence, the iteration may not converge.

Case 2: $f(x) = 2 + \frac{1}{\sqrt{x}}$

$$f'(x) = -\frac{1}{2} x^{-3/2}$$

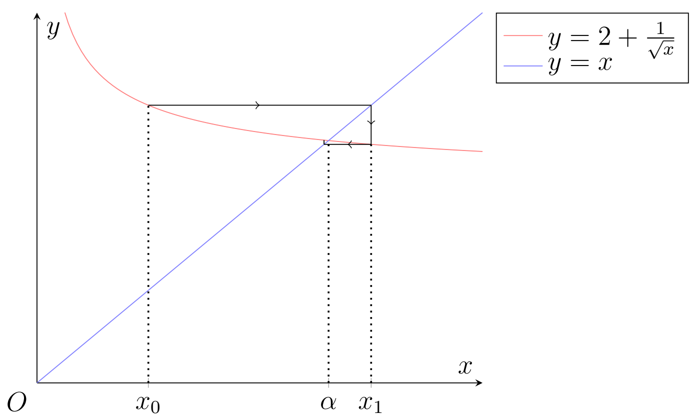
Note that for all $x \in (2, 3)$, $|f'(x)| < 1$. Hence, the iteration converges.

Case 3: $f(x) = x^{5/2} - x^{3/2}$

$$f'(x) = \frac{5}{2} x^{3/2} - \frac{3}{2} x^{1/2}$$

Note that for all $x \in (2, 3)$, $|f'(x)| > 1$. Hence, the iteration may not converge.

$$\boxed{x_{n+1} = 2 + \frac{1}{\sqrt{x_n}} \text{ converges to } \alpha.}$$



From the graph, the subsequent approximations get closer and closer to α. Hence, $x_{n+1} = 2 + \dfrac{1}{\sqrt{x_n}}$ converges to α.

Subpart (ii)

$$\begin{aligned} x_0 &= 2 \\ \implies x_1 &= 2 + \frac{1}{\sqrt{x_0}} = 2.7171068 \\ \implies x_2 &= 2 + \frac{1}{\sqrt{x_1}} = 2.6077813 \\ \implies x_3 &= 2 + \frac{1}{\sqrt{x_2}} = 2.6192477 \\ \implies x_4 &= 2 + \frac{1}{\sqrt{x_3}} = 2.6178908 \\ \implies x_5 &= 2 + \frac{1}{\sqrt{x_4}} = 2.6180509 \\ \implies x_6 &= 2 + \frac{1}{\sqrt{x_5}} = 2.6180320 \\ \implies x_7 &= 2 + \frac{1}{\sqrt{x_6}} = 2.6180342 \end{aligned}$$

Observe that $g(2.61795)\, g(2.61805) = (-1.5 \times 10^{-4})(2.9 \times 10^{-5}) < 0$. Hence, $\alpha \in (2.61795, 2.61805)$. Thus,

$$\boxed{\alpha = 2.6180 \text{ (4 d.p.)}}$$

Part (b)

Subpart (i) Note that $g'(x) = \frac{3}{2}x^{1/2} - x^{-1/2}$. Thus,

$$\begin{aligned} x_{n+1} &= x_n - \frac{g(x_n)}{g'(x_n)} \\ &= x_n - \frac{x_n^{3/2} - 2x_n^{1/2} - 1}{\frac{3}{2}x_n^{1/2} - x_n^{-1/2}} \\ &= x_n - \frac{2x_n^{3/2} - 4x_n^{1/2} - 2}{3x_n^{1/2} - 2x_n^{-1/2}} \\ &= \frac{x_n\left(3x_n^{1/2} - 2x_n^{-1/2}\right) - \left(2x_n^{3/2} - 4x_n^{1/2} - 2\right)}{3x_n^{1/2} - 2x_n^{-1/2}} \\ &= \frac{x_n^{3/2} + 2x_n^{1/2} + 2}{3x_n^{1/2} - 2x_n^{-1/2}} \end{aligned}$$

Subpart (ii)

When $x_0 = \frac{2}{3}$, we have $3x_n^{1/2} - 2x_n^{-1/2} = 0$, whence x_1 is undefined. When $x_0 < \frac{2}{3}$, $x_1 < 0$. However, the iterative formula for x_n is valid only for $x_n > 0$. Thus, x_2 will be undefined. Hence, the method fails for $x_0 \leq \frac{2}{3}$.

Problem 9.

A point P resides on the circumference of a circular gear with centre C and radius a, which rolls in an anticlockwise direction externally without slipping on the circumference of a fixed circular axle with centre O and radius a. Figure 1 shows the initial position of P at $(a, 0)$ and Figure 2 shows its position $P(x, y)$ where OC makes an angle θ with the positive x-axis.

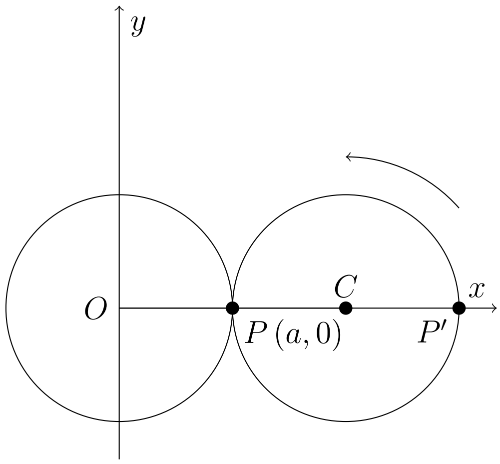


Figure 1

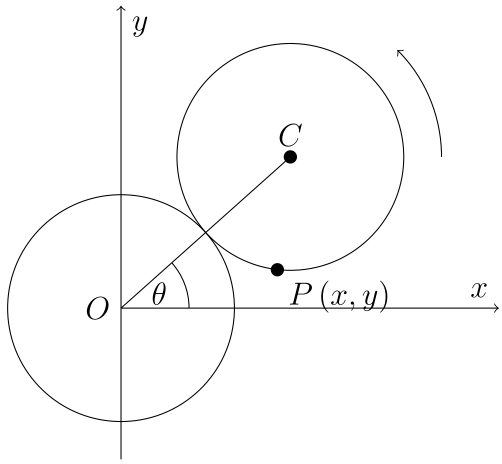


Figure 2

(a) Show that the equation of one full revolution of the path of P can be represented by

$$x = 2a\cos\theta - a\cos 2\theta$$
$$y = 2a\sin\theta - a\sin 2\theta$$

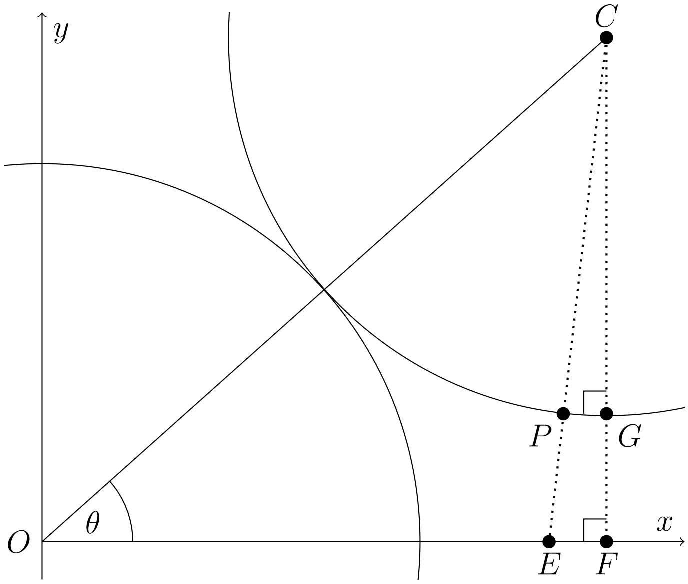
(b) Sketch the path of P for $0 \leq \theta \leq 2\pi$, indicating clearly the coordinates of the x-intercepts.

(c) Without the use of a calculator, find the length of the path of P in terms of a.

(d) A student commented that a point P' with the initial position $(3a, 0)$ (as shown in Figure 1) is further away from O than point P and therefore travels a longer path as the circular gear makes a full revolution around the fixed circular axle. Do you agree with the comment? Justify your answer.

Solution

Part (a)



Consider the above diagram. Let E be the intersection between the x-axis and CP extended. Let F be the point on the x-axis such that $CF \perp OF$. Let G be the point on the line CF such that $PG \perp CG \implies PG \parallel EF$.

By symmetry, $OE = EC$, whence $\triangle OEC$ is isosceles. Hence, $\angle COE = \angle OCE = \theta$. By the exterior angle theorem, $\angle CEF = 2\theta$. Since $PG \parallel EF$, we have $\angle CPG = \angle CEF = 2\theta$. Finally, from the angle sum of a triangle, we have $\angle ECF = \frac{\pi}{2} - \theta$.

$$\cos \angle COF = \frac{OF}{OC} \implies \cos\theta = \frac{OF}{2a} \implies OF = 2a\cos\theta$$
$$\cos \angle CPG = \frac{PG}{PC} \implies \cos 2\theta = \frac{PG}{a} \implies PG = a\cos 2\theta$$

Since $x = OF - PG$, we have

$$x = 2a\cos\theta - a\cos 2\theta$$

$$\cos \angle OCF = \frac{CF}{OC} \implies \cos\left(\frac{\pi}{2} - \theta\right) = \frac{CF}{2a} \implies CF = 2a\sin\theta$$
$$\cos \angle PCG = \frac{CG}{PC} \implies \cos\left(\frac{\pi}{2} - 2\theta\right) = \frac{CG}{a} \implies CG = a\sin 2\theta$$

Since $y = CF - CG$, we have

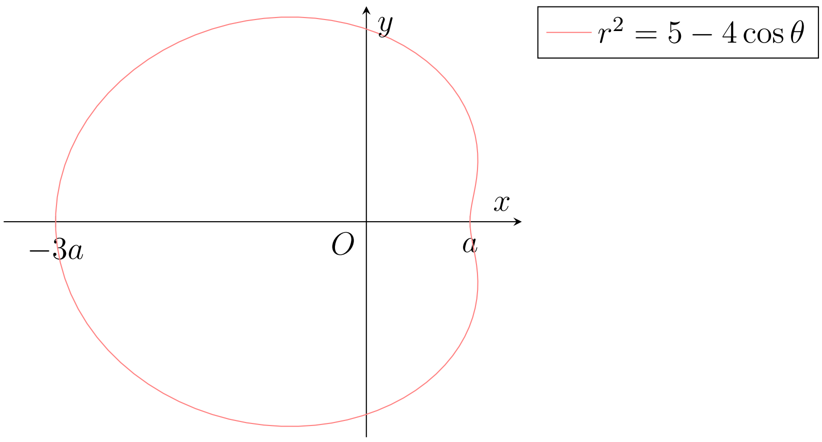
$$y = 2a\sin\theta - a\sin 2\theta$$

Hence, the path of P can be represented by

$$x = 2a\cos\theta - a\cos 2\theta$$
$$y = 2a\sin\theta - a\sin 2\theta$$

Part (b)

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= (2a\cos\theta - a\cos 2\theta)^2 + (2a\sin\theta - a\sin 2\theta)^2 \\ &= a^2\left[(2\cos\theta - \cos 2\theta)^2 + (2\sin\theta - \sin 2\theta)^2\right] \\ &= a^2\left[4\cos^2\theta - 4\cos\theta\cos 2\theta + \cos^2 2\theta + 4\sin^2\theta - 4\sin\theta\sin 2\theta + \sin^2 2\theta\right] \\ &= a^2\left[(4\cos^2\theta + 4\sin^2\theta) + (\cos^2 2\theta + \sin^2 2\theta) - 4(\cos\theta\cos 2\theta + \sin\theta\sin 2\theta)\right] \\ &= a^2(4 + 1 - 4\cos\theta) \\ &= a^2(5 - 4\cos\theta) \\ \implies r &= \pm a\sqrt{5 - 4\cos\theta} \end{aligned}$$



Part (c)

Note that we have $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -2a\sin\theta + 2a\sin 2\theta$ and $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 2a\cos\theta - 2a\cos 2\theta$. Thus,

$$\begin{aligned} &\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 \\ &\quad = (-2a\sin\theta + 2a\sin 2\theta)^2 + (2a\cos\theta - 2a\cos 2\theta)^2 \\ &\quad = 4a^2\left[(-\sin\theta + \sin 2\theta)^2 + (\cos\theta - \cos 2\theta)^2\right] \\ &\quad = 4a^2\left[\sin^2\theta - 2\sin\theta\sin 2\theta + \sin^2 2\theta + \cos^2\theta - 2\cos\theta\cos 2\theta + \cos^2 2\theta\right] \\ &\quad = 4a^2\left[\left(\sin^2\theta + \cos^2\theta\right) + \left(\sin^2 2\theta + \cos^2 2\theta\right) - 2\left(\sin\theta\sin 2\theta + \cos\theta\cos 2\theta\right)\right] \\ &\quad = 4a^2\left(1 + 1 - 2\cos\theta\right) \\ &\quad = 8a^2\left(1 - \cos\theta\right) \end{aligned}$$

Hence, the length of the path of P is given by

$$\begin{aligned} \text{Length} &= \int_0^{2\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta \\ &= \int_0^{2\pi} \sqrt{8a^2(1-\cos\theta)}\,\mathrm{d}\theta \\ &= \sqrt{8}a\int_0^{2\pi} \sqrt{1-\cos\theta}\,\mathrm{d}\theta \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{8}a \int_0^{2\pi} \sqrt{2\sin^2\left(\frac{\theta}{2}\right)} \, \mathrm{d}\theta \\
&= 4a \int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) \mathrm{d}\theta \\
&= 8a \int_0^{\pi} \sin u \, \mathrm{d}\theta \\
&= 8a \left[-\cos u\right]_0^{\pi} \\
&= 16a
\end{aligned}
$$

$$
\begin{aligned}
u &= \theta/2 \\
\mathrm{d}u &= \mathrm{d}\theta\,/2
\end{aligned}
$$

$\boxed{\text{The length of the path of } P \text{ is } 16a \text{ units.}}$

Part (d)

When the gear has rotated π radians, P ends up at $(-3a, 0)$. This is the reflection of P' in the y-axis. Indeed, P' also ends up at $(-a, 0)$, which is the reflection of P in the y-axis. Hence, the path taken by P' is exactly the path taken by P in the y-axis, but reflected in the y-axis. Hence, the total length travelled by P and P' are equal. Thus, the comment is incorrect.

Problem 10.

To promote Earth Day, the Earth Day Network organizes an event that will be held for n consecutive days. Cash prizes will be awarded to participants throughout the event, with one lucky participant chosen each day. Let the budget for the cash prizes by $\$m$.

The amount of cash prize given out on the first day is a sum of \$10 and $\frac{1}{7}$ of the remaining budget, i.e. $\$\left(10 + \frac{1}{7}(m-10)\right)$. The amount of cash prize given out on the second day is a sum of \$20 and $\frac{1}{7}$ of the remaining budget. In general, the amount of cash prize given out on the kth day is a sum of $\$10k$ and $\frac{1}{7}$ of the remaining budget. Let u_k denote the amount of cash prize given out on the kth day, with $1 \leq k \leq n$, $k \in \mathbb{Z}^+$.

(a) Write down the expression for u_k, in terms of $u_1, u_2, \ldots, u_{k-1}$.

(b) By considering $u_{k+1} - u_k$, show that $u_{k+1} = \frac{6}{7}u_k + \frac{60}{7}$.

(c) Find u_k in the form $u_k = p\left(\frac{6}{7}\right)^k (m - 360) + q$, where p and q are constants to be determined.

It is given that $m = 4000$.

(d) Find an expression of the total amount of cash prizes given out in n days. Hence or otherwise, explain if it is possible for the Earth Day Network to host this event for 2 weeks.

(e) Find the set of values of k for which the daily cash prize will be less than 5% of the initial budget.

(f) Determine the minimum budget that the Earth Day Network needs to have so that they can host this event for 2 weeks.

Solution

Part (a)

On the kth day, the total cash prizes that have already been given out is $u_1 + u_2 + \ldots + u_{k-1}$. Hence,

$$\boxed{u_k = 10k + \frac{1}{7}\Big(m - 10k - (u_1 + u_2 + \ldots + u_{k-1})\Big)}$$

Part (b)

$$\begin{aligned} u_{k+1} - u_k &= 10(k+1) + \frac{1}{7}\Big(m - 10(k+1) - (u_1 + u_2 + \ldots + u_k)\Big) \\ &\quad - \left(10k + \frac{1}{7}\Big(m - 10k - (u_1 + u_2 + \ldots + u_{k-1})\Big)\right) \\ &= 10 + \frac{1}{7}(-10 - u_k) \end{aligned}$$

$$
\begin{aligned}
&= \frac{60}{7} - \frac{1}{7}u_k \\
\implies \qquad u_{k+1} &= \frac{60}{7} + \frac{6}{7}u_k
\end{aligned}
$$

Part (c)

Let x be the constant such that $u_k + x = \frac{6}{7}(u_{k-1} + x)$. Then $-\frac{1}{7}x = \frac{60}{7} \implies x = -60$.

$$
\begin{aligned}
u_k - 60 &= \frac{6}{7}(u_{k-1} - 60) \\
&= \left(\frac{6}{7}\right)^{k-1}(u_1 - 60) \\
\implies \qquad u_k &= \left(\frac{6}{7}\right)^{k-1}\left(10 + \frac{1}{7}(m - 10) - 60\right) + 60 \\
&= \left(\frac{6}{7}\right)^{k-1}\left(\frac{1}{7}m - \frac{360}{7}\right) + 60 \\
&= \left(\frac{6}{7}\right)^{k} \cdot \frac{7}{6}\left(\frac{1}{7}m - \frac{360}{7}\right) + 60 \\
&= \frac{1}{6}\left(\frac{6}{7}\right)^{k}(m - 360) + 60
\end{aligned}
$$

$$
\boxed{u_k = \frac{1}{6}\left(\frac{6}{7}\right)^{k}(m - 360) + 60}
$$

Part (d)

Let S_n be the total amount of cash prizes given out in n days.

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} u_k \\
&= \sum_{k=1}^{n}\left(\frac{1}{6}\left(\frac{6}{7}\right)^{k}(m - 360) + 60\right) \\
&= \frac{m - 360}{6} \cdot \frac{6}{7} \cdot \frac{1 - (6/7)^n}{1 - 6/7} + 60n \\
&= (m - 360)\left(1 - \left(\frac{6}{7}\right)^{n}\right) + 60n
\end{aligned}
$$

Substituting $m = 4000$ and $n = 14$, we have

$$
\begin{aligned}
S_{14} &= (4000 - 360)\left(1 - \left(\frac{6}{7}\right)^{14}\right) + 60 \cdot 14 \\
&= 4059
\end{aligned}
$$

Since $S_{14} > 4000 = m$, there is not enough money to pay out the cash prizes for 14 days, or 2 weeks.

$$\boxed{\text{It is not possible for the Earth Day Network to host this event for 2 weeks.}}$$

Part (e)

Consider $u_k < \frac{5}{100}m = 200$.

$$\begin{aligned} u_k &< 200 \\ \implies \frac{1}{6}\left(\frac{6}{7}\right)^k (4000-360) + 60 &< 200 \\ \implies \left(\frac{6}{7}\right)^k &< \frac{3}{13} \\ \implies k &> \log_{6/7}\frac{3}{13} \\ &= 9.5 \text{ (2 s.f.)} \end{aligned}$$

Note that $S_{13} = (4000-360)\left(1-\left(\frac{6}{7}\right)^{13}\right) + 60 \cdot 13 = 3929 < 4000 = m$. Hence,

$$\boxed{\{k \in \mathbb{N} : 10 \leq k \leq 13\}}$$

Part (f)

Consider $S_{14} \leq m$.

$$\begin{aligned} S_{14} &\leq m \\ \implies (m-360)\left(1-\left(\frac{6}{7}\right)^{14}\right) + 60 \cdot 14 &\leq m \\ \implies m\left(1-\left(\frac{6}{7}\right)^{14}\right) - 360\left(1-\left(\frac{6}{7}\right)^{14}\right) + 840 &\leq m \\ \implies m - m\left(\frac{6}{7}\right)^{14} - m &\leq 360\left(1-\left(\frac{6}{7}\right)^{14}\right) - 840 \\ \implies -m\left(\frac{6}{7}\right)^{14} &\leq 360\left(1-\left(\frac{6}{7}\right)^{14}\right) - 840 \\ \implies m &\geq \frac{840 - 360\left(1-(6/7)^{14}\right)}{(6/7)^{14}} \\ &= 4514.285 \end{aligned}$$

$$\boxed{\text{Earth Day Network needs to have a minimum budget of \$4514.29.}}$$