

Problem 1.

Two biological cultures, X and Y , react with each other, and their volumes at time t are x and y respectively, in appropriate units. Their rates of growth are modelled by the simultaneous equations

$$\begin{aligned}\frac{dx}{dt} &= (2-x)y, \\ \frac{dy}{dt} &= \frac{y^2}{x}\end{aligned}$$

When $t = 0$, $x = y = 1$.

- (a) Show that $x = \frac{2y^2}{1+y^2}$.
- (b) Find and simplify expressions for y and x in terms of t .
- (c) Sketch the graph of y against x for $0 < t < \frac{\pi}{2}$.

Solution**Part (a)**

Note that $x, y > 0$ since they represent volume. Also, for $x \in (0, 2)$, we have $\frac{dx}{dt} = (2-x)y > 0$. When $x = 2$, we have $\frac{dx}{dt} = 0$. Hence, $0 < x \leq 2$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{y^2/x}{(2-x)y} \\ &= \frac{y}{x(2-x)} \\ \implies \frac{1}{y} \frac{dy}{dx} &= \frac{1}{x(2-x)} \\ \implies \int \frac{1}{y} \frac{dy}{dx} dx &= \int \frac{1}{x(2-x)} dx \\ \implies \int \frac{1}{y} dy &= \int \frac{1}{x(2-x)} dx \\ &= \frac{1}{2} \int \left(\frac{1}{x} + \frac{1}{2-x} \right) dx \\ \implies \ln y &= \frac{1}{2} [\ln x - \ln(2-x)] + C_1 \\ &= \ln \sqrt{\frac{x}{2-x}} + C_1 \\ \implies y &= C_2 \sqrt{\frac{x}{2-x}}\end{aligned}$$

At $t = 0$, $x = y = 1$. Hence, $1 = C_2\sqrt{12} - 1 \implies C_2 = 1$.

$$\begin{aligned}
 y &= \sqrt{\frac{x}{2-x}} \\
 \implies y^2 &= \frac{x}{2-x} \\
 \implies (2-x)y^2 &= x \\
 \implies 2y^2 - xy^2 &= x \\
 \implies x + xy^2 &= 2y^2 \\
 \implies x(1+y^2) &= 2y^2 \\
 \implies x &= \frac{2y^2}{1+y^2}
 \end{aligned}$$

Part (b)

$$\begin{aligned}
 \frac{dy}{dt} &= \frac{y^2}{x} \\
 &= \frac{y^2}{2y^2/(1+y^2)} \\
 &= \frac{1}{2}(1+y^2) \\
 \implies \frac{1}{1+y^2} \frac{dy}{dt} &= \frac{1}{2} \\
 \implies \int \frac{1}{1+y^2} \frac{dy}{dt} dt &= \int \frac{1}{2} dt \\
 \implies \int \frac{1}{1+y^2} dy &= \int \frac{1}{2} dt \\
 \implies \arctan y &= \frac{1}{2}t + C \\
 \implies y &= \tan\left(\frac{1}{2}t + C\right)
 \end{aligned}$$

At $t = 0$, $y = 1$. Hence, $1 = \tan C \implies C = \frac{\pi}{4}$.

$$\begin{aligned}
 y &= \tan\left(\frac{1}{2}t + \frac{\pi}{4}\right) \\
 &= \frac{1 - \cos(t + \pi/2)}{\sin(t + \pi/2)} \\
 &= \frac{1 + \sin t}{\cos t} \\
 &= \sec t + \tan t
 \end{aligned}$$

$$\boxed{y = \sec t + \tan t}$$

$$\begin{aligned}
& \frac{dx}{dt} = (2-x)y \\
& = (2-x)\sqrt{\frac{x}{2-x}} \\
& = \sqrt{x(2-x)} \\
\Rightarrow & \frac{1}{\sqrt{x(2-x)}} \frac{dx}{dt} = 1 \\
\Rightarrow & \int \frac{1}{\sqrt{x(2-x)}} \frac{dx}{dt} dt = \int dt \\
\Rightarrow & \int \frac{1}{\sqrt{x(2-x)}} dx = \int dt \\
\Rightarrow & \int \frac{1}{\sqrt{x}\sqrt{2-x}} dx = t + C_1 \\
\Rightarrow & \int \frac{2u}{u\sqrt{2-u^2}} du = t + C_1 \\
\Rightarrow & 2 \int \frac{1}{\sqrt{2-u^2}} du = t + C_1 \\
\Rightarrow & 2 \arcsin\left(\frac{u}{\sqrt{2}}\right) = t + C_1 \\
\Rightarrow & 2 \arcsin\left(\sqrt{\frac{x}{2}}\right) = t + C_1 \\
\Rightarrow & \arcsin\left(\sqrt{\frac{x}{2}}\right) = \frac{1}{2}t + C_2 \\
\Rightarrow & \sqrt{\frac{x}{2}} = \sin\left(\frac{1}{2}t + C_2\right) \\
\Rightarrow & \frac{x}{2} = \sin^2\left(\frac{1}{2}t + C_2\right) \\
\Rightarrow & x = 2 \sin^2\left(\frac{1}{2}t + C_2\right)
\end{aligned}$$

$$\begin{aligned}
u &= \sqrt{x} \\
dx &= 2u du
\end{aligned}$$

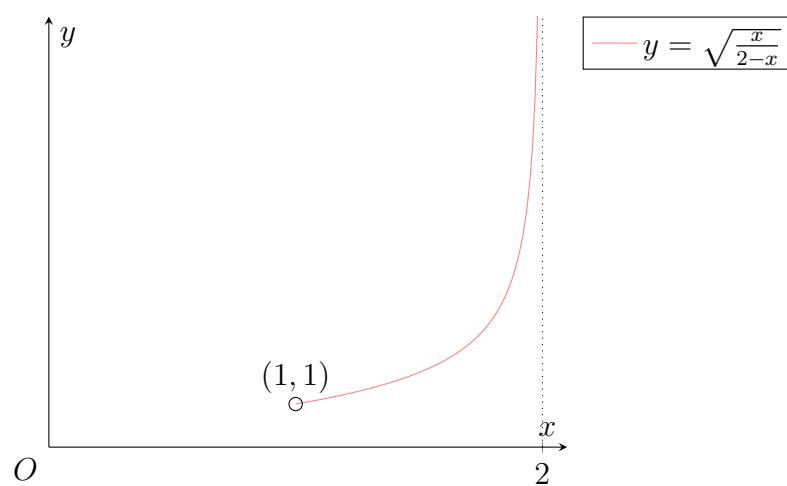
At $t = 0$, $x = 1$. Hence, $1 = 2 \sin C_2$, whence $C_2 = \frac{\pi}{4}$.

$$\begin{aligned}
x &= 2 \sin^2\left(\frac{1}{2}t + \frac{\pi}{4}\right) \\
&= 1 - \cos\left(t + \frac{\pi}{2}\right) \\
&= 1 + \sin t
\end{aligned}$$

$$\boxed{x = 1 + \sin t}$$

Part (c)

Note that $0 < t < \frac{\pi}{2} \Rightarrow 1 < x < 2$.



Problem 2.

Find the general solution of the differential equation

$$x \frac{dy}{dx} + 4y - 10x = 0,$$

Find the particular solution such that $y \rightarrow 0$ as $x \rightarrow 0$.

Show, on a single diagram, a sketch of this particular solution and one typical member of the family, F of solution curves for which $\frac{dy}{dx}$ is positive whenever x is positive.

Show that there is a straight line which passes through the maximum point of every member of F and find its equation.

Solution

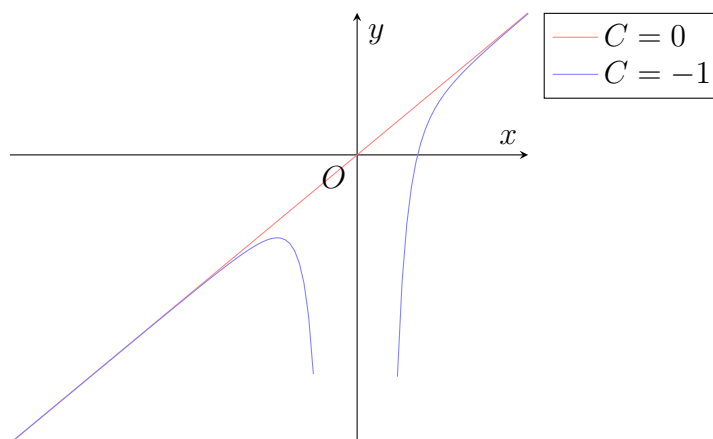
$$\begin{aligned} & x \frac{dy}{dx} + 4y - 10x = 0 \\ \Rightarrow & x^4 \frac{dy}{dx} + 4x^3 y = 10x^4 \\ \Rightarrow & \frac{d}{dx} (x^4 y) = 10x^4 \\ \Rightarrow & x^4 y = \int 10x^4 dx \\ \Rightarrow & x^4 y = 2x^5 + C \\ \Rightarrow & y = 2x + Cx^{-4} \end{aligned}$$

$$\boxed{y = 2x + Cx^{-4}}$$

As $x \rightarrow 0$, $x^{-4} \rightarrow \infty$. Hence, C must be 0.

$$\boxed{y = 2x}$$

Note that $\frac{dy}{dx} = 2 - 4Cx^{-5} > 0 \Rightarrow C < \frac{1}{2}x^5$. Since $x > 0$, we hence have the constraint $C \leq 0$ for members of F .



Consider the stationary points of members of F . For stationary points, $\frac{dy}{dx} = 0$. Hence,

$$\begin{aligned} x \frac{dy}{dx} + 4y - 10x &= 0 \\ \implies 4y - 10x &= 0 \\ \implies y &= \frac{5}{2}x \end{aligned}$$

Differentiating the original differential equation with respect to x , we obtain

$$\begin{aligned} x \frac{dy}{dx} + 4y - 10x &= 0 \\ \implies x \frac{d^2y}{dx^2} + \frac{dy}{dx} + 4 \frac{dy}{dx} - 10 &= 0 \\ \implies x \frac{d^2y}{dx^2} &= 10 \\ \implies \frac{d^2y}{dx^2} &= \frac{10}{x} \end{aligned}$$

Note that for members of F , we have that $\frac{dy}{dx} > 0$ for $x > 0$. Hence, there are no stationary points when $x > 0$. That is, any stationary point must occur when $x < 0$. (Indeed, there is a stationary point when $x = \sqrt[5]{2C} < 0$.) Furthermore, when $x < 0$, $\frac{d^2y}{dx^2} < 0$. Hence, all stationary points must be a maximum point. Thus, $\boxed{y = \frac{5}{2}x}$ passes through the maximum point of every member of F .

Problem 3.

(a) The variables x and y are related by the differential equation

$$x^2 \frac{dy}{dx} - 2xy + y = 0.$$

- (i) Find the general solution of this differential equation, expressing y in terms of x .
- (ii) Find the particular solution for which $y = -e$ when $x = 1$. Obtain the coordinates of the turning point of the solution curve of this particular solution and sketch the curve for $x > 0$.

(b) Find the general solution of the differential equation

$$\frac{dy}{dx} + xy = e^x x^2,$$

expressing y in terms of x .

Solution**Part (a)****Subpart (i)**

$$\begin{aligned} & x^2 \frac{dy}{dx} - 2xy + y = 0 \\ \implies & x^2 \frac{dy}{dx} = y(2x - 1) \\ \implies & \frac{1}{y} \frac{dy}{dx} = \frac{2x - 1}{x^2} \\ & = \frac{2}{x} - \frac{1}{x^2} \\ \implies & \int \frac{1}{y} \frac{dy}{dx} dx = \int \left(\frac{2}{x} - \frac{1}{x^2} \right) dx \\ \implies & \int \frac{1}{y} dy = \int \left(\frac{2}{x} - \frac{1}{x^2} \right) dx \\ \implies & \ln |y| = 2 \ln(x) + \frac{1}{x} + C_1 \\ \implies & y = C_2 x^2 e^{1/x} \end{aligned}$$

$$\boxed{y = C_2 x^2 e^{1/x}}$$

Subpart (ii)

When $x = 1$, $y = -e$. Hence, $-e = C_2 \cdot 1^2 \cdot e^1 \implies C_2 = -1$.

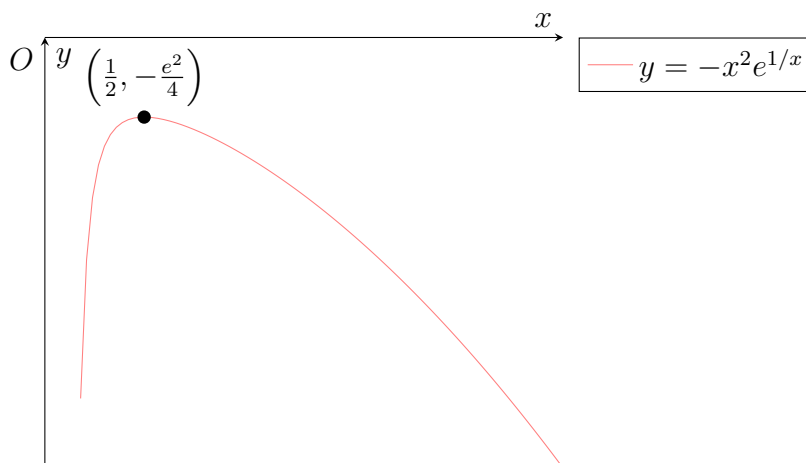
$$\boxed{y = -x^2 e^{1/x}}$$

For stationary point, $\frac{dy}{dx} = 0$. Hence, $y(2x - 1) = 0$, whence $x = \frac{1}{2}$. Note that we reject $y = 0$ since $e^{1/x} \neq 0$ and $x \neq 0$ due to the presence of a $\frac{1}{x}$ term. Hence, y has a stationary point at $\left(\frac{1}{2}, -\frac{e^2}{4}\right)$.

Differentiating the original differential equation with respect to x , we obtain

$$\begin{aligned} x^2 \frac{dy}{dx} - 2xy + y &= 0 \\ \implies x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2x \frac{dy}{dx} - 2y + \frac{dy}{dx} &= 0 \\ \implies x^2 \frac{d^2y}{dx^2} - 2y &= 0 \\ \implies \frac{d^2y}{dx^2} &= \frac{2y}{x^2} \end{aligned}$$

Hence, at $\left(\frac{1}{2}, -\frac{e^2}{4}\right)$, we have $\frac{d^2y}{dx^2} = \frac{-e^2/2}{1/4} < 0$. Thus, $\left(\frac{1}{2}, -\frac{e^2}{4}\right)$ is a maximum point and is thus a turning point.



Part (b)

$$\begin{aligned} \frac{dy}{dx} + xy &= e^x x^2 \\ \implies e^{\frac{1}{2}x^2} \frac{dy}{dx} + x e^{\frac{1}{2}x^2} y &= e^{\frac{1}{2}x^2+x} x^2 \\ \implies \frac{d}{dx} \left(e^{\frac{1}{2}x^2} y \right) &= e^{\frac{1}{2}x^2+x} x^2 \\ \implies e^{\frac{1}{2}x^2} y &= \int e^{\frac{1}{2}x^2+x} x^2 dx \end{aligned}$$

Suppose $\int e^{\frac{1}{2}x^2+x} x^2 dx = P(x) e^{\frac{1}{2}x^2+x} + C$ for some function $P(x)$. Differentiating both sides with respect to x , we obtain

$$x^2 e^{\frac{1}{2}x^2+x} = e^{\frac{1}{2}x^2+x} [(x+1)P(x) + P'(x)],$$

whence

$$x^2 = (x + 1)P(x) + P'(x).$$

Thus, $P(x)$ is a polynomial of degree 1. Let $P(x) = ax + b$. For some constants a and b . Then

$$x^2 = ax^2 + (a + b)x + (a + b).$$

Comparing coefficients of x^2 , x and constant terms, we have $a = 1$ and $a + b = 0 \implies b = -1$. Thus,

$$\int x^2 e^{\frac{1}{2}x^2+x} dx = (x - 1)e^{\frac{1}{2}x^2+x} + C.$$

Thus,

$$\boxed{y = (x - 1)e^x + Ce^{-\frac{1}{2}x^2}}$$