Problem 1.

On an Argand diagram, mark and label clearly the points P and Q representing the complex numbers p and q respectively, where

$$p = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4}, \qquad q = 2\cos\frac{\pi}{4} + 2i\sin\frac{\pi}{4}.$$

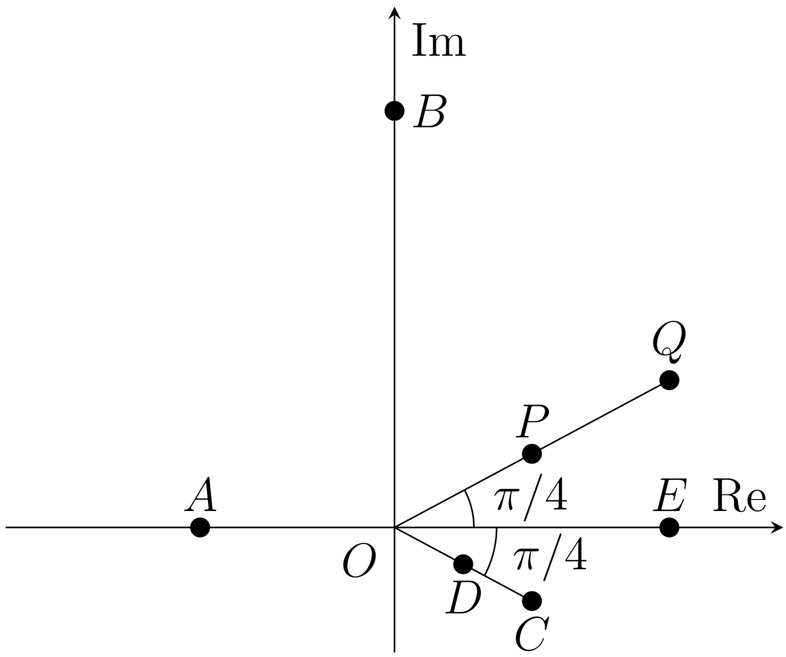
Find the moduli and arguments of the complex numbers a, b, c, d and e, where $a = p^4$, $b = q^2$, $c = -ip$, $d = \dfrac{1}{q}$, $e = p + p^*$.

On your Argand diagram, mark and label the points A, B, C, D and E representing these complex numbers.

Find the area of triangle COQ.

Find the modulus and argument of $p^{13/3}q^{45/2}$.

Solution



Note that $p = e^{i\pi/4}$ and $q = 2e^{i\pi/4}$.

$$\begin{aligned}
a &= p^4 = \left(e^{i\pi/4}\right)^4 = e^{i\pi} \\
b &= q^2 = \left(2e^{i\pi/4}\right)^2 = 4e^{i\pi/2} \\
c &= -ip = e^{-i\pi/2}e^{i\pi/4} = e^{-i\pi/4} \\
d &= \frac{1}{q} = \frac{1}{2}e^{-i\pi/4} \\
e &= p + p^* = 2\operatorname{Re} p = 2\cos\left(\frac{\pi}{4}\right) = \sqrt{2}
\end{aligned}$$

	modulus	argument
a	1	π
b	4	$\pi/2$
c	1	$-\pi/4$
d	$1/2$	$-\pi/4$
e	$\sqrt{2}$	0

Since $\angle COQ = \dfrac{\pi}{2}$, we have Area $\triangle COQ = \dfrac{1}{2} \cdot 2 \cdot 1 = 1$ units^2.

$$\boxed{\text{Area}\,\triangle COQ = 1 \text{ units}^2}$$

$$\begin{aligned} p^{13/3}q^{45/2} &= \left(e^{i\pi/4}\right)^{13/3}\left(2e^{i\pi/4}\right)^{45/2} \\ &= e^{i\pi 13/12} \cdot 2^{45/2}e^{i\pi 45/8} \\ &= 2^{45/2}e^{i\pi 161/24} \\ &= 2^{45/2}e^{i\pi 17/24} \end{aligned}$$

$$\boxed{\left|p^{13/3}q^{45/2}\right| = e^{45/2},\ \arg\left(p^{13/3}q^{45/2}\right) = \frac{17}{24}\pi}$$

Problem 2.

The complex number q is given by $q = \dfrac{e^{i2\theta}}{1 - e^{i2\theta}}$, where $0 < \theta < 2\pi$. In either order,

(a) find the real part of q,

(b) show that the imaginary part of q is $\dfrac{1}{2}\cot\theta$.

Solution

$$\begin{aligned}
q &= \frac{e^{i2\theta}}{1 - e^{i2\theta}} \\
&= \frac{e^{i\theta}}{e^{-i\theta} - e^{i\theta}} \\
&= -\frac{e^{i\theta}}{e^{i\theta} - e^{-i\theta}} \\
&= -\frac{e^{i\theta}/2i}{(e^{i\theta} - e^{-i\theta})/2i} \\
&= -\frac{\cos\theta + i\sin\theta}{2i} \cdot \frac{1}{\sin\theta} \\
&= -\frac{-i(\cos\theta + i\sin\theta)}{2} \cdot \frac{1}{\sin\theta} \\
&= \frac{-\sin\theta + i\cos\theta}{2} \cdot \frac{1}{\sin\theta} \\
&= \frac{-1 + i\cot\theta}{2} \\
&= -\frac{1}{2} + i\frac{1}{2}\cot\theta
\end{aligned}$$

$$\boxed{\operatorname{Re} q = -\frac{1}{2},\ \operatorname{Im} q = \frac{1}{2}\cot\theta}$$

Problem 3.

The complex numbers z and w are such that $z = 4\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right)$ and $w = 1 - i\sqrt{3}$. z^* denotes the conjugate of z.

(a) Find the modulus r and the argument θ of $\frac{w^2}{z^*}$, where $r > 0$ and $-\pi < \theta < \pi$.

(b) Given that $\left(\frac{w^2}{z^*}\right)^n$ is purely imaginary, find the set of values that n can take.

Solution

Part (a)

Note that $z = 4e^{i3\pi/4}$ and $w = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 2e^{-i\pi/3}$.

$$\begin{aligned}\frac{w^2}{z^*} &= \frac{\left(2e^{-i\pi/3}\right)^2}{4e^{-i3\pi/4}}\\ &= \frac{4e^{-i2\pi/3}}{4e^{-i3\pi/4}}\\ &= \frac{e^{-i2\pi/3}}{e^{-i3\pi/4}}\\ &= e^{i\pi/12}\end{aligned}$$

$$\boxed{r = 1,\ \theta = \frac{\pi}{12}}$$

Part (b)

Note that $\left(\frac{w^2}{z^*}\right)^n = \left(e^{i\pi/12}\right)^n = e^{in\pi/12}$. Since $\left(\frac{w^2}{z^*}\right)^n$ is purely imaginary, we have $\arg\left(\frac{w^2}{z^*}\right)^n = \frac{\pi}{2} + \pi k$, where $k \in \mathbb{Z}$. Thus, $\frac{n\pi}{12} = \frac{\pi}{2} + \pi k$, whence $n = 6 + 12k$.

$$\boxed{n \in \{k \in \mathbb{Z} : 6 + 12k\}}$$

Problem 4.

The complex number w has modulus $\sqrt{2}$ and argument $\dfrac{1}{4}\pi$ and the complex number z has modulus $\sqrt{2}$ and argument $\dfrac{5}{6}\pi$.

(a) By first expressing w and z in the form $x + iy$, find the exact real and imaginary parts of $w + z$.

(b) On the same Argand diagram, sketch the points P, Q, R representing the complex numbers z, w, and $z + w$ respectively. State the geometrical shape of the quadrilateral $OPRQ$.

(c) Referring the Argand diagram in part (b), find $\arg(w + z)$ and show that $\tan \dfrac{11}{24}\pi = \dfrac{a+\sqrt{2}}{\sqrt{6}+b}$, where a and b are constants to be determined.

Solution

Part (a)

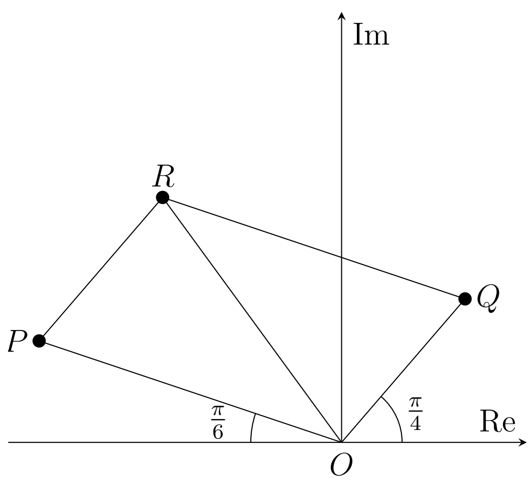
$$w = \sqrt{2}e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 1 + i$$

$$z = \sqrt{2}e^{i5\pi/6} = \sqrt{2}\left(\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\frac{\sqrt{3}}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$\implies w + z = (1+i) + (-\frac{\sqrt{3}}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \left(1 - \frac{\sqrt{3}}{\sqrt{2}}\right) + i\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$\boxed{w + z = \left(1 - \frac{\sqrt{3}}{\sqrt{2}}\right) + i\left(1 + \frac{1}{\sqrt{2}}\right)}$$

Part (b)



$\boxed{OPRQ \text{ is a parallelogram.}}$

Part (c)

Note that $\angle POQ = \pi - \frac{\pi}{6} - \frac{\pi}{4} = \frac{7}{12}\pi$. Since $|z| = |w|$, we have $OP = OQ$, whence $\angle ROQ = \frac{1}{2} \cdot \frac{7}{12}\pi = \frac{7}{24}\pi$. Hence, $\arg(w+z) = \frac{\pi}{4} + \frac{7}{24}\pi = \frac{13}{24}\pi$.

$$\boxed{\arg(w+z) = \frac{13}{24}\pi}$$

Thus,

$$\tan\left(\frac{13}{24}\pi\right) = \frac{1 + 1/\sqrt{2}}{1 - \sqrt{3}/\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}-\sqrt{3}} = \frac{2+\sqrt{2}}{2-\sqrt{6}}$$

However, $\tan\left(\frac{13}{24}\pi\right) = -\tan\left(\pi - \frac{13}{24}\right) = -\tan\left(\frac{11}{24}\pi\right)$. Hence,

$$\tan\left(\frac{11}{24}\pi\right) = -\frac{2+\sqrt{2}}{2-\sqrt{6}} = \frac{2+\sqrt{2}}{\sqrt{6}-2}$$

$$\boxed{a = 2,\ b = -2}$$

Problem 5.

The complex number z is given by $z = 2(\cos\beta + i\sin\beta)$ where $0 < \beta < \frac{\pi}{2}$.

(a) Show that $\frac{z}{4-z^2} = (k\csc\beta)i$, where k is positive real constant to be determined.

(b) State the argument of $\frac{z}{4-z^2}$, giving your reasons clearly.

(c) Given the complex number $w = -\sqrt{3} + i$, find the three smallest positive integer values of n such that $\left(\frac{z}{4-z^2}\right)(w^*)^n$ is a real number.

Solution

Part (a)

Observe that $z = 2(\cos\beta + i\sin\beta) = 2e^{i\beta}$. Hence,

$$\begin{aligned}
\frac{z}{4-z^2} &= \frac{2e^{i\beta}}{4-(2e^{i\beta})^2} \\
&= \frac{2e^{i\beta}}{4-4e^{i2\beta}} \\
&= \frac{1}{2}\cdot\frac{e^{i\beta}}{1-e^{i2\beta}} \\
&= \frac{1}{2}\cdot\frac{1}{e^{-i\beta}-e^{i\beta}} \\
&= -\frac{1}{2}\cdot\frac{1}{e^{i\beta}-e^{-i\beta}} \\
&= -\frac{1}{2}\cdot\frac{1/2i}{(e^{i\beta}-e^{-i\beta})/2i} \\
&= -\frac{1}{2}\cdot\frac{1}{2i}\cdot\frac{1}{\sin\beta} \\
&= -\frac{1}{2}\cdot -\frac{i}{2}\cdot\csc\beta \\
&= \left(\frac{1}{4}\csc\beta\right)i
\end{aligned}$$

$$\boxed{k = \frac{1}{4}}$$

Part (b)

Since $0 < \beta < \frac{\pi}{2}$, we know that $\csc\beta > 0$. Hence, $\operatorname{Im}\left(\frac{z}{4-z^2}\right) > 0$. Furthermore, $\operatorname{Re}\left(\frac{z}{4-z^2}\right) = 0$. Thus, $\arg\left(\frac{z}{4-z^2}\right) = \frac{\pi}{2}$.

$$\boxed{\arg\left(\frac{z}{4-z^2}\right) = \frac{\pi}{2}}$$

Part (c)

Note that $w = -\sqrt{3} + i = 2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = 2\left[\cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi\right] = 2e^{-i5\pi/6}$. Hence,

$$\begin{aligned}\left(\frac{z}{4-z^2}\right)(w^*)^n &= \left(\frac{1}{4}\csc\beta\right)i\cdot\left(2e^{-i5\pi/6}\right)^n \\ &= \frac{1}{4}\csc\beta 2^n \cdot e^{i\pi/2}\cdot e^{-i5n\pi/6} \\ &= \frac{1}{4}\csc\beta 2^n \cdot e^{i\pi(1/2-5n/6)}\end{aligned}$$

Hence, $\arg\left(\left(\frac{z}{4-z^2}\right)(w^*)^n\right) = \pi\left(\frac{1}{2}-\frac{5}{6}n\right)$. However, for $\left(\frac{z}{4-z^2}\right)(w^*)^n$ to be a real number, we required $\arg\left(\left(\frac{z}{4-z^2}\right)(w^*)^n\right) = \pi k$, where $k \in \mathbb{Z}$. Hence,

$$\begin{aligned} && \pi\left(\frac{1}{2}-\frac{5}{6}n\right) &= \pi k \\ \implies && \frac{1}{2}-\frac{5}{6}n &= k \\ \implies && 3-5n &= 6k \\ \implies && 3-5n &\equiv 0 \pmod 6 \\ \implies && 5n &\equiv 3 \pmod 6 \\ \implies && -1\cdot n &\equiv 3 \pmod 6 \\ \implies && n &\equiv 3 \pmod 6\end{aligned}$$

Hence, the three smallest possible values of n are 3, 9 and 15.

$$\boxed{3,\ 9,\ 15}$$