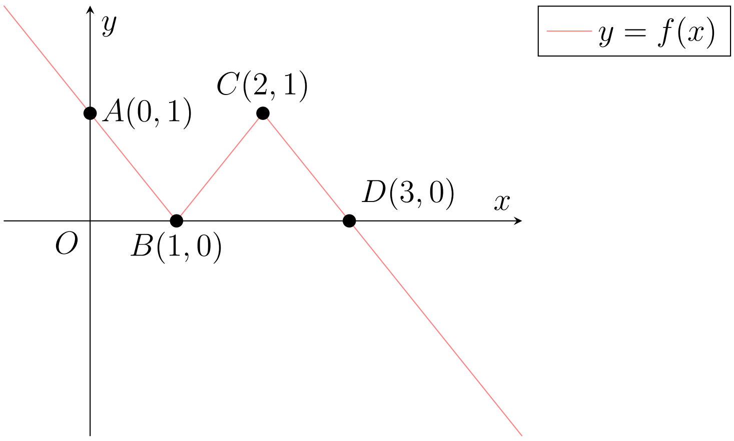
Problem 1.



The graph of $y = f(x)$ is shown here. The points A, B, C and D have coordinates $(0, 1)$, $(1, 0)$, $(2, 1)$ and $(3, 0)$ respectively. Sketch, separately, the graphs of

(a) $y = f(2x)$

(b) $y = f(x + 3)$

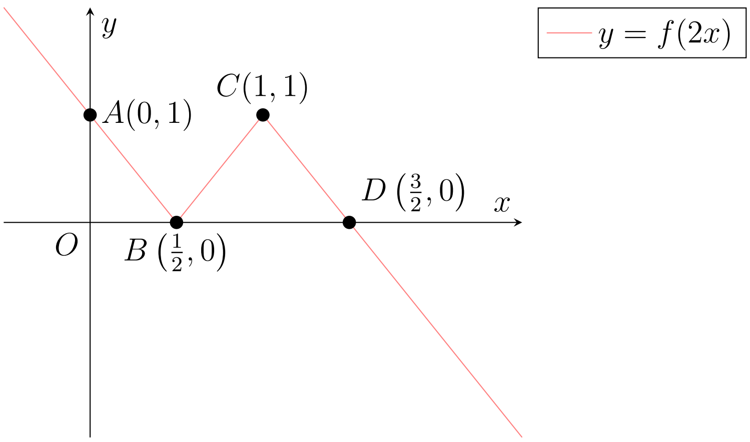
(c) $y = 1 - f(x)$

(d) $y = 3f\left(\frac{x}{2} - 1\right)$

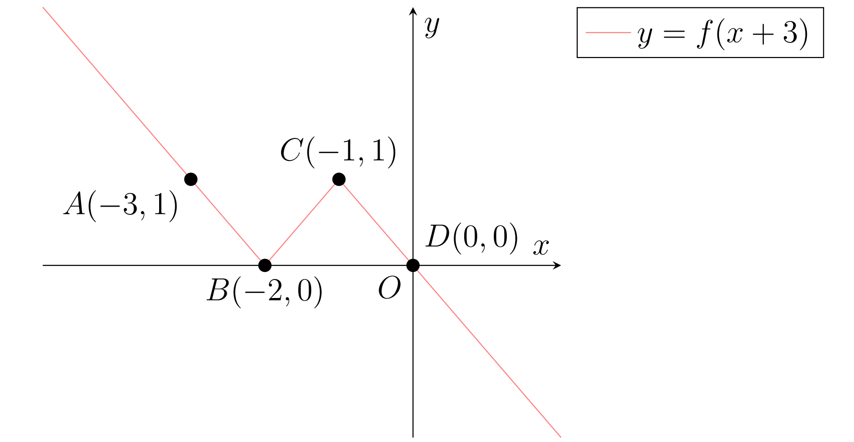
stating, in each case, the coordinates of the points corresponding to A, B, C and D.

Solution

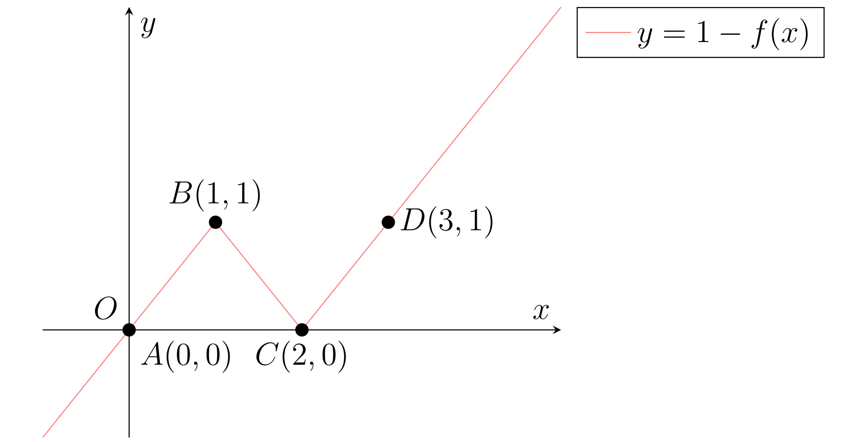
Part (a)



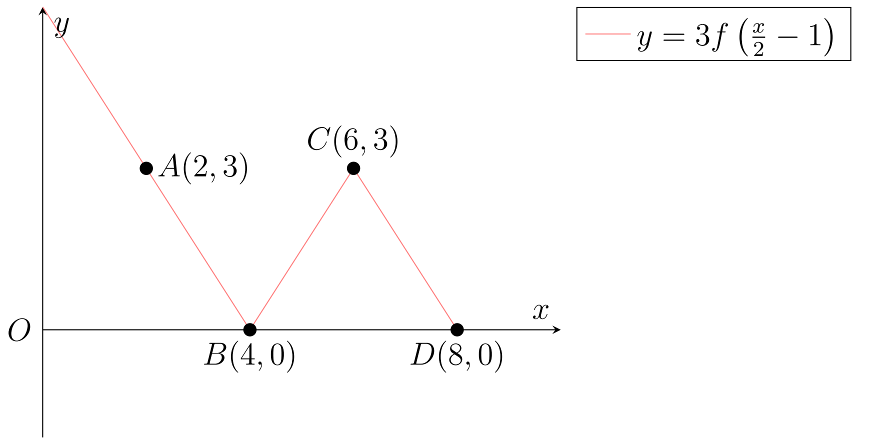
Part (b)



Part (c)



Part (d)



Problem 2.

Sketch, on a single clear diagram, the graphs of

(a) $y = x^2$

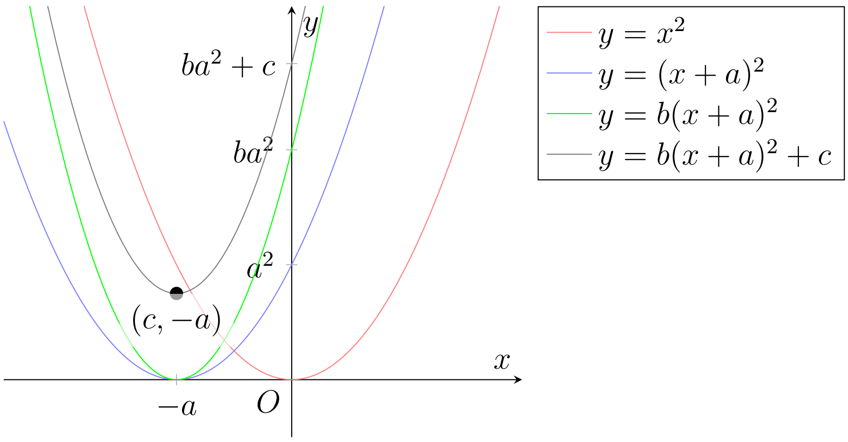
(b) $y = (x + a)^2$

(c) $y = b(x + a)^2$

(d) $y = b(x + a)^2 + c$

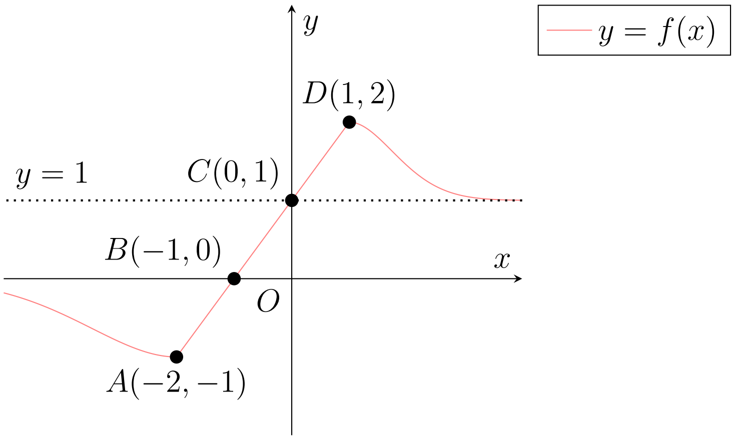
(Assume constants $a > 0$, $c > 0$ and $b > 1$)

Solution



Problem 3.

The graph below has equation $y = f(x)$. It has asymptotes $y = 1$ and $y = 0$, a maximum point at $D(1, 2)$, a minimum point at $A(-2, -1)$, cuts the x-axis at $B(-1.0)$ and cuts the y-axis at $C(0, 1)$.



Sketch on separate diagrams the graphs of the following curves, labelling each curve clearly, indicating the horizontal asymptotes and showing the coordinates of the points corresponding to points A, B, C and D.

(a) $y = f(x + 1)$

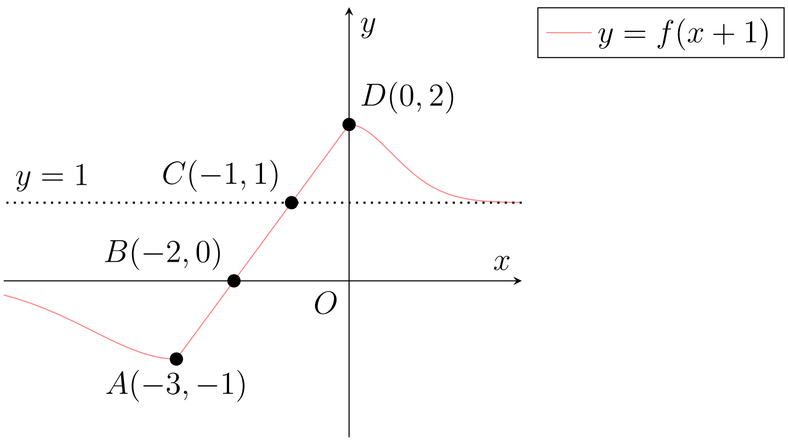
(b) $y = f\left(\frac{x}{2}\right)$

(c) $y = 2f(x) - 2$

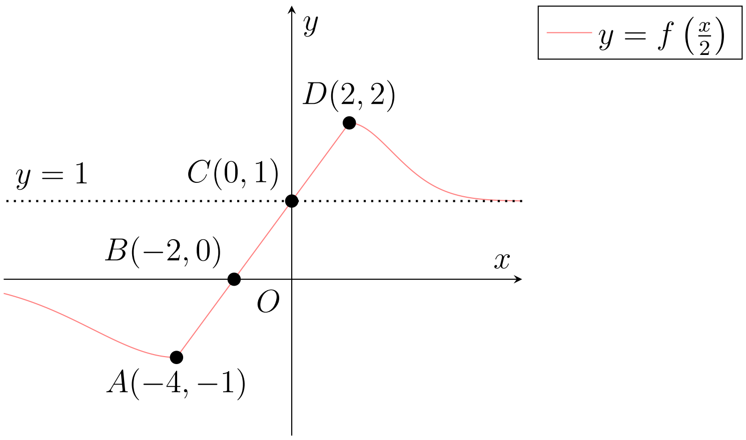
Find the number of solutions to the equation $f(x) = af(x)$ where $a \geq 2$.

Solution

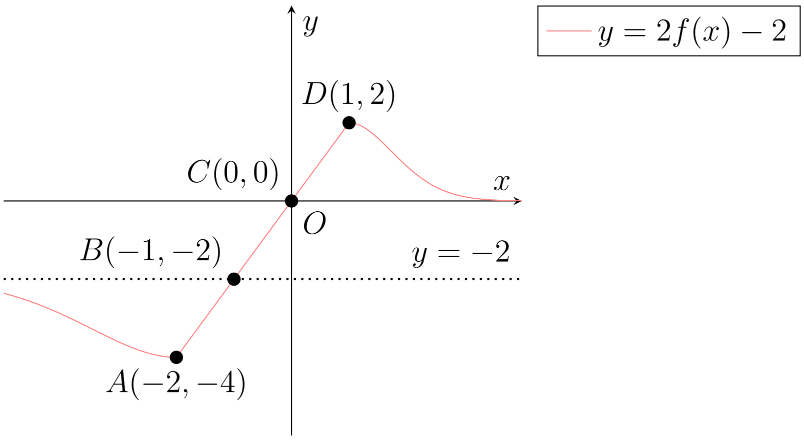
Part (a)



Part (b)



Part (c)



All points with a y-coordinate of 0 are invariant under the transformation $f(x) \mapsto af(x)$. Since there is only one such point $(B(-1, 0))$, there is only 1 solution to the equation $f(x) = af(x)$, where $a \geq 2$.

$\boxed{\text{There is 1 solution.}}$

Problem 4.

The curve with equation $y = x^2$ is tranformed by a translation of 2 units in the positive x-direction, followed by a stretch with scale factor $\frac{1}{2}$ parallel to the y-axis, followed by a translation of 6 units in the negative y-direction. Find the equation of the new curve in the form $y = f(x)$ and the exact coordinates of the points where this curve crosses the x- and y-axes.

Solution

$$\begin{array}{c} y = x^3 \\ \Big\downarrow x \mapsto x - 2 \\ y = (x-2)^3 \\ \Big\downarrow y \mapsto 2y \\ 2y = (x-2)^3 \\ \Big\downarrow y \mapsto y + 6 \\ 2(y+6) = (x-2)^3 \end{array}$$

$$\boxed{y = \frac{1}{2}(x-2)^3 - 6}$$

When $x = 0$, $y = \frac{1}{2}(-2)^3 - 6 = -10$. When $y = 0$, $x = 2 + \sqrt[3]{12}$.

$$\boxed{\text{The curve crosses the } x\text{-axis at } (2 + \sqrt[3]{12}, 0) \text{ and the } y\text{-axis at } (0, -10).}$$

Problem 5.

Find the values of the constants A and B such that $\dfrac{x^2-4x}{(x-2)^2} = A + \dfrac{B}{(x-2)^2}$ for all values of x except $x=2$.

Hence state a sequence of transformations by which the graph of $y = \dfrac{x^2-4x}{(x-2)^2}$ may be obtained from the graph of $y = \dfrac{1}{x^2}$.

Solution

$$\begin{aligned}\frac{x^2-4x}{(x-2)^2} &= \frac{(x-2)^2-4}{(x-2)^2}\\ &= 1 + \frac{-4}{(x-2)^2}\end{aligned}$$

$$\boxed{A=1,\ B=-4}$$

$$\begin{array}{c} y = \dfrac{1}{x^2} \\ \Big\downarrow x \mapsto x-2 \\ y = \dfrac{1}{(x-2)^2} \\ \Big\downarrow y \mapsto \dfrac{1}{4}y \\ y = \dfrac{4}{(x-2)^2} \\ \Big\downarrow y \mapsto -y \\ y = \dfrac{-4}{(x-2)^2} \\ \Big\downarrow y \mapsto y-1 \\ y = 1 + \dfrac{-4}{(x-2)^2} \end{array}$$

1. Translate the curve 2 units in the positive x-direction.
2. Stretch the curve with a scale factor of 4 parallel to the y-axis.
3. Reflect the curve about the x-axis.
4. Translate the curve 1 unit in the positive y-direction.

Problem 6.

The transformations A, B, C and D are given as follows:

- A: A reflection about the y-axis.
- B: A translation of 2 units in the positive x-direction.
- C: A scaling parallel to the y-axis by a factor of 3.
- D: A translation of 1 unit in the positive y-direction.

A curve undergoes the transformations A, B, C and D in succession, and the equation of the resulting curve is $y = 3\sqrt{2-x}+1$. Determine the equation of the curve before the transformations were effected.

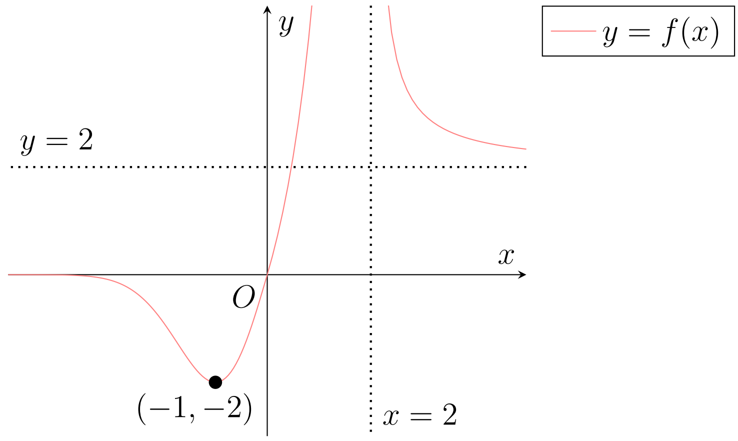
Solution

$$\begin{aligned}
A\colon x &\mapsto -x & &\implies & A^{-1}\colon x &\mapsto -x \\
B\colon x &\mapsto x-2 & &\implies & B^{-1}\colon x &\mapsto x+2 \\
C\colon y &\mapsto \frac{1}{3}y & &\implies & C^{-1}\colon y &\mapsto 3y \\
D\colon y &\mapsto y-1 & &\implies & D^{-1}\colon y &\mapsto y+1
\end{aligned}$$

$$\begin{array}{c}
y = 3\sqrt{2-x}+1 \\
\Big\downarrow D^{-1} \\
y+1 = 3\sqrt{2-x}+1 \\
\Big\downarrow C^{-1} \\
3y+1 = 3\sqrt{2-x}+1 \\
\Big\downarrow B^{-1} \\
3y+1 = 3\sqrt{2-(x+2)}+1 \\
\Big\downarrow A^{-1} \\
3y+1 = 3\sqrt{2-(-x+2)}+1
\end{array}$$

$$\boxed{\text{The original curve has equation } y = \sqrt{x}.}$$

Problem 7.



The diagram shows the graph of $y = f(x)$. The curve passes through the origin and has minimum point $(-1, -2)$. The asymptotes are $x = 2$, $y = 0$ and $y = 2$.

Sketch, on separate diagrams, the graphs of

(a) $y = f(x - 1)$

(b) $y = f(|x|)$

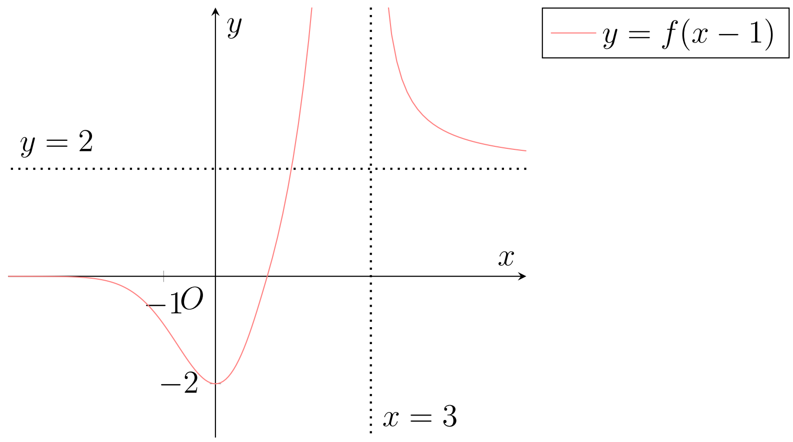
(c) $y = f(|x - 1|)$

(d) $y = |f(x)|$

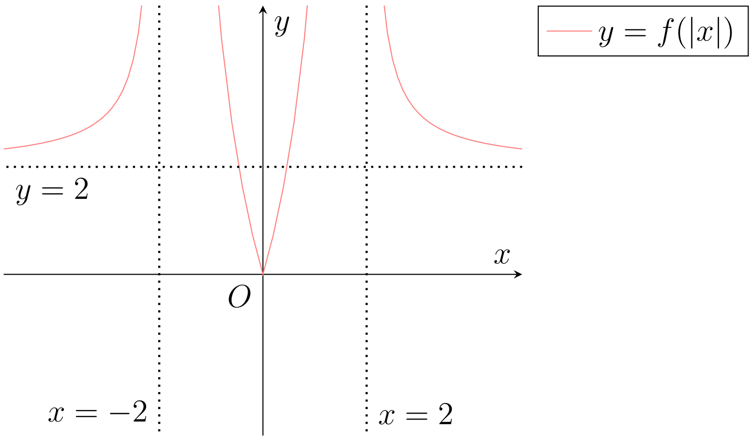
(e) $y = \dfrac{1}{f(x)}$

Solution

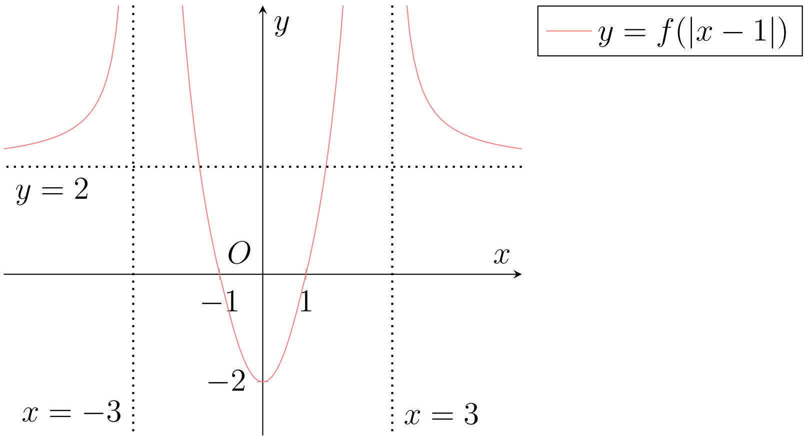
Part (a)



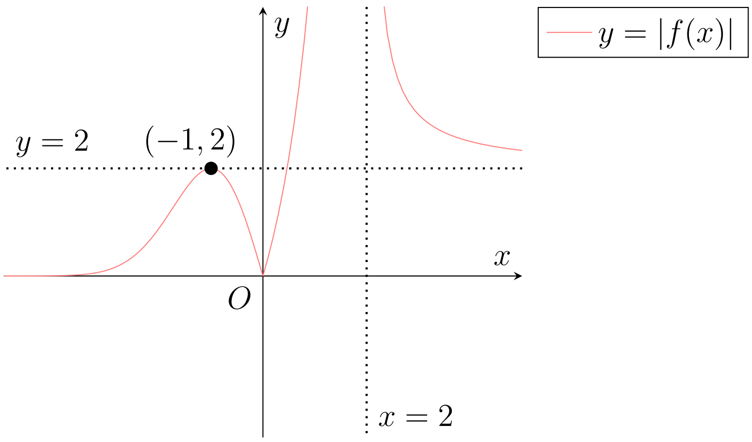
Part (b)



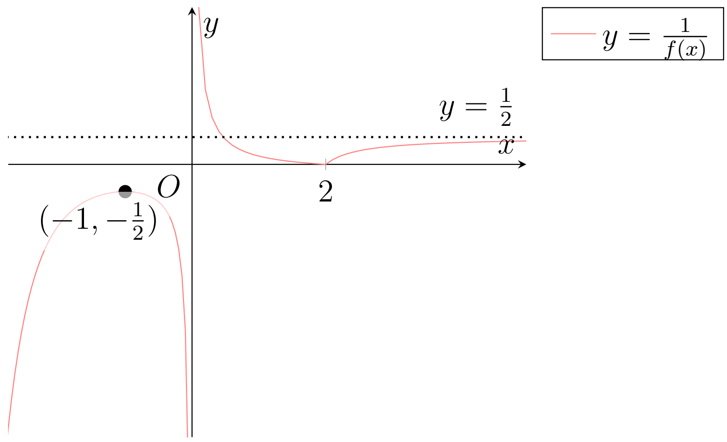
Part (c)



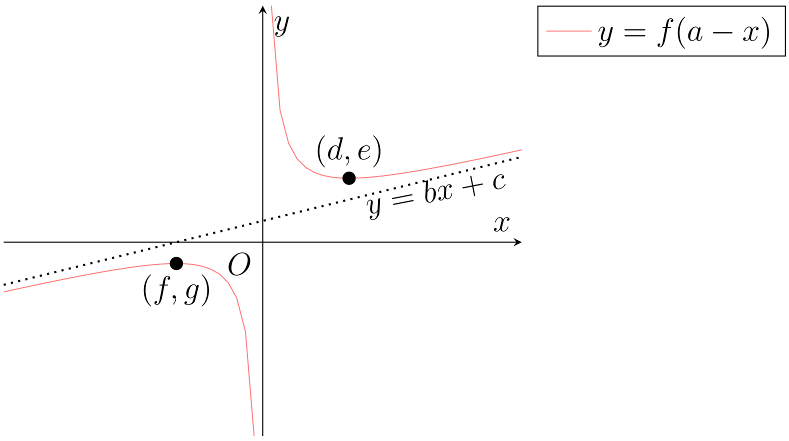
Part (d)



Part (e)



Problem 8.



The graph of $y = f(a - x)$ is shown in the figure, where $a > 0$. The curve has asymptotes $x = 0$, $y = bx + c$, a minimum point at (d, e) and a maximum point at (f, g).

Given $a > d$, sketch separately, the graphs of

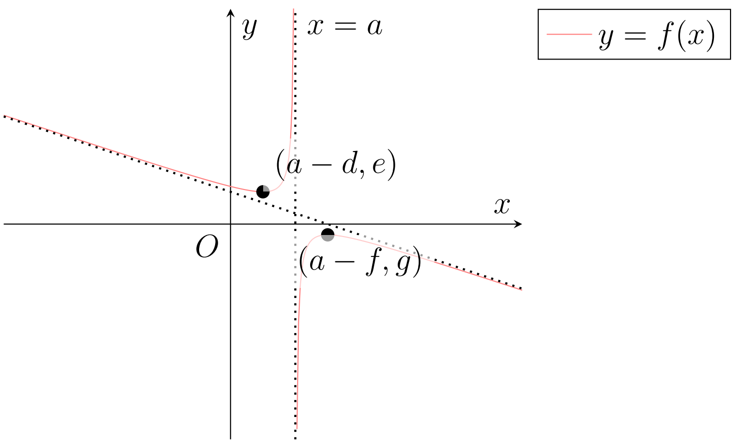
(a) $y = f(x)$

(b) $y = f(|x|)$

(c) $y = \dfrac{1}{f(x)}$

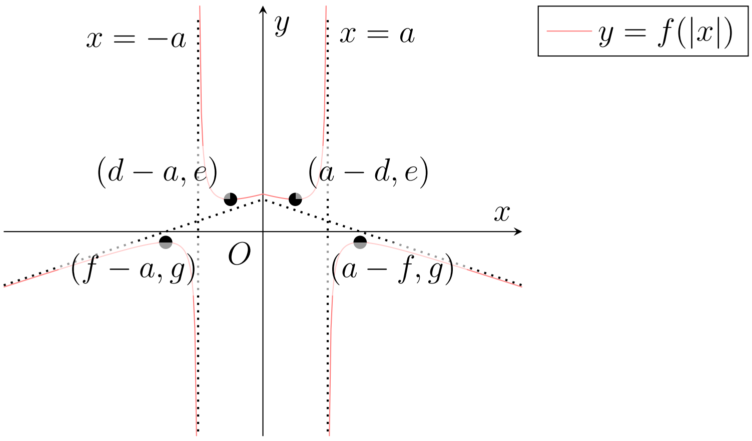
Solution

Part (a)



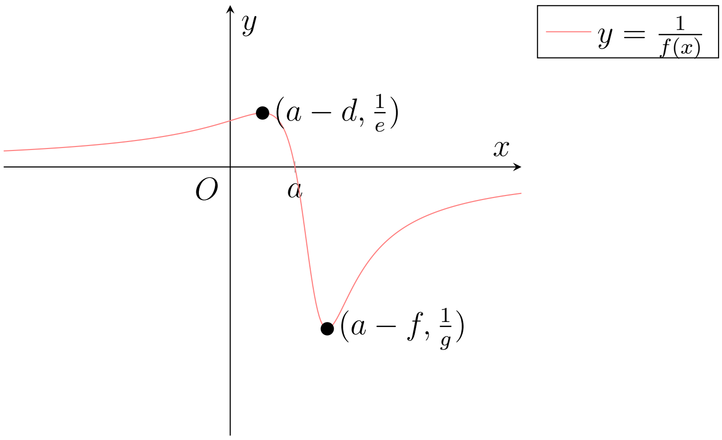
Equation of asymptote: $y = b(a - x) + c$

Part (b)



Equation of asymptotes: $y = b(a+x)+c$, $y = b(a-x)+c$

Part (c)



Problem 9.

A curve C_1 is defined by the parametric equations

$$x = t(t+2),\ y = 2(t+1)$$

(a) Find the axial intercepts of the curve.

(b) Sketch C_1.

(c) A curve C_2 is defined by the parametric equations $x = t(t+2)$, $y = t+1$. Describe a geometrical transformation which maps C_1 to C_2. Hence sketch the curve C_2 in the same diagram as C_1.

(d) Show that the Cartesian equation of the curve C_1 is given by $y^2 = 4(x+1)$.

Solution

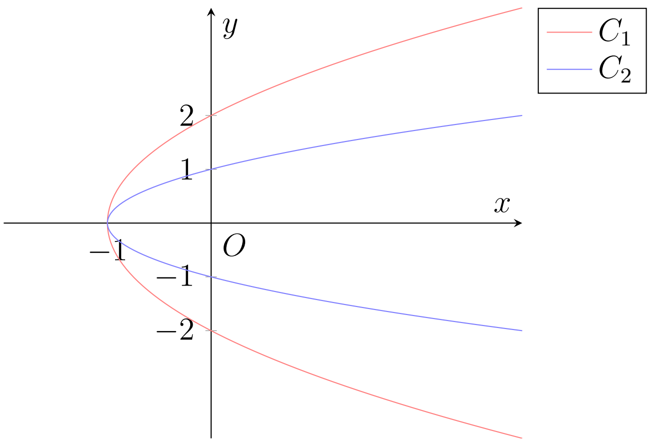
Part (a)

Consider $x = 0$. Then $t(t+2) = 0$, whence $t = 0$ or $t = -2$. When $t = 0$, $y = 2$. When $t = -2$, $y = -2$. Hence, the curve intercepts the y-axis at $(0, 2)$ and $(0, -2)$.

Consider $y = 0$. Then $t = -1$, whence $x = -1$. Hence, the curve intercepts the x-axis at $(-1, 0)$.

$$\boxed{\text{The axial intercepts of the curve are } (0,2),\ (0,-2) \text{ and } (-1,0).}$$

Part (b)



Part (c)

$$\boxed{\text{Scale by a factor of } \frac{1}{2} \text{ parallel to the } y\text{-axis.}}$$

Part (d)

$$\begin{aligned} y^2 &= (2(t+1))^2 \\ &= 4(t^2+2t+1) \\ &= 4(t(t+1)+1) \\ &= 4(x+1) \end{aligned}$$