Problem 1.

Differentiate the following with respect to x.

(a)
$$\ln \frac{x^3}{\sqrt{1+x^2}}$$

- (b) $\arctan \frac{x^2}{2}$
- (c) $e^{2x} \sec x$

Solution

Part (a)

$$\frac{d}{dx} \ln \frac{x^3}{\sqrt{1+x^2}} = \frac{d}{dx} 3 \ln x - \frac{1}{2} \ln(1+x^2)$$

$$= 3 \cdot \frac{1}{x} - \frac{1}{2} \cdot \frac{1}{1+x^2} \cdot 2x$$

$$= \frac{3}{x} - \frac{x}{1+x^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\frac{x^3}{\sqrt{1+x^2}} = \frac{3}{x} - \frac{x}{1+x^2}$$

Part (b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan\frac{x^2}{2} = \frac{1}{1 + (\frac{x^2}{2})} \cdot \frac{1}{2} \cdot 2x$$
$$= \frac{1}{1 + \frac{x^4}{4}} \cdot x$$
$$= \frac{4x}{4 + x^4}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan\frac{x^2}{2} = \frac{4x}{4+x^4}$$

Part (c)

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{2x}\sec x = e^{2x}\sec x \tan x + \sec x \cdot e^{2x} \cdot 2$$
$$= e^{2x}\sec x (\tan x + 2)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{2x}\sec x = e^{2x}\sec x \left(\tan x + 2\right)$$

Problem 2.

Find the gradient of the curve $x^3 + xy^2 = 5y$ at the point where x = 1 and 0 < y < 1, leaving your answer to 3 significant figures.

Solution

Substituting x = 1 into the given equation,

$$1^{4} + 1 \cdot y^{2} = 5y$$

$$\implies y^{2} - 5y + 1 = 0$$

$$\implies y = \frac{-(-5) \pm \sqrt{5^{2} - 4}}{2 \cdot 1}$$

$$= \frac{5 \pm \sqrt{21}}{2}$$

Since 0 < y < 1, we reject $y = \frac{5 + \sqrt{21}}{2}$ and take $y = \frac{5 - \sqrt{21}}{2} = 0.20871$ (5 s.f.). Implicitly differentiating the given equation,

$$3x^{2} + x \cdot 2y \cdot y' + y^{2} = 5y'$$

$$2xy \cdot y' - 5y' = -3x^{2} - y^{2}$$

$$\Rightarrow \qquad y'(2xy - 5) = -3x^{2} - y^{2}$$

$$\Rightarrow \qquad y' = \frac{-3x^{2} - y^{2}}{2xy - 5}$$
(2.1)

Substituting x = 1 and y = 0.20871 into Equation 2.1,

$$y' = \frac{-3 \cdot 1^2 - 0.20871^2}{2 \cdot 1 \cdot 0.20871 - 5}$$
$$= 0.664 (3 \text{ s.f.})$$

The gradient of the curve is 0.664.

Problem 3.

A curve C has parametric equations

$$x = \sin^3 \theta, \ y = 3\sin^2 \theta \cos \theta, \qquad 0 \le \theta \le \frac{1}{2}\pi$$

Show that $\frac{dy}{dx} = a \cot \theta + b \tan \theta$, where a and b are values to be determined.

Solution

$$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin^2\theta\cos\theta$$

$$\frac{dy}{d\theta} = 3\left(\sin^2\theta \cdot - \sin\theta + \cos\theta \cdot (2\sin\theta\cos\theta)\right)$$
$$= 3(2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$= \frac{dy}{d\theta} \cdot \left(\frac{dx}{d\theta}\right)^{-1}$$

$$= \frac{3(2\sin\theta\cos^2\theta - \sin^3\theta)}{3\sin^2\theta\cos\theta}$$

$$= \frac{2\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos^2\theta}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\cos\theta}{\sin\theta} - \frac{\sin\theta}{\cos\theta}$$

$$= 2\cot\theta - \tan\theta$$

a = 2, b = -1