

## Problem 1.

Find the natural domain of the function  $f$  for the following:

$$(a) f(x, y) = \sqrt{1 - x^2 - y^2}$$

$$(b) f(x, y) = \ln(x^2 - y)$$

$$(c) f(x, y) = \arcsin(x + y)$$

$$(d) f(x, y) = \frac{1}{x^2 - y^2}$$

### Solution

#### Part (a)

Observe that the argument of the square root must be non-negative. Hence,  $1 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 1$ .

$$D_f = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

#### Part (b)

Observe that the argument of the natural log must be positive. Hence,  $x^2 - y > 0 \implies y < x^2$ .

$$D_f = \{(x, y) \in \mathbb{R}^2 : y < x^2\}$$

#### Part (c)

Observe that the argument of arcsin must be within the range of sin, i.e. between  $-1$  and  $1$  inclusive. Hence,  $-1 \leq x + y \leq 1$ .

$$D_f = \{(x, y) \in \mathbb{R}^2 : -1 \leq x + y \leq 1\}$$

#### Part (d)

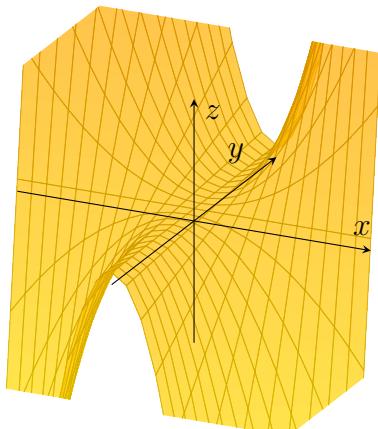
Observe that the denominator must be non-zero. Hence,  $x^2 - y^2 \neq 0 \implies y^2 \neq x^2 \implies y \neq x \vee y \neq -x$ .

$$D_f = \{(x, y) \in \mathbb{R}^2 : y \neq x \vee y \neq -x\}$$

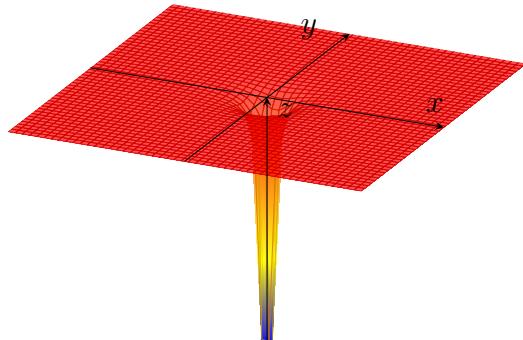
## Problem 2.

Identify the correct equations of the following surfaces in 3-D space.

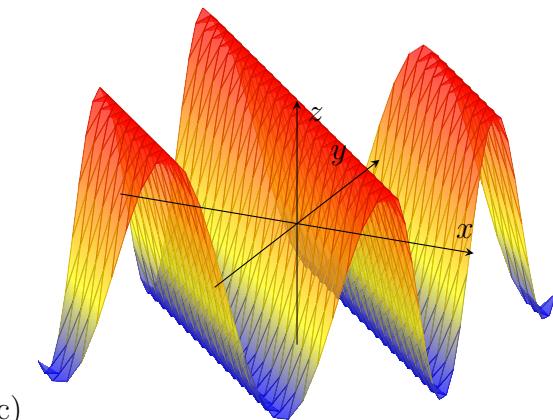
- $z = \cos(x + y)$
- $z = x^2y + 1$
- $z = 3 - x + y$
- $z = -\frac{1}{\sqrt{x^2 + y^2}}$



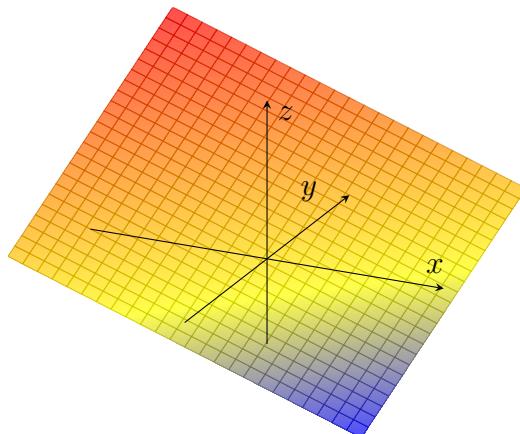
(a)



(b)



(c)



(d)

**Part (a)**

$$z = x^2y + 1$$

**Part (b)**

$$z = -\frac{1}{\sqrt{x^2 + y^2}}$$

**Part (c)**

$$z = \cos(x + y)$$

**Part (d)**

$$z = 3 - x + y$$

## Problem 3.

Let  $f(x, y) = x^2 - 2x^3 + 3xy$ . Find an equation of the level curve that passes through the point

- (a)  $(-1, 1)$
- (b)  $(2, -1)$
- (c)  $(1, 5)$

### Solution

#### Part (a)

Note that  $f(-1, 1) = 0$ . Hence, the level curve is given by  $x^2 - 2x^3 + 3xy = 0$ .

#### Part (b)

Note that  $f(2, -1) = -18$ . Hence, the level curve is given by  $x^2 - 2x^3 + 3xy = -18$ .

#### Part (c)

Note that  $f(1, 5) = 14$ . Hence, the level curve is given by  $x^2 - 2x^3 + 3xy = 14$ .

## Problem 4.

If  $V(x, y)$  is the voltage or potential at a point  $(x, y)$  in the  $xy$ -plane, then the level curves of  $V$  are called equipotential curves. Along such a curve, the voltage remains constant. Given that

$$V(x, y) = \frac{8}{\sqrt{16 + x^2 + y^2}}$$

find an equation of the equipotential curves at which

- (a)  $V = 2.0$
- (b)  $V = 1.0$
- (c)  $V = 0.5$

### Solution

Rearranging the given equation, we have

$$x^2 + y^2 = \frac{64}{V^2} - 16.$$

#### Part (a)

When  $V = 2.0$ , we have  $x^2 + y^2 = \frac{64}{2.0^2} - 16 = 0$ , whence  $x = 0$  and  $y = 0$ .

$x = 0 \wedge y = 0$

#### Part (b)

When  $V = 1.0$ , we have  $x^2 + y^2 = \frac{64}{1.0^2} - 16 = 48$ .

$x^2 + y^2 = 48$

#### Part (c)

When  $V = 0.5$ , we have  $x^2 + y^2 = \frac{64}{0.5^2} - 16 = 240$ .

$x^2 + y^2 = 240$

**Problem 5.**

Given that  $f(x, y) = x^4 \sin(xy^3)$ , find  $f_x(x, y)$ ,  $f_y(x, y)$ ,  $f_{xy}(x, y)$  and  $f_{yx}(x, y)$ .

**Solution**

Differentiating  $f$  with respect to  $x$ ,

$$f_x(x, y) = 4x^3 \sin(xy^3) + x^4 y^3 \cos(xy^3)$$

Differentiating  $f$  with respect to  $y$ ,

$$f_y(x, y) = 3x^5 y^2 \cos(xy^3)$$

Differentiating  $f_x$  with respect to  $y$ ,

$$\begin{aligned} f_{xy}(x, y) &= 12x^4 y^2 \cos(xy^3) + x^4 [3y^2 \cos(xy^3) - 3xy^5 \sin(xy^3)] \\ &= 15x^4 y^2 \cos(xy^3) - 3x^5 y^5 \sin(xy^3) \end{aligned}$$

$$f_{xy}(x, y) = 15x^4 y^2 \cos(xy^3) - 3x^5 y^5 \sin(xy^3)$$

Differentiating  $f_y$  with respect to  $x$ ,

$$\begin{aligned} f_{yx}(x, y) &= 3y^2 [5x^4 \cos(xy^3) - x^5 y^3 \sin(xy^3)] \\ &= 15x^4 y^2 \cos(xy^3) - 3x^5 y^5 \sin(xy^3) \end{aligned}$$

$$f_{yx}(x, y) = 15x^4 y^2 \cos(xy^3) - 3x^5 y^5 \sin(xy^3)$$

**Problem 6.**

Given that  $z = x^2y$ ,  $x = t^2$ ,  $y = t^3$ , use the chain rule to find  $\frac{dz}{dt}$  in terms of  $t$ .

**Solution**

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \\ &= 2xy \cdot 2t + x^2 \cdot 3t^2 \\ &= 2t^2t^3 \cdot 2t + (t^2)^2 \cdot 3t^2 \\ &= 7t^6\end{aligned}$$

$$\boxed{\frac{dz}{dt} = 7t^6}$$

## Problem 7.

Find the gradient of  $f(x, y) = 3x^2y$  at the point  $(1, 2)$  and use it to calculate the directional derivative of  $f$  at  $(1, 2)$  in the direction of the vector  $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$ .

### Solution

Note that  $f_x(x, y) = 6xy$  and  $f_y(x, y) = 3x^2$ . Hence,  $\nabla f$  at  $(1, 2)$  is given by  $\begin{pmatrix} 12 \\ 3 \end{pmatrix}$ .

The derivative at  $(1, 2)$  is  $\begin{pmatrix} 12 \\ 3 \end{pmatrix}$ .

Observe that the directional derivative of  $f$  in the direction of  $\mathbf{u}$  at  $(1, 2)$  is given by

$$\nabla f \cdot \hat{\mathbf{u}} = \begin{pmatrix} 12 \\ 3 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{48}{5}$$

The instantaneous rate of change at  $(1, 2)$  in the direction of  $\mathbf{u}$  is  $\frac{48}{5}$ .

## Problem 8.

Suppose that a point moves along the intersection of the sphere  $x^2 + y^2 + z^2 = 1$  with the plane  $x = \frac{2}{3}$ . Find the rate of  $z$  with respect to  $y$  when the point is at  $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$ .

### Solution

Note that  $x^2 + y^2 + z^2 = 1 \implies z = \pm\sqrt{1 - x^2 - y^2}$ . Given that the object we want (the rate of change of  $z$  with respect to  $y$ ) will later be evaluated when  $z = \frac{2}{3} > 0$ , we consider only the positive branch. Let  $f(x, y) = \sqrt{1 - x^2 - y^2}$ . Then  $f_y(x, y) = \frac{-y}{\sqrt{1 - x^2 - y^2}}$ . Evaluating at the desired point, we get,  $f_y\left(\frac{2}{3}, \frac{1}{3}\right) = \frac{-1/3}{\sqrt{1 - (2/3)^2 - (1/3)^2}} = -\frac{1}{2}$ .

The rate of change of  $z$  with respect to  $y$  at  $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$  is  $-\frac{1}{2}$ .

## Problem 9.

- (a) The Cauchy-Riemann equations are such that  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$  and  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$  for  $u(x, y)$  and  $v(x, y)$ . Show that  $u = e^x \cos y$ ,  $v = e^x \sin y$  satisfy the Cauchy-Riemann equations.
- (b) Show that the function  $f(x, y) = e^x \sin y + e^y \cos x$  satisfies that Laplace equation, i.e.  $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ .
- (c) If  $u(x, y)$  and  $v(x, y)$  satisfy the Cauchy-Riemann equations, state the conditions for both  $u$  and  $v$  to satisfy the Laplace equation.

## Solution

### Part (a)

Differentiating  $u$  with respect to  $x$ ,  $\frac{\partial u}{\partial x} = e^x \cos y$ . Differentiating  $v$  with respect to  $y$ ,  $\frac{\partial v}{\partial y} = e^x \cos y$ . Hence,  $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ .

Differentiating  $u$  with respect to  $y$ ,  $\frac{\partial u}{\partial y} = -e^x \sin y$ . Differentiating  $v$  with respect to  $x$ ,  $\frac{\partial v}{\partial x} = e^x \sin y$ . Hence,  $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ .

Thus,  $u$  and  $v$  satisfy the Cauchy-Riemann equations.

### Part (b)

Differentiating  $f$  twice with respect to  $x$ ,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}(e^x \sin y - e^y \sin x) = e^x \sin y - e^y \cos x$$

Differentiating  $f$  twice with respect to  $y$ ,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}(e^x \cos y + e^y \cos x) = -e^x \sin y + e^y \cos x$$

Hence,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (e^x \sin y - e^y \cos x) + (-e^x \sin y + e^y \cos x) = 0$$

Thus,  $f(x, y) = e^x \sin y + e^y \cos x$  satisfies the Laplace equation.

### Part (c)

Suppose  $u(x, y)$  and  $v(x, y)$  satisfy the Cauchy-Riemann equations. This gives

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Differentiating with respect to  $x$ , we obtain

$$\begin{cases} u_{xx} = v_{yx} \\ u_{yx} = -v_{xx} \end{cases}$$

Differentiating with respect to  $y$ , we obtain

$$\begin{cases} u_{xy} = v_{yy} \\ u_{yy} = -v_{xy} \end{cases}$$

This gives

$$\begin{cases} u_{xx} + u_{yy} = v_{yx} - v_{xy} \\ v_{xx} + v_{yy} = -u_{yx} + u_{xy} \end{cases}$$

Hence, if  $u$  and  $v$  both satisfy the Laplace equation, we require

$$\begin{cases} v_{yx} - v_{xy} = 0 \\ -u_{yx} + u_{xy} = 0 \end{cases}$$

which gives the conditions  $u_{xy} = u_{yx}$  and  $v_{xy} = v_{yx}$ .

$u_{xy} = u_{yx}, v_{xy} = v_{yx}$

## Problem 10.

Find the equation of the tangent plane to the surface  $z = x^2y$  at the point  $(2, 1, 4)$ . Hence, state the normal vector of the tangent plane.

### Solution

Let  $f(x, y) = x^2y$ . Then  $f_x(x, y) = 2xy$  and  $f_y(x, y) = x^2$ . Hence, the equation of the tangent plane at  $(2, 1, 4)$  is given by

$$\begin{aligned} z &= 4 + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1) \\ &= 4 + 4(x - 2) + 4(y - 1) \\ &= 4x + 4y - 8 \\ \implies 4x + 4y - z &= 8 \end{aligned}$$

$$4x + 4y - z = 8, \mathbf{n} = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix}$$

## Problem 11.

The volume of a right-circular cone of radius  $r$  cm and height  $h$  cm is denoted by  $V$ . If  $h$  increases from 10 cm to 10.01 cm and  $r$  decreases from 12 cm to 11.95 cm, use a linear approximation to estimate the volume of the cone after the changes.

### Solution

Let  $V(r, h) = \frac{1}{3}\pi r^2 h$  be the volume of the cone. We have  $V_r(r, h) = \frac{2}{3}\pi r h$  and  $V_h(r, h) = \frac{1}{3}\pi r^2$ . The equation of the tangent plane at  $r = 12$  and  $h = 10$  is given by

$$\begin{aligned} v &= V(12, 10) + V_r(12, 10)(r - 12) + V_h(12, 10)(h - 10) \\ &= \frac{1}{3}\pi(12^2)(10) + \frac{2}{3}\pi(12)(10)(r - 12) + \frac{1}{3}\pi(12^2)(h - 10) \\ &= 16\pi(5r + 3h - 60) \end{aligned}$$

Evaluating at  $r = 11.95$  and  $h = 10.01$ , we have

$$v = 16\pi [5 \cdot 11.95 + 3 \cdot 10.01 - 60] = 476.48\pi$$

The volume of the cone after the changes is approximately  $476.48\pi$  cm<sup>3</sup>.

## Problem 12.

The radius of a right-circular cylinder is measured with an error of at most 2%, and the height is measured with an error of at most 4%. Approximate the maximum possible percentage error in the volume of the cylinder calculated from these measurements.

### Solution

Let the volume of the cylinder be  $V = \pi r^2 h$ .

$$\begin{aligned} dV &= \frac{\partial V}{\partial r} dr + \frac{\partial f}{\partial h} dh \\ &= 2\pi rh dr + \pi r^2 dh \\ \implies \frac{dV}{V} &= \frac{2\pi rh dr + \pi r^2 dh}{\pi r^2 h} \\ &= 2\frac{dr}{r} + \frac{dh}{h} \end{aligned}$$

Note that  $\frac{dV}{V}$  measures the percentage error of the volume  $V$ , while  $\frac{dr}{r}$  and  $\frac{dh}{h}$  measure the percentage error of the radius and height respectively. Hence,

$$\max \frac{dV}{V} = 2(2\%) + 4\% = 8\%$$

The maximum percentage error in the volume of the cylinder is 8%.

## Problem 13.

On a certain mountain, the elevation  $z$  above a point  $(x, y)$  in a horizontal  $xy$ -plane that lies at sea level is  $z = 2000 - 2x^2 - 4y^2$  ft. The positive  $x$ -axis points east, and the positive  $y$ -axis points north. A climber is at the point  $(-20, 5, 1100)$ .

- (a) If the climber uses a compass reading to walk due northeast, will he ascend or descend? Find this rate.
- (b) Find the direction where the climber should walk to travel a level path.

### Solution

#### Part (a)

Let  $f(x, y) = 2000 - 2x^2 - 4y^2$ . Then  $f_x(x, y) = -4x$  and  $f_y(x, y) = -8y$ . Hence,

$$\nabla f = \begin{pmatrix} -4x \\ -8y \end{pmatrix} = -4 \begin{pmatrix} x \\ 2y \end{pmatrix}$$

Note that the vector  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  points northeast.

$$\nabla f \cdot \widehat{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = -4 \begin{pmatrix} x \\ 2y \end{pmatrix} \cdot \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2\sqrt{2}(x + 2y)$$

Evaluating at  $(-20, 5, 1100)$ , the instantaneous rate of change of the climber's altitude would be  $-2\sqrt{2}(-20 + 2 \cdot 5) = 20\sqrt{2}$  ft/s.

The climber would ascend at a rate of  $20\sqrt{2}$  feet per second.

#### Part (b)

For a level path, the instantaneous rate of change of the climber's altitude should be 0.

Let the direction of the climber be  $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix}$ . Consider  $D_{\mathbf{u}}f(x, y)|_{(-20,5)} = 0$ .

$$\begin{aligned} D_{\mathbf{u}}f(x, y)|_{(-20,5)} &= 0 \\ \implies -4 \begin{pmatrix} -20 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} &= 0 \\ \implies -2a + b &= 0 \\ \implies b &= 2a \end{aligned}$$

We hence have  $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

The climber should walk in the direction of  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ .

## Problem 14.

Find the absolute maximum and minimum values of  $f(x, y) = 3xy - 6x - 3y + 7$  on the closed triangular region  $R$  with vertices  $(0, 0)$ ,  $(3, 0)$  and  $(0, 5)$ .

### Solution

Note that  $f_x(x, y) = 3y - 6$  and  $f_y(x, y) = 3x - 3$ , whence  $f_{xx}(x, y) = f_{yy}(x, y) = 0$  and  $f_{xy} = 3$ . For stationary points,  $\nabla f = \mathbf{0} \implies \begin{pmatrix} 3y - 6 \\ 3x - 3 \end{pmatrix} = \mathbf{0} \implies x = 1, y = 2$ .

Consider the nature of the stationary point at  $(1, 2)$ . We have

$$D = f_{xx}(1, 2)f_{yy}(1, 2) - [f_{xy}(1, 2)]^2 = -9 < 0$$

Hence, by the second derivative test, we see that  $f(x, y)$  has a saddle point at  $(1, 2)$ . Thus, the extrema of  $f(x, y)$  must occur along its boundary.

Note that the boundary of  $f(x, y)$  is given by

- $x = 0, y \in [0, 5]$
- $x \in [0, 3], y = 0$
- $x \in [0, 3], y = 5 - \frac{5}{3}x$

**Case 1:**  $x = 0, y \in [0, 5]$ . We have that  $f(0, y) = -3y + 7$ , which clearly attains a maximum of 7 at  $y = 0$  and a minimum of  $-8$  at  $y = 5$ .

**Case 2:**  $x \in [0, 3], y = 0$ . We have that  $f(x, 0) = -6x + 7$ , which clearly attains a maximum of 7 at  $x = 0$  and a minimum of  $-11$  at  $x = 3$ .

**Case 3:**  $x \in [0, 3], y = 5 - \frac{5}{3}x$ . Observe that

$$\begin{aligned} f\left(x, 5 - \frac{5}{3}x\right) &= 3x\left(5 - \frac{5}{3}x\right) - 6x - 3\left(5 - \frac{5}{3}x\right) + 7 \\ &= -5x^2 + 14x - 8 \\ &= -(x - 2)(5x - 4) \end{aligned}$$

is concave down and has a turning point at  $x = 1.4$ . Hence, the function clearly attains a maximum of 1.8 when  $x = 1.4$  and a minimum of  $-11$  when  $x = 3$  (note that at  $x = 0$ , the function returns  $-8$ ).

Hence, the maximum of  $f(x, y)$  is 7, while the minimum is  $-11$ .

$\max f(x, y) = 7, \min f(x, y) = -11$
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## Problem 15.

Find the dimensions of a rectangular box, open at the top, having a volume of  $32 \text{ cm}^3$ , and requiring the least amount of material for its construction.

### Solution

Let the box have side lengths of  $x$ ,  $y$  and  $z$  cm. Given that the volume of the box is fixed at  $32 \text{ cm}^3$ , we have

$$xyz = 32 \implies z = \frac{32}{xy}$$

Let the surface area of the box be measured by  $f(x, y)$ . Then

$$\begin{aligned} f(x, y) &= xy + 2yz + 2xz \\ &= xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right) \\ &= xy + 64x^{-1} + 64y^{-1} \end{aligned}$$

Note that  $\nabla f = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix} = \begin{pmatrix} y - 64x^{-2} \\ x - 64y^{-2} \end{pmatrix}$ . For stationary points,  $\nabla f = \mathbf{0}$ . We hence obtain

$$\begin{cases} y = 64x^{-2} \\ x = 64y^{-2} \end{cases} \implies \begin{cases} xy^2 = 64 \\ xy^2 = 64 \end{cases} \implies x^3y^3 = 64^2 \implies xy = 16$$

Hence,  $x = \frac{x^2y}{xy} = \frac{64}{16} = 4$ , implying that  $y = 4$  and  $z = 2$ . Hence,  $f(x, y)$  has a stationary point at  $(4, 4, 2)$ .

We now consider the nature of this stationary point. Note that  $f_{xx}(x, y) = 128x^{-3}$ ,  $f_{yy} = 128y^{-3}$  and  $f_{xy} = 1$ . Hence,

$$D = f_{xx}(4, 4)f_{yy}(4, 4) - [f_{xy}(4, 4)]^2 = 3$$

Since  $D > 0$  and  $f_{xx}(4, 4) = 2 > 0$ , by the second derivative test,  $f(x, y)$  attains a minimum at  $(4, 4, 2)$ . Thus, the amount of material required is lowest for a box of dimension  $4 \times 4 \times 2$ .

The dimensions of the box are 4 cm by 4 cm by 2 cm.

**Problem 16.**

Find the quadratic approximation of  $f(x, y) = x^2y + xy^2$  around the point  $(1, 1)$ .

**Solution**

Taking partial derivatives, we have

$$\begin{aligned}f_x(x, y) &= 2xy + y^2 \\f_y(x, y) &= 2xy + x^2 \\f_{xx}(x, y) &= 2y \\f_{xy}(x, y) &= 2x + 2y \\f_{yy}(x, y) &= 2x\end{aligned}$$

Hence, the required quadratic approximation  $Q(x, y)$  is given by

$$\begin{aligned}Q(x, y) &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\&\quad + \frac{1}{2}f_{xx}(1, 1)(x - 1)^2 + f_{xy}(1, 1)(x - 1)(y - 1) + \frac{1}{2}f_{yy}(1, 1)(y - 1)^2 \\&= 2 + 3(x - 1) + 3(y - 1) + (x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2 \\&= 2 + (3x - 3) + (3y - 3) + (x^2 - 2x + 1) + (4xy - 4x - 4y + 4) + (y^2 - 2y + 1) \\&= 2 - 3x - 3y + 4xy + x^2 + y^2\end{aligned}$$

$$Q(x, y) = 2 - 3x - 3y + 4xy + x^2 + y^2$$