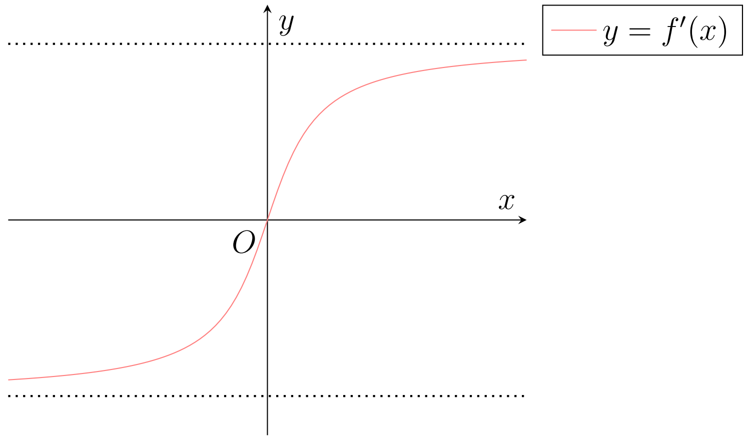
Problem 1.

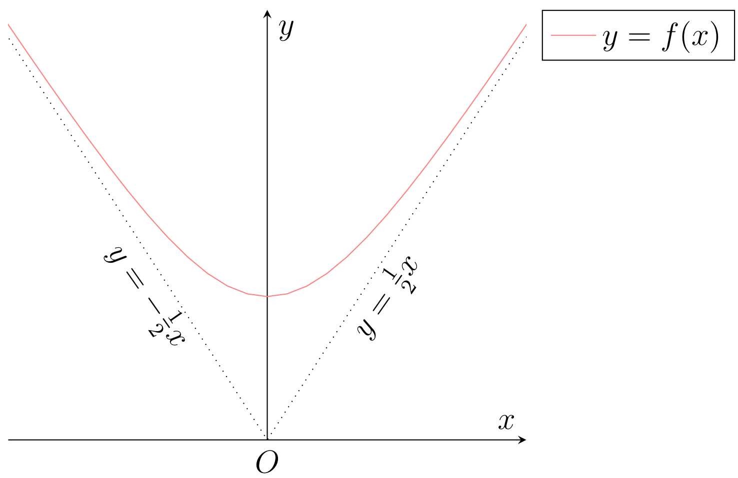
The graph of the first derivative of a function f is shown in the diagram below. It is symmetrical about the origin O and approaches the lines $y = 0.5$ and $y = -0.5$ for large values of x. Sketch the graph of $y = f(x)$ given that it has a pair of asymptotes that intersect at the origin.



Solution

$$\begin{aligned} & \lim_{x\to\pm\infty} f'(x) = \pm\frac{1}{2} \\ \implies & \lim_{x\to\pm\infty} \int f'(x)\,\mathrm{d}x = \int \pm\frac{1}{2}x\,\mathrm{d}x \\ \implies & \lim_{x\to\pm\infty} f(x) = \pm\frac{1}{2}x + C \end{aligned}$$

Hence, $f(x)$ has asymptotes $y = \pm\frac{1}{2}x + C$, for some arbitrary constant C. Since both asymptotes meet at the origin, $C = 0$, whence $f(x)$ has asymptotes $y = \pm\frac{1}{2}x$.



Problem 2.

The terms in the sequence u_0, u_1, u_2, ... satisfy the recurrence relation

$$u_{n+2} - u_{n+1} = r(u_{n+1} - u_n)$$

where r is a non-zero constant.

(a) Find the general solution of this recurrence relation.

(b) Given that $u_0 = 0$ and the sequence converges to a finite value L, find an expression for u_n in terms of L, n and r. State a necessary condition on r.

Solution

Part (a)

$$\begin{aligned} u_{n+2} - u_{n+1} &= r(u_{n+1} - u_n) \\ \implies \qquad u_{n_2} &= (1+r)u_{n+1} - ru_n \end{aligned}$$

Consider the characteristic equation of the above recurrence relation.

$$\begin{aligned} x^2 - (1+r)x + r &= 0 \\ \implies \quad (x-1)(x-r) &= 0 \end{aligned}$$

Hence, the roots of the characteristic equation are 1 and r. Thus, the general solution of the recurrence relation is given by

$$\begin{aligned} u_n &= A \cdot 1^n + B \cdot r^n \\ &= A + B \cdot r^n \end{aligned}$$

$$\boxed{u_n = A + B \cdot r^n}$$

Part (b)

When $n = 0$, we have

$$\begin{aligned} A + B &= 0 \\ \implies \quad B &= -A \end{aligned}$$

Since the sequence converges to a finite value, we know $|r| < 1$. Hence, considering $n \to \infty$, we have

$$\begin{aligned} L &= \lim_{n\to\infty} (A + Br^n) \\ &= A \end{aligned}$$

whence $B = -A = -L$. Putting everything together, we have

$$\boxed{u_n = L - Lr^n,\ |r| < 1}$$

Problem 3.

A curve is defined parametrically by $x = \dfrac{t^2}{1+t^2}$, $y = t^3 - \lambda t$, where λ is a positive constant.

(a) Sketch the curve, stating the equation of its asymptote.

(b) Find in terms of λ, the x-coordinate of the point P where the curve intersects itself.

(c) Show that the area of the region bounded by the curve between P and the origin is given by an integral of the form

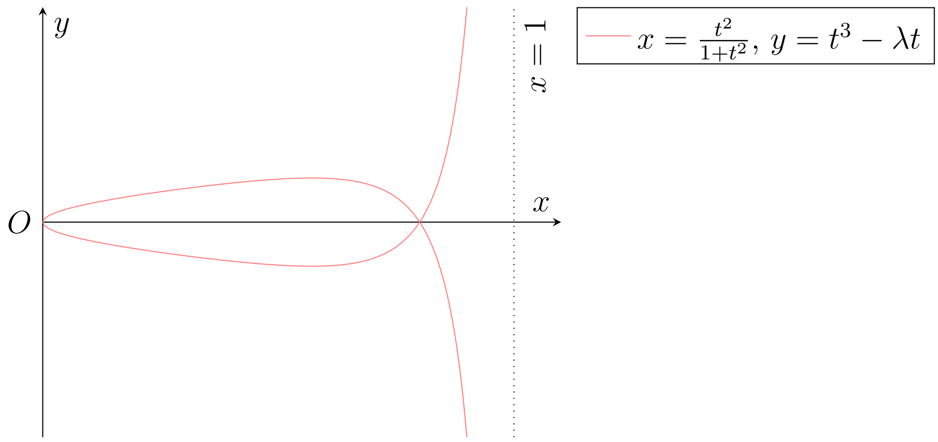
$$4 \int_0^{f(\lambda)} g(t^2) \, \mathrm{d}t$$

where $f(\lambda)$ is a function of λ and $g(t^2)$ is a function of t^2 to be determined.

Solution

Part (a)

Note that $\displaystyle\lim_{t \to \pm\infty} x = \lim_{t \to \pm\infty} \frac{t^2}{1+t^2} = 1$. Hence, the curve has a vertical asymptote with equation $x = 1$.



Part (b)

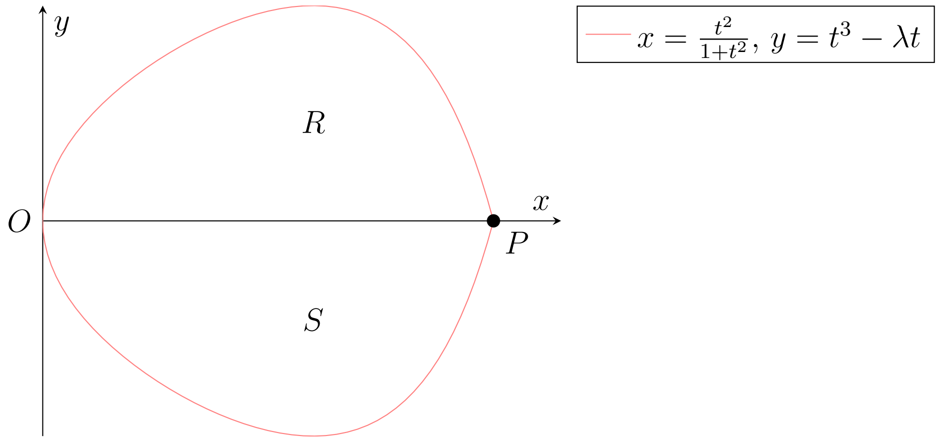
From the graph, the curve intersects itself when $y = 0$ and $x \neq 0 \implies t \neq 0$.

$$\begin{aligned} y &= 0 \\ \implies t^3 - \lambda t &= 0 \\ \implies t^2 - \lambda &= 0 \end{aligned}$$

Hence, $\lambda = t^2$, whence $x = \dfrac{t^2}{1+t^2} = \dfrac{\lambda}{1+\lambda}$.

$$\boxed{x = \frac{\lambda}{1+\lambda}}$$

Part (c)



Let the region bounded by the curve between P and the origin be A. Let R be the region of A where $y \geq 0$. Let S be the region of A where $y \leq 0$. By symmetry, Area R = Area S. Hence,

$$\text{Area}\, A = 2\,\text{Area}\, R$$

We will consider only region R for the rest of the solution. Note that R is bounded by the part of the curve where $-\sqrt{\lambda} \leq t \leq 0$. Also note that

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{t^2}{1+t^2} \\
&= \frac{(1+t^2)\cdot 2t - t^2 \cdot 2t}{(1+t^2)^2} \\
&= \frac{2t}{(1+t^2)^2} \\
\implies \mathrm{d}x &= \frac{2t}{(1+t^2)^2}\,\mathrm{d}t
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{Area}\, A &= 2\,\text{Area}\, R \\
&= 2\int_0^{-\sqrt{\lambda}} y\,\mathrm{d}x \\
&= 2\int_0^{-\sqrt{\lambda}} \left(t^3 - \lambda t\right)\cdot \frac{2t}{(1+t^2)^2}\,\mathrm{d}t \\
&= 4\int_0^{-\sqrt{\lambda}} \frac{t^4 - \lambda t^2}{(1+t^2)^2}\,\mathrm{d}t \\
&= 4\int_0^{-\sqrt{\lambda}} \frac{t^2\left(t^2 - \lambda\right)}{(1+t^2)^2}\,\mathrm{d}t
\end{aligned}$$

Hence,

$$\boxed{f(\lambda) = -\sqrt{\lambda},\; g(t^2) = \frac{t^2\left(t^2-\lambda\right)}{(1+t^2)^2}}$$

Problem 4.

It is given that the equation $1 + \cos(\pi x) - 2\sqrt{x} = 0$ has a root α in the interval $[0, 1]$.

(a) Use linear interpolation once on the interval $[0, 1]$ to obtain an approximation x_1 to α.

(b) Using x_1 as an initial estimate, apply the Newton-Raphson method to find α, correct to 2 decimal places.

(c) With the help of an appropriate graph, explain how Newton-Raphson method using another initial estimate x_1^* in the interval $[0, 1]$ fails to give an approximation to α.

Solution

Let $f(x) = 1 + \cos(\pi x) - 2\sqrt{x}$.

Part (a)

Using linear interpolation on the interval $[0, 1]$,

$$x_1 = \frac{1 \cdot f(0) - 0 \cdot f(1)}{f(0) - f(1)} = \frac{1}{2}$$

$$\boxed{x_1 = \frac{1}{2}}$$

Part (b)

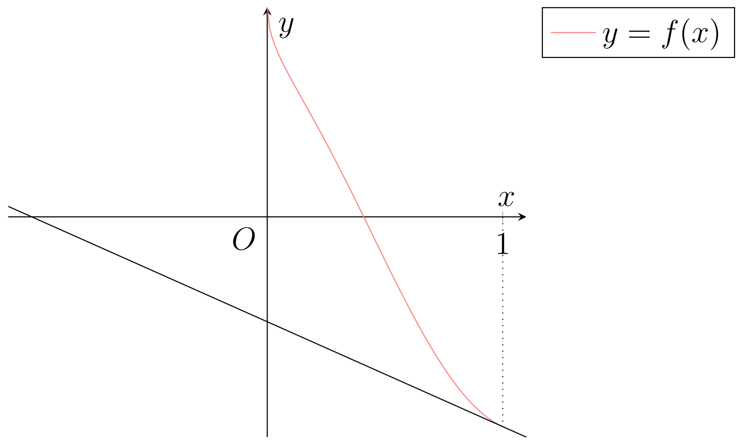
Note that $f'(x) = -\sin(\pi x) \cdot \pi - \frac{2}{2\sqrt{x}} = -\pi \sin(\pi x) - \frac{1}{\sqrt{x}}$.

$$\begin{aligned} x_1 &= \frac{1}{2} \\ \implies x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} = 0.40908 \\ \implies x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} = 0.40964 \\ \implies x_4 &= x_3 - \frac{f(x_3)}{f'(x_3)} = 0.40964 \end{aligned}$$

Since $f(0.405) = 0.02 > 0$ and $f(0.415) = -0.02 < 0$, $\alpha \in (0.405, 0.415)$. Thus,

$$\boxed{\alpha = 0.41 \text{ (2 d.p.)}}$$

Part (c)



When $x_1^* = 1$,the tangent to the curve intersects the x-axis at a negative x-value, thus giving a negative x_2. Since $f(x)$ is only defined for $x \geq 0$ due to the presence of $\sqrt{x}$, $f(x_2)$ and thus x_3 will be undefined. Hence, the Newton-Raphson method will fail to give an approximation to α.

Problem 5.

(a) For a positive constant a, there is an angle ϕ such that $\sin\phi = a$ and $\frac{\pi}{2} < \phi < \pi$.

Evaluate $\int_{-1}^{0} \frac{1}{\sqrt{1-a^2x^2}} \, \mathrm{d}x$, leaving your answer in terms of a, ϕ and π.

(b) Using the substitution $t = \tan\frac{x}{2}$, show that

$$\int \frac{\cos x}{1+\cos x - \sin x} \, \mathrm{d}x = \int \frac{1+t}{1+t^2} \, \mathrm{d}t$$

Hence determine $\int \frac{\cos x}{1+\cos x-\sin x} \, \mathrm{d}x$.

Solution

Part (a)

$$\begin{aligned}
\int_{-1}^{0} \frac{1}{\sqrt{1-a^2x^2}} \, \mathrm{d}x &= \int_{-1}^{0} \frac{1}{\sqrt{1-(ax)^2}} \, \mathrm{d}x \\
&= \frac{1}{a} \int_{-a}^{0} \frac{1}{\sqrt{1-u^2}} \, \mathrm{d}x \\
&= \frac{1}{a} \left[\arcsin u\right]_{-a}^{0} \\
&= \frac{1}{a} \cdot -\arcsin(-a) \\
&= \frac{\arcsin a}{a} \\
&= \frac{\pi-\phi}{a}
\end{aligned}$$

$$u = ax$$
$$\mathrm{d}u = a \, \mathrm{d}x$$

$$\boxed{\int_{-1}^{0} \frac{1}{\sqrt{1-a^2x^2}} \, \mathrm{d}x = \frac{\pi-\phi}{a}}$$

Part (b)

Consider the substitution $t = \tan\frac{x}{2}$.

$$\begin{aligned}
\sin x &= \frac{2\sin(x/2)\cos(x/2)}{\cos^2(x/2)+\sin^2(x/2)} \\
&= \frac{2\tan(x/2)}{1+\tan^2(x/2)} \\
&= \frac{2t}{1+t^2} \\
\cos x &= \frac{\cos^2(x/2)-\sin^2(x/2)}{\cos^2(x/2)+\sin^2(x/2)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} \\
&= \frac{1 - t^2}{1 + t^2}
\end{aligned}$$

Also note that

$$\begin{aligned}
t &= \tan \frac{x}{2} \\
\implies \mathrm{d}t &= \frac{1}{2} \sec^2 \frac{x}{2} \, \mathrm{d}x \\
\implies \mathrm{d}t &= \frac{1}{2} \left(1 + \tan^2 \frac{x}{2}\right) \mathrm{d}x \\
&= \frac{1}{2}(1 + t^2) \, \mathrm{d}x \\
\implies \mathrm{d}x &= \frac{2}{1 + t^2} \, \mathrm{d}t
\end{aligned}$$

Hence,

$$\begin{aligned}
\int \frac{\cos x}{1 + \cos x - \sin x} \, \mathrm{d}x &= \int \frac{\frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2} - \frac{2t}{1+t^2}} \cdot \frac{2}{1 + t^2} \, \mathrm{d}t \\
&= \int \frac{2 \cdot \frac{1-t^2}{1+t^2}}{(1 + t^2) + (1 - t^2) - 2t} \, \mathrm{d}t \\
&= \int \frac{2 \cdot \frac{1-t^2}{1+t^2}}{2 - 2t} \, \mathrm{d}t \\
&= \int \frac{\frac{1-t^2}{1+t^2}}{1 - t} \, \mathrm{d}t \\
&= \int \frac{\frac{(1-t)(1+t)}{1+t^2}}{1 - t} \, \mathrm{d}t \\
&= \int \frac{1 + t}{1 + t^2} \, \mathrm{d}t \quad \square \\
&= \int \left(\frac{1}{1 + t^2} + \frac{t}{1 + t^2}\right) \mathrm{d}t \\
&= \int \left(\frac{1}{1 + t^2} + \frac{1}{2} \cdot \frac{2t}{1 + t^2}\right) \mathrm{d}t \\
&= \arctan t + \frac{1}{2} \ln \left|1 + t^2\right| + C \\
&= \arctan \left(\tan \frac{x}{2}\right) + \frac{1}{2} \ln \left|1 + \tan^2 \frac{x}{2}\right| + C \\
&= \frac{x}{2} + \frac{1}{2} \ln \left|\sec^2 \frac{x}{2}\right| + C \\
&= \frac{x}{2} + \ln \left|\sec \frac{x}{2}\right| + C
\end{aligned}$$

$$\boxed{\int \frac{\cos x}{1 + \cos x - \sin x} \, \mathrm{d}x = \frac{x}{2} + \ln \left|\sec \frac{x}{2}\right| + C}$$

Problem 6.

The curve G has equation $y = \dfrac{x^2 - 2kx + k}{x - k}$, where k is a non-zero constant and $k \neq 1$.

(a) State, in terms of k, the equations of the asymptotes of G.

(b) Determine the set of values for which G as two stationary points.

(c) Give a sketch of G for $k > 1$, stating in terms of k, the coordinates of the point of intersection of its asymptotes.

(d) With the help of your sketch in part (c), determine, in exact form, the value of m ($m < 0$) such that the line $y = m(x - k)$ is a line of symmetry of G.

Solution

Part (a)

$$\begin{aligned} y &= \frac{x^2 - 2kx + k}{x - k} \\ &= \frac{x^2 - 2kx + k^2 + k - k^2}{x - k} \\ &= \frac{(x-k)^2 + k - k^2}{x - k} \\ &= x - k + \frac{k - k^2}{x - k} \end{aligned}$$

Hence, G has oblique asymptote $y = x - k$ and vertical asymptote $x = k$.

$$\boxed{y = x - k,\ x = k}$$

Part (b)

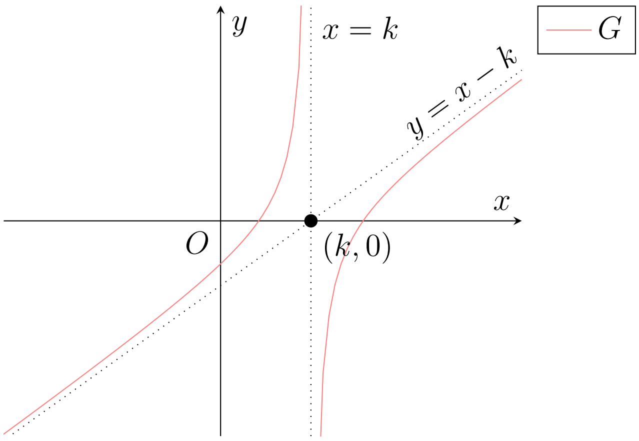
For stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\begin{aligned} && \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies && 1 - \frac{k - k^2}{(x-k)^2} &= 0 \\ \implies && (x-k)^2 &= k - k^2 \\ \implies && x - k &= \pm\sqrt{k - k^2} \end{aligned}$$

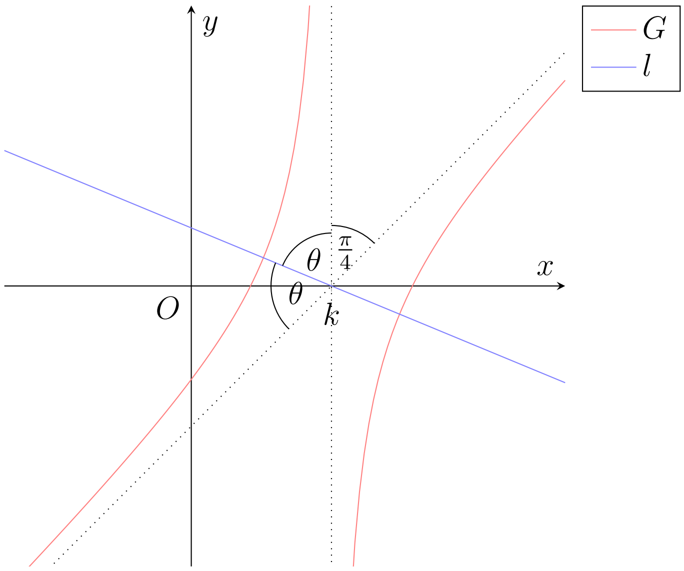
For G to have two stationary points, $k - k^2 > 0$, whence $0 < k < 1$.

$$\boxed{\{k \in \mathbb{R} : 0 < k < 1\}}$$

Part (c)



Part (d)



Let l be the line with equation $y = m(x - k)$. Since l is a line of symmetry of G, l bisects the angle between the asymptotes. Since the asymptote $y = x - k$ makes an angle $\frac{\pi}{4}$ with the point $(k, 0)$, we have

$$\theta + \theta + \frac{\pi}{4} = \pi$$

whence $\theta = \frac{3}{8}\pi$. Thus, l makes an angle $\theta + \frac{\pi}{4} = \frac{7}{8}\pi$ with the point $(k, 0)$), giving it a gradient of $\tan \frac{7}{8}\pi$. Hence,

$$\boxed{m = \tan \frac{7}{8}\pi}$$

Problem 7.

Omitted.

Problem 8.

Omitted.

Problem 9.

It is given that $I_n = \int_0^\pi \cos^n(2\theta)\,\mathrm{d}\theta$, where n is a positive integer.

(a) Without using the calculator, evaluate I_2.

(b) For $n > 3$, show that $I_n = \frac{n-1}{n} I_{n-2}$.

(c) Deduce that for all odd values of n, I_n is independent of n.

(d) For even values of n, show that

$$I_n = \frac{n!\,\pi}{2^n\left[\left(\frac{n}{2}\right)!\right]^2}$$

Solution

Part (a)

$$\begin{aligned} I_2 &= \int_0^\pi \cos^2(2\theta)\,\mathrm{d}\theta \\ &= \int_0^\pi \frac{\cos 4\theta + 1}{2}\,\mathrm{d}\theta \\ &= \int_0^{4\pi} \frac{\cos u + 1}{8}\,\mathrm{d}u \\ &= \frac{1}{8}\left[\sin u + u\right]_0^{4\pi} \\ &= \frac{\pi}{2} \end{aligned}$$

$$u = 4\theta$$
$$\mathrm{d}u = 4\,\mathrm{d}\theta$$

$$\boxed{I_2 = \frac{\pi}{2}}$$

Part (b)

$$\begin{aligned} I_n &= \int_0^\pi \cos^n 2\theta\,\mathrm{d}\theta \\ &= \int_0^\pi \cos 2\theta \cos^{n-1} 2\theta\,\mathrm{d}\theta \end{aligned}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}\theta}\cos^{n-1} 2\theta = -2(n-1)\sin 2\theta \cos^{n-2} 2\theta$. Integrating by parts, we have

	D	I
$+$	$\cos^{n-1} 2\theta$	$\cos 2\theta$
$-$	$-2(n-1)\sin 2\theta \cos^{n-2} 2\theta$	$\frac{1}{2}\sin 2\theta$

$$
\begin{aligned}
I_n &= \left[\frac{1}{2}\sin 2\theta\cos^{n-1}2\theta\right]_0^{\pi} + (n-1)\int_0^{\pi}\sin^2 2\theta\cos^{n-2}2\theta\,\mathrm{d}\theta \\
&= (n-1)\int_0^{\pi}\sin^2 2\theta\cos^{n-2}2\theta\,\mathrm{d}\theta \\
&= (n-1)\int_0^{\pi}\left(1-\cos^2 2\theta\right)\cos^{n-2}2\theta\,\mathrm{d}\theta \\
&= (n-1)\int_0^{\pi}\left(\cos^{n-2}2\theta-\cos^n 2\theta\right)\mathrm{d}\theta \\
&= (n-1)\left(I_{n-2}-I_n\right) \\
\implies nI_n &= (n-1)I_{n-2} \\
\implies I_n &= \frac{n-1}{n}I_{n-2}
\end{aligned}
$$

Part (c)

Note that $I_1 = \displaystyle\int_0^{\pi}\cos 2\theta\,\mathrm{d}\theta = \frac{1}{2}\left[\sin 2\theta\right]_0^{\pi} = 0$. For all odd n, I_n will eventually reduce to I_1 with the recurrence relation derived above. Hence, $I_n = 0$ for odd n, which is independent of n.

Part (d)

$$
\begin{aligned}
I_n &= \frac{n-1}{n}I_{n-2} \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}I_{n-4} \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}I_{n-6} \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\ldots\cdot\frac{3}{4}I_2 \\
&= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\ldots\cdot\frac{3}{4}\cdot\frac{\pi}{2} \\
&= \frac{(n-1)(n-3)(n-5)\cdot\ldots\cdot 3\cdot 1}{n(n-2)(n-4)\cdot\ldots\cdot 4\cdot 2}\cdot\pi \\
&= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)\cdot\ldots\cdot 4\cdot 3\cdot 2\cdot 1}{\Big[n(n-2)(n-4)\cdot\ldots\cdot 4\cdot 2\Big]^2}\cdot\pi \\
&= \frac{n!\,\pi}{\Big[n(n-2)(n-4)\cdot\ldots\cdot 4\cdot 2\Big]^2}
\end{aligned}
$$

However, we have

$$
\begin{aligned}
n(n-2)(n-4)\cdot\ldots\cdot 2 &= \left(2\cdot\frac{n}{2}\right)\left(2\cdot\frac{n-2}{2}\right)\left(2\cdot\frac{n-4}{2}\right)\cdot\ldots\cdot(2\cdot 1) \\
&= 2^{n/2}\left[\left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)\left(\frac{n-4}{2}\right)\cdot\ldots\cdot 1\right] \\
&= 2^{n/2}\left[\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)\left(\frac{n}{2}-2\right)\cdot\ldots\cdot 1)\right]
\end{aligned}
$$

$$= 2^{n/2}\left(\frac{n}{2}\right)!$$

Hence,

$$\begin{aligned} I_n &= \frac{n!\,\pi}{\left[2^{n/2}\left(\frac{n}{2}\right)!\right]^2} \\ &= \frac{n!\,\pi}{2^n\left[\left(\frac{n}{2}\right)!\right]^2} \end{aligned}$$

Problem 10.

In a membership drive, a fitness club is trying to recruit new members. The sales manager models the number of members that the club has at the end of each month assuming that a certain portion p ($0 < p < 1$) of its members in the previous month will be lost to competitors, and that it will recruit a constant number, k, of new members in each month.

Let M_n ($n \geq 1$) be the number of members that the club has n months after the start of the membership drive.

(a) Write down an expression for M_{n+1} in terms of M_n.

(b) Given that the club has 500 members at the end of the first month, determine M_n in terms of n, p and k.

The sales manager sets a target for the club membership to reach 750 by the end of 6 months.

(c) Given that $k = 80$, show that to meet its target, the club needs to retain approximately 95% of its members, month-by-month.

(d) Given that the club can only retain 90% of its members, month-by-month, find the least number of members it must recruit each month to meet or exceed its target.

Solution

Part (a)

$$\boxed{M_{n+1} = (1-p)M_n + k}$$

Part (b)

Let q be the constant such that $M_{n+1} + q = (1-p)(M_n + q)$. Then $(1-p)q - q = k \implies q = -\frac{k}{p}$.

$$\begin{aligned} & M_{n+1} - \frac{k}{p} = (1-p)\left(M_n - \frac{k}{p}\right) \\ \implies \quad & M_n - \frac{k}{p} = (1-p)^{n-1}\left(M_1 - \frac{k}{p}\right) \\ & = (1-p)^{n-1}\left(500 - \frac{k}{p}\right) \\ \implies \quad & M_n = (1-p)^{n-1}\left(500 - \frac{k}{p}\right) + \frac{k}{p} \end{aligned}$$

$$\boxed{M_n = (1-p)^{n-1}\left(500 - \frac{k}{p}\right) + \frac{k}{p}}$$

Part (c)

Consider $M_6 \geq 750$ with $k = 80$.

$$\begin{aligned} M_6 &\geq 750 \\ \implies (1-p)^5 \left(500 - \frac{80}{p}\right) + \frac{80}{p} &\geq 750 \end{aligned}$$

From G.C., $p = 0.0495$ (3 s.f.). Hence, the club needs to retain $(1-p) = 95.05\%$ of its members, month-by-month.

Part (d)

Consider $M_6 \geq 750$ with $p = 0.10$.

$$\begin{aligned} M_6 &\geq 750 \\ \implies (1-p)^5 \left(500 - \frac{k}{0.10}\right) + \frac{k}{0.10} &\geq 750 \end{aligned}$$

From G.C., $k > 111.05$ (2 d.p.). Since $k \in \mathbb{N}$, the least k is 112.

$$\boxed{\text{The club must recruit at least 112 members each month.}}$$

Problem 11.

Omitted.