Problem 1.

The equation of the plane Π_1 is $y+z=0$ and the equation of the line l is $\dfrac{x-2}{2} = \dfrac{y-2}{-1} = \dfrac{z-2}{3}$. Find

(a) the position vector of the point of intersection of l and Π_1,

(b) the length of the perpendicular from the origin to l,

(c) the Cartesian equation for the plane Π_2 which contains l and the origin,

(d) the acute angle between the planes Π_1 and Π_2, giving your answer correct to the nearest $0.1°$.

Solution

Note that Π_1 has equation $\mathbf{r} \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} = 0$ and l has equation $\mathbf{r} = \begin{pmatrix}5\\2\\2\end{pmatrix} + \lambda \begin{pmatrix}2\\-1\\3\end{pmatrix}$, $\lambda \in \mathbb{R}$.

Part (a)

Let P be the point of intersection of Π_1 and l. Then $\overrightarrow{OP} = \begin{pmatrix}5\\2\\2\end{pmatrix} + \lambda \begin{pmatrix}2\\-1\\3\end{pmatrix}$ for some $\lambda \in \mathbb{R}$. Also, $\overrightarrow{OP} \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} = 0$.

$$\begin{aligned}
& \left[\begin{pmatrix}5\\2\\2\end{pmatrix} + \lambda \begin{pmatrix}2\\-1\\3\end{pmatrix}\right] \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} = 0 \\
\implies & \begin{pmatrix}5\\2\\2\end{pmatrix} \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} + \lambda \begin{pmatrix}2\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}0\\1\\1\end{pmatrix} = 0 \\
\implies & 4 + 2\lambda = 0 \\
\implies & \lambda = -2 \\
\implies & \overrightarrow{OP} = \begin{pmatrix}5\\2\\2\end{pmatrix} - 2\begin{pmatrix}2\\-1\\3\end{pmatrix} \\
& = \begin{pmatrix}1\\4\\-4\end{pmatrix}
\end{aligned}$$

$$\boxed{\overrightarrow{OP} = \begin{pmatrix}1\\4\\-4\end{pmatrix}}$$

Part (b)

$$\begin{aligned}\text{Length} &= \left|\begin{pmatrix}5\\2\\2\end{pmatrix} \times \begin{pmatrix}2\\-1\\3\end{pmatrix}\right| \Big/ \left|\begin{pmatrix}2\\-1\\3\end{pmatrix}\right| \\ &= \frac{1}{\sqrt{14}}\left|\begin{pmatrix}8\\-11\\-9\end{pmatrix}\right| \\ &= \frac{\sqrt{266}}{\sqrt{14}} \\ &= \sqrt{19}\end{aligned}$$

$$\boxed{\text{The perpendicular distance from the origin to } l \text{ is } \sqrt{19} \text{ units.}}$$

Part (c)

Observe that Π_2 is parallel to $\begin{pmatrix}5\\2\\2\end{pmatrix}$ and $\begin{pmatrix}2\\-1\\3\end{pmatrix}$. Thus, $\mathbf{n} = \begin{pmatrix}5\\2\\2\end{pmatrix} \times \begin{pmatrix}2\\-1\\3\end{pmatrix} = \begin{pmatrix}8\\-11\\-9\end{pmatrix}$. Since Π_2 contains the origin, $d = 0$. Hence, Π_2 has vector equation $\mathbf{r} \cdot \begin{pmatrix}8\\-11\\-9\end{pmatrix} = 0$, which translates to $8x - 11y - 9z = 0$.

$$\boxed{\Pi_2 : 8x - 11y - 9z = 0}$$

Part (d)

Let the acute angle be θ.

$$\begin{aligned}\cos\theta &= \frac{\left|\begin{pmatrix}0\\1\\1\end{pmatrix} \cdot \begin{pmatrix}8\\-11\\-9\end{pmatrix}\right|}{\left|\begin{pmatrix}0\\1\\1\end{pmatrix}\right|\left|\begin{pmatrix}8\\-11\\-9\end{pmatrix}\right|} \\ &= \frac{20}{\sqrt{2}\sqrt{266}} \\ \implies \qquad \theta &= 29.9^\circ \text{ (1 d.p.)}\end{aligned}$$

$$\boxed{\theta = 29.9^\circ}$$

Problem 2.

The plane Π_1 has equation $\mathbf{r} \cdot (-\mathbf{i} + 2\mathbf{k}) = -4$ and the points A and P have position vectors $4\mathbf{i}$ and $\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ respectively, where $\alpha \in \mathbb{R}$.

(a) Show that A lies on Π_1, but P does not.

(b) Find, in terms of α, the position vector of N, the foot of perpendicular of P on Π_1.

The plane Π_2 contains the points A, P and N.

(c) Show that the equation of Π_2 is $\mathbf{r} \cdot (2\alpha\mathbf{i} + 5\mathbf{j} + \alpha\mathbf{k}) = 8\alpha$ and write down the equation of l, the line of the intersection of Π_1 and Π_2.

The plane Π_3 has equation $\mathbf{r} \cdot (\mathbf{i} + \mathbf{j} + 2\mathbf{k}) = 4$.

(d) By considering l, or otherwise, find the value of α for which the three planes intersect in a line.

Solution

Note that $\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -4$, $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{OP} = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix}$.

Part (a)

$$\overrightarrow{OA} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -4$$

Hence, A lies on Π_1.

$$\overrightarrow{OP} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = 1 \neq -4$$

Hence, P does not lie on Π_1.

Part (b)

Note that $\overrightarrow{NP} = \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$, and $\overrightarrow{ON} \cdot \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = -4$.

$$\begin{aligned} && \overrightarrow{NP} &= \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\ \implies && \overrightarrow{OP} - \overrightarrow{ON} &= \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \\ \implies && \begin{pmatrix} 1 \\ \alpha \\ 1 \end{pmatrix} - \overrightarrow{ON} &= \lambda \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&\implies & \left[\begin{pmatrix}1\\\alpha\\1\end{pmatrix} - \overrightarrow{ON}\right] \cdot \begin{pmatrix}-1\\0\\2\end{pmatrix} &= \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\2\end{pmatrix} \\
&\implies & \begin{pmatrix}1\\\alpha\\1\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\2\end{pmatrix} - (-4) &= \lambda \begin{pmatrix}-1\\0\\2\end{pmatrix} \cdot \begin{pmatrix}-1\\0\\2\end{pmatrix} \\
&\implies & 5 &= 5\lambda \\
&\implies & \lambda &= 1
\end{aligned}$$

Hence, $\overrightarrow{NP} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$, whence $\overrightarrow{ON} = \overrightarrow{OP} - \overrightarrow{NP} = \begin{pmatrix}2\\\alpha\\-1\end{pmatrix}$.

$$\boxed{\overrightarrow{ON} = \begin{pmatrix}2\\\alpha\\-1\end{pmatrix}}$$

Part (c)

Note that Π_2 is parallel to $\overrightarrow{NP} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$ and $\overrightarrow{AN} = \begin{pmatrix}2\\\alpha\\-1\end{pmatrix} - \begin{pmatrix}4\\0\\0\end{pmatrix} = \begin{pmatrix}-2\\\alpha\\-1\end{pmatrix}$. Hence, $\mathbf{n} = \begin{pmatrix}-1\\0\\2\end{pmatrix} \times \begin{pmatrix}-2\\\alpha\\-1\end{pmatrix} = -\begin{pmatrix}2\alpha\\5\\\alpha\end{pmatrix} \implies d = \begin{pmatrix}4\\0\\0\end{pmatrix} \cdot \begin{pmatrix}2\alpha\\5\\\alpha\end{pmatrix} = 8\alpha$. Thus, Π_2 has vector equation $\mathbf{r} \cdot \begin{pmatrix}2\alpha\\5\\\alpha\end{pmatrix} = 8\alpha$ which translates to $\mathbf{r} \cdot (2\alpha\mathbf{i} + 5\mathbf{j} + \alpha\mathbf{k}) = 8\alpha$.

$$\boxed{l : \begin{pmatrix}4\\0\\0\end{pmatrix} + \mu \begin{pmatrix}-2\\\alpha\\-1\end{pmatrix},\ \mu \in \mathbb{R}}$$

Part (d)

If the three planes intersect in a line, they must intersect at l. Hence, l lies on Π_3.

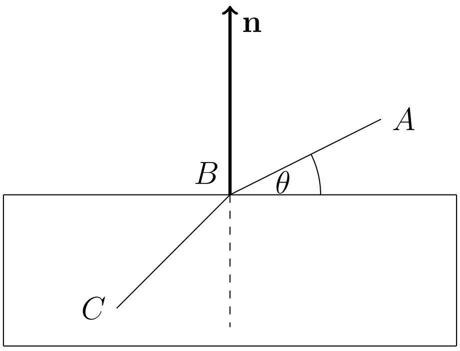
$$\begin{aligned}
& & \left[\begin{pmatrix}4\\0\\0\end{pmatrix} + \mu \begin{pmatrix}-2\\\alpha\\-1\end{pmatrix}\right] \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} &= 4 \\
&\implies & \begin{pmatrix}4\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} + \mu \begin{pmatrix}-2\\\alpha\\-1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\2\end{pmatrix} &= 4 \\
&\implies & 4 + (\alpha - 4)\mu &= 4 \\
&\implies & (\alpha - 4)\mu &= 0
\end{aligned}$$

Hence, $\alpha = 4$.

$$\boxed{\alpha = 4}$$

Problem 3.

When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1, 2, 2)$ and enters a glass object at point $B(0, 0, 2)$. The surface of the glass object is a plane with normal vector $\mathbf{n}$. The diagram shows a cross-section of the glass object in the plane of the light ray and $\mathbf{n}$.



(a) Find a vector equation of the line AB.

The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the plane.

(b) Calculate the value of θ, giving your answer in degrees.

The line BC makes an angle of 45° with the normal to the plane, and BC is parallel to the unit vector $\begin{pmatrix} -2/3 \\ p \\ q \end{pmatrix}$.

(c) By considering a vector perpendicular to the plane containing the light ray and $\mathbf{n}$, or otherwise, find the values of p and q.

The light ray leaves the glass object through a plane with equation $3x + 3z = -4$.

(d) Find the exact thickness of the glass object, taking one unit as one cm.

(e) Find the exact coordinates of the point at which the light ray leaves the glass object.

Solution

Let Π_G be the plane representing the surface of the glass object.

Part (a)

Note that $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

$$\boxed{l_{AB} : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \ \lambda \in \mathbb{R}}$$

Part (b)

Observe that Π_G has equation $\mathbf{r} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} = 2$. Hence,

$$\begin{aligned}\sin\theta &= \frac{\left|\begin{pmatrix}1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\0\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\0\\1\end{pmatrix}\right|\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\right|} \\ &= \frac{1}{\sqrt{2}\sqrt{5}} \\ \implies \qquad \theta &= 71.6^\circ \text{ (1 d.p.)}\end{aligned}$$

$$\boxed{\theta = 71.6^\circ}$$

Part (c)

Since line BC makes an angle of 45° with $\mathbf{n}_G$,

$$\begin{aligned}\sin 45^\circ &= \frac{\left|\begin{pmatrix}1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}-2/3\\p\\q\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\0\\1\end{pmatrix}\right|\left|\begin{pmatrix}-2/3\\p\\q\end{pmatrix}\right|} \\ \implies \qquad \frac{1}{\sqrt{2}} &= \frac{\left|q - \frac{2}{3}\right|}{\sqrt{2} \cdot 1} \\ \implies \left|q - \frac{2}{3}\right| &= 1\end{aligned}$$

Hence, $q = -\frac{1}{3}$. Note that we reject $q = \frac{5}{3}$ since $\begin{pmatrix}-2/3\\p\\q\end{pmatrix}$ is a unit vector, which implies that $|q| \leq 1$.

Let Π_L be the plane containing the light ray. Note that Π_L is parallel to $\overrightarrow{AB}$ and $\overrightarrow{BC}$. Hence, $\mathbf{n}_L = \begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}-2/3\\p\\q\end{pmatrix} = \begin{pmatrix}2q\\-q\\p+4/3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}6q\\-3q\\3p+4\end{pmatrix}$. Since Π_L contains $\mathbf{n}_G$, we have that $\mathbf{n}_L \perp \mathbf{n}_G$, whence $\mathbf{n}_L \cdot \mathbf{n}_G = 0$.

$$\begin{aligned}\begin{pmatrix}6q\\-3q\\3p+4\end{pmatrix} \cdot \begin{pmatrix}1\\0\\1\end{pmatrix} &= 0 \\ \implies \qquad 6q + 3p + 4 &= 0\end{aligned}$$

$$\begin{aligned}\implies \quad 6\cdot -\frac{1}{3} + 3p + 4 &= 0\\ \implies \quad p &= -\frac{2}{3}\end{aligned}$$

$$\boxed{p = -\frac{2}{3},\ q = -\frac{1}{3}}$$

Part (d)

Let Π'_G be the plane with equation $3x + 3z = -4$. Observe that Π_G is parallel to Π'_G. Also note that $\left(-\frac{4}{3}, 0, 0\right)$ is a point on Π'_G. Hence, the distance between Π_G and Π'_G is given by

$$\text{Distance} = \left| 2 - \begin{pmatrix}-4/3\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right| \Big/ \left|\begin{pmatrix}1\\0\\1\end{pmatrix}\right| = \frac{10/3}{\sqrt{2}} = \frac{10}{3\sqrt{2}}$$

$$\boxed{\text{The distance between the two planes is } \frac{10}{3\sqrt{2}} \text{ cm.}}$$

Part (e)

Observe that $\begin{pmatrix}-2/3\\p\\q\end{pmatrix} = \begin{pmatrix}-2/3\\-2/3\\-1/3\end{pmatrix} = -\frac{1}{3}\begin{pmatrix}2\\2\\1\end{pmatrix}$, whence the line BC has equation $\mathbf{r} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\1\end{pmatrix}$, $\mu \in \mathbb{R}$. Let P be the intersection between line BC and Π'_G. Note that $\overrightarrow{OP} = \begin{pmatrix}0\\0\\2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\1\end{pmatrix}$ for some $\mu \in \mathbb{R}$, and $\overrightarrow{OP}\cdot\begin{pmatrix}3\\0\\3\end{pmatrix} = -4$.

$$\begin{aligned}\left[\begin{pmatrix}0\\0\\2\end{pmatrix} + \mu\begin{pmatrix}2\\2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\0\\3\end{pmatrix} &= -4\\ \implies \begin{pmatrix}0\\0\\2\end{pmatrix}\cdot\begin{pmatrix}3\\0\\3\end{pmatrix} + \mu\begin{pmatrix}2\\2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\0\\3\end{pmatrix} &= -4\\ \implies 6 - 9\mu &= -4\\ \implies \mu &= -\frac{10}{9}\end{aligned}$$

Hence, $\overrightarrow{OP} = \begin{pmatrix}0\\0\\2\end{pmatrix} - \frac{10}{9}\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-20/9\\-20/9\\8/9\end{pmatrix}$.

$$\boxed{\text{The coordinates are } \left(-\frac{20}{9}, -\frac{20}{9}, \frac{8}{9}\right).}$$