

Problem 1.

- (a) Given that $f(x) = e^{\cos x}$, find $f(0)$, $f'(0)$ and $f''(0)$. Hence write down the first two non-zero terms in the Maclaurin series for $f(x)$. Give the coefficients in terms of e .
- (b) Given that $g(x) = \tan\left(2x + \frac{1}{4}\pi\right)$, find $g(0)$, $g'(0)$ and $g''(0)$. Hence find the first three terms in the Maclaurin series of $g(x)$.

Solution**Part (a)**

$$\begin{aligned}
 f(x) &= e^{\cos x} \\
 \implies f'(x) &= e^{\cos x} \cdot -\sin x \\
 &= -\sin x \cdot f(x) \\
 \implies f''(x) &= -\cos x \cdot f(x) - \sin x \cdot f'(x)
 \end{aligned}$$

Evaluating the above derivatives at $x = 0$,

$$\begin{aligned}
 f(0) &= e \\
 f'(0) &= 0 \\
 f''(0) &= -e
 \end{aligned}$$

Hence,

$$\begin{aligned}
 f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\
 &= \frac{e}{0!} x^0 + \frac{0}{1!} x^1 + \frac{-e}{2!} x^2 + \dots \\
 &= e - \frac{e}{2} x^2 + \dots
 \end{aligned}$$

$$f(x) = e - \frac{e}{2} x^2 + \dots$$

Part (b)

$$\begin{aligned}
 g(x) &= \tan\left(2x + \frac{1}{4}\pi\right) \\
 \implies g'(x) &= \sec^2\left(2x + \frac{1}{4}\pi\right) \cdot 2 \\
 &= 2 \left(1 + \tan^2\left(2x + \frac{1}{4}\pi\right)\right) \\
 &= 2 + 2g^2(x) \\
 \implies g''(x) &= 2 \cdot 2g(x) \cdot g'(x) \\
 &= 4g(x)g'(x)
 \end{aligned}$$

Evaluating the above derivatives at $x = 0$,

$$\begin{aligned}g(x) &= 1 \\g'(x) &= 4 \\g''(x) &= 16\end{aligned}$$

Hence,

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n \\&= \frac{1}{0!} x^0 + \frac{4}{1!} x^1 + \frac{16}{2!} x^2 + \dots \\&= 1 + 4x + 8x^2 + \dots\end{aligned}$$

$$\boxed{g(x) = 1 + 4x + 8x^2 + \dots}$$

Problem 2.

Find the first three non-zero terms of the Maclaurin series for the following in ascending powers of x . In each case, find the range of values of x for which the series is valid.

(a) $\frac{(1+3x)^4}{\sqrt{1+2x}}$

(b) $\frac{\sin 2x}{2+3x}$

Solution**Part (a)**

$$y = \frac{(1+3x)^4}{\sqrt{1+2x}} \quad (2.1)$$

$$\begin{aligned} \Rightarrow y^2 &= \frac{(1+3x)^8}{1+2x} \\ \Rightarrow (1+2x) \cdot y^2 &= (1+3x)^8 \end{aligned} \quad (2.2)$$

Implicitly differentiating Equation 2.2,

$$\begin{aligned} (1+2x) \cdot 2y \cdot y' + y^2 \cdot 2 &= 8(1+3x)^7 \cdot 3 \\ \Rightarrow (1+2x) \cdot y \cdot y' + y^2 &= 12(1+3x)^7 \\ \Rightarrow y((1+2x) \cdot y' + y) &= 12(1+3x)^7 \end{aligned} \quad (2.3)$$

Implicitly differentiating Equation 2.3,

$$\begin{aligned} y'((1+2x) \cdot y' + y) + y((1+2x) \cdot y'' + y' \cdot 2 + y') &= 12 \cdot 7(1+3x)^6 \cdot 3 \\ \Rightarrow (1+2x)(y')^2 + (1+2x)y \cdot y'' + 4y \cdot y' &= 252(1+3x)^6 \end{aligned} \quad (2.4)$$

Evaluating Equations 2.1, 2.3 and 2.4 at $x = 0$,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 11 \\ y''(0) &= 87 \end{aligned}$$

Hence,

$$\begin{aligned} \frac{(1+3x)^4}{\sqrt{1+2x}} &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= \frac{1}{0!} x^0 + \frac{11}{1!} x^1 + \frac{87}{2!} x^2 + \dots \\ &= 1 + 11x + \frac{87}{2} x^2 + \dots \end{aligned}$$

$$\frac{(1+3x)^4}{\sqrt{1+2x}} = 1 + 11x + \frac{87}{2}x^2 + \dots$$

Note that the series is valid only when $|2x| < 1 \implies -\frac{1}{2} < x < \frac{1}{2}$.

$$-\frac{1}{2} < x < \frac{1}{2}$$

Part (b)

$$y = \frac{\sin 2x}{2+3x} \tag{2.5}$$

$$\implies (2+3x)y = \sin 2x \tag{2.6}$$

Implicitly differentiating Equation 2.6,

$$\begin{aligned} (2+3x)y' + y \cdot 3 &= \cos 2x \cdot 2 \\ \implies (2+3x)y' + 3y &= 2 \cos 2x \end{aligned} \tag{2.7}$$

Implicitly differentiating Equation 2.7,

$$\begin{aligned} (2+3x)y'' + y' \cdot 3 + 3y' &= 2 \cdot -\sin 2x \cdot 2 \\ \implies (2+3x)y'' + 6y' &= -4 \sin 2x \end{aligned} \tag{2.8}$$

Implicitly differentiating Equation 2.8,

$$\begin{aligned} (2+3x)y''' + y'' \cdot 3 + 6y'' &= -4 \cdot \cos 2x \cdot 2 \\ \implies (2+3x)y''' + 9y'' &= -8 \cos 2x \end{aligned} \tag{2.9}$$

Evaluating Equations 2.5, 2.7, 2.8 and 2.9 at $x = 0$,

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \\ y''(0) &= -3 \\ y'''(0) &= \frac{19}{2} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{\sin 2x}{2+3x} &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= \frac{0}{0!} x^0 + \frac{1}{1!} x^1 + \frac{-3}{2!} x^2 + \frac{\frac{19}{2}}{3!} x^3 + \dots \\ &= x - \frac{3}{2} x^2 + \frac{19}{12} x^3 + \dots \end{aligned}$$

$$\frac{\sin 2x}{2+3x} = x - \frac{3}{2}x^2 + \frac{19}{12}x^3 + \dots$$

Note that the denominator can be rewritten as $2 \left(1 + \frac{3}{2}x\right)$. Hence, the series is only valid when $\left|\frac{3}{2}x\right| < 1 \implies -\frac{2}{3} < x < \frac{2}{3}$.

Problem 3.

Find the Maclaurin series of $\ln(1 + \cos x)$, up to and including the term in x^4 .

Solution

Recall that

$$\ln(1 + x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$$

and

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Hence,

$$\begin{aligned} \ln(1 + \cos x) &= \sum_{n=0}^{\infty} (-1)^n \frac{\cos^{n+1} x}{n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \right)^{n+1} \end{aligned}$$

Consider $\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \right)^{n+1}$, which is equivalent to

$$\underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \dots \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right)}_{(n+1) \text{ copies}}$$

The constant term is clearly 1. Now consider the coefficient of the x^2 term. The only way to obtain a x^2 term is to select a constant term (1) from n copies, and a x^2 term $(-\frac{x^2}{2!})$ from the remaining copy. There are $\binom{n+1}{1} = n+1$ ways to do this. Hence, the coefficient of the x^2 term is $(n+1) \cdot 1 \cdot -\frac{1}{2!} = -\frac{n+1}{2}$.

Now consider the coefficient of the x^4 term. There are two ways to obtain a x^4 term. The first way is to select a constant term (1) from n copies, and a x^4 term $(\frac{x^4}{4!})$ from the remaining copy. There are $\binom{n+1}{1} = n+1$ ways to do this, which contributes $(n+1) \cdot 1 \cdot \frac{1}{4!} = \frac{n+1}{24}$ to the coefficient of x^4 .

The second way to obtain a x^4 term is to select a x^2 term $(-\frac{x^2}{2!})$ from 2 copies and a constant term (1) from the remaining copies. There are $\binom{n+1}{2} = \frac{(n+1)n}{2}$ ways to do this, which further contributes $\frac{(n+1)n}{2} \cdot 1 \cdot \left(-\frac{1}{2!}\right)^2 = \frac{n(n+1)}{8}$ to the coefficient of x^4 . Hence, the coefficient of x^4 is given by $\frac{n+1}{24} + \frac{n(n+1)}{8} = \frac{(n+1)(3n+1)}{24}$.

Thus, up to and including the term in x^4 ,

$$\begin{aligned}\ln(1 + \cos x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \left(1 - \frac{n+1}{2}x^2 + \frac{(n+1)(3n+1)}{24} + \dots \right) \\ &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n+1} - \frac{1}{2}x^2 + \frac{3n+1}{24}x^4 + \dots \right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} - \frac{1}{2}x^2 \sum_{n=0}^{\infty} (-1)^n + \frac{3}{24}x^4 \sum_{n=0}^{\infty} n(-1)^n + \frac{1}{24}x^4 \sum_{n=0}^{\infty} (-1)^n + \dots\end{aligned}$$

Observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} &= \sum_{n=0}^{\infty} (-1)^n \frac{1^{n+1}}{n+1} \\ &= \ln(1+1) \\ &= \ln 2\end{aligned}$$

Now consider the Abel regularization of $\sum_{n=0}^{\infty} (-1)^n$.

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n &= \lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} (-1)^n x^n \\ &= \lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} (-x)^n \\ &= \lim_{x \rightarrow 1^-} \frac{1}{1 - (-x)} \\ &= \frac{1}{2}\end{aligned}$$

Now observe that $\sum_{n=0}^{\infty} x^n$ is absolutely convergent for $|x| < 1$. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty} nx^{n-1} &= \sum_{n=0}^{\infty} \frac{d}{dx} x^n \\ &= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \\ &= \frac{d}{dx} \frac{1}{1-x} \\ &= \frac{1}{(1-x)^2}\end{aligned}$$

Multiplying by x on both sides gives

$$\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$$

Hence, the Abel regularization of $\sum_{n=0}^{\infty} n(-1)^n$ is given by

$$\begin{aligned}\sum_{n=0}^{\infty} n(-1)^n &= \lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} n(-1)^n x^n \\ &= \lim_{x \rightarrow 1^-} \sum_{n=0}^{\infty} n(-x)^n \\ &= \lim_{x \rightarrow 1^-} \frac{-x}{(1 - (-x))^2} \\ &= -\frac{1}{4}\end{aligned}$$

Finally,

$$\begin{aligned}\ln(1 + \cos x) &= \ln 2 - \frac{1}{2}x^2 \cdot \frac{1}{2} + \frac{3}{24}x^4 \cdot -\frac{1}{4} + \frac{1}{24}x^4 \cdot \frac{1}{2} + \dots \\ &= \ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \dots\end{aligned}$$

$$\ln(1 + \cos x) = \ln 2 - \frac{1}{4}x^2 - \frac{1}{96}x^4 + \dots$$

Problem 4.

- (a) Find the first three terms of the Maclaurin series for $e^x(1 + \sin 2x)$.
- (b) It is given that the first two terms of this series are equal to the first two terms in the series expansion, in ascending powers of x , of $\left(1 + \frac{4}{3}x\right)^n$. Find n and show that the third terms in each of these series are equal.

Solution**Part (a)**

$$\begin{aligned}
 f(x) &= e^x(1 + \sin 2x) \\
 &= e^x + e^x \sin 2x \\
 &= e^x + e^x \Im \{e^{i2x}\} \\
 &= e^x + \Im \{e^x e^{i2x}\} \\
 &= e^x + \Im \{e^{x(1+2i)}\} \\
 \Rightarrow f^{(n)}(x) &= e^x + \Im \left\{ \frac{d^n}{dx^n} e^{x(1+2i)} \right\} \\
 &= e^x + \Im \{(1+2i)^n e^{x(1+2i)}\} \\
 &= e^x + \Im \left\{ \left(\sqrt{5} e^{i \arctan 2} \right)^n e^{x(1+2i)} \right\} \\
 &= e^x + \Im \left\{ 5^{\frac{n}{2}} e^{in \arctan 2} e^{x(1+2i)} \right\} \\
 &= e^x + 5^{\frac{n}{2}} e^x \Im \{e^{i(n \arctan 2 + 2x)}\} \\
 &= e^x + 5^{\frac{n}{2}} e^x \sin(n \arctan 2 + 2x) \\
 \Rightarrow f^{(n)}(0) &= 1 + 5^{\frac{n}{2}} e^x \sin(n \arctan 2)
 \end{aligned}$$

Hence,

$$\begin{aligned}
 f^{(0)}(0) &= 1 + 5^{\frac{0}{2}} e^x \sin(0 \arctan 2) \\
 &= 1 \\
 f^{(1)}(0) &= 1 + 5^{\frac{1}{2}} e^x \sin(1 \arctan 2) \\
 &= 1 + \sqrt{5} \cdot \frac{2}{\sqrt{5}} \\
 &= 3 \\
 f^{(2)}(0) &= 1 + 5^{\frac{2}{2}} e^x \sin(2 \arctan 2) \\
 &= 1 + 5 \cdot 2 \sin(\arctan 2) \cos(\arctan 2) \\
 &= 1 + 5 \cdot 2 \cdot \frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \\
 &= 5
 \end{aligned}$$

Thus,

$$\begin{aligned} e^x(1 + \sin 2x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{1}{0!} x^0 + \frac{3}{1!} x^1 + \frac{5}{2!} x^2 + \dots \\ &= 1 + 3x + \frac{5}{2} x^2 + \dots \end{aligned}$$

$$e^x(1 + \sin 2x) = 1 + 3x + \frac{5}{2} x^2 + \dots$$

Part (b)

By the Binomial Theorem,

$$\begin{aligned} \left(1 + \frac{4}{3}x\right)^n &= \sum_{k=0}^n \binom{n}{k} \left(\frac{4}{3}x\right)^k 1^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} \left(\frac{4}{3}\right)^k x^k \\ &= \binom{n}{0} \left(\frac{4}{3}\right)^0 x^0 + \binom{n}{1} \left(\frac{4}{3}\right)^1 x^1 + \dots \\ &= 1 + \frac{4}{3}nx + \dots \end{aligned}$$

Comparing the coefficient of x terms, we have $3 = \frac{4}{3}n$, whence $n = \frac{9}{4}$. Hence, the third term is in the expansion of $\left(1 + \frac{4}{3}x\right)^n$ is given by

$$\begin{aligned} \binom{\frac{9}{4}}{2} \left(\frac{4}{3}\right)^2 x^2 &= \frac{\frac{9}{4}(\frac{9}{4} - 1)}{2} \left(\frac{4}{3}\right)^2 x^2 \\ &= \frac{5}{2} x^2 \end{aligned}$$

Hence, the third terms in each of these series are equal.

Problem 5.

- (a) Show that the first three non-zero terms in the expansion of $\left(\frac{8}{x^3} - 1\right)^{\frac{1}{3}}$ in ascending powers of x are in the form $\frac{a}{x} + bx^2 + cx^5$, where a , b and c are constants to be determined.
- (b) By putting $x = \frac{2}{3}$ in your result, obtain an approximation for $\sqrt[3]{26}$ in the form of a fraction in its lowest terms.

A student put $x = 6$ into the expansion to obtain an approximation of $\sqrt[3]{26}$. Comment on the suitability of this choice of x for the approximation of $\sqrt[3]{26}$.

Solution**Part (a)**

$$\begin{aligned}
 \left(\frac{8}{x^3} - 1\right)^{\frac{1}{3}} &= \frac{2}{x} \left(1 - \frac{x^3}{8}\right)^{\frac{1}{3}} \\
 &= \frac{2}{x} \sum_{k=0}^{\infty} \binom{\frac{1}{3}}{k} \left(-\frac{x^3}{8}\right)^k \\
 &= \frac{2}{x} \left(\binom{\frac{1}{3}}{0} \left(-\frac{x^3}{8}\right)^0 + \binom{\frac{1}{3}}{1} \left(-\frac{x^3}{8}\right)^1 + \binom{\frac{1}{3}}{2} \left(-\frac{x^3}{8}\right)^2 + \dots \right) \\
 &= \frac{2}{x} \left(1 + \frac{1}{3} \cdot -\frac{x^3}{8} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2} \cdot \frac{x^6}{64} + \dots \right) \\
 &= \frac{2}{x} - \frac{x^2}{12} - \frac{x^5}{288} + \dots
 \end{aligned}$$

Part (b)

Evaluating the above equation at $x = \frac{2}{3}$,

$$\begin{aligned}
 \left(\frac{8}{\left(\frac{2}{3}\right)^3} - 1\right)^{\frac{1}{3}} &= \frac{2}{\frac{2}{3}} - \frac{\left(\frac{2}{3}\right)^2}{12} - \frac{\left(\frac{2}{3}\right)^5}{288} + \dots \\
 \Rightarrow \sqrt[3]{26} &= 3 - \frac{1}{27} - \frac{1}{2187} \\
 &= \frac{6479}{2187}
 \end{aligned}$$

$$\boxed{\sqrt[3]{26} = \frac{6479}{2187}}$$

Since $|6| > 1$, the binomial expansion of $\left(\frac{8}{x^3} - 1\right)^{\frac{1}{3}}$ does not hold. Hence, $x = 6$ is not an appropriate choice.

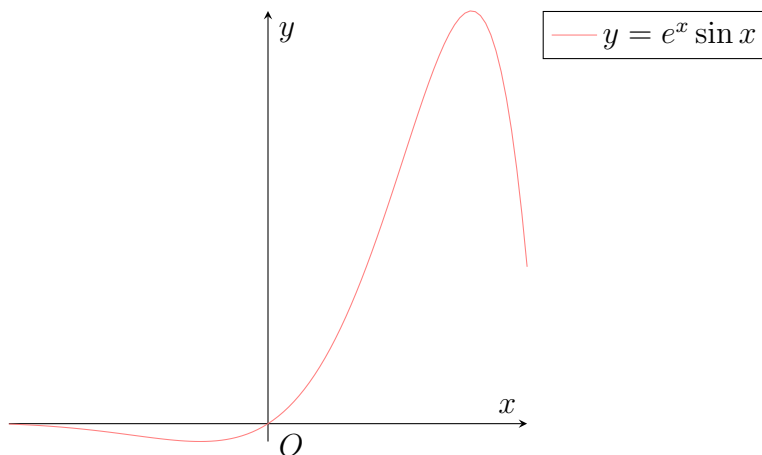
Problem 6.

Let $f(x) = e^x \sin x$.

- Sketch the graph of $y = f(x)$ for $-3 \leq x \leq 3$.
- Find the series expansion of $f(x)$ in ascending powers of x , up to and including the term in x^3 .

Denote the answer to part (b) by $g(x)$.

- On the same diagram, sketch the graph of $y = f(x)$ and $y = g(x)$. Label the two graphs clearly.
- Find, for $-3 \leq x \leq 3$, the set of values of x for which the value of $g(x)$ is within ± 0.5 of the value of $f(x)$.

Solution**Part (a)****Part (b)**

$$\begin{aligned}
 f(x) &= e^x \sin x \\
 &= e^x \Im \{ e^{ix} \} \\
 &= \Im \{ e^x e^{ix} \} \\
 &= \Im \{ e^{x(1+i)} \} \\
 \implies f^{(n)}(x) &= \Im \left\{ \frac{d^n}{dx^n} e^{x(1+i)} \right\} \\
 &= \Im \{ (1+i)^n e^{x(1+i)} \} \\
 &= \Im \left\{ \left(\sqrt{2} e^{i\frac{\pi}{4}} \right)^n e^{x(1+i)} \right\} \\
 &= \Im \left\{ 2^{\frac{n}{2}} e^x e^{i\frac{\pi}{4}n} e^{ix} \right\} \\
 &= 2^{\frac{n}{2}} e^x \Im \left\{ e^{i(\frac{\pi}{4}n+x)} \right\} \\
 &= 2^{\frac{n}{2}} e^x \sin \left(\frac{\pi}{4}n + x \right)
 \end{aligned}$$

Evaluating $f^{(n)}(x)$ at $x = 0$,

$$f^{(n)}(x) = 2^{\frac{n}{2}} \sin\left(\frac{\pi}{4}n\right)$$

Hence,

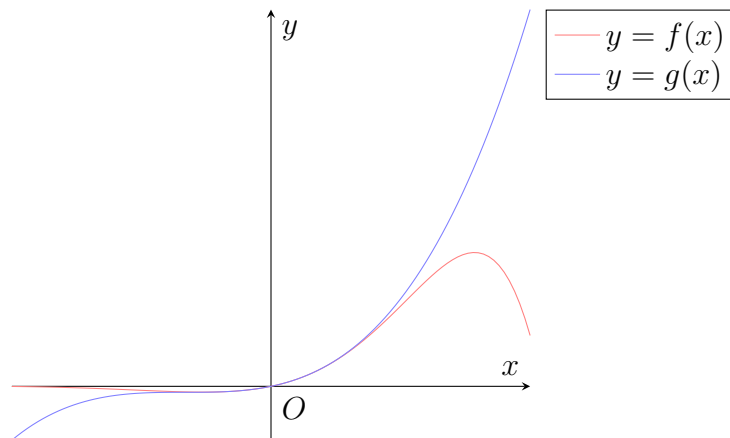
$$\begin{aligned} f(0) &= 2^{\frac{0}{2}} \sin\left(\frac{\pi}{4} \cdot 0\right) = 0 \\ f'(0) &= 2^{\frac{1}{2}} \sin\left(\frac{\pi}{4} \cdot 1\right) = 1 \\ f''(0) &= 2^{\frac{2}{2}} \sin\left(\frac{\pi}{4} \cdot 2\right) = 2 \\ f^{(3)}(0) &= 2^{\frac{3}{2}} \sin\left(\frac{\pi}{4} \cdot 2\right) = 2 \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \\ &= \frac{f^{(0)}(0)}{0!} x^0 + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \dots \\ &= x + x^2 + \frac{1}{3} x^3 + \dots \end{aligned}$$

$$f(x) = x + x^2 + \frac{1}{3} x^3 + \dots$$

Part (c)



Part (d)

Consider $|f(x) - g(x)| \leq 0.5$ for $-3 \leq x \leq 3$, where $g(x) = x + x^2 + \frac{1}{3}x^3$.

Case 1: $f(x) - g(x) \leq 0.5$

$$\begin{aligned} f(x) - g(x) &\leq 0.5 \\ \implies e^x \sin x - \left(x + x^2 + \frac{1}{3}x^3\right) &\leq 0.5 \\ \implies x &\geq -1.96 \end{aligned}$$

Case 2: $-(f(x) - g(x)) \leq 0.5$

$$\begin{aligned} & -(f(x) - g(x)) \leq 0.5 \\ \implies & g(x) - f(x) \leq 0.5 \\ \implies & x + x^2 + \frac{1}{3}x^3 - e^x \sin x \leq 0.5 \\ \implies & x \leq 1.56 \end{aligned}$$

Putting both inequalities together, we have

$$\boxed{-1.96 \leq x \leq 1.56}$$

Problem 7.

It is given that $y = \frac{1}{1 + \sin 2x}$. Show that, when $x = 0$, $\frac{d^2y}{dx^2} = 8$. Find the first three terms of the Maclaurin series for y .

- (a) Use the series to obtain an approximate value for $\int_{-0.1}^{0.1} y \, dx$, leaving your answer as a fraction in its lowest terms.
- (b) Find the first two terms of the Maclaurin series for $\frac{dy}{dx}$.
- (c) Write down the equation of the tangent at the point where $x = 0$ on the curve $y = \frac{1}{1 + \sin 2x}$.

Solution

$$y = \frac{1}{1 + \sin 2x} \quad (7.1)$$

$$\begin{aligned} \Rightarrow y' &= -\frac{1}{(1 + \sin 2x)^2} \cdot (\cos 2x \cdot 2) \\ &= -2y^2 \cos 2x \end{aligned} \quad (7.2)$$

$$\begin{aligned} \Rightarrow y'' &= -2(\cos 2x \cdot 2y \cdot y' + y^2 \cdot -\sin 2x \cdot 2) \\ &= -4(y \cdot y' \cos 2x - y^2 \sin 2x) \end{aligned} \quad (7.3)$$

From Equations 7.1, 7.2 and 7.3,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= -2 \\ y''(0) &= 8 \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{1 + \sin 2x} &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= \frac{y(0)}{0!} x^0 + \frac{y'(0)}{1!} x^1 + \frac{y''(0)}{2!} x^2 + \dots \\ &= 1 - 2x + 4x^2 + \dots \end{aligned}$$

Part (a)

$$\begin{aligned} \int_{-0.1}^{0.1} y \, dx &\approx \int_{-0.1}^{0.1} (1 - 2x + 4x^2) \, dx \\ &= \left[x - 2 \cdot \frac{1}{2} x^2 + 4 \cdot \frac{1}{3} x^3 \right] \Big|_{-0.1}^{0.1} \\ &= \frac{76}{275} \end{aligned}$$

$$\int_{-0.1}^{0.1} y \, dx \approx \frac{76}{275}$$

Part (b)

$$\begin{aligned}y' &= \frac{d}{dx}y \\&= \frac{d}{dx}(1 - 2x + 4x^2 + \dots) \\&= -2 + 8x + \dots\end{aligned}$$

$$\boxed{y' = -2 + 8x + \dots}$$

Part (c)

Using the point-slope formula,

$$\begin{aligned}y - 1 &= -2(x - 0) \\ \implies y &= -2x + 1\end{aligned}$$

$$\boxed{y = -2x + 1}$$

Problem 8.

It is given that $y = e^{\arcsin 2x}$.

- (a) Show that $(1 - 4x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} = 4y$.
- (b) By further differentiating this result, find the Maclaurin series for y in ascending powers of x , up to and including the term in x^3 .
- (c) Hence, find an approximation value of $e^{\frac{\pi}{2}}$, by substituting a suitable value of x in the Maclaurin series for y .
- (d) Suggest one way to improve the accuracy of the approximated value obtained.

Solution**Part (a)**

$$y = e^{\arcsin 2x} \quad (8.1)$$

$$\implies \ln y = \arcsin 2x \quad (8.2)$$

Implicitly differentiating Equation 8.2,

$$\begin{aligned} \frac{y'}{y} &= \frac{1}{\sqrt{1 - (2x)^2}} \cdot 2 \\ &= \frac{2}{\sqrt{1 - 4x^2}} \\ \implies y' \sqrt{1 - 4x^2} &= 2y \end{aligned} \quad (8.3)$$

Implicitly differentiating Equation 8.3,

$$\begin{aligned} y'' \sqrt{1 - 4x^2} + y' \frac{1}{2\sqrt{1 - 4x^2}} \cdot -8x &= 2y' \\ \implies (1 - 4x^2) y'' - 4xy' &= 2y' \sqrt{1 - 4x^2} \\ &= 2 \left(\frac{2y}{\sqrt{1 - 4x^2}} \right) \sqrt{1 - 4x^2} \\ &= 4y \end{aligned} \quad (8.4)$$

Part (b)

Implicitly differentiating Equation 8.4,

$$\begin{aligned} y^{(3)}(1 - 4x^2) + y'' \cdot -8x - 4(xy'' + y') &= 4y' \\ \implies y^{(3)}(1 - 4x^2) - 12xy'' - 8y' &= 0 \end{aligned} \quad (8.5)$$

From Equations 8.1, 8.3, 8.4 and 8.5,

$$\begin{aligned}y(0) &= 1 \\y'(0) &= 2 \\y''(0) &= 4 \\y^{(3)}(0) &= 16\end{aligned}$$

Hence,

$$\begin{aligned}y &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\&= \frac{y(0)}{1!} x^0 + \frac{y'(0)}{1!} x^1 + \frac{y''(0)}{2!} x^2 + \frac{y^{(3)}(0)}{3!} x^3 + \dots \\&= 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \dots\end{aligned}$$

$$y = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \dots$$

Part (c)

Consider $y = e^{\frac{\pi}{2}} \implies \arcsin 2x = \frac{\pi}{2} \implies x = \frac{1}{2} \cdot \sin \frac{\pi}{2} = \frac{1}{2}$. Hence, substituting $x = \frac{1}{2}$ in the Maclaurin series for y ,

$$\begin{aligned}e^{\frac{\pi}{2}} &\approx 1 + 2 \cdot \frac{1}{2} + 2 \left(\frac{1}{2}\right)^2 + \frac{8}{3} \left(\frac{1}{2}\right)^3 \\&= \frac{17}{6}\end{aligned}$$

$$e^{\frac{\pi}{2}} \approx \frac{17}{6}$$

Part (d)

More terms of the Maclaurin series of y could be considered.

Problem 9.

The curve $y = f(x)$ passes through the point $(0, 1)$ and satisfies the equation $\frac{dy}{dx} = \frac{6 - 2y}{\cos 2x}$.

- Find the Maclaurin series of $f(x)$, up to and including the term in x^3 .
- Using standard results given in the List of Formulae (MF27), express $\frac{1 - \sin x}{\cos x}$ as a power series of x , up to and including the term in x^3 .
- Using the two power series you have found, show to this degree of approximation, that $f(x)$ can be expressed as $a(\tan 2x - \sec 2x) + b$, where a and b are constants to be determined.

Solution**Part (a)**

$$\begin{aligned} y' &= \frac{6 - 2y}{\cos 2x} \\ \implies y' \cos 2x &= 6 - 2y \end{aligned} \tag{9.1}$$

Implicitly differentiating Equation 9.1,

$$\begin{aligned} -\sin 2x \cdot 2 \cdot y' + y'' \cos 2x &= -2y' \\ \implies -2y' \sin 2x + y'' \cos 2x &= -2y' \end{aligned} \tag{9.2}$$

Implicitly differentiating Equation 9.2,

$$\begin{aligned} -2(y'' \sin 2x + y' \cos 2x \cdot 2) + (y'' \cdot -\sin 2x \cdot 2 + y^{(3)} \cos 2x) &= -2y'' \\ \implies -4y' \cos 2x - 3y'' \sin 2x + y^{(3)} \cos 2x &= -2y'' \end{aligned} \tag{9.3}$$

Given that y passes through the point $(0, 1)$, and from Equations 9.1, 9.2 and 9.3,

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 4 \\ y''(0) &= -8 \\ y^{(3)}(0) &= 32 \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{y^{(n)}(0)}{n!} x^n \\ &= \frac{y(0)}{0!} x^0 + \frac{y'(0)}{1!} x^1 + \frac{y''(0)}{2!} x^2 + \frac{y^{(3)}(0)}{3!} x^3 + \dots \\ &= 1 + 4x - 4x^2 + \frac{16}{3}x^3 + \dots \end{aligned}$$

$$f(x) = 1 + 4x - 4x^2 + \frac{16}{3}x^3 + \dots$$

Part (b)

Observe that

$$\frac{1 - \sin x}{\cos x} = \sec x - \tan x$$

Since $\sec x$ is even, $\sec x$ only contributes even powers of x to the power series expansion of $\frac{1 - \sin x}{\cos x}$. Likewise, since $\tan x$ is odd, $\tan x$ only contributes odd powers of x to the power series expansion of $\frac{1 - \sin x}{\cos x}$.

Let $f(x) = \sec x$ and $g(x) = \tan x$.

$$\begin{aligned} f(x) &= \sec x \\ \implies f'(x) &= \ln(\sec x + \tan x) \\ &= \ln(f(x) + g(x)) \\ \implies f''(x) &= \frac{f'(x) + g'(x)}{f(x) + g(x)} \end{aligned}$$

$$\begin{aligned} g(x) &= \tan x \\ \implies g'(x) &= \sec^2(x) \\ &= f^2(x) \\ \implies g''(x) &= 2f(x)f'(x) \\ \implies g^{(3)}(x) &= 2f(x)f''(x) + 2(f'(x))^2 \end{aligned}$$

Evaluating the above derivatives at $x = 0$, we have

$$\begin{aligned} f(0) &= 1, & g(0) &= 0 \\ f'(0) &= 0, & g'(0) &= 1 \\ f''(0) &= 1, & g''(0) &= 0 \\ & & g^{(3)}(0) &= 2 \end{aligned}$$

Thus,

$$\begin{aligned} \frac{1 - \sin x}{\cos x} &= \sec x - \tan x \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n - \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} x^n \\ &= \left(1 + \frac{1}{2}x^2 + \dots\right) - \left(x + \frac{1}{3}x^3 + \dots\right) \\ &= 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots \end{aligned}$$

$\frac{1 - \sin x}{\cos x} = 1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots$

Part (c)

$$\begin{aligned}
a(\tan 2x - \sec 2x) + b &= -a(\sec 2x - \tan 2x) + b \\
&= -a \left(1 - 2x + \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3 + \dots \right) + b \\
&\approx -a \left(1 - 2x + \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3 \right) + b \\
&= -a \left(1 - 2x + 2x^2 - \frac{8}{3}x^3 \right) + b \\
&= a \left(-1 + 2x - 2x^2 + \frac{8}{3}x^3 \right) + b \\
&= a \left(-1 + \frac{1}{2}(f(x) - 1) \right) + b \\
&= -\frac{3}{2}a + b + \frac{a}{2}f(x)
\end{aligned}$$

Hence,

$$\frac{a}{2}f(x) - \frac{3}{2}a + b \approx a(\tan 2x - \sec 2x) + b$$

In order to obtain an approximation for $f(x)$, we need $\frac{a}{2} = 1$ and $-\frac{3}{2}a + b = 0$, whence $a = 2$ and $b = 3$.

$$\boxed{a = 2, b = 3}$$

Problem 10.

Given that x is sufficiently small for x^3 and higher powers of x to be neglected, and that $13 - 59 \sin x = 10(2 - \cos 2x)$, find a quadratic equation for x and hence solve for x .

Solution

$$\begin{aligned} 13 - 59 \sin x &= 10(2 - \cos 2x) \\ &= 10(2 - (1 - 2 \sin^2 x)) \\ &= 10(1 + 2 \sin^2 x) \\ &= 10 + 20 \sin^2 x \\ \implies 20 \sin^2 x + 59 \sin x - 3 &= 0 \\ \implies (20 \sin x - 1)(\sin x + 3) &= 0 \end{aligned}$$

Hence, $\sin x = \frac{1}{20}$. Note that we reject $\sin x = -3$ since $|\sin x| \leq 1$. Since x is sufficiently small for x^3 and higher powers of x to be neglected, $\sin x \approx x$. Thus, $x \approx \frac{1}{20}$.

$$\boxed{x \approx \frac{1}{20}}$$

Problem 11.

In triangle ABC , angle $A = \frac{\pi}{3}$ radians, angle $B = \left(\frac{\pi}{3} + x\right)$ radians and angle $C = \left(\frac{\pi}{3} - x\right)$ radians, where x is small. The lengths of the sides BC , CA and AB are denoted by a , b and c respectively. Show that $b - c \approx \frac{2ax}{\sqrt{3}}$.

Solution

By the sine rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Hence,

$$\begin{aligned} b &= a \cdot \frac{\sin B}{\sin A} = a \cdot \frac{\sin B}{\sqrt{3}/2} = \frac{2a}{\sqrt{3}} \sin B \\ c &= a \cdot \frac{\sin C}{\sin A} = a \cdot \frac{\sin C}{\sqrt{3}/2} = \frac{2a}{\sqrt{3}} \sin C \end{aligned}$$

This gives

$$\begin{aligned} b - c &= \frac{2a}{\sqrt{3}} (\sin B - \sin C) \\ &= \frac{2a}{\sqrt{3}} \left(\sin \left(\frac{\pi}{3} + x \right) - \sin \left(\frac{\pi}{3} - x \right) \right) \\ &= \frac{2a}{\sqrt{3}} \left(\left(\sin \frac{\pi}{3} \cos x + \cos \frac{\pi}{3} \sin x \right) - \left(\sin \frac{\pi}{3} \cos x - \cos \frac{\pi}{3} \sin x \right) \right) \\ &= \frac{2a}{\sqrt{3}} \cdot 2 \cos \frac{\pi}{3} \sin x \\ &= \frac{2a}{\sqrt{3}} \cdot 2 \cdot \frac{1}{2} \sin x \\ &= \frac{2a}{\sqrt{3}} \sin x \end{aligned}$$

Since x is small, $\sin x \approx x$. Hence, $b - c \approx \frac{2ax}{\sqrt{3}}$.

Problem 12.

D'Alembert's ratio test states that a series of the form $\sum_{r=0}^{\infty} a_r$ converges when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$, and diverges when $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$. When $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the test is inconclusive. Using the test, explain why the series $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all real values of x and state the sum to infinity of this series, in terms of x .

Solution

Let $a_n = \frac{x^n}{n!}$ and consider $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{(n+1)!} \bigg/ \frac{x^n}{n!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| \\ &= 0 \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$ for all $x \in \mathbb{R}$, it follows by D'Alembert's ratio test that $\sum_{r=0}^{\infty} \frac{x^r}{r!}$ converges for all real values of x . The sum to infinity of the series in question is e^x .