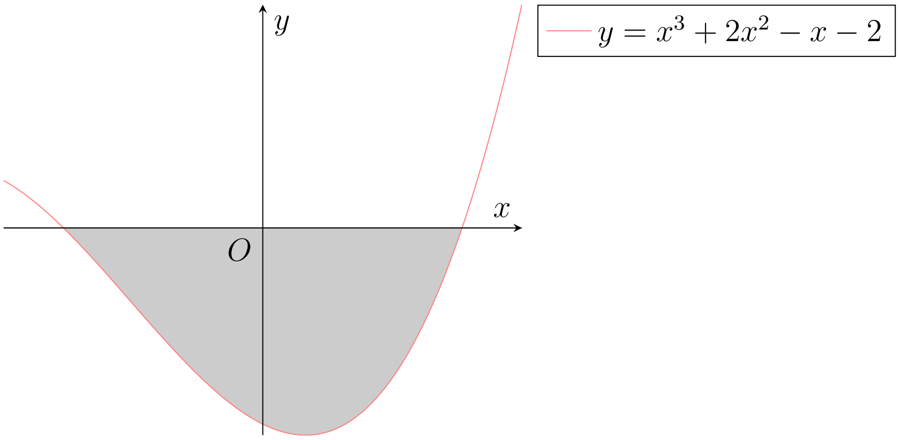
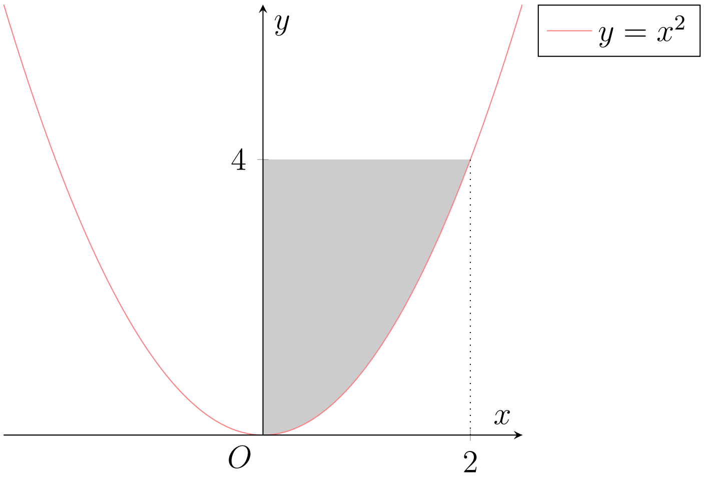
Problem 1.

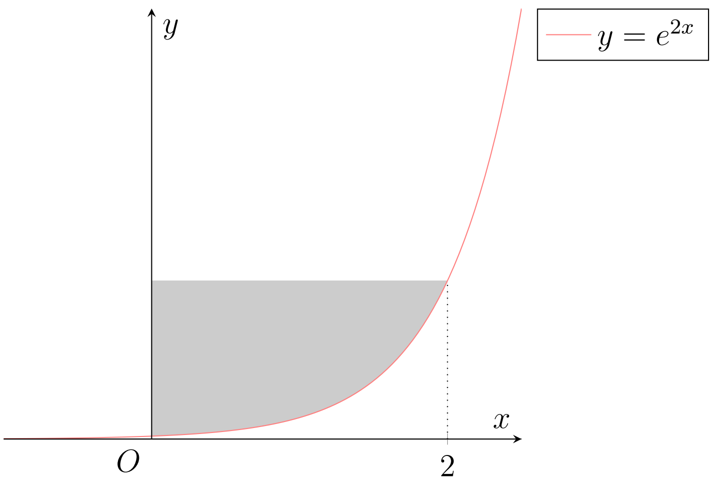
Write down the integral for the area of the shaded region for each of the figure below and use the GC to evaluate it, to 3 significant figures.



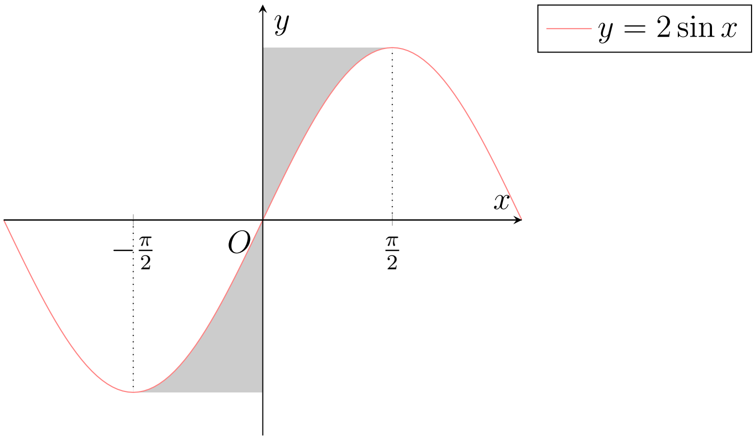
(a)



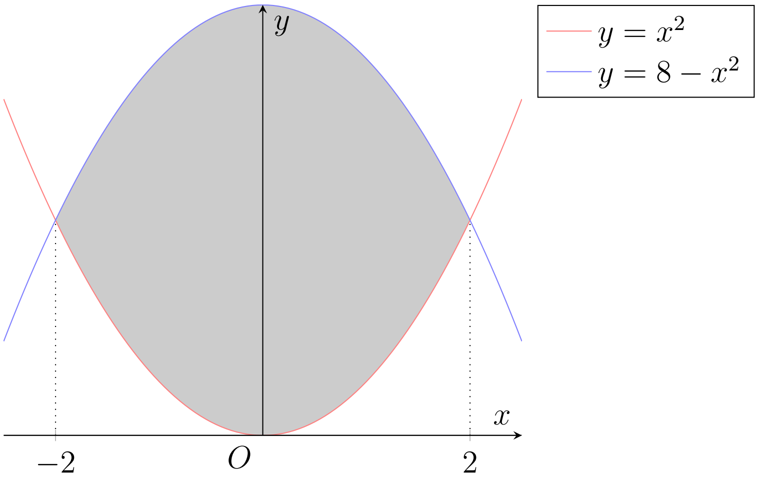
(b)



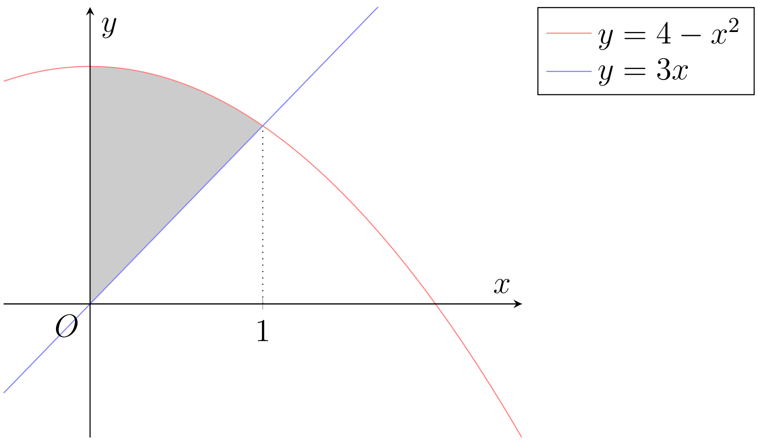
(c)



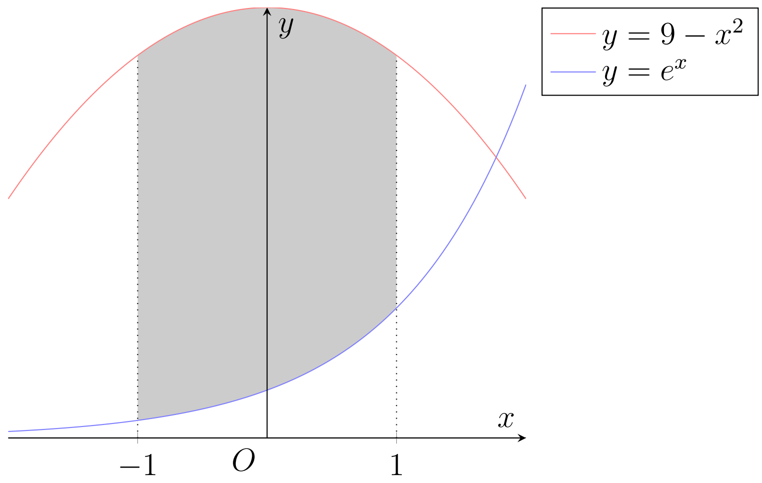
(d)



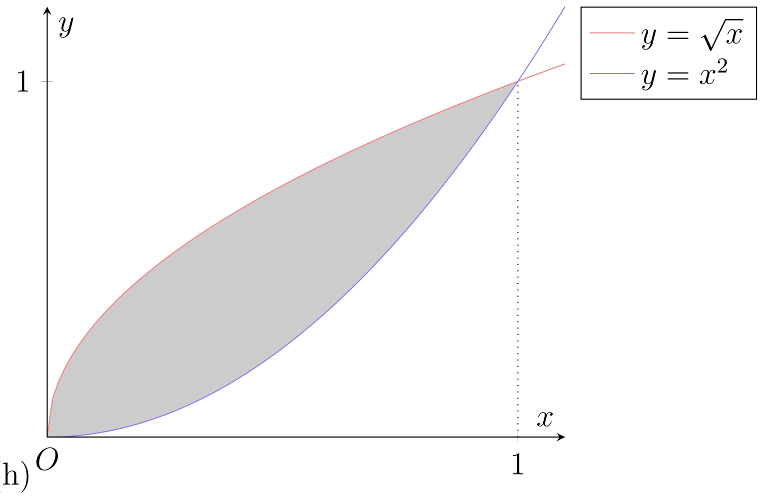
(e)



(f)



(g)



(h)

Solution

Part (a)

$$\begin{aligned}\text{Area} &= -\int_{-1}^{1} \left(x^3 + 2x^2 - x - 2\right) \mathrm{d}x \\ &= 2.67 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 2.67 \text{ units}^2.}$$

Part (b)

Note that $y = x^2 \implies x = \sqrt{y}$.

$$\begin{aligned}\text{Area} &= \int_{0}^{4} \sqrt{y} \, \mathrm{d}y \\ &= 5.33 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 5.33 \text{ units}^2.}$$

Part (c)

Note that $y = e^{2x} \implies x = \frac{1}{2}\ln y$. Also, when $x = 0$, we have $y = 1$. Further, when $x = 2$, we have $y = e^4$.

$$\begin{aligned}\text{Area} &= \int_0^{e^4} \frac{1}{2}\ln y \, \mathrm{d}y \\ &= 82.4 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 82.4 \text{ units}^2.}$$

Part (d)

Note that when $x = \frac{\pi}{2}$, we have $y = 2$.

$$\begin{aligned}\text{Area} &= 2\int_0^{2} \arcsin\frac{y}{2} \, \mathrm{d}y \\ &= 2.28 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 2.28 \text{ units}^2.}$$

Part (e)

$$\begin{aligned}\text{Area} &= \int_{-2}^{2} \left(\left(8 - x^2\right) - x^2\right) \mathrm{d}x \\ &= 21.3 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 21.3 \text{ units}^2.}$$

Part (f)

$$\begin{aligned}\text{Area} &= \int_{0}^{1} \left(\left(4 - x^2\right) - 3x\right) \mathrm{d}x \\ &= 2.17 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 2.17 \text{ units}^2.}$$

Part (g)

$$\begin{aligned}\text{Area} &= \int_{-1}^{1} \left(\left(9 - x^2\right) - e^x\right) \mathrm{d}x \\ &= 15.0 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 15.0 \text{ units}^2.}$$

Part (h)

$$\begin{aligned}\text{Area} &= \int_0^1 \left(\sqrt{x} - x^2\right) \mathrm{d}x \\ &= 0.333 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of the shaded region is } 0.333 \text{ units}^2.}$$

Problem 2.

(a) Write down the integral for the volume of the solid generated when the shaded region is rotated about the x-axis through 2π for questions 1(a), (e), (f) and (h) using the disc method and use the GC to evaluate it.

(b) Write down the integral for the volume of the solid generated when the shaded region is rotated about the y-axis through 2π for questions 1(b), (d) and (f) using the disc method and use the GC to evaluate it.

Solution

Part (a)

Subpart (i)

$$\begin{aligned}\text{Volume} &= \pi \int_{-1}^{1} \left(x^3 + 2x^2 - x - 2\right)^2 \mathrm{d}x \\ &= 13.9 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is } 13.9 \text{ units}^3.}$$

Subpart (ii)

$$\begin{aligned}\text{Volume} &= \pi \int_{-2}^{2} \left(\left(8 - x^2\right)^2 - x^2\right) \mathrm{d}x \\ &= 536 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is } 536 \text{ units}^3.}$$

Subpart (iii)

$$\begin{aligned}\text{Volume} &= \pi \int_{0}^{1} \left(\left(4 - x^2\right)^2 - (3x)^2\right) \mathrm{d}x \\ &= 33.1 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is } 33.1 \text{ units}^3.}$$

Subpart (iv)

$$\begin{aligned}\text{Volume} &= \pi \int_{0}^{1} \left(\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right) \mathrm{d}x \\ &= 0.942 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is } 0.942 \text{ units}^3.}$$

Part (b)

Subpart (i)

$$\begin{aligned}\text{Volume} &= \pi \int_0^4 \left(\sqrt{y}\right)^2 \, \mathrm{d}y \\ &= 25.1 \text{ (3 s.f.)}\end{aligned}$$

$\boxed{\text{The volume of the solid is 25.1 units}^3.}$

Subpart (ii)

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^2 \arcsin^2 \frac{y}{2} \, \mathrm{d}y \\ &= 5.87 \text{ (3 s.f.)}\end{aligned}$$

$\boxed{\text{The volume of the solid is 5.87 units}^3.}$

Subpart (iii)

$$\begin{aligned}\text{Volume} &= \pi \int_3^4 (4 - y) \, \mathrm{d}y + \frac{1}{3}\pi \cdot 1^2 \cdot 3 \\ &= 4.71 \text{ (3 s.f.)}\end{aligned}$$

$\boxed{\text{The volume of the solid is 4.71 units}^3.}$

Problem 3.

(a) Write down the integral for the volume of the solid generated when the shaded region is rotated about the x-axis through 2π for questions 1(e), (f) and (h) using the shell method and use the GC to evaluate it.

(b) Write down the integral for the volume of the solid generated when the shaded region is rotated about the y-axis through 2π for questions 1(b), (d) and (f) using the shell method and use the GC to evaluate it.

Solution

Part (a)

Subpart (i)

Note that $y = x^2 \implies x = \sqrt{y}$ and $y = 8 - x^2 \implies x = \sqrt{8-y}$ for $x > 0$.

$$\begin{aligned}\text{Volume} &= 2\left(2\pi \int_0^4 \sqrt{y} \cdot y \,\mathrm{d}y + 2\pi \int_4^8 \sqrt{8-y} \cdot y \,\mathrm{d}y\right) \\ &= 536 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is 536 units}^3.}$$

Subpart (ii)

Note that $y = 3x \implies x = \frac{1}{3}y$ and $y = 4 - x^2 \implies x = \sqrt{4-y}$ for $x > 0$.

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^3 \frac{1}{3}y \cdot y \,\mathrm{d}y + 2\pi \int_3^4 \sqrt{4-y} \cdot y \,\mathrm{d}y \\ &= 33.1 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is 33.1 units}^3.}$$

Subpart (iii)

Note that $y = \sqrt{x} \implies x = y^2$ and $y = x^2 \implies x = \sqrt{y}$ for $x > 0$.

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^1 \left(\sqrt{y} - y^2\right) y \,\mathrm{d}y \\ &= 0.942 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is 0.942 units}^3.}$$

Part (b)

Subpart (i)

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^2 x \cdot x^2 \,\mathrm{d}x \\ &= 25.1 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is 25.1 units}^3.}$$

Subpart (ii)

$$\begin{aligned}\text{Volume} &= 2 \cdot 2\pi \int_0^{\pi/2} x\,(2 - 2\sin x)\,\mathrm{d}x \\ &= 5.87 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is 5.87 units}^3.}$$

Subpart (iii)

$$\begin{aligned}\text{Volume} &= 2\pi \int_0^{1} x\left(\left(4 - x^2\right) - 3x\right)\mathrm{d}x \\ &= 4.71 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of the solid is 4.71 units}^3.}$$

Problem 4.

Calculate the area enclosed by the petals of the curve $r = \sin 2\theta$ where $r \geq 0$.

Solution

Note that $r \geq 0 \implies \sin 2\theta \geq 0 \implies r \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$.

$$\begin{aligned} u &= 2\theta \\ \mathrm{d}u &= 2\,\mathrm{d}\theta \\ \theta = 0 &\implies u = 0 \\ \theta = \frac{\pi}{2} &\implies u = \pi \end{aligned}$$

$$\begin{aligned} \text{Area} &= 2 \cdot \frac{1}{2} \int_0^{\pi/2} \sin^2 2\theta \,\mathrm{d}\theta \\ &= \frac{1}{2} \int_0^{\pi} \sin^2 u \,\mathrm{d}u \\ &= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2u}{2} \,\mathrm{d}u \\ &= \frac{1}{4} \int_0^{\pi} (1 - \cos 2u) \,\mathrm{d}u \\ &= \frac{1}{4} \left[u - \frac{\sin 2u}{2}\right]_0^{\pi} \\ &= \frac{\pi}{4} \end{aligned}$$

$$\boxed{\text{The area enclosed is } \frac{\pi}{4} \text{ units}^2.}$$

Problem 5.

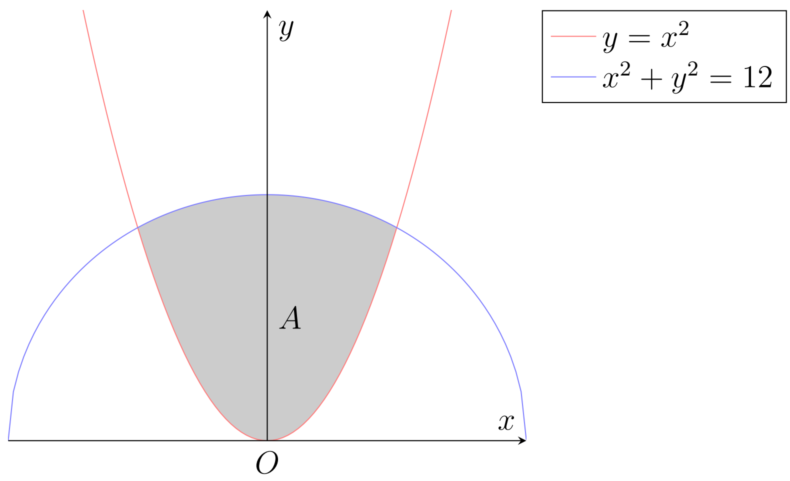
The finite region A is bounded by the curve $y = x^2$ and a minor arc of the circle $x^2 + y^2 = 12$.

(a) Find the numerical value of the area of A, correct to 2 decimal places.

(b) Find the exact volume of the solid obtained when A is rotated about the x-axis through 2π radians.

(c) Find the exact volume of the solid obtained when A is rotated about the y-axis through π radians.

Solution



Part (a)

Consider the intersections between $y = x^2$ and $x^2 + y^2 = 12$.

$$\begin{aligned} & x^2 + y^2 = 12 \\ \implies & x^2 + \left(x^2\right)^2 = 12 \\ \implies & x^4 + x^2 - 12 = 0 \\ \implies & (x^2 - 3)(x^2 + 4) = 0 \\ \implies & \left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)\left(x^2 + 4\right) = 0 \end{aligned}$$

Hence, the two curves intersect at $x = -\sqrt{3}$ and $x = \sqrt{3}$. Note that $x^2 + 4 = 0$ has no solution since $x^2 + 4 > 0$. Also note that $x^2 + y^2 = 12 \implies y = \sqrt{12 - x^2}$ for $y > 0$.

$$\begin{aligned} \text{Area} &= 2\int_0^{\sqrt{3}} \left(\sqrt{12 - x^2} - x^2\right) \mathrm{d}x \\ &= 8.02 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{A \text{ has an area of } 8.02 \text{ units}^2.}$$

Part (b)

Note that $x^2 + y^2 = 12 \implies y^2 = 12 - x^2$.

$$\begin{aligned}
\text{Volume} &= 2\pi \int_0^{\sqrt{3}} \left(\left(12 - x^2\right) - \left(x^2\right)^2 \right) \mathrm{d}x \\
&= 2\pi \left[12x - \frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_0^{\sqrt{3}} \\
&= 2\pi \cdot \frac{46\sqrt{3}}{5} \\
&= \frac{92\sqrt{3}\pi}{5}
\end{aligned}$$

$$\boxed{\text{The solid has a volume of } \frac{92\sqrt{3}\pi}{5} \text{ units}^3.}$$

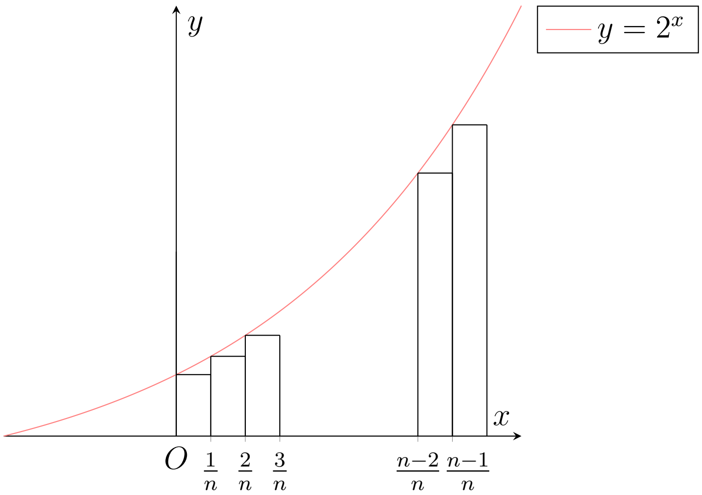
Part (c)

Note that when the curves intersect at $x = \sqrt{3}$, we have $y = 3$. Furthermore, when $x = 0$, we have $y = \sqrt{12}$. Also note that $x^2 + y^2 = 12 \implies x^2 = 12 - y^2$.

$$\begin{aligned}
\text{Volume} &= \pi \int_0^3 y \, \mathrm{d}y + \pi \int_3^{\sqrt{12}} \left(12 - y^2\right) \mathrm{d}y \\
&= \pi \left(16\sqrt{3} - \frac{45}{2} \right)
\end{aligned}$$

$$\boxed{\text{The solid has a volume of } \pi \left(16\sqrt{3} - \frac{45}{2} \right) \text{ units}^3.}$$

Problem 6.



(a) The graph of $y = 2^x$, for $0 \leq x \leq 1$ is shown in the diagram. Rectangles, each of width $\frac{1}{n}$, are drawn under the curve. Given that $\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}$, show that the total area A of all n rectangles is given by $\frac{1}{n}\left(\frac{1}{2^{\frac{1}{n}} - 1}\right)$.

(b) Find the limit of A in exact form as $n \to \infty$.

Let V be the volume of all n rectangles rotated about the x-axis.

(c) Find V in terms of n.

(d) State the limit of V in exact form as $n \to \infty$.

Solution

Part (a)

$$
\begin{aligned}
A &= \sum_{k=0}^{n-1} \frac{2^{k/n}}{n} \\
&= \frac{1}{n} \sum_{k=0}^{n-1} \left(2^{1/n}\right)^k \\
&= \frac{1}{n} \cdot \frac{1 - \left(2^{1/n}\right)^{n-1+1}}{1 - 2^{1/n}} \\
&= \frac{1}{n} \cdot \frac{1 - 2}{1 - 2^{1/n}} \\
&= \frac{1}{n} \cdot \frac{1}{2^{1/n} - 1}
\end{aligned}
$$

Part (b)

$$\begin{aligned}
\lim_{n\to\infty} A &= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{2^{1/n}-1} \\
&= \lim_{n\to\infty} \frac{1/n}{2^{1/n}-1} \\
&= \lim_{m\to 0} \frac{m}{2^m-1} \\
&= \lim_{m\to 0} \frac{1}{\ln 2 \cdot 2^m} \\
&= \frac{1}{\ln 2}
\end{aligned}$$

$$\boxed{\lim_{n\to\infty} A = \frac{1}{\ln 2}}$$

Part (c)

$$\begin{aligned}
V &= \pi \sum_{k=0}^{n-1} \left(2^{k/n}\right)^2 \cdot \frac{1}{n} \\
&= \frac{\pi}{n} \sum_{k=0}^{n-1} \left(2^{2/n}\right)^k \\
&= \frac{\pi}{n} \cdot \frac{1-\left(2^{2/n}\right)^{n-1+1}}{1-2^{2/n}} \\
&= \frac{\pi}{n} \cdot \frac{1-4}{1-2^{2/n}} \\
&= \frac{\pi}{n} \cdot \frac{3}{2^{2/n}-1} \\
&= \frac{\pi}{n} \cdot \frac{3}{4^{1/n}-1} \\
&= \frac{3\pi}{n\left(4^{1/n}-1\right)}
\end{aligned}$$

$$\boxed{V = \frac{3\pi}{n\left(4^{1/n}-1\right)}}$$

Part (d)

$$\begin{aligned}
\lim_{n\to\infty} V &= \lim_{n\to\infty} \frac{3\pi}{n\left(4^{1/n}-1\right)} \\
&= 3\pi \lim_{n\to\infty} \frac{1}{n\left(4^{1/n}-1\right)} \\
&= 3\pi \lim_{n\to\infty} \frac{1/n}{4^{1/n}-1} \\
&= 3\pi \lim_{m\to 0} \frac{m}{4^m-1} \\
&= 3\pi \lim_{m\to 0} \frac{1}{\ln 4 \cdot 4^m} \\
&= 3\pi \cdot \frac{1}{\ln 4} \\
&= \frac{3\pi}{2\ln 2}
\end{aligned}$$

$$\boxed{\lim_{n\to\infty} V = \frac{3\pi}{2\ln 2}}$$

Problem 7.

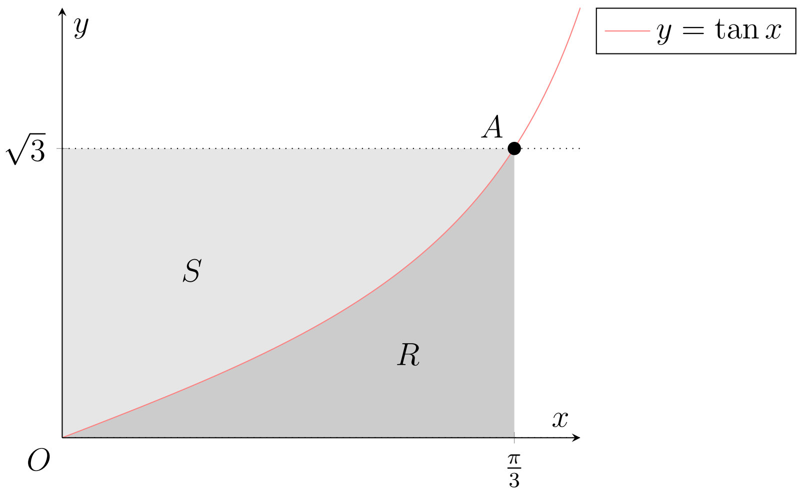
O is the origin and A is the point on the curve $y = \tan x$ where $x = \frac{1}{3}\pi$.

(a) Calculate the area of the region R enclosed by the arc OA, the x-axis and the line $x = \frac{1}{3}\pi$, giving your answer in an exact form.

(b) The region S is enclosed by the arc OA, the y-axis and the line $y = \sqrt{3}$. Find the volume of the solid of revolution formed when S is rotated through 360° about the x-axis, giving your answer in an exact form.

(c) Find $\displaystyle\int_0^{\sqrt{3}} \arctan y \,\mathrm{d}y$ in exact form.

Solution



Part (a)

$$\begin{aligned}\text{Area} &= \int_0^{\pi/3} \tan x \,\mathrm{d}x \\ &= [\ln \sec x]_0^{\frac{\pi}{3}} \\ &= \ln 2\end{aligned}$$

$$\boxed{R \text{ has an area of } \ln 2 \text{ units}^2.}$$

Part (b)

$$\begin{aligned}\text{Volume} &= \pi \int_0^{\pi/3} \left(\left(\sqrt{3}\right)^2 - \tan^2 x \right) \mathrm{d}x \\ &= \pi \int_0^{\pi/3} \left(3 - \sec^2 x + 1\right) \mathrm{d}x \\ &= \pi \left[4x - \tan x\right]_0^{\frac{\pi}{3}} \\ &= \frac{4\pi^2}{3} - \sqrt{3}\pi\end{aligned}$$

$$\boxed{\text{The solid has a volume of } \left(\frac{4\pi^2}{3} - \sqrt{3}\pi\right) \text{ units}^3.}$$

Part (c)

Observe that $\displaystyle\int_0^{\sqrt{3}} \arctan y \, \mathrm{d}y = \text{Area}\, S = \text{Area}(R+S) - \text{Area}\, R = \frac{\pi}{3} \cdot \sqrt{3} - \ln 2.$

$$\boxed{\int_0^{\sqrt{3}} \arctan y \, \mathrm{d}y = \frac{\pi}{\sqrt{3}} - \ln 2}$$

Problem 8.

A portion of the curve $ay = x^2$, where a is a positive constant, is rotated about the vertical axis Oy to form the curved surface of an open bowl. The bowl has a horizontal circular base of radius r and a horizontal circular rim of radius $3r$.

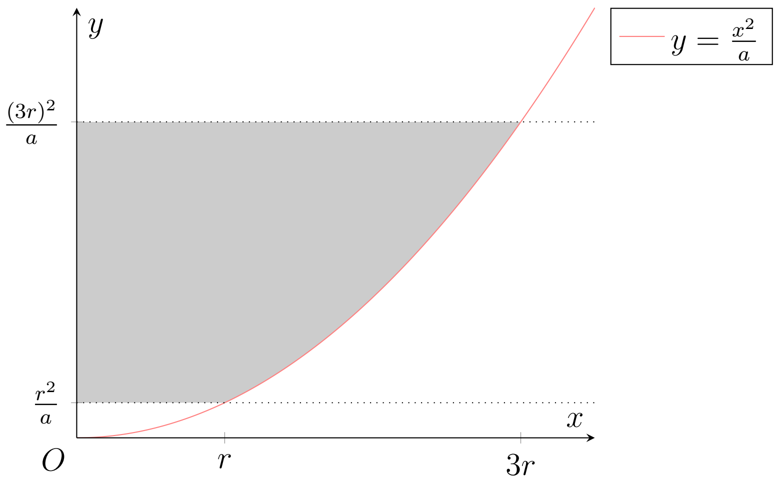
(a) Prove that the depth of the bowl is $\frac{8r^2}{a}$.

(b) Find the volume of the bowl in terms of r and a.

(c) Given that the volume of the bowl is $\frac{\pi a^3}{10}$, find the depth of the bowl in terms of a only.

Solution

Note that $ay = x^2 \implies y = \frac{x^2}{a}$.



Part (a)

$$\begin{aligned}\text{Depth of bowl} &= \frac{(3r)^2}{a} - \frac{r^2}{a}\\ &= \frac{9r^2}{a} - \frac{r^2}{a}\\ &= \frac{8r^2}{a}\end{aligned}$$

Part (b)

$$\begin{aligned}\text{Volume} &= \pi \int_{r^2/a}^{9r^2/a} ay \, \mathrm{d}y\\ &= \pi \left[\frac{a}{2}y^2\right]_{r^2/a}^{9r^2/a}\\ &= \frac{a\pi}{2}\left[\left(\frac{9r^2}{a}\right)^2 - \left(\frac{r^2}{a}\right)^2\right]\end{aligned}$$

$$= \frac{a\pi}{2} \cdot \frac{80r^4}{a^2}$$
$$= \frac{40\pi r^4}{a}$$

$$\boxed{\text{The volume of the bowl is } \frac{40\pi r^4}{a} \text{ units}^3.}$$

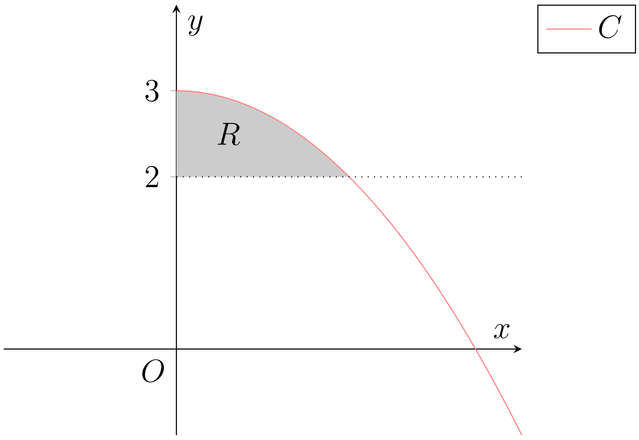
Part (c)

$$\begin{aligned}
& & \frac{40\pi r^4}{a} &= \frac{\pi a^3}{10} \\
&\implies & 400r^4 &= a^4 \\
&\implies & 20r^2 &= a^2 \\
&\implies & r^2 &= \frac{1}{20}a^2 \\
&\implies & \text{Depth of bowl} &= \frac{8 \cdot a^2/20}{a} \\
& & &= \frac{2}{5}a
\end{aligned}$$

$$\boxed{\text{The depth of the bowl is } \frac{2}{5}a \text{ units.}}$$

Problem 9.

The diagram shows the region R bounded by part of the curve C with equation $y = 3-x^2$, the y-axis and the line $y = 2$, lying in the first quadrant.



Write down the equation of the curve obtained when C is translated by 2 units in the negative y-direction.

Hence, or otherwise, show that the volume of the solid formed when R is rotated completely about the line $y = 2$ is given by $\pi \int_0^1 \left(1 - 2x^2 + x^4\right) \mathrm{d}x$ and evaluate this integral exactly.

Solution

$$\boxed{C : y = 1 - x^2}$$

Note that $3 - x^2 = 2 \implies x = \pm 1$, whence $x = 1$ since $x > 0$.

$$\begin{aligned}
\text{Volume} &= \pi \int_0^1 \left(1 - x^2\right)^2 \mathrm{d}x \\
&= \pi \int_0^1 \left(1 - 2x^2 + x^4\right) \mathrm{d}x \\
&= \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5\right]_0^1 \\
&= \frac{8}{15}\pi
\end{aligned}$$

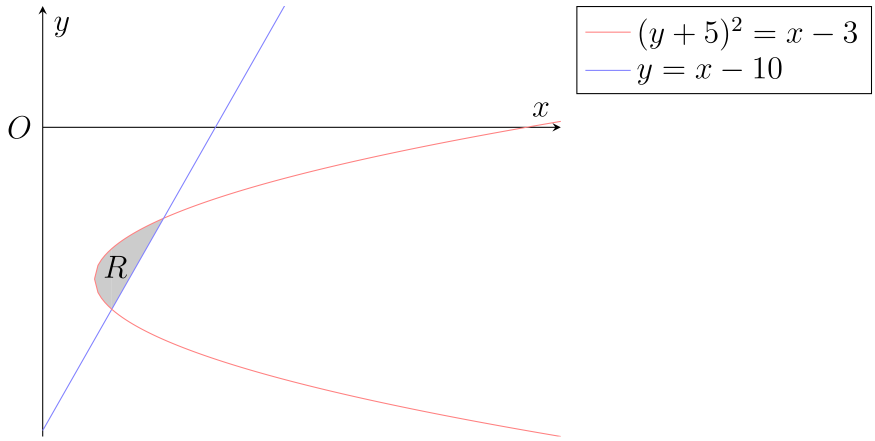
$$\boxed{\text{The volume of the solid is } \frac{8}{15}\pi \text{ units}^3.}$$

Problem 10.

The diagram below shows a region R bounded by the curve $(y+5)^2 = x-3$ and the line $y = x-10$. Find the volume of solid formed when R is rotated four right angles about

(a) the y-axis, and

(b) the x-axis.



Solution

Part (a)

Consider the intersections between $(y+5)^2 = x-3$ and $y = x-10$.

$$\begin{aligned} & (y+5)^2 = x-3 \\ \implies & (x-5)^2 = x-3 \\ \implies & x^2 - 10x + 25 = x - 3 \\ \implies & x^2 - 11x + 28 = 0 \\ \implies & (x-4)(x-7) = 0 \end{aligned}$$

Hence, $x = 4$ and $x = 7$, whence $y = -6$ and $y = -3$. Thus, the two curves intersect at $(4, -6)$ and $(7, -3)$.

Note that $(y+5)^2 = x-3 \implies x = 3 + (y+5)^2$ and $y = x - 10 \implies x = y + 10$.

$$\begin{aligned} \text{Volume} &= \pi \int_{-6}^{-3} \left((y+10)^2 - \left(3 + (y+5)^2\right)^2 \right) \mathrm{d}y \\ &= 130 \text{ (3 s.f.)} \end{aligned}$$

$$\boxed{\text{The volume of the solid is } 130 \text{ units}^3.}$$

Part (b)

Note that

$$(y+5)^2 = x-3 \implies \begin{cases} y = -5 + \sqrt{x-3}, & y \geq 5 \\ y = -5 - \sqrt{x-3}, & y < 5 \end{cases}$$

$$\begin{aligned}\text{Volume} &= \pi \int_3^4 \left(\left(-5 - \sqrt{x-3}\right)^2 - \left(-5 + \sqrt{x+3}\right)^2 \right) \mathrm{d}x \\ &\quad + \pi \int_4^7 \left((x-10)^2 - \left(-5 + \sqrt{x-3}\right)^2 \right) \mathrm{d}x \\ &= 127 \text{ (3 s.f.)}\end{aligned}$$

$\boxed{\text{The volume of the solid is 127 units}^3.}$

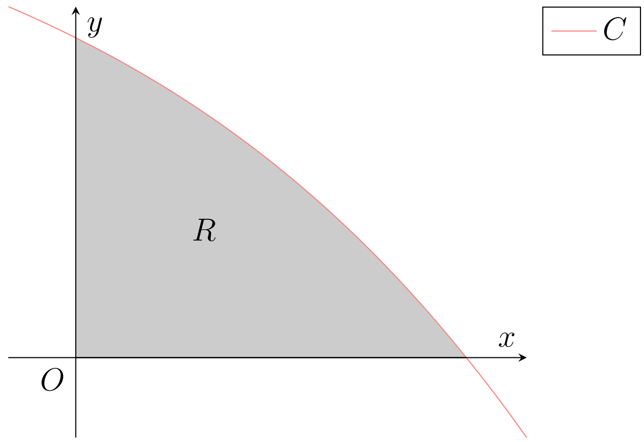
Problem 11.

The curve C is defined by the following pair of parametric equations.

$$x = t - \frac{1}{t^2},\ y = 2 - t^2, \qquad t > 0$$

Find the area of the finite region R enclosed by the curve C and the axes as well as the volume of solid obtained when R is rotated about the x-axis through 4 right-angles.

Solution



Note that when $x = 0 \implies t - \frac{1}{t^2} = 0 \implies t = 1$. Also note that when $y = 0 \implies 2 - t^2 \implies t = \sqrt{2}$, whence $x = \sqrt{2} - \frac{1}{2}$.

$$\begin{aligned}\text{Area} &= \int_0^{\sqrt{2}-\frac{1}{2}} y \, \mathrm{d}x \\ &= \int_1^{\sqrt{2}} \left(2 - t^2\right) \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\ &= \int_1^{\sqrt{2}} \left(2 - t^2\right) \left(1 + \frac{2}{t^3}\right) \mathrm{d}t \\ &= 0.526 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The area of } R \text{ is } 0.526 \text{ units}^2.}$$

$$\begin{aligned}\text{Volume} &= \pi \int_0^{\sqrt{2}-\frac{1}{2}} y^2 \, \mathrm{d}x \\ &= \pi \int_1^{\sqrt{2}} \left(2 - t^2\right) \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t \\ &= \pi \int_1^{\sqrt{2}} \left(2 - t^2\right) \left(1 + \frac{2}{t^3}\right) \mathrm{d}t \\ &= 1.19 \text{ (3 s.f.)}\end{aligned}$$

$$\boxed{\text{The volume of } R \text{ is } 1.19 \text{ units}^3.}$$

Problem 12.

Find the area enclosed by the ellipse $x = a\cos t$, $y = b\sin t$, where a and b are positive constants. Find also the volume of solid obtained when the region enclosed by the ellipse is rotated through π radians about the x-axis.

Solution

By symmetry, we only need to consider the area of the ellipse in the first quadrant. Note that $x = 0 \implies t = \pi/2$ and $x = a \implies t = 0$.

$$\begin{aligned}
\text{Area} &= 4\int_0^a y\,\mathrm{d}x \\
&= 4\int_{\pi/2}^0 y \cdot \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= 4\int_{\pi/2}^0 (b\sin t)(-a\sin t)\,\mathrm{d}t \\
&= 4ab\int_0^{\pi/2} \sin^2 t\,\mathrm{d}t \\
&= 4ab\int_0^{\pi/2} \frac{1-\cos 2t}{2}\,\mathrm{d}t \\
&= 4ab \cdot \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2} \\
&= \pi ab
\end{aligned}$$

$$\boxed{\text{The area of the ellipse is } \pi ab \text{ units}^2.}$$

$$\begin{aligned}
\text{Volume} &= 2\pi\int_0^a y^2\,\mathrm{d}x \\
&= 2\pi\int_{\pi/2}^0 y^2 \cdot \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \\
&= 2\pi\int_{\pi/2}^0 (b\sin t)^2(-a\sin t)\,\mathrm{d}t \\
&= 2\pi ab^2\int_0^{\pi/2} \sin^3 t\,\mathrm{d}t \\
&= 2\pi ab^2\int_0^{\pi/2} \frac{3\sin t - \sin 3t}{4}\,\mathrm{d}t \\
&= 2\pi ab^2 \cdot \frac{1}{4}\left[-3\cos t + \frac{1}{3}\cos 3t\right]_0^{\pi/2} \\
&= \frac{4\pi}{3}ab^2
\end{aligned}$$

$$\boxed{\text{The volume of the ellipse is } \frac{4\pi}{3}ab^2 \text{ units}^3.}$$

Problem 13.

Find the polar equation of the curve C with equation $x^5 + y^5 = 5bx^2y^2$, where b is a positive constant. Sketch the part of the curve C where $0 \leq \theta \leq \frac{\pi}{2}$. Show, using polar coordinates, that the area A of the region enclosed by this part of the curve is given by

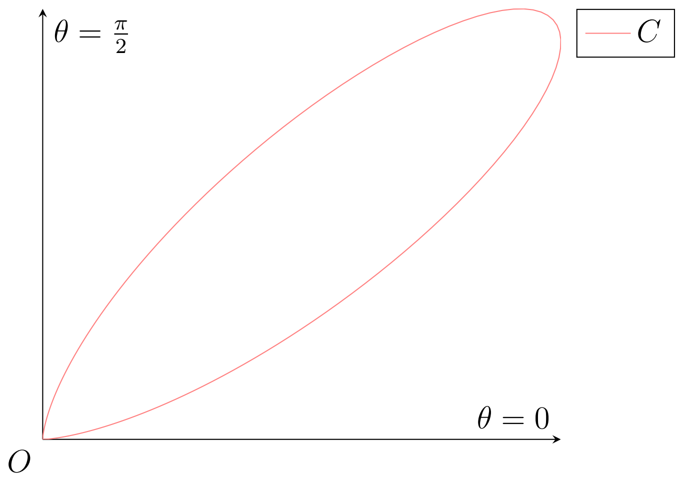
$$A = \frac{25b^2}{2} \int_0^{\frac{\pi}{2}} \frac{\sin^4\theta\cos^4\theta}{(\cos^5\theta + \sin^5\theta)^2} \, \mathrm{d}\theta$$

By differentiating $\frac{1}{1 + \tan^5\theta}$ with respect to θ, or otherwise, find the exact value of A in terms of b.

Solution

$$\begin{aligned}
x^5 + y^5 &= 5bx^2y^2 \\
\implies (r\cos\theta)^5 + (r\sin\theta)^5 &= 5b(r\cos\theta)^2(r\sin\theta)^2 \\
\implies r^5\left(\cos^5\theta + \sin^5\theta\right) &= 5br^4\cos^2\theta\sin^2\theta \\
\implies r\left(\cos^5\theta + \sin^5\theta\right) &= 5b\cos^2\theta\sin^2\theta \\
\implies r &= \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}
\end{aligned}$$

$$\boxed{r = \frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}}$$



$$\begin{aligned}
A &= \frac{1}{2}\int_0^{\pi/2} \left(\frac{5b\cos^2\theta\sin^2\theta}{\cos^5\theta + \sin^5\theta}\right)^2 \mathrm{d}\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{25b^2\cos^4\theta\sin^4\theta}{\left(\cos^5\theta + \sin^5\theta\right)^2} \mathrm{d}\theta \\
&= \frac{25b^2}{2}\int_0^{\pi/2} \frac{\cos^4\theta\sin^4\theta}{\left(\cos^5\theta + \sin^5\theta\right)^2} \mathrm{d}\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{1}{1+\tan^5\theta} &= -\frac{1}{\left(1+\tan^5\theta\right)^2}\cdot 5\tan^4\theta\cdot\sec^2\theta \\
&= -5\cdot\frac{1}{\left(1+\frac{\sin^5\theta}{\cos^5\theta}\right)^2}\cdot\frac{\sin^4\theta}{\cos^4\theta}\cdot\frac{1}{\cos^2\theta} \\
&= -5\cdot\frac{\left(\cos^5\theta\right)^2}{\left(\cos^5\theta\right)^2\left(1+\frac{\sin^5\theta}{\cos^5\theta}\right)^2}\cdot\frac{\sin^4\theta}{\cos^6\theta} \\
&= -5\cdot\frac{\cos^{10}\theta}{\cos^5\theta+\sin^5\theta}\cdot\frac{\sin^4\theta}{\cos^6\theta} \\
&= -\frac{5\cos^4\theta\sin^4\theta}{\cos^5\theta+\sin^5\theta} \\
\implies \qquad A &= \frac{-5b^2}{2}\int_0^{\pi/2} -\frac{5\cos^4\theta\sin^4\theta}{\left(\cos^5\theta+\sin^5\theta\right)^2}\,\mathrm{d}\theta \\
&= -\frac{5b^2}{2}\left[\frac{1}{1+\tan^5\theta}\right]_0^{\pi/2} \\
&= -\frac{5b^2}{2}\cdot -1 \\
&= \frac{5b^2}{2}
\end{aligned}$$

$$\boxed{A=\frac{5b^2}{2}}$$

Problem 14.

The polar equation of a curve is given by $r = e^\theta$ where $0 \leq \theta \leq \pi/2$. Cartesian axes are taken at the pole O. Express x and y in terms of θ and hence find the Cartesian equation of the tangent at $\left(e^{\pi/2}, \pi/2\right)$. The region R is bounded by the polar curve, tangent and the x-axis. Find the exact area of the region R.

Solution

$$\boxed{x = e^\theta \cos\theta,\ y = e^\theta \sin\theta}$$

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}x} \\
&= \frac{\mathrm{d}y}{\mathrm{d}\theta} \bigg/ \frac{\mathrm{d}x}{\mathrm{d}\theta} \\
&= \frac{e^\theta \cos\theta + e^\theta \sin\theta}{e^\theta(-\sin\theta) + e^\theta \cos\theta} \\
&= \frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} \\
\implies \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/2} &= \frac{\cos(\pi/2) + \sin(\pi/2)}{\cos(\pi/2) - \sin(\pi/2)} \\
&= -1
\end{aligned}$$

When $\theta = \dfrac{\pi}{2}$, we have $x = 0$ and $y = e^{\pi/2}$. Hence, the tangent is given by

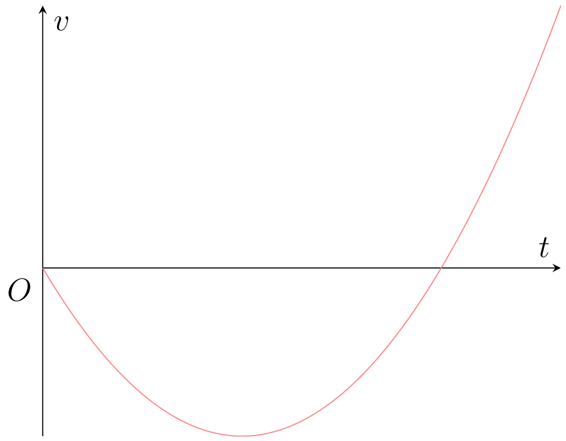
$$\begin{aligned}
y - e^{\pi/2} &= -(x - 0) \\
\implies \qquad y &= -x + e^{\pi/2}
\end{aligned}$$

$$\boxed{y = -x + e^{\pi/2}}$$

$$\begin{aligned}
\text{Area } R &= \frac{1}{2} \cdot e^{\pi/2} \cdot e^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \left(e^\theta\right)^2 \mathrm{d}\theta \\
&= \frac{e^\pi}{2} - \frac{1}{2} \int_0^{\pi/2} e^{2\theta} \,\mathrm{d}\theta \\
&= \frac{e^\pi}{2} - \frac{1}{2} \left[\frac{e^{2\theta}}{2}\right]_0^{\pi/2} \\
&= \frac{e^\pi}{2} - \frac{1}{4}\left(e^\pi - 1\right) \\
&= \frac{1}{4}\left(e^\pi + 1\right)
\end{aligned}$$

$$\boxed{\text{The area of } R \text{ is } \frac{1}{4}\left(e^\pi + 1\right) \text{ units}^2.}$$

Problem 15.



The diagram shows the velocity-time graph of a particle moving in a straight line. The equation of the curve shown is $v = t(t-10)$ where t seconds is the time and v ms^{-1} is the velocity. The particle starts at a point A on the line when $t = 0$.

Calculate

(a) the distance travelled by the particle before coming to instantaneous rest, and

(b) the time at which the particle returns to A.

Solution

Part (a)

For instantaneous rest, $v = 0$. Hence, $t(t-10) = 0$, whence $t = 10$. Note that we reject $t = 0$ since $t > 0$.

$$\begin{aligned} \text{Distance travelled} &= -\int_0^{10} v \, \mathrm{d}t \\ &= -\int_0^{10} t(t-10) \, \mathrm{d}t \\ &= -\int_0^{10} \left(t^2 - 10t\right) \mathrm{d}t \\ &= -\left[\frac{t^3}{3} - \frac{10t^2}{2}\right]_0^{10} \\ &= \frac{500}{3} \end{aligned}$$

$$\boxed{\text{The particle travelled } \frac{500}{3} \text{ m before coming to instantaneous rest.}}$$

Part (b)

When the particle returns to A, $s = 0$. Let the time at which the particle returns to A be t_0.

$$\begin{aligned} & \int_0^{t_0} v \, \mathrm{d}t = 0 \\ \implies & \int_0^{t_0} t(t-10) \, \mathrm{d}t = 0 \\ \implies & \left[\frac{t_0^3}{3} - \frac{10t_0^2}{2} \right]_0^{t_0} = 0 \\ \implies & \frac{1}{3}t_0^3 - 5t_0^2 = 0 \\ \implies & \frac{1}{3}t_0^2 \left(t_0 - 15\right) = 0 \end{aligned}$$

Thus, $t_0 = 15$. Note that we reject $t_0 = 0$ since $t_0 > 0$.

$\boxed{\text{It takes the particle 15 seconds to return to } A.}$