

Problem 1.

On a single Argand diagram, sketch the following loci.

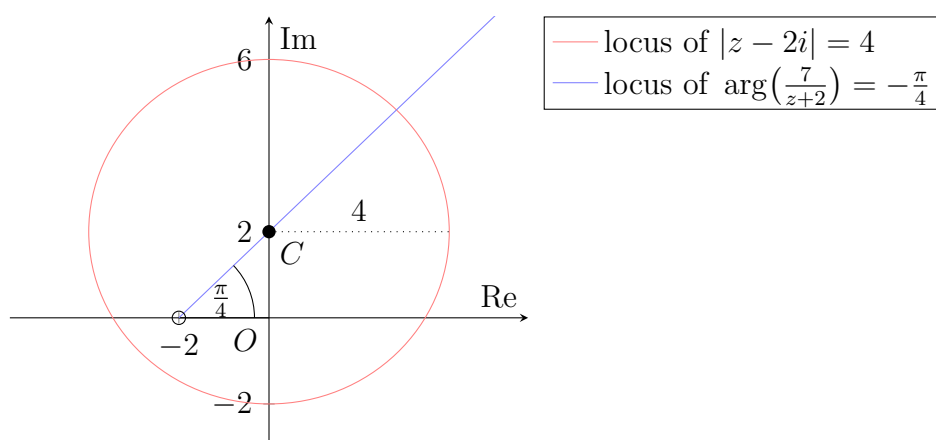
(a) $|z - 2i| = 4$.

(b) $\arg\left(\frac{7}{z+2}\right) = -\frac{\pi}{4}$.

Hence, or otherwise, find the exact value of z satisfying both equations in part (a) and (b).

Solution

Note that $\arg\left(\frac{7}{z+2}\right) = -\frac{\pi}{4} \implies \arg(z - (-2)) = \frac{\pi}{4}$.

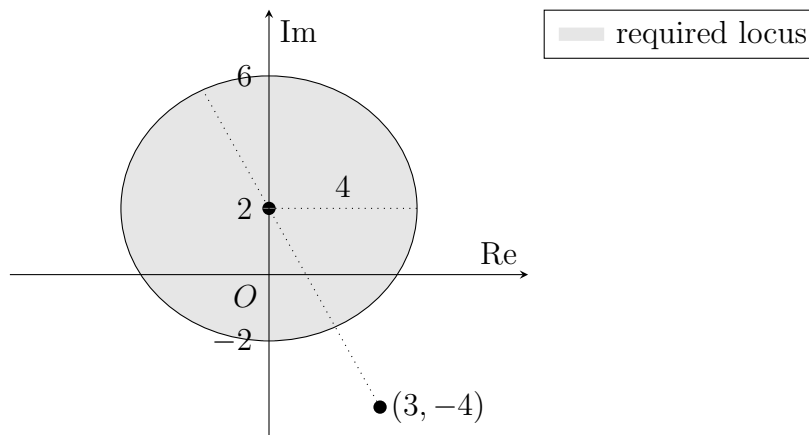


$$\begin{aligned}
 z &= 2i + \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right) \\
 &= 2i + \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \\
 &= \boxed{\frac{\sqrt{2}}{2} + \left(2 + \frac{\sqrt{2}}{2}\right)i}
 \end{aligned}$$

Problem 2.

Given that $|z - 2i| \leq 4$, illustrate the locus of the point representing the complex number z in an Argand diagram.

Hence, find the greatest and least possible value of $|z - 3 + 4i|$, given that $|z - 2i| \leq 4$.

Solution

Note that $|z - 3 + 4i| = |z - (3 - 4i)|$ represents the distance between z and the point $(3, -4)$.

By Pythagoras' Theorem, the distance between the centre of the circle $(0, 2)$ and $(3, -4)$ is $\sqrt{(0 - 3)^2 + (2 + 4)^2} = 3\sqrt{5}$. Hence, $\max |z - 3 + 4i| = 3\sqrt{5} + 4$, while $\min |z - 3 + 4i| = 3\sqrt{5} - 4$.

$$\boxed{\max |z - 3 + 4i| = 3\sqrt{5} + 4, \min |z - 3 + 4i| = 3\sqrt{5} - 4}$$

Problem 3.

The point A on an Argand diagram represents the fixed complex number a , where $0 < \arg a < \frac{\pi}{2}$. The complex numbers z and w are such that $|z - 2ia| = |a|$ and $|w| = |w + ia|$.

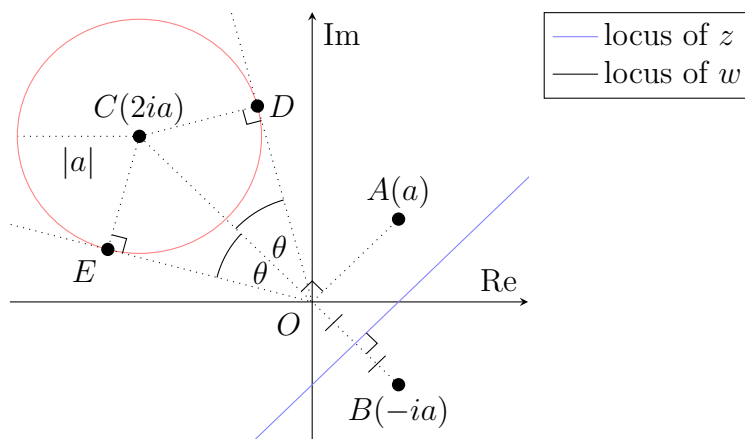
Sketch, on a single diagram, the loci of the point representing z and w .

Find

- the minimum value of $|z - w|$ in terms of $|a|$,
- the range of values of $\arg \frac{1}{z}$ in terms of $\arg a$.

Solution

Note that $|w| = |w + ia| \implies |w - 0| = |w - (-ia)|$.

**Part (a)**

Let $B(-ia)$ and $C(2ia)$. Note that $W\left(-\frac{1}{2}ia\right)$ lies on the locus of w as well as the line passing through OC . Since CW is perpendicular to the locus of w , it follows that the minimum value of $|z - w|$ is given by

$$\begin{aligned} CW - |a| &= \left| 2ia + \frac{1}{2}ia \right| - |a| \\ &= \frac{5}{2} |a| |i| - |a| \\ &= \frac{3}{2} |a| \end{aligned}$$

The minimum value of $|z - w|$ is $\frac{3}{2} |a|$.

Part (b)

Let D and E be such that OD and OE are tangent to the circle given by the locus of z . Let $\angle COD = \theta$. Observe that $\sin \theta = \frac{CD}{CO} = \frac{|a|}{|2ia|} = \frac{1}{2}$, whence $\theta = \frac{\pi}{6}$. Since

$\angle COA = \arg i = \frac{\pi}{2}$, it follows that $\angle DOA = \frac{\pi}{2} - \theta = \frac{\pi}{2} - \arcsin \frac{1}{2} = \frac{\pi}{3}$. Thus, $\min \arg z = \arg a + \angle DOA = \arg a + \frac{\pi}{3}$. Meanwhile, $\angle COE = \angle COD = \theta$, whence $\max \arg z = \arg a + \frac{\pi}{2} + \theta = \arg a + \frac{2}{3}\pi$. Since $\arg \frac{1}{z} = -\arg z$, we thus have

$$\arg \frac{1}{z} \in \left[-\left(\arg a + \frac{2}{3}\pi \right), -\left(\arg a + \frac{\pi}{3} \right) \right]$$

Problem 4.

- (a) Solve the equation

$$z^7 - (1 + i) = 0,$$

giving the roots in the form $re^{i\alpha}$, where $r > 0$ and $-\pi < \alpha \leq \pi$.

- (b) Show the roots on an Argand diagram.

- (c) The roots represented by z_1 and z_2 are such that $0 < \arg z_1 < \arg z_2 < \frac{\pi}{2}$. Explain why the locus of all points z such that $|z - z_1| = |z - z_2|$ passes through the origin. Draw this locus on your Argand diagram and find its Cartesian equation.

- (d) Describe the transformation that will map the points representing the roots of the equation $z^7 - (1 + i) = 0$ to the points representing the roots of the equation $(z - 2)^7 - (1 + i) = 0$ on the Argand diagram.

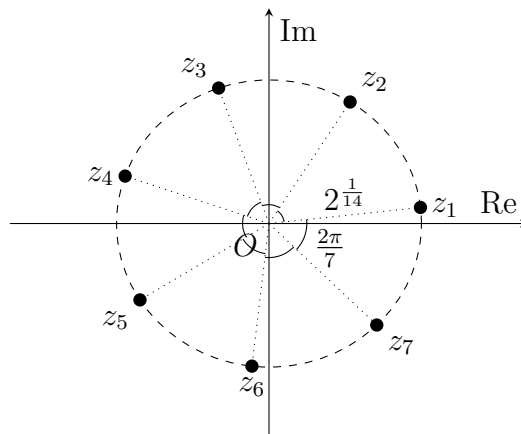
Solution**Part (a)**

Note that $1 + i = \sqrt{2}e^{i\pi/4} = 2^{\frac{1}{2}}e^{i\pi(\frac{1}{4}+2k)}$, where $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} z^7 - (1 + i) &= 0 \\ \implies z^7 &= 2^{\frac{1}{2}}e^{i\pi(\frac{1}{4}+2k)} \\ \implies z &= 2^{\frac{1}{14}}e^{i\pi(\frac{1}{4}+2k)/7} \\ &= 2^{\frac{1}{14}}e^{i\pi(1+8k)/28} \end{aligned}$$

Taking $k \in \{-3, -2, \dots, 2, 3\}$,

$$z = 2^{\frac{1}{14}}e^{-i\pi\frac{23}{28}}, 2^{\frac{1}{14}}e^{-i\pi\frac{15}{28}}, 2^{\frac{1}{14}}e^{-i\pi\frac{7}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{1}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{9}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{17}{28}}, 2^{\frac{1}{14}}e^{i\pi\frac{25}{28}}$$

Part (b)**Part (c)**

Since $|z_1| = |z_2| = 2^{\frac{1}{14}}$, the distance between z_1 and the origin and the distance between z_2 and the origin are equal. Since the locus of $|z - z_1| = |z - z_2|$ represents all points equidistant from z_1 and z_2 , it passes through the origin.

Observe that the midpoint of z_1 and z_2 will have argument $\frac{1}{2} \left(\frac{1}{28}\pi + \frac{9}{28}\pi \right) = \frac{5}{28}\pi$. Thus, the Cartesian equation of the locus of z is given by

$$y = \tan\left(\frac{5}{28}\pi\right)x$$

Part (d)

Translate the points 2 units in the positive real direction.