

**Problem 1.**

A university student has a goal of saving at least \$1 000 000 (in Singapore dollars). He begins working at the start of the year 2019. In order to achieve his goal, he saves 40% of his annual salary at the end of each year. If his annual salary in the year 2019 is \$40800, and it increases by 5% (of his previous year's salary) every year, find

- (a) his annual savings in 2027 (to the nearest dollar),
- (b) his total savings at the end of  $n$  years.

What is the minimum number of complete years for which he has to work in order to achieve his goal?

**Solution**

Let  $u_n$  be his annual salary in the  $n$ th year after 2019, with  $n \in \mathbb{N}$ . Then  $u_{n+1} = 1.05 \cdot u_n$ , with  $u_0 = 40800$ . Hence,  $u_n = 40800 \cdot 1.05^n$ .

**Part (a)**

In 2027,  $n = 8$ . Hence,

$$\begin{aligned} \text{Annual savings in 2027} &= 0.40 \cdot u_8 \\ &= 0.40 \cdot 40800 \cdot 1.05^8 \\ &= 24112 \text{ (to the nearest integer)} \end{aligned}$$

His annual savings in 2027 will be \$24112.

**Part (b)**

$$\begin{aligned} \sum_{k=0}^{n-1} 0.40 \cdot u_k &= \sum_{k=0}^{n-1} 0.40 \cdot 40800 \cdot 1.05^k \\ &= 16320 \sum_{k=0}^{n-1} 1.05^k \\ &= 16320 \sum_{k=1}^n 1.05^{k-1} \\ &= 16320 \cdot 1.05^{-1} \cdot \sum_{k=1}^n 1.05^k \\ &= 16320 \cdot 1.05^{-1} \cdot \frac{1.05 \cdot (1.05^n - 1)}{1.05 - 1} \\ &= 326400(1.05^n - 1) \end{aligned}$$

His total savings at the end of  $n$  years is \$326400(1.05^n - 1).

$$\begin{aligned}326400(1.05^n - 1) &\geq 1000000 \\1.05^n - 1 &\geq \frac{1000000}{326400} \\1.05^n &\geq \frac{1000000}{326400} + 1 \\n &\geq \log_{1.05}\left(\frac{1000000}{326400} + 1\right) \\&= 28.7 \text{ (3 s.f.)}\end{aligned}$$

Since  $n \in \mathbb{N}$ , the minimum value of  $n$  is 29.

He has to work for a minimum of 29 complete years.

**Problem 2.**

- (a) A rope of length  $200\pi$  cm is cut into pieces to form as many circles as possible, whose radii follow an arithmetic progression with common difference 0.25 cm. Given that the smallest circle has an area of  $\pi$  cm<sup>2</sup>, find the area of the largest circle in terms of  $\pi$ .
- (b) The sum of the first  $n$  terms of a sequence is given by  $S_n = \alpha^{-n} - 1$ , where  $\alpha$  is a non-zero constant,  $\alpha \neq 1$ .
- (i) Show that the sequence is a geometric progression and state its common ratio in terms of  $\alpha$ .
- (ii) Find the set of values of  $\alpha$  for which the sum to infinity of the sequence exists.
- (iii) Find the value of the sum to infinity.

**Solution****Part (a)**

Let the sequence  $r_n$  be the radius of the  $n$ th smallest circle, in centimetres. Hence,  $r_n = \frac{1}{4} + r_{n-1}$ . Since the smallest circle has area  $\pi$  cm<sup>2</sup>,  $r_1 = 1$ . Thus,  $r_n = 1 + \frac{1}{4}(n-1)$ .

Consider the  $n$ th partial sum of the circumferences.

$$\begin{aligned}
 \sum_{k=1}^n 2\pi r_k &= 2\pi \sum_{k=1}^n \left(1 + \frac{1}{4}(k-1)\right) \\
 &= 2\pi \left(n + \frac{1}{4} \cdot \frac{n(n-1)}{2} - \frac{1}{4}n\right) \\
 &= 2\pi \left(\frac{3}{4}n + \frac{1}{8}n(n-1)\right) \\
 &= \frac{1}{4}\pi (6n + n(n-1)) \\
 &= \frac{1}{4}\pi (n^2 + 5n)
 \end{aligned}$$

Since the rope has length  $200\pi$  cm, we have the inequality

$$\begin{aligned}
 \sum_{k=1}^n 2\pi r_k &\leq 200\pi \\
 \frac{1}{4}\pi (n^2 + 5n) &\leq 200\pi \\
 n^2 + 5n &\leq 800 \\
 n^2 + 5n - 800 &\leq 0 \\
 (n+32)(n-25) &\leq 0
 \end{aligned}$$

Hence,  $n \leq 25$ . Since the rope is cut to form as many circles as possible,  $n = 25$ .

Observe that  $r_{25} = 1 + \frac{1}{4}(25-1) = 7$ . Hence, the largest circle has area  $\pi \cdot 7^2 = 49\pi$  cm<sup>2</sup>.

The largest circle has area  $49\pi$  cm<sup>2</sup>.

**Part (b)**

Let  $S_n = \sum_{k=1}^n u_k$ .

$$\begin{aligned} u_{n+1} &= S_{n+1} - S_n \\ &= \alpha^{-(n+1)} - 1 - (\alpha^{-n} - 1) \\ &= \alpha^{-n} \cdot \alpha^{-1} - \alpha^{-n} \\ &= \alpha^{-n}(\alpha^{-1} - 1) \end{aligned}$$

**Subpart (i)**

**Test for Geometric Progression.**

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{\alpha^{-(n+1)}(\alpha^{-1} - 1)}{\alpha^{-n}(\alpha^{-1} - 1)} \\ &= \frac{\alpha^{-(n+1)}}{\alpha^{-n}} \\ &= \alpha^{-1} \end{aligned}$$

Since  $\alpha^{-1}$  is a constant,  $u_n$  is in geometric progression with common ratio  $\alpha^{-1}$ .

The common ratio of the sequence is  $\alpha^{-1}$ .

**Subpart (ii)**

Consider  $L = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} (\alpha^{-n} - 1)$ . For  $L$  to exist, we need  $\lim_{n \rightarrow \infty} \alpha^{-n}$  to exist. Hence,  $|\alpha^{-1}| < 1$ , whence  $|\alpha| > 1$ . Thus,  $\alpha < -1$  or  $\alpha > 1$ .

$$\{x \in \mathbb{R} : x < -1 \vee x > 1\}$$

**Subpart (iii)**

Since  $|\alpha^{-1}| < 1$ , we know  $\lim_{n \rightarrow \infty} \alpha^{-n} = 0$ . Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} S_n &= \lim_{n \rightarrow \infty} (\alpha^{-n} - 1) \\ &= -1 \end{aligned}$$

The sum to infinity of the sequence is  $-1$ .

**Problem 3.**

A sequence  $u_1, u_2, u_3, \dots$  is such that  $u_{n+1} = 2u_n + An$ , where  $A$  is a constant and  $n \geq 1$ .

(a) Given that  $u_1 = 5$  and  $u_2 = 15$ , find  $A$  and  $u_3$ .

It is known that the  $n$ th term of this sequence is given by

$$u_n = a(2^n) + bn + c$$

where  $a$ ,  $b$  and  $c$  are constants.

(b) Find  $a$ ,  $b$  and  $c$ .

**Solution****Part (a)**

$$\begin{aligned} u_2 &= 2u_1 + A \cdot 1 \\ &= 2 \cdot 5 + A \\ &= 10 + A \\ &= 15 \end{aligned}$$

Hence,  $A = 5$ .

$$\begin{aligned} u_3 &= 2u_2 + A \cdot 2 \\ &= 2 \cdot 15 + 2 \cdot 5 \\ &= 40 \end{aligned}$$

$$\boxed{A = 5, u_3 = 40}$$

**Part (b)**

Since  $u_1 = 5$ ,  $u_2 = 15$  and  $u_3 = 40$ , we have the following system

$$\begin{cases} 2a + b + c = 5 \\ 4a + 2b + c = 15 \\ 8a + 3b + c = 40 \end{cases}$$

which has solutions  $a = \frac{15}{2}$ ,  $b = -5$  and  $c = -5$

$$\boxed{a = \frac{15}{2}, b = -5, c = -5}$$

**Problem 4.**

The graphs of  $y = \frac{1}{3}(2^x)$  and  $y = x$  intersect at  $x = \alpha$  and  $x = \beta$  where  $\alpha < \beta$ . A sequence of real numbers  $x_1, x_2, x_3, \dots$  satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3}(2^{x_n}), \quad n \geq 1$$

- (a) Prove algebraically that, if the sequence converges, then it converges to either  $\alpha$  or  $\beta$ .
- (b) By using the graphs of  $y = \frac{1}{3}(2^x)$  and  $y = x$ , prove that
- if  $\alpha < x_n < \beta$ , then  $\alpha < x_{n+1} < x_n$
  - if  $x_n < \alpha$ , then  $x_n < x_{n+1} < \alpha$
  - if  $x_n > \beta$ , then  $x_n < x_{n+1}$

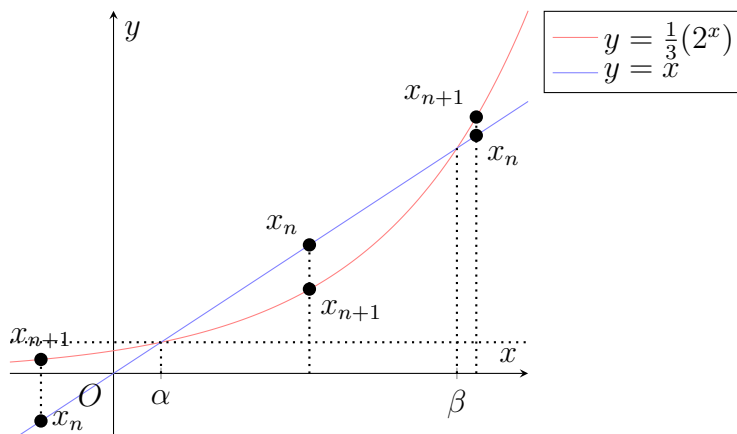
Describe the behaviour of the sequence for the three cases.

**Solution****Part (a)**

Let  $L = \lim_{n \rightarrow \infty} x_n$ .

$$\begin{aligned} x_{n+1} &= \frac{1}{3}(2^{x_n}) \\ \implies \lim_{n \rightarrow \infty} x_{n+1} &= \lim_{n \rightarrow \infty} \frac{1}{3}(2^{x_n}) \\ \implies L &= \frac{1}{3}(2^L) \end{aligned}$$

Since  $y = x$  and  $y = \frac{1}{3}(2^x)$  intersect only at  $x = \alpha$  and  $x = \beta$ , then  $\alpha$  and  $\beta$  are the only roots of  $x = \frac{1}{3}(2^x)$ . Since  $L$  is also a root of  $x = \frac{1}{3}(2^x)$ ,  $L$  must be either  $\alpha$  or  $\beta$ .

**Part (b)**

<p>If <math>\alpha &lt; x_n &lt; \beta</math>, then <math>x_n</math> is decreasing and converges to <math>\alpha</math>. If <math>x_n &lt; \alpha</math>, then <math>x_n</math> is increasing and converges to <math>\alpha</math>. If <math>x_n &gt; \beta</math>, then <math>x_n</math> is increasing and diverges.</p>
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