Problem 1.

A student claims that a unique plane can always be defined based on the given information. True or False? (Whenever a line is mentioned, assume the vector equation is known.)

	Statement	**T/F**
(a)	Any 2 vectors parallel to the plane and a point lying on the plane.	False
(b)	Any 3 distinct points lying on the plane.	False
(c)	A vector perpendicular to the plane and a point lying on the plane.	True
(d)	A line l perpendicular to the plane and a particular point on l lying on the plane.	True
(e)	A line l lying on the plane.	False
(f)	A line l and a point not on l, both lying on the plane.	True
(g)	A pair of distinct, intersecting lines, both lying on the plane.	True
(h)	A pair of distinct, parallel lines, both lying on the plane.	True
(i)	A pair of skew lines both parallel to the plane.	False
(j)	2 intersecting lines both parallel to the plane.	False

Problem 2.

Find the equations of the following planes in parametric, scalar product and Cartesian form:

(a) The plane passes through the point with position vector $7\mathbf{i} + 2\mathbf{j} - 3\mathbf{j}$ and is parallel to $\mathbf{i} + 3\mathbf{j}$ and $4\mathbf{j} - 2\mathbf{k}$.

(b) The plane passes through the points $A(2, 0, 1)$, $B(1, -1, 2)$ and $C(1, 3, 1)$.

(c) The plane passes through the point with position vector $7\mathbf{i}$ and is parallel to the plane $\mathbf{r} = (2 - p + q)\mathbf{i} + (p + 3q)\mathbf{j} + (-2 - 3q)\mathbf{k}$, $p, q \in \mathbb{R}$.

(d) The plane contains the line $l : \mathbf{r} = (-2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}) + \lambda(2\mathbf{i} + \mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$ and is perpendicular to the plane $\pi : \mathbf{r} \cdot (7\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}) = 2$.

Solution

Part (a)

Parametric. Note that $\begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$$

Scalar Product. Note that $\mathbf{n} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \implies d = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -25$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -2$$

Cartesian. Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. From the scalar product form, we have

$$-3x + y + 2z = -25$$

Form	Equation
Parametric	$\mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}, \ \lambda, \mu \in \mathbb{R}$
Scalar Product	$\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = -25$
Cartesian	$-3x + y + 2z = -25$

Part (b)

Parametric. Since the plane passes through the points A, B and C, it is parallel to both $\overrightarrow{AB} = -\begin{pmatrix}1\\1\\-1\end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix}-1\\3\\0\end{pmatrix}$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}-1\\3\\0\end{pmatrix}, \lambda, \mu \in \mathbb{R}$$

Scalar Product. Note that $\mathbf{n} = \begin{pmatrix}1\\1\\-1\end{pmatrix} \times \begin{pmatrix}-1\\3\\0\end{pmatrix} = \begin{pmatrix}3\\1\\4\end{pmatrix} \implies d = \begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix}2\\0\\1\end{pmatrix} \cdot \begin{pmatrix}3\\1\\4\end{pmatrix} = 10$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix}3\\1\\4\end{pmatrix} = 10$$

Cartesian. Let $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. From the scalar product form, we have

$$3x + y + 4z = 10$$

Form	Equation
Parametric	$\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}-1\\3\\0\end{pmatrix}, \lambda, \mu \in \mathbb{R}$
Scalar Product	$\mathbf{r} \cdot \begin{pmatrix}3\\1\\4\end{pmatrix} = 10$
Cartesian	$3x + y + 4z = 10$

Part (c)

Parametric. Note that the plane is parallel to $\mathbf{r} = \begin{pmatrix}2-p+q\\p+3q\\-2-3q\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix} + p\begin{pmatrix}-1\\1\\0\end{pmatrix} + q\begin{pmatrix}1\\3\\-3\end{pmatrix}$ and passes through $\begin{pmatrix}7\\0\\0\end{pmatrix}$. Hence, the plane has parametric form

$$\mathbf{r} = \begin{pmatrix}7\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}1\\3\\-3\end{pmatrix}, \lambda, \mu \in \mathbb{R}$$

Scalar Product. Note that $\mathbf{n} = \begin{pmatrix}-1\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\3\\-3\end{pmatrix} = \begin{pmatrix}-3\\-3\\-4\end{pmatrix} = -\begin{pmatrix}3\\3\\4\end{pmatrix} \implies d = \begin{pmatrix}7\\0\\0\end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix}7\\0\\0\end{pmatrix} \cdot -\begin{pmatrix}3\\3\\4\end{pmatrix} = -21$. Thus, the plane has scalar product form $\mathbf{r} \cdot -\begin{pmatrix}3\\3\\4\end{pmatrix} = -21$, which simplifies to

$$\mathbf{r} \cdot \begin{pmatrix}3\\3\\4\end{pmatrix} = 21$$

Cartesian. Let $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. From the scalar product form, we have

$$3x + 3y + 4z = 21$$

Form	**Equation**
Parametric	$\mathbf{r} = \begin{pmatrix}7\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu \begin{pmatrix}1\\3\\-3\end{pmatrix}, \lambda, \mu \in \mathbb{R}$
Scalar Product	$\mathbf{r} \cdot \begin{pmatrix}3\\3\\4\end{pmatrix} = 21$
Cartesian	$3x + 3y + 4z = 21$

Part (d)

Parametric. Since the plane contains the line with equation $\mathbf{r} = \begin{pmatrix}-2\\5\\-3\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\2\end{pmatrix}, \lambda \in \mathbb{R}$, the plane passes through the point with position vector $\begin{pmatrix}-2\\5\\-3\end{pmatrix}$ and is parallel to the vector $\begin{pmatrix}2\\1\\2\end{pmatrix}$. Furthermore, since the plane is perpendicular to the plane with normal $\begin{pmatrix}7\\4\\5\end{pmatrix}$, it must be parallel to said vector. Thus, the plane has the following parametric form:

$$\mathbf{r} = \begin{pmatrix}-2\\5\\-3\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\2\end{pmatrix} + \mu \begin{pmatrix}7\\4\\5\end{pmatrix}, \lambda, \mu \in \mathbb{R}$$

Scalar Product. Note that $\mathbf{n} = \begin{pmatrix}2\\1\\2\end{pmatrix} \times \begin{pmatrix}7\\4\\5\end{pmatrix} = \begin{pmatrix}-3\\4\\1\end{pmatrix} \implies d = \begin{pmatrix}-2\\5\\-3\end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix}-2\\5\\-3\end{pmatrix} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = 23$. Thus, the plane has scalar product form

$$\mathbf{r} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = 23$$

Cartesian. Let $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. From the scalar product form, we have

$$-3x + 4y + z = 23$$

Form	Equation
Parametric	$\mathbf{r} = \begin{pmatrix}-2\\5\\-3\end{pmatrix} + \lambda \begin{pmatrix}2\\1\\2\end{pmatrix} + \mu \begin{pmatrix}7\\4\\5\end{pmatrix}, \lambda, \mu \in \mathbb{R}$
Scalar Product	$\mathbf{r} \cdot \begin{pmatrix}-3\\4\\1\end{pmatrix} = 23$
Cartesian	$-3x + 4y + z = 23$

Problem 3.

The line l passes through the points A and B with coordinates $(1, 2, 4)$ and $(-2, 3, 1)$ respectively. The plane p has equation $3x - y + 2z = 17$. Find

(a) the coordinates of the point of intersection of l and p,

(b) the acute angle between l and p,

(c) the perpendicular distance from A to p, and

(d) the position vector of the foot of the perpendicular from B to p.

The line m passes through the point C with position vector $6\mathbf{i} + \mathbf{j}$ and is parallel to $2\mathbf{j} + \mathbf{k}$.

(e) Determine whether m lies in p.

Solution

Note that $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$, whence $\overrightarrow{AB} = -\begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}$. Thus, the line l has vector equation

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$$

Also note that the equation of the plane p can be written as

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 17$$

Part (a)

Let the point of intersection of l and p be P. Consider $l = p$.

$$\begin{aligned} && l &= p \\ \implies && \left[\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} &= 17 \\ \implies && \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} &= 17 \\ \implies && 9 + 16\lambda &= 17 \\ \implies && \lambda &= \frac{1}{2} \end{aligned}$$

Thus, $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 5/2 \\ 3/2 \\ 11/2 \end{pmatrix}$, whence $P\left(\frac{5}{2}, \frac{3}{2}, \frac{11}{2}\right)$.

$$\boxed{\left(\frac{5}{2}, \frac{3}{2}, \frac{11}{2}\right)}$$

Part (b)

Let θ be the acute angle between l and p.

$$\begin{aligned}
\sin\theta &= \frac{\left|\begin{pmatrix}3\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}3\\-1\\2\end{pmatrix}\right|} \\
&= \frac{16}{\sqrt{266}} \\
\implies \qquad \theta &= 1.38 \text{ (3 s.f.)}
\end{aligned}$$

$$\boxed{\theta = 1.38}$$

Part (c)

Note that $\overrightarrow{AP} = -\frac{1}{2}\begin{pmatrix}3\\-1\\3\end{pmatrix}$.

$$\begin{aligned}
\text{Perpendicular distance} &= \left|\overrightarrow{AP} \cdot \hat{\mathbf{n}}\right| \\
&= \left|-\frac{1}{2}\begin{pmatrix}3\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix}\right| \Big/ \left|\begin{pmatrix}3\\-1\\2\end{pmatrix}\right| \\
&= \frac{8}{\sqrt{14}}
\end{aligned}$$

$$\boxed{\text{The perpendicular distance from } A \text{ to } p \text{ is } \frac{8}{\sqrt{14}} \text{ units.}}$$

Part (d)

Let F be the foot of the perpendicular from B to p. Since F is on p, we have $\overrightarrow{OF} \cdot \begin{pmatrix}3\\-1\\2\end{pmatrix} = 17$. Furthermore, since BF is perpendicular to p, we have $\overrightarrow{BF} = \lambda\mathbf{n} = \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$ for some $\lambda \in \mathbb{R}$. We hence have $\overrightarrow{OF} = \overrightarrow{OB} + \overrightarrow{BF} = \begin{pmatrix}-2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}$.

$$
\begin{aligned}
&& \overrightarrow{OF} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} &= 17 \\
\implies && \left[\begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} &= 17 \\
\implies && \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} &= 17 \\
\implies && -7 + 14\lambda &= 17 \\
\implies && \lambda &= \frac{12}{7}
\end{aligned}
$$

Thus, $\overrightarrow{OF} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{12}{7} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 22 \\ 9 \\ 31 \end{pmatrix}$.

$$\boxed{\overrightarrow{OF} = \frac{1}{7} \begin{pmatrix} 22 \\ 9 \\ 31 \end{pmatrix}}$$

Part (e)

Note that m has the vector equation

$$\mathbf{r} = \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Consider $m \cdot \mathbf{n}$.

$$
\begin{aligned}
m \cdot \mathbf{n} &= \left[\begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 6 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} \\
&= 17 + 0\lambda \\
&= 17
\end{aligned}
$$

Since $m \cdot \mathbf{n} = 17$ for all $\lambda \in \mathbb{R}$, m lies in p.

$$\boxed{m \text{ lies in } p.}$$

Problem 4.

A plane contains distinct points P, Q, R and S, of which no 3 points are collinear. What can be said about the relationship between the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$?

Solution

Each of the three vectors can be expressed as a unique linear combination of the other two.

Problem 5.

(a) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors and t is a parameter.

(b) Interpret geometrically the vector equation $\mathbf{r} \cdot \mathbf{n} = d$, where $\mathbf{n}$ is a constant unit vector and d is a constant scalar, stating what d represents.

(c) Given that $\mathbf{b} \cdot \mathbf{n} \neq 0$, solve the equations $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ and $\mathbf{r} \cdot \mathbf{n} = d$ to find $\mathbf{r}$ in terms of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{n}$ and d. Interpret the solution geometrically.

Solution

Part (a)

The vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ represents a line with direction vector $\mathbf{b}$ that passes through the point with position vector $\mathbf{a}$.

Part (b)

The vector equation $\mathbf{r} \cdot \mathbf{n} = d$ represents a plane perpendicular to $\mathbf{n}$ that has a perpendicular distance of d units from the origin. Here, a negative value of d corresponds to a plane d units from the origin in the opposite direction of $\mathbf{n}$.

Part (c)

$$\begin{aligned}
& \mathbf{r} \cdot \mathbf{n} = d \\
\implies & (\mathbf{a} + t\mathbf{b}) \cdot \mathbf{n} = d \\
\implies & \mathbf{a} \cdot \mathbf{n} + t\mathbf{b} \cdot \mathbf{n} = d \\
\implies & t = \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \\
\implies & \mathbf{r} = \mathbf{a} + \frac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \mathbf{b}
\end{aligned}$$

$\mathbf{a} + \dfrac{d - \mathbf{a} \cdot \mathbf{n}}{\mathbf{b} \cdot \mathbf{n}} \mathbf{b}$ is the position vector of the point of intersection of the line and plane.

Problem 6.

The planes p_1 and p_2 have equations $\mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 1$ and $\mathbf{r} \cdot \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} = -1$ respectively, and meet in the line l.

(a) Find the acute angle between p_1 and p_2.

(b) Find a vector equation for l.

(c) The point $A(4, 3, c)$ is equidistant from the planes p_1 and p_2. Calculate the two possible values of c.

Solution

Part (a)

Let θ the acute angle between p_1 and p_2.

$$\begin{aligned} \cos\theta &= \frac{\left| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right| \left| \begin{pmatrix} -6 \\ 3 \\ 2 \end{pmatrix} \right|} \\ &= \frac{16}{21} \\ \implies \qquad \theta &= 0.705 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\theta = 0.705}$$

Part (b)

Observe that p_1 has the Cartesian equation $2x - 2y + z = 1$ and p_2 has the Cartesian equation $-6x + 3y + 2z = -1$. Consider $p_1 = p_2$. Solving both Cartesian equations simultaneously gives the solution

$$\begin{cases} x = -\frac{1}{6} + \frac{7}{6}t \\ y = -\frac{2}{3} + \frac{5}{3}t \\ z = \qquad t \end{cases}$$

for all $t \in \mathbb{R}$. The line l thus has vector equation $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{6} \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ 10 \\ 6 \end{pmatrix}$, $t \in \mathbb{R}$.

$$\boxed{\mathbf{r} = -\frac{1}{6} \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ 10 \\ 6 \end{pmatrix}, \ t \in \mathbb{R}}$$

Part (c)

Let Q be the point with position vector $-\dfrac{1}{6}\begin{pmatrix}1\\4\\0\end{pmatrix}$, whence $\overrightarrow{AQ} = -\dfrac{1}{6}\begin{pmatrix}25\\22\\6c\end{pmatrix}$. Since Q lies on l, it lies on both p_1 and p_2. Since A is equidistant to p_1 and p_2, the perpendicular distances from A to p_1 and p_2 are equal.

$$\left|\overrightarrow{AQ}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right| \Bigg/ \left|\begin{pmatrix}2\\-2\\1\end{pmatrix}\right| = \left|\overrightarrow{AQ}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right| \Bigg/ \left|\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|$$

$$\implies \quad \frac{1}{3}\left|-\frac{1}{6}\begin{pmatrix}25\\22\\6c\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right| = \frac{1}{7}\left|-\frac{1}{6}\begin{pmatrix}25\\22\\6c\end{pmatrix}\cdot\begin{pmatrix}-6\\3\\2\end{pmatrix}\right|$$

$$\implies \quad \frac{1}{3}\,|1+c| = \frac{1}{7}\,|-14+2c|$$

$$\implies \quad |7+7c| = |-42+6c|$$

Case 1: $(7+7c)(-42+6c) > 0 \implies 7+7c = -42+6c \implies c = -49$

Case 2: $(7+7c)(-42+6c) < 0 \implies 7+7c = -(-42+6c) \implies c = -\dfrac{35}{13}$

$$\boxed{c = -49 \lor c = -\frac{35}{13}}$$

Problem 7.

A plane Π has equation $\mathbf{r} \cdot (2\mathbf{i} + 3\mathbf{j}) = -6$.

(a) Find, in vector form, an equation for the line passing through the point P with position vector $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and normal to the plane Π.

(b) Find the position vector of the foot Q of the perpendicular from P to the plane Π and hence find the position vector of the image of P after the reflection in the plane Π.

(c) Find the sine of the acute angle between OQ and the plane Π.

The plane Π' has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) = 5$.

(d) Find the position vector of the point A where the planes Π, Π' and the plane with equation $\mathbf{r} \cdot \mathbf{i} = 0$ meet.

(e) Hence, or otherwise, find also the vector equation of the line of intersection of planes Π and Π'.

Solution

Part (a)

Let l be the required line. Since l is normal to Π, it is parallel to the normal vector of Π, $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. Thus, l has vector equation

$$\boxed{l : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}}$$

Part (b)

Since Q is on Π, $\overrightarrow{OQ} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6$. Furthermore, observe that Q is also on the line l. Thus, $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

$$\begin{aligned} \overrightarrow{OQ} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} &= -6 \\ \implies \quad \left[\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} &= -6 \\ \implies \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} &= -6 \end{aligned}$$

$$\begin{aligned} \implies && 7 + 13\lambda &= -6 \\ \implies && \lambda &= -1 \end{aligned}$$

Thus, $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$.

$$\boxed{\overrightarrow{OQ} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}}$$

Let the reflection of P in Π be P'. We have that $\overrightarrow{PQ} = \overrightarrow{QP'}$.

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{QP'} \\ \implies \overrightarrow{OQ} - \overrightarrow{OP} &= \overrightarrow{OP'} - \overrightarrow{OQ} \\ \implies \qquad \overrightarrow{OP'} &= 2\overrightarrow{OQ} - \overrightarrow{OP} \\ &= 2\begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ -5 \\ 4 \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{OP'} = \begin{pmatrix} -2 \\ -5 \\ 4 \end{pmatrix}}$$

Part (c)

Let θ be the acute angle between OQ and Π.

$$\begin{aligned} \sin\theta &= \frac{\left| \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right|}{\left| \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix} \right| \left| \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right|} \\ &= \frac{3}{\sqrt{65}} \end{aligned}$$

$$\boxed{\sin\theta = \frac{3}{\sqrt{65}}}$$

Part (d)

Let $\overrightarrow{OA} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. We thus have the following system:

$$\begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = -6 \implies 2x + 3y = -6 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 5 \implies x + y + z = 5 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0 \implies x = 0 \end{cases}$$

Solving, we obtain $x = 0$, $y = -2$ and $z = 7$.

$$\boxed{\overrightarrow{OA} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix}}$$

Part (e)

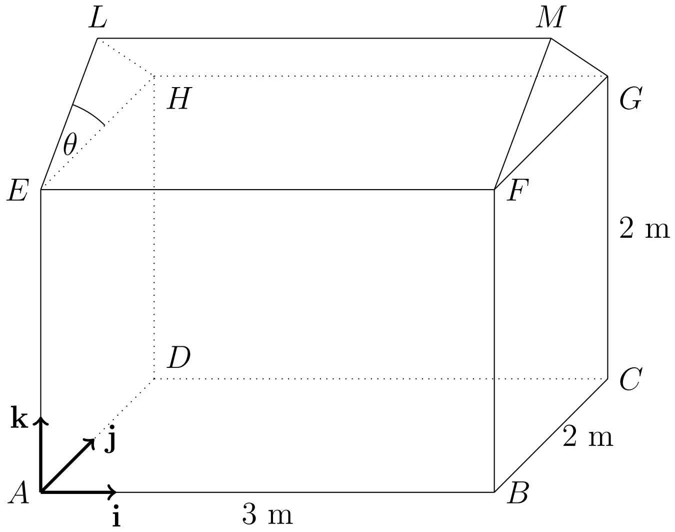
Let the line of intersection of Π and Π' be l'. Observe that A is on Π and Π' and thus lies on l'. Hence,

$$l' : \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + \lambda \mathbf{b},\ \lambda \in \mathbb{R}$$

Since l' lies on both Π and Π', $\mathbf{b}$ is perpendicular to the normals of both planes, i.e. $\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Thus, $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$.

$$\boxed{l' : \mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}}$$

Problem 8.



The diagram shows a garden shed with horizontal base $ABCD$, where $AB = 3$ m and $BC = 2$ m. There are two vertical rectangular walls $ABFE$ and $DCGH$, where $AE = BF = CG = DH = 2$ m. The roof consists of two rectangular planes $EFML$ and $HGML$, which are inclined at an angle θ to the horizontal such that $\tan\theta = \frac{3}{4}$.

The point A is taken as the origin and the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, each of length 1 m, are taken along AB, AD and AE respectively.

(a) Verify that the plane with equation $\mathbf{r} \cdot (22\mathbf{i} + 33\mathbf{j} - 12\mathbf{k}) = 66$ passes through B, D and M.

(b) Find the perpendicular distance, in metres, from A to the plane BDM.

(c) Find a vector equation of the straight line EM.

(d) Show that the perpendicular distance from C to the straight line EM is 2.91 m, correct to 3 significant figures.

Solution

Part (a)

We have $\overrightarrow{AB} = \begin{pmatrix}3\\0\\0\end{pmatrix}$, $\overrightarrow{BF} = \overrightarrow{AE} = \begin{pmatrix}0\\0\\2\end{pmatrix}$ and $\overrightarrow{FG} = \overrightarrow{AD} = \begin{pmatrix}0\\2\\0\end{pmatrix}$. Let T be the midpoint of FG. We have $\overrightarrow{FT} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\frac{TM}{FT} = \tan\theta = \frac{3}{4}$, whence $\overrightarrow{TM} = \frac{3}{4}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\3/4\end{pmatrix}$.

$$\begin{aligned}
\overrightarrow{AM} &= \overrightarrow{AB} + \overrightarrow{BF} + \overrightarrow{FT} + \overrightarrow{TM} \\
&= \begin{pmatrix}3\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\2\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\3/4\end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix}12\\4\\11\end{pmatrix}
\end{aligned}$$

Consider $\overrightarrow{AB}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix}$, $\overrightarrow{AD}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix}$ and $\overrightarrow{AM}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix}$.

$$\begin{aligned}
\overrightarrow{AB}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} &= \begin{pmatrix}3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66 \\
\overrightarrow{AD}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} &= \begin{pmatrix}0\\2\\0\end{pmatrix}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66 \\
\overrightarrow{AM}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} &= \frac{1}{4}\begin{pmatrix}12\\4\\11\end{pmatrix}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} &= 66
\end{aligned}$$

Since $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{AM}$ satisfy the equation $\mathbf{r}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix} = 66$, they all lie on the plane with said equation.

Part (b)

$$\begin{aligned}
\text{Perpendicular distance} &= \left|\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right| \\
&= \left|\begin{pmatrix}3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}22\\33\\-12\end{pmatrix}\right| \Bigg/ \left|\begin{pmatrix}22\\33\\-12\end{pmatrix}\right| \\
&= \frac{66}{\sqrt{1717}}
\end{aligned}$$

$$\boxed{\text{The perpendicular distance from } A \text{ to the plane } BDM \text{ is } \frac{66}{\sqrt{1717}} \text{ units.}}$$

Part (c)

Observe that $\overrightarrow{EM} = \overrightarrow{AM} - \overrightarrow{AE} = \frac{1}{4}\begin{pmatrix}12\\4\\3\end{pmatrix}$. Hence, the line EM has vector equation

$$\boxed{\mathbf{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}12\\4\\3\end{pmatrix},\ \lambda \in \mathbb{R}}$$

Part (d)

Note that $\overrightarrow{EC} = \overrightarrow{AC} - \overrightarrow{AE} = \begin{pmatrix}3\\2\\-2\end{pmatrix}$.

$$\begin{aligned}
\text{Perpendicular distance} &= \left|\overrightarrow{EC} \times \begin{pmatrix}12\\4\\3\end{pmatrix}\right| \Bigg/ \left|\begin{pmatrix}12\\4\\3\end{pmatrix}\right| \\
&= \frac{1}{13}\left|\begin{pmatrix}3\\2\\-2\end{pmatrix} \times \begin{pmatrix}12\\4\\3\end{pmatrix}\right| \\
&= \frac{1}{13}\left|\begin{pmatrix}14\\-33\\-12\end{pmatrix}\right| \\
&= \frac{\sqrt{1429}}{13} \\
&= 2.91 \text{ (3 s.f.)}
\end{aligned}$$

Problem 9.

The planes π_1 and π_2 have equations

$$x + y - z = 0 \text{ and } 2x - 4y + z + 12 = 0$$

respectively. The point P has coordinates $(3, 8, 2)$ and O is the origin.

(a) Verify that the vector $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$ is parallel to both π_1 and π_2.

(b) Find the equation of the plane which passes through P and is perpendicular to both π_1 and π_2.

(c) Verify that $(0, 4, 4)$ is a point common to both π_1 and π_2, and hence or otherwise, find the equation of the line of intersection of π_1 and π_2, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, $\lambda \in \mathbb{R}$.

(d) Find the coordinates of the point in which the line OP meets π_2.

(e) Find the length of projection of OP on π_1.

Solution

Note that π_1 and π_2 have the vector equations

$$\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0 \text{ and } \mathbf{r} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = -12$$

respectively.

Part (a)

Observe that $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = 0$. Thus, the vector $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ is perpendicular to the normal vectors of both π_1 and π_2 and is hence parallel to them.

Part (b)

Let the required plane be π_3. Since π_3 is perpendicular to both π_1 and π_2, its normal vector is parallel to both planes. Thus, $\mathbf{n} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \implies d = \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 15$.

π_3 hence has the vector equation

$$\boxed{\mathbf{r} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 15}$$

Part (c)

Since $\begin{pmatrix}0\\4\\4\end{pmatrix} \cdot \begin{pmatrix}1\\1\\-1\end{pmatrix} = 0$ and $\begin{pmatrix}0\\4\\4\end{pmatrix} \cdot \begin{pmatrix}2\\-4\\1\end{pmatrix} = -12$, $(0, 4, 4)$ satisfies the vector equation of both π_1 and π_2 and thus lies on both planes.

Let l be the line of intersection of π_1 and π_2. Since $(0, 4, 4)$ is a point common to both planes, l passes through it. Furthermore, since l lies on both π_1 and π_2, it is perpendicular to the normal vector of both planes and hence has direction vector $\begin{pmatrix}1\\1\\-1\end{pmatrix} \times \begin{pmatrix}2\\-4\\1\end{pmatrix} = -3\begin{pmatrix}1\\1\\2\end{pmatrix}$. Thus, l can be expressed as

$$\boxed{l : \mathbf{r} = \begin{pmatrix}0\\4\\4\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\2\end{pmatrix}, \lambda \in \mathbb{R}}$$

Part (d)

Note that the line OP, denoted l_{OP} has equation

$$l_{OP} : \mathbf{r} = \lambda\begin{pmatrix}3\\8\\2\end{pmatrix}, \lambda \in \mathbb{R}$$

Consider $l_{OP} = \pi_2$.

$$\begin{aligned} & l_{OP} = \pi_2 \\ \implies & \mu\begin{pmatrix}3\\8\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-4\\1\end{pmatrix} = -12 \\ \implies & -24\mu = -12 \\ \implies & \mu = \frac{1}{2} \end{aligned}$$

Hence, OP meets π_2 at $\left(\frac{3}{2}, \frac{8}{2}, \frac{2}{2}\right) = \left(\frac{3}{2}, 4, 1\right)$.

$$\boxed{\left(\frac{3}{2}, 4, 1\right)}$$

Part (e)

$$\begin{aligned}
\text{Length of projection} &= \overrightarrow{OP} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Bigg/ \left| \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right| \\
&= \frac{1}{\sqrt{3}} \left| \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right| \\
&= \frac{1}{\sqrt{3}} \left| \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} \right| \\
&= \frac{1}{\sqrt{3}} \cdot 5\sqrt{6} \\
&= 5\sqrt{2}
\end{aligned}$$

$$\boxed{\text{The length of projection of } OP \text{ on } \pi_1 \text{ is } 5\sqrt{2} \text{ units.}}$$

Problem 10.

The line l_1 passes through the point A, whose position vector is $3\mathbf{i}-5\mathbf{j}-4\mathbf{k}$, and is parallel to the vector $3\mathbf{i}+4\mathbf{j}+2\mathbf{k}$. The line l_2 passes through the point B, whose position vector is $2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, and is parallel to the vector $\mathbf{i}-\mathbf{j}-4\mathbf{k}$. The point P on l_1 and Q on l_2 are such that PQ is perpendicular to both l_1 and l_2. The plane Π contains PQ and l_1.

(a) Find a vector parallel to PQ.

(b) Find the equation of Π in the forms $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$, $\lambda,\mu\in\mathbb{R}$ and $\mathbf{r}\cdot\mathbf{n}=D$.

(c) Find the perpendicular distance from B to Π.

(d) Find the acute angle between Π and l_2.

(e) Find the position vectors of P and Q.

Solution

Part (a)

Note that l_1 and l_2 have vector equations

$$\mathbf{r}=\begin{pmatrix}3\\-5\\-4\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\2\end{pmatrix},\ \lambda\in\mathbb{R} \text{ and } \mathbf{r}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\-4\end{pmatrix},\ \mu\in\mathbb{R}$$

respectively. Since PQ is perpendicular to both l_1 and l_2, it is parallel to $\begin{pmatrix}3\\4\\2\end{pmatrix}\times\begin{pmatrix}1\\-1\\-4\end{pmatrix}=\begin{pmatrix}-14\\14\\-7\end{pmatrix}=-7\begin{pmatrix}2\\-2\\1\end{pmatrix}$.

$$\boxed{PQ \text{ is parallel to } \begin{pmatrix}2\\-2\\1\end{pmatrix}.}$$

Part (b)

Since Π contains PQ and l_1, it is parallel to $\begin{pmatrix}2\\-2\\1\end{pmatrix}$ and $\begin{pmatrix}3\\4\\2\end{pmatrix}$. Also note that Π contains $\begin{pmatrix}3\\-5\\-4\end{pmatrix}$. Thus,

$$\boxed{\Pi:\mathbf{r}=\begin{pmatrix}3\\-5\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}+\mu\begin{pmatrix}3\\4\\2\end{pmatrix},\ \lambda,\mu\in\mathbb{R}}$$

Note that $\begin{pmatrix}2\\-2\\1\end{pmatrix} \times \begin{pmatrix}3\\4\\2\end{pmatrix} = \begin{pmatrix}-8\\-1\\14\end{pmatrix} = -\begin{pmatrix}8\\1\\-14\end{pmatrix}$. We hence take $\mathbf{n} = \begin{pmatrix}8\\1\\-14\end{pmatrix}$, whence $d = \begin{pmatrix}3\\-5\\-4\end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix}3\\-5\\-4\end{pmatrix} \cdot \begin{pmatrix}8\\1\\-14\end{pmatrix} = 75$.

$$\boxed{\Pi : \mathbf{r} \cdot \begin{pmatrix}8\\1\\-14\end{pmatrix} = 75}$$

Part (c)

Note that $\overrightarrow{AB} = \begin{pmatrix}-1\\8\\9\end{pmatrix}$. Hence,

$$\begin{aligned}\text{Perpendicular distance} &= \frac{\left|\begin{pmatrix}-1\\8\\9\end{pmatrix} \cdot \begin{pmatrix}8\\1\\-14\end{pmatrix}\right|}{\left|\begin{pmatrix}8\\1\\-14\end{pmatrix}\right|} \\ &= \frac{126}{\sqrt{261}}\end{aligned}$$

$$\boxed{\text{The perpendicular distance from } B \text{ to } \Pi \text{ is } \frac{126}{\sqrt{261}} \text{ units.}}$$

Part (d)

Let θ be the acute angle between Π and l_2.

$$\begin{aligned}\sin\theta &= \frac{\left|\begin{pmatrix}1\\-1\\-4\end{pmatrix} \cdot \begin{pmatrix}8\\1\\-14\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\-4\end{pmatrix}\right|\left|\begin{pmatrix}8\\1\\-14\end{pmatrix}\right|} \\ &= \frac{7}{\sqrt{58}} \\ \implies \quad \theta &= 1.17 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\theta = 1.17}$$

Part (e)

Since P is on l_1, we have $\overrightarrow{OP} = \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$. Similarly, since Q is on l_2, we have $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$ for some $\mu \in \mathbb{R}$. Thus,

$$\begin{aligned}
\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\
&= \left[\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}\right] - \left[\begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}\right] \\
&= \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix}
\end{aligned}$$

Recall that PQ is parallel to $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Hence, $\overrightarrow{PQ}$ can be expressed as $\nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ for some $\nu \in \mathbb{R}$.

$$\begin{aligned}
\begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} &= \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \\
\implies \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \mu \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} + \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} \\
\implies \lambda \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + \nu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} &= \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix}
\end{aligned}$$

This gives the following system:

$$\begin{cases} 3\lambda - \mu + 2\nu = -1 \\ 4\lambda + \mu - 2\nu = 8 \\ 2\lambda + 4\mu + \nu = 9 \end{cases}$$

which has the unique solution $\lambda = 1$, $\mu = 2$ and $\nu = -1$. Thus,

$$\begin{aligned}
\overrightarrow{OP} &= \begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} \\
\overrightarrow{OQ} &= \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}
\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}, \overrightarrow{OQ} = \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix}}$$

Problem 11.

The equations of three planes p_1, p_2 and p_3 are

$$\begin{aligned} 2x - 5y + 3z &= 3 \\ 3x + 2y - 5z &= -5 \\ 5x + \lambda y + 17z &= \mu \end{aligned}$$

respectively, where λ and μ are constants. The planes p_1 and p_2 intersect in a line l.

(a) Find a vector equation of l.

(b) Given that all three planes meet in the line l, find λ and μ.

(c) Given instead that the three planes have no point in common, what can be said about the values of λ and μ?

(d) Find the Cartesian equation of the plane which contains l and the point $(1, -1, 3)$.

Solution

Part (a)

Consider $p_1 = p_2$.

$$\begin{cases} 2x - 5y + 3z = 3 \\ 3x + 2y - 5z = -5 \end{cases}$$

The above system has solution

$$\begin{cases} x = -1 + t \\ y = -1 + t \\ z = \quad t \end{cases}$$

for all $t \in \mathbb{R}$. Thus, the line l has vector equation

$$\begin{aligned} \mathbf{r} &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &= \begin{pmatrix} -1 + t \\ -1 + t \\ t \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ t \in \mathbb{R} \end{aligned}$$

$$\boxed{l : \mathbf{r} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ t \in \mathbb{R}}$$

Part (b)

Since all three planes meet in the line l, l must satisfy the equation of p_3. Substituting the above solution to the given equation, we have

$$\begin{aligned} & 5(-1+t)+\lambda(-1+t)+17t=\mu \\ \implies \quad & (22+\lambda)t-(5+\lambda+\mu)=0 \end{aligned}$$

Comparing the coefficients of t and the constant terms, we have the following system:

$$\begin{cases} \lambda \qquad + 22 = 0 \\ \lambda + \mu + \ \ 5 = 0 \end{cases}$$

which has the unique solution $\lambda = -22$ and $\mu = 17$.

$$\boxed{\lambda = -22,\ \mu = 17}$$

Part (c)

If the three planes have no point in common, we have

$$(22+\lambda)t-(5+\lambda+\mu)\neq 0$$

for all $t \in \mathbb{R}$. To satisfy this relation, we need $22+\lambda = 0$ and $5+\lambda+\mu \neq 0$, whence $\lambda = -22$ and $\mu \neq 17$.

$$\boxed{\lambda = -22,\ \mu \neq 17}$$

Part (d)

Note that $\begin{pmatrix}-1\\-1\\0\end{pmatrix}$ lies on l and is thus contained on the required plane. Observe that $\begin{pmatrix}-1\\-1\\0\end{pmatrix} - \begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}-2\\0\\-3\end{pmatrix}$. Thus, the required plane is parallel to $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}-2\\0\\-3\end{pmatrix}$ and hence has vector equation

$$\mathbf{r} = \begin{pmatrix}-1\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} + \mu\begin{pmatrix}-2\\0\\-3\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$$

Observe that $\mathbf{n} = \begin{pmatrix}1\\1\\1\end{pmatrix} \times \begin{pmatrix}-2\\0\\3\end{pmatrix} = \begin{pmatrix}-3\\1\\2\end{pmatrix}$, whence $d = \begin{pmatrix}-1\\-1\\0\end{pmatrix}\cdot\mathbf{n} = \begin{pmatrix}-1\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix} = 2$. Thus, the required plane has the equation

$$\mathbf{r}\cdot\begin{pmatrix}-3\\1\\2\end{pmatrix} = 2$$

Let $\mathbf{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$. It follows that the plane has Cartesian equation

$$\boxed{-3x+y+2=2}$$

Problem 12.

The planes p_1 and p_2, which meet in line l, have equations $x-2y+2z=0$ and $2x-2y+z=0$ respectively.

(a) Find an equation of l in Cartesian form.

The plane p_3 has equation $(x-2y+2z)+c(2x-2y+z)=d$.

(b) Given that $d=0$, show that all 3 planes meet in the line l for any constant c.

(c) Given instead that the 3 planes have no point in common, what can be said about the value of d?

Solution

Part (a)

Consider $p_1 = p_2$. This gives the system

$$\begin{cases} x-2y+2z=0 \\ 2x-2y+\ \ z=0 \end{cases}$$

which has solution $x=t$, $y=\frac{3}{2}t$ and $z=$. Thus, l has Cartesian equation

$$\boxed{x=\frac{2}{3}y=z}$$

Part (b)

When $d=0$, p_3 has equation

$$(x-2y+2z)+c(2x-2y+z)=0$$

Observe that the line l satisfies the equations $x-2y+2z=0$ and $2x-2y+z=0$. Hence, l also satisfies the equation that gives p_3 for all c. Thus, p_3 contains l, implying that all 3 planes meet in the line l.

Part (c)

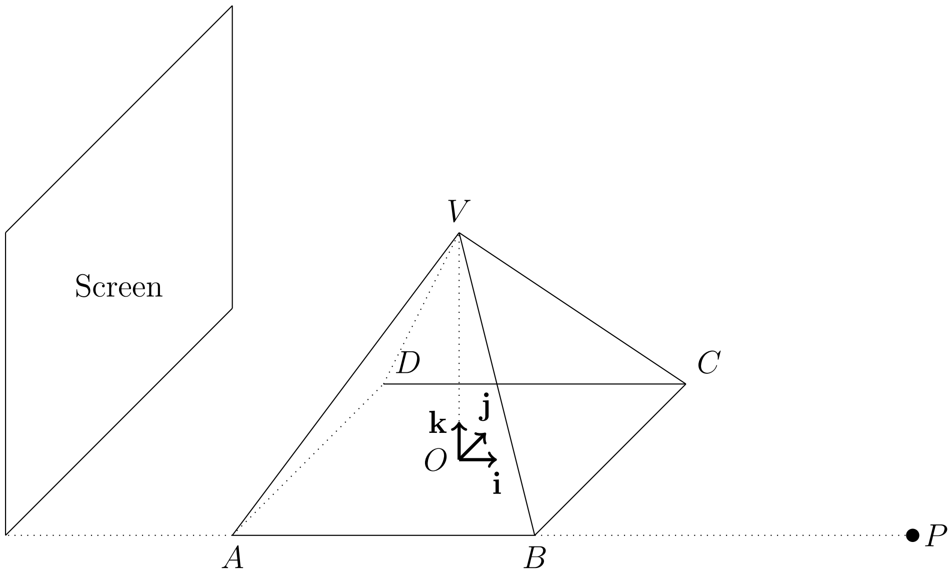
If the 3 planes have no point in common, then l does not have any point in common with p_3. That is, all points on l satisfy the relation

$$(x-2y+2z)+c(2x-2y+z)\neq d$$

Since $x-2y+2z=0$ and $2x-2y+z=0$ for all points on l, the LHS simplifies to 0. Thus, to satisfy the above relation, we require $d\neq 0$.

$$\boxed{d\neq 0}$$

Problem 13.



A right opaque pyramid with square base $ABCD$ and vertex V is placed at ground level for a shadow display, as shown in the diagram. O is the centre of the square base $ABCD$, and the perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are in the directions of $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OV}$ respectively. The length of AB is 8 units and the length of OV is $2h$ units.

A point light source for this shadow display is placed at the point $P(20, -4, 0)$ and a screen of height 35 units is placed with its base on the ground such that the screen lies on a plane with vector equation $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \alpha$, where $\alpha < -4$.

(a) Find a vector equation of the line depicting the path of the light ray from P to V in terms of h.

(b) Find an inequality between α and h so that the shadow of the pyramid cast on the screen will not exceed the height of the screen.

The point light source is now replaced by a parallel light source whose light rays are perpendicular to the screen. It is also given that $h = 10$.

(c) Find the exact length of the shadow cast by the edge VB on the screen.

A mirror is placed on the plane VBC to create a special effect during the display.

(d) Find a vector equation of the plane VBC and hence find the angle of inclination made by the mirror with the ground.

Solution

Part (a)

Note that $\overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 2h \end{pmatrix}$ and $\overrightarrow{OP} = \begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix}$, whence $\overrightarrow{PV} = \begin{pmatrix} -20 \\ 4 \\ 2h \end{pmatrix} = 2 \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix}$. Thus, the line from P to V, denoted l_{PV}, has vector equation

$$\boxed{l_{PV} : \mathbf{r} = \begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix}, \lambda \in \mathbb{R}}$$

Part (b)

Let the point of intersection between l_{PV} and the screen be I.

$$\begin{aligned} \left[\begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \alpha \\ \implies \begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} &= \alpha \\ \implies 20 - 10\lambda &= \alpha \\ \implies \lambda &= \frac{20 - \alpha}{10} \end{aligned}$$

Hence, $\overrightarrow{OI} = \begin{pmatrix} 20 \\ -4 \\ 0 \end{pmatrix} + \frac{20 - \alpha}{10} \begin{pmatrix} -10 \\ 2 \\ h \end{pmatrix}$. To prevent the shadow from exceeding the screen, we require the $\mathbf{k}$ component of $\overrightarrow{OI}$ to be less than the height of the screen, i.e. 35 units. This gives the inequality $\frac{20 - \alpha}{10} \cdot h \leq 35$, whence

$$\boxed{h \leq \frac{350}{20 - \alpha}}$$

Part (c)

Since the light rays emitted by the light source are now perpendicular to the screen, the image of some point with coordinates (a, b, c) on the screen is given by (α, b, c). Thus, the image of $B(4, -4, 0)$ and $V(0, 0, 20)$ on the screen have coordinates $(\alpha, -4, 0)$ and $(\alpha, 0, 20)$. The length of the shadow cast by VB is thus given by

$$\sqrt{(\alpha - \alpha)^2 + (-4 - 0)^2 + (0 - 20)^2} = 4\sqrt{26}$$

$$\boxed{\text{The shadow cast by the edge } VB \text{ on the screen is } 4\sqrt{26} \text{ units long.}}$$

Part (d)

Note that $\overrightarrow{BV} = 4 \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$ and $\overrightarrow{BC} = 8 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Hence, the plane VBC is parallel to $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Note that $\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$. Thus, $\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$, whence $d = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix} \cdot \mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 20$. Thus, the plane VBC has vector equation

$$\boxed{\mathbf{r} \cdot \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 20}$$

Observe that the ground is given by the vector equation

$$\mathbf{r} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

Let θ be the angle of inclination made by the mirror with the ground.

$$\begin{aligned} \cos\theta &= \frac{\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}{\left|\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}\right| \left|\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right|} \\ &= \frac{1}{\sqrt{26}} \\ \implies \qquad \theta &= 1.37 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\theta = 1.37}$$