Problem 1.

Given that $y = 1$ when $x = 1$, find the particular solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x}$.

Solution

$$\begin{aligned}
&& \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y^2}{x} \\
\implies && \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x} \\
\implies && \int \frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int \frac{1}{x}\,\mathrm{d}x \\
\implies && \int \frac{1}{y^2}\,\mathrm{d}y &= \int \frac{1}{x}\,\mathrm{d}x \\
\implies && -\frac{1}{y} &= \ln|x| + C_1 \\
\implies && y &= -\frac{1}{\ln|x| + C_1} \\
&& &= \frac{1}{C - \ln|x|}
\end{aligned}$$

At $x = 1$, $y = 1$, $1 = \frac{1}{C - \ln|1|} \implies C = 1$.

$$\boxed{y = \frac{1}{1 - \ln|x|}}$$

Problem 2.

Two variables x and t are connected by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{kx}{10-x}$, where $0 < x < 10$ and where k is a constant. It is given that $x = 1$ when $t = 0$ and that $x = 2$ when $t = 1$. Find the value of t when $x = 5$, given your answer to three s.f.

Solution

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{kx}{10-x} \\
\implies \frac{10-x}{x}\frac{\mathrm{d}x}{\mathrm{d}t} &= k \\
\implies \int \frac{10-x}{x}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= k\int \mathrm{d}t \\
\implies \int \frac{10-x}{x}\,\mathrm{d}x &= k\int \mathrm{d}t \\
\implies \int \left(\frac{10}{x} - 1\right)\mathrm{d}x &= kt + C \\
\implies 10\ln x - x &= kt + C
\end{aligned}$$

At $t = 0$, $-x = 1$, $C = -1$.

At $t = 1$, $x = 2$, $10\ln 2 - 2 = k - 1 \implies k = \ln 2 - 1$.

When $x = 5$, $10\ln 5 - 5 = (\ln 2 - 1)t - 1 \implies t = \frac{10\ln 5 - 4}{10\ln 2 - 1} = 2.04$ (3 s.f.).

$$\boxed{\text{When } x = 5,\ t = 2.04}$$

Problem 3.

Use the substitution $y = u - 2x$ to find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{8x+4y+1}{4x+2y+1}$.

Solution

Note that $y = u - 2x \implies u = y + 2x$. Also, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} - 2$.

$$\begin{aligned}
& & \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{8x+4y+1}{4x+2y+1} \\
&\implies & \frac{\mathrm{d}u}{\mathrm{d}x} - 2 &= -\frac{8x+4(u-2x)+1}{4x+2(u-2x)+1} \\
& & &= -\frac{4u+1}{2u+1} \\
&\implies & \frac{\mathrm{d}u}{\mathrm{d}x} &= 2 - \frac{4u+1}{2u+1} \\
& & &= \frac{4u+2}{2u+1} - \frac{4u+1}{2u+1} \\
& & &= \frac{1}{2u+1} \\
&\implies & (2u+1)\frac{\mathrm{d}u}{\mathrm{d}x} &= 1 \\
&\implies & \int (2u+1)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x &= \int \mathrm{d}x \\
&\implies & \int (2u+1)\,\mathrm{d}u &= \int \mathrm{d}x \\
&\implies & u^2 + u &= x + C \\
&\implies & (y+2x)^2 + (y+2x) &= x + C \\
&\implies & (y+2x)^2 + y + x &= C
\end{aligned}$$

$$\boxed{(y+2x)^2 + y + x = C}$$

Problem 4.

By using the substitution $z = ye^{2x}$, find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = xe^{-2x}$.

Find the particular solution of the differential equation given that $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Solution

Note that $z = ye^{2x} \implies \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}e^{2x} + 2ye^{2x} = \frac{\mathrm{d}y}{\mathrm{d}x}e^{2x} + 2z$. Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}x}e^{-2x} - 2y$.

$$
\begin{aligned}
& \frac{\mathrm{d}y}{\mathrm{d}x} + 2y = xe^{-2x} \\
\implies & \frac{\mathrm{d}z}{\mathrm{d}x}e^{-2x} - 2y + 2y = xe^{-2x} \\
\implies & \frac{\mathrm{d}z}{\mathrm{d}x} = x \\
\implies & z = \frac{x^2}{2} + C \\
\implies & ye^{2x} = \frac{x^2}{2} + C \\
\implies & y = \frac{x^2}{2e^{2x}} + \frac{C}{e^{2x}}
\end{aligned}
$$

$$\boxed{y = \frac{x^2}{2e^{2x}} + \frac{C}{e^{2x}}}$$

At $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$. Hence, $1 + 2y = 0 \implies y = -\frac{1}{2}$. Thus, $C = -\frac{1}{2}$.

$$\boxed{y = \frac{x^2 - 1}{2e^{2x}}}$$

Problem 5.

Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 6xy^3$.

Find its particular solution given that $y = 0.5$ when $x = 0$.

Determine the interval of validity for the particular solution.

Solution

$$\begin{aligned}
&& \frac{\mathrm{d}y}{\mathrm{d}x} &= 6xy^3 \\
\implies && \frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x} &= 6x \\
\implies && \int \frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= 6\int x\,\mathrm{d}x \\
\implies && \int \frac{1}{y^3}\,\mathrm{d}y &= 6\int x\,\mathrm{d}x \\
\implies && -\frac{1}{2}\frac{1}{y^2} &= 3x^2 + C_1 \\
\implies && \frac{1}{y^2} &= C - 6x^2 \\
\implies && y^2 &= \frac{1}{C - 6x^2}
\end{aligned}$$

$$\boxed{y^2 = \frac{1}{C - 6x^2}}$$

At $x = 0$, $y = 0.5$, $\frac{1}{4} = \frac{1}{C} \implies C = 4$.

$$\boxed{y^2 = \frac{1}{4 - 6x^2}}$$

Observe that we require $4 - 6x^2 > 0$, whence $-\sqrt{\frac{2}{3}} < x < \sqrt{\frac{2}{3}}$.

$$\boxed{\text{The interval of validity is } \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right).}$$

Problem 6.

(a) Find the general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x}{x^2+1}$.

(b) What can you say about the gradient of every solution as $x \to \pm\infty$?

(c) Find the particular solution of the differential equation for which $y = 2$ when $x = 0$. Hence sketch the graph of this solution.

Solution

Part (a)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3x}{x^2+1} \\ &= \frac{3}{2}\frac{2x}{x^2+1} \\ \implies \quad y &= \frac{3}{2}\ln\left(x^2+1\right) + C \end{aligned}$$

$$\boxed{y = \frac{3}{2}\ln(x^2+1) + C}$$

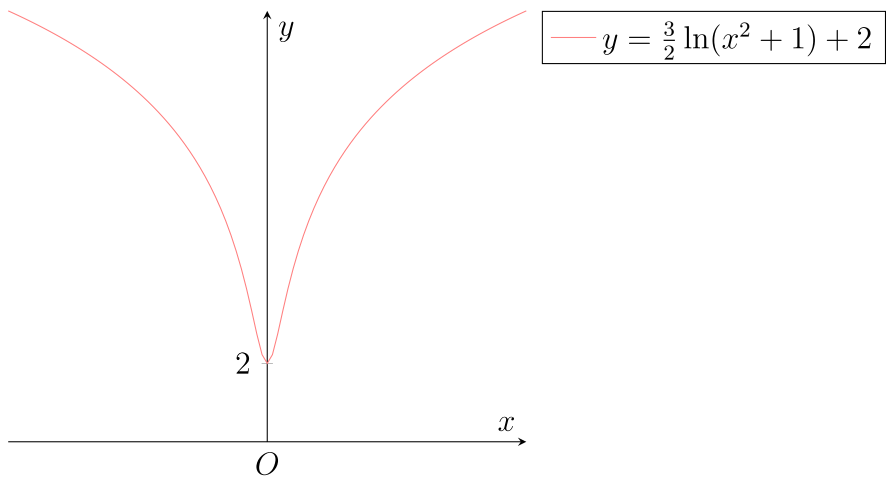
Part (b)

As $x \to \pm\infty$, $\frac{3x}{x^2+1} \to 0^+$. Hence, the gradient of every solution approaches 0 from above.

Part (c)

When $x = 0$ and $y = 2$, $C = 2$.

$$\boxed{y = \frac{3}{2}\ln(x^2+1) + 2}$$



Problem 7.

The variables x, y and z are connected by the following differential equations.

$$\frac{\mathrm{d}z}{\mathrm{d}x} = 3 - 2z \tag{7.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = z \tag{7.2}$$

(a) Given that $z < \frac{3}{2}$, solve equation 7.1 to find z in terms of x.

(b) Hence find y in terms of x.

(c) Use the result in part (b) to show that

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = a\frac{\mathrm{d}y}{\mathrm{d}x} + b$$

for constants a and b to be determined.

(d) The curve of the solution in part (b) passes through the points $(0, 1)$ and $(2, 3 + e^{-4})$. Sketch this curve, indicating its axial intercept and asymptote (if any).

Solution

Part (a)

$$\begin{aligned}
& & \frac{\mathrm{d}z}{\mathrm{d}x} &= 3 - 2z \\
&\implies & \frac{1}{3-2z}\frac{\mathrm{d}z}{\mathrm{d}x} &= 1 \\
&\implies & \int \frac{1}{3-2z}\frac{\mathrm{d}z}{\mathrm{d}x}\,\mathrm{d}x &= \int \mathrm{d}x \\
&\implies & \int \frac{1}{3-2z}\,\mathrm{d}z &= \int \mathrm{d}x \\
&\implies & -\frac{1}{2}\ln(3-2z) &= x + C_1 \\
&\implies & \ln(3-2z) &= C_2 - 2x \\
&\implies & 3 - 2z &= C_3 e^{-2x} \\
&\implies & z &= \frac{3}{2} - C_4 e^{-2x}
\end{aligned}$$

$$\boxed{z = \frac{3}{2} - Ae^{-2x},\ A > 0}$$

Part (b)

$$\begin{aligned}
& & \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{2} - C_4 e^{-2x} \\
&\implies & y &= \int \left(\frac{3}{2} - C_4 e^{-2x}\right)\mathrm{d}x \\
& & &= \frac{3}{2}x - \frac{C_4}{2}e^{-2x} + C_6
\end{aligned}$$

$$\boxed{y = \frac{3}{2}x - \frac{A}{2}e^{-2x} + B}$$

Part (c)

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{2} + C_4 e^{-2x} \\ \implies \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -2C_4 e^{-2x} \\ &= -2\left(\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{3}{2}\right) \\ &= -2\frac{\mathrm{d}y}{\mathrm{d}x} + 3 \end{aligned}$$

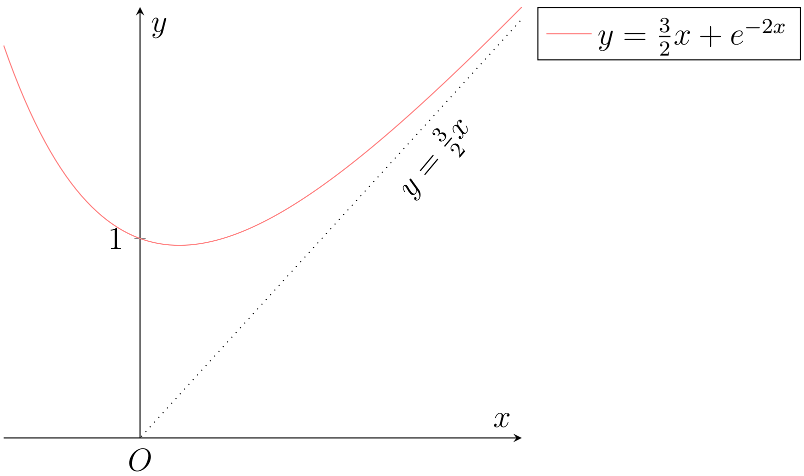
$$\boxed{a = -2,\ b = 3}$$

Part (d)

At $(0, 1)$, we obtain $1 = -\frac{A}{2} + B$.

At $(2, 3 + e^{-4})$, we obtain $3 + e^{-4} = 3 - \frac{A}{2}e^{-4} + B \implies 1 = -\frac{A}{2} + Be^{4}$.

Hence, $B = Be^4$, whence $B = 0$ and $A = -2$. The curve thus has equation $y = \frac{3}{2}x + e^{-2x}$.



Problem 8.

A bottle containing liquid is taken from a refrigerator and placed in a room where the termperature is a constant 20 °C. As the liquid warms up, the rate of increase of its temperature θ °C after time t minutes is proportional to the temperature difference $(20-\theta)$ °C. Initially the temperature of the liquid is 10 °C and the rate of increase of the temperature is 1 °C per minute. By setting up and solving a differential equation, show that $\theta = 20 - 10e^{-t/10}$.

Find the time it takes the liquid to reach a temperature of 15 °C, and state what happens to θ for large values of t. Sketch a graph of θ against t.

Solution

Since $\frac{\mathrm{d}\theta}{\mathrm{d}t} \propto (20-\theta)$, we have $\frac{\mathrm{d}\theta}{\mathrm{d}t} = k(20-\theta)$, where k is a constant. We now solve for θ.

$$\begin{aligned}
&& \frac{\mathrm{d}\theta}{\mathrm{d}t} &= k(20-\theta) \\
&\implies & \frac{1}{20-\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= k \\
&\implies & \int \frac{1}{20-\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\,\mathrm{d}t &= k\int \mathrm{d}t \\
&\implies & \int \frac{1}{20-\theta}\,\mathrm{d}\theta &= k\int \mathrm{d}t \\
&\implies & -\ln(20-\theta) &= kt + C_1 \\
&\implies & \ln(20-\theta) &= C_2 - kt \\
&\implies & 20-\theta &= Ce^{-kt} \\
&\implies & \theta &= 20 - Ce^{-kt}
\end{aligned}$$

At $t=0$, $\theta = 10$. Hence, $10 = 20 - C \implies C = 10$. We also have $\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right|_0 = 1$. Hence, $1 = k\left[20 - (20 - 10e^0)\right] = 10k \implies k = \frac{1}{10}$. Thus,

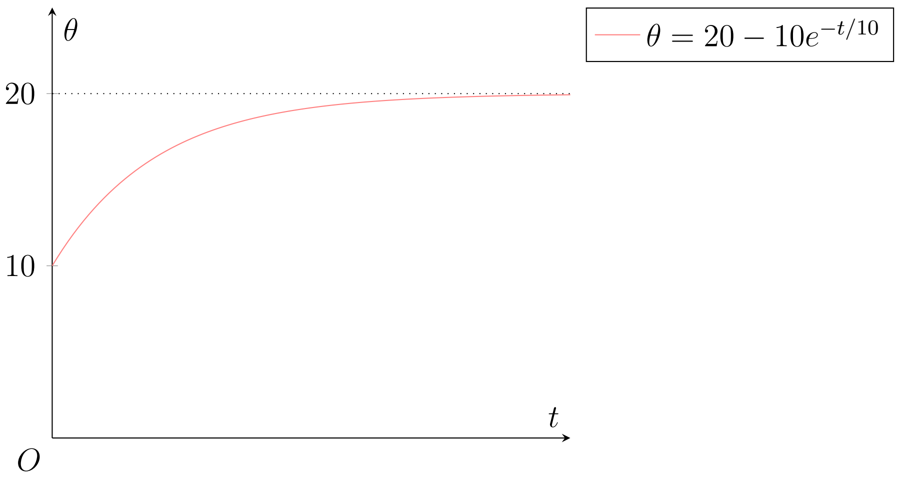
$$\theta = 20 - 10e^{-t/10}.$$

Consider $\theta = 15$.

$$\begin{aligned}
&& 15 &= 20 - 10e^{-t/10} \\
&\implies & e^{-t/10} &= \frac{1}{2} \\
&\implies & -\frac{t}{10} &= -\ln 2 \\
&\implies & t &= 10\ln 2
\end{aligned}$$

$$\boxed{\text{It takes } 10\ln 2 \text{ minutes for the liquid to reach a temperature of } 15\,^\circ\text{C.}}$$

$$\boxed{\text{As } t \to \infty,\ \theta \to 20.}$$


θ
20
10
t
O
$\theta = 20 - 10e^{-t/10}$

Problem 9.

(a) Find $\displaystyle\int \frac{1}{100 - v^2} \, \mathrm{d}x$.

(b) A stone is dropped from a stationary balloon. It leaves the balloon with zero speed, and t seconds later its speed v metres per second satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 10 - 0.1v^2.$$

(i) Find t in terms of v. Hence find the exact tiem the stone takes to reach a speed of 5 metres per second.

(ii) Find the speed of the stone after 1 second.

(iii) What happens to the speed of the stone for large values of t?

Solution

Part (a)

$$\begin{aligned}\int \frac{1}{100 - v^2} \, \mathrm{d}v &= \frac{1}{2(10)} \ln\left(\frac{10+v}{10-v}\right) + C \\ &= \frac{1}{20} \ln\left(\frac{10+v}{10-v}\right) + C\end{aligned}$$

$$\boxed{\int \frac{1}{100 - v^2} \, \mathrm{d}v = \frac{1}{20} \ln\left(\frac{10+v}{10-v}\right) + C}$$

Part (b)

Subpart (i)

$$\begin{aligned} && \frac{\mathrm{d}v}{\mathrm{d}t} &= 10 - 0.1v^2 \\ && &= \frac{1}{10}(100 - v^2) \\ \implies && \frac{1}{100 - v^2} \frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1}{10} \\ \implies && \int \frac{1}{100 - v^2} \frac{\mathrm{d}v}{\mathrm{d}t} \, \mathrm{d}t &= \frac{1}{10} \int \mathrm{d}t \\ \implies && \int \frac{1}{100 - v^2} \, \mathrm{d}v &= \frac{1}{10} \int \mathrm{d}t \\ \implies && \frac{1}{20} \ln\left(\frac{10+v}{10-v}\right) + C &= \frac{1}{10} t \\ \implies && t &= \frac{1}{2} \ln\left(\frac{10+v}{10-v}\right) + C \end{aligned}$$

At $t = 0$, $v = 0$. Hence, $C = 0$.

$$\boxed{t = \frac{1}{2}\ln\left(\frac{10+v}{10-v}\right)}$$

Subpart (ii)

Consider $t = 1$.

$$\begin{aligned}
& \frac{1}{2}\ln\left(\frac{10+v}{10-v}\right) = 1 \\
\implies & \ln\left(\frac{10+v}{10-v}\right) = 2 \\
\implies & \frac{10+v}{10-v} = e^2 \\
\implies & 10+v = e^2(10-v) \\
\implies & v(1+e^2) = 10(e^2-1) \\
\implies & v = \frac{10(e^2-1)}{e^2+1}
\end{aligned}$$

$$\boxed{\text{After 1 second, the speed of the stone is } \frac{10(e^2-1)}{e^2+1} \text{ m/s.}}$$

Subpart (iii)

As $t \to \infty$, we have $\ln\left(\frac{10+v}{10-v}\right) \to \infty \implies \frac{10+v}{10-v} \to \infty$. Thus, $v \to 10^-$.

$$\boxed{\text{For large values of } t\text{, the speed of the stone approaches 10 m/s.}}$$

Problem 10.

Two scientists are investigating the change of a certain population of an animal species of size n thousand at time t years. It is known that due to its inability to reproduce effectively, the species is unable to replace itself in the long run.

(a) One scientist suggests that n and t are related by the differential equation $\frac{\mathrm{d}^2 n}{\mathrm{d}t^2} = 10 - 6t$. Given that $n = 100$ when $t = 0$, show that the general solution of this differential equation is $n = 5t^2 - t^3 + Ct + 100$, where C is a constant. Sketch the solution curve of the particular solution when $C = 0$, stating the axial intercepts clearly.

(b) The other scientist suggests that n and t are related by the differential equation $\frac{\mathrm{d}n}{\mathrm{d}t} = 3 - 0.02n$. Find n in terms of t, given again that $n = 100$ when $t = 0$. Explain in simple terms what will eventually happen to the population using this model.

Which is a more appropriate model in modeling the population of the animal species?

Solution

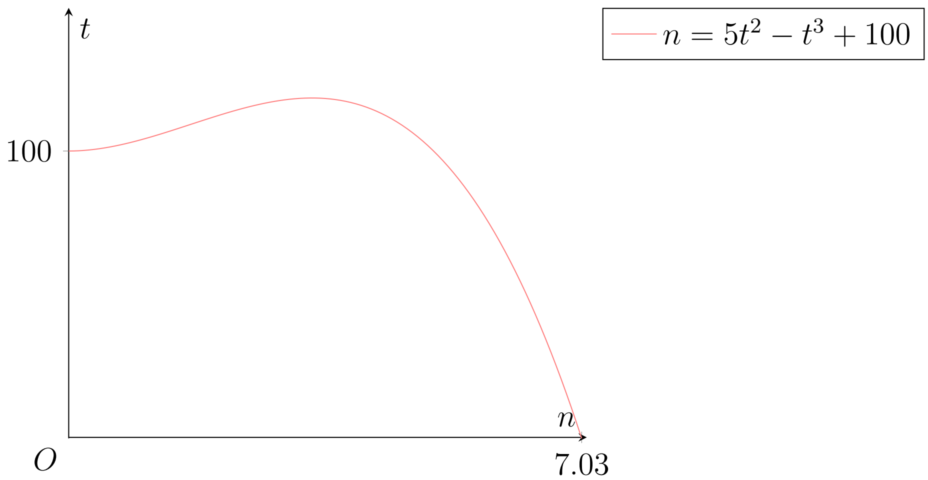
Part (a)

$$\begin{aligned} \frac{\mathrm{d}^2 n}{\mathrm{d}t^2} &= 10 - 6t \\ \implies \frac{\mathrm{d}n}{\mathrm{d}t} &= \int (10 - 6t)\, \mathrm{d}\theta \\ &= 10t - 3t^2 + C \\ \implies n &= \int \left(10t - 3t^2 + C\right) \mathrm{d}t \\ &= 5t^2 - t^3 + Ct + C' \end{aligned}$$

When $t = 0$ and $n = 100$, we have $C' = 100$. Thus,

$$n = 5t^2 - t^3 + Ct + 100.$$

When $C = 0$, $n = 5t^2 - t^3 + 100$.



Part (b)

$$\begin{aligned}
&& \frac{\mathrm{d}n}{\mathrm{d}t} &= 3 - 0.02n \\
&& &= \frac{150-n}{50} \\
\implies && \frac{1}{150-n}\frac{\mathrm{d}n}{\mathrm{d}t} &= \frac{1}{50} \\
\implies && \int \frac{1}{150-n}\frac{\mathrm{d}n}{\mathrm{d}t}\,\mathrm{d}t &= \frac{1}{50}\int \mathrm{d}t \\
\implies && \int \frac{1}{150-n}\,\mathrm{d}n &= \frac{1}{50}\int \mathrm{d}t \\
\implies && -\ln(150-n) &= \frac{1}{50}t + C_1 \\
\implies && \ln(150-n) &= C_2 - \frac{1}{50}t \\
\implies && 150-n &= Ce^{-t/50} \\
\implies && n &= 150 - Ce^{-t/50}
\end{aligned}$$

When $t = 0$ and $n = 100$, we have $C = 50$.

$$\boxed{n = 150 - 50e^{-t/50}}$$

As $t \to \infty$, $n \to 150$. Hence, the population will decrease before plateauing at 150 thousand.

The first model is more appropriate, as it account for the fact that the species will eventually go extinct ($n = 0$) due to the fact that they cannot replace itself in the long run.

Problem 11.

A rectangular tank has a horizontal base. Water is flowing into the tank at a constant rate, and flows out at a rate which is proportional to the depth of water in the tank. At time t seconds, the depth of water in the tank is x metres. If the depth is 0.5 m, it remains at this constant value. Show that $\frac{\mathrm{d}x}{\mathrm{d}t} = -k(2x-1)$, where k is a positive constant. When $t = 0$, the depth of water in the tank is 0.75 m and is decreasing at a rate of 0.01 m s^{-1}. Find the time at which the depth of water is 0.55 m.

Solution

Let V_i m^3/s be the rate at which water is flowing into the tank. Note that $V_i \geq 0$. Let the rate at which water is flowing out of the tank be $V_o x$ m^3/s. Let the base of the container be A m^2. Then $\mathrm{d}x/\mathrm{d}t = 1/A \cdot (V_i - V_o x)$. At $x = 0.5$, the volume of water in the tank is constant, i.e. $\left.\mathrm{d}x/\mathrm{d}t\right|_{0.5} = 0$. This gives $V_i - 0.5V_o = 0$, whence $V_o = 2V_i$. Thus, $\mathrm{d}x/\mathrm{d}t = 1/A \cdot (V_i - 2V_i x) = -V_i/A \cdot (2x-1)$. Letting $k = V_i/A$, we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -k(2x-1)$$

as desired.

We now solve for t.

$$\begin{aligned}
& & \frac{\mathrm{d}x}{\mathrm{d}t} &= -k(2x-1) \\
&\implies & \frac{1}{2x-1}\frac{\mathrm{d}x}{\mathrm{d}t} &= -k \\
&\implies & \int \frac{1}{2x-1}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= -k\int \mathrm{d}t \\
&\implies & \int \frac{1}{2x-1}\,\mathrm{d}x &= -k\int \mathrm{d}t \\
&\implies & \frac{1}{2}\ln(2x-1) + C_1 &= -kt \\
&\implies & \ln(2x-1) + C_2 &= -2kt \\
&\implies & t &= -\frac{1}{2k}\big(\ln(2x-1) + C_2\big)
\end{aligned}$$

At $t = 0$, we have $x = 0.75$. This gives $0 = \ln(2 \cdot 0.75 - 1) + C_2$, whence $C_2 = \ln 2$. We also have $\left.\mathrm{d}x/\mathrm{d}t\right|_0 = -0.01$. We thus obtain $-0.01 = -k(2 \cdot 0.75 - 1)$, whence $k = 0.02$. Thus,

$$t = -\frac{1}{0.04}\big(\ln(2x-1) + \ln 2\big) = -25\ln(4x-2).$$

Hence, when $x = 0.55$, we have $t = -25\ln 0.2 = 25\ln 5$.

$$\boxed{\text{The depth of the water is 0.55 m when } t = 25\ln 5 \text{ s.}}$$

Problem 12.

In a model of mortgage repayment, the sum of money owned to the Building Society is denoted by x and the time is denoted by t. Both x and t are taken to be continuous variables. The sum of money owned to the Building Society increases, due to interest, at a rate proportional to the sum of money owed. Money is also repaid at a constant rate r.

When $x = a$, interest and repayment balance. Show that, for $x > 0$, $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{r}{a}(x - a)$.

Given that, when $t = 0$, $x = A$, find x in terms of t, r, a and A.

On a single, clearly labelled sketch, show the graph of x against t in the two cases:

(a) $A > a$.

(b) $A < a$.

State the circumstances under which the loan is repaid in a finite time T and show that, in this case, $T = \dfrac{a}{r} \ln \dfrac{a}{a - A}$.

Solution

Let the rate at which money is owned to the Building Society be kx. Then $\dfrac{\mathrm{d}x}{\mathrm{d}t} = kx - r$. At $x = a$, interest and repayment balance, i.e. $\mathrm{d}x/\mathrm{d}t|_a = 0$. This gives $ka - r = 0 \implies k = r/a$. Thus,

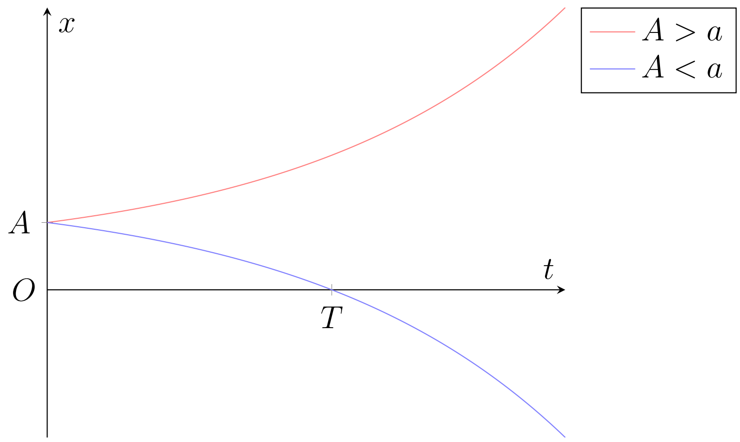
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{r}{a}x - r = \frac{r}{a}(x - a).$$

We now solve for x.

$$\begin{aligned}
& & \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{r}{a}(x - a) \\
&\implies & \frac{1}{x-a}\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{r}{a} \\
&\implies & \int \frac{1}{x-a}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t &= \frac{r}{a}\int \mathrm{d}t \\
&\implies & \int \frac{1}{x-a}\,\mathrm{d}x &= \frac{r}{a}\int \mathrm{d}t \\
&\implies & \ln(x-a) &= \frac{r}{a}t + C_1 \\
&\implies & x - a &= Ce^{rt/a} \\
&\implies & x &= Ce^{rt/a} + a
\end{aligned}$$

When $t = 0$, we have $x = A$. This gives $A = C + a$, whence $C = A - a$. Thus,

$$\boxed{x = (A - a)e^{rt/a} + a}$$



For the loan to be repaid in finite time, $A < a$. At time T, the loan has been repaid, i.e. $x = 0$. Note that $C_1 = \ln C = \ln(A - a)$. Hence,

$$\begin{aligned} & \frac{r}{a}T + \ln(A-a) = \ln(0-a) \\ \implies & \frac{r}{a}T = \ln a - \ln(A-a) \\ & = \ln \frac{a}{A-a} \\ \implies & T = \frac{a}{r} \ln \frac{a}{A-a} \end{aligned}$$