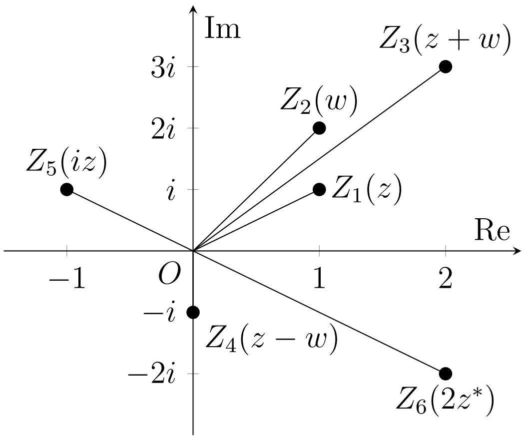
Problem 1.

Given that $z = 1 + i$ and $w = 1 + 2i$, mark on an Argand diagram, the positions representing: z, w, $z + w$, $z - w$, iz and $2z^*$.

Solution



Problem 2.

(a) Write down the exact values of the modulus and the argument of the complex number $\frac{1}{2} + \frac{\sqrt{3}}{2}i$.

(b) The complex numbers z and w satisfy the equation

$$z^2 - zw + w^2 = 0$$

Find z in terms of w. In an Argand diagram, the points O, A and B represent the complex numbers 0, z and w respectively. Show that $\triangle OAB$ is an equilateral triangle.

Solution

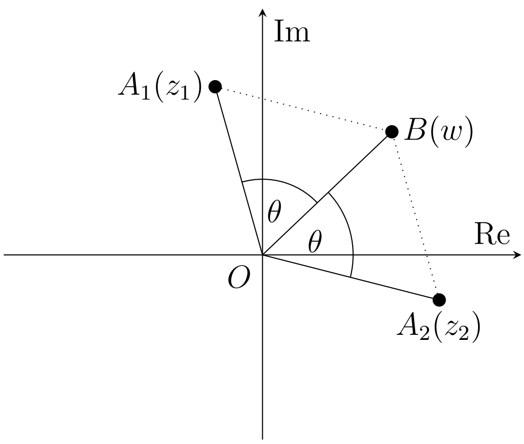
Part (a)

We have $r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 \implies r = 1$ and $\tan\theta = \frac{\sqrt{3}/2}{1/2} \implies \theta = \frac{\pi}{3}$.

$$\boxed{\left|\frac{1}{2} + \frac{\sqrt{3}}{2}i\right| = 1,\ \arg\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = \frac{\pi}{3}}$$

Part (b)

From the quadratic formula, we have $z = \frac{w \pm \sqrt{w^2 - 4w^2}}{2} = w\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right)$.



Since $\left|\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right| = 1$, we have that $OB = OA_1 = OA_2$. Further, since $\arg\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i\right) = \pm\frac{\pi}{3}$, we know $\angle A_1OB = \angle A_2OB = \frac{\pi}{3}$, whence $\triangle A_1OB$ and $\triangle A_2OB$ are both equilateral.

Problem 3.

Find the exact roots of the equations

(a) $z^3 = 1$

(b) $(z-1)^4 = -16$

in the form $x + iy$.

Solution

Part (a)

Since $1 = e^{i2\pi n}$, $n \in \mathbb{Z}$, we have $z^3 = e^{i2\pi n}$, whence $z = e^{i2\pi n/3} = \cos\frac{2\pi n}{3} + i\sin\frac{2\pi n}{3}$. Evaluating z in the $n = 0, 1, 2$ cases,

$$\begin{aligned}
n = 0 &: z = \cos 0 + i \sin 0 = 1 \\
n = 1 &: z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\
n = 2 &: z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\end{aligned}$$

$$\boxed{1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i}$$

Part (b)

Observe that $-16 = 16e^{i\pi + 2\pi n} = 16e^{i\pi(2n+1)}$, $n \in \mathbb{Z}$. Hence,

$$\begin{aligned}
(z-1)^4 &= 16e^{i\pi(2n+1)} \\
\implies \quad z - 1 &= 2e^{i\pi(2n+1)/4} \\
\implies \quad z &= 1 + 2e^{i\pi(2n+1)/4} \\
&= 1 + 2\left[\cos\left(\frac{2n+1}{4}\pi\right) + i\sin\left(\frac{2n+1}{4}\pi\right)\right]
\end{aligned}$$

Evaluating z in the $n = 0, 1, 2, 3$ cases,

$$\begin{aligned}
n = 0 &: z = 1 + 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = (1+\sqrt{2}) + i\sqrt{2} \\
n = 1 &: z = 1 + 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = (1-\sqrt{2}) + i\sqrt{2} \\
n = 2 &: z = 1 + 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = (1-\sqrt{2}) - i\sqrt{2} \\
n = 3 &: z = 1 + 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = (1+\sqrt{2}) - i\sqrt{2}
\end{aligned}$$

$$\boxed{(1+\sqrt{2}) \pm i\sqrt{2},\ (1-\sqrt{2}) \pm i\sqrt{2}}$$

Problem 4.

(a) Write down the 5 roots of the equation $z^5 - 1 = 0$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$.

(b) Show that the roots of the equation $(5+z)^5 - (5-z)^5 = 0$ can be written in the form $5i \tan \dfrac{k\pi}{5}$, where $k = 0, \pm 1, \pm 2$.

Solution

Part (a)

Observe that $1 = e^{2\pi n}$, $n \in \mathbb{Z}$. Hence, $z^5 = e^{2\pi n} \implies z = \epsilon^{2\pi n/5}$. Since $-\pi < \theta \leq \pi$, we have $z = e^{-i4\pi/5}, e^{-i2\pi/5}, 1, e^{i2\pi/5}, e^{i4\pi/5}$.

$$\boxed{e^{-i4\pi/5},\ e^{-i2\pi/5},\ 1,\ e^{i2\pi/5},\ e^{i4\pi/5}}$$

Part (b)

$$\begin{aligned}
& & (5+z)^5 - (5-z)^5 &= 0 \\
&\implies & \left(\frac{5+z}{5-z}\right)^5 - 1 &= 0 \\
&\implies & \frac{5+z}{5-z} &= e^{i2k\pi/5} \\
&\implies & 5+z &= e^{i2k\pi/5}(5-z) \\
&\implies & z(1+e^{i2k\pi/5}) &= 5(e^{i2k\pi/5}-1) \\
&\implies & z &= 5 \cdot \frac{e^{i2k\pi/5}-1}{e^{i2k\pi/5}+1} \\
& & &= 5 \cdot \frac{e^{ik\pi/5}-e^{-ik\pi/5}}{e^{ik\pi/5}+e^{-ik\pi/5}} \\
& & &= 5i \cdot \frac{\left(e^{ik\pi/5}-e^{-ik\pi/5}\right)/2i}{\left(e^{ik\pi/5}+e^{-ik\pi/5}\right)/2} \\
& & &= 5i \cdot \frac{\sin k\pi/5}{\cos k\pi/5} \\
& & &= 5i \tan \frac{k\pi}{5}
\end{aligned}$$

Problem 5.

De Moivre's theorem for a positive integral exponent states that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Use de Moivre's theorem to show that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

Hence obtain the roots of the equation

$$128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0$$

in the form $\cos q\pi$, where q is a rational number.

Solution

Taking $n = 7$, we have $\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$, whence $\cos 7\theta = \operatorname{Re}(\cos\theta + i\sin\theta)^7$

$$\begin{aligned}\cos 7\theta &= \operatorname{Re}\left(\cos\theta + i\sin\theta\right)^7 \\ &= \operatorname{Re}\sum_{k=0}^{7}\binom{7}{k}(i\sin\theta)^k\cos^{7-k}\theta \\ &= \operatorname{Re}\sum_{k=0}^{7}\binom{7}{k} i^k \sin^k\theta\cos^{7-k}\theta\end{aligned}$$

Note that $\operatorname{Re} i^k$ is given by

$$\operatorname{Re} i^k = \begin{cases} 0, & k = 1, 3 \pmod 4 \\ 1, & k = 0 \pmod 4 \\ -1, & k = 2 \pmod 4 \end{cases}$$

Hence,

$$\begin{aligned}\cos 7\theta &= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta \\ &= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3 \\ &= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1 - 2\cos^2\theta + \cos^4\theta) \\ &\quad - 7\cos\theta(1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) \\ &= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta - 70\cos^5\theta + 35\cos^7\theta \\ &\quad - 7\cos\theta + 21\cos^3\theta - 21\cos^5\theta + 7\cos^7\theta \\ &= 64\cos^7\theta - 112\cos^5 + 56\cos^3\theta - 7\cos\theta\end{aligned}$$

$$\begin{aligned} & 128x^7 - 224x^5 + 112x^3 - 14x + 1 = 0 \\ \implies \quad & 64x^7 - 112x^5 + 56x^3 - 7x = -\frac{1}{2} \qquad \boxed{x = \cos\theta}\end{aligned}$$

$$\begin{aligned} \implies && \cos 7\theta &= -\frac{1}{2} \\ \implies && 7\theta &= \frac{2}{3}\pi + 2\pi n && n \in \mathbb{Z} \\ \implies && \theta &= \frac{2\pi}{21}(3n+1) \end{aligned}$$

Taking $0 \leq n < 7$,

$$\begin{aligned} x &= \cos\frac{2\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{20\pi}{21}, \cos\frac{26\pi}{21}, \cos\frac{32\pi}{21}, \cos\frac{38\pi}{21} \\ &\equiv \cos\frac{2\pi}{21}, \cos\frac{4\pi}{21}, \cos\frac{8\pi}{21}, \cos\frac{10\pi}{21}, \cos\frac{14\pi}{21}, \cos\frac{16\pi}{21}, \cos\frac{20\pi}{21} \end{aligned}$$

$$\boxed{\cos\frac{2\pi}{21},\ \cos\frac{4\pi}{21},\ \cos\frac{8\pi}{21},\ \cos\frac{10\pi}{21},\ \cos\frac{14\pi}{21},\ \cos\frac{16\pi}{21},\ \cos\frac{20\pi}{21}}$$

Problem 6.

By considering $\sum_{n=1}^{N} z^{2n-1}$, where $z = e^{i\theta}$, or by any method, show that

$$\sum_{n=1}^{N} \sin(2n-1)\theta = \frac{\sin^2 N\theta}{\sin\theta}$$

provided $\sin\theta \neq 0$.

Solution

$$\begin{aligned}
\sum_{n=1}^{N} \sin(2n-1)\theta &= \operatorname{Im} \sum_{n=1}^{N} \Big[\cos(2n-1)\theta + i\sin(2n-1)\theta \Big] \\
&= \operatorname{Im} \sum_{n=1}^{N} z^{2n-1} \\
&= \operatorname{Im} \frac{1}{z} \sum_{n=1}^{N} \left(z^2\right)^n \\
&= \operatorname{Im} \frac{1}{z} \cdot \frac{z^2 \left[\left(z^2\right)^N - 1 \right]}{z^2 - 1} \\
&= \operatorname{Im} \frac{z^{2N} - 1}{z - z^{-1}} \\
&= \operatorname{Im} \frac{\left(z^{2N} - 1\right)/2i}{\left(z - z^{-1}\right)/2i} \\
&= \operatorname{Im} \frac{\left(z^{2N} - 1\right)/2i}{\sin\theta} \\
&= \frac{1}{2\sin\theta} \operatorname{Im} \frac{\left(z^{2N} - 1\right)}{i} \\
&= \frac{1}{2\sin\theta} \operatorname{Im} \left[-i \left(z^{2N} - 1\right) \right] \\
&= -\frac{1}{2\sin\theta} \left[\operatorname{Re} z^{2N} - 1 \right] \\
&= -\frac{1}{2\sin\theta} \left[\operatorname{Re} \left(\cos N\theta + i\sin N\theta\right)^2 - 1 \right] \\
&= -\frac{1}{2\sin\theta} \left[\operatorname{Re} \left(\cos^2 N\theta + 2i\cos N\theta \sin N\theta - \sin^2 N\theta\right) - 1 \right] \\
&= -\frac{1}{2\sin\theta} \left[\cos^2 N\theta - \sin^2 N\theta - 1 \right] \\
&= -\frac{1}{2\sin\theta} \left[\cos^2 N\theta - \sin^2 N\theta - \left(\cos^2 N\theta + \sin^2 N\theta\right) \right] \\
&= -\frac{1}{2\sin\theta} \left[-2\sin^2 N\theta \right] \\
&= \frac{\sin^2 N\theta}{\sin\theta}
\end{aligned}$$

Problem 7.

By considering the series $\sum_{n=0}^{N}\left(e^{2i\theta}\right)^n$, show that, provided $\sin\theta \neq 0$,

$$\sum_{n=0}^{N}\cos 2n\theta = \frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}$$

and deduce that

$$\sum_{n=0}^{N}\sin^2 n\theta = \frac{N}{2}+\frac{1}{2}-\frac{\sin(N+1)\theta\cos N\theta}{2\sin\theta}$$

Solution

$$\begin{aligned}
\sum_{n=0}^{N}\cos 2n\theta &= \operatorname{Re}\sum_{n=0}^{N}\Big[\cos 2n\theta + i\sin 2n\theta\Big] \\
&= \operatorname{Re}\sum_{n=0}^{N} e^{i2n\theta} \\
&= \operatorname{Re}\sum_{n=0}^{N}\left(e^{2i\theta}\right)^n \\
&= \operatorname{Re}\frac{\left(e^{2i\theta}\right)^{N+1}-1}{e^{2i\theta}-1} \\
&= \operatorname{Re}\left(e^{-i\theta}\cdot\frac{\left(e^{2i\theta}\right)^{N+1}-1}{e^{i\theta}-e^{-i\theta}}\right) \\
&= \operatorname{Re}\left(\frac{e^{-i\theta}}{2i}\cdot\frac{\left(e^{2i\theta}\right)^{N+1}-1}{\left(e^{i\theta}-e^{-i\theta}\right)/2i}\right) \\
&= \operatorname{Re}\left(\frac{-ie^{-i\theta}}{2}\cdot\frac{\left(e^{2i\theta}\right)^{N+1}-1}{\sin\theta}\right) \\
&= -\frac{1}{2\sin\theta}\operatorname{Re}\left(ie^{-i\theta}\left[\left(e^{2i\theta}\right)^{N+1}-1\right]\right) \\
&= \frac{1}{2\sin\theta}\operatorname{Im}\left(e^{-i\theta}\left[\left(e^{2i\theta}\right)^{N+1}-1\right]\right) \\
&= \frac{1}{2\sin\theta}\operatorname{Im}\left(e^{-i\theta}\left[\left(e^{i\theta(N+1)}\right)^{2}-1\right]\right) \\
&= \frac{1}{2\sin\theta}\operatorname{Im}\left((\cos\theta - i\sin\theta)\left[\Big(\cos(N+1)\theta + i\sin(N+1)\theta\Big)^2-1\right]\right) \\
&= \frac{1}{2\sin\theta}\operatorname{Im}\Big((\cos\theta - i\sin\theta)\Big[\cos^2(N+1)\theta + 2i\cos(N+1)\theta\sin(N+1)\theta \\
&\qquad\qquad - \sin^2(N+1)\theta - 1\Big]\Big) \\
&= \frac{1}{2\sin\theta}\operatorname{Im}\Big((\cos\theta - i\sin\theta)\Big[\cos^2(N+1)\theta + 2i\cos(N+1)\theta\sin(N+1)\theta \\
&\qquad\qquad - \sin^2(N+1)\theta - (\cos^2(N+1)\theta + \sin^2(N+1)\theta)\Big]\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\sin\theta}\,\mathrm{Im}\left(\,(\cos\theta - i\sin\theta)\left[2i\cos(N+1)\theta\sin(N+1)\theta - 2\sin^2(N+1)\theta\right]\right) \\
&= \frac{1}{2\sin\theta}\left[2\cos\theta\cos(N+1)\theta\sin(N+1)\theta + 2\sin\theta\sin^2(N+1)\theta\right] \\
&= \frac{\sin(N+1)\theta}{\sin\theta}\left[\cos\theta\cos(N+1)\theta + \sin\theta\sin(N+1)\theta\right] \\
&= \frac{\sin(N+1)\theta}{\sin\theta}\cdot\cos\Big((N+1)\theta - \theta\Big) \\
&= \frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}
\end{aligned}
$$

Recall that $\cos 2n\theta = 1 - 2\sin^2 n\theta \implies \sin^2 n\theta = \dfrac{1}{2}(1 - 2\cos 2n\theta)$.

$$
\begin{aligned}
\sum_{n=0}^{N}\sin^2 n\theta &= \sum_{n=0}^{N}\frac{1}{2}\left(1 - \cos 2n\theta\right) \\
&= \frac{1}{2}\sum_{n=0}^{N}(1 - \cos 2n\theta) \\
&= \frac{1}{2}\left((N+1) - \frac{\sin(N+1)\theta\cos N\theta}{\sin\theta}\right) \\
&= \frac{N}{2} + \frac{1}{2} - \frac{\sin(N+1)\theta\cos N\theta}{2\sin\theta}
\end{aligned}
$$

Problem 8.

Given that $z = e^{i\theta}$, show that $z^k + \dfrac{1}{z^k} = 2\cos k\theta$, $k \in \mathbb{Z}$.

Hence, show that $\cos^8\theta = \dfrac{1}{128}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35)$.

Find, correct to three decimal places, the values of θ such that $0 < \theta < \dfrac{1}{2}\pi$ and $\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 1 = 0$.

Solution

$$\begin{aligned} z^k + \frac{1}{z^k} &= z^k + z^{-k} \\ &= \left(e^{i\theta}\right)^k + \left(e^{-i\theta}\right)^k \\ &= e^{ik\theta} + e^{-ik\theta} \\ &= [\cos(k\theta) + i\sin(k\theta)] + (\cos(-k\theta) + i\sin(-k\theta)) \\ &= \cos(k\theta) + i\sin(k\theta) + \cos(k\theta) - i\sin(k\theta) \\ &= 2\cos(k\theta) \end{aligned}$$

$$\begin{aligned} \cos^8\theta &= \frac{1}{256}(2\cos\theta)^8 \\ &= \frac{1}{256}(z + z^{-1})^8 \\ &= \frac{1}{256}z^{-8}\left(z^2+1\right)^8 \\ &= \frac{1}{256}z^{-8}\left(1 + 8z^2 + 28z^4 + 56z^6 + 70z^8 + 56z^{10} + 28z^{12} + 8z^{14} + z^{16}\right) \\ &= \frac{1}{256}\left(z^{-8} + 8z^{-6} + 28z^{-4} + 56z^{-2} + 70 + 56z^2 + 28z^4 + 8z^6 + z^8\right) \\ &= \frac{1}{256}\left[\left(z^8 + z^{-8}\right) + 8\left(z^6 + 8z^{-6}\right) + 28\left(z^4 + z^{-4}\right) + 56\left(z^2 + z^{-2}\right) + 70\right] \\ &= \frac{2}{256}\left[\left(\frac{z^{-8}+z^8}{2}\right) + 8\left(\frac{z^6+z^{-6}}{2}\right) + 28\left(\frac{z^4+z^{-4}}{2}\right) + 56\left(\frac{z^2+z^{-2}}{2}\right) + \frac{70}{2}\right] \\ &= \frac{1}{128}(\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35) \end{aligned}$$

$$\begin{aligned} & \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 1 = 0 \\ \implies & \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35 = 35 \\ \implies & 128\cos^8\theta = 34 \\ \implies & \cos\theta = \sqrt[8]{\frac{34}{128}} \\ \implies & \theta = 0.560 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\theta = 0.560}$$