Problem 1.

The curve $y = f(x)$ passes through the origin and has gradient given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 4x + 1}{2y - 5}.$$

(a) Find $f(x)$.

(b) By considering $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, deduce the coordinates of the point on the curve where it is tangent to the x-axis.

(c) Determine the interval of validity for the solution.

Solution

Part (a)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2 - 4x + 1}{2y - 5}$$

$$\implies \quad (2y - 5)\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 4x + 1$$

$$\implies \quad \int (2y - 5)\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int 3x^2 - 4x + 1\,\mathrm{d}x$$

$$\implies \quad \int (2y - 5)\,\mathrm{d}y = \int 3x^2 - 4x + 1\,\mathrm{d}x$$

$$\implies \quad y^2 - 5y = x^3 - 2x^2 + x + C_1$$
$$= x(x-1)^2 + C_1$$

$$\implies y^2 - 5y - x(x-1)^2 + C_2 = 0$$

$$\implies \quad y = \frac{5 \pm \sqrt{25 - 4(-x(x-1)^2 + C_2)}}{2}$$

$$= \frac{5 \pm \sqrt{4x(x-1)^2 + C_3}}{2}$$

At $x = 0$, $y = 0$, we have $0 = \dfrac{5 \pm \sqrt{C_3}}{2}$, whence $C_3 = 25$. We also reject the positive branch since the square root function is non-negative. Thus,

$$\boxed{f(x) = \frac{5 - \sqrt{4x(x-1)^2 + 25}}{2}}$$

Part (b)

When the curve is tangent to the x-axis, we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $y = 0$. Note that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies 3x^2 - 4x + 1 = 0 \implies x = \dfrac{1}{3} \vee 1$. Also note that $y = 0 \implies 4x(x-1)^2 = 0 \implies x = 0 \vee 1$. Hence, the required point is $\boxed{(1, 0)}$.

Part (c)

Since the square root function is defined only on the non-negative reals, we require $4x(x-1)^2 + 25 \geq 0$, whence $x \geq -1.24$. Thus, the interval of validity is $\boxed{[-1.24, \infty)}$.

Problem 2.

(a) Using the substitution $y = ux$, find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{x},$$

where $x > 0$.

(b) Find the particular solution of the differential equation for which $y = -1$ when $x = 1$.

(c) Without sketching the curve of the solution in (b), determine the number of stationary points the solution curve has.

Solution

Part (a)

Note that $y = ux \implies \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x}x + u.$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{x}$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}x}x + u = \frac{x + ux}{x}$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}\left(\frac{x + ux}{x} - u\right)$$
$$= \frac{1}{x}$$
$$\implies u = \ln(x) + C \quad (\because x > 0)$$
$$\implies y = x\ln x + Cx$$

$$\boxed{y = x\ln x + Cx}$$

Part (b)

When $x = 1$, $y = -1$, we have $-1 = 1\ln 1 + C \implies C = -1$. Thus,

$$\boxed{y = x\ln -x}$$

Part (c)

Note that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x + y}{x} = \dfrac{x + x\ln x - x}{x} = \ln x$. Since $\ln x$ only has one root (at $x = 1$), the solution curve has only $\boxed{1}$ stationary point.

Problem 3.

As a tree grows, the rate of increase of its height, h m, with respect to time, t years after planting, is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{10}\sqrt{16 - \frac{1}{2}h}.$$

The tree is planted as a seedling of negligible height, so that $h = 0$ when $t = 0$.

(a) State the maximimum height of the tree, according to this model.

(b) Find an expression for t in terms of h, and hence find the time the tree takes to reach half of its maximum height.

Solution

Part (a)

Note that $\dfrac{\mathrm{d}h}{\mathrm{d}t} \geq 0 \implies h \leq 32$. Thus, the maximum height of the tree is $\boxed{32 \text{ m}}$.

Part (b)

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{10}\sqrt{16 - \frac{1}{2}h}$$

$$\implies \quad 10\left(16 - \frac{1}{2}h\right)^{-1/2} \frac{\mathrm{d}h}{\mathrm{d}t} = 1$$

$$\implies 10\int \left(16 - \frac{1}{2}h\right)^{-1/2} \frac{\mathrm{d}h}{\mathrm{d}t}\,\mathrm{d}t = \int \mathrm{d}t$$

$$\implies \quad 10\int \left(16 - \frac{1}{2}h\right)^{-1/2} \mathrm{d}h = \int \mathrm{d}t$$

$$\implies \quad -10\sqrt{16 - \frac{1}{2}h} + C = t$$

At $t = 0$, $h = 0$, whence $-10\sqrt{16} + C = 0 \implies C = 40$. Thus,

$$\boxed{t = 40 - 10\sqrt{16 - \frac{1}{2}h}}$$

When $h = 32/2 = 16$, $t = 40 - 10\sqrt{16 - \frac{1}{2} \cdot 16} = 11.7$

$$\boxed{\text{It takes 11.7 years for the tree to reach half its maximimum height.}}$$

Problem 4.

(a) Find $\displaystyle\int \frac{1}{x(1000-x)}\,\mathrm{d}x$.

(b) A communicable disease is spreading within a small community with a population of 1000 people. A scientist found out that the rate at which the disease spreads is proportional to the product of the number of people who are infected with the disease and the number of people who are not infected with the disease. It is known that one person in this community is infected initially and five days later, 12% of the population is infected.

Given that the infected population is x at time t days after the start of the spread of the disease, show that it takes less than 8 days for half the population to contract the disease.

(c) State an assumption made by the scientist.

Solution

Part (a)

$$\int \frac{1}{x(1000-x)}\,\mathrm{d}x = \int \frac{1}{1000}\left(\frac{1}{x} + \frac{1}{1000-x}\right)\mathrm{d}x$$

$$= \frac{1}{1000}\left(\ln|x| - \ln|1000-x|\right) + C$$

$$= \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| + C$$

$$\boxed{\int \frac{1}{x(1000-x)}\,\mathrm{d}x = \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| + C}$$

Part (b)

Note that $\dfrac{\mathrm{d}x}{\mathrm{d}t} \propto x(1000-x) \implies \dfrac{\mathrm{d}x}{\mathrm{d}t} = kx(1000-x)$ for some $k \in \mathbb{R}^+$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = kx(1000-x)$$

$$\implies \frac{1}{x(1000-x)}\frac{\mathrm{d}x}{\mathrm{d}t} = k$$

$$\implies \int \frac{1}{x(1000-x)}\frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t = \int k\,\mathrm{d}t$$

$$\implies \int \frac{1}{x(1000-x)}\,\mathrm{d}x = \int k\,\mathrm{d}t$$

$$\implies \frac{1}{1000}\ln\left(\frac{x}{1000-x}\right) + C = kt \quad (\because 0 \le x \le 1000)$$

At $t = 0$, $x = 1$. Hence, $\dfrac{1}{1000}\ln\left(\dfrac{1}{999}\right) + C = 0 \implies C = -\dfrac{1}{1000}\ln\left(\dfrac{1}{999}\right) = \dfrac{\ln 999}{1000}$.

At $t = 5$, $x = 120$. Hence, $\dfrac{1}{1000} \ln\left(\dfrac{120}{880}\right) + \dfrac{\ln 999}{1000} = 5k \implies k = \dfrac{1}{5000}\left(\ln \dfrac{3}{22} + \ln 999\right)$.

Thus,

$$
\begin{aligned}
t &= \left(\frac{1}{5000}\left(\ln \frac{3}{22} + \ln 999\right)\right)^{-1}\left[\frac{1}{1000}\ln\left(\frac{x}{1000 - x}\right) + \frac{\ln 999}{1000}\right] \\
&= \frac{5}{\ln(3/22) + \ln 999}\left[\ln\left(\frac{x}{1000 - x}\right) + \ln 999\right]
\end{aligned}
$$

Hence, when half the population is infected, i.e. $x = 500$, we have $t = 7.03 < 8$. Thus, it takes less than 8 days for half the population to contract the disease.

Part (c)

The assumption is that there are no measures taken by the population to limit the spread of the disease (e.g. quarantine).