Problem 1.

In an auction at a charity gala dinner, a group of wealthy businessmen are competing with each other to be the highest bidder. Each time one of them makes a bid amount, another counter-bids by 50% more, less a service charge of ten dollars (e.g. If A bids \$1000, then B will bid \$1490). Let u_n be the amount at the nth bid and u_1 be the initial amount.

(a) Write down a recurrence relation that describes the bidding process.

(b) Show that $u_n = \$(1.5^{n-1}(u_1 - 20) + 20)$.

(c) The target amount to be raised is \$1 234 567 and the bidding stops when the bid amount meets or crosses this target amount. Given that $u_1 = 111$,

  (i) state the least number of bids required to meet this amount.

  (ii) find the winning bid amount, correct to the nearest thousand dollars.

Solution

Part (a)

$$\boxed{u_{n+1} = 1.5u_n - 10}$$

Part (b)

Let k be the constant such that $u_{n+1} + k = 1.5(u_n + k) \implies 0.5k = -10 \implies k = -20$. Hence, $u_{n+1} - 20 = 1.5(u_n - 20)$.

$$\begin{aligned} u_{n+1} - 20 &= 1.5(u_n - 20) \\ \implies \quad u_n - 20 &= 1.5^{n-1}(u_1 - 20) \\ \implies \quad u_n &= 1.5^{n-1}(u_1 - 20) + 20 \end{aligned}$$

Part (c)

Subpart (i)

Let m be the least integer such that $u_m \geq 1234567$.

$$\begin{aligned} u_m &\geq 1234567 \\ \implies 1.5^{m-1}(111 - 20) + 20 &\geq 1234567 \\ \implies \qquad m &\geq 1 + \log_{1.5} \frac{1234567 - 20}{111 - 20} \\ &= 24.5 \text{ (3 s.f.)} \end{aligned}$$

Hence, $m = 25$.

$$\boxed{\text{It takes at least 25 bids to meet this amount.}}$$

Subpart (ii)

$$\begin{aligned} u_{25} &= 1.5^{25-1}(111 - 20) \\ &= 1532000 \text{ (to the nearest thousand)} \end{aligned}$$

$\boxed{\text{The winning bid is \$1 532 000.}}$

Problem 2.

Solve these recurrence relations together with the initial conditions.

(a) $u_{n+2} = -u_n + 2u_{n+1}$, for $n \geq 0$, $u_0 = 5$, $u_1 = -1$.

(b) $4u_n = 4u_{n-1} + u_{n-2}$, for $n \geq 2$, $u_0 = a$, $u_1 = b$, $a, b \in \mathbb{R}$.

Solution

Part (a)

Consider the characteristic equation of u_n.

$$\begin{aligned} & x^2 - 2x + 1 = 0 \\ \implies \quad & (x-1)^2 = 0 \end{aligned}$$

Hence, the only root of the characteristic equation is 1. Thus,

$$\begin{aligned} u_n &= (A + Bn) \cdot 1^n \\ &= A + Bn \end{aligned}$$

Since $u_0 = 5$,

$$\begin{aligned} 5 &= A + B \cdot 0 \\ \implies A &= 5 \end{aligned}$$

Since $u_1 = -1$,

$$\begin{aligned} -1 &= A + B \cdot 1 \\ \implies \quad B &= -1 - A \\ &= -6 \end{aligned}$$

Thus,

$$\boxed{u_n = 5 - 6n}$$

Part (b)

$$\begin{aligned} 4u_n &= 4u_{n-1} + u_{n-2} \\ \implies \quad u_n &= u_{n-1} + \frac{1}{4}u_{n-2} \end{aligned}$$

Consider the characteristic equation of u_n.

$$\begin{aligned} x^2 - x - \frac{1}{4} &= 0 \\ \implies \left(x - \frac{1}{2}\right)^2 - \frac{1}{2} &= 0 \\ \implies \qquad x &= \frac{1}{2} \pm \sqrt{\frac{1}{2}} \\ &= \frac{1 \pm \sqrt{2}}{2} \end{aligned}$$

Hence, the roots of the characteristic equation are $x = \dfrac{1+\sqrt{2}}{2}$ and $x = \dfrac{1-\sqrt{2}}{2}$. Thus,

$$u_n = A\left(\frac{1+\sqrt{2}}{2}\right)^n + B\left(\frac{1-\sqrt{2}}{2}\right)^n$$

Since $u_0 = a$,

$$\begin{aligned} a &= A\left(\frac{1+\sqrt{2}}{2}\right)^0 + B\left(\frac{1-\sqrt{2}}{2}\right)^0 \\ &= A + B \\ \implies B &= a - A \end{aligned}$$

Since $u_1 = b$,

$$\begin{aligned} b &= A\left(\frac{1+\sqrt{2}}{2}\right)^1 + B\left(\frac{1-\sqrt{2}}{2}\right)^1 \\ &= \frac{1}{2}\left(A + B + \sqrt{2}(A - B)\right) \\ &= \frac{1}{2}\left(a + \sqrt{2}(A - (a - A))\right) \\ &= \frac{1}{2}\left(a + \sqrt{2}(2A - a)\right) \\ \implies A &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}(2b - a) + a\right) \\ &= \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b \\ \implies B &= a - \left(\frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b\right) \\ &= \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b \end{aligned}$$

Thus,

$$\boxed{u_n = \left(\frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b\right)\left(\frac{1+\sqrt{2}}{2}\right)^n + \left(\frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b\right)\left(\frac{1-\sqrt{2}}{2}\right)^n}$$

Problem 3.

A passcode is generated using the digits 1 to 5, with repetitions allowed. The passcodes are classified into two types. A Type A passcode has an even number of the digit 1, while a Type B passcode has an odd number of the digit 1. For example, a Type A passcode is 1231, and a Type B passcode is 1541213. Let a_n and b_n denote the number of n-digit Type A and Type B passcodes respectively.

(a) State the values of a_1 and a_2.

(b) By considering the relationship between a_n and b_n, show that

$$a_n = xa_{n-1} + y^{n-1}, \qquad n \geq 2$$

where x and y are constants to be determined.

(c) Using the substitution $c_n = za_n + y^n$, where z is a constant to be determined, find a first order linear recurrence relation for c_n. Hence, find the general term formula for a_n.

Solution

Part (a)

$$\boxed{a_1 = 4,\ a_2 = 17}$$

Part (b)

Let P be an n-digit passcode with Type T, where $T \in \{A, B\}$. We also let Type T' be such that $T' \in \{A, B\} \setminus T$.

By concatenating a digit from 1 to 5 to P, 5 $(n+1)$-digit passcodes can be created. Let P' denote a new passcode that is created via this process. If the digit 1 is concatenated, then P' is of Type T'. If the digit 1 is not concatenated, then P' is of Type T. There are 4 choices for such a case. This hence gives the recurrence relations

$$\begin{aligned} a_n &= 4a_{n-1} + b_{n-1} \\ b_n &= 4b_{n-1} + a_{n-1} \end{aligned}$$

Note that $a_n + b_n = 5(a_{n+1} + b_{n+1})$. Thus, $a_n + b_n = 5^{n-1}(a_1 + b_1) = 5^{n-1}(4 + 1) = 5^n$. Hence,

$$\begin{aligned} a_n &= 4a_{n-1} + b_{n-1} \\ &= 3a_{n-1} + a_{n-1} + b_{n-1} \\ &= 3a_{n-1} + 5^{n-1} \end{aligned}$$

whence $x = 3$ and $y = 5$.

Part (c)

$$\begin{aligned} c_n &= za_n + y^n \\ &= za_n + 5^n \\ &= z\left(3a_{n-1} + 5^{n-1}\right) + 5^n \\ &= 3za_{n-1} + z5^{n-1} + 5 \cdot 5^{n-1} \\ &= 3\left(za_{n-1} + 5^{n-1}\right) + z5^{n-1} + 5 \cdot 5^{n-1} - 3 \cdot 5^{n-1} \\ &= 3c_{n-1} + (2+z)5^{n-1} \end{aligned}$$

Let $z = -2$. Then,

$$\begin{aligned} c_n &= 3c_{n-1} \\ &= 3^{n-1}c_1 \\ &= 3^{n-1}\left(-2a_1 + 5^1\right) \\ &= -3 \cdot 3^{n-1} \end{aligned}$$

Note that $a_n = \frac{1}{z}\left(c_n - y^n\right)$. Thus,

$$\begin{aligned} a_n &= -\frac{1}{2}\left(-3 \cdot 3^{n-1} - 5^n\right) \\ &= \frac{1}{2}\left(3^n + 5^n\right) \end{aligned}$$

$$\boxed{a_n = \frac{1}{2}\left(3^n + 5^n\right)}$$