Problem 1.

For each of the following, write down a vector equivalent of the line l and convert it to parametric and Cartesian forms.

(a) l passes through the point with position vector $-\mathbf{i}+\mathbf{k}$ and is parallel to the vector $\mathbf{i}+\mathbf{j}$.

(b) l passes through the points $P(1,-1,3)$ and $Q(2,1,-2)$.

(c) l passes through the origin and is parallel to the line $m : \mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\lambda \in \mathbb{R}$.

(d) l is the x-axis.

(e) l passes through the point $C(4,-1,2)$ and is parallel to the z-axis.

Solution

Part (a)

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix}-1\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda - 1 \\ y = \lambda \\ z = 1 \end{cases}$
Cartesian	$x+1=y,\ z=1$

Part (b)

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix}1\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-5\end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda + 1 \\ y = 2\lambda - 1 \\ z = -5\lambda + 3 \end{cases}$
Cartesian	$x-1 = \dfrac{y+1}{2} = \dfrac{3-z}{5}$

Part (c)

Form	Expression
Vector	$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda \\ y = 2\lambda \\ z = 3\lambda \end{cases}$
Cartesian	$x = \frac{y}{2} = \frac{z}{3}$

Part (d)

Form	Expression
Vector	$\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = \lambda \\ y = 0 \\ z = 0 \end{cases}$
Cartesian	$x \in \mathbb{R}$, $y = 0$, $z = 0$

Part (e)

Form	Expression
Vector	$\mathbf{r} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$
Parametric	$\begin{cases} x = 4 \\ y = -1 \\ z = \lambda + 2 \end{cases}$
Cartesian	$x = 4$, $y = -1$, $z \in \mathbb{R}$

Problem 2.

For each of the following, determine if l_1 and l_2 are parallel, intersecting or skew. In the case of intersecting lines, find the position vector of the point of intersection. In addition, find the acute angle between the lines l_1 and l_2.

(a) $l_1 : x - 1 = -y = z - 2$ and $l_2 : \dfrac{x-2}{2} = -\dfrac{y+1}{2} = \dfrac{z-4}{2}$

(b) $l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix}$, $\alpha \in \mathbb{R}$ and $l_2 : \mathbf{r} = \begin{pmatrix}0\\10\\1\end{pmatrix} + \beta\begin{pmatrix}3\\8\\1\end{pmatrix}$

(c) $l_1 : \mathbf{r} = (\mathbf{i} - 5\mathbf{k}) + \lambda(\mathbf{i} - \mathbf{j} + \mathbf{k})$, $\lambda \in \mathbb{R}$ and $l_2 : \mathbf{r} = (\mathbf{i} - \mathbf{j} + \mathbf{k}) + \mu(5\mathbf{i} - 4\mathbf{j} - \mathbf{k})$, $\mu \in \mathbb{R}$

Solution

Part (a)

Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix},\ \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}2\\1\\4\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\2\end{pmatrix},\ \mu \in \mathbb{R}$$

Since $\begin{pmatrix}2\\-2\\2\end{pmatrix} = 2\begin{pmatrix}1\\-1\\1\end{pmatrix}$, l_1 and l_2 are parallel. Since $\begin{pmatrix}1\\0\\2\end{pmatrix} \neq \begin{pmatrix}2\\1\\4\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\2\end{pmatrix}$ for all real μ, l_1 and l_2 are distinct.

$$\boxed{\text{Distinct parallel lines. } \theta = 0.}$$

Part (b)

Since $\begin{pmatrix}4\\-2\\3\end{pmatrix} \neq \beta\begin{pmatrix}3\\8\\1\end{pmatrix}$ for all real β, l_1 and l_2 are not parallel.

Consider $l_1 = l_2$.

$$\begin{aligned} & l_1 = l_2 \\ \implies & \begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix} = \begin{pmatrix}0\\10\\1\end{pmatrix} + \beta\begin{pmatrix}3\\8\\1\end{pmatrix} \\ \implies & \alpha\begin{pmatrix}4\\-2\\-3\end{pmatrix} - \beta\begin{pmatrix}3\\8\\1\end{pmatrix} = \begin{pmatrix}-1\\10\\1\end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 4\alpha - 3\beta &= -1 \\ -2\alpha - 8\beta &= 10 \\ -3\alpha - \beta &= 1 \end{cases}$$

There are no solutions to the above system. Hence, l_1 and l_2 do not intersect and are hence skew.

Let θ be the acute angle between l_1 and l_2.

$$\begin{aligned} \cos\theta &= \frac{\left|\begin{pmatrix}4\\-2\\-3\end{pmatrix}\cdot\begin{pmatrix}3\\8\\1\end{pmatrix}\right|}{\left|\begin{pmatrix}4\\-2\\-3\end{pmatrix}\right|\left|\begin{pmatrix}3\\8\\1\end{pmatrix}\right|} \\ &= \frac{7}{\sqrt{2146}} \\ \implies \quad \theta &= 1.42 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{Skew lines. } \theta = 1.42.}$$

Part (c)

Note that l_1 and l_2 have vector form

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}5\\-4\\-1\end{pmatrix}$$

Since $\begin{pmatrix}1\\-1\\1\end{pmatrix} \neq \mu\begin{pmatrix}5\\-4\\-1\end{pmatrix}$ for all real μ, l_1 and l_2 are not parallel.

Consider $l_1 = l_2$.

$$\begin{aligned} & l_1 = l_2 \\ \implies \quad & \begin{pmatrix}1\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\end{pmatrix} + \mu\begin{pmatrix}5\\-4\\-1\end{pmatrix} \\ \implies \lambda & \begin{pmatrix}1\\-1\\1\end{pmatrix} - \mu\begin{pmatrix}5\\-4\\-1\end{pmatrix} = \begin{pmatrix}0\\-1\\6\end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} \lambda - 5\mu &= 0 \\ -\lambda + 4\mu &= -1 \\ \lambda + \mu &= 6 \end{cases}$$

The above system has the unique solution $\lambda = 5$ and $\mu = 1$. Hence, l_1 and l_2 intersect at $\begin{pmatrix}1\\0\\-5\end{pmatrix} + 5\begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}6\\-5\\0\end{pmatrix}$.

Let θ be the acute angle between l_1 and l_2.

$$\begin{aligned}
\cos\theta &= \frac{\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}5\\-4\\-1\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\1\end{pmatrix}\right|\left|\begin{pmatrix}5\\-4\\-1\end{pmatrix}\right|} \\
&= \frac{8}{3\sqrt{14}} \\
\implies \qquad \theta &= 0.777 \text{ (3 s.f.)}
\end{aligned}$$

$$\boxed{\text{Intersecting lines. } \begin{pmatrix}6\\-5\\0\end{pmatrix}.\ \theta = 0.777.}$$

Problem 3.

(a) Find the shortest distance from the point $(1, 2, 3)$ to the line with equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} + \lambda(\mathbf{i} + 2\mathbf{j} + 2\mathbf{k})$, $\lambda \in \mathbb{R}$.

(b) Find the length of projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\dfrac{x+5}{4} = \dfrac{y-5}{3} = 10 - 2z$.

(c) Find the projection of $4\mathbf{i} - 5\mathbf{j} + 6\mathbf{k}$ onto the line with equation $\dfrac{x+5}{4} = \dfrac{y-5}{3} = 10 - 2z$.

Solution

Part (a)

Let $\overrightarrow{OP} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\overrightarrow{OA} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$. We have that A is on the line with equation

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \lambda \in \mathbb{R}$$

Note that $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix}$.

$$\begin{aligned} \text{Shortest distance} &= \frac{\left| \begin{pmatrix} -2 \\ 0 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right|} \\ &= \frac{1}{\sqrt{1^2 + 2^2 + 2^2}} \left| \begin{pmatrix} 2 \\ -3 \\ -4 \end{pmatrix} \right| \\ &= \frac{\sqrt{2^2 + (-3)^2 + (-4)^2}}{3} \\ &= \frac{\sqrt{29}}{3} \end{aligned}$$

$$\boxed{\text{The shortest distance is } \frac{\sqrt{29}}{3} \text{ units.}}$$

Part (b)

Note that the line has vector form

$$\begin{aligned}\mathbf{r} &= \begin{pmatrix}-5\\5\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\3\\-\frac{1}{2}\end{pmatrix}\\ &= \begin{pmatrix}-5\\5\\5\end{pmatrix} + \lambda\begin{pmatrix}8\\6\\-1\end{pmatrix}, \lambda \in \mathbb{R}\end{aligned}$$

$$\begin{aligned}\text{Length of projection} &= \frac{\left|\begin{pmatrix}4\\-5\\6\end{pmatrix} \cdot \begin{pmatrix}8\\6\\-1\end{pmatrix}\right|}{\left|\begin{pmatrix}8\\6\\-1\end{pmatrix}\right|}\\ &= \frac{4}{\sqrt{101}}\end{aligned}$$

$$\boxed{\text{The length of projection is } \frac{4}{\sqrt{101}} \text{ units.}}$$

Part (c)

$$\begin{aligned}\text{Projection} &= \frac{\begin{pmatrix}4\\-5\\6\end{pmatrix} \cdot \begin{pmatrix}8\\6\\-1\end{pmatrix}}{\left|\begin{pmatrix}8\\6\\-1\end{pmatrix}\right|} \cdot \frac{\begin{pmatrix}8\\6\\-1\end{pmatrix}}{\left|\begin{pmatrix}8\\6\\-1\end{pmatrix}\right|}\\ &= \frac{-4}{101}\begin{pmatrix}8\\6\\-1\end{pmatrix}\end{aligned}$$

Problem 4.

The points P and Q have coordinates $(0, -1, -1)$ and $(3, 0, 1)$ respectively, and the equations of the lines l_1 and l_2 are given by

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \lambda \in \mathbb{R} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \mu \in \mathbb{R}$$

(a) Show that P lies on l_1 but not on l_2.

(b) Determine if l_2 passes through Q.

(c) Find the coordinates of the foot of the perpendicular from P to l_2. Hence, or otherwise, find the perpendicular distance from P to l_2.

(d) Find the length of projection of $\overrightarrow{PQ}$ onto l_2.

Solution

We have that $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$.

Part (a)

When $\lambda = -2$, we have $\begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \overrightarrow{OP}$. Hence, P lies on l_1.

Observe that all points on l_2 have a z-coordinate of 1. Since P has a z-coordinate of -1, P does not lie on l_2.

Part (b)

When $\mu = 3$, we have $\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \overrightarrow{OQ}$. Hence, l_2 passes through Q.

$$\boxed{l_2 \text{ passes through } Q.}$$

Part (c)

Let the foot of the perpendicular from P to l_2 be F. Since F is on l_2, we have that $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ for some real μ. We also have that $\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0$.

$$\overrightarrow{PF} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$\begin{aligned}
\implies && \left(\overrightarrow{OF} - \overrightarrow{OP}\right) \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\implies && \left(\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\implies && \left(\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right) \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\implies && \begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = 0 \\
\implies && -10 + 5m = 0 \\
\implies && m = 2
\end{aligned}$$

Hence, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

$$\boxed{F(1, 1, 1)}$$

$$\begin{aligned}
\text{Perpendicular distance} &= \left|\overrightarrow{PF}\right| \\
&= \left|\overrightarrow{OF} - \overrightarrow{OP}\right| \\
&= \left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}\right| \\
&= \left|\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}\right| \\
&= \sqrt{1^2 + 2^2 + 2^2} \\
&= 3
\end{aligned}$$

$$\boxed{\text{The perpendicular distance from } P \text{ to } l_2 \text{ is 3 units.}}$$

Part (d)

Note that $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$.

$$\text{Length of projection} = \frac{\left|\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}\right|}$$

$$\begin{aligned} &= \frac{|6 - 1 + 0|}{\sqrt{2^2 + (-1)^2 + 0^2}} \\ &= \frac{5}{\sqrt{5}} \\ &= \sqrt{5} \end{aligned}$$

$$\boxed{\text{The length of projection of } \overrightarrow{PQ} \text{ onto } l_2 \text{ is } \sqrt{5} \text{ units.}}$$

Problem 5.

The lines l_1 and l_2 have equations

$$\mathbf{r} = \begin{pmatrix}0\\1\\2\end{pmatrix} + s\begin{pmatrix}1\\0\\3\end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix}-2\\3\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}$$

respectively. Find the position vectors of the points P on l_1 and Q on l_2 such that O, P and Q are collinear, where O is the origin.

Solution

We have that $\overrightarrow{OP} = \begin{pmatrix}0\\1\\2\end{pmatrix} + s\begin{pmatrix}1\\0\\3\end{pmatrix}$ and $\overrightarrow{OQ} = \begin{pmatrix}-2\\3\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}$ for some reals s and t.

For O, P and Q to be collinear, we need $\overrightarrow{OP} = \lambda\overrightarrow{OQ}$ for some real λ.

$$\begin{aligned} \begin{pmatrix}0\\1\\2\end{pmatrix} + s\begin{pmatrix}1\\0\\3\end{pmatrix} &= \lambda\left(\begin{pmatrix}-2\\3\\1\end{pmatrix} + t\begin{pmatrix}2\\1\\0\end{pmatrix}\right) \\ \implies \qquad \begin{pmatrix}s\\1\\2+3s\end{pmatrix} &= \lambda\begin{pmatrix}-2+2t\\3+t\\1\end{pmatrix} \end{aligned}$$

This gives use the system:

$$\begin{cases} s = \lambda(-2+2t) \\ 1 = \lambda(3+t) \\ 2+3s = \lambda \end{cases}$$

Substituting the third equation into the first two gives the reduced system:

$$\begin{cases} s = (2+3s)(-2+2t) \\ 1 = (2+3s)(3+t) \end{cases}$$

Subtracting twice of the second equation from the first yields

$$\begin{aligned} s-2 &= (2+3s)(-2+2t) - 2(2+3s)(3+t) \\ &= (2+3s)(-2+2t) - (2+3s)(6+2t) \\ &= (2+3s)(-2+2t-(6+2t)) \\ &= -8(2+3s) \\ &= -16-24s \\ \implies \quad 25s &= -14 \\ \implies \quad s &= -\frac{14}{25} \end{aligned}$$

It quickly follows that $t = \frac{1}{8}$. Hence,

$$\begin{aligned}
\overrightarrow{OP} &= \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{14}{25} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \\
&= \frac{1}{25} \begin{pmatrix} 0 \\ 25 \\ 50 \end{pmatrix} - \frac{1}{25} \begin{pmatrix} 14 \\ 0 \\ 42 \end{pmatrix} \\
&= \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix} \\
\overrightarrow{OQ} &= \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{8} \begin{pmatrix} -16 \\ 24 \\ 8 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}
\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \frac{1}{25} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}, \overrightarrow{OQ} = \frac{1}{8} \begin{pmatrix} -14 \\ 25 \\ 8 \end{pmatrix}}$$

Problem 6.

Relative to the origin O, the points A, B and C have position vectors $5\mathbf{i}+4\mathbf{j}+10\mathbf{k}$, $-4\mathbf{i}+4\mathbf{j}-2\mathbf{k}$ and $-5\mathbf{i}+9\mathbf{j}+5\mathbf{k}$ respectively.

(a) Find the Cartesian equation of the line AB.

(b) Find the length of projection of $\overrightarrow{AC}$ onto the line AB. Hence, find the perpendicular distance from C to the line AB.

(c) Find the position vector of the foot N of the perpendicular from C to the line AB.

(d) The point D is such that it is a reflection of point C about the line AB. Find the position vector of D.

Solution

We have that $\overrightarrow{OA}=\begin{pmatrix}5\\4\\10\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-4\\4\\-2\end{pmatrix}$ and $\overrightarrow{OC}=\begin{pmatrix}-5\\9\\5\end{pmatrix}$.

Part (a)

Note that $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-4\\4\\-2\end{pmatrix}-\begin{pmatrix}5\\4\\10\end{pmatrix}=\begin{pmatrix}-9\\0\\-12\end{pmatrix}=-3\begin{pmatrix}3\\0\\4\end{pmatrix}$. The line AB hence has the vector form

$$\mathbf{r}=\begin{pmatrix}5\\4\\10\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\4\end{pmatrix},\ \lambda\in\mathbb{R}$$

The line AB thus has the Cartesian form

$$\boxed{\frac{x-5}{3}=\frac{z-10}{4},\ y=4}$$

Part (b)

Note that $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=\begin{pmatrix}-5\\9\\5\end{pmatrix}-\begin{pmatrix}5\\4\\10\end{pmatrix}=\begin{pmatrix}-10\\5\\-5\end{pmatrix}=-5\begin{pmatrix}2\\-1\\1\end{pmatrix}$.

$$\begin{aligned}\text{Length of projection}&=\frac{\left|\overrightarrow{AC}\cdot\overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|}\\&=\frac{1}{-3\sqrt{3^2+0^2+4^2}}\left|-5\begin{pmatrix}2\\-1\\1\end{pmatrix}\cdot-3\begin{pmatrix}3\\0\\4\end{pmatrix}\right|\\&=10\end{aligned}$$

$$\boxed{\text{The perpendicular distance from } C \text{ to the line } AB \text{ is 10 units.}}$$

Part (c)

Let $\overrightarrow{AN} = \lambda \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix}$ for some real λ such that $\left|\overrightarrow{AN}\right| = 10$.

$$\begin{aligned} \overrightarrow{AN} &= 10 \\ \implies \lambda \cdot -3\sqrt{3^2 + 0^2 + 4^2} &= 10 \\ \implies \qquad\qquad \lambda &= \frac{2}{3} \end{aligned}$$

Hence, $\overrightarrow{AN} = \frac{2}{3} \begin{pmatrix} -9 \\ 0 \\ -12 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix}$. Thus, $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \begin{pmatrix} -6 \\ 0 \\ -8 \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}$.

$$\boxed{\overrightarrow{ON} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}}$$

Part (d)

Note that $\overrightarrow{NC} = \overrightarrow{OC} - \overrightarrow{ON} = \begin{pmatrix} -5 \\ 9 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix}$. Since D is the reflection of C about AB, we have that $\overrightarrow{ND} = -\overrightarrow{NC}$.

$$\begin{aligned} \overrightarrow{OD} &= \overrightarrow{ON} + \overrightarrow{ND} \\ &= \overrightarrow{ON} - \overrightarrow{NC} \\ &= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} -4 \\ 5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix} \end{aligned}$$

$$\boxed{\overrightarrow{OD} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}}$$

Problem 7.

The points A and B have coordinates $(0, 9, c)$ and $(d, 5, -2)$ respectively, where c and d are constants. The line l has equation $\dfrac{x+3}{-1} = \dfrac{y-1}{4} = \dfrac{z-5}{3}$.

(a) Given that $d = \dfrac{22}{7}$ and the line AB intersects l, find the value of c. Find also the coordinates of the foot of the perpendicular from A to l.

(b) Given instead that the lines AB and l are parallel, state the value of c and d and find the shortest distance between the lines AB and l.

Solution

We have that $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 9 \\ c \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} d \\ 5 \\ -2 \end{pmatrix}$. We also have that l is given by the vector $\mathbf{r} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix}$. Hence, the line AB is given by the vector $\mathbf{r}_{AB} = \begin{pmatrix} d \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix}$ for $\mu \in \mathbb{R}$.

Part (a)

Consider the direction vectors of AB and l. Since $\begin{pmatrix} \frac{22}{7} \\ -4 \\ -2-c \end{pmatrix} \neq \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ for all real λ and c, the lines AB and l are not parallel. Hence, AB and l intersect at only one point. Thus, there must be a unique solution to $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{aligned}
& \mathbf{r} = \mathbf{r}_{AB} \\
\implies & \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{22}{7} \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} \frac{22}{7} \\ -4 \\ -2-c \end{pmatrix} \\
\implies & \begin{pmatrix} -21 \\ 7 \\ 35 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 22 \\ 35 \\ -14 \end{pmatrix} + \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} \\
\implies & \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \mu \begin{pmatrix} 22 \\ -28 \\ -14-7c \end{pmatrix} = \begin{pmatrix} 43 \\ 28 \\ -49 \end{pmatrix}
\end{aligned}$$

This gives the following system:

$$\begin{cases} -\lambda - 22\mu &= 43 \\ 4\lambda + 28\mu &= 28 \\ 3\lambda + (14+7c)\mu &= -49 \end{cases}$$

Solving the first two equations gives $\lambda = \dfrac{91}{3}$ and $\mu = -\dfrac{10}{3}$. It follows from the third equation that $c = 4$.

$$\boxed{c = 4}$$

Let F be the foot of the perpendicular from A to l. We have that $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ for some real λ. We also have that $\overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = 0$.

$$\begin{aligned}
& & \overrightarrow{AF} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= 0 \\
&\implies & \left(\overrightarrow{OF} - \overrightarrow{OA}\right) \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= 0 \\
&\implies & \left(\begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 9 \\ 4 \end{pmatrix}\right) \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= 0 \\
&\implies & \left(\begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}\right) \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= 0 \\
&\implies & \begin{pmatrix} -3 \\ -8 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} &= 0 \\
&\implies & -26 + 26\lambda &= 0 \\
&\implies & \lambda &= 1
\end{aligned}$$

Hence, $\overrightarrow{OF} = \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \\ 8 \end{pmatrix}$.

$$\boxed{\text{The foot of the perpendicular from } A \text{ to } l \text{ has coordinates } (-4, 5, 8).}$$

Part (b)

Given that AB is parallel to l, one of their direction vectors must be a scalar multiple of the other. Hence, for some real λ,

$$\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = \lambda \begin{pmatrix} d \\ -4 \\ -2-c \end{pmatrix}$$

It is obvious that $\lambda = -1$, whence $c = 1$ and $d = 1$.

$$\boxed{c = 1,\ d = 1}$$

Note that the direction vector of l and AB is $\begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$. Also note that $(-3, 1, 5)$ is on l and $(1, 5, -2)$ is on AB.

$$\begin{aligned}
\text{Shortest distance between } AB \text{ and } l &= \frac{\left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \left(\begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix} - \begin{pmatrix} -3 \\ 1 \\ 5 \end{pmatrix} \right) \right|}{\left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \right|} \\
&= \frac{1}{\sqrt{(-1)^2 + 4^2 + 3^2}} \left| \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix} \right| \\
&= \frac{1}{\sqrt{26}} \left| \begin{pmatrix} -40 \\ -5 \\ -20 \end{pmatrix} \right| \\
&= \frac{1}{\sqrt{26}} \left| -5 \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix} \right| \\
&= \frac{5\sqrt{8^2 + 1^2 + 4^2}}{\sqrt{26}} \\
&= \frac{45}{\sqrt{26}}
\end{aligned}$$

$$\boxed{\text{The shortest distance between } AB \text{ and } l \text{ is } \frac{45}{\sqrt{26}} \text{ units.}}$$

Problem 8.

The equation of the line L is $\mathbf{r} = \begin{pmatrix}1\\3\\7\end{pmatrix} + t\begin{pmatrix}2\\-1\\5\end{pmatrix}$, $t \in \mathbb{R}$. The points A and B have position vectors $\begin{pmatrix}9\\3\\26\end{pmatrix}$ and $\begin{pmatrix}13\\9\\\alpha\end{pmatrix}$ respectively. The line L intersects the line through A and B at P.

(a) Find α and the acute angle between line L and AB.

The point C has position vector $\begin{pmatrix}2\\5\\1\end{pmatrix}$ and the foot of the perpendicular from C to L is Q.

(b) Find the position vector of Q. Hence, find the shortest distance from C to L.

(c) Find the position vector of the point of reflection of the point C about the line L. Hence, find the reflection of the line passing through C and the point $(1, 3, 7)$ about the line L.

Solution

Part (a)

We have that $\overrightarrow{OA} = \begin{pmatrix}9\\3\\26\end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix}13\\9\\\alpha\end{pmatrix}$. Hence, $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix}4\\6\\\alpha-26\end{pmatrix}$.

The line AB is thus given by $\mathbf{r}_{AB} = \begin{pmatrix}9\\3\\26\end{pmatrix} + u\begin{pmatrix}4\\6\\\alpha-26\end{pmatrix}$ for $u \in \mathbb{R}$. Note that AB is not parallel to L. Hence, $\overrightarrow{OP}$ is the only solution to the equation $\mathbf{r} = \mathbf{r}_{AB}$.

$$\begin{aligned} \begin{pmatrix}1\\3\\7\end{pmatrix} + t\begin{pmatrix}2\\-1\\5\end{pmatrix} &= \begin{pmatrix}9\\3\\26\end{pmatrix} + u\begin{pmatrix}4\\6\\\alpha-26\end{pmatrix} \\ \implies t\begin{pmatrix}2\\-1\\5\end{pmatrix} - u\begin{pmatrix}4\\6\\\alpha-26\end{pmatrix} &= \begin{pmatrix}8\\0\\19\end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} 2t - 4u &= 8 \\ -t - 6u &= 0 \\ 5t - (\alpha - 26)u &= 19 \end{cases}$$

Solving the first two equations gives $t = 3$ and $u = -\frac{1}{2}$. It follows from the third equation that $\alpha = 34$.

$$\boxed{\alpha = 34}$$

Let the acute angle between L and AB be θ.

$$\begin{aligned}
\cos\theta &= \frac{\left|\begin{pmatrix}2\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}4\\6\\8\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\-1\\5\end{pmatrix}\right|\left|\begin{pmatrix}4\\6\\8\end{pmatrix}\right|} \\
&= \frac{42}{\sqrt{30}\sqrt{116}} \\
\implies \qquad \theta &= \arccos\frac{42}{\sqrt{30}\sqrt{116}} \\
&= 44.6^\circ \text{ (1 d.p.)}
\end{aligned}$$

$$\boxed{\theta = 44.6^\circ \text{ (1 d.p.)}}$$

Part (b)

Since Q is on L, we have that $\overrightarrow{OQ} = \begin{pmatrix}1\\3\\7\end{pmatrix} + t\begin{pmatrix}2\\-1\\5\end{pmatrix}$ for some real t. Further, since $\overrightarrow{CQ} \perp L$, we have that $\overrightarrow{CQ}\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix} = 0$.

$$\begin{aligned}
&& \overrightarrow{CQ}\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix} &= 0 \\
\implies && \left(\overrightarrow{OQ}-\overrightarrow{OC}\right)\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix} &= 0 \\
\implies && \left(\begin{pmatrix}1\\3\\7\end{pmatrix}+t\begin{pmatrix}2\\-1\\5\end{pmatrix}-\begin{pmatrix}2\\5\\1\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix} &= 0 \\
\implies && \left(\begin{pmatrix}-1\\-2\\6\end{pmatrix}+t\begin{pmatrix}2\\-1\\5\end{pmatrix}\right)\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix} &= 0 \\
\implies && \begin{pmatrix}-1\\-2\\6\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix}+t\begin{pmatrix}2\\-1\\5\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\5\end{pmatrix} &= 0 \\
\implies && 30+30t &= 0 \\
\implies && t &= 1
\end{aligned}$$

Hence, $\overrightarrow{OQ} = \begin{pmatrix}1\\3\\7\end{pmatrix} + \begin{pmatrix}2\\-1\\5\end{pmatrix} = \begin{pmatrix}-1\\4\\2\end{pmatrix}$.

$$\boxed{\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix}}$$

$$\begin{aligned}
\text{Shortest distance from } C \text{ to } L &= \left|\overrightarrow{CQ}\right| \\
&= \left|\begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}\right| \\
&= \left|\begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}\right| \\
&= \sqrt{(-3)^2 + (-1)^2 + 1^2} \\
&= \sqrt{11}
\end{aligned}$$

$$\boxed{\text{The shortest distance from } C \text{ to } L \text{ is } \sqrt{11} \text{ units.}}$$

Part (c)

Let C' be the reflection of C about L.

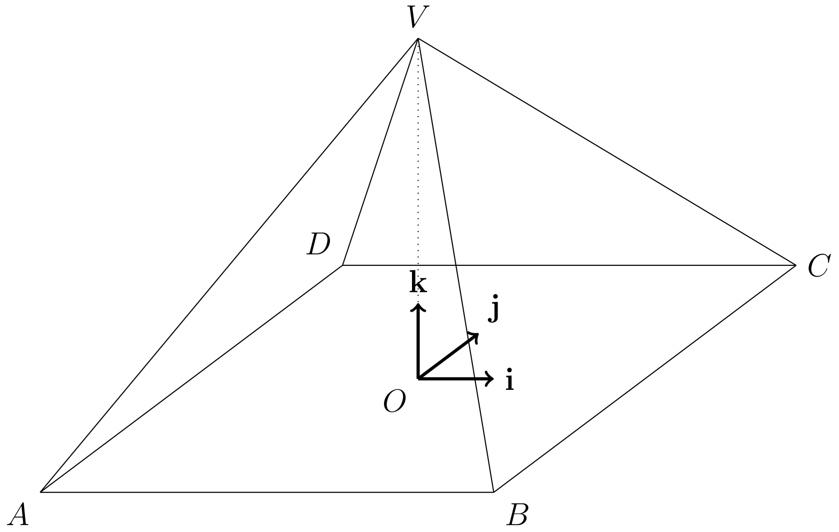
$$\begin{aligned}
\overrightarrow{OC'} &= \overrightarrow{OQ} - \overrightarrow{QC} \\
&= \overrightarrow{OQ} + \overrightarrow{CQ} \\
&= \begin{pmatrix} -1 \\ 4 \\ 2 \end{pmatrix} + \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}
\end{aligned}$$

$$\boxed{\overrightarrow{OC'} = \begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix}}$$

Note that $(1, 3, 7)$ is on L and is hence invariant under a reflection about L. Let the reflection about L of the line passing through C and $(1, 3, 7)$ be L'. Since $\begin{pmatrix} -4 \\ 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} = \begin{pmatrix} -5 \\ 0 \\ -4 \end{pmatrix} = -\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$, L' has direction vector $\begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$. Thus, L' is given by $\mathbf{r}' = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$ for $\lambda \in \mathbb{R}$.

$$\boxed{L' : \mathbf{r}' = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} + \lambda \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}, \ \lambda \in \mathbb{R}}$$

Problem 9.



In the diagram, O is the origin of the square base $ABCD$ of a right pyramid with vertex V. The perpendicular unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are parallel to AB, AD and OV respectively. The length of AB is 4 units and the length of OV is $2h$ units. P, Q, M and N are the mid-points of AB, BC, CV and VA respectively. The point O is taken as the origin for position vectors.

Show that the equation of the line PM may be expressed as $\mathbf{r} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ h \end{pmatrix}$, where t is a parameter.

(a) Find an equation for the line QN.

(b) Show that the lines PM and QN intersect and that the position vector $\overrightarrow{OX}$ of their point of intersection is $\mathbf{r} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix}$.

(c) Given that OX is perpendicular to VB, find the value of h and calculate the acute angle between PM and QN, giving your answer correct to the nearest 0.1°.

Solution

We are given that $\overrightarrow{OP} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ and $\overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 2h \end{pmatrix}$. Hence, $\overrightarrow{CV} = \overrightarrow{OV} - \overrightarrow{OC} = \begin{pmatrix} -2 \\ -2 \\ 2h \end{pmatrix}$. Thus, $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CV} = \begin{pmatrix} -1 \\ -1 \\ h \end{pmatrix}$. Since $\overrightarrow{OM} = \overrightarrow{OC} + \overrightarrow{CM} = \begin{pmatrix} 1 \\ 1 \\ h \end{pmatrix}$, we have that $\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = \begin{pmatrix}1\\3\\h\end{pmatrix}$. Thus, PM is given by

$$\mathbf{r} = \begin{pmatrix}0\\-2\\0\end{pmatrix} + t\begin{pmatrix}1\\3\\h\end{pmatrix},\ t \in \mathbb{R}$$

Part (a)

Since $\overrightarrow{OM} = \begin{pmatrix}1\\1\\h\end{pmatrix}$, by symmetry, $\overrightarrow{ON} = \begin{pmatrix}-1\\-1\\h\end{pmatrix}$. Given that $\overrightarrow{OQ} = \begin{pmatrix}2\\0\\0\end{pmatrix}$, we have that $\overrightarrow{QN} = \overrightarrow{ON} - \overrightarrow{OQ} = \begin{pmatrix}-3\\-1\\h\end{pmatrix}$. Thus, QN is given by

$$\boxed{\mathbf{r} = \begin{pmatrix}2\\0\\0\end{pmatrix} + u\begin{pmatrix}-3\\-1\\h\end{pmatrix},\ u \in \mathbb{R}}$$

Part (b)

Consider $PM = QN$.

$$\begin{aligned} & PM = QN \\ \implies\ & \begin{pmatrix}0\\-2\\0\end{pmatrix} + t\begin{pmatrix}1\\3\\h\end{pmatrix} = \begin{pmatrix}2\\0\\0\end{pmatrix} + u\begin{pmatrix}-3\\-1\\h\end{pmatrix} \\ \implies\ & t\begin{pmatrix}1\\3\\h\end{pmatrix} - u\begin{pmatrix}-3\\-1\\h\end{pmatrix} = \begin{pmatrix}2\\2\\0\end{pmatrix} \end{aligned}$$

This gives the following system:

$$\begin{cases} t + 3u &= 2 \\ 3t + u &= 2 \\ th - uh &= 0 \end{cases}$$

From the first two equations, we see that $t = \frac{1}{2}$ and $u = \frac{1}{2}$, which is consistent with the third equation. Hence, $\overrightarrow{OX} = \begin{pmatrix}0\\-2\\0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\3\\h\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\-1\\h\end{pmatrix}$.

Part (c)

Note that $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -2 \\ 6 \end{pmatrix}$, whence $\overrightarrow{VB} = \overrightarrow{OB} - \overrightarrow{OV} = \begin{pmatrix} 2 \\ -2 \\ -2h \end{pmatrix}$. Since OX is perpendicular to VB, we have that $\overrightarrow{OX} \cdot \overrightarrow{VB} = 0$.

$$\begin{aligned} & \overrightarrow{OX} \cdot \overrightarrow{VB} = 0 \\ \implies & \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ h \end{pmatrix} \cdot 2\begin{pmatrix} 1 \\ -1 \\ -h \end{pmatrix} = 0 \\ \implies & 1 + 1 - h^2 = 0 \\ \implies & h^2 = 2 \end{aligned}$$

We hence have that $h = \sqrt{2}$. Note that we reject $h = -\sqrt{2}$ since $h > 0$.

$$\boxed{h = \sqrt{2}}$$

Let the acute angle between PM and QN be θ.

$$\begin{aligned} \cos\theta &= \frac{\left|\overrightarrow{PM} \cdot \overrightarrow{QN}\right|}{\left|\overrightarrow{PM}\right| \left|\overrightarrow{QN}\right|} \\ &= \frac{1}{\sqrt{1^2 + 3^2 + \sqrt{2}^2}} \cdot \frac{1}{\sqrt{(-3)^2 + (-1)^2 + \sqrt{2}^2}} \cdot \left|\begin{pmatrix} 1 \\ 3 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} -3 \\ -1 \\ \sqrt{2} \end{pmatrix}\right| \\ &= \frac{1}{\sqrt{12}} \cdot \frac{1}{\sqrt{12}} \cdot \left|-3 - 3 + 2\right| \\ &= \frac{1}{3} \\ \implies \qquad \theta &= \arccos\frac{1}{3} \\ &= 70.5^\circ \text{ (1 d.p.)} \end{aligned}$$

$$\boxed{\theta = 70.5^\circ \text{ (1 d.p.)}}$$