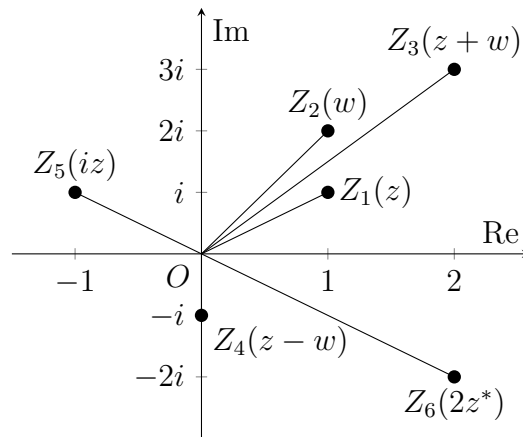


**Problem 1.**

Given that  $z = 1 + i$  and  $w = 1 + 2i$ , mark on an Argand diagram, the positions representing:  $z$ ,  $w$ ,  $z + w$ ,  $z - w$ ,  $iz$  and  $2z^*$ .

**Solution**

**Problem 2.**

- (a) Write down the exact values of the modulus and the argument of the complex number  $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ .
- (b) The complex numbers  $z$  and  $w$  satisfy the equation

$$z^2 - zw + w^2 = 0$$

Find  $z$  in terms of  $w$ . In an Argand diagram, the points  $O$ ,  $A$  and  $B$  represent the complex numbers  $0$ ,  $z$  and  $w$  respectively. Show that  $\triangle OAB$  is an equilateral triangle.

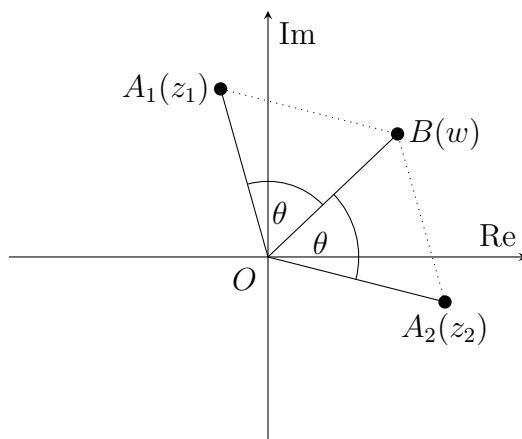
**Solution****Part (a)**

We have  $r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow r = 1$  and  $\tan \theta = \frac{\sqrt{3}/2}{1/2} \Rightarrow \theta = \frac{\pi}{3}$ .

$$\left| \frac{1}{2} + \frac{\sqrt{3}}{2}i \right| = 1, \arg\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{\pi}{3}$$

**Part (b)**

From the quadratic formula, we have  $z = \frac{w \pm \sqrt{w^2 - 4w^2}}{2} = w \left( \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right)$ .



Since  $\left| \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right| = 1$ , we have that  $OB = OA_1 = OA_2$ . Further, since  $\arg\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) = \pm \frac{\pi}{3}$ , we know  $\angle A_1OB = \angle A_2OB = \frac{\pi}{3}$ , whence  $\triangle A_1OB$  and  $\triangle A_2OB$  are both equilateral.

**Problem 3.**

Find the exact roots of the equations

(a)  $z^3 = 1$

(b)  $(z - 1)^4 = -16$

in the form  $x + iy$ .

**Solution****Part (a)**

Since  $1 = e^{i2\pi n}$ ,  $n \in \mathbb{Z}$ , we have  $z^3 = e^{i2\pi n}$ , whence  $z = e^{i2\pi n/3} = \cos \frac{2\pi n}{3} + i \sin \frac{2\pi n}{3}$ . Evaluating  $z$  in the  $n = 0, 1, 2$  cases,

$$n = 0 : z = \cos 0 + i \sin 0 = 1$$

$$n = 1 : z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$n = 2 : z = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\boxed{1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

**Part (b)**

Observe that  $-16 = 16e^{i\pi+2\pi n} = 16e^{i\pi(2n+1)}$ ,  $n \in \mathbb{Z}$ . Hence,

$$\begin{aligned} (z - 1)^4 &= 16e^{i\pi(2n+1)} \\ \implies z - 1 &= 2e^{i\pi(2n+1)/4} \\ \implies z &= 1 + 2e^{i\pi(2n+1)/4} \\ &= 1 + 2 \left[ \cos \left( \frac{2n+1}{4} \pi \right) + i \sin \left( \frac{2n+1}{4} \pi \right) \right] \end{aligned}$$

Evaluating  $z$  in the  $n = 0, 1, 2, 3$  cases,

$$n = 0 : z = 1 + 2 \left[ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] = (1 + \sqrt{2}) + i\sqrt{2}$$

$$n = 1 : z = 1 + 2 \left[ \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right] = (1 - \sqrt{2}) + i\sqrt{2}$$

$$n = 2 : z = 1 + 2 \left[ \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right] = (1 - \sqrt{2}) - i\sqrt{2}$$

$$n = 3 : z = 1 + 2 \left[ \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right] = (1 + \sqrt{2}) - i\sqrt{2}$$

$$\boxed{(1 + \sqrt{2}) \pm i\sqrt{2}, (1 - \sqrt{2}) \pm i\sqrt{2}}$$

**Problem 4.**

- (a) Write down the 5 roots of the equation  $z^5 - 1 = 0$  in the form  $re^{i\theta}$ , where  $r > 0$  and  $-\pi < \theta \leq \pi$ .
- (b) Show that the roots of the equation  $(5 + z)^5 - (5 - z)^5 = 0$  can be written in the form  $5i \tan \frac{k\pi}{5}$ , where  $k = 0, \pm 1, \pm 2$ .

**Solution****Part (a)**

Observe that  $1 = e^{2\pi n}$ ,  $n \in \mathbb{Z}$ . Hence,  $z^5 = e^{2\pi n} \implies z = e^{2\pi n/5}$ . Since  $-\pi < \theta \leq \pi$ , we have  $z = e^{-i4\pi/5}, e^{-i2\pi/5}, 1, e^{i2\pi/5}, e^{i4\pi/5}$ .

$$\boxed{e^{-i4\pi/5}, e^{-i2\pi/5}, 1, e^{i2\pi/5}, e^{i4\pi/5}}$$

**Part (b)**

$$\begin{aligned}
 & (5 + z)^5 - (5 - z)^5 = 0 \\
 \implies & \left( \frac{5 + z}{5 - z} \right)^5 - 1 = 0 \\
 \implies & \frac{5 + z}{5 - z} = e^{i2k\pi/5} \\
 \implies & 5 + z = e^{i2k\pi/5}(5 - z) \\
 \implies & z(1 + e^{i2k\pi/5}) = 5(e^{i2k\pi/5} - 1) \\
 \implies & z = 5 \cdot \frac{e^{i2k\pi/5} - 1}{e^{i2k\pi/5} + 1} \\
 & = 5 \cdot \frac{e^{ik\pi/5} - e^{-ik\pi/5}}{e^{ik\pi/5} + e^{-ik\pi/5}} \\
 & = 5i \cdot \frac{(e^{ik\pi/5} - e^{-ik\pi/5})/2i}{(e^{ik\pi/5} + e^{-ik\pi/5})/2} \\
 & = 5i \cdot \frac{\sin k\pi/5}{\cos k\pi/5} \\
 & = 5i \tan \frac{k\pi}{5}
 \end{aligned}$$

**Problem 5.**

De Moivre's theorem for a positive integral exponent states that

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Use de Moivre's theorem to show that

$$\cos 7\theta = 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta$$

Hence obtain the roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

in the form  $\cos q\pi$ , where  $q$  is a rational number.

**Solution**

Taking  $n = 7$ , we have  $\cos 7\theta + i \sin 7\theta = (\cos \theta + i \sin \theta)^7$ , whence  $\cos 7\theta = \operatorname{Re}(\cos \theta + i \sin \theta)^7$

$$\begin{aligned} \cos 7\theta &= \operatorname{Re} (\cos \theta + i \sin \theta)^7 \\ &= \operatorname{Re} \sum_{k=0}^7 \binom{7}{k} (i \sin \theta)^k \cos^{7-k} \theta \\ &= \operatorname{Re} \sum_{k=0}^7 \binom{7}{k} i^k \sin^k \theta \cos^{7-k} \theta \end{aligned}$$

Note that  $\operatorname{Re} i^k$  is given by

$$\operatorname{Re} i^k = \begin{cases} 0, & k = 1, 3 \pmod{4} \\ 1, & k = 0 \pmod{4} \\ -1, & k = 2 \pmod{4} \end{cases}$$

Hence,

$$\begin{aligned} \cos 7\theta &= \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta \\ &= \cos^7 \theta - 21 \cos^5 \theta (1 - \cos^2 \theta) + 35 \cos^3 \theta (1 - \cos^2 \theta)^2 - 7 \cos \theta (1 - \cos^2 \theta)^3 \\ &= \cos^7 \theta - 21 \cos^5 \theta (1 - \cos^2 \theta) + 35 \cos^3 \theta (1 - 2 \cos^2 \theta + \cos^4 \theta) \\ &\quad - 7 \cos \theta (1 - 3 \cos^2 \theta + 3 \cos^4 \theta - \cos^6 \theta) \\ &= \cos^7 \theta - 21 \cos^5 \theta + 21 \cos^7 \theta + 35 \cos^3 \theta - 70 \cos^5 \theta + 35 \cos^7 \theta \\ &\quad - 7 \cos \theta + 21 \cos^3 \theta - 21 \cos^5 \theta + 7 \cos^7 \theta \\ &= 64 \cos^7 \theta - 112 \cos^5 \theta + 56 \cos^3 \theta - 7 \cos \theta \end{aligned}$$

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$$\begin{aligned} &128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0 \\ \implies &64x^7 - 112x^5 + 56x^3 - 7x = -\frac{1}{2} \end{aligned}$$

$$x = \cos \theta$$

$$\begin{aligned}
&\Rightarrow \cos 7\theta = -\frac{1}{2} \\
&\Rightarrow 7\theta = \frac{2}{3}\pi + 2\pi n \\
&\Rightarrow \theta = \frac{2\pi}{21}(3n+1)
\end{aligned}$$

$$n \in \mathbb{Z}$$

Taking  $0 \leq n < 7$ ,

$$\begin{aligned}
x &= \cos \frac{2\pi}{21}, \cos \frac{8\pi}{21}, \cos \frac{14\pi}{21}, \cos \frac{20\pi}{21}, \cos \frac{26\pi}{21}, \cos \frac{32\pi}{21}, \cos \frac{38\pi}{21} \\
&\equiv \cos \frac{2\pi}{21}, \cos \frac{4\pi}{21}, \cos \frac{8\pi}{21}, \cos \frac{10\pi}{21}, \cos \frac{14\pi}{21}, \cos \frac{16\pi}{21}, \cos \frac{20\pi}{21}
\end{aligned}$$

$ \cos \frac{2\pi}{21}, \cos \frac{4\pi}{21}, \cos \frac{8\pi}{21}, \cos \frac{10\pi}{21}, \cos \frac{14\pi}{21}, \cos \frac{16\pi}{21}, \cos \frac{20\pi}{21} $
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**Problem 6.**

By considering  $\sum_{n=1}^N z^{2n-1}$ , where  $z = e^{i\theta}$ , or by any method, show that

$$\sum_{n=1}^N \sin(2n-1)\theta = \frac{\sin^2 N\theta}{\sin \theta}$$

provided  $\sin \theta \neq 0$ .

**Solution**

$$\begin{aligned}
 \sum_{n=1}^N \sin(2n-1)\theta &= \operatorname{Im} \sum_{n=1}^N \left[ \cos(2n-1)\theta + i \sin(2n-1)\theta \right] \\
 &= \operatorname{Im} \sum_{n=1}^N z^{2n-1} \\
 &= \operatorname{Im} \frac{1}{z} \sum_{n=1}^N (z^2)^n \\
 &= \operatorname{Im} \frac{1}{z} \cdot \frac{z^2 \left[ (z^2)^N - 1 \right]}{z^2 - 1} \\
 &= \operatorname{Im} \frac{z^{2N} - 1}{z - z^{-1}} \\
 &= \operatorname{Im} \frac{(z^{2N} - 1) / 2i}{(z - z^{-1}) / 2i} \\
 &= \operatorname{Im} \frac{(z^{2N} - 1) / 2i}{\sin \theta} \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} \frac{(z^{2N} - 1)}{i} \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} [-i (z^{2N} - 1)] \\
 &= -\frac{1}{2 \sin \theta} [\operatorname{Re} z^{2N} - 1] \\
 &= -\frac{1}{2 \sin \theta} [\operatorname{Re} (\cos N\theta + i \sin N\theta)^2 - 1] \\
 &= -\frac{1}{2 \sin \theta} [\operatorname{Re} (\cos^2 N\theta + 2i \cos N\theta \sin N\theta - \sin^2 N\theta) - 1] \\
 &= -\frac{1}{2 \sin \theta} [\cos^2 N\theta - \sin^2 N\theta - 1] \\
 &= -\frac{1}{2 \sin \theta} [\cos^2 N\theta - \sin^2 N\theta - (\cos^2 N\theta + \sin^2 N\theta)] \\
 &= -\frac{1}{2 \sin \theta} [-2 \sin^2 N\theta] \\
 &= \frac{\sin^2 N\theta}{\sin \theta}
 \end{aligned}$$

**Problem 7.**

By considering the series  $\sum_{n=0}^N (e^{2i\theta})^n$ , show that, provided  $\sin \theta \neq 0$ ,

$$\sum_{n=0}^N \cos 2n\theta = \frac{\sin(N+1)\theta \cos N\theta}{\sin \theta}$$

and deduce that

$$\sum_{n=0}^N \sin^2 n\theta = \frac{N}{2} + \frac{1}{2} - \frac{\sin(N+1)\theta \cos N\theta}{2 \sin \theta}$$

**Solution**

$$\begin{aligned}
 \sum_{n=0}^N \cos 2n\theta &= \operatorname{Re} \sum_{n=0}^N [\cos 2n\theta + i \sin 2n\theta] \\
 &= \operatorname{Re} \sum_{n=0}^N e^{i2n\theta} \\
 &= \operatorname{Re} \sum_{n=0}^N (e^{2i\theta})^n \\
 &= \operatorname{Re} \frac{(e^{2i\theta})^{N+1} - 1}{e^{2i\theta} - 1} \\
 &= \operatorname{Re} \left( e^{-i\theta} \cdot \frac{(e^{2i\theta})^{N+1} - 1}{e^{i\theta} - e^{-i\theta}} \right) \\
 &= \operatorname{Re} \left( \frac{e^{-i\theta}}{2i} \cdot \frac{(e^{2i\theta})^{N+1} - 1}{(e^{i\theta} - e^{-i\theta})/2i} \right) \\
 &= \operatorname{Re} \left( \frac{-ie^{-i\theta}}{2} \cdot \frac{(e^{2i\theta})^{N+1} - 1}{\sin \theta} \right) \\
 &= -\frac{1}{2 \sin \theta} \operatorname{Re} \left( ie^{-i\theta} [(e^{2i\theta})^{N+1} - 1] \right) \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} \left( e^{-i\theta} [(e^{2i\theta})^{N+1} - 1] \right) \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} \left( e^{-i\theta} [(e^{i\theta(N+1)})^2 - 1] \right) \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} \left( (\cos \theta - i \sin \theta) \left[ (\cos(N+1)\theta + i \sin(N+1)\theta)^2 - 1 \right] \right) \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} \left( (\cos \theta - i \sin \theta) \left[ \cos^2(N+1)\theta + 2i \cos(N+1)\theta \sin(N+1)\theta \right. \right. \\
 &\quad \left. \left. - \sin^2(N+1)\theta - 1 \right] \right) \\
 &= \frac{1}{2 \sin \theta} \operatorname{Im} \left( (\cos \theta - i \sin \theta) \left[ \cos^2(N+1)\theta + 2i \cos(N+1)\theta \sin(N+1)\theta \right. \right. \\
 &\quad \left. \left. - \sin^2(N+1)\theta - (\cos^2(N+1)\theta + \sin^2(N+1)\theta) \right] \right)
 \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{2 \sin \theta} \operatorname{Im} \left( (\cos \theta - i \sin \theta) \left[ 2i \cos(N+1)\theta \sin(N+1)\theta - 2 \sin^2(N+1)\theta \right] \right) \\
&= \frac{1}{2 \sin \theta} \left[ 2 \cos \theta \cos(N+1)\theta \sin(N+1)\theta + 2 \sin \theta \sin^2(N+1)\theta \right] \\
&= \frac{\sin(N+1)\theta}{\sin \theta} \left[ \cos \theta \cos(N+1)\theta + \sin \theta \sin(N+1)\theta \right] \\
&= \frac{\sin(N+1)\theta}{\sin \theta} \cdot \cos((N+1)\theta - \theta) \\
&= \frac{\sin(N+1)\theta \cos N\theta}{\sin \theta}
\end{aligned}$$


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Recall that  $\cos 2n\theta = 1 - 2 \sin^2 n\theta \implies \sin^2 n\theta = \frac{1}{2}(1 - \cos 2n\theta)$ .

$$\begin{aligned}
\sum_{n=0}^N \sin^2 n\theta &= \sum_{n=0}^N \frac{1}{2} (1 - \cos 2n\theta) \\
&= \frac{1}{2} \sum_{n=0}^N (1 - \cos 2n\theta) \\
&= \frac{1}{2} \left( (N+1) - \frac{\sin(N+1)\theta \cos N\theta}{\sin \theta} \right) \\
&= \frac{N}{2} + \frac{1}{2} - \frac{\sin(N+1)\theta \cos N\theta}{2 \sin \theta}
\end{aligned}$$

**Problem 8.**

Given that  $z = e^{i\theta}$ , show that  $z^k + \frac{1}{z^k} = 2 \cos k\theta$ ,  $k \in \mathbb{Z}$ .

Hence, show that  $\cos^8 \theta = \frac{1}{128} (\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35)$ .

Find, correct to three decimal places, the values of  $\theta$  such that  $0 < \theta < \frac{1}{2}\pi$  and  $\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 1 = 0$ .

**Solution**

$$\begin{aligned}
 z^k + \frac{1}{z^k} &= z^k + z^{-k} \\
 &= (e^{i\theta})^k + (e^{-i\theta})^k \\
 &= e^{ik\theta} + e^{-ik\theta} \\
 &= [\cos(k\theta) + i \sin(k\theta)] + [\cos(-k\theta) + i \sin(-k\theta)] \\
 &= \cos(k\theta) + i \sin(k\theta) + \cos(k\theta) - i \sin(k\theta) \\
 &= 2 \cos(k\theta)
 \end{aligned}$$


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$$\begin{aligned}
 \cos^8 \theta &= \frac{1}{256} (2 \cos \theta)^8 \\
 &= \frac{1}{256} (z + z^{-1})^8 \\
 &= \frac{1}{256} z^{-8} (z^2 + 1)^8 \\
 &= \frac{1}{256} z^{-8} (1 + 8z^2 + 28z^4 + 56z^6 + 70z^8 + 56z^{10} + 28z^{12} + 8z^{14} + z^{16}) \\
 &= \frac{1}{256} (z^{-8} + 8z^{-6} + 28z^{-4} + 56z^{-2} + 70 + 56z^2 + 28z^4 + 8z^6 + z^8) \\
 &= \frac{1}{256} \left[ (z^8 + z^{-8}) + 8(z^6 + z^{-6}) + 28(z^4 + z^{-4}) + 56(z^2 + z^{-2}) + 70 \right] \\
 &= \frac{2}{256} \left[ \left( \frac{z^8 + z^{-8}}{2} \right) + 8 \left( \frac{z^6 + z^{-6}}{2} \right) + 28 \left( \frac{z^4 + z^{-4}}{2} \right) + 56 \left( \frac{z^2 + z^{-2}}{2} \right) + \frac{70}{2} \right] \\
 &= \frac{1}{128} (\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35)
 \end{aligned}$$


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$$\begin{aligned}
 &\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 1 = 0 \\
 \implies &\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35 = 35 \\
 \implies &128 \cos^8 \theta = 34 \\
 \implies &\cos \theta = \sqrt[8]{\frac{34}{128}} \\
 \implies &\theta = 0.560 \text{ (3 s.f.)}
 \end{aligned}$$

$\theta = 0.560$