Problem 1.

The equation of a curve is $y = 2x^3 + 3x^2 + 6x + 4$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and hence show that y is increasing for all real values of x.

Solution

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= 2 \cdot 3x^2 + 3 \cdot 2x + 6 \\ &= 6x^2 + 6x + 6\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 6x + 6}$$

Observe that $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^2 + 6x + 6 = 6\left(x + \frac{1}{2}\right)^2 + \frac{18}{4}$. For all $x \in \mathbb{R}$, we have $\left(x + \frac{1}{2}\right)^2 \geq 0$. Hence, $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$. Thus, y is increasing for all real values of x.

Problem 2.

Find, by differentiation, the x-coordinates of all the stationary points on the curve $y = \dfrac{x^3}{(x+1)^2}$ stating, with reasons, the nature of each point.

Solution

$$\begin{aligned} && y &= \frac{x^3}{(x+1)^2} \\ \implies && (x+1)^2 y &= x^3 \\ \implies && (x+1)^2 \cdot y' + y \cdot 2(x+1) &= 3x^2 \end{aligned}$$

For stationary points, $y' = 0$.

$$\begin{aligned} \implies && y \cdot 2(x+1) &= 3x^2 \\ \implies && \frac{x^3}{(x+1)^2} \cdot 2(x+1) &= 33x^2 \\ \implies && \frac{2x^3}{x+1} &= 3x^2 \\ \implies && 2x^3 &= 3x^2(x+1) \\ \implies && 2x^3 &= 3x^3 + 3x^2 \\ \implies && x^3 + 3x^2 &= 0 \\ \implies && x^2(x+3) &= 0 \end{aligned}$$

Hence, $x = 0$ or $x = -3$.

$$\boxed{\text{The } x\text{-coordinates of the stationary points are } x = 0 \text{ and } x = -3.}$$

x	0^-	0	0^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	+ve	0	+ve

Thus, there is a stationary point of inflexion at $x = 0$.

x	$(-3)^-$	-3	$(-3)^+$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	+ve	0	-ve

Thus, there is a maximum point at $x = -3$.

$$\boxed{\begin{array}{l} \text{At } x = 0, \text{ there is a stationary point of inflexion.} \\ \text{At } x = -3, \text{ there is a maximum point.} \end{array}}$$

Problem 3.

Differentiate $f(x) = 8\sin\dfrac{x}{2} - 4x$ with respect to x and deduce that $f(x) < 0$ for $x > 0$.

Solution

$$\begin{aligned} f'(x) &= 8\cos\frac{x}{2}\cdot\frac{1}{2} - \cos x - 4 \\ &= 4\cos\frac{x}{2} - \cos x - 4 \end{aligned}$$

$$\boxed{f'(x) = 4\cos\frac{x}{2} - \cos x - 4}$$

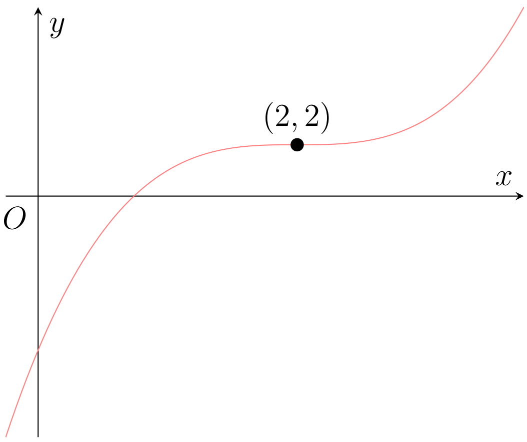
$$\begin{aligned} f'(x) &= 4\cos\frac{x}{2} - \cos x - 4 \\ &= 4\cos\frac{x}{2} - \left(2\cos^2\frac{x}{2} - 1\right) - 4 \\ &= -2\left(\cos\frac{x}{2} - 1\right)^2 - 1 \end{aligned}$$

Observe that for all $x \in \mathbb{R}$, $\left(\cos\dfrac{x}{2} - 1\right)^2 \geq 0$. Hence, $f'(x) < 0$ for all real values of x. Thus, $f(x)$ is strictly decreasing on $\mathbb{R}$.

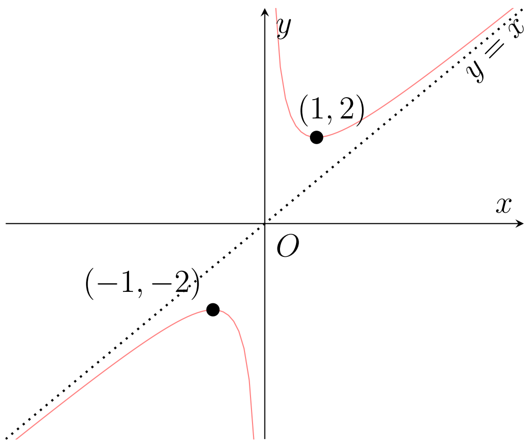
Note that $f(0) = 8\sin 0 - \sin 0 - 4\cdot 0 = 0$. Since $f(x)$ is strictly decreasing, for all $x > 0$, $f(x) < f(0) = 0$.

Problem 4.

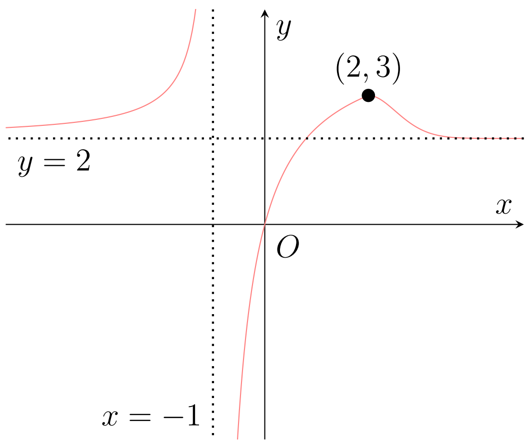
Sketch the graphs of the derivative functions for each of the graphs of the following functions below. In each graph, the point(s) labelled in coordinate form are stationary points.



(a)

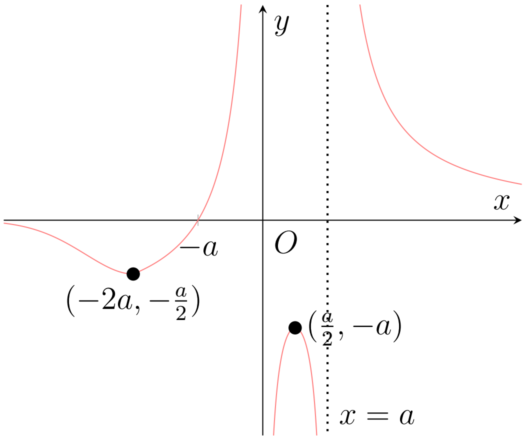


(b)



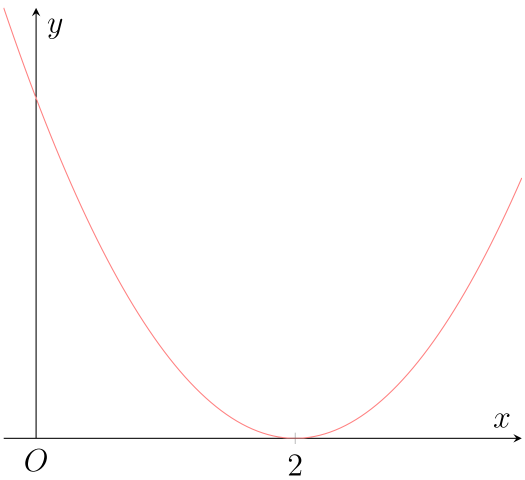
(c)

(d)

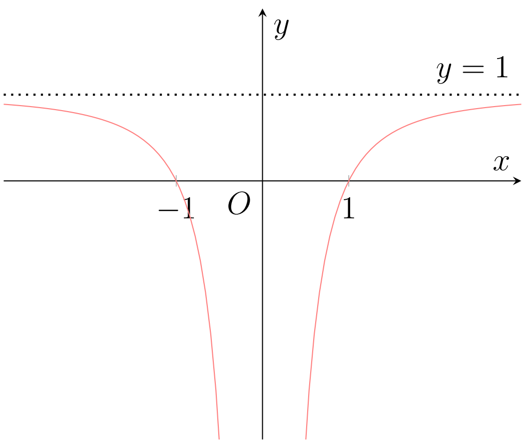


Solution

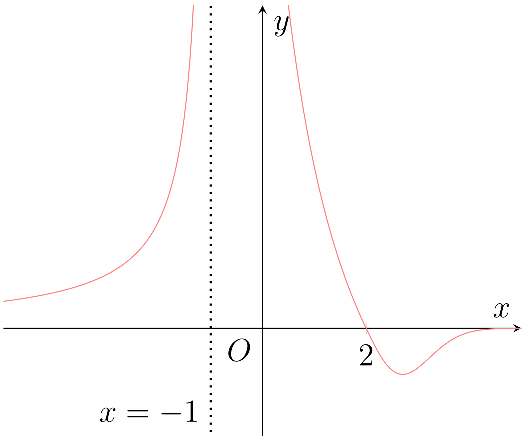
Part (a)



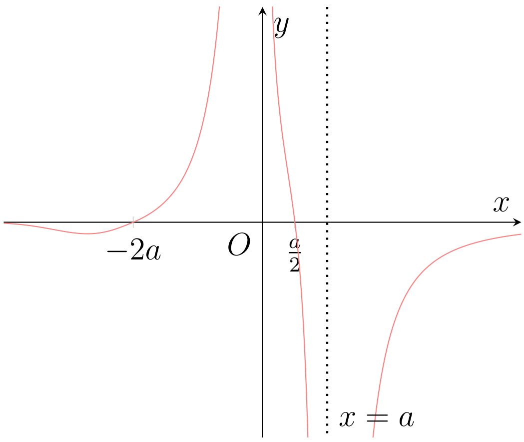
Part (b)



Part (c)



Part (d)



Problem 5.

(a) Given that $y = ax\sqrt{x+2}$ where $a > 0$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$, expressing your answer as a single algebraic fraction. Hence show that the curve $y = ax\sqrt{x+2}$ has only one turning point, and state its coordinates in exact form.

(b) Sketch the graph of $y = f'(x)$, where $f(x) = ax\sqrt{x+2}$, where $a > 0$.

Solution

Part (a)

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= a\left(x \cdot \frac{1}{2\sqrt{x+2}} + \sqrt{x+2}\right)\\ &= a\left(\frac{x}{2\sqrt{x+2}} + \frac{2(x+2)}{2\sqrt{x+2}}\right)\\ &= \frac{a(3x+4)}{2\sqrt{x+2}}\end{aligned}$$

$$\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a(3x+4)}{2\sqrt{x+2}}}$$

Consider the stationary points of $y = ax\sqrt{x+2}$. For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\begin{aligned}&\frac{\mathrm{d}y}{\mathrm{d}x} = 0\\ \implies &\frac{a(3x+4)}{2\sqrt{x+2}} = 0\\ \implies &a(3x+4) = 0\end{aligned}$$

Since $a > 0$, we have $3x + 4 = 0$, whence $x = -\frac{4}{3}$.

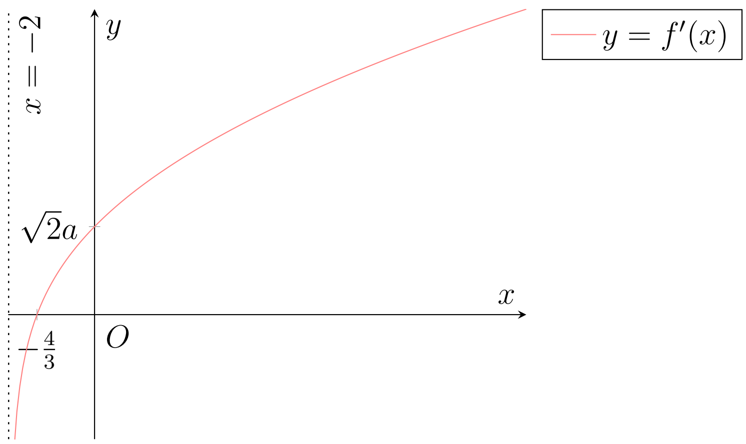
x	$\left(-\frac{4}{3}\right)^-$	$-\frac{4}{3}$	$\left(-\frac{4}{3}\right)^+$
$\frac{\mathrm{d}y}{\mathrm{d}x}$	-ve	0	+ve

Hence, at $x = -\frac{4}{3}$, there is a turning point (minimum point). Thus, $y = ax\sqrt{x+2}$ has only one turning point.

Substituting $x = -\frac{4}{3}$ into $y = ax\sqrt{x+2}$, we see that $y = -\frac{4a}{3}\sqrt{\frac{2}{3}}$. Hence, the coordinate of the turning point is $(-\frac{4}{3}, -\frac{4a}{3}\sqrt{\frac{2}{3}})$.

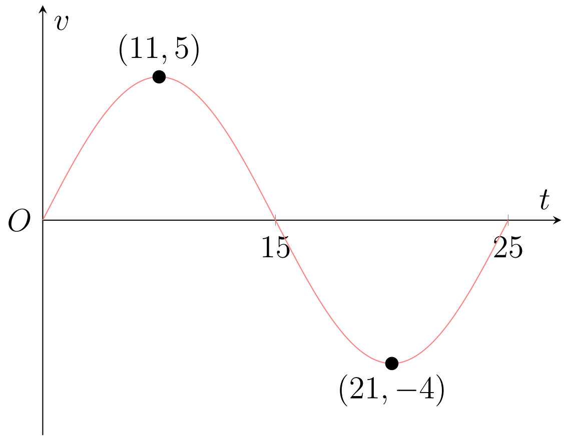
$$\boxed{\left(-\frac{4}{3}, -\frac{4a}{3}\sqrt{\frac{2}{3}}\right)}$$

Part (b)



Problem 6.

A particle P moves along the x-axis. Initially, P is at the origin O. At time t s, the velocity is v ms^{-1} and the acceleration is a ms^{-2}. Below is the velocity-time graph of the particle for $0 \leq t \leq 25$.



(a) Describe the motion of the particle for $0 \leq t \leq 25$.

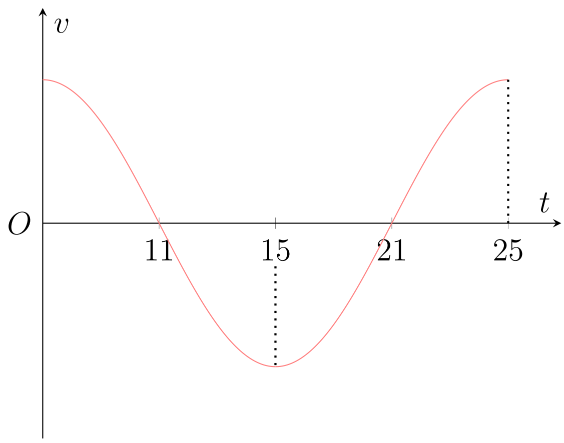
(b) Sketch the acceleration-time graph of the particle P.

Solution

Part (a)

From $t = 0$ to $t = 11$, P speeds up and reaches a top speed of 5 ms^{-1}. From $t = 11$ to $t = 15$, P slows down. At $t = 15$, P is instantaneously at rest. From $t = 15$ to $t = 21$, P speeds up and moves in the opposite direction, reaching a top speed of 4 ms^{-1}. From $t = 21$ to $t = 25$, P slows down. At $t = 25$, P is instantaneously at rest.

Part (b)



Problem 7.

The function f defined by $f(x) = \ln x - 2\left(x - \frac{1}{2}\right)$, where $x \in \mathbb{R}, x > 0$. Find $f'(x)$ and show that the function is decreasing for $x > \frac{1}{2}$. Hence show that for $x > \frac{1}{2}$, $2\left(x - \frac{1}{2}\right) - \ln x > \ln 2$.

Solution

$$f' = \frac{1}{x} - 2$$

When $x > \frac{1}{2}$, $\frac{1}{x} < 2 \implies \frac{1}{x} - 2 < 0$. Thus, $f'(x) < 0$, whence $f(x)$ is decreasing.

Note that $f\left(\frac{1}{2}\right) = \ln \frac{1}{2} - 2\left(\frac{1}{2} - \frac{1}{2}\right) = -\ln 2$. Since $f(x)$ is decreasing for all $x > \frac{1}{2}$, $f(x) < f\left(\frac{1}{2}\right) = -\ln 2 \implies \ln x - 2\left(x - \frac{1}{2}\right) < -\ln 2 \implies 2\left(x - \frac{1}{2}\right) - \ln x > \ln 2$.