Problem 1.

Find the position vector of the foot of the perpendicular from the point with position vector $\mathbf{c}$ to the line with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$. Leave your answers in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Solution

Let the foot of the perpendicular be F. We have that $\overrightarrow{OF} = \mathbf{a} + \lambda\mathbf{b}$ for some real λ, and $\overrightarrow{CF} \cdot \mathbf{b} = 0$.

$$\begin{aligned}
& \overrightarrow{CF} \cdot \mathbf{b} = 0 \\
\implies & \left(\overrightarrow{OF} - \overrightarrow{OC}\right) \cdot \mathbf{b} = 0 \\
\implies & (\mathbf{a} + \lambda\mathbf{b} - \mathbf{c}) \cdot \mathbf{b} = 0 \\
\implies & \lambda\mathbf{b} \cdot \mathbf{b} + (\mathbf{a} - \mathbf{c}) \cdot \mathbf{b} = 0 \\
\implies & \lambda|\mathbf{b}|^2 = (\mathbf{c} - \mathbf{a}) \cdot \mathbf{b} \\
\implies & \lambda = \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \\
\implies & \overrightarrow{OF} = \mathbf{a} + \frac{(\mathbf{c} - \mathbf{a}) \cdot \mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} \\
& = \mathbf{a} + \left((\mathbf{c} - \mathbf{a}) \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}
\end{aligned}$$

$$\boxed{\overrightarrow{OF} = \mathbf{a} + \left((\mathbf{c} - \mathbf{a}) \cdot \hat{\mathbf{b}}\right)\hat{\mathbf{b}}}$$

Problem 2.

The point O is the origin, and points A, B, C have position vectors given by $\overrightarrow{OA} = 6\mathbf{i}$, $\overrightarrow{OB} = 3\mathbf{j}$, $\overrightarrow{OC} = 4\mathbf{k}$. The point P is on line AB between A and B, and is such that $AP = 2PB$. The point Q has position vector given by $\overrightarrow{OQ} = q\mathbf{i}$, where q is a scalar.

(a) Express, in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, the vector $\overrightarrow{CP}$.

(b) Show that the line BQ has equation $\mathbf{r} = 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j})$, where t is a parameter. Give an equation of the line CP in a similar form.

(c) Find the value of q for which the lines CP and BQ are perpendicular.

(d) Find the sine of the acute angle between the lines CP and BQ in terms of q.

Solution

We have that $\overrightarrow{OA} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

Part (a)

By the Ratio Theorem,

$$\begin{aligned} \overrightarrow{OP} &= \frac{2\,\overrightarrow{OB} + \overrightarrow{OA}}{1+2} \\ &= \frac{1}{3}\left(2\begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \\ \implies \overrightarrow{CP} &= \overrightarrow{OP} - \overrightarrow{OC} \\ &= \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{CP} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix}}$$

Part (b)

Note that $\overrightarrow{BQ} = \overrightarrow{OQ} - \overrightarrow{OB} = \begin{pmatrix} q \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix}$. Thus, BQ is given by

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix}, \ t \in \mathbb{R} \\ \implies \mathbf{r} &= 3\mathbf{j} + t(q\mathbf{i} - 3\mathbf{j}), \ t \in \mathbb{R} \end{aligned}$$

Note that $\overrightarrow{CP} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$. Hence, CP is given by

$$\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + u \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \ u \in \mathbb{R}$$

$$\boxed{CP : \mathbf{r} = 4\mathbf{k} + u(\mathbf{i} + \mathbf{j} - 2\mathbf{k}), \ u \in \mathbb{R}}$$

Part (c)

Since CP is perpendicular to BQ, we have $\overrightarrow{CP} \cdot \overrightarrow{BQ} = 0$.

$$\begin{aligned} \overrightarrow{CP} \cdot \overrightarrow{BQ} &= 0 \\ \implies 2 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} &= 0 \\ \implies \quad q - 3 + 0 &= 0 \\ \implies \quad q &= 3 \end{aligned}$$

$$\boxed{q = 3}$$

Part (d)

Let θ be the acute angle between CP and BQ.

$$\begin{aligned} \left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} \right| &= \left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} \right| \sin\theta \\ \implies \quad \left| \begin{pmatrix} -6 \\ 2q \\ 3 - q \end{pmatrix} \right| &= \left| \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right| \left| \begin{pmatrix} q \\ -3 \\ 0 \end{pmatrix} \right| \sin\theta \\ \implies \sqrt{(-6)^2 + (2q)^2 + (3-q)^2} &= \sqrt{1^2 + 1^2 + (-2)^2} \cdot \sqrt{q^2 + (-3)^2 + 0^2} \cdot \sin\theta \\ \implies \quad \sqrt{36 + 4q^2 + 9 - 6q + q^2} &= \sqrt{6} \cdot \sqrt{q^2 + 9} \cdot \sin\theta \\ \implies \quad \sqrt{5q^2 - 6q + 45} &= \sqrt{6(q^2 + 9)} \sin\theta \end{aligned}$$

$$\begin{aligned}
\implies \qquad \sqrt{5q^2 - 6q + 45} &= \sqrt{6q^2 + 54}\sin\theta \\
\implies \qquad \sin\theta &= \frac{\sqrt{5q^2 - 6q + 45}}{\sqrt{6q^2 + 54}} \\
&= \sqrt{\frac{5q^2 - 6q + 45}{6q^2 + 54}}
\end{aligned}$$

$$\boxed{\sin\theta = \sqrt{\frac{5q^2 - 6q + 45}{6q^2 + 54}}}$$

Problem 3.

Line l_1 passes through the point A with position vector $3\mathbf{i}-2\mathbf{k}$ and is parallel to $-2\mathbf{i}+4\mathbf{j}-\mathbf{j}$. Line l_2 has Cartesian equation given by $\dfrac{x-1}{2} = y = z+3$.

(a) Show that the two lines intersect and find the coordinates of their point of intersection.

(b) Find the acute angle between the two lines l_1 and l_2. Hence, or otherwise, find the shortest distance from point A to line l_2.

(c) Find the position vector of the foot N of the perpendicular from A to the line l_2. The point B lies on the line AN produced and is such that N is the mid-point of AB. Find the position vector of B.

Solution

We have that

$$l_1 : \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R}$$

and

$$l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \mu \in \mathbb{R}$$

Part (a)

Consider $l_1 = l_2$.

$$\begin{aligned} l_1 &= l_2 \\ \implies \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\ \implies \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \lambda \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} &= \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 2\mu + 2\lambda &= 2 \\ \mu + 4\lambda &= 0 \\ \mu - \lambda &= 1 \end{cases}$$

which has the unique solution $\mu = \dfrac{4}{5}$ and $\lambda = -\dfrac{1}{5}$. Thus, the intersection point P has position vector $\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} - \dfrac{1}{5} \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} = \dfrac{1}{5} \begin{pmatrix} 17 \\ -4 \\ -4 \end{pmatrix}$ and thus has coordinates $\left(\dfrac{17}{5}, -\dfrac{4}{5}, -\dfrac{9}{5}\right)$.

$$\boxed{\left(\frac{17}{5}, -\frac{4}{5}, -\frac{9}{5}\right)}$$

Part (b)

Let θ be the acute angle between l_1 and l_2.

$$\begin{aligned}
\cos\theta &= \frac{\left|\begin{pmatrix}-2\\4\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}-2\\4\\1\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}\\
&= \frac{|-4+4-1|}{\sqrt{21}\cdot\sqrt{6}}\\
&= \frac{1}{\sqrt{126}}\\
\implies \qquad \theta &= \arccos\frac{1}{\sqrt{126}}\\
&= 84.9^\circ \text{ (1 d.p.)}
\end{aligned}$$

$$\boxed{\theta = 84.9^\circ}$$

Note that

$$\begin{aligned}
AP &= \sqrt{\left(\frac{17}{5}-3\right)^2+\left(-\frac{4}{5}-0\right)^2+\left(-\frac{9}{5}-(-2)\right)^2}\\
&= \sqrt{\frac{21}{25}}\\
&= \frac{\sqrt{21}}{5}
\end{aligned}$$

Since $\sin\theta = \dfrac{AN}{AP}$, we have that $AN = AP\sin\theta$.

$$\begin{aligned}
AN &= \frac{\sqrt{21}}{5}\sin\arccos\frac{1}{\sqrt{126}}\\
&= \frac{\sqrt{21}}{5}\cdot\frac{\sqrt{\left(\sqrt{126}\right)^2-1}}{\sqrt{126}}\\
&= \frac{\sqrt{21}}{5}\cdot\frac{\sqrt{125}}{\sqrt{126}}\\
&= \frac{\sqrt{21}}{5}\cdot\frac{5\sqrt{5}}{\sqrt{6}\cdot\sqrt{21}}\\
&= \frac{\sqrt{5}}{\sqrt{6}}\\
&= \sqrt{\frac{5}{6}}
\end{aligned}$$

$$\boxed{\text{The shortest distance between } A \text{ and } l_2 \text{ is } \sqrt{\frac{5}{6}} \text{ units.}}$$

Part (c)

Since N is on l_2, we have that $\overrightarrow{ON} = \begin{pmatrix} 1 \\ 0 \\ 03 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ for some real μ.

$$
\begin{aligned}
& & \overrightarrow{AN} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= 0 \\
\implies & & \left(\overrightarrow{ON} - \overrightarrow{OA}\right) \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} 1 &= 0 \\
\implies & & \left(\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= 0 \\
\implies & & \begin{pmatrix} -2 + 2\mu \\ \mu \\ -1 + \mu \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} &= 0 \\
\implies & & 2(-2 + 2\mu) + \mu + (-1 + \mu) &= 0 \\
\implies & & \mu &= \frac{5}{6} \\
\implies & & \overrightarrow{ON} &= \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \frac{5}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \\
& & &= \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}
\end{aligned}
$$

$$\boxed{\overrightarrow{ON} = \frac{1}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix}}$$

By the Ratio Theorem,

$$
\begin{aligned}
\overrightarrow{ON} &= \frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} \\
\implies \overrightarrow{OB} &= 2\overrightarrow{ON} - \overrightarrow{OA} \\
&= \frac{2}{6} \begin{pmatrix} 16 \\ 5 \\ -13 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \\
&= \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}
\end{aligned}
$$

$$\boxed{\overrightarrow{OB} = \frac{1}{3} \begin{pmatrix} 7 \\ 5 \\ -7 \end{pmatrix}}$$