Problem 1.

A university student has a goal of saving at least \$1 000 000 (in Singapore dollars). He begins working at the start of the year 2019. In order to achieve his goal, he saves 40% of his annual salary at the end of each year. If his annual salary in the year 2019 is \$40800, and it increases by 5% (of his previous year's salary) every year, find

(a) his annual savings in 2027 (to the nearest dollar),

(b) his total savings at the end of n years.

What is the minimum number of complete years for which he has to work in order to achieve his goal?

Solution

Let $\$u_n$ be his annual salary in the nth year after 2019, with $n \in \mathbb{N}$. Then $u_{n+1} = 1.05 \cdot u_n$, with $u_0 = 40800$. Hence, $u_n = 40800 \cdot 1.05^n$.

Part (a)

In 2027, $n = 8$. Hence,

$$\begin{aligned}\text{Annual savings in 2027} &= 0.40 \cdot u_8 \\ &= 0.40 \cdot 40800 \cdot 1.05^8 \\ &= 24112 \text{ (to the nearest integer)}\end{aligned}$$

$$\boxed{\text{His annual savings in 2027 will be \$24112.}}$$

Part (b)

$$\begin{aligned}\sum_{k=0}^{n-1} 0.40 \cdot u_k &= \sum_{k=0}^{n-1} 0.40 \cdot 40800 \cdot 1.05^k \\ &= 16320 \sum_{k=0}^{n-1} 1.05^k \\ &= 16320 \sum_{k=1}^{n} 1.05^{k-1} \\ &= 16320 \cdot 1.05^{-1} \cdot \sum_{k=1}^{n} 1.05^k \\ &= 16320 \cdot 1.05^{-1} \cdot \frac{1.05 \cdot (1.05^k - 1)}{1.05 - 1} \\ &= 326400(1.05^k - 1)\end{aligned}$$

$$\boxed{\text{His total savings at the end of } n \text{ years is } \$326400(1.05^k - 1).}$$

$$
\begin{aligned}
326400(1.05^n - 1) &\geq 1000000 \\
1.05^n - 1 &\geq \frac{1000000}{326400} \\
1.05^n &\geq \frac{1000000}{326400} + 1 \\
n &\geq \log_{1.05}(\frac{1000000}{326400} + 1) \\
&= 28.7 \text{ (3 s.f.)}
\end{aligned}
$$

Since $n \in \mathbb{N}$, the minimum value of n is 29.

$\boxed{\text{He has to work for a minimum of 29 complete years.}}$

Problem 2.

(a) A rope of length 200π cm is cut into pieces to form as many circles as possible, whose radii follow an arithmetic progression with common difference 0.25 cm. Given that the smallest circle has an area of π cm^2, find the area of the largest circle in terms of π.

(b) The sum of the first n terms of a sequence is given by $S_n = \alpha^{-n} - 1$, where α is a non-zero constant, $\alpha \neq 1$.

(i) Show that the sequence is a geometric progression and state its common ratio in terms of α.

(ii) Find the set of values of α for which the sum to infinity of the sequence exists.

(iii) Find the value of the sum to infinity.

Solution

Part (a)

Let the sequence r_n be the radius of the nth smallest circle, in centimetres. Hence, $r_n = \frac{1}{4} + r_{n-1}$. Since the smallest circle has area π cm^2, $r_1 = 1$. Thus, $r_n = 1 + \frac{1}{4}(n-1)$.

Consider the nth partial sum of the circumferences.

$$\begin{aligned}\sum_{k=1}^{n} 2\pi r_k &= 2\pi \sum_{k=1}^{n} \left(1 + \frac{1}{4}(n-1)\right) \\ &= 2\pi \left(n + \frac{1}{4} \cdot \frac{n(n+1)}{2} - \frac{1}{4}n\right) \\ &= 2\pi \left(\frac{3}{4}n + \frac{1}{8}n(n+1)\right) \\ &= \frac{1}{4}\pi \left(6n + n(n+1)\right) \\ &= \frac{1}{4}\pi(n^2 + 7n)\end{aligned}$$

Since the rope has length 200π cm, we have the inequality

$$\begin{aligned}\sum_{k=1}^{n} 2\pi r_k &\leq 200\pi \\ \frac{1}{4}\pi(n^2 + 7n) &\leq 200\pi \\ n^2 + 7n &\leq 800 \\ n^2 + 7n - 800 &\leq 0 \\ (n+32)(n-25) &\leq 0\end{aligned}$$

Hence, $n \leq 25$. Since the rope is cut to form as many circles as possible, $n = 25$.

Observe that $r_25 = 1 + \frac{1}{4}(25-1) = 7$. Hence, the largest circle has area $\pi \cdot 7^2 = 49\pi$ cm^2.

$$\boxed{\text{The largest circle has area } 49\pi \text{ cm}^2.}$$

Part (b)

Let $S_n = \sum_{k=1}^{n} u_k$.

$$\begin{aligned} u_{n+1} &= S_{n+1} - S_n \\ &= \alpha^{-(n+1)} - 1 - (\alpha^{-n} - 1) \\ &= \alpha^{-n} \cdot \alpha^{-1} - \alpha^{-n} \\ &= \alpha^{-n}(\alpha^{-1} - 1) \end{aligned}$$

Subpart (i)
Test for Geometric Progression

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{\alpha^{-(n+1)}(\alpha^{-1} - 1)}{\alpha^{-n}(\alpha^{-1} - 1)} \\ &= \frac{\alpha^{-(n+1)}}{\alpha^{-n}} \\ &= \alpha^{-1} \end{aligned}$$

Since α^{-1} is a constant, u_n is in geometric progression with common ratio α^{-1}.

$$\boxed{\text{The common ratio of the sequence is } \alpha^{-1}.}$$

Subpart (ii)
Consider $L = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (\alpha^{-n} - 1)$. For L to exist, we need $\lim_{n\to\infty} \alpha^{-n}$ to exist. Hence, $|\alpha^{-1}| < 1$, whence $|\alpha| > 1$. Thus, $\alpha < -1$ or $\alpha > 1$.

$$\boxed{\{x \in \mathbb{R} \colon x < -1 \lor x > 1\}}$$

Subpart (iii)
Since $|\alpha^{-1}| < 1$, we know $\lim_{n\to infty} \alpha^{-n} = 0$. Hence,

$$\begin{aligned} \lim_{n\to\infty} S_n &= \lim_{n\to\infty} (\alpha^{-n} - 1) \\ &= -1 \end{aligned}$$

$$\boxed{\text{The sum to infinity of the sequence is } -1.}$$

Problem 3.

A sequence $u_1, u_2, u_3, \ldots$ is such that $u_{n+1} = 2u_n + An$, where A is a constant and $n \geq 1$.

(a) Given that $u_1 = 5$ and $u_2 = 15$, find A and u_3.

It is known that the nth term of this sequence is given by

$$u_n = a(2^n) + bn + c$$

where a, b and c are constants.

(b) Find a, b and c.

Solution

Part (a)

$$\begin{aligned} u_2 &= 2u_1 + A \cdot 1 \\ &= 2 \cdot 5 + A \\ &= 10 + A \\ &= 15 \end{aligned}$$

Hence, $A = 5$.

$$\begin{aligned} u_3 &= 2u_2 + A \cdot 2 \\ &= 2 \cdot 15 + 2 \cdot 5 \\ &= 40 \end{aligned}$$

$$\boxed{A = 15,\ u_3 = 40}$$

Part (b)

Since $u_1 = 5$, $u_2 = 15$ and $u_3 = 40$, we have the following system

$$\begin{cases} 2a + b + c = 5 \\ 4a + 2b + c = 15 \\ 8a + 3b + c = 40 \end{cases}$$

which has solutions $a = \dfrac{15}{2}$, $b = -5$ and $c = -5$

$$\boxed{a = \frac{15}{2},\ b = -5,\ c = -5}$$

Problem 4.

The graphs of $y = \frac{1}{3}(2^x)$ and $y = x$ intersect at $x = \alpha$ and $x = \beta$ where $\alpha < \beta$. A sequence of real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{1}{3}(2^{x_n}), \qquad n \geq 1$$

(a) Prove algebraically that, if the sequence converges, then it converges to either α or β.

(b) By using the graphs of $y = \frac{1}{3}(2^x)$ and $y = x$, prove that

- if $\alpha < x_n < \beta$, then $\alpha < x_{n+1} < x_n$
- if $x_n < \alpha$, then $x_n < x_{n+1} < \alpha$
- if $x_n > \beta$, then $x_n < x_{n+1}$

Describe the behaviour of the sequence for the three cases.

Solution

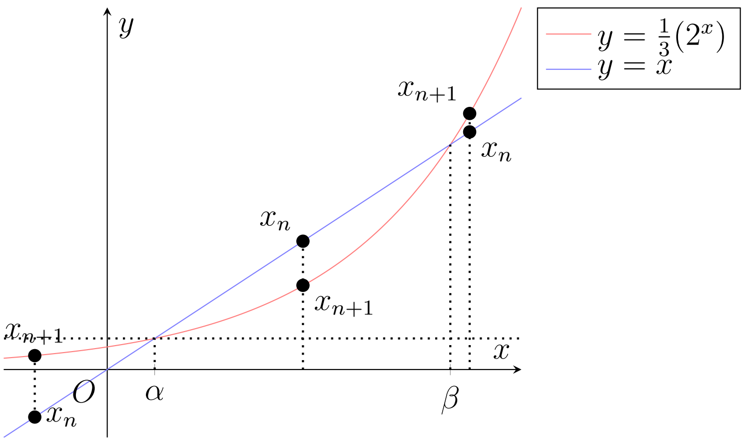
Part (a)

Let $L = \lim_{n\to\infty} x_n$.

$$\begin{aligned} x_{n+1} &= \frac{1}{3}(2^{x_n}) \\ \implies \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \frac{1}{3}(2^{x_n}) \\ \implies \qquad L &= \frac{1}{3}(2^L) \end{aligned}$$

Since $y = x$ and $y = \frac{1}{3}(2^x)$ intersect only at $x = \alpha$ and $x = \beta$, then α and β are the only roots of $x = \frac{1}{3}(2^x)$. Since L is also a root of $x = \frac{1}{3}(2^x)$, L must be either α or β.

Part (b)



$$\boxed{\begin{array}{c}\text{If } \alpha < x_n < \beta \text{, then } x_n \text{ is decreasing and converges to } \alpha. \\ \text{If } x_n < \alpha \text{, then } x_n \text{ is increasing and converges to } \alpha. \\ \text{If } x_n > \beta \text{, then } x_n \text{ is increasing and diverges.}\end{array}}$$