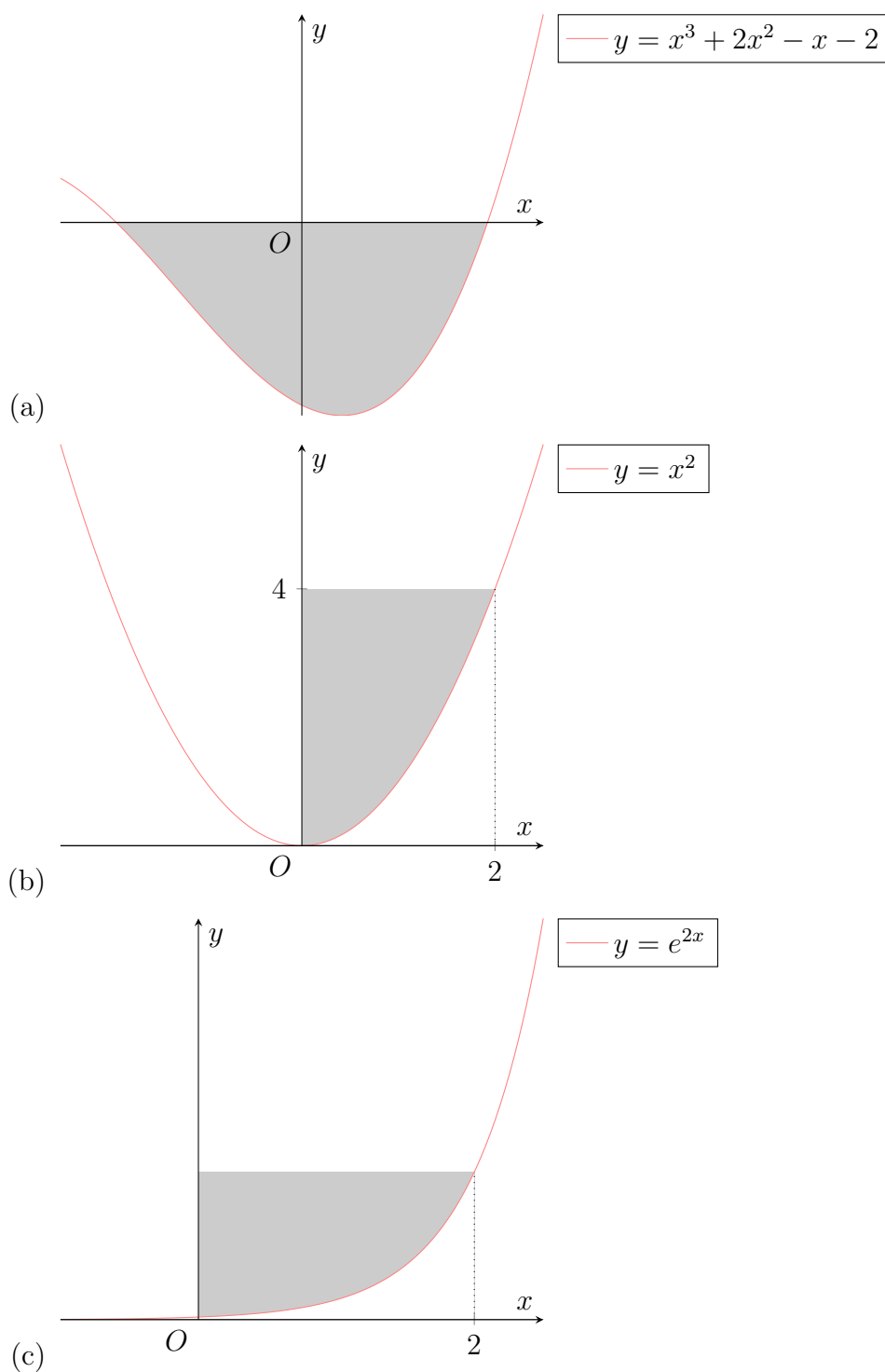
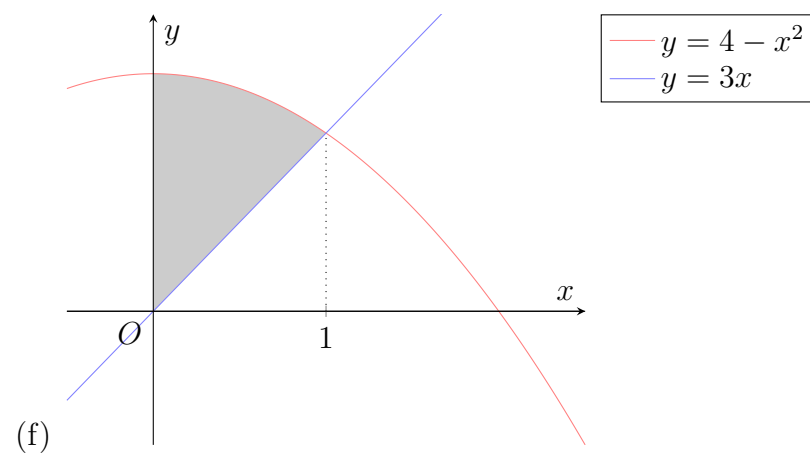
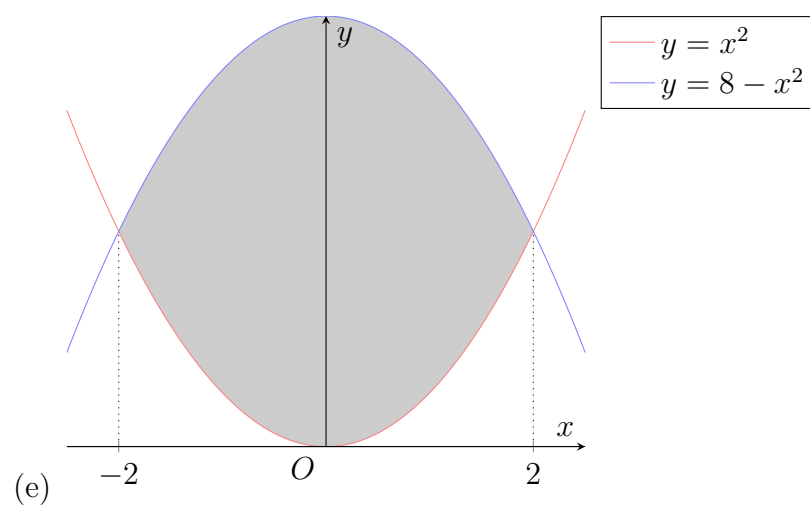
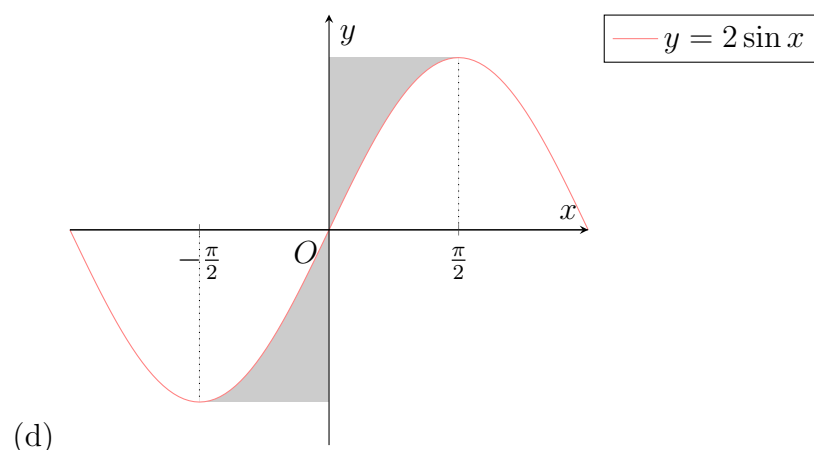
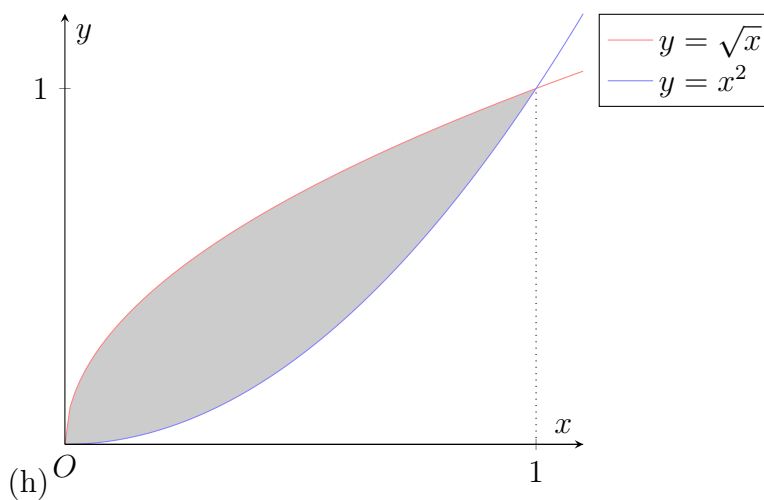
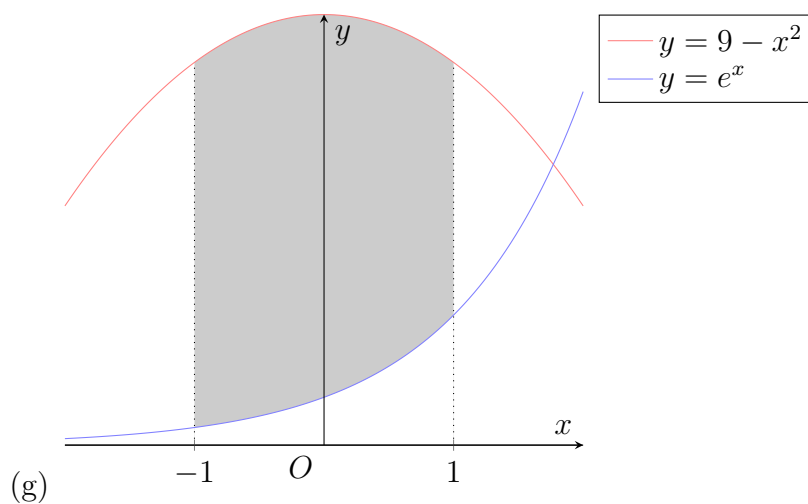


Problem 1.

Write down the integral for the area of the shaded region for each of the figure below and use the GC to evaluate it, to 3 significant figures.







Solution

Part (a)

$$\begin{aligned} \text{Area} &= - \int_{-1}^1 (x^3 + 2x^2 - x - 2) \, dx \\ &= 2.67 \text{ (3 s.f.)} \end{aligned}$$

The area of the shaded region is 2.67 units².

Part (b)

Note that $y = x^2 \implies x = \sqrt{y}$.

$$\begin{aligned} \text{Area} &= \int_0^4 \sqrt{y} \, dy \\ &= 5.33 \text{ (3 s.f.)} \end{aligned}$$

The area of the shaded region is 5.33 units².

Part (c)

Note that $y = e^{2x} \implies x = \frac{1}{2} \ln y$. Also, when $x = 0$, we have $y = 1$. Further, when $x = 2$, we have $y = e^4$.

$$\begin{aligned}\text{Area} &= \int_0^{e^4} \frac{1}{2} \ln y \, dy \\ &= 82.4 \text{ (3 s.f.)}\end{aligned}$$

The area of the shaded region is 82.4 units².**Part (d)**

Note that when $x = \frac{\pi}{2}$, we have $y = 2$.

$$\begin{aligned}\text{Area} &= 2 \int_0^2 \arcsin \frac{y}{2} \, dy \\ &= 2.28 \text{ (3 s.f.)}\end{aligned}$$

The area of the shaded region is 2.28 units².**Part (e)**

$$\begin{aligned}\text{Area} &= \int_{-2}^2 ((8 - x^2) - x^2) \, dx \\ &= 21.3 \text{ (3 s.f.)}\end{aligned}$$

The area of the shaded region is 21.3 units².**Part (f)**

$$\begin{aligned}\text{Area} &= \int_0^1 ((4 - x^2) - 3x) \, dx \\ &= 2.17 \text{ (3 s.f.)}\end{aligned}$$

The area of the shaded region is 2.17 units².**Part (g)**

$$\begin{aligned}\text{Area} &= \int_{-1}^1 ((9 - x^2) - e^x) \, dx \\ &= 15.0 \text{ (3 s.f.)}\end{aligned}$$

The area of the shaded region is 15.0 units².

Part (h)

$$\begin{aligned}\text{Area} &= \int_0^1 (\sqrt{x} - x^2) \, dx \\ &= 0.333 \text{ (3 s.f.)}\end{aligned}$$

The area of the shaded region is 0.333 units².

Problem 2.

- (a) Write down the integral for the volume of the solid generated when the shaded region is rotated about the x -axis through 2π for questions 1(a), (e), (f) and (h) using the disc method and use the GC to evaluate it.
- (b) Write down the integral for the volume of the solid generated when the shaded region is rotated about the y -axis through 2π for questions 1(b), (d) and (f) using the disc method and use the GC to evaluate it.

Solution**Part (a)****Subpart (i)**

$$\begin{aligned}\text{Volume} &= \pi \int_{-1}^1 (x^3 + 2x^2 - x - 2)^2 dx \\ &= 13.9 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 13.9 units³.

Subpart (ii)

$$\begin{aligned}\text{Volume} &= \pi \int_{-2}^2 ((8 - x^2)^2 - x^2) dx \\ &= 536 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 536 units³.

Subpart (iii)

$$\begin{aligned}\text{Volume} &= \pi \int_0^1 ((4 - x^2)^2 - (3x)^2) dx \\ &= 33.1 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 33.1 units³.

Subpart (iv)

$$\begin{aligned}\text{Volume} &= \pi \int_0^1 ((\sqrt{x})^2 - (x^2)^2) dx \\ &= 0.942 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 0.942 units³.

Part (b)**Subpart (i)**

$$\begin{aligned}\text{Volume} &= \pi \int_0^4 (\sqrt{y})^2 \, dy \\ &= 25.1 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 25.1 units³.

Subpart (ii)

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^2 \arcsin^2 \frac{y}{2} \, dy \\ &= 5.87 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 5.87 units³.

Subpart (iii)

$$\begin{aligned}\text{Volume} &= \pi \int_3^4 (4 - y) \, dy + \frac{1}{3}\pi \cdot 1^2 \cdot 3 \\ &= 4.71 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 4.71 units³.

Problem 3.

- (a) Write down the integral for the volume of the solid generated when the shaded region is rotated about the x -axis through 2π for questions 1(e), (f) and (h) using the shell method and use the GC to evaluate it.
- (b) Write down the integral for the volume of the solid generated when the shaded region is rotated about the y -axis through 2π for questions 1(b), (d) and (f) using the shell method and use the GC to evaluate it.

Solution**Part (a)****Subpart (i)**

Note that $y = x^2 \implies x = \sqrt{y}$ and $y = 8 - x^2 \implies x = \sqrt{8 - y}$ for $x > 0$.

$$\begin{aligned}\text{Volume} &= 2 \left(2\pi \int_0^4 \sqrt{y} \cdot y \, dy + 2\pi \int_4^8 \sqrt{8 - y} \cdot y \, dy \right) \\ &= 536 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 536 units³.

Subpart (ii)

Note that $y = 3x \implies x = \frac{1}{3}y$ and $y = 4 - x^2 \implies x = \sqrt{4 - y}$ for $x > 0$.

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^3 \frac{1}{3}y \cdot y \, dy + 2\pi \int_3^4 \sqrt{4 - y} \cdot y \, dy \\ &= 33.1 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 33.1 units³.

Subpart (iii)

Note that $y = \sqrt{x} \implies x = y^2$ and $y = x^2 \implies x = \sqrt{y}$ for $x > 0$.

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^1 (\sqrt{y} - y^2) y \, dy \\ &= 0.942 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 0.942 units³.

Part (b)**Subpart (i)**

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^2 x \cdot x^2 \, dx \\ &= 25.1 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 25.1 units³.

Subpart (ii)

$$\begin{aligned}\text{Volume} &= 2 \cdot 2\pi \int_0^{\pi/2} x (2 - 2 \sin x) \, dx \\ &= 5.87 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 5.87 units³.

Subpart (iii)

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^1 x ((4 - x^2) - 3x) \, dx \\ &= 4.71 \text{ (3 s.f.)}\end{aligned}$$

The volume of the solid is 4.71 units³.

Problem 4.

Calculate the area enclosed by the petals of the curve $r = \sin 2\theta$ where $r \geq 0$.

Solution

Note that $r \geq 0 \implies \sin 2\theta \geq 0 \implies r \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$.

$$\text{Area} = 2 \cdot \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$\begin{aligned} &= \frac{1}{2} \int_0^{\pi} \sin^2 u \, du \\ &= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2u}{2} \, du \\ &= \frac{1}{4} \int_0^{\pi} (1 - \cos 2u) \, du \\ &= \frac{1}{4} \left[u - \frac{\sin 2u}{2} \right]_0^{\pi} \\ &= \frac{\pi}{4} \end{aligned}$$

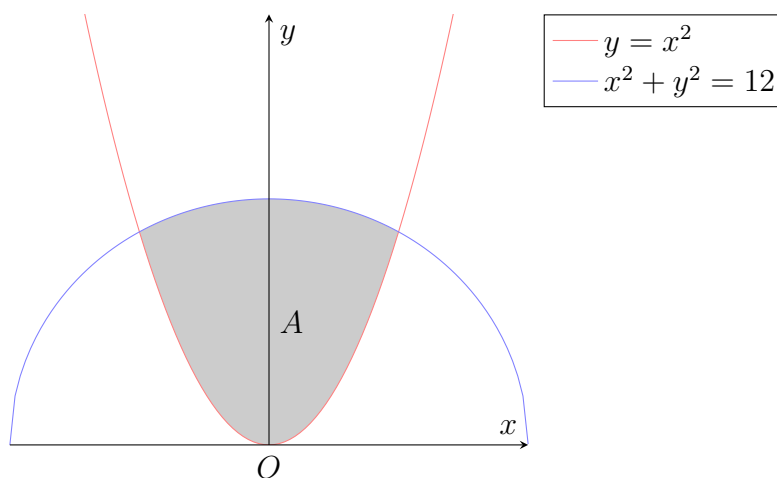
The area enclosed is $\frac{\pi}{4}$ units².

$$\begin{aligned} u &= 2\theta \\ du &= 2 \, d\theta \\ \theta = 0 &\implies u = 0 \\ \theta = \frac{\pi}{2} &\implies u = \pi \end{aligned}$$

Problem 5.

The finite region A is bounded by the curve $y = x^2$ and a minor arc of the circle $x^2 + y^2 = 12$.

- Find the numerical value of the area of A , correct to 2 decimal places.
- Find the exact volume of the solid obtained when A is rotated about the x -axis through 2π radians.
- Find the exact volume of the solid obtained when A is rotated about the y -axis through π radians.

Solution**Part (a)**

Consider the intersections between $y = x^2$ and $x^2 + y^2 = 12$.

$$\begin{aligned}
 & x^2 + y^2 = 12 \\
 \Rightarrow & x^2 + (x^2)^2 = 12 \\
 \Rightarrow & x^4 + x^2 - 12 = 0 \\
 \Rightarrow & (x^2 - 3)(x^2 + 4) = 0 \\
 \Rightarrow & (x - \sqrt{3})(x + \sqrt{3})(x^2 + 4) = 0
 \end{aligned}$$

Hence, the two curves intersect at $x = -\sqrt{3}$ and $x = \sqrt{3}$. Note that $x^2 + 4 = 0$ has no solution since $x^2 + 4 > 0$. Also note that $x^2 + y^2 = 12 \Rightarrow y = \sqrt{12 - x^2}$ for $y > 0$.

$$\begin{aligned}
 \text{Area} &= 2 \int_0^{\sqrt{3}} (\sqrt{12 - x^2} - x^2) dx \\
 &= 8.02 \text{ (3 s.f.)}
 \end{aligned}$$

A has an area of 8.02 units².

Part (b)

Note that $x^2 + y^2 = 12 \implies y^2 = 12 - x^2$.

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^{\sqrt{3}} \left((12 - x^2) - (x^2)^2 \right) dx \\ &= 2\pi \left[12x - \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^{\sqrt{3}} \\ &= 2\pi \cdot \frac{46\sqrt{3}}{5} \\ &= \frac{92\sqrt{3}\pi}{5}\end{aligned}$$

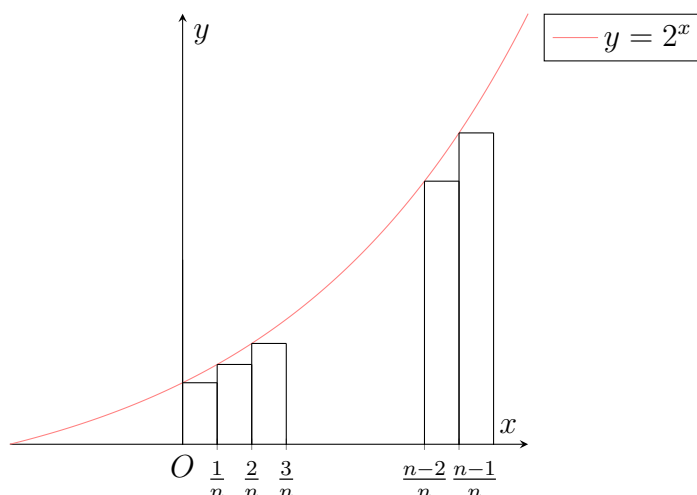
The solid has a volume of $\frac{92\sqrt{3}\pi}{5}$ units³.

Part (c)

Note that when the curves intersect at $x = \sqrt{3}$, we have $y = 3$. Furthermore, when $x = 0$, we have $y = \sqrt{12}$. Also note that $x^2 + y^2 = 12 \implies x^2 = 12 - y^2$.

$$\begin{aligned}\text{Volume} &= \pi \int_0^3 y \, dy + \pi \int_3^{\sqrt{12}} (12 - y^2) \, dy \\ &= \pi \left(16\sqrt{3} - \frac{45}{2} \right)\end{aligned}$$

The solid has a volume of $\pi \left(16\sqrt{3} - \frac{45}{2} \right)$ units³.

Problem 6.

- (a) The graph of $y = 2^x$, for $0 \leq x \leq 1$ is shown in the diagram. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve. Given that $\sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x}$, show that the total area A of all n rectangles is given by $\frac{1}{n} \left(\frac{1}{2^{\frac{1}{n}} - 1} \right)$.

- (b) Find the limit of A in exact form as $n \rightarrow \infty$.

Let V be the volume of all n rectangles rotated about the x -axis.

- (c) Find V in terms of n .
- (d) State the limit of V in exact form as $n \rightarrow \infty$.

Solution**Part (a)**

$$\begin{aligned}
 A &= \sum_{k=0}^{n-1} \frac{2^{k/n}}{n} \\
 &= \frac{1}{n} \sum_{k=0}^{n-1} (2^{1/n})^k \\
 &= \frac{1}{n} \cdot \frac{1 - (2^{1/n})^{n-1+1}}{1 - 2^{1/n}} \\
 &= \frac{1}{n} \cdot \frac{1 - 2}{1 - 2^{1/n}} \\
 &= \frac{1}{n} \cdot \frac{1}{2^{1/n} - 1}
 \end{aligned}$$

Part (b)

$$\begin{aligned}
\lim_{n \rightarrow \infty} A &= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{1}{2^{1/n} - 1} \\
&= \lim_{n \rightarrow \infty} \frac{1/n}{2^{1/n} - 1} \\
&= \lim_{m \rightarrow 0} \frac{m}{2^m - 1} \\
&= \lim_{m \rightarrow 0} \frac{1}{\ln 2 \cdot 2^m} \\
&= \frac{1}{\ln 2}
\end{aligned}$$

$$\boxed{\lim_{n \rightarrow \infty} A = \frac{1}{\ln 2}}$$

Part (c)

$$\begin{aligned}
V &= \pi \sum_{k=0}^{n-1} (2^{k/n})^2 \cdot \frac{1}{n} \\
&= \frac{\pi}{n} \sum_{k=0}^{n-1} (2^{2/n})^k \\
&= \frac{\pi}{n} \cdot \frac{1 - (2^{2/n})^{n-1+1}}{1 - 2^{2/n}} \\
&= \frac{\pi}{n} \cdot \frac{1 - 4}{1 - 2^{2/n}} \\
&= \frac{\pi}{n} \cdot \frac{3}{2^{2/n} - 1} \\
&= \frac{\pi}{n} \cdot \frac{3}{4^{1/n} - 1} \\
&= \frac{3\pi}{n(4^{1/n} - 1)}
\end{aligned}$$

$$\boxed{V = \frac{3\pi}{n(4^{1/n} - 1)}}$$

Part (d)

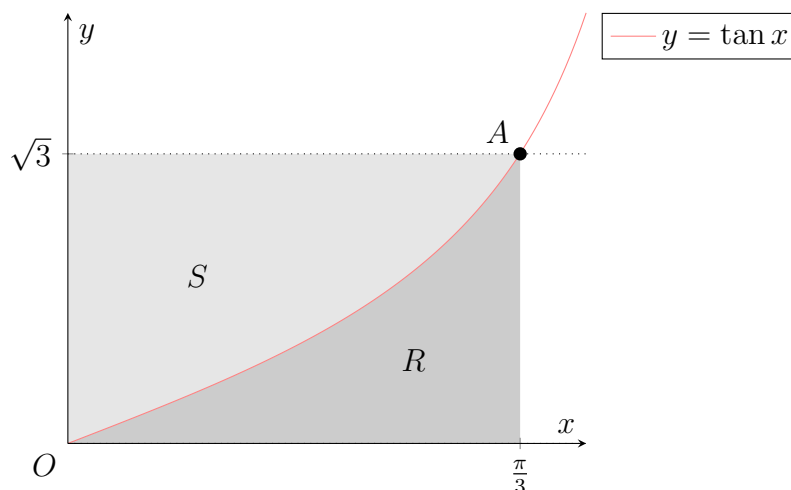
$$\begin{aligned}\lim_{n \rightarrow \infty} V &= \lim_{n \rightarrow \infty} \frac{3\pi}{n(4^{1/n} - 1)} \\&= 3\pi \lim_{n \rightarrow \infty} \frac{1}{n(4^{1/n} - 1)} \\&= 3\pi \lim_{n \rightarrow \infty} \frac{1/n}{4^{1/n} - 1} \\&= 3\pi \lim_{m \rightarrow 0} \frac{m}{4^m - 1} \\&= 3\pi \lim_{m \rightarrow 0} \frac{1}{\ln 4 \cdot 4^m} \\&= 3\pi \cdot \frac{1}{\ln 4} \\&= \frac{3\pi}{2 \ln 2}\end{aligned}$$

$\lim_{n \rightarrow \infty} V = \frac{3\pi}{2 \ln 2}$
--

Problem 7.

O is the origin and A is the point on the curve $y = \tan x$ where $x = \frac{1}{3}\pi$.

- (a) Calculate the area of the region R enclosed by the arc OA , the x -axis and the line $x = \frac{1}{3}\pi$, giving your answer in an exact form.
- (b) The region S is enclosed by the arc OA , the y -axis and the line $y = \sqrt{3}$. Find the volume of the solid of revolution formed when S is rotated through 360° about the x -axis, giving your answer in an exact form.
- (c) Find $\int_0^{\sqrt{3}} \arctan y \, dy$ in exact form.

Solution**Part (a)**

$$\begin{aligned} \text{Area} &= \int_0^{\pi/3} \tan x \, dx \\ &= [\ln \sec x]_0^{\pi/3} \\ &= \ln 2 \end{aligned}$$

R has an area of $\ln 2$ units².

Part (b)

$$\begin{aligned} \text{Volume} &= \pi \int_0^{\pi/3} \left((\sqrt{3})^2 - \tan^2 x \right) dx \\ &= \pi \int_0^{\pi/3} (3 - \sec^2 x + 1) dx \\ &= \pi [4x - \tan x]_0^{\pi/3} \\ &= \frac{4\pi^2}{3} - \sqrt{3}\pi \end{aligned}$$

The solid has a volume of $\left(\frac{4\pi^2}{3} - \sqrt{3}\pi\right)$ units³.

Part (c)

	D	I
+	$\arctan y$	1
−	$\frac{1}{1+y^2}$	y

$$\int_0^{\sqrt{3}} \arctan y \, dy = [y \arctan y]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{y}{1+y^2} \, dy$$

$$= \frac{\pi}{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{y}{1+y^2} \, dy$$

$$= \frac{\pi}{\sqrt{3}} - \frac{1}{2} \int_0^3 \frac{1}{1+u} \, du$$

$$= \frac{\pi}{\sqrt{3}} - \frac{1}{2} [\ln(1+u)]_0^3$$

$$= \frac{\pi}{\sqrt{3}} - \ln 2$$

$$\int_0^{\sqrt{3}} \arctan y \, dy = \frac{\pi}{\sqrt{3}} - \ln 2$$

$$\begin{aligned} u &= y^2 \\ du &= 2y \, dy \\ y = 0 &\implies u = 0 \\ y = \sqrt{3} &\implies u = 3 \end{aligned}$$

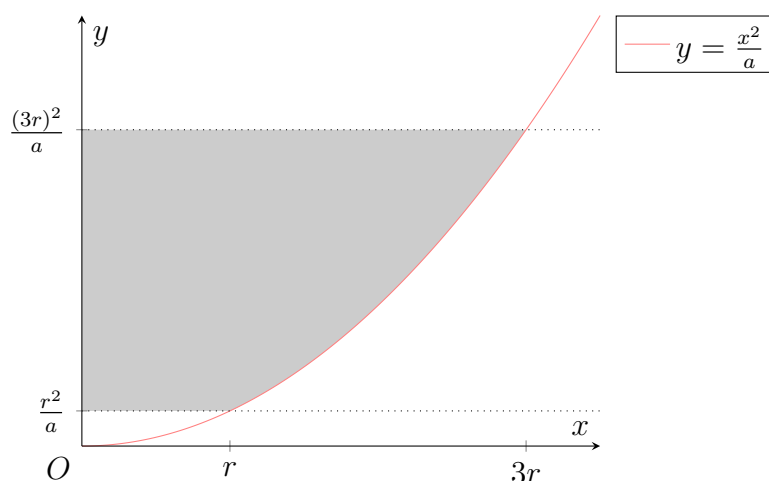
Problem 8.

A portion of the curve $ay = x^2$, where a is a positive constant, is rotated about the vertical axis Oy to form the curved surface of an open bowl. The bowl has a horizontal circular base of radius r and a horizontal circular rim of radius $3r$.

- (a) Prove that the depth of the bowl is $\frac{8r^2}{a}$.
- (b) Find the volume of the bowl in terms of r and a .
- (c) Given that the volume of the bowl is $\frac{\pi a^3}{10}$, find the depth of the bowl in terms of a only.

Solution

Note that $ay = x^2 \implies y = \frac{x^2}{a}$.

**Part (a)**

$$\begin{aligned}
 \text{Depth of bowl} &= \frac{(3r)^2}{a} - \frac{r^2}{a} \\
 &= \frac{9r^2}{a} - \frac{r^2}{a} \\
 &= \frac{8r^2}{a}
 \end{aligned}$$

Part (b)

$$\begin{aligned}
 \text{Volume} &= \pi \int_{r^2/a}^{9r^2/a} ay \, dy \\
 &= \pi \left[\frac{a}{2} y^2 \right]_{r^2/a}^{9r^2/a} \\
 &= \frac{a\pi}{2} \left(\left(\frac{9r^2}{a} \right)^2 - \left(\frac{r^2}{a} \right)^2 \right)
 \end{aligned}$$

$$\begin{aligned} &= \frac{a\pi}{2} \cdot \frac{80r^4}{a^2} \\ &= \frac{40\pi r^4}{a} \end{aligned}$$

The volume of the bowl is $\frac{40\pi r^4}{a}$ units³.

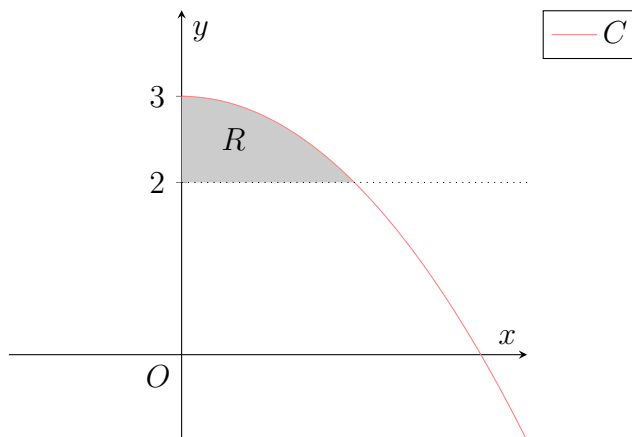
Part (c)

$$\begin{aligned} &\frac{40\pi r^4}{a} = \frac{\pi a^3}{10} \\ \Rightarrow &400r^4 = a^4 \\ \Rightarrow &20r^2 = a^2 \\ \Rightarrow &r^2 = \frac{1}{20}a^2 \\ \Rightarrow \text{Depth of bowl} &= \frac{8 \cdot \frac{1}{20}a^2}{a} \\ &= \frac{2}{5}a \end{aligned}$$

The depth of the bowl is $\frac{2}{5}a$ units.

Problem 9.

The diagram shows the region R bounded by part of the curve C with equation $y = 3 - x^2$, the y -axis and the line $y = 2$, lying in the first quadrant.



Write down the equation of the curve obtained when C is translated by 2 units in the negative y -direction.

Hence, or otherwise, show that the volume of the solid formed when R is rotated completely about the line $y = 2$ is given by $\pi \int_0^1 (1 - 2x^2 + x^4) dx$ and evaluate this integral exactly.

Solution

$$C : y = 1 - x^2$$

Note that $3 - x^2 = 2 \implies x = \pm 1$, whence $x = 1$ since $x > 0$.

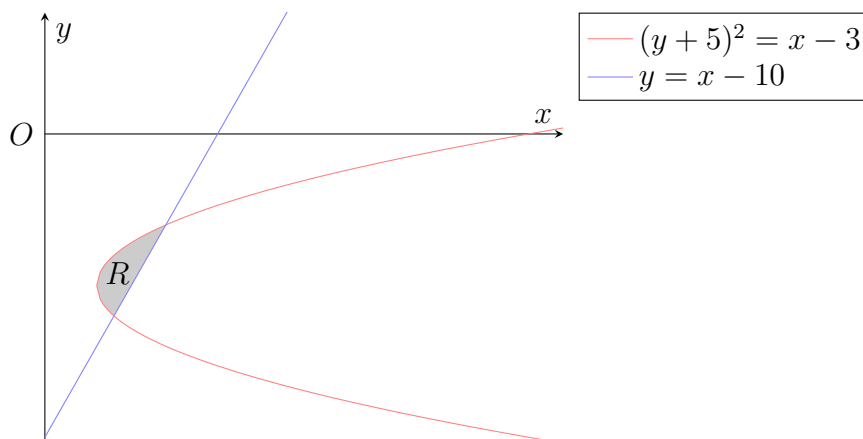
$$\begin{aligned} \text{Volume} &= \pi \int_0^1 (1 - x^2)^2 dx \\ &= \pi \int_0^1 (1 - 2x^2 + x^4) dx \\ &= \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_0^1 \\ &= \frac{8}{15}\pi \end{aligned}$$

$$\text{The volume of the solid is } \frac{8}{15}\pi \text{ units}^3.$$

Problem 10.

The diagram below shows a region R bounded by the curve $(y + 5)^2 = x - 3$ and the line $y = x - 10$. Find the volume of solid formed when R is rotated four right angles about

- (a) the y -axis, and
(b) the x -axis.

**Solution****Part (a)**

Consider the intersections between $(y + 5)^2 = x - 3$ and $y = x - 10$.

$$\begin{aligned}
 & (y + 5)^2 = x - 3 \\
 \implies & (x - 5)^2 = x - 3 \\
 \implies & x^2 - 10x + 25 = x - 3 \\
 \implies & x^2 - 11x + 28 = 0 \\
 \implies & (x - 4)(x - 7) = 0
 \end{aligned}$$

Hence, $x = 4$ and $x = 7$, whence $y = -6$ and $y = -3$. Thus, the two curves intersect at $(4, -6)$ and $(7, -3)$.

Note that $(y + 5)^2 = x - 3 \implies x = 3 + (y + 5)^2$ and $y = x - 10 \implies x = y + 10$.

$$\begin{aligned}
 \text{Volume} &= \pi \int_{-6}^{-3} \left((y + 10)^2 - (3 + (y + 5)^2)^2 \right) dy \\
 &= 130 \text{ (3 s.f.)}
 \end{aligned}$$

The volume of the solid is 130 units³.

Part (b)

Note that

$$(y + 5)^2 = x - 3 \implies \begin{cases} y = -5 + \sqrt{x - 3}, & y \geq -5 \\ y = -5 - \sqrt{x - 3}, & y < -5 \end{cases}$$

$$\begin{aligned}\text{Volume} &= \pi \int_3^4 \left((-5 - \sqrt{x-3})^2 - (-5 + \sqrt{x+3})^2 \right) dx \\ &\quad + \pi \int_4^7 \left((x-10)^2 - (-5 + \sqrt{x-3})^2 \right) dx \\ &= 127 \text{ (3 s.f.)}\end{aligned}$$

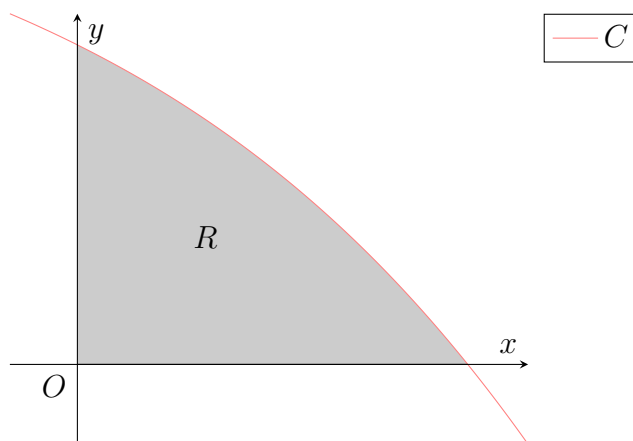
The volume of the solid is 127 units ³ .

Problem 11.

The curve C is defined by the following pair of parametric equations.

$$x = t - \frac{1}{t^2}, \quad y = 2 - t^2, \quad t > 0$$

Find the area of the finite region R enclosed by the curve C and the axes as well as the volume of solid obtained when R is rotated about the x -axis through 4 right-angles.

Solution

Note that when $x = 0 \implies t - \frac{1}{t^2} = 0 \implies t = 1$. Also note that when $y = 0 \implies 2 - t^2 \implies t = \sqrt{2}$, whence $x = \sqrt{2} - \frac{1}{2}$.

$$\begin{aligned} \text{Area} &= \int_0^{\sqrt{2}-\frac{1}{2}} y \, dx \\ &= \int_1^{\sqrt{2}} (2 - t^2) \frac{dx}{dt} \, dt \\ &= \int_1^{\sqrt{2}} (2 - t^2) \left(1 + \frac{2}{t^3}\right) \, dt \\ &= 0.526 \text{ (3 s.f.)} \end{aligned}$$

The area of R is 0.526 units².

$$\begin{aligned} \text{Volume} &= \pi \int_0^{\sqrt{2}-\frac{1}{2}} y^2 \, dx \\ &= \pi \int_1^{\sqrt{2}} (2 - t^2)^2 \frac{dx}{dt} \, dt \\ &= \pi \int_1^{\sqrt{2}} (2 - t^2)^2 \left(1 + \frac{2}{t^3}\right) \, dt \\ &= 1.19 \text{ (3 s.f.)} \end{aligned}$$

The volume of R is 1.19 units³.

Problem 12.

Find the area enclosed by the ellipse $x = a \cos t$, $y = b \sin t$, where a and b are positive constants. Find also the volume of solid obtained when the region enclosed by the ellipse is rotated through π radians about the x -axis.

Solution

By symmetry, we only need to consider the area of the ellipse in the first quadrant. Note that $x = 0 \implies t = \frac{\pi}{2}$ and $x = a \implies t = 0$.

$$\begin{aligned}
 \text{Area} &= 4 \int_0^a y \, dx \\
 &= 4 \int_{\pi/2}^0 y \cdot \frac{dx}{dt} \, dt \\
 &= 4 \int_{\pi/2}^0 (b \sin t)(-a \sin t) \, dt \\
 &= 4ab \int_0^{\pi/2} \sin^2 t \, dt \\
 &= 4ab \int_0^{\pi/2} \frac{1 - \cos 2t}{2} \, dt \\
 &= 4ab \left[\frac{1}{2}t - \frac{\sin 2t}{4} \right]_0^{\pi/2} \\
 &= \pi ab
 \end{aligned}$$

The area of the ellipse is πab units².

$$\begin{aligned}
 \text{Volume} &= 2\pi \int_0^a y^2 \, dx \\
 &= 2\pi \int_{\pi/2}^0 y^2 \cdot \frac{dx}{dt} \, dt \\
 &= 2\pi \int_{\pi/2}^0 (b \sin t)^2 (-a \sin t) \, dt \\
 &= 2\pi ab^2 \int_0^{\pi/2} \sin^3 t \, dt \\
 &= 2\pi ab^2 \int_0^{\pi/2} \frac{3 \sin t - \sin 3t}{4} \, dt \\
 &= 2\pi ab^2 \left[-\frac{3}{4} \cos t + \frac{1}{2} \cos 3t \right]_0^{\pi/2} \\
 &= \frac{4\pi}{3} ab^2
 \end{aligned}$$

The volume of the ellipse is $\frac{4\pi}{3} ab^2$ units³.

Problem 13.

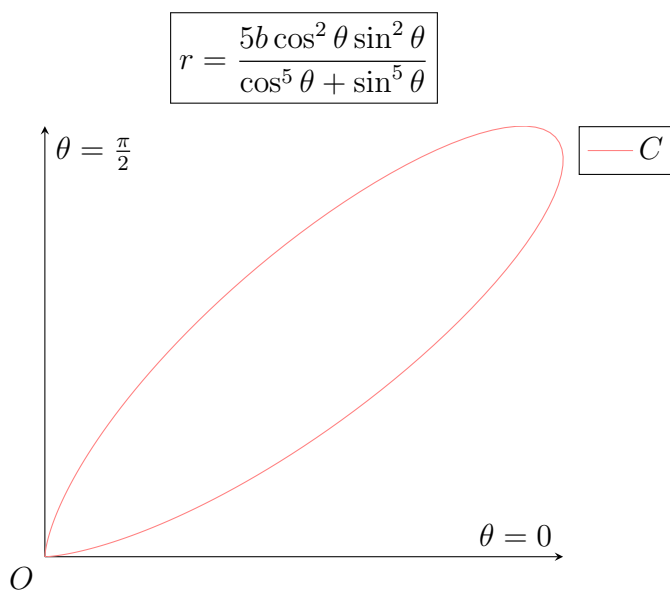
Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \leq \theta \leq \frac{\pi}{2}$. Show, using polar coordinates, that the area A of the region enclosed by this part of the curve is given by

$$A = \frac{25b^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^4 \theta \cos^4 \theta}{(\cos^5 \theta + \sin^5 \theta)^2} d\theta$$

By differentiating $\frac{1}{1 + \tan^5 \theta}$ with respect to θ , or otherwise, find the exact value of A in terms of b .

Solution

$$\begin{aligned} x^5 + y^5 &= 5bx^2y^2 \\ \Rightarrow (r \cos \theta)^5 + (r \sin \theta)^5 &= 5b(r \cos \theta)^2(r \sin \theta)^2 \\ \Rightarrow r^5 (\cos^5 \theta + \sin^5 \theta) &= 5br^4 \cos^2 \theta \sin^2 \theta \\ \Rightarrow r (\cos^5 \theta + \sin^5 \theta) &= 5b \cos^2 \theta \sin^2 \theta \\ \Rightarrow r &= \frac{5b \cos^2 \theta \sin^2 \theta}{\cos^5 \theta + \sin^5 \theta} \end{aligned}$$



$$\begin{aligned} A &= \frac{1}{2} \int_0^{\pi/2} \left(\frac{5b \cos^2 \theta \sin^2 \theta}{\cos^5 \theta + \sin^5 \theta} \right)^2 d\theta \\ &= \frac{1}{2} \int_0^{\pi/2} \frac{25b^2 \cos^4 \theta \sin^4 \theta}{(\cos^5 \theta + \sin^5 \theta)^2} d\theta \\ &= \frac{25b^2}{2} \int_0^{\pi/2} \frac{\cos^4 \theta \sin^4 \theta}{(\cos^5 \theta + \sin^5 \theta)^2} d\theta \end{aligned}$$

$$\begin{aligned}
\frac{d}{d\theta} \frac{1}{1 + \tan^5 \theta} &= -\frac{1}{(1 + \tan^5 \theta)^2} \cdot 5 \tan^4 \theta \cdot \sec^2 \theta \\
&= -5 \cdot \frac{1}{\left(1 + \frac{\sin^5 \theta}{\cos^5 \theta}\right)^2} \cdot \frac{\sin^4 \theta}{\cos^4 \theta} \cdot \frac{1}{\cos^2 \theta} \\
&= -5 \cdot \frac{(\cos^5 \theta)^2}{(\cos^5 \theta)^2 \left(1 + \frac{\sin^5 \theta}{\cos^5 \theta}\right)^2} \cdot \frac{\sin^4 \theta}{\cos^6 \theta} \\
&= -5 \cdot \frac{\cos^{10} \theta}{\cos^5 \theta + \sin^5 \theta} \cdot \frac{\sin^4 \theta}{\cos^6 \theta} \\
&= -\frac{5 \cos^4 \theta \sin^4 \theta}{\cos^5 \theta + \sin^5 \theta} \\
\Rightarrow A &= \frac{-5b^2}{2} \int_0^{\pi/2} -\frac{5 \cos^4 \theta \sin^4 \theta}{(\cos^5 \theta + \sin^5 \theta)^2} d\theta \\
&= -\frac{5b^2}{2} \left[\frac{1}{1 + \tan^5 \theta} \right]_0^{\pi/2} \\
&= -\frac{5b^2}{2} \cdot -1 \\
&= \frac{5b^2}{2}
\end{aligned}$$

$$A = \frac{5b^2}{2}$$

Problem 14.

The polar equation of a curve is given by $r = e^\theta$ where $0 \leq \theta \leq \frac{\pi}{2}$. Cartesian axes are taken at the pole O . Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(e^{\frac{\pi}{2}}, \frac{\pi}{2}\right)$. The region R is bounded by the polar curve, tangent and the x -axis. Find the exact area of the region R .

Solution

$$x = e^\theta \cos \theta, \quad y = e^\theta \sin \theta$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \cdot \frac{d\theta}{dx} \\ &= \frac{dy}{d\theta} \bigg/ \frac{dx}{d\theta} \\ &= \frac{e^\theta \cos \theta + e^\theta \sin \theta}{e^\theta(-\sin \theta) + e^\theta \cos \theta} \\ &= \frac{\cos \theta + \sin \theta}{\cos \theta - \sin \theta} \\ \Rightarrow \frac{dy}{dx} \bigg|_{\theta=\pi/2} &= \frac{\cos \frac{\pi}{2} + \sin \frac{\pi}{2}}{\cos \frac{\pi}{2} - \sin \frac{\pi}{2}} \\ &= -1 \end{aligned}$$

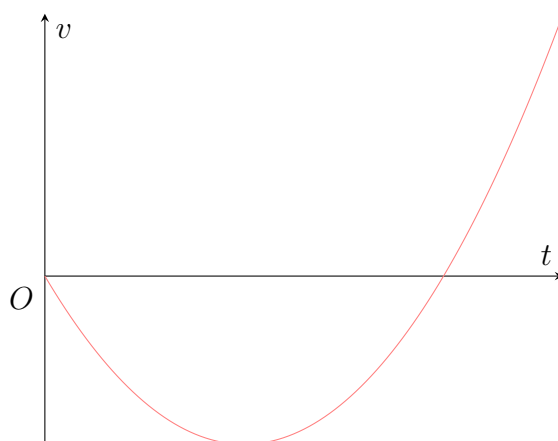
Note that when $\theta = \frac{\pi}{2}$, we have that $x = 0$ and $y = e^{\frac{\pi}{2}}$. Hence, by the point-slope formula, the tangent is given by

$$\begin{aligned} y - e^{\frac{\pi}{2}} &= -(x - 0) \\ \Rightarrow y &= -x + e^{\frac{\pi}{2}} \end{aligned}$$

$$y = -x + e^{\frac{\pi}{2}}$$

$$\begin{aligned} \text{Area} &= \frac{1}{2} \cdot e^{\frac{\pi}{2}} \cdot e^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\pi/2} (e^\theta)^2 d\theta \\ &= \frac{e^\pi}{2} - \frac{1}{2} \left[\frac{e^{2\theta}}{2} \right]_0^{\pi/2} \\ &= \frac{e^\pi}{2} - \frac{1}{4} (e^\pi - 1) \\ &= \frac{1}{4} (e^\pi + 1) \end{aligned}$$

$$\text{The area of } R \text{ is } \frac{1}{4} (e^\pi + 1) \text{ units}^2.$$

Problem 15.

The diagram shows the velocity-time graph of a particle moving in a straight line. The equation of the curve shown is $v = t(t - 10)$ where t seconds is the time and $v \text{ ms}^{-1}$ is the velocity. The particle starts at a point A on the line when $t = 0$.

Calculate

- (a) the distance travelled by the particle before coming to instantaneous rest, and
- (b) the time at which the particle returns to A .

Solution**Part (a)**

For instantaneous rest, $v = 0$. Hence, $t(t - 10) = 0$, whence $t = 10$. Note that we reject $t = 0$ since $t > 0$.

$$\begin{aligned}\text{Distance travelled} &= - \int_0^{10} v \, dt \\ &= - \int_0^{10} t(t - 10) \, dt \\ &= - \int_0^{10} (t^2 - 10t) \, dt \\ &= - \left[\frac{t^3}{3} - \frac{10t^2}{2} \right]_0^{10} \\ &= \frac{500}{3}\end{aligned}$$

The particle travelled $\frac{500}{3}$ m before coming to instantaneous rest.

Part (b)

When the particle returns to A , $s = 0$. Let the time at which the particle returns to A be t_0 .

$$\begin{aligned}\int_0^{t_0} v \, dt &= 0 \\ \implies \int_0^{t_0} t(t - 10) \, dt &= 0 \\ \implies \left[\frac{t_0^3}{3} - \frac{10t_0^2}{2} \right]_0^{t_0} &= 0 \\ \implies \frac{1}{3}t_0^3 - 5t_0^2 &= 0 \\ \implies \frac{1}{3}t_0^2(t_0 - 15) &= 0\end{aligned}$$

Thus, $t_0 = 15$. Note that we reject $t_0 = 0$ since $t_0 > 0$.

It takes the particle 15 seconds to return to A .