Problem 1.

A complex number z is represented in an Argand diagram by the point P. Sketch, on separate Argand diagrams, the locus of P. Describe geometrically the locus of P and determine its Cartesian equation.

(a) $|2z - 6 - 8i| = 10$

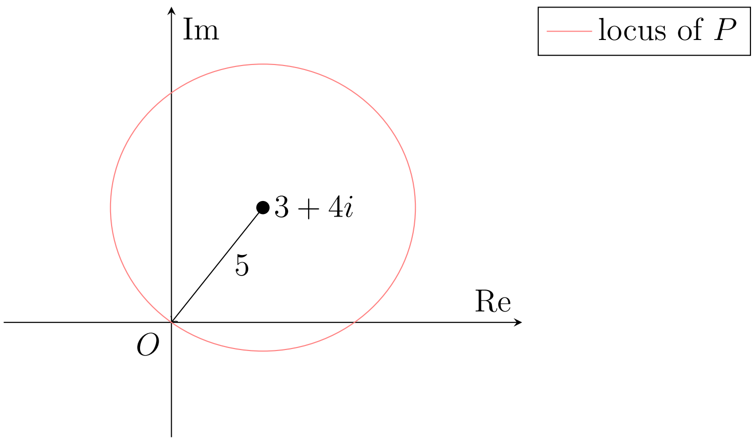
(b) $|z + 2| = |z - i|$

(c) $\arg(z + 2 - i) = -\frac{\pi}{4}$

Solution

Part (a)

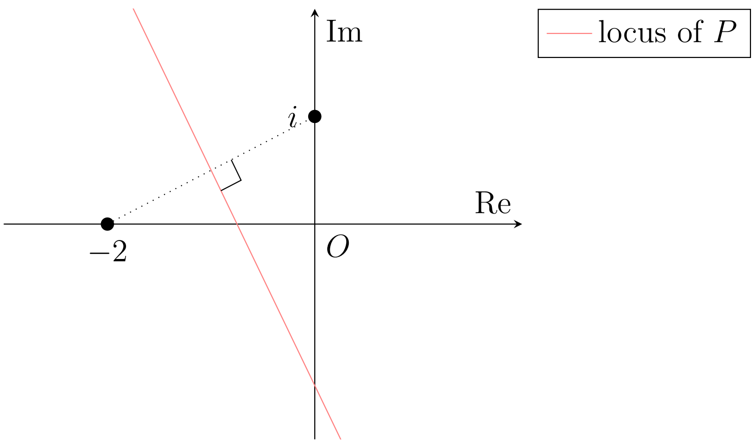
Note that $|2z - 6 - 8i| = 10 \implies |z - (3 + 4i)| = 5$.



The locus of P is a circle with centre $(3, 4)$ and radius 5.
Its Cartesian equation is $(x - 3)^2 + (y - 4)^2 = 5^2$.

Part (b)

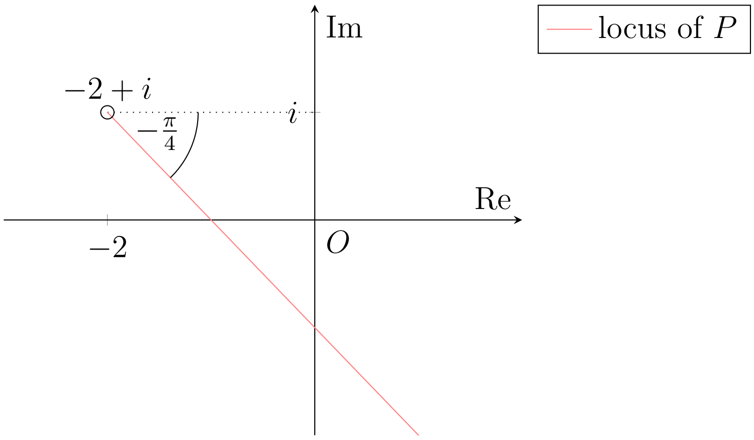
Note that $|z + 2| = |z - i| \implies |z - (-2)| = |z - i|$.



The locus of P is the perpendicular bisector of the line segment joining $(-2, 0)$ and $(0, 1)$. Its Cartesian equation is $y = -2x - 1.5$.

Part (c)

Note that $\arg(z+2-i) = -\frac{\pi}{4} \implies \arg(z-(-2+i)) = -\frac{\pi}{4}$.



The locus of P is the half-line starting from $(-2, 1)$ and inclined at an angle $-\frac{\pi}{4}$ to the positive real axis. Its Cartesian equation is $y = -x - 1$.

Problem 2.

Sketch the following loci on separate Argand diagrams.

(a) $\operatorname{Re}(z^2) = 1$

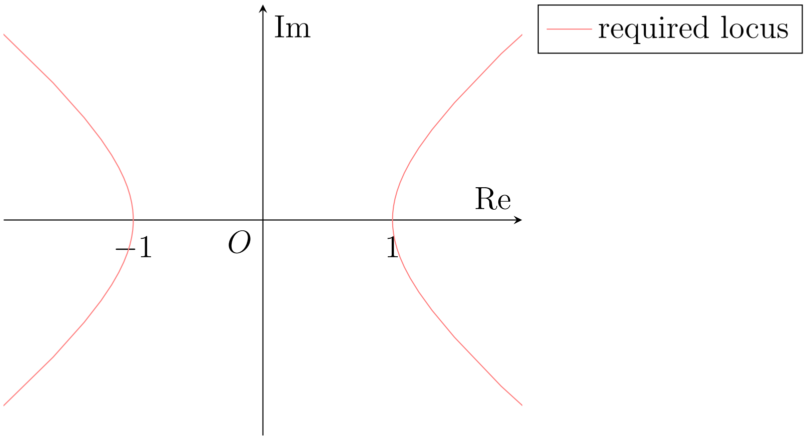
(b) $|6 - iz| = 2$,

(c) $\arg\left(\dfrac{iz}{1-\sqrt{3}i}\right) = \pi$

Solution

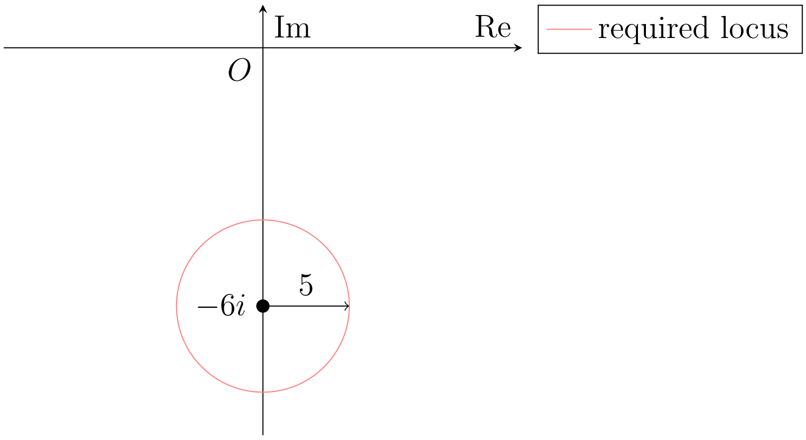
Part (a)

Let $z = r(\cos\theta + i\sin\theta)$. Then $\operatorname{Re}(z^2) = 1 \implies r^2\cos 2\theta = 1 \implies r^2 = \sec 2\theta$.



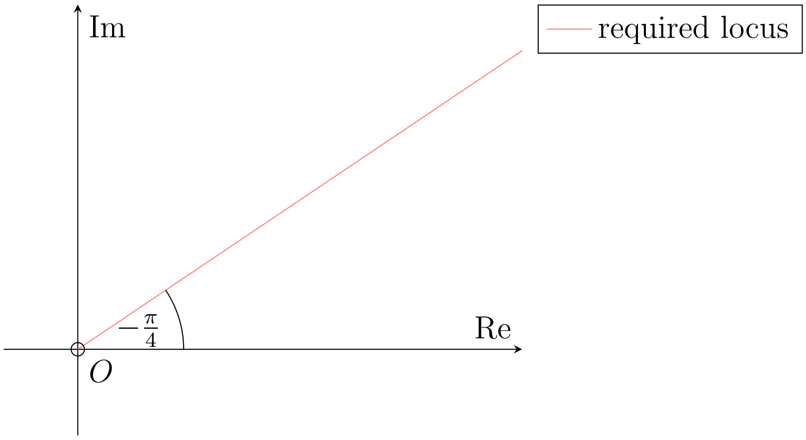
Part (b)

Note $|6 - iz| = 2 \implies |-i(z + 6i)| = 2 \implies |z + 6i| = 2 \implies |z - (6i)| = 2$.



Part (c)

Note $\arg\left(\dfrac{iz}{1-\sqrt{3}i}\right) = \pi \implies \arg(i) + \arg(z) - \arg\left(1-\sqrt{3}i\right) = \pi \implies \dfrac{\pi}{2} + \arg(z) + \dfrac{\pi}{3} \implies \arg(z) = \dfrac{\pi}{6}$.


Im
required locus
$-\frac{\pi}{4}$
Re
O

Problem 3.

Sketch, on separate Argand diagrams, the set of points satisfying the following inequalities.

(a) $2 < |z - 2i| \leq |3 - 4i|$

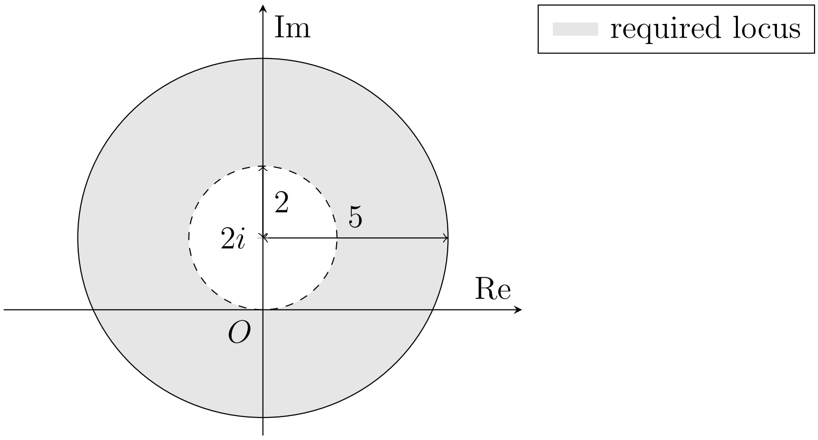
(b) $|z + i| > |z + 1 - i|$

(c) $\dfrac{\pi}{4} < \arg\left(\dfrac{1}{z}\right) \leq \dfrac{\pi}{2}$

Solution

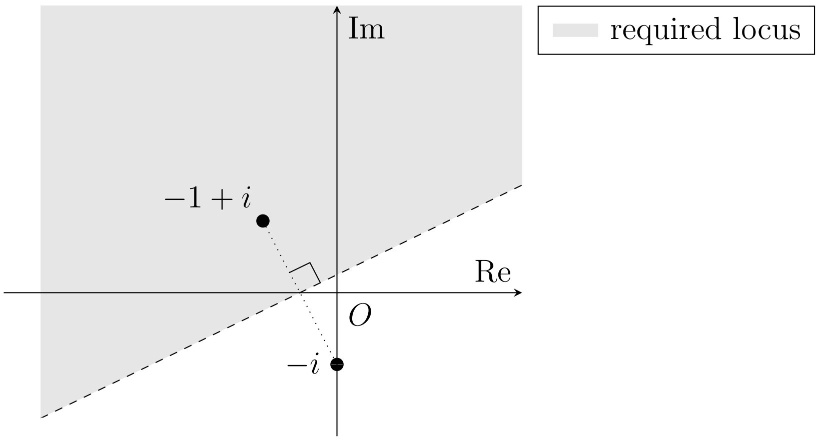
Part (a)

Note $2 < |z - 2i| \leq |3 - 4i| \implies 2 < |z - 2i| \leq 5$.



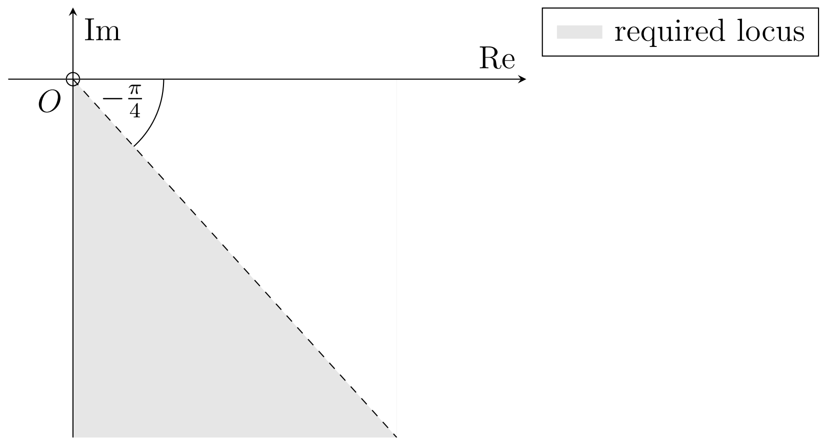
Part (b)

Note $|z + i| > |z + 1 - i| \implies |z - (-i)| > |z - (-1 + i)|$.



Part (c)

Note $\dfrac{\pi}{4} < \arg\left(\dfrac{1}{z}\right) \leq \dfrac{\pi}{2} \implies \dfrac{\pi}{4} < -\arg(z) \leq \dfrac{\pi}{2} \implies -\dfrac{\pi}{2} \geq \arg(z) > -\dfrac{\pi}{4}$.


Im
Re
required locus
O
$-\frac{\pi}{4}$

Problem 4.

Sketch on separate Argand diagrams for (a) and (b) the set of points representing all complex numbers z satisfying both of the following inequalities.

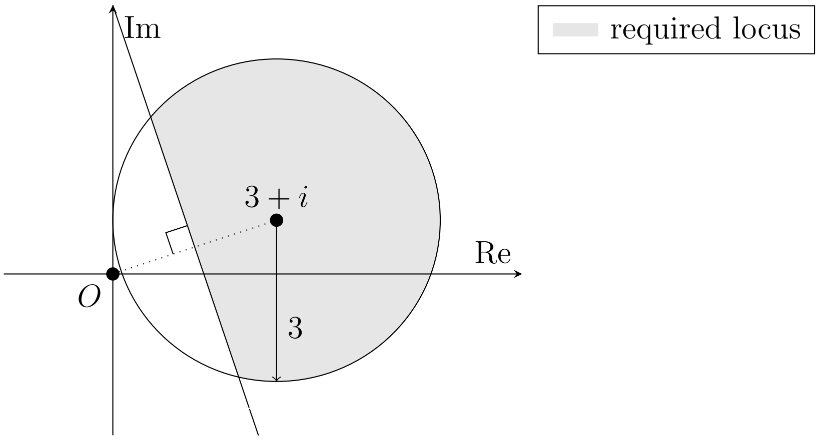
(a) $|z-3-i| \leq 3$ and $|z| \geq |z-3-i|$

(b) $\dfrac{\pi}{2} < \arg(z+1) \leq \dfrac{2}{3}\pi$ and $3\,\mathrm{Im}\,(z) > 2$

Solution

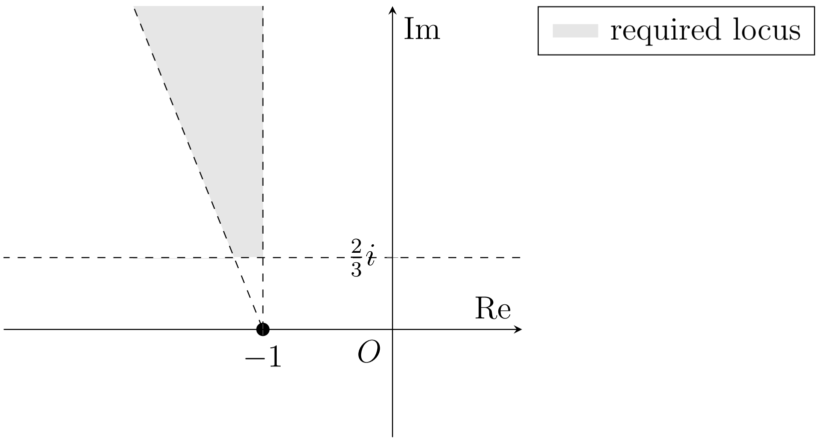
Part (a)

Note $|z-3-i| \leq 3 \implies |z-(3+i)| \leq 3$ and $|z| \geq |z-3-i| \implies |z| \geq |z-(3+i)|$.



Part (b)

Note $\dfrac{\pi}{2} < \arg(z+1) < \dfrac{2}{3}\pi \implies \dfrac{\pi}{2} < \arg(z-(-1)) < \dfrac{2}{3}\pi$ and $3\,\mathrm{Im}\,(z) > 2 \implies \mathrm{Im}\,(z) > \dfrac{2}{3}$.



Problem 5.

Illustrate, in separate Argand diagrams, the set of points z for which

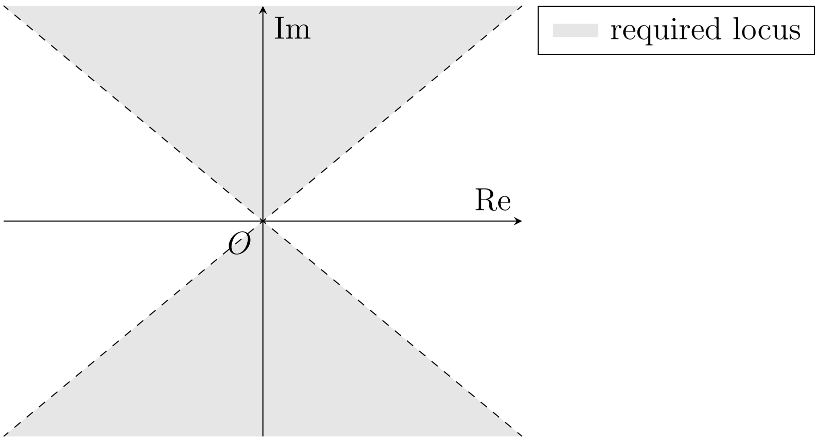
(a) $\operatorname{Re}(z^2) < 0$

(b) $\operatorname{Im}(z^3) > 0$

Solution

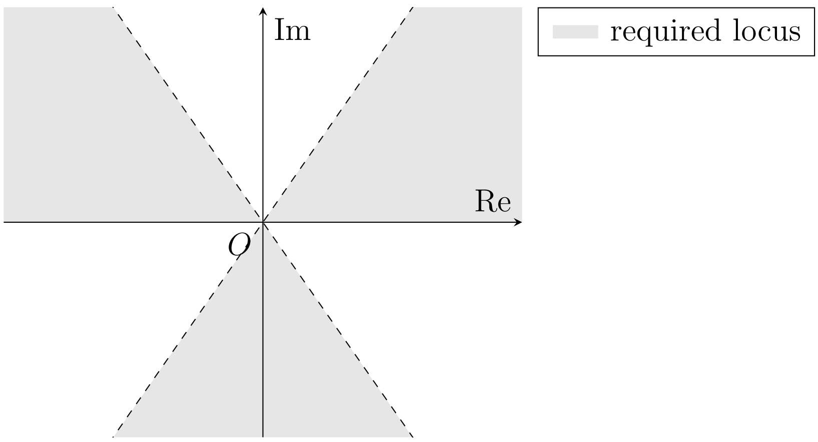
Part (a)

Let $z = r(\cos\theta + i\sin\theta)$, $0 \leq \theta < 2\pi$. Then $\operatorname{Re}(z^2) < 0 \implies r^2\cos 2\theta < 0 \implies \cos 2\theta < 0 \implies 2\theta \in \left(\frac{1}{2}\pi, \frac{3}{2}\pi\right) \cup \left(\frac{5}{2}\pi, \frac{7}{2}\pi\right) \implies \theta \in \left(\frac{1}{4}\pi, \frac{3}{4}\pi\right) \cup \left(\frac{5}{4}\pi, \frac{7}{4}\pi\right)$.



Part (b)

Let $z = r(\cos\theta + i\sin\theta)$, $0 \leq \theta < 2\pi$. Then $\operatorname{Im}(z^3) > 0 \implies r^3\sin 3\theta > 0 \implies \sin 3\theta > 0 \implies 3\theta \in (0, \pi) \cup (2\pi, 3\pi) \cup (4\pi, 5\pi) \implies \theta \in \left(0, \frac{1}{3}\pi\right) \cup \left(\frac{2}{3}\pi, \pi\right) \cup \left(\frac{4}{3}\pi, \frac{5}{3}\pi\right)$.



Problem 6.

The complex number z satisfies $|z+4-4i| = 3$.

(a) Describe, with the aid of a sketch, the locus of the point which represents z in an Argand diagram.

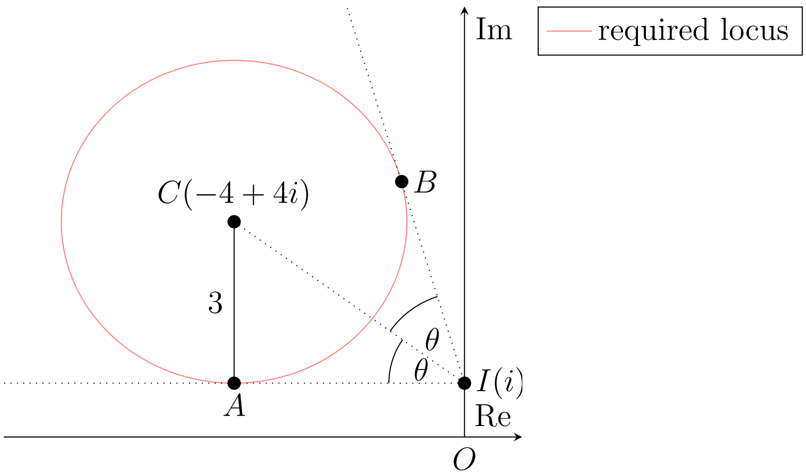
(b) Find the least possible value of $|z-i|$.

(c) Find the range of values of $\arg(z-i)$.

Solution

Part (a)

Note $|z+4-4i| = 3 \implies |z-(-4+4i)| = 3$.



Part (b)

Observe that the distance CI is equal to the sum of the radius of the circle and $\min |z-i|$. Hence,

$$\min |z-i| = \sqrt{(-4-0)^2 + (4-1)^2} - 3 = 2$$

$$\boxed{\min |z-i| = 2}$$

Part (c)

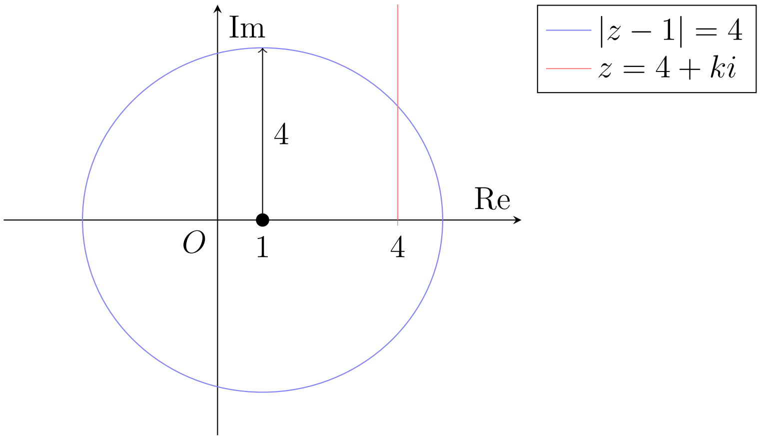
Let A and B be points on the circle such that AI and BI are tangent to the circle. Let $\angle CIA = \theta$. Then $\tan\theta = \frac{3}{4} \implies \theta = \arctan\frac{3}{4}$. By symmetry, we also have $\angle CIB = \theta$, whence $\angle AIB = 2\theta = 2\arctan\frac{3}{4}$. Hence, $\min \arg(z-i) = \pi - 2\arctan\frac{3}{4}$ (at B) and $\max \arg(z-i) = \pi$ (at A).

$$\boxed{\pi - 2\arctan\frac{3}{4} \le \arg(z-i) \le \pi}$$

Problem 7.

Sketch, on the same Argand diagram, the two loci representing the complex number z for which $z = 4 + ki$, where k is a positive real variable, and $|z - 1| = 4$. Write down, in the form $x + iy$, the complex number satisfying both conditions.

Solution



Note that z is of the form $4 + ki$, $k \in \mathbb{R}^+$. Since $|z - 1| = 4$, we have $|3 + ki| = 4 \implies 3^2 + k^2 = 4 \implies k = \sqrt{7}$. Note that we reject $k = -\sqrt{7}$ since $k > 0$.

$$\boxed{z = 4 + \sqrt{7}i}$$

Problem 8.

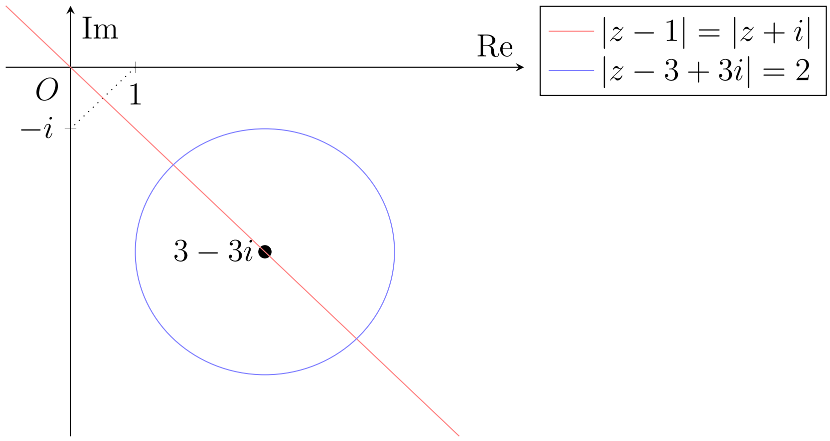
Describe, in geometrical terms, the loci given by $|z-1| = |z+i|$ and $|z-3+3i| = 2$ and sketch both loci on the same diagram.

Obtain, in the form $a+ib$, the complex numbers representing the points of intersection of the loci, giving the exact values of a and b.

Solution

Note that $|z-1| = |z+i| \implies |z-1| = |z-(-i)|$ and $|z-3+3i| = 2 \implies |z-(3-3i)| = 2$.

$$\boxed{\begin{array}{c}\text{The locus given by } |z-1| = |z+i| \text{ is the perpendicular bisector of the line segment joining } 1 \text{ and } -i. \\ \text{The locus given by } |z-3+3i| = 2 \text{ is a circle with centre } 3-3i \text{ and radius } 2.\end{array}}$$



Observe that the locus of $|z-1| = |z+i|$ has Cartesian equation $y = -x$ and the locus of $|z-3+3i| = 2$ has Cartesian equation $(x-3)^2 + (y+3)^2 = 2^2$. Solving both equations simultaneously, we have

$$\begin{aligned}
& & (x-3)^2 + (y+3)^2 &= 2^2 \\
&\implies & (x-3)^2 + (3-x)^2 &= 4 \\
&\implies & x^2 - 6x + 9 + 9 - 6x + x^2 &= 4 \\
&\implies & 2x^2 - 12x + 14 &= 0 \\
&\implies & x^2 - 6x + 7 &= 0 \\
&\implies & x &= \frac{6 \pm \sqrt{8}}{2} \\
& & &= 3 \pm \sqrt{2} \\
&\implies & y &= -(3 \pm \sqrt{2}) \\
& & &= -3 \mp \sqrt{2}
\end{aligned}$$

Hence, the complex numbers representing the points of intersections of the loci are $(3+\sqrt{2}) + (-3-\sqrt{2})i$ and $(3-\sqrt{2}) + (-3+\sqrt{2})i$.

$$\boxed{(3+\sqrt{2}) + (-3-\sqrt{2})i,\ (3-\sqrt{2}) + (-3+\sqrt{2})i}$$

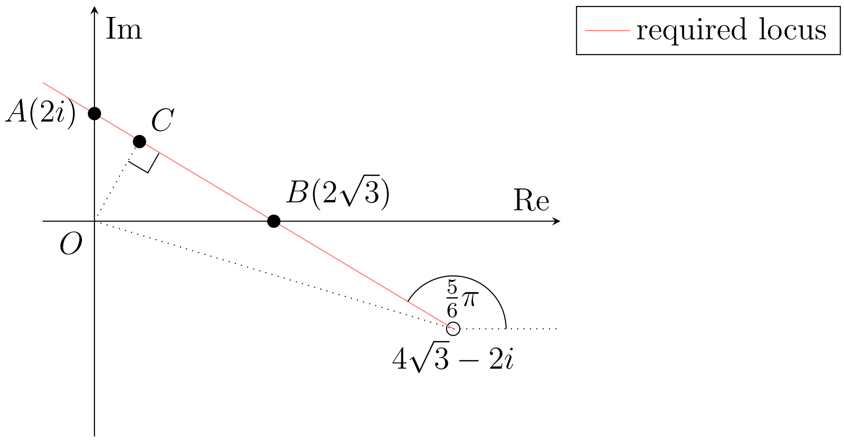
Problem 9.

Sketch the locus for $\arg\big(z - (4\sqrt{3} - 2i)\big) = \dfrac{5}{6}\pi$ in an Argand diagram.

(a) Verify that the points $2i$ and $2\sqrt{3}$ lie on it.

(b) Find the minimum value of $|z|$ and the range of values of $\arg(z)$.

Solution



Part (a)

$$\arg\Big(2i - (4\sqrt{3} - 2i)\Big) = \arg\Big(-\sqrt{3} + i\Big) = \arctan\frac{1}{-\sqrt{3}} = \frac{5}{6}\pi$$

$$\arg\Big(2\sqrt{3} - (4\sqrt{3} - 2i)\Big) = \arg\Big(-\sqrt{3} + i\Big) = \arctan\frac{1}{-\sqrt{3}} = \frac{5}{6}\pi$$

Hence, the points $2i$ and $2\sqrt{3}$ satisfy the equation $\arg\big(z - (4\sqrt{3} - 2i)\big) = \dfrac{5}{6}\pi$ and thus lie on the locus.

Part (b)

Let $A(2i)$ and $B(2\sqrt{3})$. Let C be the point on the required locus such that $OC \perp AB$. Observe that $\triangle OAB$, $\triangle COB$ and $\triangle CAO$ are all similar to one another. Hence,

$$\frac{OC}{CB} = \frac{AO}{BO} = \frac{1}{\sqrt{3}} \implies AC = \frac{1}{\sqrt{3}}OC$$

$$\frac{OC}{CA} = \frac{BO}{OA} = \frac{\sqrt{3}}{1} \implies BC = \sqrt{3}OC$$

Hence, $AB = AC + CB = \left(\sqrt{3} + \dfrac{1}{\sqrt{3}}\right) OC \implies \min|z| = OC = \dfrac{AB}{\sqrt{3} + 1/\sqrt{3}} = \dfrac{\sqrt{2^2 + (2\sqrt{3})^2}}{\sqrt{3} + 1\sqrt{3}} = \dfrac{4\sqrt{3}}{4} = \sqrt{3}.$

$$\boxed{\min|z| = \sqrt{3}}$$

Observe that $\max \arg(z) = \frac{5}{6}\pi$ and $\min \arg(z) = \min \arg\left(4\sqrt{3} - 2i\right) = \arctan \frac{-2}{4\sqrt{3}} = -\arctan \frac{1}{2\sqrt{3}}$.

$$\boxed{-\arctan \frac{1}{2\sqrt{3}} < \arg(z) \leq \frac{5}{6}\pi}$$

Problem 10.

The complex number z satisfies $|z-3-3i| \geq |z-1-i|$ and $\dfrac{\pi}{6} < \arg(z) \leq \dfrac{\pi}{3}$.

(a) On an Argand diagram, sketch the region in which the point representing z can lie.

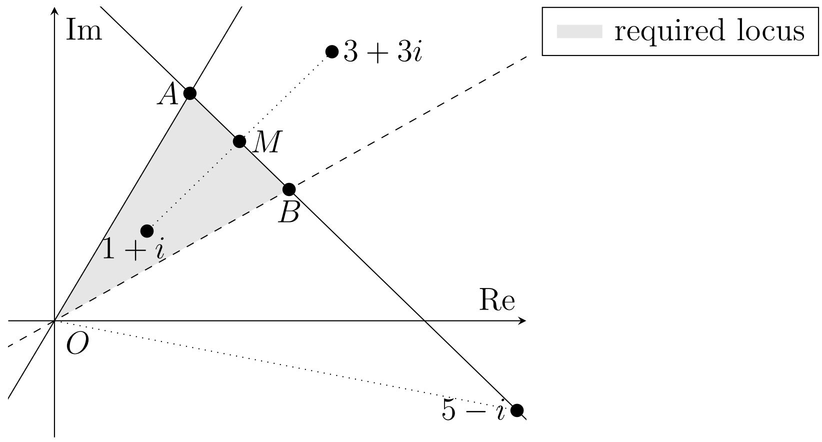
(b) Find the area of the region in part (a).

(c) Find the range of values of $\arg(z-5+i)$.

Solution

Part (a)

Note that $|z-3-3i| \leq |z-1-i| \implies |z-(3+3i)| \leq |z-(1+i)|$.



Part (b)

Note that the locus of $|z-3-3i| = |z-1-i|$ has Cartesian equation $y=-x+4$, while the loci of $\dfrac{\pi}{6} = \arg(z)$ and $\arg(z) = \dfrac{\pi}{3}$ have Cartesian equations $y = \dfrac{1}{\sqrt{3}}x$ and $y=\sqrt{3}x$ respectively. Let A and B be the intersections between $y=-x+4$ with $y=\sqrt{3}x$ and $y=\dfrac{1}{\sqrt{3}}x$ respectively.

At A, we have $y=\sqrt{3}x=-x+4$, whence $A\left(\dfrac{4}{1+\sqrt{3}}, \dfrac{4\sqrt{3}}{1+\sqrt{3}}\right)$. At B, we have $y=\dfrac{1}{\sqrt{3}}x=-x+4$, whence $B\left(\dfrac{4\sqrt{3}}{1+\sqrt{3}}, \dfrac{4}{1+\sqrt{3}}\right)$. Observe that $M\,(2,2)$ is the midpoint of AB. Then the required area is given by $\dfrac{1}{2}\cdot AB\cdot OM$.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\cdot AB\cdot OM \\ &= \frac{1}{2}\cdot\sqrt{\left(\frac{4}{1+\sqrt{3}}-\frac{4\sqrt{3}}{1+\sqrt{3}}\right)^2+\left(\frac{4\sqrt{3}}{1+\sqrt{3}}-\frac{4}{1+\sqrt{3}}\right)^2}\cdot\sqrt{2^2+2^2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \sqrt{2\left(\frac{4}{1+\sqrt{3}} - \frac{4\sqrt{3}}{1+\sqrt{3}}\right)^2} \cdot 2\sqrt{2} \\
&= 2 \cdot \left|\frac{4}{1+\sqrt{3}} - \frac{4\sqrt{3}}{1+\sqrt{3}}\right| \\
&= 8 \cdot \left|\frac{1-\sqrt{3}}{1+\sqrt{3}}\right| \\
&= 8 \cdot \left|\frac{(1-\sqrt{3})^2}{(1+\sqrt{3})(1-\sqrt{3})}\right| \\
&= 8 \cdot \left|\frac{(1-\sqrt{3})^2}{-2}\right| \\
&= 4(1-\sqrt{3})^2
\end{aligned}
$$

$$\boxed{\text{The area of the region is } 4(1-\sqrt{3})^2 \text{ units}^2.}$$

Part (c)

Note that $\arg(z-5+i) = \arg(z-(5-i))$. Observe that $\min \arg(z-(5-i)) = \frac{3}{4}\pi$ and $\max \arg(z-(5-i)) = \arctan \frac{-1}{5} + \pi = \pi - \arctan \frac{1}{5}$.

$$\boxed{\frac{3}{4}\pi \leq \arg(z-5+i) < \pi - \arctan \frac{1}{5}}$$

Problem 11.

Sketch on an Argand diagram the set of points representing all complex numbers z satisfying both inequalities

$$|iz - 2i - 2| \leq 2 \qquad \text{and} \qquad \operatorname{Re}(z) > \left|1 + \sqrt{3}i\right|$$

Find

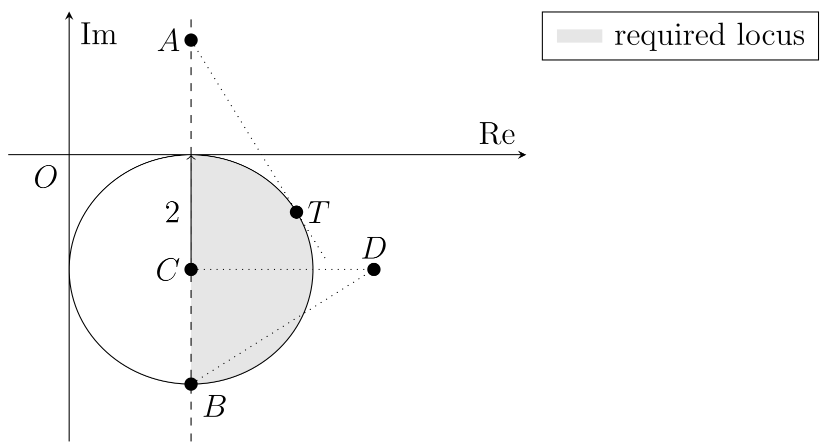
(a) the range of $\arg(z - 2 - 2i)$,

(b) the complex number z where $\arg(z - 2 - 2i)$ is a maximum.

The locus of the complex number w is defined by $|w - 5 + 2i| = k$, where k is a real and positive constant. Find the range of values of k such that the loci of w and z will intersect.

Solution

Note $|iz - 2i - 2| \leq 2 \implies |i(z - 2 + 2i)| \leq 2 \implies |z - (2 - 2i)| \leq 2$ and $\operatorname{Re}(z) > \left|1 + \sqrt{3}i\right| = 2$.



Part (a)

Note $|z - 2 - 2i| = \arg(z - (2 + 2i))$. Let $A(2 + 2i)$ and $C(2 - 2i)$. Let T be the point at which AT is tangent to the circle. Then $\angle ATC = \frac{\pi}{2}$, $AC = 4$ and $TC = 2$. Hence, $\angle CAT = \arcsin \frac{2}{4} = \frac{\pi}{6}$. Thus, min $\arg(z - 2 - 2i) = -\frac{\pi}{2}$ and max $\arg(z - 2 - 2i) =$ min $\arg(z - 2 - 2i) + \angle CAT = -\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$.

$$\boxed{-\frac{\pi}{2} < \arg(z - 2 - 2i) \leq -\frac{\pi}{3}}$$

Part (b)

Relative to C, T is given by $2\left(\cos\frac{\pi}{6} + i \sin\frac{\pi}{6}\right) = \sqrt{3} + i$. Thus, $T = (\sqrt{3} + i) + (2 - 2i) = 2 + \sqrt{3} - i$.

$$\boxed{2 + \sqrt{3} - i}$$

Note $|w-5+2i|=k \implies |w-(5-2i)|=k$. Let $D(5-2i)$. Observe that CD is given by the sum of the radius of the circle and $\min k$. Hence, $\min k = 3-2=1$. Let $B(2-4i)$. Then $\max k$ is given by the distance between B and D. By the Pythagorean Theorem, we have $\max k = \sqrt{(5-2)^2+(-2-(-4))^2} = \sqrt{13}$.

$$\boxed{1 \leq k \leq \sqrt{13}}$$