Problem 1.

Given that $y = e^{-x}\cos x$, show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = -2\left(y + \dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$. By further differentiation, find the series expansion of y, in ascending powers of x, up to and including the term in x^3. Use the series to obtain an approximate value for $\displaystyle\int_0^{0.2} \frac{\cos x^2}{e^{x^2}}\,\mathrm{d}x$, giving your answer correct to 4 decimal places.

Using the trapezium rule with 4 trapezia of equal width, find another approximation for $\displaystyle\int_0^{0.2} \frac{\cos x^2}{e^{x^2}}\,\mathrm{d}x$, giving your answer correct to 4 decimal places.

Solution

Note that $y = e^{-x}\cos x = e^{-x}\operatorname{Re} e^{ix} = \operatorname{Re} e^{(i-1)x}$. Hence, we have $y' = \operatorname{Re}\left((i-1)e^{(i-1)x}\right)$ and $y'' = \operatorname{Re}\left((1-i)^2 e^{(i-1)x}\right)$. Hence,

$$\begin{aligned}
y'' &= \operatorname{Re}\left((i^2 - 2i + 1^2)e^{(i-1)x}\right) \\
&= -2\operatorname{Re}\left(ie^{(i-1)x}\right) \\
&= -2\operatorname{Re}\left((i-1)e^{(i-1)x} + e^{(i-1)x}\right) \\
&= -2\left(\operatorname{Re}(i-1)e^{(i-1)x} + \operatorname{Re}\, e^{(i-1)x}\right) \\
&= -2\left(y' + y\right)
\end{aligned}$$

Further differentiating, we obtain $y^{(3)} = -2(y' + y'')$. Evaluating y and its derivatives at 0,

$$\begin{aligned}
y(0) &= 1 \\
y'(0) &= -1 \\
y''(0) &= 0 \\
y^{(3)}(0) &= 2
\end{aligned}$$

Hence, we have

$$\boxed{e^{-x}\cos x = 1 - x + \frac{1}{3}x^3 + \dots}$$

$$\begin{aligned}
\int_0^{0.2} \frac{\cos x^2}{e^{x^2}}\,\mathrm{d}x &= \int_0^{0.2} e^{-x^2}\cos x^2\,\mathrm{d}x \\
&\approx \int_0^{0.2} \left(1 - x^2 + \frac{1}{3}(x^2)^3\right)\mathrm{d}x \\
&= \int_0^{0.2} \left(1 - x^2 + \frac{1}{3}x^6\right)\mathrm{d}x \\
&= 0.1973 \text{ (4 d.p.)}
\end{aligned}$$

$$\boxed{\int_0^{0.2} \frac{\cos x^2}{e^{x^2}}\,\mathrm{d}x \approx 0.1973}$$

Let $g(x) = \dfrac{\cos x^2}{e^{x^2}}$. By the trapezium rule, we have

$$\begin{aligned}\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} \, \mathrm{d}x &\approx \frac{1}{2} \cdot \frac{0.2 - 0}{4} \left[g(0) + 2g(0.05) + 2g(0.1) + 2g(0.15) + g(0.2) \right] \\ &= 0.1973 \text{ (4 d.p.)}\end{aligned}$$

$$\boxed{\int_0^{0.2} \frac{\cos x^2}{e^{x^2}} \, \mathrm{d}x \approx 0.1973}$$

Problem 2.

The curve C has equation $y^2 = \dfrac{x}{\sqrt{1+x^2}}$, $y \geq 0$.

The finite region R is bounded by C, the x-axis and the lines $x = 0$ and $x = 2$. R is rotated through 2π radians about the x-axis.

(a) Find the exact volume of the solid formed.

An estimate for the volume in (a) is found using the trapezium rule with 7 ordinates.

(b) Find the percentage error resulting from using this estimate, giving your answer to 3 decimal places.

Explain, with the help of a sketch, why the estimate given by the trapezium rule is less than the actual value.

Solution

Part (a)

$$\begin{aligned}
\text{Volume} &= \pi \int_0^2 y^2 \, \mathrm{d}x \\
&= \pi \int_0^2 \frac{x}{\sqrt{1+x^2}} \, \mathrm{d}x \\
&= \frac{\pi}{2} \int_1^5 \frac{1}{\sqrt{u}} \, \mathrm{d}u \\
&= \frac{\pi}{2} \left[2\sqrt{u} \right]_1^5 \\
&= \pi(\sqrt{5} - 1)
\end{aligned}$$

$$\begin{aligned} u &= 1 + x^2 \\ \mathrm{d}u &= 2x \, \mathrm{d}x \end{aligned}$$

$$\boxed{\text{The volume of the solid formed is } \pi(\sqrt{5}-1) \text{ units}^3.}$$

Part (b)

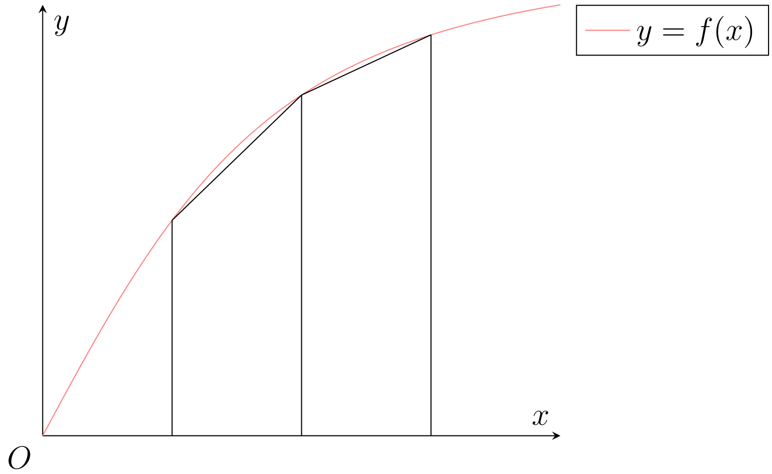
Let $f(x) = \dfrac{x}{\sqrt{1+x^2}}$. By the trapezium rule,

$$\begin{aligned}
\text{Volume} &= \pi \int_0^2 f(x) \, \mathrm{d}x \\
&\approx \pi \cdot \frac{1}{2} \cdot \frac{2-0}{6} \sum_{n=0}^{5} \left[f\left(\frac{n}{3}\right) + f\left(\frac{n+1}{3}\right) \right] \\
&= 3.8566 \text{ (5 s.f.)}
\end{aligned}$$

Hence, the percentage error is $\left| \dfrac{\pi(\sqrt{5}-1) - 3.8566}{\pi(\sqrt{5}-1)} \right| = 0.686\%$ (3 d.p.).

$$\boxed{\text{The percentage error of the estimate is } 0.686\%.}$$

Consider the following graph of $y = f(x)$.



From the graph, the curve $y = f(x)$ is clearly concave downwards. Hence, the approximation given by the trapezium rule is an underestimate and is thus less than the actual value.

Problem 3.

Prove that $\int_{-h}^{h} f(x)\,\mathrm{d}x = \frac{1}{3}h(y_{-1} + 4y_0 + y_1)$, where $y = f(x)$ is the quadratic curve passing through the points $(-h, y_{-1})$, $(0, y_0)$ and (h, y_1).

Use Simpson's rule with 5 ordinates to find an approximation to

$$\int_{-3}^{1} \left(x^4 - 7x^3 + 3x^2 + 6x + 4\right)^{1/3} \mathrm{d}x$$

Find another approximation to the same integral using the trapezium rule with 5 ordinates.

Which of these approximations would you expect to be more accurate? Justify your answer.

Solution

Let $f(x) = ax^2 + bx + c$ be the quadratic such that the graph $y = f(x)$ passes through the points $(-h, y_{-1})$, $(0, y_0)$ and (h, y_1).

Note that we have $y_0 = f(0) = a \cdot 0^2 + b \cdot 0 + c = c$. We also have

$$y_{-1} + y_1 = f(-h) + f(h) = \left[a(-h)^2 + b(-h) + c\right] + \left[ah^2 + bh + c\right] = 2ah^2 + 2c$$

Hence,

$$\begin{aligned}
\int_{-h}^{h} f(x)\,\mathrm{d}x &= \int_{-h}^{h} (ax^2 + bx + c)\,\mathrm{d}x \\
&= \left[\frac{1}{3}x^3 + \frac{1}{2}bx^2 + cx\right]_{-h}^{h} \\
&= \frac{2}{3}h^3 + 2ch \\
&= \frac{1}{3}h\left(2h^2 + 6c\right) \\
&= \frac{1}{3}h\left(2h^2 + 2c + 4c\right) \\
&= \frac{1}{3}h\left(y_{-1} + y_1 + 4y_0\right) \\
&= \frac{1}{3}h\left(y_{-1} + 4y_0 + y_1\right)
\end{aligned}$$

Let $f(x) = (x^4 - 7x^3 + 3x^2 + 6x + 4)^{1/3}$. By Simpson's rule,

$$\begin{aligned}
&\int_{-3}^{1} \left(x^4 - 7x^3 + 3x^2 + 6x + 4\right)^{1/3} \mathrm{d}x \\
&\approx \frac{1}{3} \cdot \frac{1 - (-3)}{4} \left[f(-3) + 4f(-2) + 2f(-1) + 4f(0) + f(1)\right] \\
&= 11.977 \text{ (5 s.f.)}
\end{aligned}$$

$$\boxed{\text{By Simpson's rule, } \int_{-3}^{1} \left(x^4 - 7x^3 + 3x^2 + 6x + 4\right)^{1/3} \mathrm{d}x = 11.977}$$

By the trapezium rule,

$$\begin{aligned} &\int_{-3}^{1} \left(x^4 - 7x^3 + 3x^2 + 6x + 4\right)^{1/3} \mathrm{d}x \\ &\qquad \approx \frac{1}{2} \cdot \frac{1-(-3)}{4} \left[f(-3) + 2f(-2) + 2f(-1) + 2f(0) + f(1)\right] \\ &\qquad = 12.142 \text{ (5 s.f.)} \end{aligned}$$

$$\boxed{\text{By the trapezium rule, } \int_{-3}^{1} \left(x^4 - 7x^3 + 3x^2 + 6x + 4\right)^{1/3} \mathrm{d}x = 12.142}$$

The approximation given by Simpson's rule should be more accurate as Simpson's rule accounts for the concavity of the curve $y = f(x)$.

Problem 4.

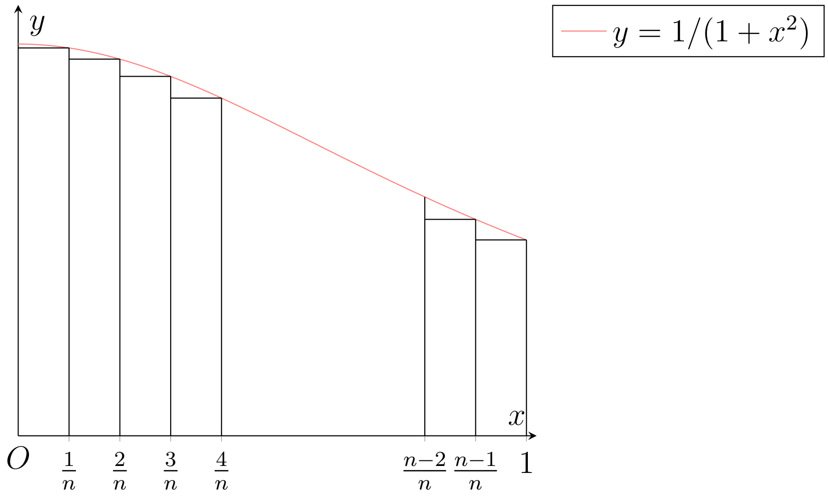
(a) Find the exact value of $\displaystyle\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$.

(b) The graph of $y = \dfrac{1}{1+x^2}$ is shown in the diagram below. Rectangles, each of width $\dfrac{1}{n}$, are drawn under the curve.

Show that the total area A of all n rectangles is given by

$$A = \frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{2}{n}\right)^2} + \frac{1}{1+\left(\frac{3}{n}\right)^2} + \ldots + \frac{1}{2}\right]$$

State the limit of A as $n \to \infty$.



(c) It is given that

$$B = \frac{1}{n}\left[\frac{1}{1+\left(\frac{1}{n}\right)^4} + \frac{1}{1+\left(\frac{2}{n}\right)^4} + \frac{1}{1+\left(\frac{3}{n}\right)^4} + \ldots + \frac{1}{2}\right]$$

Find an approximation for the limit of B as $n \to \infty$ by considering an appropriate graph and using the trapezium rule with 5 intervals. Given your answer correct to 2 decimal places.

Solution

Part (a)

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^1 = \frac{\pi}{4}$$

$$\boxed{\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \frac{\pi}{4}}$$

Part (b)

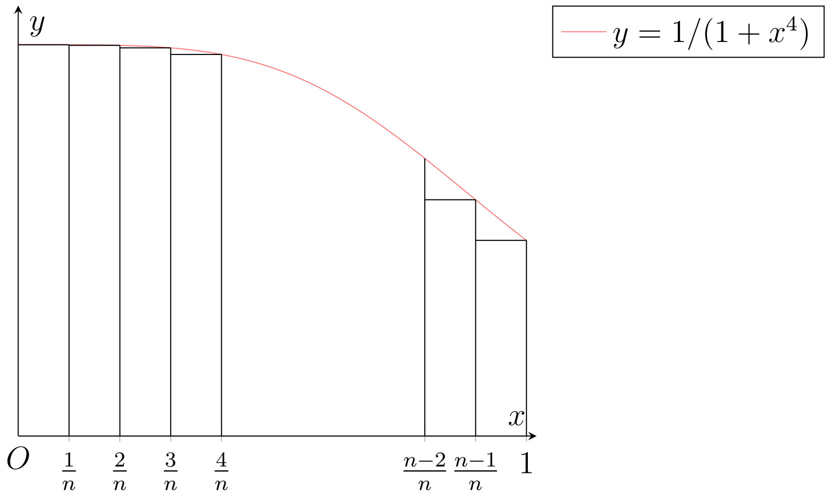
Observe that the kth rectangle has height $\dfrac{1}{1+(k/n)^2}$ and width $1/n$. Hence,

$$\begin{aligned}
A &= \sum_{k=1}^{n} \frac{1}{n} \cdot \frac{1}{1+(k/n)^2} \\
&= \frac{1}{n} \sum_{k=1}^{n} \frac{1}{1+(k/n)^2} \\
&= \frac{1}{n} \left[\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{2}{n}\right)^2} + \frac{1}{1+\left(\frac{3}{n}\right)^2} + \ldots + \frac{1}{1+\left(\frac{n}{n}\right)^2} \right] \\
&= \frac{1}{n} \left[\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{2}{n}\right)^2} + \frac{1}{1+\left(\frac{3}{n}\right)^2} + \ldots + \frac{1}{2} \right]
\end{aligned}$$

$$\boxed{\text{As } n \to \infty,\ A \to \int_0^1 \frac{1}{1+x^2} = \frac{\pi}{4}.}$$

Part (c)

Consider the following graph of $y = \dfrac{1}{1+x^4}$.



Using a similar line of logic presented in part (b), we have that B is the total area of the rectangles above. Hence, as $n \to \infty$, $B \to \displaystyle\int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x$.

Let $f(x) = \dfrac{1}{1+x^4}$. Using the trapezium rule,

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x &\approx \frac{1}{2} \cdot \frac{1-0}{5} \left[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1) \right] \\
&= 0.86 \text{ (2 d.p.)}
\end{aligned}$$

$$\boxed{\text{As } n \to \infty,\ B \to 0.86.}$$