Problem 1.

Estimate, using the trapezium rule, the values of the following definite integrals, taking the number ordinates given in each case.

(a) $\displaystyle\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta$, 3 ordinates

(b) $\displaystyle\int_{-0.4}^{0.2} \frac{x^2-4x+1}{4x-4}$, 4 ordinates

Solution

Part (a)

Let $f(\theta) = \dfrac{1}{1+\cos\theta}$.

$$\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta \approx \frac{1}{2}\cdot\frac{0-(-\pi/2)}{3-1}\cdot\left[f\left(-\frac{\pi}{2}\right)+2f\left(-\frac{\pi}{4}\right)+f(0)\right] = 1.05$$

$$\boxed{\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta \approx 1.05}$$

Part (b)

Let $f(x) = \dfrac{x^2-4x+1}{4x-4}$.

$$\int_{-0.4}^{0.2} \frac{x^2-4x+1}{4x-4}\,\mathrm{d}x \approx \frac{1}{2}\cdot\frac{0.2-(-0.4)}{4-1}\cdot\Big[f(-0.4)+2\big(f(-0.2)+f(0)\big)+f(0.2)\Big] = -0.183$$

$$\boxed{\int_{-0.4}^{0.2} \frac{x^2-4x+1}{4x-4}\,\mathrm{d}x \approx -0.183}$$

Problem 2.

Use the trapezium rule with intervals of width 0.5 to obtain an approximation to $\displaystyle\int_2^{3.5} \ln\frac{1}{x}\,\mathrm{d}x$, giving your answer to 2 decimal places.

Solution

$$\begin{aligned}
\int_2^{3.5} \ln\frac{1}{x}\,\mathrm{d}x &= -\int_2^{3.5} \ln x\,\mathrm{d}x \\
&\approx -\left(\frac{1}{2}\cdot\frac{3.5-2}{4-1}\cdot[\ln 2 + 2(\ln 2.5 + \ln 3) + \ln 3.5]\right) \\
&= -1.49 \text{ (2 d.p.)}
\end{aligned}$$

$$\boxed{\int_2^{3.5} \ln\frac{1}{x}\,\mathrm{d}x = -1.49}$$

Problem 3.

Estimate, using Simpson's rule, the values of the following definite integrals, taking the number of ordinates given in each case.

(a) $\displaystyle\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta$, 3 ordinates

(b) $\displaystyle\int_{0}^{0.4} \sqrt{1-x^2}\,\mathrm{d}x$, 5 ordinates

Solution

Part (a)

Let $f(\theta) = \dfrac{1}{1+\cos\theta}$.

$$\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta \approx \frac{1}{3}\cdot\frac{0-(-\pi/2)}{3-1}\cdot\left[f\left(-\frac{\pi}{2}\right)+4f\left(-\frac{\pi}{4}\right)+f(0)\right] = 1.01$$

$$\boxed{\int_{-\pi/2}^{0} \frac{1}{1+\cos\theta}\,\mathrm{d}\theta \approx 1.01}$$

Part (b)

Let $f(x) = \sqrt{1-x^2}$.

$$\int_{0}^{0.4} \sqrt{1-x^2}\,\mathrm{d}x \approx \frac{1}{3}\cdot\frac{0.4-0}{5-1}\cdot\Big[f(0)+4f(0.1)+2f(0.2)+4f(0.3)+f(0.4)\Big] = 0.389$$

$$\boxed{\int_{0}^{0.4} \sqrt{1-x^2}\,\mathrm{d}x \approx 0.389}$$

Problem 4.

Show, by means of substitution $u = \sqrt{x}$, that

$$\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} \,\mathrm{d}x = \int_0^{0.5} 2e^{-u^2} \,\mathrm{d}u$$

Use the trapezium rule, with ordinates at $u = 0$, $u = 0.1$, $u = 0.2$, $u = 0.3$, $u = 0.4$ and $u = 0.5$, to estimate the value of $I = \displaystyle\int_0^{0.5} 2e^{-u^2} \,\mathrm{d}u$, giving three decimal places in your answer.

Explain briefly why the trapezium rule cannot be used directly to estimate the value of $\displaystyle\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} \,\mathrm{d}x$.

By using the first four terms of the expansion of e^{-x}, obtain an estimate for the integral $\displaystyle\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} \,\mathrm{d}x$, giving three decimal places in your answer.

Solution

Note that $u = \sqrt{x} \implies u^2 = x \implies 2u \,\mathrm{d}u = \mathrm{d}x$. Furthermore, $x = 0 \implies u = 0$ and $x = 0.25 \implies u = 0.5$.

$$\begin{aligned}\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} \,\mathrm{d}x &= \int_0^{0.5} \frac{1}{u} e^{-u^2} \cdot 2u \,\mathrm{d}u \\ &= \int_0^{0.5} 2e^{-u^2} \,\mathrm{d}u\end{aligned}$$

Let $f(u) = 2e^{-u^2}$. Using the trapezium rule,

$$\begin{aligned}\int_0^{0.5} 2e^{-u^2} \,\mathrm{d}u &\approx \frac{1}{2} \cdot \frac{0.5 - 0}{5} \cdot \Big[f(0) + 2\big(f(0.1) + f(0.2) + f(0.3) + f(0.4)\big) + f(0.5)\Big] \\ &= 0.921 \text{ (3 d.p.)}\end{aligned}$$

$$\boxed{I \approx 0.921}$$

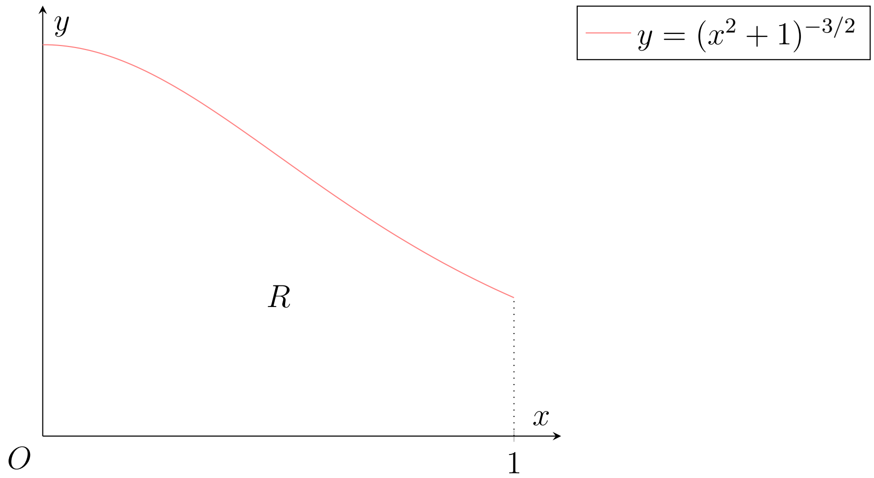
At $x = 0$, $\dfrac{1}{\sqrt{x}} e^{-x}$ is undefined. Hence, the trapezium rule cannot be used.

Recall that $e^{-x} = 1 - x + \dfrac{1}{2}x^2 - \dfrac{1}{6}x^3 + \dots$.

$$\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} \,\mathrm{d}x \approx \int_0^{0.25} \frac{1}{\sqrt{x}} \left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3\right) \mathrm{d}x = 0.923 \text{ (3 d.p.)}$$

$$\boxed{\int_0^{0.25} \frac{1}{\sqrt{x}} e^{-x} \,\mathrm{d}x \approx 0.923}$$

Problem 5.



The diagram (not to scale) show the region R bounded by the axes, the curve $y = (x^2+1)^{-3/2}$ and the line $x = 1$. The integral $\int_0^1 (x^2+1)^{-3/2}$ is denoted by I.

(a) Use the trapezium rule and Simpson's rule, with ordinates at $x = 0$, $x = 0.5$ and $x = 1$, to estimate the value of I correct to 4 significant figures.

(b) Use the substitution $x = \tan\theta$ to show that $I = \frac{1}{2}\sqrt{2}$. Comment on the approximations using the 2 rules and give a reason why one gives a better approximation than the other.

(c) By using the trapezium rule, with the same ordinates as in part (a), or otherwise, estimate the volume of the solid formed when R is rotated completely about the x-axis, giving your answer to 2 significant figures.

Solution

Part (a)

Let $f(x) = (x^2+1)^{-3/2}$. Using the trapezium rule,

$$I \approx \frac{1}{2} \cdot \frac{1-0}{3-1} \cdot \Big[f(0) + 2f(0.5) + f(1)\Big] = 0.6962 \text{ (4 s.f.)}$$

$$\boxed{I \approx 0.6962}$$

Using Simpson's rule,

$$I \approx \frac{1}{3} \cdot \frac{1-0}{3-1} \cdot \Big[f(0) + 4f(0.5) + f(1)\Big] = 0.7026 \text{ (4 s.f.)}$$

$$\boxed{I \approx 0.7026}$$

Part (b)

$$x = \tan\theta \qquad \mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$$

$$\begin{aligned}
\int_0^1 (x^2+1)^{-3/2} &= \int_0^{\pi/4} \left(\tan^2\theta + 1\right)^{-3/2} \sec^2\theta\,\mathrm{d}\theta \\
&= \int_0^{\pi/4} (\sec^2\theta)^{-3/2} \sec^2\theta\,\mathrm{d}\theta \\
&= \int_0^{\pi/4} \frac{1}{\sec\theta}\,\mathrm{d}\theta \\
&= \int_0^{\pi/4} \cos\theta\,\mathrm{d}\theta \\
&= [\sin\theta]_0^{\pi/4} \\
&= \frac{1}{2}\sqrt{2}
\end{aligned}$$

The curve $y = (x^2+1)^{-3/2}$ resembles a quadratic more than a line on the interval $[0, 1]$. Hence, Simpson's rule, which uses quadratic curves to approximate y, gives a better approximation for I than the trapezium rule, which uses lines to approximate y.

Part (c)

Let $g(x) = (x^2+1)^{-3}$.

$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 y^2\,\mathrm{d}x \\
&= \pi \int_0^1 (x^2+1)^{-3}\,\mathrm{d}x \\
&\approx \pi \left(\frac{1}{2} \cdot \frac{1-0}{3-1} \cdot \Big[g(0) + 2g(0.5) + g(1)\Big]\right) \\
&= 1.7 \text{ (2 s.f.)}
\end{aligned}$$

$$\boxed{\text{The volume of the solid formed is 1.7 units}^3.}$$

Problem 6.

It is given that $f(x) = \dfrac{1}{\sqrt{1+\sqrt{x}}}$, and the integral $\displaystyle\int_0^1 f(x)\,\mathrm{d}x$ is denoted by I.

(a) Using the trapezium rule, with four trapezia of equal width, obtain an approximation I_1 to the value of I, giving 3 decimal places in your answer.

(b) Explain, with the aid of a sketch, why $I < I_1$.

(c) Evaluate I_2, where $I_2 = \dfrac{1}{3}\displaystyle\sum_{r=1}^{3} f\left(\frac{1}{3}r\right)$, giving 3 decimal places in your answer, and use the sketch in (b) to justify the inequality $I > I_2$.

(d) By means of a substitution $\sqrt{x} = u - 1$, show that the value of I is $\dfrac{4}{3}(2-\sqrt{2})$.

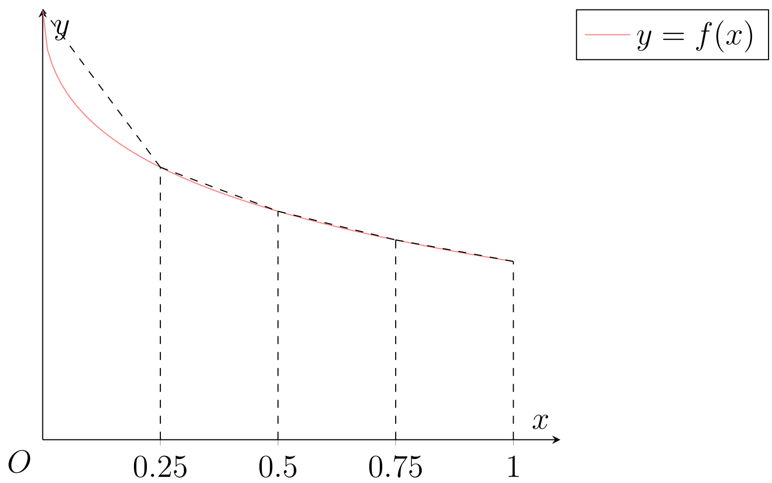
Solution

Part (a)

$$I_1 = \frac{1}{2}\cdot\frac{1-0}{4}\cdot\Big[f(0) + 2\big(f(0.25)+f(0.5)+f(0.75)\big)+f(1)\Big] = 0.792 \text{ (3 d.p.)}$$

$$\boxed{I_1 = 0.792}$$

Part (b)

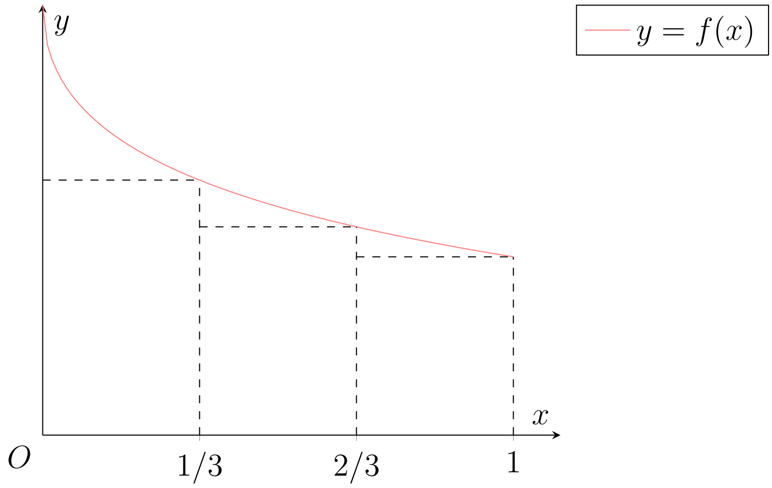


I is the area under the curve $y = f(x)$, while I_1 is the sum of the areas of the trapeziums. Hence, from the sketch, $I_1 > I$.

Part (c)

$$\begin{aligned} I_2 &= \frac{1}{3}\sum_{r=1}^{3} f\left(\frac{1}{3}r\right) \\ &= \frac{1}{3}\left[f\left(\frac{1}{3}\right) + f\left(\frac{2}{3}\right) + f(1)\right] \\ &= 0.748 \text{ (3 d.p.)} \end{aligned}$$

$$\boxed{I_2 = 0.748}$$



I is the area under the curve $y = f(x)$, while I_2 is the sum of the areas of the rectangles. Hence, from the sketch, $I_2 < I$.

Part (d)

Note $\sqrt{x} = u - 1 \implies x = u^2 - 2u + 1 \implies \mathrm{d}x = (2u-2)\,\mathrm{d}u$. Furthermore, $x = 0 \implies u = 1$ and $x = 1 \implies u = 2$.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{1+\sqrt{x}}}\,\mathrm{d}x &= \int_1^2 \frac{1}{\sqrt{1+(u-1)}} \cdot (2u-2)\,\mathrm{d}u \\
&= 2\int_1^2 \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \\
&= 2\left[\frac{2}{3}u^{3/2} - 2u^{1/2}\right]_1^2 \\
&= \frac{4}{3}(2-\sqrt{2})
\end{aligned}$$

Problem 7.

For $0 < x < \pi$, the curve C has the equation $y = \ln \sin x$. The region of the plane bounded by C, the x-axis and the lines $x = \dfrac{\pi}{4}$ and $x = \dfrac{\pi}{2}$ is rotated through 2π radians about the x-axis.

Show that the surface area of the solid generated in this way is given by S, where

$$S = 2\pi \int_{\pi/4}^{\pi/2} \left| \frac{\ln \sin x}{\sin x} \right| \mathrm{d}x$$

Use Simpson's rule with 5 ordinates to find an approximate value of S, giving your answer to 3 decimal places.

Solution

Note that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\sin x} \cdot \cos x = \cot x \implies 1 + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1 + \cot^2 x = \csc^2 x$.

$$\begin{aligned} S &= 2\pi \int_{\pi/4}^{\pi/2} y \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \, \mathrm{d}x \\ &= 2\pi \int_{\pi/4}^{\pi/2} \ln \sin x \sqrt{\csc^2 x} \, \mathrm{d}x \\ &= 2\pi \int_{\pi/4}^{\pi/2} \ln \sin x \left| \csc x \right| \mathrm{d}x \\ &= 2\pi \int_{\pi/4}^{\pi/2} \left| \ln \sin x \right| \left| \frac{1}{\sin x} \right| \mathrm{d}x \\ &= 2\pi \int_{\pi/4}^{\pi/2} \left| \frac{\ln \sin x}{\sin x} \right| \mathrm{d}x \end{aligned}$$

Let $f(x) = \left| \dfrac{\ln \sin x}{\sin x} \right|$.

$$\begin{aligned} S &\approx 2\pi \left(\frac{1}{3} \cdot \frac{\pi/2 - \pi/4}{5 - 1} \cdot \left[f\left(\frac{4}{16}\pi\right) + 4f\left(\frac{5}{16}\pi\right) + 2f\left(\frac{6}{16}\pi\right) + 4f\left(\frac{7}{16}\pi\right) + f\left(\frac{8}{16}\pi\right) \right] \right) \\ &= 0.670 \text{ (3 d.p.)} \end{aligned}$$

$$\boxed{S \approx 0.670}$$

Problem 8.

The value of the integral $\int_{0.2}^{0.4} f(x)\,\mathrm{d}x$ is to be estimated from information in the table below.

x	0.2	0.3	0.4
$f(x)$	1.2030	1.2441	1.2777

(a) Find the best possible estimate for the integral using the trapezium rule.

(b) Using the table of values above, find an approximate value for $f''(0.3)$ and use your answer to explain why the estimate found in part (a) is likely to be smaller than the actual value.

(c) Estimate the integral using Simpson's rule and determine the equation of the curve used in this method.

Solution

Part (a)

$$\int_{0.2}^{0.4} f(x)\,\mathrm{d}x \approx \frac{1}{2} \cdot \frac{0.4-0.2}{3-1} \cdot \Big[f(0.2) + 2f(0.3) + f(0.4)\Big] = 0.248$$

$$\boxed{\int_{0.2}^{0.4} f(x)\,\mathrm{d}x \approx 0.248}$$

Part (b)

Note that $f'(0.25) \approx \dfrac{f(0.3)-f(0.2)}{0.3-0.2} = 0.411$ and $f'(0.35) \approx \dfrac{f(0.4)-f(0.3)}{0.4-0.3} = 0.336$. Hence,

$$f''(0.30) \approx \frac{f'(0.35)-f'(0.25)}{0.35-0.25} = -0.75$$

$$\boxed{f''(0.30) \approx -0.75}$$

Since $f''(0.3) < 0$, $f(x)$ is concave downwards around $x = 0.3$. Hence, the estimate is likely to be smaller than the actual value.

Part (c)

$$\int_{0.2}^{0.4} f(x)\,\mathrm{d}x \approx= \frac{1}{3} \cdot \frac{0.4-0.2}{3-1} \cdot \Big[f(0.2) + 4f(0.3) + f(0.4)\Big] = 0.249$$

$$\boxed{\int_{0.2}^{0.4} f(x)\,\mathrm{d}x \approx 0.249}$$

Let the equation of the quadratic used be $P(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Since $P(0.2) = f(0.2)$, $P(0.3) = f(0.3)$ and $P(0.4) = f(0.4)$, we obtain the system

$$\begin{cases} (0.2)^2 a + 0.2b + c = 1.2030 \\ (0.3)^2 a + 0.3b + c = 1.2441 \\ (0.4)^2 a + 0.4b + c = 1.2777 \end{cases}$$

which has the unique solution $a = -0.375$, $b = 0.5985$, $c = 1.0983$. Thus, the required equation is $y = -0.375x^2 + 0.5985x + 1.0983$.

$$\boxed{y = -0.375x^2 + 0.5985x + 1.0983}$$

Problem 9.

The curve C is given by $y = \frac{1}{x}$, where $x > 0$.

(a) Apply the trapezium rule with ordinates at unit intervals to the function $f : x \mapsto \frac{1}{x}$, $x \in \mathbb{R}^+$, to show that $\ln n < \frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r}$ where $n \geq 3$.

(b) Obtain the area of the trapezium bounded by the axis, the lines $x = r \pm \frac{1}{2}$, and the tangent to the curve $y = \frac{1}{x}$ at the point $\left(r, \frac{1}{r}\right)$.

Hence, show that $\sum_{r=2}^{n-1} \frac{1}{r} < \ln\left(\frac{2n-1}{3}\right)$, where $n \geq 3$.

(c) From these results, obtain numerical values between which the value of $\sum_{r=2}^{99} \frac{1}{r}$ lies, and show that $4.110 < \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{100} < 4.205$.

Solution

Part (a)

$$\begin{aligned}
\int_1^n \frac{1}{x} \, \mathrm{d}x &\approx \frac{1}{2} \cdot 1 \cdot \left[\frac{1}{1} + 2\left(\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right) + \frac{1}{n}\right] \\
&= \frac{1}{2}\left(1 + \frac{1}{n} + 2\sum_{r=2}^{n-1} \frac{1}{r}\right) \\
&= \frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r}
\end{aligned}$$

Note that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} \frac{1}{x} = \frac{2}{x^3} > 0$ for $x > 0$. Hence, $y = \frac{1}{x}$ is concave upwards. Thus,

$$\frac{1}{2} + \frac{1}{2n} + \sum_{r=2}^{n-1} \frac{1}{r} > \int_1^n \frac{1}{x} \, \mathrm{d}x = \ln n$$

Part (b)

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{x^2}$, the equation of the tangent at $x = r$ is given by

$$y - \frac{1}{r} = -\frac{1}{r^2}(x - r) \implies y = -\frac{1}{r^2}x + \frac{2}{r}$$

Area of trapezium centred at r

$$
\begin{aligned}
&= \int_{r-1/2}^{r+1/2} \left(-\frac{1}{r^2}x + \frac{2}{r}\right) \mathrm{d}x \\
&= \left[-\frac{1}{r^2} \cdot \frac{1}{2}x^2 + \frac{2}{r}x\right]_{r-1/2}^{r+1/2} \\
&= \left[-\frac{1}{2r^2}\left(r + \frac{1}{2}\right)^2 + \frac{2}{r}\left(r + \frac{1}{2}\right)\right] - \left[-\frac{1}{r^2} \cdot \frac{1}{2}\left(r - \frac{1}{2}\right)^2 + \frac{2}{r}\left(r - \frac{1}{2}\right)\right] \\
&= \frac{1}{2r^2}\left[\left(r - \frac{1}{2}\right)^2 - \left(r + \frac{1}{2}\right)^2\right] + \frac{2}{r}\left[\left(r + \frac{1}{2}\right) - \left(r - \frac{1}{2}\right)\right] \\
&= \frac{1}{2r^2} \cdot 2r \cdot -1 + \frac{2}{r} \cdot 1 \\
&= \frac{-1}{r} + \frac{2}{r} \\
&= \frac{1}{r}
\end{aligned}
$$

$$\boxed{\text{The area of the trapezium is } \frac{1}{r} \text{ units}^2.}$$

Observe that the area of the trapezium centred at r is less than the area under the curve $y = \dfrac{1}{x}$ from $r - \dfrac{1}{2}$ to $r + \dfrac{1}{2}$. That is,

$$\frac{1}{r} < \int_{r-1/2}^{r+1/2} \frac{1}{x} \,\mathrm{d}x = \ln\left(r + \frac{1}{2}\right) - \ln\left(r - \frac{1}{2}\right)$$

Summing from $r = 2$ to $n - 1$,

$$
\begin{aligned}
\sum_{r=2}^{n-1} \frac{1}{r} &< \sum_{r=2}^{n-1} \left[\ln\left(r + \frac{1}{2}\right) - \ln\left(r - \frac{1}{2}\right)\right] \\
&= \sum_{r=2}^{n-1} \ln\left(r + \frac{1}{2}\right) - \sum_{r=2}^{n-1} \ln\left(r - \frac{1}{2}\right) \\
&= \sum_{r=3}^{n} \ln\left(r - \frac{1}{2}\right) - \sum_{r=2}^{n-1} \ln\left(r - \frac{1}{2}\right) \\
&= \ln\left(n - \frac{1}{2}\right) - \ln\left(2 - \frac{1}{2}\right) \\
&= \ln\left(\frac{n - 1/2}{3/2}\right) \\
&= \ln\left(\frac{2n - 1}{3}\right)
\end{aligned}
$$

Part (c)

Taking $n = 100$, we have

$$\frac{1}{2} + \frac{1}{2 \cdot 100} + \sum_{r=2}^{100-1} \frac{1}{r} > \ln 100 \implies \sum_{r=2}^{99} \frac{1}{r} > \ln 100 - \frac{1}{2} - \frac{1}{200} = 4.100$$

We also have

$$\sum_{r=2}^{100-1} \frac{1}{r} < \ln\left(\frac{2 \cdot 100 - 1}{3}\right) \implies \sum_{r=2}^{100-1} \frac{1}{r} < \ln\left(\frac{199}{3}\right) = 4.195$$

Putting both inequalities together, we obtain

$$\boxed{4.100 < \sum_{r=2}^{99} \frac{1}{r} < 2.195}$$

Adding $\frac{1}{100} = 0.01$ to all sides of the inequality, we see that

$$4.110 < \sum_{r=2}^{100} \frac{1}{r} < 2.205$$