Problem 1.

Is the following true or false in general?

(a) $|w^2| = |w|^2$

(b) $|z + 2w| = |z| + |2w|$

Solution

Part (a)

Let $w = re^{i\theta}$, where $r, \theta \in \mathbb{R}$. Note that $\left|e^{i\theta}\right| = \left|e^{2i\theta}\right| = 1$.

$$\left|w^2\right| = \left|r^2 e^{2i\theta}\right| = r^2 \left|e^{2i\theta}\right| = r^2 = r^2 \left|e^{i\theta}\right|^2 = \left|re^{i\theta}\right|^2 = |w|^2$$

$$\boxed{\text{The statement } |w^2| = |w|^2 \text{ is true in general.}}$$

Part (b)

Take $z = 1$ and $w = -1$.

$$|z + 2w| = |1 - 2| = 1 \neq 3 = |1| + |2 \cdot -1| = |z| + |2w|$$

$$\boxed{\text{The statement } |z + 2w| = |z| + |2w| \text{ is false in general.}}$$

Problem 2.

Express the following complex numbers z in polar form $r(\cos\theta + i\sin\theta)$ with exact values.

(a) $z = 2 - 2i$

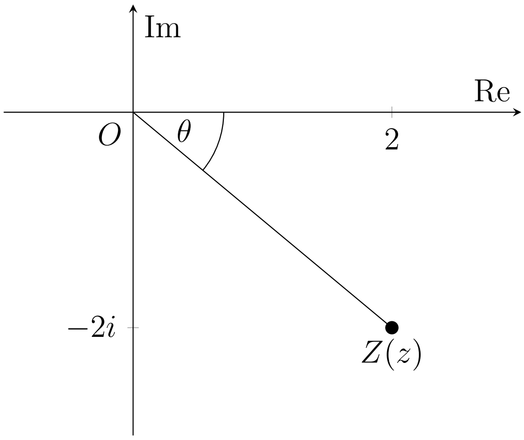
(b) $z = -1 + i\sqrt{3}$

(c) $z = -5i$

(d) $z = -2\sqrt{3} - 2i$

Solution

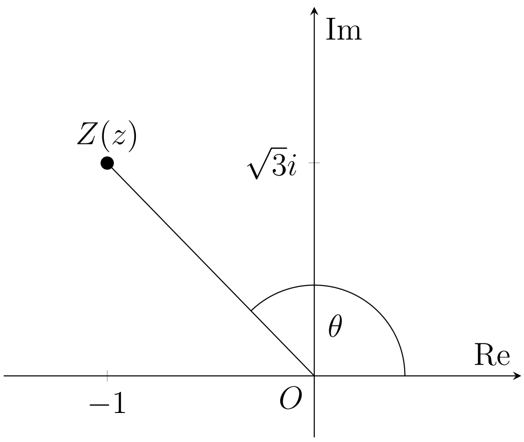
Part (a)



We have $r^2 = 2^2 + (-2)^2 \implies r = 2\sqrt{2}$ and $\tan\theta = \dfrac{-2}{2} \implies \theta = -\dfrac{\pi}{4}$.

$$\boxed{2 - 2i = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right]}$$

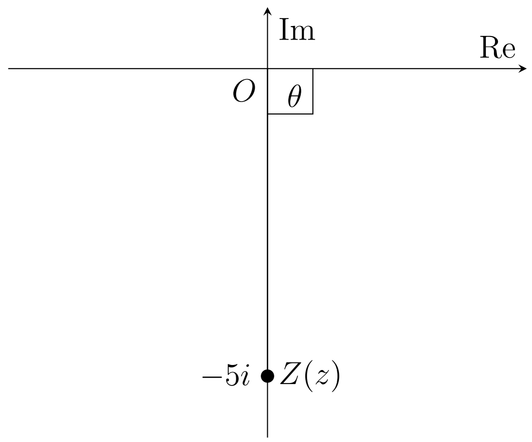
Part (b)



We have $r^2 = (-1)^2 + (\sqrt{3})^2 \implies r = 2$ and $\tan t = \dfrac{\sqrt{3}}{-1} \implies \theta = \dfrac{2}{3}\pi$.

$$\boxed{-1 + \sqrt{3}i = 2\left[\cos\left(\frac{2}{3}\pi\right) + i\sin\left(\frac{2}{3}\pi\right)\right]}$$

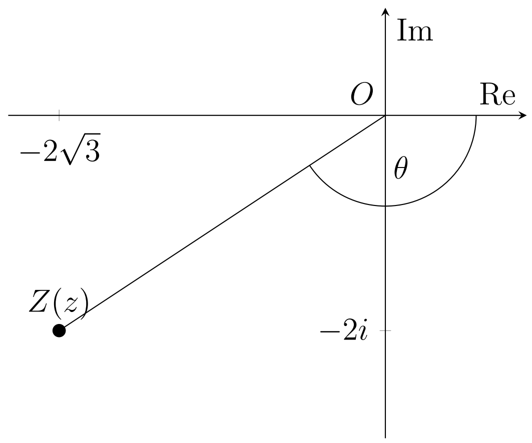
Part (c)



We have $r = 5$ and $\theta = -\frac{\pi}{2}$.

$$\boxed{-5i = 5\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]}$$

Part (d)



We have $r^2 = (-2\sqrt{3})^2 + (-2)^2 \implies r = 4$ and $\tan t = \dfrac{-2}{-2\sqrt{3}} \implies \theta = -\dfrac{5}{6}\pi$.

$$\boxed{-2\sqrt{3} - 2i = 4\left[\cos\left(-\frac{5}{6}\pi\right) + i\sin\left(-\frac{5}{6}\pi\right)\right]}$$

Problem 3.

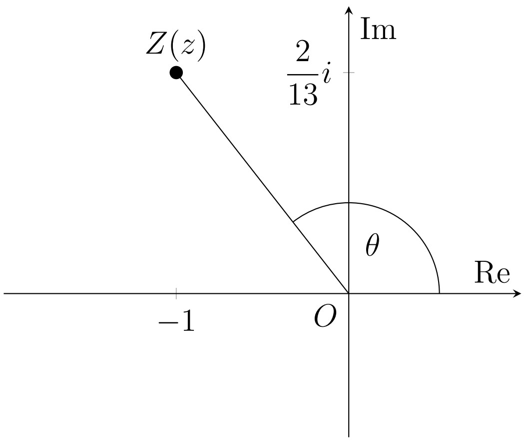
Express the following complex numbers z in exponential form $re^{i\theta}$.

(a) $z = -1 + \dfrac{2}{13}i$

(b) $z = \cos 50^\circ + i \sin 50^\circ$

Solution

Part (a)



We have $r^2 = (-1)^2 + \left(\dfrac{2}{13}\right)^2 \implies r = 1.02$ (3 s.f.) and $\tan t = \dfrac{2/13}{-1} \implies \theta = 2.99$ (3 s.f.).

$$\boxed{-1 + \frac{2}{13}i = 1.02e^{2.99i}}$$

Part (b)

We have $r = 1$ and $\theta = -50^\circ = -\dfrac{50}{180}\pi = -\dfrac{5}{18}\pi$.

$$\boxed{\cos 50^\circ + i \sin 50^\circ = e^{-i\frac{5}{18}\pi}}$$

Problem 4.

Express the following complex numbers z in Cartesian form.

(a) $z = 7e^{1-5i}$

(b) $z = 6\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$

Solution

Part (a)

$$\begin{aligned} z &= 7e^{1-5i} \\ &= 7e \cdot e^{-5i} \\ &= 7e\,[\,\cos(-5) + i\sin(-5)] \\ &= 5.40 + 18.2i \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{7e^{1-5i} = 5.40 + 18.2i}$$

Part (b)

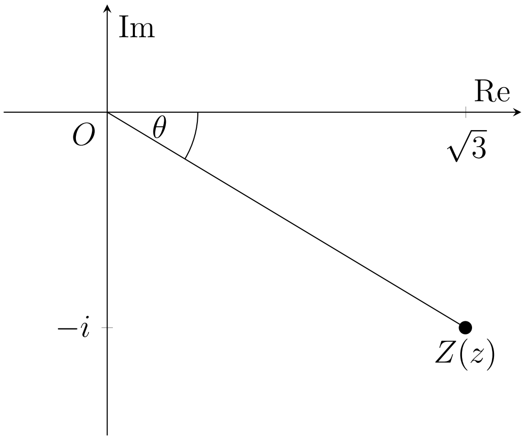
$$\begin{aligned} z &= 6\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right) \\ &= 5.54 - 2.30i \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{6\left(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right) = 5.54 - 2.30i}$$

Problem 5.

Given that $z = \sqrt{3} - i$, find the exact modulus and argument of z. Hence, find the exact modulus and argument of $\frac{1}{z^2}$ and z^{10}.

Solution



We have $r^2 = (\sqrt{3})^2 + (-1)^2 \implies r = 2$ and $\tan t = \frac{-1}{\sqrt{3}} \implies \theta = -\frac{\pi}{6}$.

$$\boxed{|z| = 2, \arg z = -\frac{\pi}{6}}$$

Note that $\left|\frac{1}{z^2}\right| = |z|^{-2} = 2^{-2} = \frac{1}{4}$. Also, $\arg\left(\frac{1}{z^2}\right) = -2\arg z = -2 \cdot -\frac{\pi}{6} = \frac{\pi}{3}$.

$$\boxed{\left|\frac{1}{z^2}\right| = \frac{1}{4},\ \arg\left(\frac{1}{z^2}\right) = \frac{\pi}{3}}$$

Note that$|z^{10}| = |z|^1 0 = 2^1 0 = 1024$. Also, $\arg z^{10} = 10 \arg z = 10 \cdot -\frac{\pi}{6} = -\frac{5}{3}\pi \equiv \frac{\pi}{3}$.

$$\boxed{|z^{10}| = 1024,\ \arg(z^{10}) = \frac{\pi}{3}}$$

Problem 6.

If $\arg\left(z - \frac{1}{2}\right) = \frac{\pi}{5}$, determine $\arg(2z-1)$.

Solution

$$\arg(2z-1) = \arg\left(\frac{1}{2}\left[z - \frac{1}{2}\right]\right) = \arg\left(z - \frac{1}{2}\right) = \frac{\pi}{5}$$

$$\boxed{\arg(2z-1) = \frac{\pi}{5}}$$

Problem 7.

In an Argand diagram, points P and Q represent the complex numbers $z = 1 + i$ and $w = 1 + 2i$ respectively, and O is the origin.

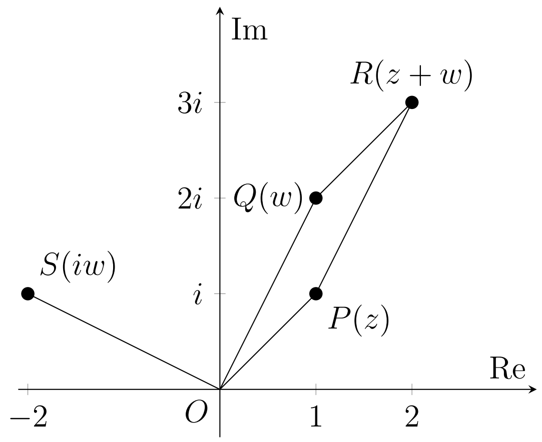
(a) Mark on the Argand diagram the points P and Q, and the points R and S which represent $z + w$ and iw respectively.

(b) What is the geometrical shape of $OPRQ$?

(c) State the angle SOP.

Solution

Part (a)



Part (b)

$$\boxed{OPRQ \text{ is a parallelogram.}}$$

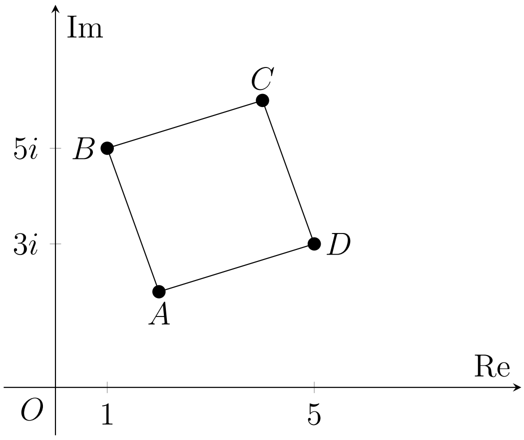
Part (c)

$$\boxed{\angle SOP = \frac{\pi}{2}}$$

Problem 8.

B and D are points in the Argand diagram representing the complex numbers $1 + 5i$ and $5 + 3i$ respectively. Given that BD is a diagonal of the square $ABCD$, calculate the complex numbers represented by A and C.

Solution



Let $A(x + iy)$. Since $AB \perp AD$, we have $b - a = i(d - a)$.

$$\begin{aligned} b - a &= i(d - a) \\ \implies (1 + 5i) - (x + iy) &= i\left[(5 + 3i) - (x + iy)\right] \\ \implies (1 - x) + (5 - y)i &= (-3 + y) + (5 - x)i \\ \implies (x + y) + (y - x)i &= 4 \end{aligned}$$

Comparing real and imaginary parts, we obtain $x = y = 2$. Hence, $A(2 + 2i)$.

Let $C(u + iv)$. Since $CB \perp CD$, we have $d - c = i(b - c)$.

$$\begin{aligned} d - c &= i(b - c) \\ \implies (5 + 3i) - (u + iv) &= i\left[(1 + 5i) - (u + iv)\right] \\ \implies (5 - u) + (3 - v)i &= (-5 + v) + (1 - u)i \\ \implies (u + v) + (v - u)i &= 10 + 2i \end{aligned}$$

Comparing real and imaginary parts, we obtain $u = 4$ and $v = 6$. Hence, $C(4 + 6i)$.

$$\boxed{A(2 + 2i),\ C(4 + 6i)}$$

Problem 9.

(a) Given that $u = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $w = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$, find the modulus and argument of $\frac{u^*}{w^3}$ in exact form.

(b) Let z be the complex number $-1 + i\sqrt{3}$. Find the value of the real number a such that $\arg(z^2 + az) = -\frac{\pi}{2}$.

Solution

Part (a)

Note that $|u| = 2$, $\arg u = \frac{\pi}{6}$, $|w| = 4$ and $\arg w = -\frac{\pi}{3}$.

$$\left|\frac{u^*}{w^3}\right| = \frac{|u^*|}{|w^3|} = \frac{|u|}{|w|^3} = \frac{2}{4^3} = \frac{1}{32}$$

$$\arg\frac{u^*}{w^3} = \arg u^* - \arg w^3 = -\arg u - 3\arg w = -\frac{\pi}{6} - 3 \cdot -\frac{\pi}{3} = \frac{5}{6}\pi$$

$$\boxed{\left|\frac{u^*}{w^3}\right| = \frac{1}{32},\ \arg\frac{u^*}{w^3} = \frac{5}{6}\pi}$$

Part (b)

Since $\arg(z^2 + az) = -\frac{\pi}{2}$, we have that $z^2 + az$ is purely imaginary, with a negative imaginary part. Note that $z^2 = (-1 + i\sqrt{3})^2 = -2 - 2\sqrt{3}i$.

$$\begin{aligned} & \operatorname{Re}\left(z^2 + az\right) = 0 \\ \implies & \operatorname{Re}\left((-2 - 2\sqrt{3}i) + a(-1 + i\sqrt{3})\right) = 0 \\ \implies & -2 - a = 0 \\ \implies & a = -2 \end{aligned}$$

$$\boxed{a = -2}$$

Problem 10.

The complex number w has modulus r and argument θ, where $0 < \theta < \pi/2$, and w^* denotes the conjugate of w. State the modulus and argument of p, where $p = \dfrac{w}{w^*}$. Given that p^5 is real and positive, find the possible values of θ.

Solution

$$\boxed{|p| = 1, \arg p = 2\theta}$$

Since $p^5 = 0$, we have $\arg p^5 = 2\pi n$, where $n \in \mathbb{Z}$. Thus, $\arg p = \dfrac{2\pi n}{5} = 2\theta \implies \theta = \dfrac{\pi n}{5}$. Since $0 < \theta < \dfrac{\pi}{2}$, the possible values of θ are $\dfrac{1}{5}\pi$ and $\dfrac{2}{5}\pi$.

$$\boxed{\theta = \frac{1}{5}\pi, \frac{2}{5}\pi}$$

Problem 11.

The complex number w has modulus $\sqrt{2}$ and argument $-\frac{3}{4}\pi$, and the complex number z has modulus 2 and argument $-\frac{\pi}{3}$. Find the modulus and argument of wz, giving each answer exactly.

By first expressing w and z in the form $x + iy$, find the exact real and imaginary parts of wz.

Hence, show that $\sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Solution

$$|wz| = |w|\,|z| = 2\sqrt{2}$$

$$\arg(wz) = \arg w + \arg z = -\frac{3}{4}\pi - \frac{1}{3}\pi = -\frac{13}{12}\pi \equiv \frac{11}{12}\pi$$

$$\boxed{|wz| = 2\sqrt{2},\ \arg(wz) = \frac{11}{12}\pi}$$

$$w = \sqrt{2}\left[\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right] = \sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -1 - i$$

$$z = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right] = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 1 - \sqrt{3}i$$

$$\implies wz = (-1 - i)(1 - \sqrt{3}i) = (-1 + \sqrt{3} - i - \sqrt{3}) = (-1 - \sqrt{3}) + (\sqrt{3} - 1)i$$

$$\boxed{\operatorname{Re}(wz) = -1 - \sqrt{3},\ \operatorname{Im}(wz) = \sqrt{3} - 1}$$

From the first part, we have that $wz = 2\sqrt{2}\left[\cos\left(\frac{11}{12}\pi\right) + i\sin\left(\frac{11}{12}\pi\right)\right]$. Thus, $\operatorname{Im}(wz) = 2\sqrt{2}\sin\left(\frac{11}{12}\pi\right) = 2\sqrt{2}\sin\frac{\pi}{12}$. Equating the result for $\operatorname{Im}(wz)$ found in the second part, we have $2\sqrt{2}\sin\frac{\pi}{12} = \sqrt{3} - 1 \implies \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

Problem 12.

Given that $\dfrac{5+z}{5-z} = e^{i\theta}$, show that z can be written as $5i\tan\dfrac{\theta}{2}$.

Solution

Note that $\dfrac{5+z}{5-z} = e^{i\theta} \implies 5+z = e^{i\theta}(5-z) \implies z + e^{i\theta}z = 5e^{i\theta} - 5 \implies z = 5\cdot\dfrac{e^{i\theta}-1}{e^{i\theta}+1}$.

$$
\begin{aligned}
z &= 5\cdot\frac{e^{i\theta}-1}{e^{i\theta}+1} \\
&= 5\cdot\frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} \\
&= 5\cdot\frac{\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] + i\left[2\sin(\theta/2)\cos(\theta/2)\right] - \left[\cos^2(\theta/2) + \sin^2(\theta/2)\right]}{\left[\cos^2(\theta/2) - \sin^2(\theta/2)\right] + i\left[2\sin(\theta/2)\cos(\theta/2)\right] + \left[\cos^2(\theta/2) + \sin^2(\theta/2)\right]} \\
&= 5\cdot\frac{-2\sin^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2) + 2i\sin(\theta/2)\cos(\theta/2)} \\
&= 5\cdot\frac{-\sin^2(\theta/2) + i\sin(\theta/2)\cos(\theta/2)}{\cos^2(\theta/2) + i\sin(\theta/2)\cos(\theta/2)} \\
&= 5\cdot\frac{-\tan(\theta/2) + i}{\cot(\theta/2) + i} \\
&= 5\cdot\frac{i^2\tan(\theta/2) + i\tan(\theta/2)\cot(\theta/2)}{\cot(\theta/2) + i} \\
&= 5\cdot\frac{i\tan(\theta/2)\left[i + \cot(\theta/2)\right]}{\cot(\theta/2) + i} \\
&= 5i\tan\frac{\theta}{2}
\end{aligned}
$$

Problem 13.

The polynomial $P(z)$ has real coefficients. The equation $P(z) = 0$ has a root $re^{i\theta}$, where $r > 0$ and $0 < \theta < \pi$.

(a) Write down a second root in terms of r and θ, and hence show that a quadratic factor of $P(z)$ is $z^2 - 2rz\cos\theta + r^2$.

(b) Given that 3 roots of the equation $z^6 = -64$ are $2e^{i\frac{\pi}{6}}$, $2e^{i\frac{\pi}{2}}$ and $2e^{-i\frac{5\pi}{6}}$, express $z^6 + 64$ as a product of three quadratic factors with real coefficients, giving each factor in non-trigonometric form.

(c) Represent all roots of $z^6 = -64$ on an Argand diagram and interpret the geometrical shape formed by joining the roots.

Solution

Part (a)

Since $P(z)$ has real coefficients, $(re^{i\theta})^* = re^{-i\theta}$ is also a root of $P(z)$.

$$\boxed{\text{A second root is } re^{-i\theta}.}$$

$$\begin{aligned} P(z) &= Q(z)(z - re^{i\theta})(z - re^{-i\theta}) \\ &= Q(z)(z^2 - rz(e^{i\theta} + e^{-i\theta}) + r^2 e^{i\theta} e^{-i\theta}) \\ &= Q(z)(z^2 - rz \cdot 2\,\mathrm{Re}\left(e^{i\theta}\right) + r^2) \\ &= Q(z)(z^2 - 2rz\cos\theta + r^2) \end{aligned}$$

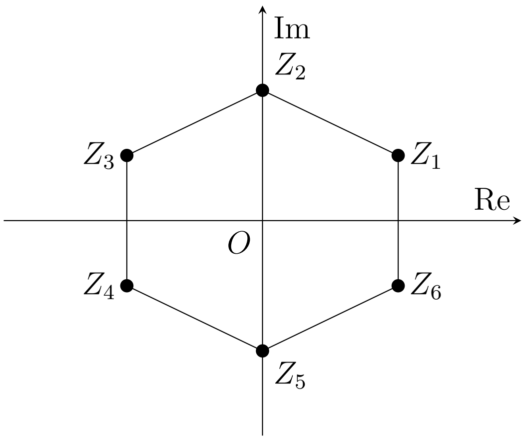
Part (b)

Let $r_1 = r_2 = r_3 = 2$ and $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{2}$ and $\theta_3 = -\frac{5}{6}\pi$.

$$\begin{aligned} z^6 + 64 &= \left(z^2 - 2r_1 z\cos\theta_1 + r_1^2\right)\left(z^2 - 2r_2 z\cos\theta_2 + r_2^2\right)\left(z^2 - 2r_3 z\cos\theta_3 + r_3^2\right) \\ &= \left(z^2 - 4z\cos\left(\frac{\pi}{6}\right) + 4\right)\left(z^2 - 4z\cos\left(\frac{\pi}{2}\right) + 4\right)\left(z^2 - 4z\cos\left(-\frac{5}{6}\pi\right) + 4\right) \\ &= \left(z^2 - 2\sqrt{3}z + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right) \end{aligned}$$

$$\boxed{z^6 + 64 = \left(z^2 - 2\sqrt{3}z + 4\right)\left(z^2 + 4\right)\left(z^2 + 2\sqrt{3}z + 4\right)}$$

Part (c)



$\boxed{\text{The geometrical shape formed is a regular hexagon.}}$