Problem 1.

Differentiate the following with respect to x.

(a) $\ln \dfrac{x^3}{\sqrt{1+x^2}}$

(b) $\arctan\left(\dfrac{x^2}{2}\right)$

(c) $e^{2x} \sec x$

Solution

Part (a)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{x^3}{\sqrt{1+x^2}} &= \frac{\mathrm{d}}{\mathrm{d}x} 3 \ln x - \frac{1}{2} \ln\left(1+x^2\right) \\ &= 3 \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x \\ &= \frac{3}{x} - \frac{x}{1+x^2}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \ln \frac{x^3}{\sqrt{1+x^2}} = \frac{3}{x} - \frac{x}{1+x^2}}$$

Part (b)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \arctan\left(\frac{x^2}{2}\right) &= \frac{1}{1+(x^2/2)^2} \cdot \frac{1}{2} \cdot 2x \\ &= \frac{1}{1+x^4/4} \cdot x \\ &= \frac{4x}{4+x^4}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} \arctan\left(\frac{x^2}{2}\right) = \frac{4x}{4+x^4}}$$

Part (c)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} e^{2x} \sec x &= e^{2x} \sec x \tan x + \sec x \cdot e^{2x} \cdot 2 \\ &= e^{2x} \sec x \,(\tan x + 2)\end{aligned}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x} e^{2x} \sec x = e^{2x} \sec x \,(\tan x + 2)}$$

Problem 2.

Find the gradient of the curve $x^3 + xy^2 = 5y$ at the point where $x = 1$ and $0 < y < 1$, leaving your answer to 3 significant figures.

Solution

Substituting $x = 1$ into the given equation,

$$\begin{aligned}
&1^4 + 1 \cdot y^2 = 5y \\
\implies &y^2 - 5y + 1 = 0 \\
\implies &\qquad y = \frac{-(-5) \pm \sqrt{5^2 - 4}}{2 \cdot 1} \\
&\qquad\;\; = \frac{5 \pm \sqrt{21}}{2}
\end{aligned}$$

Since $0 < y < 1$, we reject $y = \dfrac{5 + \sqrt{21}}{2}$ and take $y = \dfrac{5 - \sqrt{21}}{2} = 0.20871$ (5 s.f.).

Implicitly differentiating the given equation,

$$\begin{aligned}
3x^2 + x \cdot 2y \cdot y' + y^2 &= 5y' \\
2xy \cdot y' - 5y' &= -3x^2 - y^2 \\
\implies \qquad y'(2xy - 5) &= -3x^2 - y^2 \\
\implies \qquad y' &= \frac{-3x^2 - y^2}{2xy - 5} \qquad (2.1)
\end{aligned}$$

Substituting $x = 1$ and $y = 0.20871$ into Equation 2.1,

$$\begin{aligned}
y' &= \frac{-3 \cdot 1^2 - 0.20871^2}{2 \cdot 1 \cdot 0.20871 - 5} \\
&= 0.664 \text{ (3 s.f.)}
\end{aligned}$$

$$\boxed{\text{The gradient of the curve is } 0.664.}$$

Problem 3.

A curve C has parametric equations

$$x = \sin^3\theta,\ y = 3\sin^2\theta\cos\theta, \qquad 0 \le \theta \le \frac{1}{2}\pi$$

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = a\cot\theta + b\tan\theta$, where a and b are values to be determined.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin^2\theta\cos\theta$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}\theta} &= 3\left(\sin^2\theta\cdot -\sin\theta + \cos\theta\cdot(2\sin\theta\cos\theta)\right)\\ &= 3(2\sin\theta\cos^2\theta - \sin^3\theta)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}x}\\ &= \frac{\mathrm{d}y}{\mathrm{d}\theta}\cdot\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^{-1}\\ &= \frac{3(2\sin\theta\cos^2\theta - \sin^3\theta)}{3\sin^2\theta\cos\theta}\\ &= \frac{2\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}\\ &= \frac{2\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}\\ &= \frac{2\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}\\ &= 2\cot\theta - \tan\theta\end{aligned}$$

$$\boxed{a = 2,\ b = -1}$$