Problem 1.

The curve C has parametric equations

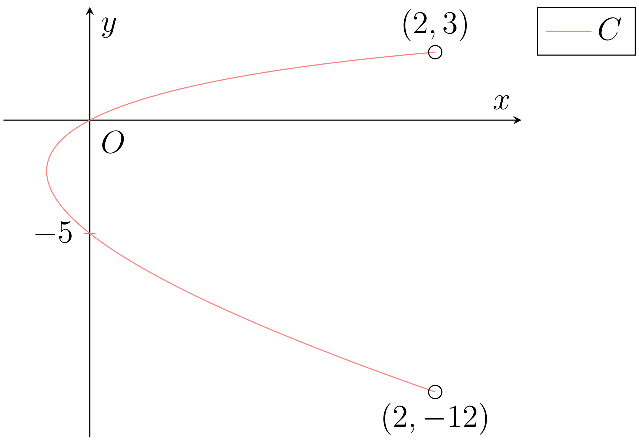
$$x = t^2 + t,\ y = 4t - t^2,\ -2 < t < 1$$

(a) Sketch C, indicating the coordinates of the end-points and the axial intercepts (if any) of this curve.

(b) Find the coordinates of the point(s) of intersection between C and the line $8y - 12x = 5$.

Solution

Part (a)



Part (b)

$$\begin{aligned}
& & 8y - 12x &= 5 \\
&\implies & 8(4t - t^2) - 12(t^2 + t) &= 5 \\
&\implies & 32t - 8t^2 - 12t^2 - 12t - 5 &= 0 \\
&\implies & -20t^2 + 20t - 5 &= 0 \\
&\implies & t^2 - t + \frac{1}{4} &= 0 \\
&\implies & \left(t - \frac{1}{2}\right)^2 &= 0 \\
&\implies & t &= \frac{1}{2}
\end{aligned}$$

When $t = \dfrac{1}{2}$, we have that $x = \dfrac{3}{4}$ and $y = \dfrac{7}{4}$. Thus, C and the line $8y - 12x = 5$ intersect at $\left(\dfrac{3}{4}, \dfrac{7}{4}\right)$.

$$\boxed{\left(\frac{3}{4}, \frac{7}{4}\right)}$$

Problem 2.

(a) Without using a calculator, solve $\dfrac{4}{3+2x-x^2} \leq 1$.

(b) Hence, solve $\dfrac{4}{3+2|x|-x^2} \leq 1$.

Solution

Part (a)

$$\begin{aligned}
& \frac{4}{3+2x-x^2} \leq 1 \\
\implies \quad & \frac{4}{x^2-2x-3} \geq -1 \\
\implies \quad & \frac{4}{(x-3)(x+1)} + 1 \geq 0 \\
\implies \quad & \frac{4+(x-3)(x+1)}{(x-3)(x+1)} \geq 0 \\
\implies \quad & \frac{4+(x^2-2x-3)}{(x-3)(x+1)} \geq 0 \\
\implies \quad & \frac{x^2-2x+1}{(x-3)(x+1)} \geq 0 \\
\implies \quad & \frac{(x-1)^2}{(x-3)(x+1)} \geq 0
\end{aligned}$$

We thus have that $x = 1$ is a solution. In the case when $(x-1)^2 > 0$,

$$\begin{aligned}
& \frac{1}{(x-3)(x+1)} \geq 0 \\
\implies \quad & (x-3)(x+1) \geq 0
\end{aligned}$$

whence $x < 1$ or $x > 3$. Putting everything together, we have

$$\boxed{x < -1 \lor x = 1 \lor x > 3}$$

Part (b)

$$\begin{aligned}
& \frac{4}{3+2|x|-x^2} \leq 1 \\
\implies \quad & \frac{4}{3+2|x|-|x|^2} \leq 1
\end{aligned}$$

From part (a), we have that $|x| < -1$, $|x| = 1$ or $|x| > 3$.
Case 1: $|x| < -1$. Since $|x| \geq 0$ this case yields no solutions.
Case 2: $|x| = 1$. We have $x = 1$ or $x = -1$.
Case 3: $|x| > 3$. We have $x > 3$ or $x < -3$.

$$\boxed{x < -3 \lor x = -1 \lor x = 1 \lor x > 3}$$

Problem 3.

The curve C_1 has equation

$$y = \frac{2x^2 + 2x - 2}{x - 1}$$

(a) Sketch the graph of C_1, stating the equations of any asymptotes and the coordinates of any axial intercepts and/or turning points.

The curve C_2 has equation

$$\frac{(x-a)^2}{1^2} + \frac{(y-6)^2}{b^2} = 1$$

where $b > 0$. It is given that C_1 and C_2 have no points in common for all $a \in \mathbb{R}$.

(b) By adding an appropriate curve in part (a), state the range of values of b, explaining your answer.

(c) The function f is defined by

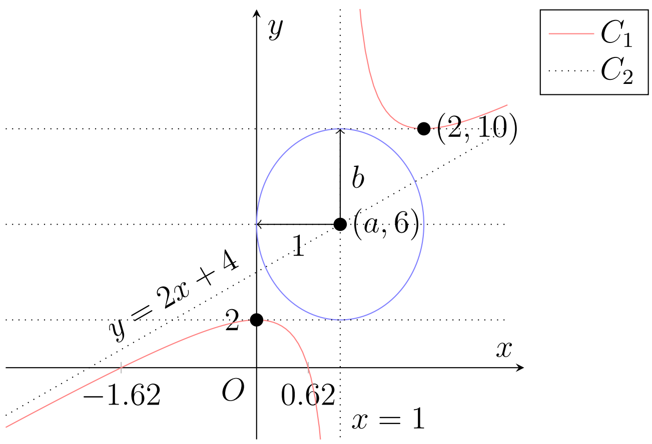
$$f(x) = \frac{2x^2 + 2x - 2}{x - 1},\ x < 1$$

(i) By using the graph in part (a) or otherwise, explain why the inverse function f^{-1} does not exist.

(ii) The domain of f is restricted to $[c, 1)$ such that c is the least value for which the inverse function f^{-1} exists. State the value of c and define f^{-1} clearly.

Solution

Part (a)



Part (b)

Observe that C_2 describes an ellipse with vertical radius b and horizontal radius 1. Furthermore, the ellipse is centered at $(a, 6)$. Since C_1 and C_2 have no points in common for all $a \in \mathbb{R}$, the maximum y-value of the ellipse corresponds to the y-value of the minimum point $(2, 10)$ of C_1. Similarly, the minimum y-value of the ellipse corresponds to the y-value of the maximum point $(0, 2)$ of C_1. Thus, $2 < y < 10$, whence $b < \min\{|6-2|, |6-10|\} = 4$. Thus,

$$\boxed{0 < b < 4.}$$

Part (c)

Subpart (i)

Observe that $f(-1.62) = f(0.618) = 0$. Hence, there exist two different values of x in D_f that have the same image under f. Thus, f is not one-one. Hence, f^{-1} does not exist.

Subpart (ii)

$$\boxed{c = 0}$$

$$
\begin{aligned}
& f(x) = \frac{2x^2 + 2x - 2}{x - 1} \\
\implies & (x-1)f(x) = 2x^2 + 2x - 2 \\
\implies & xf(x) - x = 2x^2 + 2x - 2 \\
\implies & 2x^2 + 2x - xf(x) - 2 + f(x) = 0 \\
\implies & 2x^2 + \big(2 - f(x)\big)x + \big(f(x) - 2\big) = 0 \\
\implies & x = \frac{-\big(2 - f(x)\big) \pm \sqrt{\big(2 - f(x)\big)^2 - 4 \cdot 2 \cdot \big(f(x) - 2\big)}}{2 \cdot 2} \\
& = \frac{f(x) - 2 \pm \sqrt{f(x)^2 - 12f(x) + 20}}{4} \\
\implies & f^{-1}(x) = \frac{x - 2 \pm \sqrt{x^2 - 12x + 20}}{4}
\end{aligned}
$$

Note that $D_f = R_{f^{-1}} = [0, 1)$. We thus take the positive root. Further note that $R_f = D_{f^{-1}} = (-\infty, 2]$.

$$\boxed{f^{-1} : x \mapsto \frac{x - 2 + \sqrt{x^2 - 12x + 20}}{4},\ x \in \mathbb{R},\ x \leq 2}$$