# Problem 1.

Determine whether each of the following systems of equations has a unique solution, infinitely many solutions, or no solutions. Find the solutions, where appropriate.

(a) $\begin{cases} a + 2b - 3c = -5 \\ -2a - 4b - 6c = 10 \\ 3a + 7b - 2c = -13 \end{cases}$

(b) $\begin{cases} x - y + 3z = 3 \\ 4x - 8y + 32z = 24 \\ 2x - 3y + 11z = 4 \end{cases}$

(c) $\begin{cases} x_1 + x_2 = 5 \\ 2x_1 + x_2 + x_3 = 13 \\ 4x_1 + 3x_2 + x_3 = 23 \end{cases}$

(d) $\begin{cases} \dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = 5 \\ \dfrac{2}{p} - \dfrac{3}{q} - \dfrac{4}{r} = -11 \\ \dfrac{3}{p} + \dfrac{2}{q} - \dfrac{1}{r} = -6 \end{cases}$

(e) $\begin{cases} 2\sin\alpha - \cos\beta + 3\tan\gamma = 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma = 9 \end{cases}$ , where $0 \le \alpha \le 2\pi$, $0 \le \beta \le 2\pi$, and $0 \le \gamma < \pi$.

## Solution

### Part (a)

$$\boxed{\text{Unique solution: } a = -9,\ b = 2,\ c = 0}$$

### Part (b)

$$\boxed{\text{No solution.}}$$

### Part (c)

$$\boxed{\text{Infinitely many solutions: } x_1 = 8 - t,\ x_2 = t - 3,\ x_3 = t}$$

**Part (d)**

$$\frac{1}{p} = 2, \ \frac{1}{q} = -3, \ \frac{1}{r} = 6$$

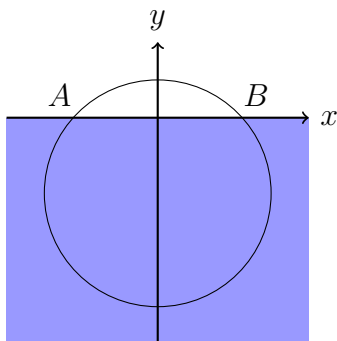Unique solution: $p = \dfrac{1}{2}, \ q = -\dfrac{1}{3}, \ r = \dfrac{1}{6}$

**Part (e)**

$$\sin \alpha = 1, \ \cos \beta = -1, \ \tan \gamma = 0$$

Unique solution: $\alpha = \dfrac{\pi}{2}, \ \beta = \pi, \ \gamma = 0$

# Problem 2.

The following figure shows the circular cross section of a uniform log floating in a canal.



With respect to the axes shown, the circular outline of the log can be modelled by the equation

$$x^2 + y^2 + ax + by + c = 0 \tag{2.1}$$

$A$ and $B$ are points on the outline that lie on the water surface. Given that the highest point of the log is 1-cm above the water surface when $AB$ is 40 cm apart horizontally, determine the values of $a$, $b$ and $c$ by forming a system of linear equations.

## Solution

Since $AB = 40$, we have $A(-20, 0)$ and $B(20, 0)$. We also know $(0, 10)$ lies on the circle. Substituting these points into Equation 2.1,

$$\begin{cases} -20a + c = -400 \\ 20a + c = -400 \\ 10b + c = -100 \end{cases}$$

Solving the system of equations,

$$\boxed{a = 0, \ b = 30, \ c = -400}$$

# Problem 3.

Find the exact solution set of the following inequalities.

(a) $x^2 - 2 \geq 0$

(b) $4x^2 - 12x + 10 > 0$

(c) $x^2 + 4x + 13 < 0$

(d) $x^3 < 6x - x^2$

(e) $x^2(x-1)(x+3) \geq 0$

## Solution

**Part (a)**

$$x^2 - 2 \geq 0$$
$$\implies \quad x^2 \geq 2$$
$$\implies \quad x \leq -\sqrt{2} \vee x \geq \sqrt{2}$$

Solution set: $\left\{ x \in \mathbb{R} : x \leq -\sqrt{2} \vee x \geq \sqrt{2} \right\}$

**Part (b)**

$$4x^2 - 12x + 10 > 0$$
$$\implies \quad x^2 - 3x + \frac{5}{2} > 0$$
$$\implies x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + \frac{5}{2} > 0$$
$$\implies \quad \left(x - \frac{3}{2}\right)^2 + \frac{19}{4} > 0$$

Since $\left(x - \frac{3}{2}\right)^2 \geq 0$, all $x \in \mathbb{R}$ satisfy the inequality.
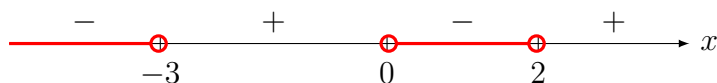
Solution set: $\mathbb{R}$

**Part (c)**

$$x^2 + 4x + 13 < 0$$
$$\implies x^2 + 4x + 4 + 9 < 0$$
$$\implies \quad (x + 2)^2 + 9 < 0$$

Since $(x + 2)^2 \geq 0$, there is no solution to the inequality.

$$\boxed{\text{Solution set: } \varnothing}$$

**Part (d)**

$$x^3 < 6x - x^2$$
$$\implies x(x + 3)(x - 2) < 0$$



$$\implies x < -3 \vee 0 < x < 2$$

$$\boxed{\text{Solution set: } \{x \in \mathbb{R}: x < -3 \vee 0 < x < 2\}}$$

**Part (e)**

$$x^2(x - 1)(x + 3) \geq 0$$
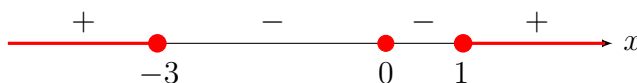


$$\implies x \leq -3 \vee x = 0 \vee x \geq 1$$

$$\boxed{\text{Solution set: } \{x \in \mathbb{R}: x \leq -3 \vee x = 0 \vee x \geq 1\}}$$

# Problem 4.

Find the exact solution set of the following inequalities.

(a) $|3x + 5| < 4$

(b) $|x - 2| < 2x$

## Solution

**Part (a)**

$$|3x + 5| < 4$$

**Case 1:** $3x + 5 < 4$

$$3x + 5 < 4$$
$$\implies \quad 3x < -1$$
$$\implies \quad x < -\frac{1}{3}$$

**Case 2:** $-(3x + 5) < 4$

$$-(3x + 5) < 4$$
$$\implies \quad -3x - 5 < 4$$
$$\implies \quad -3x < 9$$
$$\implies \quad x > -3$$

Combining both inequalities, we have $-3 < x < -\frac{1}{3}$.

$$\boxed{\text{Solution set: } \left\{ x \in \mathbb{R} \colon -3 < x < -\frac{1}{3} \right\}}$$

**Part (b)**

**Case 1:** $x - 2 < 2x$

$$x - 2 < 2x$$
$$\implies \quad x > -2$$

**Case 2:** $-(x - 2) < 2x$

$$-(x - 2) < 2x$$
$$\implies \quad -x + 2 < 2x$$
$$\implies \quad 3x > 2$$
$$\implies \quad x > \frac{2}{3}$$

Tutorial A1

Eytan Chong · · · · · · · · · · · · · · · · · · · Equations and Inequalities · · · · · · · · · · · · · · · · · · · 2024-02-20

Combining both inequalities, we have $x > \dfrac{2}{3}$.

$$\boxed{\text{Solution set: } \left\{ x \in \mathbb{R} \colon x > \dfrac{2}{3} \right\}}$$

# Problem 5.

It is given that $p(x) = x^4 + ax^3 + bx^2 + cx + d$, where $a$, $b$, $c$ and $d$ are constants. Given that the curve with equation $y = p(x)$ is symmetrical about the $y$-axis, and that it passes through the points with coordinates $(1, 2)$ and $(2, 11)$, find the values of $a$, $b$, $c$ and $d$.

## Solution

We know that $(1, 2)$ and $(2, 11)$ lie on the curve. Since $y = p(x)$ is symmetrical about the $y$-axis, we have that $(-1, 2)$ and $(-2, 11)$ also lie on the curve. Substituting these points into $y = p(x)$, we obtain the following system of equations.

$$\begin{cases} a + b + c + d = 1 \\ a - b + c - d = -1 \\ 8a + 4b + 2c + d = -5 \\ 8a - 4b + 2c - d = 5 \end{cases}$$

Solving the system of equations,

$$\boxed{a = 0,\ b = -2,\ c = 0,\ d = 3}$$

# Problem 6.

Mr Mok invested $50,000 in three funds A, B and C. Each fund has a different risk level and offers a different rate of return.

In 2016, the rates of return for funds A, B and C were 6%, 8%, and 10% respectively and Mr Mok attained a total return of $3,700. He invested twice as much money in Fund A as in Fund C. How much did he invest in each of the funds in 2016?

## Solution

Let $a$, $b$ and $c$ be the amount of money Mr Mok invested in Funds A, B and C respectively, in dollars. We thus have the following system of equations.

$$\begin{cases} a + b + c = 50000 \\ \dfrac{6}{100}a + \dfrac{8}{100}b + \dfrac{10}{100}c = 3700 \\ a = 2c \end{cases}$$

Solving the system of equations, we have $a = 30000$, $b = 5000$ and $c = 15000$.

> Mr Mok invested $30,000, $5,000 and $15,000 in Funds A, B and C respectively.

# Problem 7.

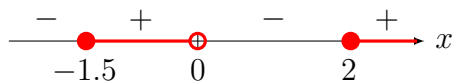Solve the following inequalities with exact answers.

(a) $2x - 1 \geq \dfrac{6}{x}$

(b) $x - \dfrac{1}{x} < 1$

(c) $-1 < \dfrac{2x+3}{x-1} < 1$
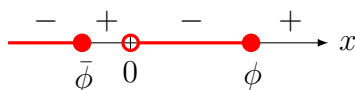
## Solution

**Part (a)**

$$2x - 1 \geq \frac{6}{x}, \qquad x \neq 0$$
$$\implies \qquad x^2(2x - 1) \geq 6x$$
$$\implies \quad x^2(2x - 1) - 6x \geq 0$$
$$\implies x(x(2x - 1) - 6) \geq 0$$
$$\implies \quad x\left(2x^2 - x - 6\right) \geq 0$$
$$\implies x(2x + 3)(x - 2) \geq 0$$



$$-\frac{3}{2} \leq x < 0 \lor x \geq 2$$

**Part (b)**

$$x - \frac{1}{x} < 1, \qquad x \neq 0$$
$$\implies \qquad x^3 - x < x^2$$
$$\implies \qquad x^3 - x^2 - x < 0$$
$$\implies \quad x\left(x^2 - x - 1\right) < 0$$
$$\implies x(x - \phi)(x - \bar{\phi}) < 0$$



$$x \leq \bar{\phi} \lor 0 < x \leq \phi$$

**Part (c)**

$$-1 < \frac{2x+3}{x-1} < 1$$
$$\implies -1 < \frac{(2x-2)+5}{x-1} < 1$$
$$\implies -1 < 2 + \frac{5}{x-1} < 1$$
$$\implies -3 < \frac{5}{x-1} < -1$$
$$\implies -\frac{3}{5} < \frac{1}{x-1} < -\frac{1}{5}$$
$$\implies -5 < x-1 < -\frac{5}{3}$$
$$\implies -4 < x < -\frac{2}{3}$$

$$\boxed{-4 < x < -\frac{2}{3}}$$

# Problem 8.

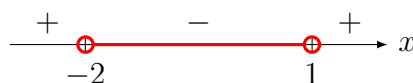Without using a calculator, solve the inequality $\dfrac{x^2 + x + 1}{x^2 + x - 2} < 0$.

## Solution

Observe that

$$x^2 + x + 1 = x^2 + x + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$$
$$= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$$
$$> 0$$

Thus, the inequality reduces to $\dfrac{1}{x^2 + x - 2} < 0$

$$\frac{1}{x^2 + x - 2} < 0$$
$$\implies \quad x^2 + x - 1 < 0$$
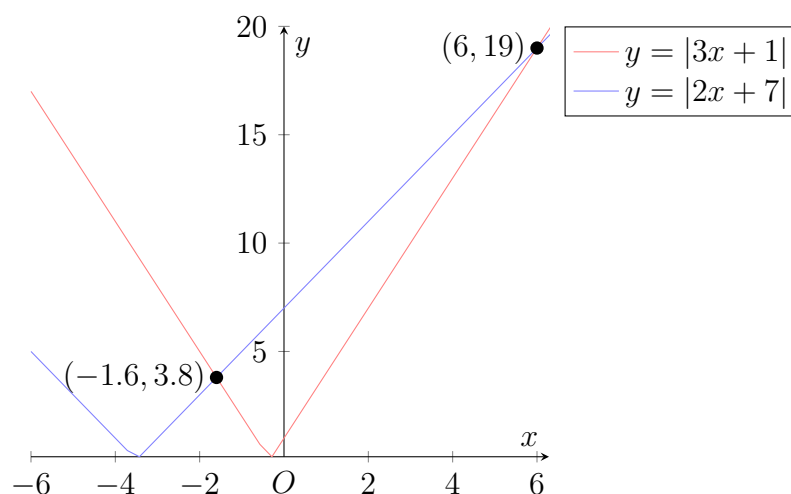$$\implies (x - 1)(x + 2) < 0$$
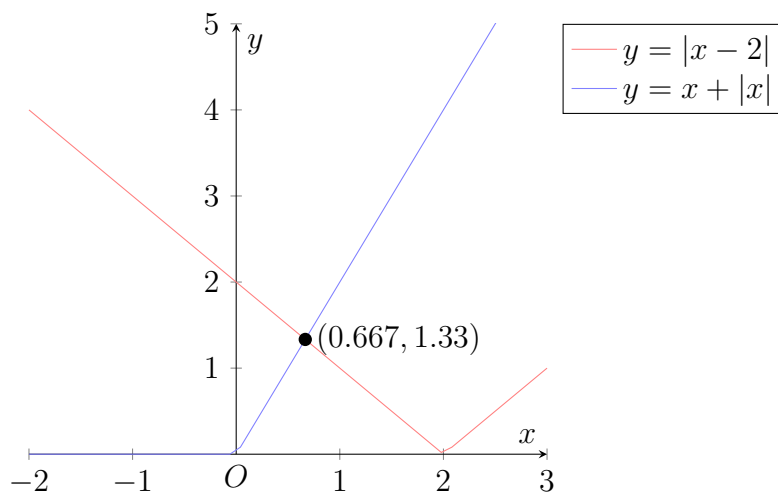


$$\boxed{-2 < x < 1}$$

## Problem 9.

Solve the following inequalities using a graphical method.

(a) $|3x + 1| < (4x + 3)^2$

(b) $|3x + 1| \geq |2x + 7|$

(c) $|x - 2| \geq x + |x|$

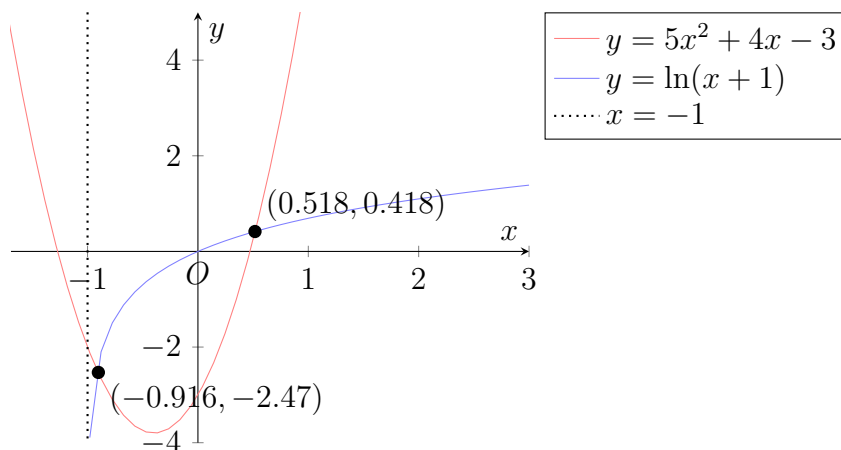(d) $5x^2 + 4x - 3 > \ln(x + 1)$

## Solution

**Part (a)**



$$\boxed{x < -1.14 \lor x > -0.549}$$
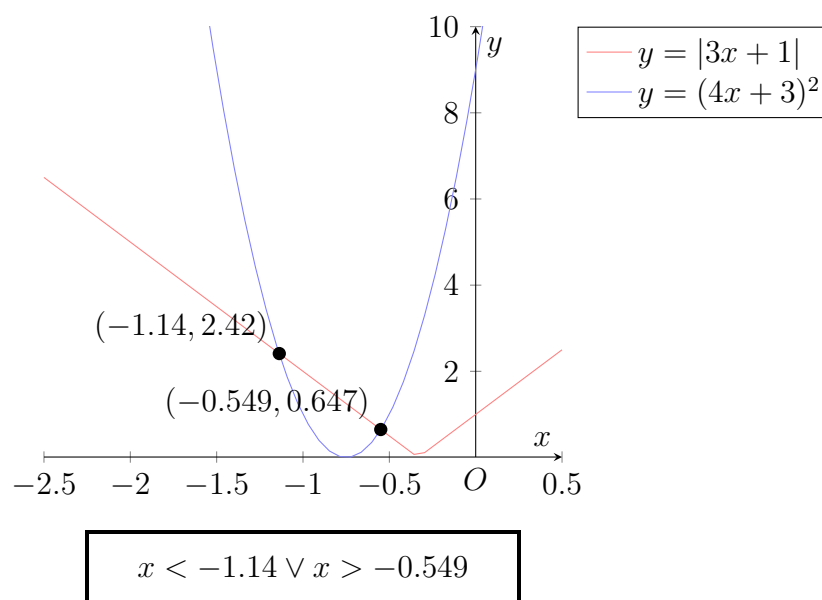
**Part (b)**

$$x \leq -1.6 \vee x \geq 6$$

**Part (c)**
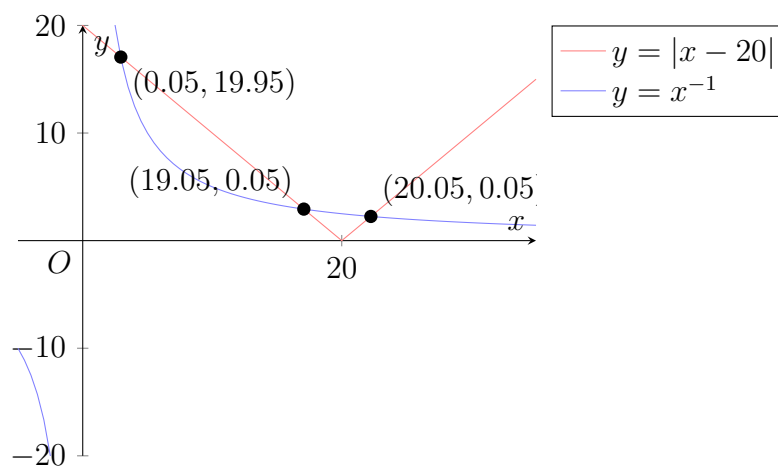


$$x \leq 0.667$$

**Part (d)**



$$-1 < x < -0.916 \vee x > 0.518$$

# Problem 10.

Sketch the graphs of $y = |x - 20|$ and $y = \dfrac{1}{x}$ on the same diagram. Hence or otherwise, solve the inequality $|x - 20| < \dfrac{1}{x}$, leaving your anwers correct to 2 decimal places.

## Solution



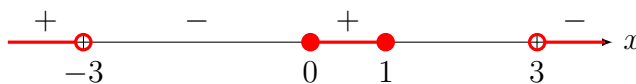$$0 < x < 0.05 \lor 19.95 < x < 20.05$$

# Problem 11.

Solve the inequality $\dfrac{x-9}{x^2-9} \le 1$. Hence, solve the inequalities

(a) $\dfrac{|x|-9}{x^2-9} \le 1$

(b) $\dfrac{x+9}{x^2-9} \ge -1$

## Solution

$$\frac{x-9}{x^2-9} \le 1$$
$$\implies \quad (x-9)\left(x^2-9\right) \le \left(x^2-9\right)^2$$
$$\implies \quad x^3 - 9x^2 - 9x + 81 \le x^4 - 18x^2 + 81$$
$$\implies \quad x^4 - x^3 - 9x^2 + 9x \ge 0$$
$$\implies (x+3)x(x-1)(x-3) \ge 0$$



$$\boxed{x < -3 \lor 0 \le x \le 1 \lor x > 3}$$

**Part (a)**

Consider the substitution $x \mapsto |x|$ on the inequality $\dfrac{x-9}{x^2-9} \le 1$. This yields our desired inequality $\dfrac{|x|-9}{x^2-9} \le 1$. Hence,

$$|x| < -3 \lor 0 \le |x| \le 1 \lor |x| > 3$$

**Case 1:** $|x| < -3$
  No solution.

**Case 2:** $0 \le |x| \le 1$

$$-1 \le x \le 1$$

**Case 3:** $|x| > 3$

$$x < -3 \lor x > 3$$

  Combining all three cases, we finally have

$$\boxed{x < -3 \lor -1 \le x \le 1 \lor x > 3}$$

**Part (b)**

Consider the substitution $x \mapsto -x$ on the inequality $\dfrac{x-9}{x^2-9} \leq 1$. This yields $\dfrac{-x-9}{x^2-9} \leq 1$, which is equivalent to our desired inequality $\dfrac{x+9}{x^2-9} \geq -1$. Hence,

$$-x < -3 \vee 0 \leq -x \leq 1 \vee -x > 3$$

**Case 1:** $-x < -3$

$$x > 3$$

**Case 2:** $0 \leq -x \leq 1$

$$-1 \leq x \leq 0$$

**Case 3:** $-x > 3$

$$x < -3$$

Combining all three cases, we finally have

$$\boxed{x < -3 \vee -1 \leq x \leq 0 \vee x > 3}$$

# Problem 12.

Solve the inequality $\dfrac{x-5}{1-x} \geq 1$. Hence, solve $0 < \dfrac{1 - \ln x}{\ln x - 5} \leq 1$.

## Solution

$$\frac{x-5}{1-x} \geq 1, \qquad x \neq 1$$
$$\implies \quad (x-5)(1-x) \geq (1-x)^2$$
$$\implies \quad 2x^2 - 8x + 6 \leq 0$$
$$\implies 2(x-1)(x-3) \leq 0$$



$$\boxed{1 < x \leq 3}$$

Consider the substitution $x \mapsto \ln x$ on the inequality $\dfrac{x-5}{1-x} \geq 1$. This yields our desired inequality $0 < \dfrac{1 - \ln x}{\ln x - 5} \leq 1$. Hence,
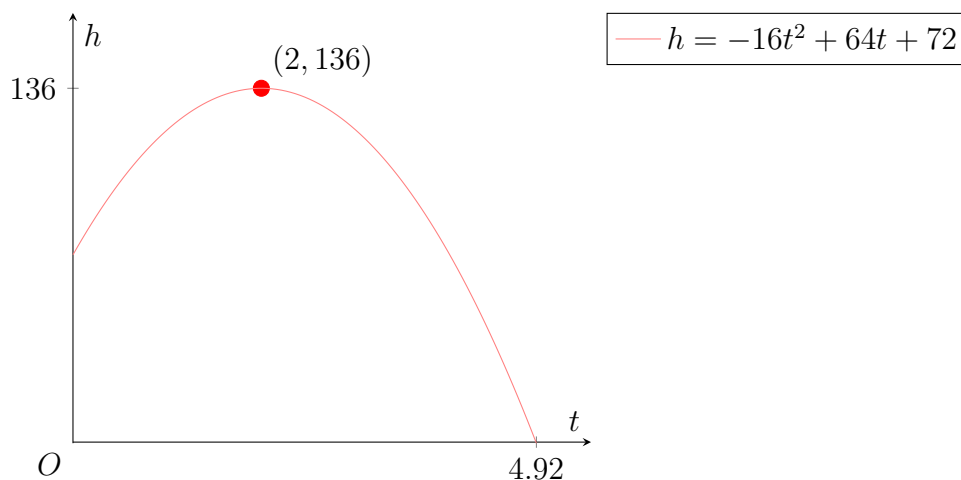
$$1 < \ln x \leq 3$$
$$\implies \quad e < x \leq e^3$$

$$\boxed{e < x \leq e^3}$$

# Problem 13.

A small rocket is launched from a height of 72 m from the ground. The heigh of the rocket in metres, $h$, is represented by the equation $h = -16t^2 + 64t + 72$, where $t$ is the time in seconds after the launch.

(a) Sketch the graph of $h$ against $t$.

(b) Determine the number of seconds that the rocket will remain at or above 100 m from the ground.

## Solution

**Part (a)**



**Part (b)**

$$h \geq 100$$
$$\implies -16t^2 + 64t + 72 \geq 100$$
$$\implies -16t^2 + 64t - 28 \geq 0$$
$$\implies 16t^2 - 64t + 28 \geq 0$$
$$\implies 4(2t - 1)(2t - 7) \leq 0$$
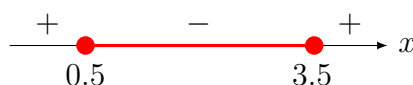


The rocket will remain at or above 100 m from the ground for $\dfrac{7}{2} - \dfrac{1}{2} = 3$ seconds.

# Problem 14.

Xinxin, a new graduate, starts work at a company with an initial monthly pay of \$2,000. For every subsequent quarter that she works, she will get a pay increase of 5%, leading to a new monthly pay of $2000(1.05)^{n-1}$ dollars in the $n$th quarter, where $n$ is a positive integer. She also gives a regular donation of \$300$n$ in the $n$th quarter that she works. However, she will stop the donation when her monthly pay falls below the donation amount. At which quarter will this first happen?

## Solution

Consider the curves $y = 2000(1.05)^{x-1}$ and $y = 300x$.
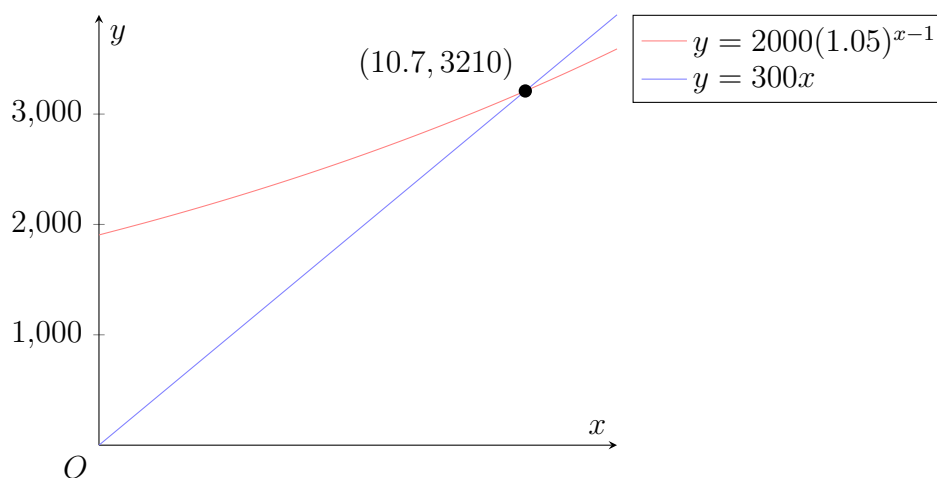


Xinxin will stop donating in the $\lceil 10.7 \rceil = 11$th quarter.