Problem 1.

Find $\sum_{r=0}^{n}\left(n^2+1-3r\right)$ in terms of n, giving your answer in factorized form.

Solution

$$\begin{aligned}\sum_{r=0}^{n}\left(n^2+1-3r\right) &= (n+1)(n^2+1)+3\cdot\frac{n(n+1)}{2}\\ &= (n+1)\left(n^2+\frac{3}{2}n+1\right)\\ &= \frac{1}{2}(n+1)(2n^2+3n+2)\end{aligned}$$

$$\boxed{\sum_{r=0}^{n}\left(n^2+1-3r\right) = \frac{1}{2}(n+1)(2n^2+3n+2)}$$

Problem 2.

Given that $\sum_{k=1}^{n} k!\left(k^2+1\right) = (n+1)!\,n$, find $\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right)$.

Solution

$$\begin{aligned}
\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right) &= \sum_{k-1=1}^{k-1=n-1}((k-1)+1)!\left((k-1)^2+2(k-1)+2\right) \\
&= \sum_{k=2}^{n} k!\left(k^2+1\right) \\
&= \sum_{k=1}^{n} k!\left(k^2+1\right) - 1!\left(1^2+1\right) \\
&= (n+1)!\,n-2
\end{aligned}$$

$$\boxed{\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right) = (n+1)!\,n-2}$$

Problem 3.

Given that $\displaystyle\sum_{r=1}^{n} = \frac{1}{6}n(n+1)(2n+1)$, find $\displaystyle\sum_{r=N+1}^{2N}\left(7^{r+1}+3r^2\right)$ in terms of N, simplifying your answer.

Solution

$$\begin{aligned}
\sum_{r=N+1}^{2N}\left(7^{r+1}+3r^2\right) &= \sum_{r=N+1}^{2N} 7^{r+1} + 3\sum_{r=N+1}^{2N} r^2 \\
&= \frac{7^{(N+1)+1}(7^N-1)}{7-1} + 3\left(\sum_{r=1}^{2N} r^2 - \sum_{r=1}^{N} r^2\right) \\
&= \frac{7^{N+2}(7^N-1)}{6} + 3\left(\frac{1}{6}(2N)(2N+1)(2\cdot 2N+1) - \frac{1}{6}N(N+1)(2N+1)\right) \\
&= \frac{7^{N+2}(7^N-1)}{6} + \frac{1}{2}\left(2N(2N+1)(4N+1) - N(N+1)(2N+1)\right) \\
&= \frac{7^{N+2}(7^N-1)}{6} + \frac{1}{2}N(2N+1)(2\cdot(4N+1)-(N+1)) \\
&= \frac{7^{N+2}(7^N-1)}{6} + \frac{1}{2}N(2N+1)(7N+1)
\end{aligned}$$

$$\boxed{\sum_{r=N+1}^{2N}\left(7^{r+1}+3r^2\right) = \frac{7^{N+2}(7^N-1)}{6} + \frac{1}{2}N(2N+1)(7N+1)}$$

Problem 4.

Let $f(r) = \dfrac{3}{r-1}$.

(a) Show that $f(r+1) - f(r) = -\dfrac{3}{r(r-1)}$.

(b) Hence, find in terms of N, the sum of the series $S_N = \displaystyle\sum_{r=2}^{N} \frac{1}{r(r-1)}$.

(c) Explain why $\displaystyle\sum_{r=2}^{\infty} \frac{1}{r(r-1)}$ is a convergent series, and find the value of the sum to infinity.

(d) Using the result from part (b), find $\displaystyle\sum_{r=2}^{N} \frac{1}{r(r+1)}$.

Solution

Part (a)

$$\begin{aligned}
f(r+1) - f(r) &= \frac{3}{(r+1)-1} - \frac{3}{r-1} \\
&= \frac{3(r-1) - 3r}{r(r-1)} \\
&= -\frac{3}{r(r-1)}
\end{aligned}$$

Part (b)

$$\begin{aligned}
S_N = \sum_{r=2}^{N} \frac{1}{r(r-1)} &= -\frac{1}{3} \sum_{r=2}^{N} -\frac{3}{r(r-1)} \\
&= -\frac{1}{3}\left(\sum_{r=2}^{N} f(r+1) - \sum_{r=2}^{N} f(r)\right) \\
&= -\frac{1}{3}\left(\sum_{r=3}^{N+1} f(r) - \sum_{r=2}^{N} f(r)\right) \\
&= -\frac{1}{3}\left(\left(\sum_{r=3}^{N} f(r) + f(N+1)\right) - \left(f(2) + \sum_{r=3}^{N} f(r)\right)\right) \\
&= -\frac{1}{3}\left(f(N+1) - f(2)\right) \\
&= -\frac{1}{3}\left(\frac{3}{N+1-1} - \frac{3}{2-1}\right) \\
&= 1 - \frac{1}{N}
\end{aligned}$$

$$\boxed{S_N = 1 - \frac{1}{N}}$$

Part (c)

Consider $\lim_{n\to\infty} S_n$.

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty}\left(1 - \frac{1}{N}\right) \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

Since 1 is a constant, S_N is a convergent series.

$$\boxed{\text{The value of the sum to infinity is 1.}}$$

Part (d)

$$\begin{aligned}
\sum_{r=2}^{N} \frac{1}{r(r+1)} &= \sum_{r=3}^{N+1} \frac{1}{(r-1)r} \\
&= \sum_{r=2}^{N} \frac{1}{r(r-1)} - \frac{1}{2(2-1)} + \frac{1}{(N+1)N} \\
&= 1 - \frac{1}{N} - \frac{1}{2} + 1\frac{1}{N(N+1)} \\
&= \frac{1}{2} + \frac{1-(N+1)}{N(N+1)} \\
&= \frac{1}{2} - \frac{N}{N(N+1)} \\
&= \frac{1}{2} - \frac{1}{N+1}
\end{aligned}$$

$$\boxed{\sum_{r=2}^{N} \frac{1}{r(r+1)} = \frac{1}{2} - \frac{1}{N+1}}$$

Problem 5.

Sketch the graph of $y = \dfrac{4\lambda - x^2}{x^2 + \lambda}$ in each of the following two cases:

(a) $\lambda > 0$,

(b) $\lambda < 0$.

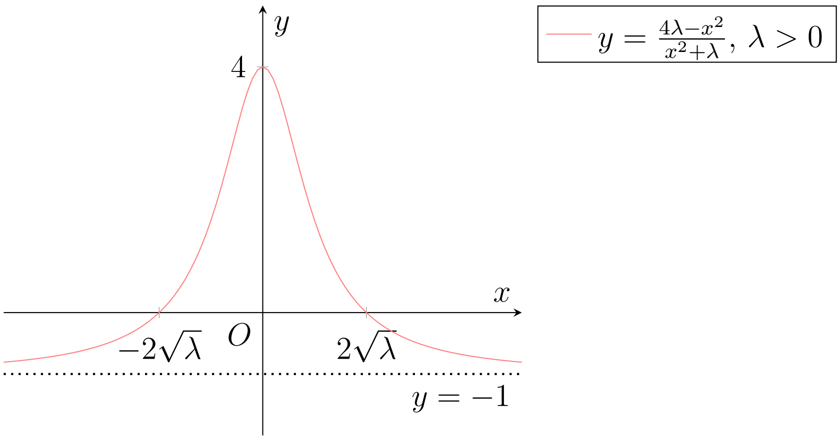
By using your graph in part (b) and considering a suitable graph whose Cartesian equation is to be stated, find the positive value of h such that the equation

$$\left(\frac{4\lambda - x^2}{hx^2 + h\lambda}\right)^2 = 1 + \frac{x^2}{\lambda}$$

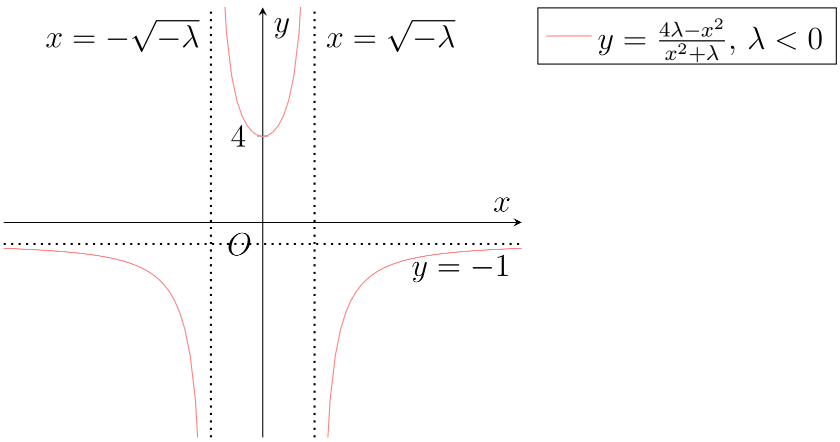
where $\lambda < 0$, has only one real root. State the value of the real root for this value of h.

Solution

Part (a)



Part (b)

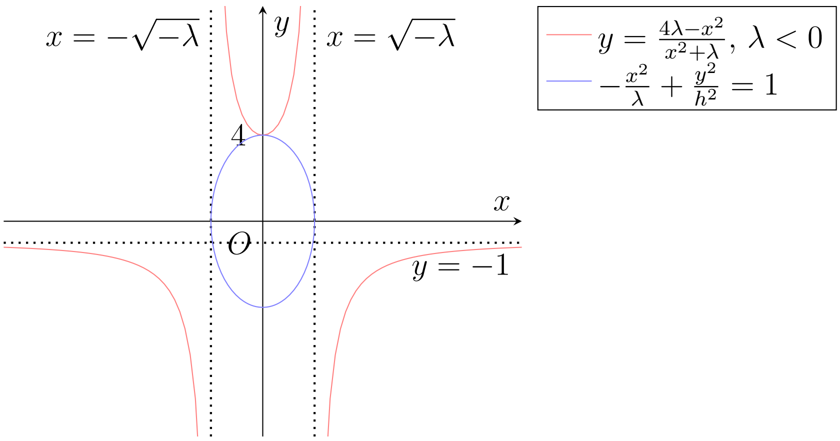


Consider $\left(\dfrac{4\lambda - x^2}{hx^2 + h\lambda}\right)^2 = 1 + \dfrac{x^2}{\lambda}$.

$$
\begin{aligned}
\left(\frac{4\lambda - x^2}{hx^2 + h\lambda}\right)^2 &= 1 + \frac{1}{x^2}\lambda \\
\implies \left(\frac{1}{h} \cdot \frac{4\lambda - x^2}{x^2 + \lambda}\right)^2 &= 1 + \frac{x^2}{\lambda} \\
\implies \left(\frac{1}{h} \cdot y\right)^2 &= 1 + \frac{x^2}{\lambda} \\
\implies \frac{y^2}{h^2} &= 1 + \frac{x^2}{\lambda} \\
\implies -\frac{x^2}{\lambda} + \frac{y^2}{h^2} &= 1 \\
\implies -\frac{x^2}{\sqrt{\lambda}^2} + \frac{y^2}{h^2} &= 1
\end{aligned}
$$

We hence plot an ellipse centred at the origin with horizontal radius $\sqrt{\lambda}$ and vertical radius h.

For only one real root, the ellipse must meet the above graph at exactly one point. Hence, $h = 4$ and the corresponding root is $x = 0$.



$\boxed{h = 4,\ x = 0}$