Problem 1.

(a) Explain why the Euler method will fail for the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\cos\sqrt{x}, \qquad y(0) = 0,$$

where $y = y(x)$ satisfies that differential equation and is not a constant.

(b) Suppose the initial condition for the problem in part (a) is now $y(0) = 10$. Use the improved Euler method with a step size of 0.1 to find, to three decimal places, an estimate for $y(0.1)$.

(c) Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}y(2-x), \qquad y > 0, \quad y(0) = 10,$$

expressing y in terms of x, and simplifying your answer as far as possible.

(d) Explain why the solution found in part (c) will give a reasonable estimate for $y(0.1)$ in part (b).

Solution

Part (a)

By the Euler method,

$$y_1 = y_0 + \Delta x(y_0 \cos\sqrt{x_0}) = 0 + \Delta x(0\cos 0) = 0.$$

It follows that $y_n = 0$ for all $n \in \mathbb{N}$, whence y is the zero function. However, because y is not a constant function, y cannot be the zero function, a contradiction. Hence, the Euler method fails.

Part (b)

Let $\Delta x = 0.1$, $y_0 = 10$ and $x_n = n\Delta x$.

$$\begin{aligned}\widetilde{y_1} &= y_0 + \Delta x(y_0\cos\sqrt{x_0}) = 11\\ y_1 &= y_0 + \frac{1}{2}\Delta x\left[y_0\cos\sqrt{x_0} + \widetilde{y_1}\cos\sqrt{x_1}\right] = 11.023 \text{ (3 d.p.)}\end{aligned}$$

$$\boxed{y(0.1) \approx 11.023}$$

Part (c)

$$\begin{aligned}&& \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}y(2-x)\\ \Longrightarrow && \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{2}(2-x)\\ \Longrightarrow && \int \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x &= \int \frac{1}{2}(2-x)\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}
\implies \quad \int \frac{1}{y}\,\mathrm{d}y &= \int \frac{1}{2}(2-x)\,\mathrm{d}x \\
\implies \quad \ln y &= \frac{1}{2}\left[2x - \frac{1}{2}x^2\right] + C_1 \\
&= x - \frac{1}{4}x^2 + C_1 \\
\implies \quad y &= C\exp\left(x - \frac{1}{4}x^2\right)
\end{aligned}$$

Since $y(0) = 10$, we have $C = 10$. Thus,

$$\boxed{y = 10\exp\left(x - \frac{1}{4}x^2\right)}$$

Part (d)

For small x, we have that $\cos\sqrt{x} \approx 1 - \frac{1}{2}(\sqrt{x})^2 = \frac{1}{2}(2-x)$. Thus,

$$y\cos\sqrt{x} \approx \frac{1}{2}y(2-x),$$

whence the two differential equations and thus their solutions are approximately equal. Since $x = 0.1$ is small, the solution found in part (c) will give a reasonable estimate for $y(0.1)$ in part (b).

Problem 2.

A particle moves along a straight line which passes through a fixed point O. It is acted on by two resistive forces, one of which is proportional to its displacement x from O while the other is proportional to its speed v. As a result, the motion of the particle is governed by the differential equation

$$v\frac{\mathrm{d}v}{\mathrm{d}x} = -7x - 24v.$$

Given that $v = 121$ when $x = 0$, estimate the value of v when $x = 1$ using

(a) one iteration of the Euler method,

(b) one iteration of the improved Euler method.

Hence, explain why v is approximately a linear function of x for $0 \leq x \leq 1$.

By considering the values of $\frac{x}{v}$ for $0 \leq x \leq 1$, use the given differential equatoin to find an expression for this linear function.

Solution

Rewriting the given differential equation, we obtain

$$\frac{\mathrm{d}v}{\mathrm{d}x} = -\frac{7x}{v} - 24.$$

Let $f(x, v) = -\frac{7x}{v} - 24$, $\Delta x = 1$, $v_0 = 121$, and $x_n = n\Delta x$.

Part (a)

By the Euler method,

$$v_1 = v_0 + \Delta x f(x_0, v_0) = 97$$

$$\boxed{y(1) \approx 97}$$

Part (b)

By the improved Euler method,

$$\begin{aligned} \widetilde{v_1} &= v_0 + \Delta x f(x_0, v_0) = 97 \\ v_1 &= v_0 + \frac{1}{2}\Delta x \left[f(x_0, v_0) + f(x_1, \widetilde{v_1})\right] = 96.964 \end{aligned}$$

$$\boxed{y(1) \approx 96.964}$$

The gradient of v at $x = 0$ is $f(x_0, v_0) = -24$, which is very close to the gradient of v at $x = 1$, $f(x_1, \widetilde{v_1}) = -24.072$. Since the gradient of v is approximately constant for $0 \leq x \leq 1$, we have that v is approximately a linear function on that interval.

Observe that for $0 \leq x \leq 1$, $x/v \approx 0$ since $x \in [0, 1]$, while $v \geq 96$. Thus, $\mathrm{d}v/\mathrm{d}x \approx -24$, whence $v = -24x + C$. Since $v = 121$ when $x = 0$, we have $\boxed{v \approx -24x + 121}$.

Problem 3.

The function $y = y(x)$ satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5}(\tan x + x^3 y).$$

The value of $y(h)$ is to be found, where h is a small positive number, and $y(0) = 0$.

(a) Use one step of the improved Euler method to find an alternative approximation to $y(h)$ in terms of h.

(b) It can be shown that $y = y(x)$ satisfies

$$y(h) = e^{0.05h^4} \int_0^h \frac{\tan x}{5} e^{-0.05x^4} \, \mathrm{d}x \, .$$

Assume that h is small and hence find another approximation to $y(h)$ in terms of h.

(c) Discuss the relative merits of these two methods employed to obtain these approximations.

Solution

Part (a)

Let $f(x, y) = \frac{1}{5}(\tan x + x^3 y)$, $\Delta x = h$, $y_0 = 0$ and $x_n = n\Delta x$. By the improved Euler method,

$$\begin{aligned}
\widetilde{y_1} &= y_0 + \Delta x f(x_0, y_0) = 0 \\
y_1 &= y_0 + \frac{1}{2}\Delta x \left[f(x_0, y_0) + f(x_1, \widetilde{y_1}) \right] \\
&= 0 + \frac{1}{2} h \left[0 + \frac{1}{5}(\tan h + 0) \right] \\
&= \frac{h \tan h}{10}
\end{aligned}$$

$$\boxed{y(h) \approx \frac{h \tan h}{10}}$$

Part (b)

Let $g(x) = \frac{\tan x}{5} e^{-0.05x^4}$. Consider Simpson's rule with abscissae 0, $h/2$ and h.

$$\begin{aligned}
e^{0.05h^4} \int_0^h \frac{\tan x}{5} e^{-0.05x^4} \, \mathrm{d}x &\approx e^{0.05h^4} \cdot \frac{h - 0}{6} \left[g(0) + g(h/2) + g(h) \right] \\
&= e^{0.05h^4} \cdot \frac{h}{6} \left[0 + \frac{\tan(h/2)}{5} e^{-0.05(h/2)^4} + \frac{\tan h}{5} e^{-0.05h^4} \right] \\
&= e^{0.05h^4} \cdot \frac{h}{30} \left[\tan\left(\frac{h}{2}\right) e^{-h^4/320} + \tan(h) e^{-0.05h^4} \right] \\
&= \frac{h}{30} \left[\tan\left(\frac{h}{2}\right) e^{3h^4/64} + \tan(h) \right]
\end{aligned}$$

$$\boxed{y(h) \approx \frac{h}{30}\left[\tan\left(\frac{h}{2}\right)e^{3h^4/64} + \tan(h)\right]}$$

Part (c)

Simpson's rule requires less computations, while the improved Euler method gives a nicer approximation.