Problem 1.

(a) The fixed-point iteration $g(x_n) = x_{n+1}$ is used for iterating to a root α of the equation $f(x) = 0$. Explain, with the aid of a diagram, when this procedure might not converge to α even when the initial approximation x_0 is close to α.

(b) Let $f(x) = \ln(3x) - \dfrac{k}{x}$, where $x > 0$ and k is a real positive constant.

(i) Show that the equation $f(x) = 0$ has exactly one root α.

(ii) Find the exact range of values of k such that α lies in the interval $(2, 3)$.

Let $k = 5$ for the rest of the question.

(iii) If α lies in the interval $(2, 3)$, obtain, by linear interpolation over the interval $(2, 3)$, a first approximation α_1 to α. Give your answer correct to 3 decimal places.

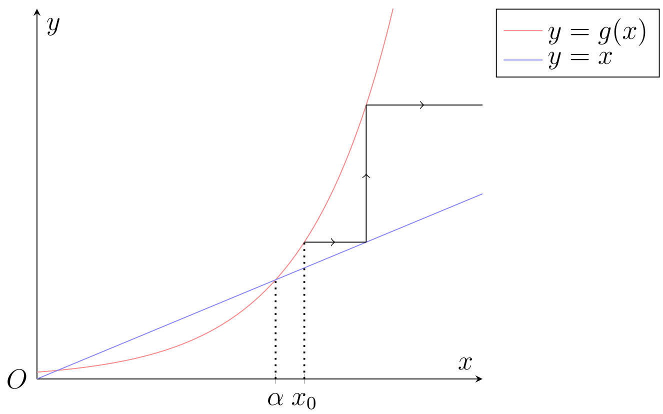
(iv) Explain, with a sketch of $y = f(x)$ for $2 < x < 3$, why α_1 is greater than α.

(v) Using the Newton-Raphson method, and $\alpha_1 = 2.5$ as the first estimate, find an approximation to the root α. Give your answer correct to 3 decimal places.

(vi) Without further calculation, determine, with justification, if Newton-Raphson can be used to estimate the root if the first approximation used is $\alpha_1 = 7$.

Solution

Part (a)



From the above diagram, although the initial approximation x_0 is close to α, if the gradient of $y = g(x)$ is too steep compared to $y = x$ near α, the procedure will not converge to α.

Part (b)

Subpart (i)

Observe that f is a continuous function. Note that $f\left(\frac{1}{3}\right) = \ln\left(3 \cdot \frac{1}{3}\right) - \frac{k}{\frac{1}{3}} = -3k < 0$. Furthermore, as x approaches infinity, $f(x)$ also approaches infinity. Thus, there is at least one root to the equation $f(x) = 0$.

Note that $f'(x) = \frac{1}{x} + \frac{k}{x^2}$, which is greater than 0 for all real x. Thus, f is an increasing function. Hence, there is only one root to the equation $f(x) = 0$.

Subpart (ii)
Since $f(x)$ is strictly increasing and continuous, if $\alpha \in (2, 3)$, we need $f(2) < 0$ and $f(3) > 0$.

Case 1: $f(2) < 0 \implies \ln 6 - \frac{k}{2} < 0 \implies k > 2\ln 6$

Case 2: $f(3) > 0 \implies \ln 9 - \frac{k}{3} > 0 \implies k < 3\ln 9$

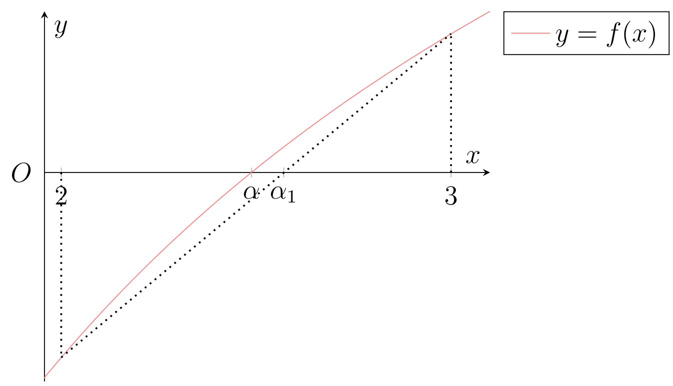
$$\boxed{2\ln 6 < k < 3\ln 9}$$

Subpart (iii)
Using linear interpolation on $(2, 3)$,

$$\begin{aligned}\alpha_1 &= \frac{2f(3) - 3f(2)}{f(3) - f(2)}\\ &= 2.572 \text{ (3 d.p.)}\end{aligned}$$

$$\boxed{\alpha_1 = 2.572 \text{ (3 d.p.)}}$$

Subpart (iv)



The graph of $y = f(x)$ is increasing and concave downwards. Thus, $\alpha_1 > \alpha$.

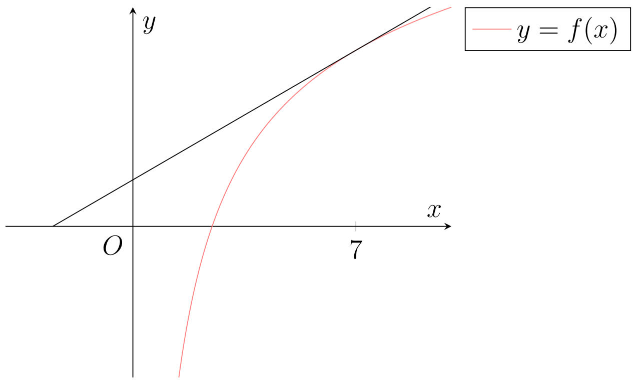
Subpart (v)

$$\begin{aligned}&\alpha_1 = 2.5\\ \implies &\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)} = 2.48758\\ \implies &\alpha_3 = \alpha_2 - \frac{f(\alpha_2)}{f'(\alpha_2)} = 2.48763\end{aligned}$$

Since $f(2.4875) < 0$ and $f(2.4885) > 0$, we know $\alpha \in (2.4875, 2.4885)$. Hence, $\alpha = 2.488$ (3 d.p.).

$$\boxed{\alpha = 2.488 \text{ (3 d.p.)}}$$

Subpart (vi)



Newton-Raphson cannot be used to estimate the root if the first approximation used is $\alpha_1 = 7$. From the diagram, the gradient at $x = 7$ is very small. As the iteration continues, the second approximation α_2 will be a negative number, causing it to be impossible to continue with the Newton-Raphson iteration as $f(x)$ will not be defined.