

**Problem 1.**

Show that

$$\sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) = \frac{a}{b} \left( 729 \cdot 9^{(n-1)^2} - 1 \right) - c(n-1)^3 - d(n-1)$$

where  $a$ ,  $b$ ,  $c$  and  $d$  are constants to be determined.

**Solution**

$$\begin{aligned} \sum_{r=1}^{(n-1)^2+3} (3^{2r} - n + 1) &= \sum_{r=1}^{(n-1)^2+3} 9^r - \sum_{r=1}^{(n-1)^2+3} (n-1) \\ &= \frac{9 \left( 9^{(n-1)^2+3} - 1 \right)}{9 - 1} - (n-1) \left( (n-1)^2 + 3 \right) \\ &= \frac{9}{8} \left( 729 \cdot 9^{(n-1)^2} - 1 \right) - (n-1)^3 - 3(n-1) \end{aligned}$$

**Problem 2.****Do not use a calculator in answering this question.**The sequence of positive numbers,  $u_n$ , satisfies the recurrence relation:

$$u_{n+1} = \sqrt{2u_n + 3}, \quad n = 1, 2, 3, \dots$$

- (a) If the sequence converges to  $m$ , find the value of  $m$ .
- (b) By using a graphical approach, explain why  $m < u_{n+1} < u_n$  when  $u_n > u_m$ . Hence, determine the behaviour of the sequence when  $u_1 > m$ .

**Solution****Part (a)**

Observe that

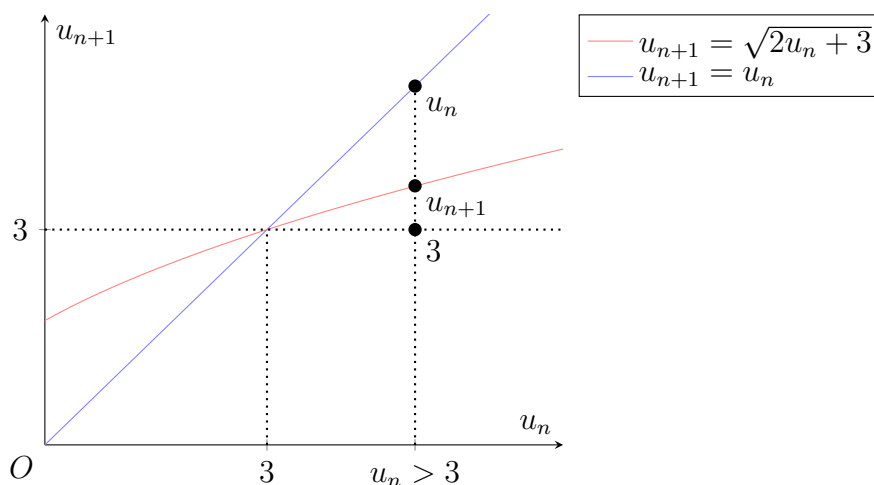
$$\begin{aligned} \lim_{n \rightarrow \infty} u_n &= \lim_{n \rightarrow \infty} \sqrt{2u_{n-1} + 3} \\ &= \sqrt{2 \lim_{n \rightarrow \infty} u_{n-1} + 3} \\ &= \sqrt{2 \lim_{n \rightarrow \infty} u_n + 3} \end{aligned}$$

Since the sequence converges to  $m$ , we have  $\lim_{n \rightarrow \infty} u_n = m$ . Thus,

$$\begin{aligned} m &= \sqrt{2m + 3} \\ \implies m^2 &= 2m + 3 \\ \implies m^2 - 2m - 3 &= 0 \\ \implies (m - 3)(m + 1) &= 0 \end{aligned}$$

Thus,  $m = 3$  or  $m = -1$ . Since  $u_n$  is always positive, we take  $m = 3$ .

$$\boxed{m = 3}$$

**Part (b)**

From the graph, if  $u_n > 3$ , then  $3 < u_{n+1} < u_n$ .

The sequence decreases and converges to 3.
--

**Problem 3.**

Two expedition teams are to climb a vertical distance of 8100 m from the foot to the peak of a mountain. Team *A* plans to cover a vertical distance of 400 m on the first day. On each subsequent day, the vertical distance covered is 5 m less than the vertical distance covered in the previous day. Team *B* plans to cover a vertical distance of 800 m on the first day. On each subsequent day, the vertical distance covered is 90% of the vertical distance covered in the previous day.

- (a) Find the number of days required for Team *A* to reach the peak.
- (b) Explain why Team *B* will never be able to reach the peak.
- (c) At the end of the 15th day, Team *B* decided to modify their plan, such that on each subsequent day, the vertical distance covered is 95% of the vertical distance covered in the previous day. Which team will be the first to reach the peak of the mountain? Justify your answer.

**Solution****Part (a)**

The vertical distance Team *A* plans to cover in a day can be described as a sequence in arithmetic progression with first term 400 and common difference  $-5$ . In order to reach the peak, the total vertical distance covered by Team *A* has to be greater than 8100 m. Hence,

$$\frac{n}{2}(2(400) + (n-1)(-5)) \geq 8100$$

From the graphing calculator,  $n \geq 24$ . Hence, Team *A* requires 24 days to reach the peak.

24 days.

**Part (b)**

The vertical distance Team *B* plans to cover in the  $n$ th day can be described by the sequence  $U_n$  in geometric progression with first term 800 and common ratio  $r = 0.9$ . Let  $S_n^U$  be the  $n$ th partial sum of  $U_n$ . Since  $|r| < 1$ , the sum to infinity exists and is equal to

$$S_\infty^U = \frac{800}{1 - 0.9} = 8000$$

Hence, Team *B* will never be able to surpass 8 km in height. Thus, they will not reach the peak no matter how long they take.

**Part (c)**

The new vertical distance covered by Team *B* after Day 15 can be described by the sequence  $V_n$  in geometric progression with first term  $U_{15}$  and common ratio  $r = 0.95$ . Let  $S_n^V$  be the  $n$ th partial sum of  $V_n$ . Then,

$$S_n^V = \frac{U_{15} \cdot 0.95(1 - (0.95)^n)}{1 - 0.95}$$

Note that

$$S_n^U = \frac{800(1 - (0.9)^n)}{1 - 0.9}$$

Hence, after Day 15, Team  $B$  has to climb another  $8000 - S_{15}^U = 1747.13$  metres. Since  $U_{15} = 183.01$ , we have the inequality

$$\frac{183.01 \cdot 0.95(1 - (0.95)^n)}{1 - 0.95} \geq 1747.13$$

Using the graphing calculator,  $n \geq 14$ . Hence, Team  $B$  will need at least  $15 + 14 = 29$  days to reach the peak.

Team  $A$  will reach the peak first.

**Problem 4.**

The function  $f$  is given by  $f(x) = x^2 - 3x + 2 - e^{-x}$ . It is known from graphical work that this equation has 2 roots  $x = \alpha$  and  $x = \beta$ , where  $\alpha < \beta$ .

- (a) Show that  $f(x) = 0$  has at least one root in the interval  $[0, 1]$ .

It is known that there is exactly one root in  $[0, 1]$  where  $x = \alpha$ .

- (b) Starting with  $x_0 = 0.5$ , use an iterative method based on the form

$$x_{n+1} = p(x_n^2 + q - e^{-x_n})$$

where  $p$  and  $q$  are real numbers to be determined, to find the value of  $\alpha$  correct to 3 decimal places. You should demonstrate that your answer has the required accuracy.

It is known that the other root  $x = \beta$  lies in the interval  $[2, 3]$ .

- (c) With the aid of a clearly labelled diagram, explain why the method in (b) will fail to obtain any reasonable approximation to  $\beta$ , where  $x_0$  is chosen such that  $x_0 \in [2, 3]$ ,  $x_0 \neq \beta$ .

To obtain an approximation to  $\beta$ , another approach is used.

- (d) Use linear interpolation once in the interval  $[2, 3]$  to find a first approximation to  $\beta$ , giving your answer to 2 decimal places. Explain whether this approximate is an overestimate or underestimate.
- (e) With your answer in (d) as the initial approximate, use the Newton-Raphson method to obtain  $\beta$  correct to 3 decimal places. Your process should terminate when you have two successive iterates that are equal when rounded to 3 decimal places.

**Solution****Part (a)**

Observe that  $f(0) = 1 > 0$  and  $f(1) = -e^{-1} < 0$ . Since  $f$  is continuous and  $f(0)f(1) < 0$ , there must be at least one root to  $f(x) = 0$  in the interval  $[0, 1]$ .

**Part (b)**

Let  $f(x) = 0$ . Then,

$$\begin{aligned} x^2 - 3x + 2 - e^{-x} &= 0 \\ \implies x^2 + 2 - e^{-x} &= 3x \\ \implies x &= \frac{1}{3}(x^2 + 2 - e^{-x}) \end{aligned}$$

Hence, we should use an iterative method based on the form

$$x_{n+1} = \frac{1}{3}(x_n^2 + 2 - e^{-x_n})$$

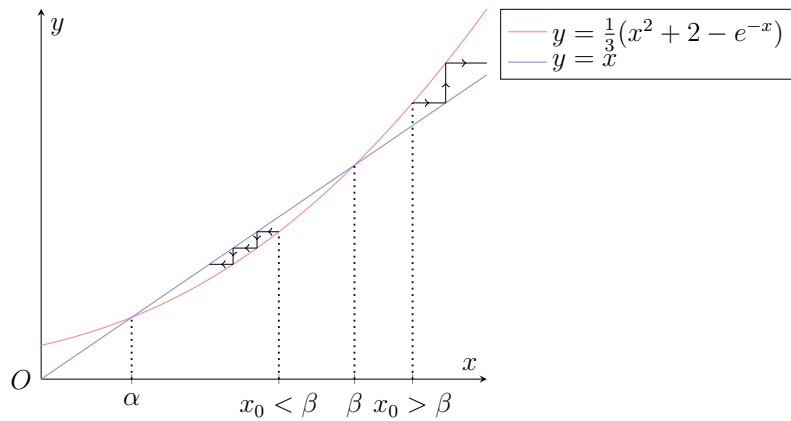
Starting with  $x_0 = 0.5$ ,

$$\begin{aligned}
 x_1 &= \frac{1}{3} (x_0^2 + 2 - e^{-x_0}) = 0.54782 \\
 \implies x_2 &= \frac{1}{3} (x_1^2 + 2 - e^{-x_1}) = 0.57396 \\
 \implies x_3 &= \frac{1}{3} (x_2^2 + 2 - e^{-x_2}) = 0.58871 \\
 \implies x_4 &= \frac{1}{3} (x_3^2 + 2 - e^{-x_3}) = 0.59718 \\
 \implies x_5 &= \frac{1}{3} (x_4^2 + 2 - e^{-x_4}) = 0.60208 \\
 \implies x_6 &= \frac{1}{3} (x_5^2 + 2 - e^{-x_5}) = 0.60494 \\
 \implies x_7 &= \frac{1}{3} (x_6^2 + 2 - e^{-x_6}) = 0.60662 \\
 \implies x_8 &= \frac{1}{3} (x_7^2 + 2 - e^{-x_7}) = 0.60759 \\
 \implies x_9 &= \frac{1}{3} (x_8^2 + 2 - e^{-x_8}) = 0.60817 \\
 \implies x_{10} &= \frac{1}{3} (x_9^2 + 2 - e^{-x_9}) = 0.60851 \\
 \implies x_{11} &= \frac{1}{3} (x_{10}^2 + 2 - e^{-x_{10}}) = 0.60870
 \end{aligned}$$

Since  $f(0.6085) = 0.000606 > 0$  and  $f(0.6095) = -0.000632 < 0$ , we have that  $\alpha \in (0.6085, 0.6095)$ . Hence,

$\alpha = 0.609 \text{ (3 d.p.)}$

### Part (c)



From the diagram, we see that whether we chose  $x_0 < \beta$  or  $x_0 > \beta$ , the approximates move away from the root  $\beta$ . In fact, if we choose  $x_0 < \beta$ , the approximates converge to the root  $\alpha$  instead.

**Part (d)**

Using linear interpolation on the interval  $[2, 3]$ ,

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = 2.06 \text{ (2 d.p.)}$$

$$\boxed{\beta = 2.06 \text{ (2 d.p.)}}$$

Observe that  $f(2.06) = -0.039 < 0$  and  $f(3) = 1.950 > 0$ . Hence,  $\beta \in (2.06, 3)$ . Thus,

$$\boxed{\beta = 2.06 \text{ is an underestimate.}}$$

**Part (e)**

Observe that  $f'(x) = 2xx - 3 + e^{-x}$ . Using the Newton-Raphson method with  $x_1 = 2.06$ ,

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 2.11118 = 2.111 \text{ (3 d.p.)} \\ \implies x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 2.10935 = 2.109 \text{ (3 d.p.)} \\ \implies x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 2.10935 = 2.109 \text{ (3 d.p.)} \end{aligned}$$

Hence,

$$\boxed{\beta = 2.109 \text{ (3 d.p.)}}$$