Problem 1.

Find the natural domain of the function f for the following:

(a) $f(x, y) = \sqrt{1 - x^2 - y^2}$

(b) $f(x, y) = \ln(x^2 - y)$

(c) $f(x, y) = \arcsin(x + y)$

(d) $f(x, y) = \dfrac{1}{x^2 - y^2}$

Solution

Part (a)

Observe that the argument of the square root must be non-negative. Hence, $1 - x^2 - y^2 \geq 0 \implies x^2 + y^2 \leq 1$.

$$\boxed{D_f = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}}$$

Part (b)

Observe that the argument of the natural log must be positive. Hence, $x^2 - y > 0 \implies y < x^2$.

$$\boxed{D_f = \{(x, y) \in \mathbb{R}^2 : y < x^2\}}$$

Part (c)

Observe that the argument of arcsin must be within the range of sin, i.e. between -1 and 1 inclusive. Hence, $-1 \leq x + y \leq 1$.

$$\boxed{D_f = \{(x, y) \in \mathbb{R}^2 : -1 \leq x + y \leq 1\}}$$

Part (d)

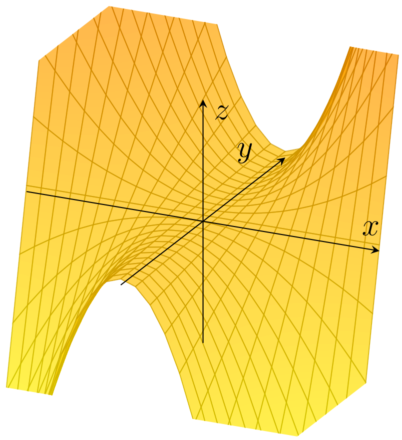
Observe that the denominator must be non-zero. Hence, $x^2 - y^2 \neq 0 \implies y^2 \neq x^2 \implies y \neq x \vee y \neq -x$.

$$\boxed{D_f = \{(x, y) \in \mathbb{R}^2 : y \neq x \vee y \neq -x\}}$$

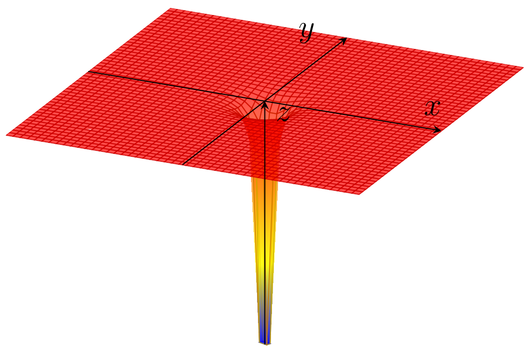
Problem 2.

Identify the correct equations of the following surfaces in 3-D space.

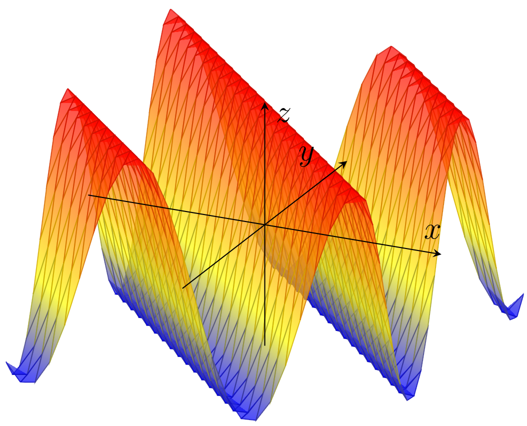
- $z = \cos(x + y)$
- $z = x^2y + 1$
- $z = 3 - x + y$
- $z = -\dfrac{1}{\sqrt{x^2 + y^2}}$



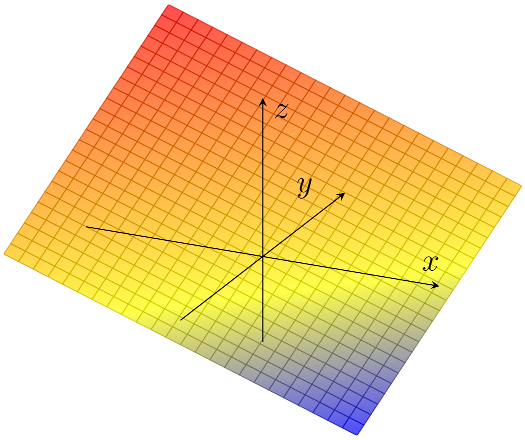
(a)



(b)



(c)



(d)

Part (a)

$$\boxed{z = x^2 y + 1}$$

Part (b)

$$\boxed{z = -\frac{1}{\sqrt{x^2 + y^2}}}$$

Part (c)

$$\boxed{z = \cos(x + y)}$$

Part (d)

$$\boxed{z = 3 - x + y}$$

Problem 3.

Let $f(x, y) = x^2 - 2x^3 + 3xy$. Find an equation of the level curve that passes through the point

(a) $(-1, 1)$

(b) $(2, -1)$

(c) $(1, 5)$

Solution

Part (a)

Note that $f(-1, 1) = 0$. Hence, the level curve is given by $\boxed{x^2 - 2x^3 + 3xy = 0}$.

Part (b)

Note that $f(2, -1) = -18$. Hence, the level curve is given by $\boxed{x^2 - 2x^3 + 3xy = -18}$.

Part (c)

Note that $f(1, 5) = 14$. Hence, the level curve is given by $\boxed{x^2 - 2x^3 + 3xy = 14}$.

Problem 4.

If $V(x, y)$ is the voltage or potential at a point (x, y) in the xy-plane, then the level curves of V are called equipotential curves. Along such a curve, the voltage remains constant. Given that

$$V(x, y) = \frac{8}{\sqrt{16 + x^2 + y^2}}$$

find an equation of the equipotential curves at which

(a) $V = 2.0$

(b) $V = 1.0$

(c) $V = 0.5$

Solution

Rearranging the given equation, we have

$$x^2 + y^2 = \frac{64}{V^2} - 16.$$

Part (a)

When $V = 2.0$, we have $x^2 + y^2 = \dfrac{64}{2.0^2} - 16 = 0$, whence $x = 0$ and $y = 0$.

$$\boxed{x = 0 \wedge y = 0}$$

Part (b)

When $V = 1.0$, we have $x^2 + y^2 = \dfrac{64}{1.0^2} - 16 = 48$.

$$\boxed{x^2 + y^2 = 48}$$

Part (c)

When $V = 0.5$, we have $x^2 + y^2 = \dfrac{64}{0.5^2} - 16 = 240$.

$$\boxed{x^2 + y^2 = 240}$$

Problem 5.

Given that $f(x,y) = x^4 \sin(xy^3)$, find $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$ and $f_{yx}(x,y)$.

Solution

Differentiating f with respect to x,

$$\boxed{f_x(x,y) = 4x^3 \sin(xy^3) + x^4y^3 \cos(xy^3)}$$

Differentiating f with respect to y,

$$\boxed{f_y(x,y) = 3x^5y^2 \cos(xy^3)}$$

Differentiating f_x with respect to y,

$$\begin{aligned} f_{xy}(x,y) &= 12x^4y^2 \cos\left(xy^3\right) + x^4 \left[3y^2 \cos\left(xy^3\right) - 3xy^5 \sin\left(xy^3\right)\right] \\ &= 15x^4y^2 \cos\left(xy^3\right) - 3x^5y^5 \sin\left(xy^3\right) \end{aligned}$$

$$\boxed{f_{xy}(x,y) = 15x^4y^2 \cos(xy^3) - 3x^5y^5 \sin(xy^3)}$$

Differentiating f_y with respect to x,

$$\begin{aligned} f_{yx}(x,y) &= 3y^2 \left[5x^4 \cos\left(xy^3\right) - x^5y^3 \sin\left(xy^3\right)\right] \\ &= 15x^4y^2 \cos\left(xy^3\right) - 3x^5y^5 \sin\left(xy^3\right) \end{aligned}$$

$$\boxed{f_{yx}(x,y) = 15x^4y^2 \cos(xy^3) - 3x^5y^5 \sin(xy^3)}$$

Problem 6.

Given that $z = x^2 y$, $x = t^2$, $y = t^3$, use the chain rule to find $\dfrac{\mathrm{d}z}{\mathrm{d}t}$ in terms of t.

Solution

$$\begin{aligned}
\frac{\mathrm{d}z}{\mathrm{d}t} &= \frac{\partial z}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y} \cdot \frac{\mathrm{d}y}{\mathrm{d}t} \\
&= 2xy \cdot 2t + x^2 \cdot 3t^2 \\
&= 2t^2 t^3 \cdot 2t + (t^2)^2 \cdot 3t^2 \\
&= 7t^6
\end{aligned}$$

$$\boxed{\frac{\mathrm{d}z}{\mathrm{d}t} = 7t^6}$$

Problem 7.

Find the gradient of $f(x, y) = 3x^2y$ at the point $(1, 2)$ and use it to calculate the directional derivative of f at $(1, 2)$ in the direction of the vector $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$.

Solution

Note that $f_x(x, y) = 6xy$ and $f_y(x, y) = 3x^2$. Hence, ∇f at $(1, 2)$ is given by $\begin{pmatrix} 12 \\ 3 \end{pmatrix}$.

$$\boxed{\text{The derivative at } (1, 2) \text{ is } \begin{pmatrix} 12 \\ 3 \end{pmatrix}.}$$

Observe that the directional derivative of f in the direction of $\mathbf{u}$ at $(1, 2)$ is given by

$$\nabla f \cdot \hat{\mathbf{u}} = \begin{pmatrix} 12 \\ 3 \end{pmatrix} \cdot \frac{1}{5} \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \frac{48}{5}$$

$$\boxed{\text{The instantaneous rate of change at } (1, 2) \text{ in the direction of } \mathbf{u} \text{ is } \frac{48}{5}.}$$

Problem 8.

Suppose that a point moves along the intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x = \frac{2}{3}$. Find the rate of z with respect to y when the point is at $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$.

Solution

Note that $x^2 + y^2 + z^2 = 1 \implies z = \pm\sqrt{1 - x^2 - y^2}$. Given that the object we want (the rate of change of z with respect to y) will later be evaluated when $z = \frac{2}{3} > 0$, we consider only the positive branch. Let $f(x, y) = \sqrt{1 - x^2 - y^2}$. Then $f_y(x, y) = \dfrac{-y}{\sqrt{1 - x^2 - y^2}}$. Evaluating at the desired point, we get, $f_y\left(\frac{2}{3}, \frac{1}{3}\right) = \dfrac{-1/3}{\sqrt{1 - (2/3)^2 - (1/3)^2}} = -\frac{1}{2}$.

$$\boxed{\text{The rate of change of } z \text{ with respect to } y \text{ at } \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right) \text{ is } -\frac{1}{2}.}$$

Problem 9.

(a) The Cauchy-Riemann equations are such that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ for $u(x, y)$ and $v(x, y)$. Show that $u = e^x \cos y$, $v = e^x \sin y$ satisfy the Cauchy-Riemann equations.

(b) Show that the function $f(x, y) = e^x \sin y + e^y \cos x$ satisfies that Laplace equation, i.e. $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$.

(c) If $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations, state the conditions for both u and v to satisfy the Laplace equation.

Solution

Part (a)

Differentiating u with respect to x, $\dfrac{\partial u}{\partial x} = e^x \cos y$. Differentiating v with respect to y, $\dfrac{\partial v}{\partial y} = e^x \cos y$. Hence, $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$.

Differentiating u with respect to y, $\dfrac{\partial u}{\partial y} = -e^x \sin y$. Differentiating v with respect to x, $\dfrac{\partial v}{\partial x} = e^x \sin y$. Hence, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Thus, u and v satisfy the Cauchy-Riemann equations.

Part (b)

Differentiating f twice with respect to x,

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}(e^x \sin y - e^y \sin x) = e^x \sin y - e^y \cos x$$

Differentiating f twice with respect to y,

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}(e^x \cos y + e^y \cos x) = -e^x \sin y + e^y \cos x$$

Hence,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (e^x \sin y - e^y \cos x) + (-e^x \sin y + e^y \cos x) = 0$$

Thus, $f(x, y) = e^x \sin y + e^y \cos x$ satisfies the Laplace equation.

Part (c)

Suppose $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations. This gives

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Differentiating with respect to x, we obtain

$$\begin{cases} u_{xx} = v_{yx} \\ u_{yx} = -v_{xx} \end{cases}$$

Differentiating with respect to y, we obtain

$$\begin{cases} u_{xy} = v_{yy} \\ u_{yy} = -v_{xy} \end{cases}$$

This gives

$$\begin{cases} u_{xx} + u_{yy} = v_{yx} - v_{xy} \\ v_{xx} + v_{yy} = -u_{yx} + u_{xy} \end{cases}$$

Hence, if u and v both satisfy the Laplace equation, we require

$$\begin{cases} v_{yx} - v_{xy} = 0 \\ -u_{yx} + u_{xy} = 0 \end{cases}$$

which gives the conditions $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$.

$$\boxed{u_{xy} = u_{yx},\ v_{xy} = v_{yx}}$$

Problem 10.

Find the equation of the tangent plane to the surface $z = x^2 y$ at the point $(2, 1, 4)$. Hence, state the normal vector of the tangent plane.

Solution

Let $f(x, y) = x^2 y$. Then $f_x(x, y) = 2xy$ and $f_y(x, y) = x^2$. Hence, the equation of the tangent plane at $(2, 1, 4)$ is given by

$$\begin{aligned} z &= 4 + f_x(2,1)(x-2) + f_y(y-1) \\ &= 4 + 4(x-2) + 4(y-1) \\ &= 4x + 4y - 8 \\ \implies 4x + 4y - z &= 8 \end{aligned}$$

$$\boxed{4x + 4y - z = 8,\ \mathbf{n} = \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix}}$$

Problem 11.

The volume of a right-circular cone of radius r cm and height h cm is denoted by V. If h increases from 10 cm to 10.01 cm and r decreases from 12 cm to 11.95 cm, use a linear approximation to estimate the volume of the cone after the changes.

Solution

Let $V(r,h) = \dfrac{1}{3}\pi r^2 h$ be the volume of the cone. We have $V_r(r,h) = \dfrac{2}{3}\pi r h$ and $V_h(r,h) = \dfrac{1}{3}\pi r^2$. The equation of the tangent plane at $r = 12$ and $h = 10$ is given by

$$\begin{aligned} v &= V(12,10) + V_r(12,10)(r-12) + V_h(12,10)(h-10) \\ &= \frac{1}{3}\pi(12^2)(10) + \frac{2}{3}\pi(12)(10)(r-12) + \frac{1}{3}\pi(12^2)(h-10) \\ &= 16\pi(5r + 3h - 60) \end{aligned}$$

Evaluating at $r = 11.95$ and $h = 10.01$, we have

$$v = 16\pi\left[5 \cdot 11.95 + 3 \cdot 10.01 - 60\right] = 476.48\pi$$

$$\boxed{\text{The volume of the cone after the changes is approximately } 476.48\pi \text{ cm}^3.}$$

Problem 12.

The radius of a right-circular cylinder is measured with an error of at most 2%, and the height is measured with an error of at most 4%. Approximate the maximum possible percentage error in the volume of the cylinder calculated from these measurements.

Solution

Let the volume of the cylinder be $V = \pi r^2 h$.

$$\begin{aligned} \mathrm{d}V &= \frac{\partial V}{\partial r}\,\mathrm{d}r + \frac{\partial f}{\partial h}\,\mathrm{d}h \\ &= 2\pi r h\,\mathrm{d}r + \pi r^2\,\mathrm{d}h \\ \implies \frac{\mathrm{d}V}{V} &= \frac{2\pi r h\,\mathrm{d}r + \pi r^2\,\mathrm{d}h}{\pi r^2 h} \\ &= 2\frac{\mathrm{d}r}{r} + \frac{\mathrm{d}h}{h} \end{aligned}$$

Note that $\frac{\mathrm{d}V}{V}$ measures the percentage error of the volume V, while $\frac{\mathrm{d}r}{r}$ and $\frac{\mathrm{d}h}{h}$ measure the percentage error of the radius and height respectively. Hence,

$$\max \frac{\mathrm{d}V}{V} = 2(2\%) + 4\% = 8\%$$

$\boxed{\text{The maximum percentage error in the volume of the cylinder is 8\%.}}$

Problem 13.

On a certain mountain, the elevation z above a point (x, y) in a horizontal xy-plane that lies at sea level is $z = 2000 - 2x^2 - 4y^2$ ft. The positive x-axis points east, and the positive y-axis points north. A climber is at the point $(-20, 5, 1100)$.

(a) If the climber uses a compass reading to walk due northeast, will he ascend or descend? Find this rate.

(b) Find the direction where the climber should walk to travel a level path.

Solution

Part (a)

Let $f(x, y) = 2000 - 2x^2 - 4y^2$. Then $f_x(x, y) = -4x$ and $f_y(x, y) = -8y$. Hence,

$$\nabla f = \begin{pmatrix} -4x \\ -8y \end{pmatrix} = -4 \begin{pmatrix} x \\ 2y \end{pmatrix}$$

Note that the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ points northeast.

$$\nabla f \cdot \widehat{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = -4 \begin{pmatrix} x \\ 2y \end{pmatrix} \cdot \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2\sqrt{2}\,(x + 2y)$$

Evaluating at $(-20, 5, 1100)$, the instantaneous rate of change of the climber's altitude would be $-2\sqrt{2}\,(-20 + 2 \cdot 5) = 20\sqrt{2}$ ft/s.

$$\boxed{\text{The climber would ascend at a rate of } 20\sqrt{2} \text{ feet per second.}}$$

Part (b)

For a level path, the instantaneous rate of change of the climber's altitude should be 0. Let the direction of the climber be $u = \begin{pmatrix} a \\ b \end{pmatrix}$. Consider $D_{\mathbf{u}} f(x, y)|_{(-20,5)} = 0$.

$$\begin{aligned} D_{\mathbf{u}} f(x, y)|_{(-20,5)} &= 0 \\ \implies -4 \begin{pmatrix} -20 \\ 10 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} &= 0 \\ \implies \qquad -2a + b &= 0 \\ \implies \qquad b &= 2a \end{aligned}$$

We hence have $\mathbf{u} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 2a \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

$$\boxed{\text{The climber should walk in the direction of } \begin{pmatrix} 1 \\ 2 \end{pmatrix}.}$$

Problem 14.

Find the absolute maximum and minimum values of $f(x,y) = 3xy - 6x - 3y + 7$ on the closed triangular region R with vertices $(0,0)$, $(3,0)$ and $(0,5)$.

Solution

Note that $f_x(x,y) = 3y - 6$ and $f_y(x,y) = 3x - 3$, whence $f_{xx}(x,y) = f_{yy}(x,y) = 0$ and $f_{xy} = 3$. For stationary points, $\nabla f = \mathbf{0} \implies \begin{pmatrix} 3y-6 \\ 3x-3 \end{pmatrix} = \mathbf{0} \implies x = 1, y = 2$.

Consider the nature of the stationary point at $(1,2)$. We have

$$D = f_{xx}(1,2)f_{yy}(1,2) - [f_{xy}(1,2)]^2 = -9 < 0$$

Hence, by the second derivative test, we see that $f(x,y)$ has a saddle point at $(1,2)$. Thus, the extrema of $f(x,y)$ must occur along its boundary.

Note that the boundary of $f(x,y)$ is given by

- $x = 0$, $y \in [0,5]$
- $x \in [0,3]$, $y = 0$
- $x \in [0,3]$, $y = 5 - \frac{5}{3}x$

Case 1: $x = 0$, $y \in [0,5]$. We have that $f(0,y) = -3y + 7$, which clearly attains a maximum of 7 at $y = 0$ and a minimum of -8 at $y = 5$.

Case 2: $x \in [0,3]$, $y = 0$. We have that $f(x,0) = -6x + 7$, which clearly attains a maximum of 7 at $x = 0$ and a minimum of -11 at $x = 3$.

Case 3: $x \in [0,3]$, $y = 5 - \frac{5}{3}x$. Observe that

$$\begin{aligned} f\left(x, 5 - \frac{5}{3}x\right) &= 3x\left(5 - \frac{5}{3}x\right) - 6x - 3\left(5 - \frac{5}{3}x\right) + 7 \\ &= -5x^2 + 14x - 8 \\ &= -(x-2)(5x-4) \end{aligned}$$

is concave down and has a turning point at $x = 1.4$. Hence, the function clearly attains a maximum of 1.8 when $x = 1.4$ and a minimum of -11 when $x = 3$ (note that at $x = 0$, the function returns -8).

Hence, the maximum of $f(x,y)$ is 7, while the minimum is -11.

$$\boxed{\max f(x,y) = 7, \min f(x,y) = -11}$$

Problem 15.

Find the dimensions of a rectangular box, open at the top, having a volume of 32 cm^3, and requiring the least amount of material for its construction.

Solution

Let the box have side lengths of x, y and z cm. Given that the volume of the box is fixed at 32 cm^3, we have

$$xyz = 32 \implies z = \frac{32}{xy}$$

Let the surface area of the box be measured by $f(x, y)$. Then

$$\begin{aligned} f(x,y) &= xy + 2yz + 2xz \\ &= xy + 2y\left(\frac{32}{xy}\right) + 2x\left(\frac{32}{xy}\right) \\ &= xy + 64x^{-1} + 64y^{-1} \end{aligned}$$

Note that $\nabla f = \begin{pmatrix} f_x(x,y) \\ f_y(x,y) \end{pmatrix} = \begin{pmatrix} y - 64x^{-2} \\ x - 64y^{-2} \end{pmatrix}$. For stationary points, $\nabla f = \mathbf{0}$. We hence obtain

$$\begin{cases} y = 64x^{-2} \\ x = 64y^{-2} \end{cases} \implies \begin{cases} yx^2 = 64 \\ xy^2 = 64 \end{cases} \implies x^3y^3 = 64^2 \implies xy = 16$$

Hence, $x = \dfrac{x^2y}{xy} = \dfrac{64}{16} = 4$, implying that $y = 4$ and $z = 2$. Hence, $f(x, y)$ has a stationary point at $(4, 4, 2)$.

We now consider the nature of this stationary point. Note that $f_{xx}(x, y) = 128x^{-3}$, $f_{yy} = 128y^{-3}$ and $f_{xy} = 1$. Hence,

$$D = f_{xx}(4,4)f_{yy}(4,4) - [f_{xy}(4,4)]^2 = 3$$

Since $D > 0$ and $f_{xx}(4, 4) = 2 > 0$, by the second derivative test, $f(x, y)$ attains a minimum at $(4, 4, 2)$. Thus, the amount of material required is lowest for a box of dimension $4 \times 4 \times 2$.

$\boxed{\text{The dimensions of the box are 4 cm by 4 cm by 2 cm.}}$

Problem 16.

Find the quadratic approximation of $f(x, y) = x^2y + xy^2$ around the point $(1, 1)$.

Solution

Taking partial derivatives, we have

$$\begin{aligned}
f_x(x, y) &= 2xy + y^2 \\
f_y(x, y) &= 2xy + x^2 \\
f_{xx}(x, y) &= 2y \\
f_{xy}(x, y) &= 2x + 2y \\
f_{yy}(x, y) &= 2x
\end{aligned}$$

Hence, the required quadratic approximation $Q(x, y)$ is given by

$$\begin{aligned}
Q(x, y) &= f(1, 1) + f_x(1, 1)(x - 1) + f_y(1, 1)(y - 1) \\
&\qquad + \frac{1}{2}f_{xx}(1, 1)(x - 1)^2 + f_{xy}(1, 1)(x - 1)(y - 1) + \frac{1}{2}f_{yy}(1, 1)(y - 1)^2 \\
&= 2 + 3(x - 1) + 3(y - 1) + (x - 1)^2 + 4(x - 1)(y - 1) + (y - 1)^2 \\
&= 2 + (3x - 3) + (3y - 3) + (x^2 - 2x + 1) + (4xy - 4x - 4y + 4) + (y^2 - 2y + 1) \\
&= 2 - 3x - 3y + 4xy + x^2 + y^2
\end{aligned}$$

$$\boxed{Q(x, y) = 2 - 3x - 3y + 4xy + x^2 + y^2}$$