

**Problem 1.**

In an auction at a charity gala dinner, a group of wealthy businessmen are competing with each other to be the highest bidder. Each time one of them makes a bid amount, another counter-bids by 50% more, less a service charge of ten dollars (e.g. If A bids \$1000, then B will bid \$1490). Let  $u_n$  be the amount at the  $n$ th bid and  $u_1$  be the initial amount.

- (a) Write down a recurrence relation that describes the bidding process.
- (b) Show that  $u_n = \$(1.5^{n-1}(u_1 - 20) + 20)$ .
- (c) The target amount to be raised is \$1 234 567 and the bidding stops when the bid amount meets or crosses this target amount. Given that  $u_1 = 111$ ,
  - (i) state the least number of bids required to meet this amount.
  - (ii) find the winning bid amount, correct to the nearest thousand dollars.

**Solution****Part (a)**

$$u_{n+1} = 1.5u_n - 10$$

**Part (b)**

Let  $k$  be the constant such that  $u_{n+1} + k = 1.5(u_n + k) \implies 0.5k = -10 \implies k = -20$ . Hence,  $u_{n+1} - 20 = 1.5(u_n - 20)$ .

$$\begin{aligned} u_{n+1} - 20 &= 1.5(u_n - 20) \\ \implies u_n - 20 &= 1.5^{n-1}(u_1 - 20) \\ \implies u_n &= 1.5^{n-1}(u_1 - 20) + 20 \end{aligned}$$

**Part (c)****Subpart (i)**

Let  $m$  be the least integer such that  $u_m \geq 1234567$ .

$$\begin{aligned} u_m &\geq 1234567 \\ \implies 1.5^{m-1}(111 - 20) + 20 &\geq 1234567 \\ \implies m &\geq 1 + \log_{1.5} \frac{1234567 - 20}{111 - 20} \\ &= 24.5 \text{ (3 s.f.)} \end{aligned}$$

Hence,  $m = 25$ .

It takes at least 25 bids to meet this amount.

**Subpart (ii)**

$$\begin{aligned} u_{25} &= 1.5^{25-1}(111 - 20) \\ &= 1532000 \text{ (to nearest thousand)} \end{aligned}$$

The winning bid is \$1 532 000.

**Problem 2.**

Solve these recurrence relations together with the initial conditions.

(a)  $u_{n+2} = -u_n + 2u_{n+1}$ , for  $n \geq 0$ ,  $u_0 = 5$ ,  $u_1 = -1$ .

(b)  $4u_n = 4u_{n-1} + u_{n-2}$ , for  $n \geq 2$ ,  $u_0 = a$ ,  $u_1 = b$ ,  $a, b \in \mathbb{R}$ .

**Solution****Part (a)**

Consider the characteristic equation of  $u_n$ .

$$\begin{aligned} x^2 - 2x + 1 &= 0 \\ \implies (x - 1)^2 &= 0 \end{aligned}$$

Hence, the only root of the characteristic equation is 1. Thus,

$$\begin{aligned} u_n &= (A + Bn) \cdot 1^n \\ &= A + Bn \end{aligned}$$

Since  $u_0 = 5$ ,

$$\begin{aligned} 5 &= A + B \cdot 0 \\ \implies A &= 5 \end{aligned}$$

Since  $u_1 = -1$ ,

$$\begin{aligned} -1 &= A + B \cdot 1 \\ \implies B &= -1 - A \\ &= -6 \end{aligned}$$

Thus,

$$\boxed{u_n = 5 - 6n}$$

**Part (b)**

$$\begin{aligned} 4u_n &= 4u_{n-1} + u_{n-2} \\ \implies u_n &= u_{n-1} + \frac{1}{4}u_{n-2} \end{aligned}$$

Consider the characteristic equation of  $u_n$ .

$$\begin{aligned} x^2 - x - \frac{1}{4} &= 0 \\ \implies \left(x - \frac{1}{2}\right)^2 - \frac{1}{2} &= 0 \\ \implies x &= \frac{1}{2} \pm \sqrt{\frac{1}{2}} \\ &= \frac{1 \pm \sqrt{2}}{2} \end{aligned}$$

Hence, the roots of the characteristic equation are  $x = \frac{1+\sqrt{2}}{2}$  and  $x = \frac{1-\sqrt{2}}{2}$ .  
Thus,

$$u_n = A \left( \frac{1+\sqrt{2}}{2} \right)^n + B \left( \frac{1-\sqrt{2}}{2} \right)^n$$

Since  $u_0 = a$ ,

$$\begin{aligned} a &= A \left( \frac{1+\sqrt{2}}{2} \right)^0 + B \left( \frac{1-\sqrt{2}}{2} \right)^0 \\ &= A + B \\ \implies B &= a - A \end{aligned}$$

Since  $u_1 = b$ ,

$$\begin{aligned} b &= A \left( \frac{1+\sqrt{2}}{2} \right)^1 + B \left( \frac{1-\sqrt{2}}{2} \right)^1 \\ &= \frac{1}{2} (A + B + \sqrt{2}(A - B)) \\ &= \frac{1}{2} (a + \sqrt{2}(A - (a - A))) \\ &= \frac{1}{2} (a + \sqrt{2}(2A - a)) \\ \implies A &= \frac{1}{2} \left( \frac{1}{\sqrt{2}}(2b - a) + a \right) \\ &= \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b \\ \implies B &= a - \left( \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b \right) \\ &= \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b \end{aligned}$$

Thus,

$$u_n = \left( \frac{\sqrt{2}-1}{2\sqrt{2}}a + \frac{1}{\sqrt{2}}b \right) \left( \frac{1+\sqrt{2}}{2} \right)^n + \left( \frac{\sqrt{2}+1}{2\sqrt{2}}a - \frac{1}{\sqrt{2}}b \right) \left( \frac{1-\sqrt{2}}{2} \right)^n$$

**Problem 3.**

A passcode is generated using the digits 1 to 5, with repetitions allowed. The passcodes are classified into two types. A Type  $A$  passcode has an even number of the digit 1, while a Type  $B$  passcode has an odd number of the digit 1. For example, a Type  $A$  passcode is 1231, and a Type  $B$  passcode is 1541213. Let  $a_n$  and  $b_n$  denote the number of  $n$ -digit Type  $A$  and Type  $B$  passcodes respectively.

- (a) State the values of  $a_1$  and  $a_2$ .
- (b) By considering the relationship between  $a_n$  and  $b_n$ , show that

$$a_n = xa_{n-1} + y^{n-1}, \quad n \geq 2$$

where  $x$  and  $y$  are constants to be determined.

- (c) Using the substitution  $c_n = za_n + y^n$ , where  $z$  is a constant to be determined, find a first order linear recurrence relation for  $c_n$ . Hence find the general term formula for  $a_n$ .

**Solution****Part (a)**

$$a_1 = 4, a_2 = 17$$

**Part (b)**

Let  $P$  be a  $n$ -digit passcode with Type  $T$ , where  $T \in \{A, B\}$ . We also let Type  $T'$  be such that  $T' \in \{A, B\} \setminus T$ .

By concatenating a digit from 1 to 5 to  $P$ , 5  $(n+1)$ -digit passcodes can be created. Let  $P'$  denote a newly passcode that is created via this process. If the digit 1 is concatenated, then  $P'$  is of Type  $T'$ . If the digit 1 is not concatenated, then  $P'$  is of Type  $T$ . There are 4 choices for such a case. This hence gives the recurrence relations

$$\begin{aligned} a_n &= 4a_{n-1} + b_{n-1} \\ b_n &= 4b_{n-1} + a_{n-1} \end{aligned}$$

Note that  $a_n + b_n = 5(a_{n-1} + b_{n-1})$ . Thus,  $a_n + b_n = 5^{n-1}(a_1 + b_1) = 5^{n-1}(4 + 1) = 5^n$ . Hence,

$$\begin{aligned} a_n &= 4a_{n-1} + b_{n-1} \\ &= 3a_{n-1} + a_{n-1} + b_{n-1} \\ &= 3a_{n-1} + 5^{n-1} \end{aligned}$$

whence  $x = 3$  and  $y = 5$ .

**Part (c)**

$$\begin{aligned}
c_n &= za_n + y^n \\
&= za_n + 5^n \\
&= z(3a_{n-1} + 5^{n-1}) + 5^n \\
&= 3za_{n-1} + z5^{n-1} + 5 \cdot 5^{n-1} \\
&= 3(z a_{n-1} + 5^{n-1}) + z5^{n-1} + 5 \cdot 5^{n-1} - 3 \cdot 5^{n-1} \\
&= 3c_{n-1} + (2 + z)5^{n-1}
\end{aligned}$$

Let  $z = -2$ . Then,

$$\begin{aligned}
c_n &= 3c_{n-1} \\
&= 3^{n-1}c_1 \\
&= 3^{n-1}(-2a_1 + 5^1) \\
&= -3 \cdot 3^{n-1}
\end{aligned}$$

Note that  $a_n = \frac{1}{z}(c_n - y^n)$ . Thus,

$$\begin{aligned}
a_n &= -\frac{1}{2}(-3 \cdot 3^{n-1} - 5^n) \\
&= \frac{1}{2}(3^n - 5^n)
\end{aligned}$$

$$\boxed{a_n = \frac{1}{2}(3^n - 5^n)}$$