Problem 1.

Functions f and g are defined as follows:

$$\begin{aligned} f &\colon x \mapsto (x-3)^2+6, && x\in\mathbb{R},\ x\le 2 \\ g &\colon x \mapsto \ln(x-2), && x\in\mathbb{R},\ x>3 \end{aligned}$$

(a) Show that f^{-1} exists and define f^{-1} in a similar form.

(b) Sketch, on the same diagram, the graphs of f, f^{-1} and ff^{-1}.

(c) Find fg and gf if they exist, and find their ranges (where applicable).

Solution

Part (a)

Note that $f' = 2(x-3) < 0$ for all $x \le 2$. Thus, f is strictly decreasing. Since f is also continuous, f is one-one. Thus, f^{-1} exists.

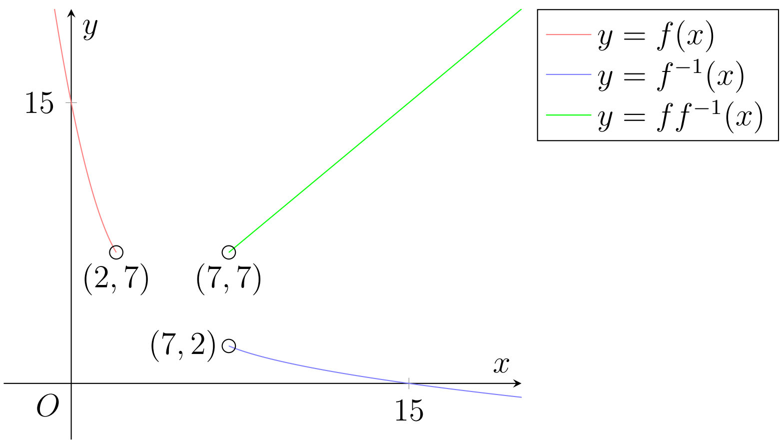
Let $y = f(x) \implies x = f^{-1}(y)$.

$$\begin{aligned} && y &= f(x) \\ \implies && y &= (x-3)^2+6 \\ \implies && (x-3)^2 &= y-6 \\ \implies && x-3 &= -\sqrt{y-6} && (\text{rej. } x-3=\sqrt{y-6} \because x-3<0) \\ \implies && x &= 3-\sqrt{y-6} \end{aligned}$$

Hence, $f^{-1}(x) = 3-\sqrt{x-6}$. Observe that $D_{f^{-1}} = R_f = [f(2),\infty) = [7,\infty)$.

$$\boxed{f^{-1}\colon x\mapsto 3-\sqrt{x-6},\ x\in\mathbb{R},\ x\ge 7}$$

Part (b)



Part (c)

Note that $R_g = (0, \infty)$ and $D_f = (-\infty, 2]$. Hence, $R_g \nsubseteq D_f$. Thus, fg does not exist. Note that $R_f = [7, \infty)$ and $D_g = (3, \infty)$. Hence, $R_f \subseteq D_g$. Thus, gf exists.

Since $\ln x$ is a strictly increasing function, we have that g is also strictly increasing. Hence, $R_{gf} = [\,\ln(7-2), \infty) = [\ln 5, \infty)$.

$$\boxed{R_{gf} = [\ln 5, \infty)}$$

Problem 2.

The function f is defined as follows:

$$f\colon x \mapsto \frac{1}{x^2-1}, \qquad x \in \mathbb{R},\, x \neq -1,\, x \neq 1$$

(a) Sketch the graph of $y = f(x)$.

(b) If the domain of f is further restricted to $x \geq k$, state with a reason the least value of k for which the function f^{-1} exists.

In the rest of the question, the domain of f is $x \in \mathbb{R}$, $x \neq -1$, $x \neq 1$, as originally defined.

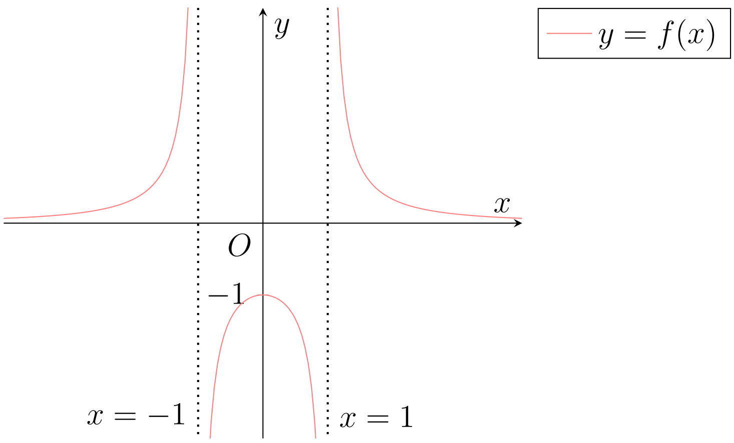
The function g is defined as follows:

$$g\colon x \mapsto \frac{1}{x-3}, \qquad x \in \mathbb{R},\, x \neq 2,\, x \neq 3, x \neq 4$$

(c) Find the range of fg.

Solution

Part (a)



Part (b)

If the domain of f is further restricted to $x \geq 0$, f would pass the horizontal line test, whence f^{-1} would exist.

$$\boxed{\min k = 0}$$

Part (c)

Observe that $R_g = \mathbb{R} \setminus \{g(2), g(4)\} = \mathbb{R} \setminus \{-1, 1\}$. Hence, $R_{fg} = R_f = \mathbb{R} \setminus (-1, 0\,]$.

$$\boxed{R_{fg} = \mathbb{R} \setminus (-1, 0\,]}$$

Problem 3.

The function f is defined by

$$f \colon x \mapsto \frac{x}{x^2-1}, \qquad x \in \mathbb{R},\, x \neq -1,\, x \neq 1$$

(a) Explain why f does not have an inverse.

(b) The function f has an inverse if the domain is restricted to $x \leq k$. State the largest value of k.

The function g is defined by

$$g \colon x \mapsto \ln x - 1, \qquad x \in \mathbb{R},\, 0 < x < 1$$

(c) Find an expression for $h(x)$ for each of the following cases:

(i) $gh(x) = x$

(ii) $hg(x) = x^2 + 1$

Solution

Part (a)

Observe that $f\left(\frac{1}{2}\right) = -\frac{2}{3}$ and $f(-2) = -\frac{2}{3}$. Hence, $f\left(\frac{1}{2}\right) = f(-2)$. Since $\frac{1}{2} \neq -2$, f is not one-one. Thus, f does not have an inverse.

Part (b)

$$\boxed{\max k = 0}$$

Part (c)

Subpart (i)

Note that $gh(x) = x \implies h(x) = g^{-1}(x)$. Hence, consider $y = g(x) \implies x = h(y)$.

$$\begin{aligned} & y = g(x) \\ \implies & y = \ln x - 1 \\ \implies & \ln x = y + 1 \\ \implies & x = e^{y+1} \end{aligned}$$

Hence, $h(x) = e^{x+1}$.

$$\boxed{h(x) = e^{x+1}}$$

Subpart (ii)

Let $h = h_2 \circ h_1$ such that $h_1 g(x) = x \implies h_1(x) = g^{-1}(x) \implies h_1(x) = e^{x+1}$.

$$
\begin{aligned}
hg(x) &= x^2 + 1 \\
\implies h_2 h_1 g(x) &= x^2 + 1 \\
\implies \quad h_2(x) &= x^2 + 1
\end{aligned}
$$

Hence, $h(x) = h_2 h_1(x) = h_2(e^{x+1}) = \left(e^{x+1}\right)^2 + 1 = e^{2x+2} + 1$

$$\boxed{h(x) = e^{2x+2} + 1}$$