Problem 1.

The diagram shows the region R, which is bounded by the axes and the part of the curve $y^2 = 4a(a - x)$ lying in the first quadrant.

Find, in terms of a, the volume, V_x, of the solid formed when R is rotated completely about the x-axis.

The volume of the solid formed when R is rotated completely about the y-axis is V_y. Show that $V_y = \frac{8}{15}V_x$.

The region S, lying in the first quadrant, is bounded by the curve $y^2 = 4a(a - x)$ and the lines $x = a$ and $y = 2a$. Find, in terms of a, the volume of the solid formed when S is rotated completely about the y-axis.

Solution

Problem 2.

The region bounded by the axes and the curve $y = \cos x$ from $x = 0$ to $x = \dfrac{1}{2}\pi$ is divided into two parts, of areas A_1 and A_2, by the curve $y = \sin x$.

(a) Prove that $A_2 = \sqrt{2}A_1$.

(b) Prove Find the volume of the solid obtained when the region with area A_2 is rotated about the y-axis through 2π radians. Give your answer in exact form.

Solution

Part (a)

Part (b)

Problem 3.

A curve has parametric equations

$$x = \cos^2 t,\ y = \sin^3 t,\ 0 \leq t \leq \frac{1}{2}\pi$$

(a) Sketch the curve.

(b) Show that the area under the curve for $0 \leq t \leq \frac{1}{2}\pi$ is $2\int_0^{\pi/2} \cos t \sin^4 t \, \mathrm{d}t$, and find the exact value of the area.

(c) Find the volume of the solid obtained when the region in (b) is rotated about the y-axis through 2π radians.

Solution

Part (a)

Part (b)

Part (c)

Problem 4.

(a) Given that f is a continuous function, explain, with the aid of a sketch, why the value of

$$\lim_{n\to\infty} \frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\ldots+f\left(\frac{n}{n}\right)\right]$$

is $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$.

(b) Hence, evaluate $\displaystyle\lim_{n\to\infty}\frac{1}{n}\left(\frac{\sqrt[3]{1}+\sqrt[3]{2}+\ldots+\sqrt[3]{n}}{\sqrt[3]{n}}\right)$.

Solution

Part (a)

Part (b)

Problem 5.

The function f satisfies $f'(x) > 0$ for $a \leq x \leq b$, and g is the inverse of f. By making a suitable change of variable, prove that

$$\int_a^b f(x)\,\mathrm{d}x = b\beta - a\alpha - \int_\alpha^\beta g(y)\,\mathrm{d}y$$

where $\alpha = f(a)$ and $\beta = f(b)$. Interpret this formula geometrically by means of a sketch where α and a are positive. Verify this result for the case where $f(x) = e^{2x}$, $a = 0$, $b = 1$.

Prove similarly and interpret geometrically the formula

$$2\pi \int_a^b x f(x)\,\mathrm{d}x = \pi(b^2\beta - a^2\alpha) - \pi \int_\alpha^\beta [g(y)]^2\,\mathrm{d}y$$

Solution