CHAPTER 8
Advanced Counting Techniques

SECTION 8.1 Applications of Recurrence Relations

2. **a)** A permutation of a set with n elements consists of a choice of a first element (which can be done in n ways), followed by a permutation of a set with $n-1$ elements. Therefore $P_n = nP_{n-1}$. Note that $P_0 = 1$, since there is just one permutation of a set with no objects, namely the empty sequence.

b) $P_n = nP_{n-1} = n(n-1)P_{n-2} = \cdots = n(n-1)\cdots 2 \cdot 1 \cdot P_0 = n!$

4. This is similar to Exercise 3 and solved in exactly the same way. The recurrence relation is $a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10} + a_{n-20} + a_{n-50} + a_{n-100}$. It would be quite tedious to write down the 100 initial conditions.

6. **a)** Let s_n be the number of such sequences. A string ending in n must consist of a string ending in something less than n, followed by an n as the last term. Therefore the recurrence relation is $s_n = s_{n-1} + s_{n-2} + \cdots + s_2 + s_1$. Here is another approach, with a more compact form of the answer. A sequence ending in n is either a sequence ending in $n-1$, followed by n (and there are clearly s_{n-1} of these), or else it does not contain $n-1$ as a term at all, in which case it is *identical* to a sequence ending in $n-1$ in which the $n-1$ has been replaced by an n (and there are clearly s_{n-1} of these as well). Therefore $s_n = 2s_{n-1}$. Finally we notice that we can derive the second form from the first (or vice versa) algebraically (for example, $s_4 = 2s_3 = s_3 + s_3 = s_3 + s_2 + s_2 = s_3 + s_2 + s_1$).

b) We need two initial conditions if we use the second formulation above, $s_1 = 1$ and $s_2 = 1$ (otherwise, our argument is invalid, because the first and last terms are the same). There is one sequence ending in 1, namely the sequence with just this 1 in it, and there is only the sequence $1, 2$ ending in 2. If we use the first formulation above, then we can get by with just the initial condition $s_1 = 1$.

c) Clearly the solution to this recurrence relation and initial condition is $s_n = 2^{n-2}$ for all $n \geq 2$.

8. This is very similar to Exercise 7, except that we need to go one level deeper.

a) Let a_n be the number of bit strings of length n containing three consecutive 0's. In order to construct a bit string of length n containing three consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-3$ containing three consecutive 0's, or we could start with a 000 and follow with any string of length $n-3$. These four cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3} + 2^{n-3}$.

b) There are no bit strings of length 0, 1, or 2 containing three consecutive 0's, so the initial conditions are $a_0 = a_1 = a_2 = 0$.

c) We will compute a_3 through a_7 using the recurrence relation:

$$a_3 = a_2 + a_1 + a_0 + 2^0 = 0 + 0 + +0 + 1 = 1$$
$$a_4 = a_3 + a_2 + a_1 + 2^1 = 1 + 0 + 0 + 2 = 3$$
$$a_5 = a_4 + a_3 + a_2 + 2^2 = 3 + 1 + 0 + 4 = 8$$
$$a_6 = a_5 + a_4 + a_3 + 2^3 = 8 + 3 + 1 + 8 = 20$$
$$a_7 = a_6 + a_5 + a_4 + 2^4 = 20 + 8 + 3 + 16 = 47$$

Thus there are 47 bit strings of length 7 containing three consecutive 0's.

10. First let us solve this problem without using recurrence relations at all. It is clear that the only strings that do not contain the string 01 are those that consist of a string of 1's follows by a string of 0's. The string can consist of anywhere from 0 to n 1's, so the number of such strings is $n+1$. All the rest have at least one occurrence of 01. Therefore the number of bit strings that contain 01 is $2^n - (n+1)$. However, this approach does not meet the instructions of this exercise.

a) Let a_n be the number of bit strings of length n that contain 01. If we want to construct such a string, we could start with a 1 and follow it with a bit string of length $n-1$ that contains 01, and there are a_{n-1} of these. Alternatively, for any k from 1 to $n-1$, we could start with k 0's, follow this by a 1, and then follow this by any $n-k-1$ bits. For each such k there are 2^{n-k-1} such strings, since the final bits are free. Therefore the number of such strings is $2^0 + 2^1 + 2^2 + \cdots + 2^{n-2}$, which equals $2^{n-1} - 1$. Thus our recurrence relation is $a_n = a_{n-1} + 2^{n-1} - 1$. It is valid for all $n \geq 2$.

b) The initial conditions are $a_0 = a_1 = 0$, since no string of length less than 2 can have 01 in it.

c) We will compute a_2 through a_7 using the recurrence relation:

$$a_2 = a_1 + 2^1 - 1 = 0 + 2 - 1 = 1$$
$$a_3 = a_2 + 2^2 - 1 = 1 + 4 - 1 = 4$$
$$a_4 = a_3 + 2^3 - 1 = 4 + 8 - 1 = 11$$
$$a_5 = a_4 + 2^4 - 1 = 11 + 16 - 1 = 26$$
$$a_6 = a_5 + 2^5 - 1 = 26 + 32 - 1 = 57$$
$$a_7 = a_6 + 2^6 - 1 = 57 + 64 - 1 = 120$$

Thus there are 120 bit strings of length 7 containing 01. Note that this agrees with our nonrecursive analysis, since $2^7 - (7+1) = 120$.

12. This is identical to Exercise 11, one level deeper.

a) Let a_n be the number of ways to climb n stairs. In order to climb n stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in a_{n-1} ways) or else start with a step of two stairs and then climb $n-2$ stairs (and this can be done in a_{n-2} ways) or else start with a step of three stairs and then climb $n-3$ stairs (and this can be done in a_{n-3} ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

b) The initial conditions are $a_0 = 1$, $a_1 = 1$, and $a_2 = 2$, since there is one way to climb no stairs (do nothing), clearly only one way to climb one stair, and two ways to climb two stairs (one step twice or two steps at once). Note that the recurrence relation is the same as that for Exercise 9.

c) Each term in our sequence $\{a_n\}$ is the sum of the previous three terms, so the sequence begins $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 7$, $a_5 = 13$, $a_6 = 24$, $a_7 = 44$, $a_8 = 81$. Thus a person can climb a flight of 8 stairs in 81 ways under the restrictions in this problem.

14. a) Let a_n be the number of ternary strings that contain two consecutive 0's. To construct such a string we could start with either a 1 or a 2 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-1}$ ways), or we could start with 01 or 02 and follow with a string containing two consecutive 0's (and this can be done in $2a_{n-2}$ ways), we could start with 00 and follow with any ternary string of length $n-2$ (of which there are clearly 3^{n-2}). Therefore the recurrence relation, valid for all $n \geq 2$, is $a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$.

b) Clearly $a_0 = a_1 = 0$.

c) We will compute a_2 through a_6 using the recurrence relation:

$$\begin{aligned} a_2 &= 2a_1 + 2a_0 + 3^0 = 2 \cdot 0 + 2 \cdot 0 + 1 = 1 \\ a_3 &= 2a_2 + 2a_1 + 3^1 = 2 \cdot 1 + 2 \cdot 0 + 3 = 5 \\ a_4 &= 2a_3 + 2a_2 + 3^2 = 2 \cdot 5 + 2 \cdot 1 + 9 = 21 \\ a_5 &= 2a_4 + 2a_3 + 3^3 = 2 \cdot 21 + 2 \cdot 5 + 27 = 79 \\ a_6 &= 2a_5 + 2a_4 + 3^4 = 2 \cdot 79 + 2 \cdot 21 + 81 = 281 \end{aligned}$$

Thus there are 281 bit strings of length 6 containing two consecutive 0's.

16. **a)** Let a_n be the number of ternary strings that contain either two consecutive 0's or two consecutive 1's. To construct such a string we could start with a 2 and follow with a string containing either two consecutive 0's or two consecutive 1's, and this can be done in a_{n-1} ways. There are other possibilities, however. For each k from 0 to $n-2$, the string could start with $n-1-k$ alternating 0's and 1's, followed by a 2, and then be followed by a string of length k containing either two consecutive 0's or two consecutive 1's. The number of such strings is $2a_k$, since there are two ways for the initial part to alternate. The other possibility is that the string has no 2's at all. Then it must consist $n-k-2$ alternating 0's and 1's, followed by a pair of 0's or 1's, followed by any string of length k. There are $2 \cdot 3^k$ such strings. Now the sum of these quantities as k runs from 0 to $n-2$ is (since this is a geometric progression) $3^{n-1} - 1$. Putting this all together, we have the following recurrence relation, valid for all $n \geq 2$: $a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0 + 3^{n-1} - 1$. (By subtracting this recurrence relation from the same relation with $n-1$ substituted for n, we can obtain the following closed form recurrence relation for this problem: $a_n = 2a_{n-1} + a_{n-2} + 2 \cdot 3^{n-2}$.)

b) Clearly $a_0 = a_1 = 0$.

c) We will compute a_2 through a_6 using the recurrence relation:

$$\begin{aligned} a_2 &= a_1 + 2a_0 + 3^1 - 1 = 0 + 2 \cdot 0 + 3 - 1 = 2 \\ a_3 &= a_2 + 2a_1 + 2a_0 + 3^2 - 1 = 2 + 2 \cdot 0 + 2 \cdot 0 + 9 - 1 = 10 \\ a_4 &= a_3 + 2a_2 + 2a_1 + 2a_0 + 3^3 - 1 = 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 27 - 1 = 40 \\ a_5 &= a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^4 - 1 = 40 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 81 - 1 = 144 \\ a_6 &= a_5 + 2a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 3^5 - 1 \\ &= 144 + 2 \cdot 40 + 2 \cdot 10 + 2 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 + 243 - 1 = 490 \end{aligned}$$

Thus there are 490 ternary strings of length 6 containing two consecutive 0's or two consecutive 1's.

18. **a)** Let a_n be the number of ternary strings that contain two consecutive symbols that are the same. We will develop a recurrence relation for a_n by exploiting the symmetry among the three symbols. In particular, it must be the case that $a_n/3$ such strings start with each of the three symbols. Now let us see how we might specify a string of length n satisfying the condition. We can choose the first symbol in any of three ways. We can follow this by a string that starts with a different symbol but has in it a pair of consecutive symbols; by what we have just said, there are $2a_{n-1}/3$ such strings. Alternatively, we can follow the initial symbol by another copy of itself and then any string of length $n-2$; there are clearly 3^{n-2} such strings. Thus the recurrence relation is $a_n = 3 \cdot ((2a_{n-1}/3) + 3^{n-2}) = 2a_{n-1} + 3^{n-1}$. It is valid for all $n \geq 2$.

b) Clearly $a_0 = a_1 = 0$.

c) We will compute a_2 through a_6 using the recurrence relation:

$$\begin{aligned}
a_2 &= 2a_1 + 3^1 = 2\cdot 0 + 3 = 3\\
a_3 &= 2a_2 + 3^2 = 2\cdot 3 + 9 = 15\\
a_4 &= 2a_3 + 3^3 = 2\cdot 15 + 27 = 57\\
a_5 &= 2a_4 + 3^4 = 2\cdot 57 + 81 = 195\\
a_6 &= 2a_5 + 3^5 = 2\cdot 195 + 243 = 633
\end{aligned}$$

Thus there are 633 bit strings of length 6 containing two consecutive 0's, 1's, or 2's.

20. We let a_n be the number of ways to pay a toll of $5n$ cents. (Obviously there is no way to pay a toll that is not a multiple of 5 cents.)

a) This problem is isomorphic to Exercise 11, so the answer is the same: $a_n = a_{n-1}+a_{n-2}$, with $a_0 = a_1 = 1$.

b) Iterating, we find that $a_9 = 55$.

22. a) We start by computing the first few terms to get an idea of what's happening. Clearly $R_1 = 2$, since the equator, say, splits the sphere into two hemispheres. Also, $R_2 = 4$ and $R_3 = 8$. Let's try to analyze what happens when the n^{th} great circle is added. It must intersect each of the other circles twice (at diametrically opposite points), and each such intersection results in one prior region being split into two regions, as in Exercise 21. There are $n-1$ previous great circles, and therefore $2(n-1)$ new regions. Therefore $R_n = R_{n-1} + 2(n-1)$. If we impose the initial condition $R_1 = 2$, then our values of R_2 and R_3 found above are consistent with this recurrence. Note that $R_4 = 14$, $R_5 = 22$, and so on.

b) We follow the usual technique, as in Exercise 17 in Section 2.4. In the last line we use the familiar formula for the sum of the first $n-1$ positive integers. Note that the formula agrees with the values computed above.

$$\begin{aligned}
R_n &= 2(n-1) + R_{n-1}\\
&= 2(n-1) + 2(n-2) + R_{n-2}\\
&= 2(n-1) + 2(n-2) + 2(n-3) + R_{n-3}\\
&\ \ \vdots\\
&= 2(n-1) + 2(n-2) + 2(n-3) + 2\cdot 1 + R_1\\
&= n(n-1) + 2 = n^2 - n + 2
\end{aligned}$$

24. Let e_n be the number of bit sequences of length n with an even number of 0's. Note that therefore there are $2^n - e_n$ bit sequences with an odd number of 0's. There are two ways to get a bit string of length n with an even number of 0's. It can begin with a 1 and be followed by a bit string of length $n-1$ with an even number of 0's, and there are e_{n-1} of these; or it can begin with a 0 and be followed by a bit string of length $n-1$ with an odd number of 0's, and there are $2^{n-1} - e_{n-1}$ of these. Therefore $e_n = e_{n-1} + 2^{n-1} - e_{n-1}$, or simply $e_n = 2^{n-1}$. See also Exercise 31 in Section 6.4.

26. Let a_n be the number of coverings.

a) We follow the hint. If the right-most domino is positioned vertically, then we have a covering of the left-most $n-1$ columns, and this can be done in a_{n-1} ways. If the right-most domino is positioned horizontally, then there must be another domino directly beneath it, and these together cover the last two columns. The first $n-2$ columns therefore will need to contain a covering by dominoes, and this can be done in a_{n-2} ways. Thus we obtain the Fibonacci recurrence $a_n = a_{n-1} + a_{n-2}$.

b) Clearly $a_1 = 1$ and $a_2 = 2$.

c) The sequence we obtain is just the Fibonacci sequence, shifted by one. The sequence is thus 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, ..., so the answer to this part is 2584.

28. The initial conditions are of course true. We prove the recurrence relation by induction on n, starting with base cases $n = 5$ and $n = 6$, in which cases we find $5f_1 + 3f_0 = 5 = f_5$ and $5f_2 + 3f_1 = 8 = f_6$. Assume the inductive hypothesis. Then we have $5f_{n-4} + 3f_{n-5} = 5(f_{n-5} + f_{n-6}) + 3(f_{n-6} + f_{n-7}) = (5f_{n-5}+3f_{n-6})+(5f_{n-6}+3f_{n-7}) = f_{n-1}+f_{n-2} = f_n$ (we used both the inductive hypothesis and the recursive definition of the Fibonacci numbers). Finally, we prove that f_{5n} is divisible by 5 by induction on n. It is true for $n = 1$, since $f_5 = 5$ is divisible by 5. Assume that it is true for f_{5n}. Then $f_{5(n+1)} = f_{5n+5} = 5f_{5n+1}+3f_{5n}$ is divisible by 5, since both summands in this expression are divisible by 5.

30. **a)** We do this systematically, based on the position of the outermost dot, working from left to right:

$$x_0 \cdot (x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$$
$$x_0 \cdot (x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$$
$$x_0 \cdot ((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$$
$$x_0 \cdot ((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$$
$$x_0 \cdot (((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$$
$$(x_0 \cdot x_1) \cdot (x_2 \cdot (x_3 \cdot x_4))$$
$$(x_0 \cdot x_1) \cdot ((x_2 \cdot x_3) \cdot x_4)$$
$$(x_0 \cdot (x_1 \cdot x_2)) \cdot (x_3 \cdot x_4)$$
$$((x_0 \cdot x_1) \cdot x_2) \cdot (x_3 \cdot x_4)$$
$$(x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))) \cdot x_4$$
$$(x_0 \cdot ((x_1 \cdot x_2) \cdot x_3)) \cdot x_4$$
$$((x_0 \cdot x_1) \cdot (x_2 \cdot x_3)) \cdot x_4$$
$$((x_0 \cdot (x_1 \cdot x_2)) \cdot x_3) \cdot x_4$$
$$(((x_0 \cdot x_1) \cdot x_2) \cdot x_3) \cdot x_4$$

b) We know from Example 5 that $C_0 = 1$, $C_1 = 1$, and $C_3 = 5$. It is also easy to see that $C_2 = 2$, since there are only two ways to parenthesize the product of three numbers. Therefore the recurrence relation tells us that $C_4 = C_0C_3 + C_1C_2 + C_2C_1 + C_3C_0 = 1 \cdot 5 + 1 \cdot 2 + 2 \cdot 1 + 5 \cdot 1 = 14$. We have the correct number of solutions listed above.

c) Here $n = 4$, so the formula gives $\frac{1}{5}C(8, 4) = \frac{1}{5} \cdot 8 \cdot 7 \cdot 6 \cdot 5/4! = 14$.

32. We let a_n be the number of moves required for this puzzle.

a) In order to move the bottom disk off peg 1, we must have transferred the other $n - 1$ disks to peg 3 (since we must move the bottom disk to peg 2); this will require a_{n-1} steps. Then we can move the bottom disk to peg 2 (one more step). Our goal, though, was to move it to peg 3, so now we must move the other $n - 1$ disks from peg 3 back to peg 1, leaving the bottom disk quietly resting on peg 2. By symmetry, this again takes a_{n-1} steps. One more step lets us move the bottom disk from peg 2 to peg 3. Now it takes a_{n-1} steps to move the remaining disks from peg 1 to peg 3. So our recurrence relation is $a_n = 3a_{n-1} + 2$. The initial condition is of course that $a_0 = 0$.

b) Computing the first few values, we find that $a_1 = 2$, $a_2 = 8$, $a_3 = 26$, and $a_4 = 80$. It appears that $a_n = 3^n - 1$. This is easily verified by induction: The base case is $a_0 = 3^0 - 1 = 1 - 1 = 0$, and $3a_{n-1} + 2 = 3 \cdot (3^{n-1} - 1) + 2 = 3^n - 3 + 2 = 3^n - 1 = a_n$.

c) The only choice in distributing the disks is which peg each disk goes on, since the order of the disks on a given peg is fixed. Since there are three choices for each disk, the answer is 3^n.

d) The puzzle involves $1 + a_n = 3^n$ arrangements of disks during its solution—the initial arrangement and the arrangement after each move. None of these arrangements can repeat a previous arrangement, since if it did so, there would have been no point in making the moves between the two occurrences of the same arrangement. Therefore these 3^n arrangements are all distinct. We saw in part **(c)** that there are exactly 3^n arrangements, so every arrangement was used.

34. If we follow the hint, then it certainly looks as if $J(n) = 2k+1$, where k is the amount left over after the largest possible power of 2 has been subtracted from n (i.e., $n = 2^m + k$ and $k < 2^m$).

36. The basis step is trivial, since when $n = 1 = 2^0 + 0$, the conjecture in Exercise 34 states that $J(n) = 2 \cdot 0 + 1 = 1$, which is correct. For the inductive step, we look at two cases, depending on whether there are an even or an odd number of players. If there are $2n$ players, suppose that $2n = 2^m + k$, as in the hint for Exercise 34. Then k must be even and we can write $n = 2^{m-1} + (k/2)$, and $k/2 < 2^{m-1}$. By the inductive hypothesis, $J(n) = 2(k/2) + 1 = k + 1$. Then by the recurrence relation from Exercise 35, $J(2n) = 2J(n) - 1 = 2(k+1) - 1 = 2k + 1$, as desired. For the other case, assume that there are $2n+1$ players, and again write $2n+1 = 2^m + k$, as in the hint for Exercise 34. Then k must be odd and we can write $n = 2^{m-1} + (k-1)/2$, where $(k-1)/2 < 2^{m-1}$. By the inductive hypothesis, $J(n) = 2((k-1)/2) + 1 = k$. Then by the recurrence relation from Exercise 35, $J(2n+1) = 2J(n) + 1 = 2k + 1$, as desired.

38. Since we can only move one disk at a time, we need one move to lift the smallest disk off the middle disk, and another to lift the middle disk off the largest. Similarly, we need two moves to rejoin these disks. And of course we need at least one move to get the largest disk off peg 1. Therefore we can do no better than five moves. To see that this is possible, we just make the obvious moves (disk 1 is the smallest, and $a \xrightarrow{b} c$ means to move disk b from peg a to peg c: $1 \xrightarrow{1} 2$, $1 \xrightarrow{2} 3$, $1 \xrightarrow{3} 4$, $3 \xrightarrow{2} 4$, $2 \xrightarrow{1} 4$.

40. In our notation (see Exercise 38), disk 1 is the smallest, disk n is the largest, and $a \xrightarrow{b} c$ means to move disk b from peg a to peg c.

a) According to the algorithm, we take $k = 3$, since 5 is between the triangular numbers $t_2 = 3$ and $t_3 = 6$. The moves are to first move $5 - 3 = 2$ disks from peg 1 to peg 2 ($1 \xrightarrow{1} 3$, $1 \xrightarrow{2} 2$, $3 \xrightarrow{1} 2$), then working with pegs 1, 3, and 4 move disks 3, 4, and 5 to peg 4 ($1 \xrightarrow{3} 4$, $1 \xrightarrow{4} 3$, $4 \xrightarrow{3} 3$, $1 \xrightarrow{5} 4$, $3 \xrightarrow{3} 1$, $3 \xrightarrow{4} 4$, $1 \xrightarrow{3} 4$), and then move disks 1 and 2 from peg 2 to peg 4 ($2 \xrightarrow{1} 3$, $2 \xrightarrow{2} 4$, $3 \xrightarrow{1} 4$). Note that this took 13 moves in all.

b) According to the algorithm, we take $k = 3$, since 6 is between the triangular numbers $t_2 = 3$ and $t_3 = 6$. The moves are to first move $6 - 3 = 3$ disks from peg 1 to peg 2 ($1 \xrightarrow{1} 3$, $1 \xrightarrow{2} 4$, $1 \xrightarrow{3} 2$, $4 \xrightarrow{2} 2$, $3 \xrightarrow{1} 2$), then working with pegs 1, 3, and 4 move disks 4, 5, and 6 to peg 4 ($1 \xrightarrow{4} 4$, $1 \xrightarrow{5} 3$, $4 \xrightarrow{4} 3$, $1 \xrightarrow{6} 4$, $3 \xrightarrow{4} 1$, $3 \xrightarrow{5} 4$, $1 \xrightarrow{4} 4$), and then move disks 1, 2, and 3 from peg 2 to peg 4 ($2 \xrightarrow{1} 3$, $2 \xrightarrow{2} 1$, $2 \xrightarrow{3} 4$, $1 \xrightarrow{2} 4$, $3 \xrightarrow{1} 4$). Note that this took 17 moves in all.

c) According to the algorithm, we take $k = 4$, since 7 is between the triangular numbers $t_3 = 6$ and $t_4 = 10$. The moves are to first move $7 - 4 = 3$ disks from peg 1 to peg 2 (five moves, as in part **(b)**), then working with pegs 1, 3, and 4 move disks 4, 5, 6, and 7 to peg 4 (15 moves, using the usual Tower of Hanoi algorithm), and then move disks 1, 2, and 3 from peg 2 to peg 4 (again five moves, as in part **(b)**). Note that this took 25 moves in all.

d) According to the algorithm, we take $k = 4$, since 8 is between the triangular numbers $t_3 = 6$ and $t_4 = 10$. The moves are to first move $8 - 4 = 4$ disks from peg 1 to peg 2 (nine moves, as in Exercise 39, with peg 2 playing the role of peg 4), then working with pegs 1, 3, and 4 move disks 5, 6, 7, and 8 to peg 4 (15 moves, using the usual Tower of Hanoi algorithm), and then move disks 1, 2, 3, and 4 from peg 2 to peg 4 (again nine moves, as above). Note that this took 33 moves in all.

42. To clarify the problem, we note that k is chosen to be the smallest nonnegative integer such that $n \le k(k+1)/2$. If $n - 1 \ne k(k-1)/2$, then this same value of k applies to $n - 1$ as well; otherwise the value for $n - 1$ is $k - 1$. If $n - 1 \ne k(k-1)/2$, it also follows by subtracting k from both sides of the inequality that the smallest nonnegative integer m such that $n-k \le m(m+1)/2$ is $m = k-1$, so $k-1$ is the value selected by the Frame–Stewart algorithm for $n-k$. Now we proceed by induction, the basis steps being trivial. There are two cases for the inductive step. If $n-1 \ne k(k-1)/2$, then we have from the recurrence relation in Exercise 41 that $R(n) = 2R(n-k) + 2^k - 1$ and $R(n-1) = 2R(n-k-1) + 2^k - 1$. Subtracting yields $R(n) - R(n-1) = 2(R(n-k) - R(n-k-1))$. Since $k-1$ is the value selected for $n-k$, the inductive hypothesis tells us that this difference is $2 \cdot 2^{k-2} = 2^{k-1}$, as desired. On the other hand, if $n-1 = k(k-1)/2$, then $R(n) - R(n-1) = 2R(n-k) + 2^k - 1 - (2R(n-1-(k-1)) + 2^{k-1} - 1 = 2^{k-1}$.

44. Since the Frame–Stewart algorithm solves the puzzle, the number of moves it uses, $R(n)$, is an upper bound to the number of moves needed to solve the puzzle. By Exercise 43 we have a recurrence or formula for these numbers. The table below shows n, the corresponding k and t_k, and $R(n)$.

n	k	t_k	$R(n)$
1	1	1	1
2	2	3	3
3	2	3	5
4	3	6	9
5	3	6	13
6	3	6	17
7	4	10	25
8	4	10	33
9	4	10	41
10	4	10	49
11	5	15	65
12	5	15	81
13	5	15	97
14	5	15	113
15	5	15	129
16	6	21	161
17	6	21	193
18	6	21	225
19	6	21	257
20	6	21	289
21	6	21	321
22	7	28	353
23	7	28	417
24	7	28	481
25	7	28	545

46. **a)** $\nabla a_n = 4 - 4 = 0$ **b)** $\nabla a_n = 2n - 2(n-1) = 2$
c) $\nabla a_n = n^2 - (n-1)^2 = 2n - 1$ **d)** $\nabla a_n = 2^n - 2^{n-1} = 2^{n-1}$

48. This follows immediately (by algebra) from the definition.

50. We prove this by induction on k. The case $k = 1$ was Exercise 48. Assume the inductive hypothesis, that a_{n-k} can be expressed in terms of a_n, ∇a_n, $\ldots$, $\nabla^k a_n$, for all n. We will show that $a_{n-(k+1)}$ can be expressed in terms of a_n, ∇a_n, $\ldots$, $\nabla^k a_n$, $\nabla^{k+1} a_n$. Note from the definitions that $a_{n-1} = a_n - \nabla a_n$ and that $\nabla^i a_{n-1} = \nabla^i a_n - \nabla^{i+1} a_n$ for all i. By the inductive hypothesis, we know that $a_{(n-1)-k}$ (which is just $a_{n-(k+1)}$ rewritten) can be expressed as $f(a_{n-1}, \nabla a_{n-1}, \ldots, \nabla^k a_{n-1}) = f(a_n - \nabla a_n, \nabla a_n - \nabla^2 a_n, \ldots, \nabla^k a_n - \nabla^{k+1} a_n)$—exactly what we wished to show. Note that in fact all the equations involved are linear.

52. By Exercise 50, each a_{n-i} can be so expressed (as a linear function), so the entire recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ can be written as $a_n = c_1 f_1 + c_2 f_2 + \cdots + c_k f_k$, where each f_i is a linear expression involving a_n, ∇a_n, ..., $\nabla^k a_n$. This gives us the desired difference equation.

54. This problem was solved in Exercise 55.

56. **a)** If all the terms are nonnegative, then the more terms we have, the larger the sum. A sequence such as $5, -2$ shows that the maximum might not be achieved by taking all the terms if some are negative; in this example the maximum is achieved by taking just the first term, and taking all the terms gives a smaller sum.

b) If the string of consecutive terms must end at a_k, then either it consists just of a_k or it consists of a string of consecutive terms ending at a_{k-1} followed by a_k. If we want the largest such sum in the second case, then we must take the largest sum of consecutive terms ending at a_{k-1}. Therefore the given recurrence relation must hold.

c) We compute and store the values $M(k)$ using the recurrence relation in part **(b)**. We could also store, for each k, the starting point of the string of numbers ending at position k that achieves the maximum sum. This would not only give us the sum but also tell us which terms to add to achieve it. Note also that the max function will choose the first argument if and only if $M(k-1)$ is positive (or nonnegative).

```
procedure max sum(a_1, a_2, ..., a_n : real numbers)
M(1) := a_1
for k := 2 to n
        M(k) := max(M(k − 1) + a_k, a_k)
return M(n)
```

d) The successive values for $M(k)$ are 2, -1 (because $-3+2 > -3$), 4 (because $4 > -1+4$), 5 (because $4+1 > 1$), 3 (because $5+(-2) > -2$), and 6 (because $3+3 > 3$).

e) The algorithm has just the one loop containing a few arithmetic steps, iterated $O(n)$ times.

SECTION 8.2 Solving Linear Recurrence Relations

2. **a)** linear, homogeneous, with constant coefficients; degree 2

b) linear with constant coefficients but not homogeneous

c) not linear

d) linear, homogeneous, with constant coefficients; degree 3

e) linear and homogeneous, but not with constant coefficients

f) linear with constant coefficients, but not homogeneous

g) linear, homogeneous, with constant coefficients; degree 7

4. For each problem, we first write down the characteristic equation and find its roots. Using this we write down the general solution. We then plug in the initial conditions to obtain a system of linear equations. We solve these equations to determine the arbitrary constants in the general solution, and finally we write down the unique answer.

a) $r^2 - r - 6 = 0 \qquad r = -2, 3$

$$a_n = \alpha_1(-2)^n + \alpha_2 3^n$$

$$3 = \alpha_1 + \alpha_2$$

$$6 = -2\alpha_1 + 3\alpha_2$$

$$\alpha_1 = 3/5 \qquad \alpha_2 = 12/5$$

$$a_n = (3/5)(-2)^n + (12/5)3^n$$

b) $r^2 - 7r + 10 = 0 \qquad r = 2, 5$

$a_n = \alpha_1 2^n + \alpha_2 5^n$

$2 = \alpha_1 + \alpha_2$

$1 = 2\alpha_1 + 5\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = -1$

$a_n = 3 \cdot 2^n - 5^n$

c) $r^2 - 6r + 8 = 0 \qquad r = 2, 4$

$a_n = \alpha_1 2^n + \alpha_2 4^n$

$4 = \alpha_1 + \alpha_2$

$10 = 2\alpha_1 + 4\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = 1$

$a_n = 3 \cdot 2^n + 4^n$

d) $r^2 - 2r + 1 = 0 \qquad r = 1, 1$

$a_n = \alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$

$4 = \alpha_1$

$1 = \alpha_1 + \alpha_2$

$\alpha_1 = 4 \qquad \alpha_2 = -3$

$a_n = 4 - 3n$

e) $r^2 - 1 = 0 \qquad r = -1, 1$

$a_n = \alpha_1(-1)^n + \alpha_2 1^n = \alpha_1(-1)^n + \alpha_2$

$5 = \alpha_1 + \alpha_2$

$-1 = -\alpha_1 + \alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = 2$

$a_n = 3 \cdot (-1)^n + 2$

f) $r^2 + 6r + 9 = 0 \qquad r = -3, -3$

$a_n = \alpha_1(-3)^n + \alpha_2 n(-3)^n$

$3 = \alpha_1$

$-3 = -3\alpha_1 - 3\alpha_2$

$\alpha_1 = 3 \qquad \alpha_2 = -2$

$a_n = 3(-3)^n - 2n(-3)^n = (3 - 2n)(-3)^n$

g) $r^2 + 4r - 5 = 0 \qquad r = -5, 1$

$a_n = \alpha_1(-5)^n + \alpha_2 1^n = \alpha_1(-5)^n + \alpha_2$

$2 = \alpha_1 + \alpha_2$

$8 = -5\alpha_1 + \alpha_2$

$\alpha_1 = -1 \qquad \alpha_2 = 3$

$a_n = -(-5)^n + 3$

6. The model is the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-2} = a_{n-1} + 2a_{n-2}$, with $a_0 = a_1 = 1$ (see the technique of Exercise 19 in Section 8.1). To solve this, we use the characteristic equation $r^2 - r - 2 = 0$, which has roots -1 and 2. Therefore the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 2^n$. Plugging in the initial conditions gives the equations $1 = \alpha_1 + \alpha_2$ and $1 = -\alpha_1 + 2\alpha_2$, which solve to $\alpha_1 = 1/3$ and $\alpha_2 = 2/3$. Therefore in n microseconds $(1/3)(-1)^n + (2/3)2^n$ messages can be transmitted.

8. a) The recurrence relation is, by the definition of average, $L_n = (1/2)L_{n-1} + (1/2)L_{n-2}$.

b) The characteristic equation is $r^2 - (1/2)r - (1/2) = 0$, which gives us $r = -1/2$ and $r = 1$. Therefore the general solution is $L_n = \alpha_1(-1/2)^n + \alpha_2$. Plugging in the initial conditions $L_1 = 100000$ and $L_2 = 300000$ gives $100000 = (-1/2)\alpha_1 + \alpha_2$ and $300000 = (1/4)\alpha_1 + \alpha_2$. Solving these yields $\alpha_1 = 800000/3$ and $\alpha_2 = 700000/3$. Therefore the answer is $L_n = (800000/3)(-1/2)^n + (700000/3)$.

10. The proof may be found in textbooks such as *Introduction to Combinatorial Mathematics* by C. L. Liu (McGraw-Hill, 1968), Chapter 3. It is similar to the proof of Theorem 1.

12. The characteristic equation is $r^3 - 2r^2 - r + 2 = 0$. This factors as $(r-1)(r+1)(r-2) = 0$, so the roots are 1, -1, and 2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3$, $6 = \alpha_1 - \alpha_2 + 2\alpha_3$, and $0 = \alpha_1 + \alpha_2 + 4\alpha_3$. The solution to this system of equations is $\alpha_1 = 6$, $\alpha_2 = -2$, and $\alpha_3 = -1$. Therefore the answer is $a_n = 6 - 2(-1)^n - 2^n$.

14. The characteristic equation is $r^4 - 5r^2 + 4 = 0$. This factors as $(r^2-1)(r^2-4) = (r-1)(r+1)(r-2)(r+2) = 0$, so the roots are 1, -1, 2, and -2. Therefore the general solution is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 2^n + \alpha_4(-2)^n$. Plugging in initial conditions gives $3 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, $2 = \alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4$, $6 = \alpha_1 + \alpha_2 + 4\alpha_3 + 4\alpha_4$, and $8 = \alpha_1 - \alpha_2 + 8\alpha_3 - 8\alpha_4$. The solution to this system of equations is $\alpha_1 = \alpha_2 = \alpha_3 = 1$ and $\alpha_4 = 0$. Therefore the answer is $a_n = 1 + (-1)^n + 2^n$.

16. This requires some linear algebra, but follows the same basic idea as the proof of Theorem 1. See textbooks such as *Introduction to Combinatorial Mathematics* by C. L. Liu (McGraw-Hill, 1968), Chapter 3.

18. This is a third degree recurrence relation. The characteristic equation is $r^3 - 6r^2 + 12r - 8 = 0$. By the rational root test, the possible rational roots are $\pm1, \pm2, \pm4$. We find that $r = 2$ is a root. Dividing $r - 2$ into $r^3 - 6r^2 + 12r - 8$, we find that $r^3 - 6r^2 + 12r - 8 = (r-2)(r^2 - 4r + 4)$. By inspection we factor the rest, obtaining $r^3 - 6r^2 + 12r - 8 = (r-2)^3$. Hence the only root is 2, with multiplicity 3, so the general solution is (by Theorem 4) $a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3 n^2 2^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned} -5 &= a_0 = \alpha_1 \\ 4 &= a_1 = 2\alpha_1 + 2\alpha_2 + 2\alpha_3 \\ 88 &= a_2 = 4\alpha_1 + 8\alpha_2 + 16\alpha_3 \,. \end{aligned}$$

Solving this system of equations, we get $\alpha_1 = -5$, $\alpha_2 = 1/2$, and $\alpha_3 = 13/2$. Therefore the answer is $a_n = -5 \cdot 2^n + (n/2) \cdot 2^n + (13n^2/2) \cdot 2^n = -5 \cdot 2^n + n \cdot 2^{n-1} + 13n^2 \cdot 2^{n-1}$.

20. This is a fourth degree recurrence relation. The characteristic polynomial is $r^4 - 8r^2 + 16$, which factors as $(r^2-4)^2$, which then further factors into $(r-2)^2(r+2)^2$. The roots are 2 and -2, each with multiplicity 2. Thus we can write down the general solution as usual: $a_n = \alpha_1 2^n + \alpha_2 n \cdot 2^n + \alpha_3(-2)^n + \alpha_4 n \cdot (-2)^n$.

22. This is similar to Example 6. We can immediately write down the general solution using Theorem 4. In this case there are four distinct roots, so $t = 4$. The multiplicities are 3, 2, 2, and 1. So the general solution is $a_n = (\alpha_{1,0} + \alpha_{1,1}n + \alpha_{1,2}n^2)(-1)^n + (\alpha_{2,0} + \alpha_{2,1}n)2^n + (\alpha_{3,0} + \alpha_{3,1}n)5^n + \alpha_{4,0}7^n$.

24. **a)** We compute the right-hand side of the recurrence relation: $2(n-1)2^{n-1} + 2^n = (n-1)2^n + 2^n = n2^n$, which is the left-hand side.

b) The solution of the associated homogeneous equation $a_n = 2a_{n-1}$ is easily found to be $a_n = \alpha 2^n$. Therefore the general solution of the inhomogeneous equation is $a_n = \alpha 2^n + n2^n$.

c) Plugging in $a_0 = 2$, we obtain $\alpha = 2$. Therefore the solution is $a_n = 2 \cdot 2^n + n2^n = (n+2)2^n$.

26. We need to use Theorem 6, and so we need to find the roots of the characteristic polynomial of the associated homogeneous recurrence relation. The characteristic equation is $r^3 - 6r^2 + 12r - 8 = 0$, and as we saw in Exercise 18, $r = 2$ is the only root, and it has multiplicity 3.

a) Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_2n^2 + p_1n + p_0$. In the notation of Theorem 6, $s = 1$ here.

b) Since 2 is a root with multiplicity 3 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^3p_02^n$.

c) Since 2 is a root with multiplicity 3 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^3(p_1n+p_0)2^n$.

d) Since -2 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_0(-2)^n$.

e) Since 2 is a root with multiplicity 3 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^3(p_2n^2+p_1n+p_0)2^n$.

f) Since -2 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $(p_3n^3+p_2n^2+p_1n+p_0)(-2)^n$.

g) Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form p_0. In the notation of Theorem 6, $s=1$ here.

28. a) The associated homogeneous recurrence relation is $a_n=2a_{n-1}$. We easily solve it to obtain $a_n^{(h)}=\alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n=p_2n^2+p_1n+p_0$. (Note that $s=1$ here, and 1 is not a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $p_2n^2+p_1n+p_0=2(p_2(n-1)^2+p_1(n-1)+p_0)+2n^2$. We rewrite this by grouping terms with equal powers of n, obtaining $(-p_2-2)n^2+(4p_2-p_1)n+(-2p_2+2p_1-p_0)=0$. In order for this equation to be true for all n, we must have $p_2=-2$, $4p_2=p_1$, and $-2p_2+2p_1-p_0=0$. This tells us that $p_1=-8$ and $p_0=-12$. Therefore the particular solution we seek is $a_n^{(p)}=-2n^2-8n-12$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n=\alpha 2^n-2n^2-8n-12$.

b) We plug the initial condition into our solution from part **(a)** to obtain $4=a_1=2\alpha-2-8-12$. This tells us that $\alpha=13$. So the solution is $a_n=13\cdot 2^n-2n^2-8n-12$.

30. a) The associated homogeneous recurrence relation is $a_n=-5a_{n-1}-6a_{n-2}$. To solve it we find the characteristic equation $r^2+5r+6=0$, find that $r=-2$ and $r=-3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)}=\alpha(-2)^n+\beta(-3)^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n=c\cdot 4^n$. We plug this into our recurrence relation and obtain $c\cdot 4^n=-5c\cdot 4^{n-1}-6c\cdot 4^{n-2}+42\cdot 4^n$. We divide through by 4^{n-2}, obtaining $16c=-20c-6c+42\cdot 16$, whence with a little simple algebra $c=16$. Therefore the particular solution we seek is $a_n^{(p)}=16\cdot 4^n=4^{n+2}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n=\alpha(-2)^n+\beta(-3)^n+4^{n+2}$.

b) We plug the initial conditions into our solution from part **(a)** to obtain $56=a_1=-2\alpha-3\beta+64$ and $278=a_2=4\alpha+9\beta+256$. A little algebra yields $\alpha=1$ and $\beta=2$. So the solution is $a_n=(-2)^n+2(-3)^n+4^{n+2}$.

32. The associated homogeneous recurrence relation is $a_n=2a_{n-1}$. We easily solve it to obtain $a_n^{(h)}=\alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n=cn\cdot 2^n$. We plug this into our recurrence relation and obtain $cn\cdot 2^n=2c(n-1)2^{n-1}+3\cdot 2^n$. We divide through by 2^{n-1}, obtaining $2cn=2c(n-1)+6$, whence with a little simple algebra $c=3$. Therefore the particular solution we seek is $a_n^{(p)}=3n\cdot 2^n$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n=\alpha 2^n+3n\cdot 2^n=(3n+\alpha)2^n$.

34. The associated homogeneous recurrence relation is $a_n=7a_{n-1}-16a_{n-2}+12a_{n-3}$. To solve it we find the characteristic equation $r^3-7r^2+16r-12=0$. By the rational root test we soon discover that $r=2$ is a root and factor our equation into $(r-2)^2(r-3)=0$. Therefore the general solution of the homogeneous relation is $a_n^{(h)}=\alpha 2^n+\beta n\cdot 2^n+\gamma 3^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = (cn+d)4^n$, since the coefficient of 4^n in our given relation is a linear function of n, and 4 is not a root of the characteristic equation. We plug this into our recurrence relation and obtain $(cn+d)4^n = 7(cn-c+d)4^{n-1} - 16(cn-2c+d)4^{n-2} + 12(cn-3c+d)4^{n-3} + n\cdot 4^n$. We divide through by 4^{n-2}, expand and collect terms (a tedious process, to be sure), obtaining $(c-16)n + (5c+d) = 0$. Therefore $c = 16$ and $d = -80$, so the particular solution we seek is $a_n^{(p)} = (16n-80)4^n$. Thus the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + \beta n\cdot 2^n + \gamma 3^n + (16n-80)4^n$. Next we plug in the initial conditions to obtain $-2 = a_0 = \alpha + \gamma - 80$, $0 = a_1 = 2\alpha + 2\beta + 3\gamma - 256$, and $5 = a_2 = 4\alpha + 8\beta + 9\gamma - 768$. We solve this system of three linear equations in three unknowns by standard methods to obtain $\alpha = 17$, $\beta = 39/2$, and $\gamma = 61$. So the solution is $a_n = 17\cdot 2^n + 39n\cdot 2^{n-1} + 61\cdot 3^n + (16n-80)4^n$. As a check of our work (it would be too much to hope that we could always get this far without making an algebraic error), we can compute a_3 both from the recurrence and from the solution, and we find that $a_3 = 203$ both ways.

36. Obviously the n^{th} term of the sequence comes from the $(n-1)^{\text{st}}$ term by adding n^2; in symbols, $a_{n-1} + n^2 = \left(\sum_{k=1}^{n-1} k^2\right) + n^2 = \sum_{k=1}^{n} k^2 = a_n$. Also, the sum of the first square is clearly 1. To solve this recurrence relation, we easily see that the homogeneous solution is $a_n = \alpha$, so since the nonhomogeneous term is a second degree polynomial, we need a particular solution of the form $a_n = cn^3 + dn^2 + en$. Plugging this into the recurrence relation gives $cn^3 + dn^2 + en = c(n-1)^3 + d(n-1)^2 + e(n-1) + n^2$. Expanding and collecting terms, we have $(3c-1)n^2 + (-3c+2d)n + (c-d+e) = 0$, whence $c = 1/3$, $d = 1/2$, and $e = 1/6$. Thus $a_n^{(h)} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. So the general solution is $a_n = \alpha + \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$. It is now a simple matter to plug in the initial condition to see that $\alpha = 0$. Note that we can find a common denominator and write our solution in the familiar form $a_n = n(n+1)(2n+1)/6$, as was noted in Table 2 of Section 2.4 and proved by mathematical induction in Exercise 3 of Section 5.1.

38. **a)** The characteristic equation is $r^2 - 2r + 2 = 0$, whose roots are, by the quadratic formula, $1 \pm \sqrt{-1}$, in other words, $1+i$ and $1-i$.

b) The general solution is, by part **(a)**, $a_n = \alpha_1(1+i)^n + \alpha_2(1-i)^n$. Plugging in the initial conditions gives us $1 = \alpha_1 + \alpha_2$ and $2 = (1+i)\alpha_1 + (1-i)\alpha_2$. Solving these linear equations tells us that $\alpha_1 = \frac{1}{2} - \frac{1}{2}i$ and $\alpha_2 = \frac{1}{2} + \frac{1}{2}i$. Therefore the solution is $a_n = (\frac{1}{2} - \frac{1}{2}i)(1+i)^n + (\frac{1}{2} + \frac{1}{2}i)(1-i)^n$.

40. First we reduce this system to a recurrence relation and initial conditions involving only a_n. If we subtract the two equations, we obtain $a_n - b_n = 2a_{n-1}$, which gives us $b_n = a_n - 2a_{n-1}$. We plug this back into the first equation to get $a_n = 3a_{n-1} + 2(a_{n-1} - 2a_{n-2}) = 5a_{n-1} - 4a_{n-2}$, our desired recurrence relation in one variable. Note also that the first of the original equations gives us the necessary second initial condition, namely $a_1 = 3a_0 + 2b_0 = 7$. We now solve this problem for $\{a_n\}$ in the usual way. The roots of the characteristic equation $r^2 - 5r + 4 = 0$ are 1 and 4, and the solution, after solving for the arbitrary constants, is $a_n = -1 + 2\cdot 4^n$. Finally, we plug this back into the equation $b_n = a_n - 2a_{n-1}$ to find that $b_n = 1 + 4^n$.

42. We can prove this by induction on n. If $n = 1$, then the assertion is $a_1 = s\cdot f_0 + t\cdot f_1 = s\cdot 0 + t\cdot 1 = t$, which is given; and if $n = 2$, then the assertion is $a_2 = s\cdot f_1 + t\cdot f_2 = s\cdot 1 + t\cdot 1 = s + t$, which is true, since $a_2 = a_1 + a_0 = t + s$. Having taken care of the base cases, we assume the inductive hypothesis, that the statement is true for values less than n. Then $a_n = a_{n-1} + a_{n-2} = (sf_{n-2} + tf_{n-1}) + (sf_{n-3} + tf_{n-2}) = s(f_{n-2} + f_{n-3}) + t(f_{n-1} + f_{n-2}) = sf_{n-1} + tf_n$, as desired.

44. We can compute the first few terms by hand. For $n = 1$, the matrix is just the number 2, so $d_1 = 2$. For

$n = 2$, the matrix is $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, and its determinant is clearly $d_2 = 4 - 1 = 3$. For $n = 3$ the matrix is

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

and we get $d_3 = 4$ after a little arithmetic. For the general case, our matrix is

$$\mathbf{A}_n = \begin{bmatrix} 2 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 2 & 1 & 0 & \ldots & 0 \\ 0 & 1 & 2 & 1 & \ldots & 0 \\ 0 & 0 & 1 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 2 \end{bmatrix}.$$

To compute the determinant, we expand along the top row. This gives us a value of 2 times the determinant of the matrix obtained by deleting the first row and first column minus the determinant of the matrix obtained by deleting the first row and second column. The first of these smaller matrices is just $\mathbf{A}_{n-1}$, with determinant d_{n-1}. The second of these smaller matrices has just one nonzero entry in its first column, so we expand its determinant along the first column and see that it equals d_{n-2}. Therefore our recurrence relation is $d_n = 2d_{n-1} - d_{n-2}$, with initial conditions as computed at the start of this solution. If we compute a few more terms we are led to the conjecture that $d_n = n + 1$. If we show that this satisfies the recurrence, then we have proved that it is indeed the solution. And sure enough, $n + 1 = 2n - (n - 1)$. (Of course, we could have also dragged out the machinery of this section to solve the recurrence relation and initial conditions.)

46. Let a_n represent the number of goats on the island at the start of the n^{th} year.

a) The initial condition is $a_1 = 2$; we are told that at the beginning of the first year there are two goats. During each subsequent year (year n, with $n \geq 2$), the goats who were on the island the year before (year $n - 1$) double in number, and an extra 100 goats are added in. So $a_n = 2a_{n-1} + 100$.

b) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$, whose solution is $a_n^{(h)} = \alpha 2^n$. The particular solution is a polynomial of degree 0, namely a constant, $a_n = c$. Plugging this into the recurrence relation gives $c = 2c + 100$, whence $c = -100$. So the particular solution is $a_n^{(p)} = -100$ and the general solution is $a_n = \alpha 2^n - 100$. Plugging in the initial condition and solving for α gives us $2 = 2\alpha - 100$, or $\alpha = 51$. Hence the desired formula is $a_n = 51 \cdot 2^n - 100$. There are $51 \cdot 2^n - 100$ goats on the island at the start of the n^{th} year.

c) We are told that $a_1 = 2$, but that is not the relevant initial condition. Instead, since the first two years are special (no goats are removed), the relevant initial condition is $a_2 = 4$. During each subsequent year (year n, with $n \geq 3$), the goats who were on the island the year before (year $n - 1$) double in number, and n goats are removed. So $a_n = 2a_{n-1} - n$. (We assume that the removal occurs after the doubling has occurred; if we assume that the removal takes place first, then we'd have to write $a_n = 2(a_{n-1} - n) = 2a_{n-1} - 2n$.)

d) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$, whose solution is $a_n^{(h)} = \alpha 2^n$. The particular solution is a polynomial of degree 1, say $a_n = cn + d$. Plugging this into the recurrence relation and grouping like terms gives $(-c + 1)n + (2c - d) = 0$, whence $c = 1$ and $d = 2$. So the particular solution is $a_n^{(p)} = n + 2$ and the general solution is $a_n = \alpha 2^n + n + 2$. Plugging in the initial condition $a_2 = 4$ and solving for α gives us $4 = 4\alpha + 4$, or $\alpha = 0$. Hence the desired formula is simply $a_n = n + 2$ for all $n \geq 2$ (and $a_1 = 2$). There are $n + 2$ goats on the island at the start of the n^{th} year, for all $n \geq 2$.

48. a) This is just a matter of keeping track of what all the symbols mean. First note that $Q(n + 1) = Q(n)f(n)/g(n+1)$. Now the left-hand side of the desired equation is $b_n = g(n+1)Q(n+1)a_n = Q(n)f(n)a_n$. The right-hand side is $b_{n-1} + Q(n)h(n) = g(n)Q(n)a_{n-1} + Q(n)h(n) = Q(n)(g(n)a_{n-1} + h(n))$. That the two sides are the same now follows from the original recurrence relation, $f(n)a_n = g(n)a_{n-1} + h(n)$. Note that the initial condition for $\{b_n\}$ is $b_0 = g(1)Q(1)a_0 = g(1)(1/g(1))a_0 = a_0 = C$, since it is conventional to view an empty product as the number 1.

b) Since $\{b_n\}$ satisfies the trivial recurrence relation shown in part **(a)**, we see immediately that

$$\begin{aligned} b_n &= Q(n)h(n) + b_{n-1} = Q(n)h(n) + Q(n-1)h(n-1) + b_{n-2} = \cdots \\ &= \sum_{i=1}^{n} Q(i)h(i) + b_0 = \sum_{i=1}^{n} Q(i)h(i) + C\,. \end{aligned}$$

The value of a_n follows from the definition of b_n given in part **(a)**.

50. **a)** We can show this by proving that $nC_n - (n+1)C_{n-1} = 2n$, so let us calculate, using the given recurrence:

$$\begin{aligned} nC_n - (n+1)C_{n-1} &= nC_n - (n-1)C_{n-1} - 2C_{n-1} \\ &= n^2 + n + 2\sum_{k=0}^{n-1} C_k - (n-1)\left(n + \frac{2}{n-1}\sum_{k=0}^{n-2} C_k\right) - 2C_{n-1} \\ &= n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1} - n^2 + n - 2\sum_{k=0}^{n-2} C_k \ - 2C_{n-1} = 2n. \end{aligned}$$

b) We use the formula given in Exercise 48. Note first that $f(n) = n$, $g(n) = n+1$, and $h(n) = 2n$. Thus $Q(n) = \dfrac{(n-1)!}{(n+1)!} = \dfrac{1}{n(n+1)}$. Plugging this into the formula gives

$$\frac{0 + \sum_{i=1}^{n} \dfrac{2i}{i(i+1)}}{(n+2)\cdot\dfrac{1}{(n+1)(n+2)}} = 2(n+1)\sum_{i=1}^{n}\frac{1}{i+1}.$$

There is no nice closed form way to write this sum (the harmonic series), but we can check that both this formula and the recurrence yield the same values of C_n for small n (namely, $C_1 = 2$, $C_2 = 5$, $C_3 = 26/3$, and so on).

52. A proof of this theorem can be found in textbooks such as *Discrete Mathematics with Applications* by H. E. Mattson, Jr. (Wiley, 1993), Chapter 11.

SECTION 8.3 Divide-and-Conquer Algorithms and Recurrence Relations

2. The recurrence relation here is $f(n) = 2f(n/2) + 2$, where $f(1) = 0$, since no comparisons are needed for a set with 1 element. Iterating, we find that $f(2) = 2\cdot 0 + 2 = 2$, $f(4) = 2\cdot 2 + 2 = 6$, $f(8) = 2\cdot 6 + 2 = 14$, $f(16) = 2\cdot 14 + 2 = 30$, $f(32) = 2\cdot 30 + 2 = 62$, $f(64) = 2\cdot 62 + 2 = 126$, and $f(128) = 2\cdot 126 + 2 = 254$.

4. In this algorithm we assume that $a = (a_{2n-1}a_{2n-2}\ldots a_1a_0)_2$ and $b = (b_{2n-1}b_{2n-2}\ldots b_1b_0)_2$.

procedure *fast multiply*$(a, b :$ nonnegative integers$)$
if $a \le 1$ and $b \le 1$ **then return** ab
else
  $A_1 := \lfloor a/2^n \rfloor$
  $A_0 := a - 2^n A_1$
  $B_1 := \lfloor b/2^n \rfloor$
  $B_0 := b - 2^n B_1$
  {we assume that these four numbers have length n; pad if necessary}
  $x :=$ *fast multiply*(A_1, B_1)
  answer $:= (x$ shifted left $2n$ places$) + (x$ shifted left n places$)$
  $x :=$ *fast multiply*(A_0, B_0)
  answer $:=$ *answer* $+ x + (x$ shifted left n places$)$
  if $A_1 \ge A_0$ **then** $A_2 := A_1 - A_0$ **else** $A_2 := A_0 - A_1$
  if $B_0 \ge B_1$ **then** $B_2 := B_0 - B_1$ **else** $B_2 := B_1 - B_0$
  $x :=$ *fast multiply*(A_2, B_2) shifted left n places
  if $(A_1 \ge A_0 \wedge B_0 \ge B_1) \vee (A_1 < A_0 \wedge B_0 < B_1)$ **then** *answer* $:=$ *answer* $+ x$
  else *answer* $:=$ *answer* $- x$
  return *answer*

6. The recurrence relation is $f(n) = 7f(n/2) + 15n^2/4$, with $f(1) = 1$. Thus we have, iterating, $f(2) = 7\cdot 1 + 15\cdot 2^2/4 = 22$, $f(4) = 7\cdot 22 + 15\cdot 4^2/4 = 214$, $f(8) = 7\cdot 214 + 15\cdot 8^2/4 = 1738$, $f(16) = 7\cdot 1738 + 15\cdot 16^2/4 = 13126$, and $f(32) = 7\cdot 13126 + 15\cdot 32^2/4 = 95{,}722$.

8. **a)** $f(2) = 2\cdot 5 + 3 = 13$ **b)** $f(4) = 2\cdot 13 + 3 = 29$, $f(8) = 2\cdot 29 + 3 = 61$
c) $f(16) = 2\cdot 61 + 3 = 125$, $f(32) = 2\cdot 125 + 3 = 253$, $f(64) = 2\cdot 253 + 3 = 509$
d) $f(128) = 2\cdot 509 + 3 = 1021$, $f(256) = 2\cdot 1021 + 3 = 2045$, $f(512) = 2\cdot 2045 + 3 = 4093$, $f(1024) = 2\cdot 4093 + 3 = 8189$

10. Since f increases one for each factor of 2 in n, it is clear that $f(2^k) = k + 1$.

12. An exact formula comes from the proof of Theorem 1, namely $f(n) = [f(1) + c/(a-1)]n^{\log_b a} - c/(a-1)$, where $a = 2$, $b = 3$, and $c = 4$ in this exercise. Therefore the answer is $f(n) = 5n^{\log_3 2} - 4$.

14. If there is only one team, then no rounds are needed, so the base case is $R(1) = 0$. Since it takes one round to cut the number of teams in half, we have $R(n) = 1 + R(n/2)$.

16. The solution of this recurrence relation for $n = 2^k$ is $R(2^k) = k$, for the same reason as in Exercise 10.

18. **a)** Our recursive algorithm will take a sequence of $2n$ names (two different names provided by each of n voters) and determine whether the two top vote-getters occur on our list more than $n/2$ times each, and if so, who they are. We assume that our list has the votes of each voter adjacent (the first voter's choices are in positions 1 and 2, the second voter's choices are in positions 3 and 4, and so on). Note that it is possible for more than two candidates to receive more than $n/2$ votes; for example, three voters could have choices AB, AC, and BC, and then all three would qualify. However, there cannot be more than three candidates qualifying, since the sum of four numbers each larger than $n/2$ is larger than $2n$, the total number of votes cast. If $n = 1$, then the two people on the list are both winners. For the recursive step, divide the list into two parts of even size—the first half and the second half—as equally as possible. As is pointed out in the hint in Exercise 17, no one could have gotten a majority (here that means more than $n/2$ votes) on the whole list without having a majority in one half or the other, since if a candidate got approval from less than or equal to half of the voters in each half, then he got approval from less than or equal to half of the voters in all (this is essentially just the distributive law). Apply the algorithm recursively to each half to come up with at most six names (three from each half). Then run through the entire list to count the number of occurrences of each of those names to decide which, if any, are the winners. This requires at most $12n$ additional comparisons for a list of length $2n$. At the outermost stage of this recursion (i.e., when dealing with the entire list), we have to compare the actual numbers of votes each of the candidates in the running got, since only the top two can be declared winners (subject to the anomaly of three people tied, as illustrated above).

b) We apply the master theorem with $a = 2$, $b = 2$, $c = 12$, and $d = 1$. Since $a = b^d$, we know that the number of comparisons is $O(n^d \log n) = O(n \log n)$.

20. **a)** We compute a^n **mod** m, when n is even, by first computing $y := a^{n/2}$ **mod** m recursively and then doing one modular multiplication, namely $y \cdot y$. When n is odd, we first compute $y := a^{(n-1)/2}$ recursively and then do two multiplications, namely $y \cdot y \cdot a$. So if $f(n)$ is the number of multiplications required, assuming the worst, then we have essentially $f(n) = f(n/2) + 2$.

b) By the master theorem, with $a = 1$, $b = 2$, $c = 2$, and $d = 0$, we see that $f(n)$ is $O(n^0 \log n) = O(\log n)$.

22. **a)** $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$

b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2$, $b = 2$, $c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this says that our function is $O(\log n \cdot \log \log n)$.

24. To carry this down to its base level would require applying the algorithm three times, so we will show only the outermost step. The points are already sorted for us, and so we divide them into two groups, using x coordinate. The left side will have the first four points listed in it (they all have x coordinates less than 2.5), and the right side will have the rest, all of which have x coordinates greater than 2.5. Thus our vertical line will be taken to be $x = 2.5$. Now assume that we have already applied the algorithm recursively to find the minimum distance between two points on the left, and the minimum distance on the right. It turns out that $d_L = \sqrt{2}$ and $d_R = \sqrt{5}$, so $d = \sqrt{2}$. This is achieved by the points $(1, 3)$ and $(2, 4)$. Thus we want to concentrate on the strip from $x = 2.5 - \sqrt{2} \approx 1.1$ to $x = 2.5 + \sqrt{2} \approx 3.9$ of width $2d$. The only points in this strip are $(2, 4)$, $(2, 9)$, $(3, 1)$, and $(3, 5)$, Working from the bottom up, we compute distances from these points to points as much as $d = \sqrt{2} \approx 1.4$ vertical units above them. According to the discussion in the text, there can never be more than seven such computations for each point in the strip. In this case there is in fact only one, namely $\overline{(2, 4)(3, 5)}$. This distance is again $\sqrt{2}$, and it ties the minimum distance already obtained. So the minimum distance is $\sqrt{2}$.

26. In our algorithm d contains the shortest distance and is the value returned by the algorithm. We assume a function *dist* that computes Euclidean distance given two points (a, b) and (c, d), namely $\sqrt{(a - c)^2 + (b - d)^2}$. We also assume that some global preprocessing has been done to sort the points in nondecreasing order of x coordinates before calling this program, and to produce a separate list P of the points in nondecreasing order of y coordinates, but having an identification as to which points in the original list they are.

procedure *closest*$((x_1, y_1), \ldots, (x_n, y_n)$: points in the plane)
if $n = 2$ **then** $d := dist((x_1, y_1), (x_2, y_2))$
else
  $m := (x_{\lfloor n/2 \rfloor} + x_{\lceil n/2 \rceil})/2$
  $d_L := closest((x_1, y_1), \ldots, (x_{\lfloor n/2 \rfloor}, y_{\lfloor n/2 \rfloor}))$
  $d_R := closest((x_{\lceil n/2 \rceil}, y_{\lceil n/2 \rceil}), \ldots, (x_n, y_n))$
  $d := \min(d_L, d_R)$
  form the sublist P' of P consisting of those points whose x-coordinates are within d of m
  for each point (x, y) in P'
    for each point (x', y') in P' after (x, y) such that $y' - y < d$
      if $dist((x, y), (x', y')) < d$ then $d := dist((x, y), (x', y'))$
return d $\{d$ is the minimum distance between the points in the list$\}$

28. **a)** We follow the discussion given here. At each stage, we ask the question twice, "Is x in this part of the set?" if the two answers agree, then we know that they are truthful, and we proceed recursively on the half we then know contains the number. If the two answers disagree, then we ask the question a third time to determine the truth (the first person cannot lie twice, so the third answer is truthful). After we have detected the lie, we no longer need to ask each question twice, since all answers have to be truthful. If the lie occurs on our last query, however, then we have used a full $2 \log n + 1$ questions (the last 1 being the third question when the lie was detected).

b) Divide the set into four (nearly) equal-sized parts, A, B, C, and D. To determine which of the four subsets contains the first person's number, ask these questions: "Is your number in $A \cup B$?" and "Is your number in $A \cup C$?" If the answers are both "yes," then we can eliminate D, since we know that at least one of these answers was truthful and therefore the secret number is in $A \cup B \cup C$. By similar reasoning, if both answers are "no," then we can eliminate A; if the answers are first "yes" and then "no," then we can eliminate C; and if the answers are first "no" and then "yes," then we can eliminate B. Therefore after two questions we have a problem of size about $3n/4$ (exactly this when $4 \mid n$).

c) Since we reduce the problem to one problem of size $3n/4$ at each stage, the number $f(n)$ of questions satisfies $f(n) = f(3n/4) + 2$ when n is divisible by 4.

d) Using iteration, we solve the recurrence relation in part **(c)**. We have $f(n) = 2 + f((3/4)n) = 2 + 2 + f((3/4)^2 n) = 2 + 2 + 2 + f((3/4)^3 n) = \cdots = 2 + 2 + \cdots + 2$, where there are about $\log_{4/3} n$ 2's in the sum. Noting that $\log_{4/3} n = \log n / \log 4/3 \approx 2.4 \log n$, we have that $f(n) \approx 4.8 \log n$.

e) The naive way is better, with fewer than half the number of questions. Another way to see this is to observe that after four questions in the second method, the size of our set is down to $9/16$ of its original size, but after only two questions in the first method, the size of the set is even smaller $(1/2)$.

30. The second term obviously dominates the first. Also, $\log_b n$ is just a constant times $\log n$. The statement now follows from the fact that f is increasing.

32. If $a < b^d$, then $\log_b a < d$, so the first term dominates. The statement now follows from the fact that f is increasing.

34. From Exercise 31 (note that here $a = 5$, $b = 4$, $c = 6$, and $d = 1$) we have $f(n) = -24n + 25n^{\log_4 5}$.

36. From Exercise 31 (note that here $a = 8$, $b = 2$, $c = 1$, and $d = 2$) we have $f(n) = -n^2 + 2n^{\log 8} = -n^2 + 2n^3$.

SECTION 8.4 Generating Functions

2. The generating function is $f(x) = 1 + 4x + 16x^2 + 64x^3 + 256x^4$. Since the i^{th} term in this sequence (the coefficient of x^i) is 4^i for $0 \le i \le 4$, we can also write the generating function as

$$f(x) = \sum_{i=0}^{4} (4x)^i = \frac{1-(4x)^5}{1-4x}\,.$$

4. We will use Table 1 in much of this solution.

a) Apparently all the terms are 0 except for the seven -1's shown. Thus $f(x) = -1-x-x^2-x^3-x^4-x^5-x^6$. This is already in closed form, but we can also write it more compactly as $f(x) = -(1-x^7)/(1-x)$, making use of the identity from Example 2.

b) This sequence fits the pattern in Table 1 for $1/(1-ax)$ with $a = 3$. Therefore the generating function is $1/(1-3x)$.

c) We can factor out $3x^2$ and write the generating function as $3x^2(1-x+x^2-x^3+\cdots) = 3x^2/(1+x)$, again using the identity in Table 1.

d) Except for the extra x (the coefficient of x is 2 rather than 1), the generating function is just $1/(1-x)$. Therefore the answer is $x + (1/(1-x))$.

e) From Table 1, we see that the binomial theorem applies and we can write this as $(1+2x)^7$.

f) We can factor out -3 and write the generating function as $-3(1-x+x^2-x^3+\cdots) = -3/(1+x)$, using the identity in Table 1.

g) We can factor out x and write the generating function as $x(1-2x+4x^2-8x^3+\cdots) = x/(1+2x)$, using the sixth identity in Table 1 with $a = -2$.

h) From Table 1 we see that the generating function here is $1/(1-x^2)$.

6. a) Since the sequence with $a_n = 1$ for all n has generating function $1/(1-x)$, this sequence has generating function $-1/(1-x)$.

b) By Table 1, the generating function for the sequence in which $a_n = 2^n$ for *all* n is $1/(1-2x)$. Here we can either think of subtracting out the missing constant term (since $a_0 = 0$) or factoring out $2x$. Therefore the answer can be written as either $1/(1-2x)-1$ or $2x/(1-2x)$, which are of course algebraically equivalent.

c) We need to split this into two parts. Since we know that the generating function for the sequence $\{n+1\}$ is $1/(1-x)^2$, we write $n-1 = (n+1)-2$. Therefore the generating function is $(1/(1-x)^2) - (2/(1-x))$. We can combine terms and write this function as $(2x-1)/(1-x)^2$, but there is no particular reason to prefer that form in general.

d) The power series for the function e^x is $\sum_{n=0}^{\infty} x^n/n!$. That is almost what we have here; the difference is that the denominator is $(n+1)!$ instead of $n!$. So we have

$$\sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

by a change of variable. This last sum is $e^x - 1$ (only the first term is missing), so our answer is $(e^x-1)/x$.

e) Let $f(x)$ be the generating function we seek. From Table 1 we know that $1/(1-x)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, and that is almost what we have here. To transform this to $f(x)$ need to factor out x^2 and change the variable of summation:

$$\frac{1}{(1-x)^3} = \sum_{n=0}^{\infty} C(n+2,2)x^n = \frac{1}{x^2}\sum_{n=0}^{\infty} C(n+2,2)x^{n+2} = \frac{1}{x^2}\sum_{n=2}^{\infty} C(n,2)x^n = \frac{1}{x^2}\cdot(f(x)-f(0)-f(1))$$

Noting that $f(0) = f(1) = 0$ by definition, we have $f(x) = x^2/(1-x)^3$.

f) We again use Table 1:

$$\sum_{n=0}^{\infty} C(10, n+1)x^n = \sum_{n=1}^{\infty} C(10, n)x^{n-1} = \frac{1}{x}\sum_{n=1}^{\infty} C(10, n)x^n = \frac{1}{x}((1+x)^{10} - 1)$$

8. **a)** By the binomial theorem (the third line of Table 1) we get $a_{2n} = C(3, n)$ for $n = 0, 1, 2, 3$, and the other coefficients are all 0. Alternatively, we could just multiply out this finite polynomial and note the nonzero coefficients: $a_0 = 1$, $a_2 = 3$, $a_4 = 3$, $a_6 = 1$.

b) This is like part **(a)**. First we need to factor out -1 and write this as $-(1-3x)^3$. Then by the binomial theorem (the second line of Table 1) we get $a_n = -C(3, n)(-3)^n$ for $n = 0, 1, 2, 3$, and the other coefficients are all 0. Alternatively, we could (by hand or with *Maple*) just multiply out this finite polynomial and note the nonzero coefficients: $a_0 = -1$, $a_1 = 9$, $a_2 = -27$, $a_3 = 27$.

c) This problem requires a combination of the results of the sixth and seventh identities in Table 1. The coefficient of x^{2n} is 2^n, and the odd coefficients are all 0.

d) We know that $x^2/(1-x)^3 = x^2\sum_{n=0}^{\infty} C(n+2, 2)x^n = \sum_{n=0}^{\infty} C(n+2, 2)x^{n+2} = \sum_{n=2}^{\infty} C(n, 2)x^n$. Therefore $a_n = C(n, 2) = n(n-1)/2$ for $n \geq 2$ and $a_0 = a_1 = 0$. (Actually, since $C(0, 2) = C(1, 2) = 0$, we really don't need to make a special statement for $n < 2$.)

e) The last term gives us, from Table 1, $a_n = 3^n$. We need to adjust this for $n = 0$ and $n = 1$ because of the first two terms. Thus $a_0 = -1 + 3^0 = 0$, and $a_1 = 1 + 3^1 = 4$.

f) We split this into two parts and proceed as in part **(d)**:

$$\begin{aligned}
\frac{1}{(1+x)^3} + \frac{x^3}{(1+x)^3} &= \sum_{n=0}^{\infty}(-1)^n C(n+2, 2)x^n + x^3\sum_{n=0}^{\infty}(-1)^n C(n+2, 2)x^n \\
&= \sum_{n=0}^{\infty}(-1)^n C(n+2, 2)x^n + \sum_{n=0}^{\infty}(-1)^n C(n+2, 2)x^{n+3} \\
&= \sum_{n=0}^{\infty}(-1)^n C(n+2, 2)x^n + \sum_{n=3}^{\infty}(-1)^{n-3} C(n-1, 2)x^n
\end{aligned}$$

Note that n and $n-3$ have opposite parities. Therefore $a_n = (-1)^n C(n+2, 2) + (-1)^{n-3}C(n-1, 2) = (-1)^n(C(n+2, 2) - C(n-1, 2)) = (-1)^n 3n$ for $n \geq 3$ and $a_n = (-1)^n C(n+2, 2) = (-1)^n(n+2)(n+1)/2$ for $n < 3$. This answer can be confirmed using the `series` command in *Maple*.

g) The key here is to recall the algebraic identity $1 - x^3 = (1-x)(1+x+x^2)$. Therefore the given function can be rewritten as $x(1-x)/(1-x^3)$, which can then be split into $x/(1-x^3)$ plus $-x^2/(1-x^3)$. From Table 1 we know that $1/(1-x^3) = 1 + x^3 + x^6 + x^9 + \cdots$. Therefore $x/(1-x^3) = x + x^4 + x^7 + x^{10} + \cdots$, and $-x^2/(1-x^3) = -x^2 - x^5 - x^8 - x^{11} - \cdots$. Thus we see that a_n is 0 when n is a multiple of 3, it is 1 when n is 1 greater than a multiple of 3, and it is -1 when n is 2 greater than a multiple of 3. One can check this answer with *Maple*.

h) From Table 1 we know that $e^x = 1 + x + x^2/2! + x^3/3! + \cdots$. It follows that

$$e^{3x^2} = 1 + 3x^2 + \frac{(3x^2)^2}{2!} + \frac{(3x^2)^3}{3!} + \cdots.$$

We can therefore read off the coefficients of the generating function for $e^{3x^2} - 1$. First, clearly $a_0 = 0$. Second, $a_n = 0$ when n is odd. Finally, when n is even, we have $a_{2m} = 3^m/m!$.

10. Different approaches are possible for obtaining these answers. One can use brute force algebra and just multiply everything out, either by hand or with computer algebra software such as *Maple*. One can view the problem as asking for the solution to a particular combinatorial problem and solve the problem by other means (e.g., listing all the possibilities). Or one can get a closed form expression for the coefficients, using the generating function theory developed in this section.

a) First we view this combinatorially. By brute force we can list the ten ways to obtain x^9 when this product is multiplied out (where "ijk" means choose an x^i term from the first factor, an x^j term from the second factor, and an x^k term from the third factor): 009, 036, 063, 090, 306, 333, 360, 603, 630, 900. Second, it is clear that we can view this problem as asking for the coefficient of x^3 in $(1+x+x^2+x^3+\cdots)^3$, since each x^3 in the original is playing the role of x here. Since $(1+x+x^2+x^3+\cdots)^3 = 1/(1-x)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, the answer is clearly $C(3+2,2) = C(5,2) = 10$. A third way to get the answer is to ask *Maple* to expand $(1+x^3+x^6+x^9)^3$ and look at the coefficient of x^9, which will turn out to be 10. Note that we don't have to go beyond x^9 in each factor, because the higher terms can't contribute to an x^9 term in the answer.

b) If we factor out x^2 from each factor, we can write this as $x^6(1+x+x^2+\cdots)^3$. Thus we are seeking the coefficient of x^3 in $(1+x+x^2+\cdots)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, so the answer is $C(3+2,2) = 10$. The other two methods explained in part **(a)** work here as well.

c) If we factor out as high a power of x from each factor as we can, then we can write this as

$$x^7(1+x^2+x^3)(1+x)(1+x+x^2+x^3+\cdots)\,,$$

and so we seek the coefficient of x^2 in $(1+x^2+x^3)(1+x)(1+x+x^2+x^3+\cdots)$. We could do this by brute force, but let's try it more analytically. We write our expression in closed form as

$$\frac{(1+x^2+x^3)(1+x)}{1-x} = \frac{1+x+x^2+\text{higher order terms}}{1-x} = \frac{1}{1-x} + x\cdot\frac{1}{1-x} + x^2\cdot\frac{1}{1-x} + \text{irrelevant terms.}$$

The coefficient of x^2 in this power series comes either from the coefficient of x^2 in the first term in the final expression displayed above, or from the coefficient of x^1 in the second factor of the second term of that expression, or from the coefficient of x^0 in the second factor of the third term. Each of these coefficients is 1, so our answer is 3. This could also be confirmed by having *Maple* multiply out ("`expand`") the original expression (truncating the last factor at x^3).

d) The easiest approach here is simply to note that there are only two combinations of terms that will give us an x^9 term in the product: $x\cdot x^8$ and $x^7\cdot x^2$. So the answer is 2.

e) The highest power of x appearing in this expression when multiplied out is x^6. Therefore the answer is 0.

12. These can all be checked by using the `series` command in *Maple*.

a) By Table 1, the coefficient of x^n in this power series is $(-3)^n$. Therefore the answer is $(-3)^{12} = 531{,}441$.

b) By Table 1, the coefficient of x^n in this power series is $2^nC(n+1,1)$. Thus the answer is $2^{12}C(12+1,1) = 53{,}248$.

c) By Table 1, the coefficient of x^n in this power series is $(-1)^nC(n+7,7)$. Therefore the answer is $(-1)^{12}C(12+7,7) = 50{,}388$.

d) By Table 1, the coefficient of x^n in this power series is $4^nC(n+2,2)$. Thus the answer is $4^{12}C(12+2,2) = 1{,}526{,}726{,}656$.

e) This is really asking for the coefficient of x^9 in $1/(1+4x)^2$. Following the same idea as in part **(d)**, we see that the answer is $(-4)^9C(9+1,1) = -2{,}621{,}440$.

14. Each child will correspond to a factor in our generating function. We can give 0, 1, 2, or 3 figures to the child; therefore the generating function for each child is $1+x+x^2+x^3$. We want to find the coefficient of x^{12} in the expansion of $(1+x+x^2+x^3)^5$. We can multiply this out (preferably with a computer algebra package such as *Maple*), and the coefficient of x^{12} turns out to be 35. To solve it analytically, we write our generating function as

$$\left(\frac{1-x^4}{1-x}\right)^5 = \frac{1-5x^4+10x^8-10x^{12}+\text{higher order terms}}{(1-x)^5}.$$

There are four contributions to the coefficient of x^{12}, one for each term in the numerator, from the power series for $1/(1-x)^5$. Since the coefficient of x^n in $1/(1-x)^5$ is $C(n+4,4)$, our answer is $C(12+4,4) - 5C(8+4,4) + 10C(4+4,4) - 10C(0+4,4) = 1820 - 2475 + 700 - 10 = 35$.

16. The factors in the generating function for choosing the egg and plain bagels are both $x^2 + x^3 + x^4 + \cdots$. The factor for choosing the salty bagels is $x^2 + x^3$. Therefore the generating function for this problem is $(x^2 + x^3 + x^4 + \cdots)^2(x^2 + x^3)$. We want to find the coefficient of x^{12}, since we want 12 bagels. This is equivalent to finding the coefficient of x^6 in $(1 + x + x^2 + \cdots)^2(1 + x)$ This function is $(1 + x)/(1 - x)^2$, so we want the coefficient of x^6 in $1/(1 - x)^2$, which is 7, plus the coefficient of x^5 in $1/(1 - x)^2$, which is 6. Thus the answer is 13.

18. Without changing the answer, we can assume that the jar has an infinite number of balls of each color; this will make the algebra easier. For the red and green balls the generating function is $1 + x + x^2 + \cdots$, but for the blue balls the generating function is $x^3 + x^4 + \cdots + x^{10}$, so the generating function for the whole problem is $(1 + x + x^2 + \cdots)^2(x^3 + x^4 + \cdots + x^{10})$. We seek the coefficient of x^{14}. This is the same as the coefficient of x^{11} in

$$(1 + x + x^2 + \cdots)^2(1 + x + \cdots + x^7) = \frac{1 - x^8}{(1 - x)^3}.$$

Since the coefficient of x^n in $1/(1 - x)^3$ is $C(n + 2, 2)$, and we have two contributing terms determined by the numerator, our answer is $C(11 + 2, 2) - C(3 + 2, 2) = 68$.

20. We want the coefficient of x^k to be the number of ways to make change for k pesos. Ten-peso bills contribute 10 each to the exponent of x. Thus we can model the choice of the number of 10-peso bills by the choice of a term from $1 + x^{10} + x^{20} + x^{30} + \cdots$. Twenty-peso bills contribute 20 each to the exponent of x. Thus we can model the choice of the number of 20-peso bills by the choice of a term from $1 + x^{20} + x^{40} + x^{60} + \cdots$. Similarly, 50-peso bills contribute 50 each to the exponent of x, so we can model the choice of the number of 50-peso bills by the choice of a term from $1 + x^{50} + x^{100} + x^{150} + \cdots$. Similar reasoning applies to 100-peso bills. Thus the generating function is $f(x) = (1 + x^{10} + x^{20} + x^{30} + \cdots)(1 + x^{20} + x^{40} + x^{60} + \cdots)(1 + x^{50} + x^{100} + x^{150} + \cdots)(1 + x^{100} + x^{200} + x^{300} + \cdots)$, which can also be written as

$$f(x) = \frac{1}{(1 - x^{10})(1 - x^{20})(1 - x^{50})(1 - x^{100})}$$

by Table 1. Note that $c_k = 0$ unless k is a multiple of 10, and the power series has no terms whose exponents are not powers of 10.

22. Let e_i, for $i = 1, 2, \ldots, n$, be the exponent of x taken from the i^{th} factor in forming a term x^6 in the expansion. Thus $e_1 + e_2 + \cdots + e_n = 6$. The coefficient of x^6 is therefore the number of ways to solve this equation with nonnegative integers, which, from Section 6.5, is $C(n + 6 - 1, 6) = C(n + 5, 6)$. Its value, of course, depends on n.

24. **a)** The restriction on x_1 gives us the factor $x^3 + x^4 + x^5 + \cdots$. The restriction on x_2 gives us the factor $x + x^2 + x^3 + x^4 + x^5$. The restriction on x_3 gives us the factor $1 + x + x^2 + x^3 + x^4$. And the restriction on x_4 gives us the factor $x + x^2 + x^3 + \cdots$. Thus the answer is the product of these:

$$(x^3 + x^4 + x^5 + \cdots)(x + x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3 + x^4)(x + x^2 + x^3 + \cdots)$$

We can use algebra to rewrite this in closed form as $x^5(1 + x + x^2 + x^3 + x^4)^2/(1 - x)^2$.

b) We want the coefficient of x^7 in this series, which is the same as the coefficient of x^2 in the series for

$$\frac{(1 + x + x^2 + x^3 + x^4)^2}{(1 - x)^2} = \frac{1 + 2x + 3x^2 + \text{higher order terms}}{(1 - x)^2}.$$

Since the coefficient of x^n in $1/(1 - x)^2$ is $n + 1$, our answer is $1 \cdot 3 + 2 \cdot 2 + 3 \cdot 1 = 10$.

26. **a)** On each roll, we can get a total of one pip, two pips, ..., six pips. So the generating function for each roll is $x + x^2 + x^3 + x^4 + x^5 + x^6$. The exponent on x gives the number of pips. If we want to achieve a total of k pips in n rolls, then we need the coefficient of x^k in $(x + x^2 + x^3 + x^4 + x^5 + x^6)^n$. Since n is free to vary here, we must add these generating functions for all possible values of n. Therefore the generating function for this problem is $\sum_{n=0}^{\infty}(x + x^2 + x^3 + x^4 + x^5 + x^6)^n$. By the formula for summing a geometric series, this is the same as $1/(1 - (x + x^2 + x^3 + x^4 + x^5 + x^6)) = 1/(1 - x - x^2 - x^3 - x^4 - x^5 - x^6)$.

b) We seek the coefficient of x^8 in the power series for our answer to part **(a)**. The best way to get the answer is probably asking *Maple* or another computer algebra package to find this power series, which it will probably do using calculus. If we do so, the answer turns out to be 125 (the series starts out $1 + x + 2x^2 + 4x^3 + 8x^4 + 16x^5 + 32x^6 + 63x^7 + 125x^8 + 248x^9$).

28. In each case, the generating function for the choice of pennies is $1 + x + x^2 + \cdots = 1/(1 - x)$ or some portion of this to account for restrictions on the number of pennies used. Similarly, the generating function for the choice of nickels is $1 + x^5 + x^{10} + \cdots = 1/(1 - x^5)$ (or some portion); and similarly for the dimes and quarters. For each part we will write down the generating function (a product of the generating functions for each coin) and then invoke a computer algebra system to get the answer.

a) The generating function for the pennies is $1 + x + x^2 + \cdots + x^{10} = (1 - x^{11})/(1 - x)$. Thus our entire generating function is

$$\frac{1 - x^{11}}{1 - x} \cdot \frac{1}{1 - x^5} \cdot \frac{1}{1 - x^{10}} \cdot \frac{1}{1 - x^{25}}.$$

Maple says that the coefficient of x^{100} in this is 79.

b) This is just like part **(a)**, except that now the generating function is

$$\frac{1 - x^{11}}{1 - x} \cdot \frac{1 - (x^5)^{11}}{1 - x^5} \cdot \frac{1}{1 - x^{10}} \cdot \frac{1}{1 - x^{25}}.$$

This time *Maple* reports that the answer is 58.

c) This problem can be solved by using a generating function with two variables, one for the number of coins (say y) and one for the values (say x). Then the generating function for nickels, for instance, is

$$1 + x^5y + x^{10}y^2 + \cdots = \frac{1}{1 - x^5y}.$$

We multiply the four generating functions together, for the four different denominations, and get a function of x and y. Then we ask *Maple* to expand this as a power series and get the coefficient of x^{100}. This coefficient is a polynomial in y. We ask *Maple* to extract and simplify this polynomial and it turns out to be $y^4 + y^6 + 2y^7 + 2y^8 + 2y^9 + 4y^{10}$ plus higher order terms that we don't want, since we need the number of coins (which is what the exponent on y tells us) to be less than 11. Since the total of these coefficients is 12, the answer is 12, which can be confirmed by brute force enumeration.

30. **a)** Multiplication distributes over addition, even when we are talking about infinite sums, so the generating function is just $2G(x)$.

b) What used to be the coefficient of x^0 is now the coefficient of x^1, and similarly for the other terms. The way that happened is that the whole series got multiplied by x. Therefore the generating function for this series is $xG(x)$. In symbols,

$$a_0x + a_1x^2 + a_2x^3 + \cdots = x(a_0 + a_1x + a_2x^2 + \cdots) = xG(x).$$

c) The terms involving a_0 and a_1 are missing; $G(x) - a_0 - a_1x = a_2x^2 + a_3x^3 + \cdots$. Here, however, we want a_2 to be the coefficient of x^4, not x^2 (and similarly for the other powers), so we must throw in an extra factor. Thus the answer is $x^2(G(x) - a_0 - a_1x)$.

d) This is just like part **(c)**, except that we slide the powers down. Thus the answer is $(G(x) - a_0 - a_1x)/x^2$.

e) Following the hint, we differentiate $G(x) = \sum_{n=0}^{\infty} a_n x^n$ to obtain $G'(x) = \sum_{n=0}^{\infty} n\, a_n x^{n-1}$. By a change of variable this becomes $\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n = a_1 + 2a_2x + 3a_3x^2 + \cdots$, which is the generating function for precisely the sequence we are given. Thus $G'(x)$ is the generating function for this sequence.

f) If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x)\cdot G(x)$.

32. This problem is like Example 16. First let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$). Thus

$$G(x) - 7xG(x) = \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 7a_{k-1}x^k = a_0 + \sum_{k=1}^{\infty}(a_k - 7a_{k-1})x^k = a_0 + 0 = 5\,,$$

because of the given recurrence relation and initial condition. Thus $G(x)(1-7x) = 5$, so $G(x) = 5/(1-7x)$. From Table 1 we know then that $a_k = 5\cdot 7^k$.

34. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$). Thus

$$\begin{aligned} G(x) - 3xG(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1}x^k = a_0 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k = 1 + \sum_{k=1}^{\infty} 4^{k-1}x^k \\ &= 1 + x\sum_{k=1}^{\infty} 4^{k-1}x^{k-1} = 1 + x\sum_{k=0}^{\infty} 4^k x^k = 1 + x\cdot\frac{1}{1-4x} = \frac{1-3x}{1-4x}\,. \end{aligned}$$

Thus $G(x)(1-3x) = (1-3x)/(1-4x)$, so $G(x) = 1/(1-4x)$. Therefore $a_k = 4^k$, from Table 1.

36. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and $x^2G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$. Thus

$$\begin{aligned} G(x) - xG(x) - 2x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} a_{k-1}x^k - \sum_{k=2}^{\infty} 2a_{k-2}x^k = a_0 + a_1x - a_0x + \sum_{k=2}^{\infty} 2^k\cdot x^k \\ &= 4 + 8x + \frac{1}{1-2x} - 1 - 2x = \frac{4-12x^2}{1-2x}\,, \end{aligned}$$

because of the given recurrence relation, the initial conditions, Table 1, and algebra. Since the left-hand side of this equation factors as $G(x)(1-2x)(1+x)$, we have $G(x) = (4-12x^2)/((1+x)(1-2x)^2)$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{4-12x^2}{(1+x)(1-2x)^2} = \frac{A}{1+x} + \frac{B}{1-2x} + \frac{C}{(1-2x)^2}\,,$$

multiplying through by the common denominator, and equating coefficients, we find that $A = -8/9$, $B = 38/9$, and $C = 2/3$. Thus

$$G(x) = \frac{-8/9}{1+x} + \frac{38/9}{1-2x} + \frac{2/3}{(1-2x)^2} = \sum_{k=0}^{\infty}\left(-\frac{8}{9}(-1)^k + \frac{38}{9}\cdot 2^k + \frac{2}{3}(k+1)2^k\right)x^k$$

(from Table 1). Therefore $a_k = (-8/9)(-1)^k + (38/9)2^k + (2/3)(k+1)2^k$. Incidentally, it would be wise to check our answers, either with a computer algebra package, or by computing the next term of the sequence from both the recurrence and the formula (here $a_2 = 24$ both ways).

38. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from k to $k+1$), and similarly $x^2G(x) = \sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$. Thus

$$\begin{aligned}
G(x) - 2xG(x) - 3x^2G(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 2a_{k-1}x^k - \sum_{k=2}^{\infty} 3a_{k-2}x^k = a_0 + a_1 x - 2a_0 x + \sum_{k=2}^{\infty}(4^k+6)\cdot x^k \\
&= 20 + 20x + \frac{1}{1-4x} + \frac{6}{1-x} - 7 - 10x = 13 + 10x + \frac{1}{1-4x} + \frac{6}{1-x} \\
&= \frac{20 - 80x + 2x^2 + 40x^3}{(1-4x)(1-x)},
\end{aligned}$$

because of the given recurrence relation, the initial conditions, and Table 1. Since the left-hand side of this equation factors as $G(x)(1-3x)(1+x)$, we know that

$$G(x) = \frac{20 - 80x + 2x^2 + 40x^3}{(1-4x)(1-x)(1+x)(1-3x)}.$$

At this point we must use partial fractions to break up the denominator. Setting this last expression equal to

$$\frac{A}{1-4x} + \frac{B}{1-x} + \frac{C}{1+x} + \frac{D}{1-3x},$$

multiplying through by the common denominator, and equating coefficients, we find that $A = 16/5$, $B = -3/2$, $C = 31/20$, and $D = 67/4$. Thus

$$G(x) = \frac{16/5}{1-4x} + \frac{-3/2}{1-x} + \frac{31/20}{1+x} + \frac{67/4}{1-3x} = \sum_{k=0}^{\infty}\left(\frac{16}{5}\cdot 4^k - \frac{3}{2} + \frac{31}{20}(-1)^k + \frac{67}{4}\cdot 3^k\right)x^k$$

(from Table 1). Therefore $a_k = (16/5)4^k - (3/2) + (31/20)(-1)^k + (67/4)3^k$. We check our answer by computing the next term of the sequence from both the recurrence and the formula (here $a_2 = 202$ both ways). Alternatively, we ask *Maple* for the solution:

```
rsolve({a(k) = 2 * a(k - 1) + 3 * a(k - 2) + 4^k + 6, a(0) = 20, a(1) = 60}, a(k));
```

40. a) By definition,

$$\begin{aligned}
\binom{-1/2}{n} &= \frac{(-1/2)(-3/2)(-5/2)\cdots(-(2n-1)/2)}{n!} \\
&= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!} \\
&= (-1)^n \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n\, n!}\cdot\frac{2\cdot 4\cdot 6\cdot(2n)}{2^n\, n!} \\
&= (-1)^n \frac{(2n)!}{n!\,n!\,4^n} \\
&= (-1)^n \binom{2n}{n}\frac{1}{4^n} = \binom{2n}{n}\frac{1}{(-4)^n}
\end{aligned}$$

b) By the extended binomial theorem (Theorem 2), with $-4x$ in place of x and $u = -1/2$, we have

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4x)^n = \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{(-4)^n}(-4x)^n = \sum_{n=0}^{\infty}\binom{2n}{n}x^n.$$

42. First we note, as the hint suggests, that $(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)^{n-1} + x(1+x)^{n-1}$. Expanding both sides of this equality using the binomial theorem, we have

$$\begin{aligned}
\sum_{r=0}^{n} C(n,r)x^r &= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=0}^{n-1} C(n-1,r)x^{r+1} \\
&= \sum_{r=0}^{n-1} C(n-1,r)x^r + \sum_{r=1}^{n} C(n-1,r-1)x^r.
\end{aligned}$$

Thus

$$1+\left(\sum_{r=1}^{n-1} C(n,r)x^r\right)+x^n = 1+\left(\sum_{r=1}^{n-1}(C(n-1,r)+C(n-1,r-1))x^r\right)+x^n .$$

Comparing these two expressions, coefficient by coefficient, we see that $C(n,r)$ must equal $C(n-1,r)+C(n-1,r-1)$ for $1\le r\le n-1$, as desired.

44. Let $G(x)=\sum_{n=0}^{\infty} a_n x^n$ be the generating function for the sequence $\{a_n\}$, where $a_n = 1^2+2^2+3^2+\cdots+n^2$.
a) We use the method of generating functions to solve the recurrence relation and initial condition that our sequence satisfies: $a_n = a_{n-1}+n^2$ with $a_0=0$ (as in, for example, Exercise 34):

$$G(x)-xG(x)=\sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1}x^n = \sum_{n=0}^{\infty} n^2 x^n .$$

By Exercise 37, the generating function for $\{n^2\}$ is

$$\frac{2}{(1-x)^3}-\frac{3}{(1-x)^2}+\frac{1}{1-x}=\frac{x^2+x}{(1-x)^3},$$

so $(1-x)G(x)=(x^2+x)/(1-x)^3$. Dividing both sides by $1-x$ gives the desired expression for $G(x)$.
b) We split the generating function we found for $G(x)=\sum_{n=0}^{\infty} a_n x^n$ into two pieces and use Table 1:

$$\begin{aligned}\frac{x^2}{(1-x)^4}+\frac{x}{(1-x)^4} &= \sum_{n=0}^{\infty} C(n+3,3)x^{n+2}+\sum_{n=0}^{\infty} C(n+3,3)x^{n+1}\\ &= \sum_{n=0}^{\infty} C(n+1,3)x^n+\sum_{n=0}^{\infty} C(n+2,3)x^n\\ &= \sum_{n=0}^{\infty}\frac{(n+1)n(n-1)+(n+2)(n+1)n}{6}x^n\\ &= \sum_{n=0}^{\infty}\frac{n(n+1)(2n+1)}{6}x^n ,\end{aligned}$$

as desired. (Note that we did not need to change the limits of summation in line 3 because $C(1,3)=C(2,3)=0$.)

46. We will make heavy use of the identity $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!}x^n$.

a) $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}x^n = 2\sum_{n=0}^{\infty}\frac{1}{n!}(-2x)^n = e^{-2x}$

b) $\displaystyle\sum_{n=0}^{\infty}\frac{-1}{n!}x^n = -\sum_{n=0}^{\infty}\frac{1}{n!}x^n = -e^x$

c) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{n!}x^n = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = xe^x$, by a change of variable (This could also be done using calculus.)

d) This generating function can be obtained either with calculus or without. To do it without calculus, write $\displaystyle\sum_{n=0}^{\infty} n(n-1)\frac{x^n}{n!} = \sum_{n=2}^{\infty}\frac{x^n}{(n-2)!} = x^2\sum_{n=0}^{\infty}\frac{x^n}{n!} = x^2e^x$, by a change of variable. To do it with calculus, start with $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and differentiate both sides twice to obtain $e^x=\displaystyle\sum_{n=0}^{\infty}\frac{n(n-1)}{n!}x^{n-2}=\frac{1}{x^2}\sum_{n=0}^{\infty}n(n-1)\frac{x^n}{n!}$. Therefore $\displaystyle\sum_{n=0}^{\infty} n(n-1)\frac{x^n}{n!} = x^2e^x$.

e) This generating function can be obtained either with calculus or without. To do it without calculus, write

$$\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^n}{(n+2)!}=\frac{1}{x^2}\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)!}=\frac{1}{x^2}\sum_{n=2}^{\infty}\frac{x^n}{n!}=\frac{1}{x^2}(e^x-x-1)\,.$$

To do it with calculus, integrate $e^s=\sum_{n=0}^{\infty}\frac{s^n}{n!}$ from 0 to t to obtain

$$e^t-1=\sum_{n=0}^{\infty}\frac{t^{n+1}}{n+1}\cdot\frac{1}{n!}\,.$$

Then differentiate again, from 0 to x, to obtain

$$e^x-x-1=\sum_{n=0}^{\infty}\frac{x^{n+2}}{(n+2)(n+1)n!}=x^2\sum_{n=0}^{\infty}\frac{x^n}{(n+2)(n+1)n!}\,.$$

Thus $\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\cdot\frac{x^n}{n!}=(e^x-x-1)/x^2$.

48. In many of these cases, it's a matter of plugging the exponent of e into the generating function for e^x. We let a_n denote the n^{th} term of the sequence whose generating function is given.

a) The generating function is $e^{3x}=\sum_{n=0}^{\infty}\frac{(3x)^n}{n!}=\sum_{n=0}^{\infty}3^n\frac{x^n}{n!}$, so the sequence is $a_n=3^n$.

b) The generating function is $2e^{-3x+1}=(2e)e^{-3x}=2e\sum_{n=0}^{\infty}\frac{(-3x)^n}{n!}=\sum_{n=0}^{\infty}(2e(-3)^n)\frac{x^n}{n!}$, so the sequence is $a_n=2e(-3)^n$.

c) The generating function is $e^{4x}+e^{-4x}=\sum_{n=0}^{\infty}\frac{(4x)^n}{n!}+\sum_{n=0}^{\infty}\frac{(-4x)^n}{n!}=\sum_{n=0}^{\infty}(4^n+(-4)^n)\frac{x^n}{n!}$, so the sequence is $a_n=4^n+(-4)^n$.

d) The sequence whose exponential generating function is e^{3x} is clearly $\{3^n\}$, as in part **(a)**. Since

$$1+2x=\frac{1}{0!}x^0+\frac{2}{1!}x^1+\sum_{n=2}^{\infty}\frac{0}{n!}x^n\,,$$

we know that $a_n=3^n$ for $n\ge 2$, with $a_1=3^1+2=5$ and $a_0=3^0+1=2$.

e) We know that

$$\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^nx^n=\sum_{n=0}^{\infty}\frac{(-1)^nn!}{n!}x^n\,,$$

so the sequence for which $1/(1+x)$ is the exponential generating function is $\{(-1)^nn!\}$. Combining this with the rest of the function (where the generating function is just $\{1\}$), we have $a_n=1-(-1)^nn!$.

f) Note that

$$xe^x=\sum_{n=0}^{\infty}x\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}=\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}=\sum_{n=1}^{\infty}n\cdot\frac{x^n}{n!}=\sum_{n=0}^{\infty}n\cdot\frac{x^n}{n!}\,.$$

(We changed variable in the middle.) Therefore $a_n=n$, as in Exercise 46c.

g) First we note that

$$\begin{aligned}e^{x^3}=\sum_{n=0}^{\infty}\frac{(x^3)^n}{n!}&=1+\frac{x^3}{1!}+\frac{x^6}{2!}+\frac{x^9}{3!}+\cdots\\&=\frac{x^0}{0!}\cdot\frac{0!}{0!}+\frac{x^3}{3!}\cdot\frac{3!}{1!}+\frac{x^6}{6!}\cdot\frac{6!}{2!}+\frac{x^9}{9!}\cdot\frac{9!}{3!}+\cdots\,.\end{aligned}$$

Therefore we see that $a_n=0$ if n is not a multiple of 3, and $a_n=n!/(n/3)!$ if n is a multiple of 3.

50. **a)** Since all 4^n base-four strings of length n fall into one of the four categories counted by a_n, b_n, c_n, and d_n, obviously $d_n = 4^n - a_n - b_n - c_n$. Next let's see how a string of various types of length $n+1$ can be obtained from a string of length n by adding one digit. To get a string of length $n+1$ with an even number of 0s and an even number of 1s, we can take a string of length n with these same parities and append a 2 or a 3 (thus there are $2a_n$ such strings of this type), or we can take a string of length n with an even number of 0s and an odd number of 1s and append a 1 (thus there are b_n such strings of this type), or we can take a string of length n with an odd number of 0s and an even number of 1s and append a 0 (thus there are c_n such strings of this type). Therefore we have $a_{n+1} = 2a_n + b_n + c_n$. In the same way we find that $b_{n+1} = 2b_n + a_n + d_n$, which equals $b_n - c_n + 4^n$ after substituting the identity with which we began this solution. Similarly, $c_{n+1} = 2c_n + a_n + d_n = c_n - b_n + 4^n$.

b) The strings of length 1 are 0, 1, 2, and 3. So clearly $a_1 = 2$, $b_1 = c_1 = 1$, and $d_1 = 0$. (Note that 0 is an even number.) In fact we can also say that $a_0 = 1$ (the empty string) and $b_0 = c_0 = d_0 = 0$.

c) We apply the recurrences from part **(a)** twice:

$$a_2 = 2\cdot 2 + 1 + 1 = 6 \qquad a_3 = 2\cdot 6 + 4 + 4 = 20$$
$$b_2 = 1 - 1 + 4 = 4 \qquad b_3 = 4 + 16 - 4 = 16$$
$$c_2 = 1 - 1 + 4 = 4 \qquad c_3 = 4 + 16 - 4 = 16$$
$$d_2 = 16 - 6 - 4 - 4 = 2 \qquad d_3 = 64 - 20 - 16 - 16 = 12$$

d) Before proceeding as the problem asks, we note a shortcut. By symmetry, b_n must be the same as c_n. Substituting this into our recurrences, we find immediately that $b_n = c_n = 4^{n-1}$ for $n \geq 1$. Therefore $a_n = 2a_{n-1} + 2\cdot 4^{n-2}$. This recurrence with the initial condition $a_1 = 2$ can easily be solved by the methods of either this section or Section 8.2 to give $a_n = 2^{n-1} + 4^{n-1}$. But let's proceed as instructed.

Let $A(x)$, $B(x)$, and $C(x)$ be the desired generating functions. Then $xA(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$ and similarly for B and C, so we have

$$A(x) - xB(x) - xC(x) - 2xA(x) = \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} b_{n-1}x^n - \sum_{n=1}^{\infty} c_{n-1}x^n - \sum_{n=1}^{\infty} 2a_{n-1}x^n = a_0 = 1\,.$$

Similarly,

$$\begin{aligned} B(x) - xB(x) + xC(x) &= \sum_{n=0}^{\infty} b_n x^n - \sum_{n=1}^{\infty} b_{n-1}x^n + \sum_{n=1}^{\infty} c_{n-1}x^n \\ &= b_0 + \sum_{n=1}^{\infty} 4^{n-1}x^n = 0 + x\sum_{n=0}^{\infty} 4^n x^n = \frac{x}{1-4x}\,. \end{aligned}$$

Obviously C satisfies the same equation. Therefore our system of three equations (suppressing the arguments on A, B, and C) is

$$\begin{aligned} (1-2x)A - xB - xC &= 1 \\ (1-x)B + xC &= \frac{x}{1-4x} \\ xB + (1-x)C &= \frac{x}{1-4x}\,. \end{aligned}$$

e) Subtracting the third equation in part **(d)** from the second shows that $B = C$, and then plugging that back into the second equation immediately gives

$$B(x) = C(x) = \frac{x}{1-4x}\,.$$

Plugging these into the first equation yields

$$(1-2x)A - 2x\cdot\frac{x}{1-4x} = 1\,,$$

and solving for A gives us

$$A(x) = \frac{1-4x+2x^2}{(1-2x)(1-4x)}\,.$$

Now that we know the generating functions, we can recover the coefficients. For B and C (using Table 1) we immediately get a coefficient of 4^{n-1} for all $n \geq 1$, with $b_0 = c_0 = 0$. We rewrite $A(x)$ using partial fractions as

$$A(x) = \frac{1}{4} + \frac{1/2}{1-2x} + \frac{1/4}{1-4x},$$

so we have $a_n = \frac{1}{2} \cdot 2^n + \frac{1}{4} \cdot 4^n = 2^{n-1} + 4^{n-1}$ for $n \geq 1$, with $a_0 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$.

52. To form a partition of n using only odd-sized parts, we must choose some 1s, some 3s, some 5s, and so on. The generating function for choosing 1s is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

(the exponent gives the number so obtained). Similarly, the generating function for choosing 3s is

$$1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}$$

(again the exponent gives the number so obtained). The other choices have analogous generating functions. Therefore the generating function for the entire problem, so that the coefficient of x^n will give $p_o(n)$, the number of partitions of n into odd-sized part, is the infinite product

$$\frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots.$$

54. We need to carefully organize our work so as not to miss any of the partitions. We start with largest-sized parts first in all cases. For $n = 1$, we have $1 = 1$ as the only partition of either type, and so $p_o(1) = p_d(1) = 1$. For $n = 2$, we have $2 = 2$ as the only partition into distinct parts, and $2 = 1 + 1$ as the only partition into odd parts, so $p_o(1) = p_d(1) = 1$. For $n = 3$, we have $3 = 3$ and $3 = 2 + 1$ as the only partitions into distinct parts, and $3 = 3$ and $3 = 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 2$. For $n = 4$, we have $4 = 4$ and $4 = 3 + 1$ as the only partitions into distinct parts, and $4 = 3 + 1$ and $4 = 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 2$. For $n = 5$, we have $5 = 5$, $5 = 4 + 1$, and $5 = 3 + 2$ as the only partitions into distinct parts, and $5 = 5$, $5 = 3 + 1 + 1$, and $5 = 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 3$. For $n = 6$, we have $6 = 6$, $6 = 5 + 1$, $6 = 4 + 2$, and $6 = 3 + 2 + 1$ as the only partitions into distinct parts, and $6 = 5 + 1$, $6 = 3 + 3$, $6 = 3 + 1 + 1 + 1$, and $6 = 1 + 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 4$. For $n = 7$, we have $7 = 7$, $7 = 6 + 1$, $7 = 5 + 2$, $7 = 4 + 3$, and $7 = 4 + 2 + 1$ as the only partitions into distinct parts, and $7 = 7$, $7 = 5 + 1 + 1$, $7 = 3 + 3 + 1$, $7 = 3 + 1 + 1 + 1 + 1$, and $7 = 1 + 1 + 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 5$. Finally, for $n = 8$, we have $8 = 8$, $8 = 7 + 1$, $8 = 6 + 2$, $8 = 5 + 3$, $8 = 5 + 2 + 1$, and $8 = 4 + 3 + 1$ as the only partitions into distinct parts, and $8 = 7 + 1$, $8 = 5 + 3$ $8 = 5 + 1 + 1 + 1$, $8 = 3 + 3 + 1 + 1$, $8 = 3 + 1 + 1 + 1 + 1 + 1$, and $8 = 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1$ as the only partitions into odd parts, so $p_o(1) = p_d(1) = 6$. As we will prove in Exercise 55, it is no coincidence that these numbers all agree.

56. This is a very difficult problem. A solution can be found in *The Theory of Partitions* by George Andrews (Addison-Wesley, 1976), Chapter 6.

58. **a)** In order to have the first success on the n^{th} trial, where $n \geq 1$, we must have $n - 1$ failures followed by a success. Therefore $p(X = n) = q^{n-1}p$, where p is the probability of success and $q = 1 - p$ is the probability of failure. Therefore the probability generating function is

$$G(x) = \sum_{n=1}^{\infty} q^{n-1} p x^n = px \sum_{n=1}^{\infty} (qx)^{n-1} = px \sum_{n=0}^{\infty} (qx)^n = \frac{px}{1-qx}.$$

b) By Exercise 57, $E(X)$ is the derivative of $G(x)$ at $x = 1$. Here we have

$$G'(x) = \frac{p}{(1-qx)^2}, \quad \text{so} \quad G'(1) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}.$$

From the same exercise, we know that the variance is $G''(1) + G'(1) - G'(1)^2$; so we compute:

$$G''(x) = \frac{2pq}{(1-qx)^3}, \quad \text{so} \quad G''(1) = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2},$$

and therefore

$$V(X) = G''(1) + G'(1) - G'(1)^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2}.$$

60. We start with the definition and then use the fact that the only way for the sum of two nonnegative integers to be k is for one of them to be i and the other to be $k-i$, for some i between 0 and k, inclusive. We then invoke independence, and finally the definition of multiplication of infinite series:

$$\begin{aligned} G_{X+Y}(x) &= \sum_{k=0}^{\infty} p(X+Y=k)x^k \\ &= \sum_{k=0}^{\infty}\Bigg(\sum_{i=0}^{k} p(X=i \text{ and } Y=k-i)\Bigg)x^k \\ &= \sum_{k=0}^{\infty}\Bigg(\sum_{i=0}^{k} p(X=i)\cdot p(Y=k-i)\Bigg)x^k \\ &= G_X(x)\cdot G_Y(x) \end{aligned}$$

SECTION 8.5 Inclusion–Exclusion

2. $|C \cup D| = |C| + |D| - |C \cap D| = 345 + 212 - 188 = 369$

4. $|P \cap S| = |P| + |S| - |P \cup S| = 650{,}000 + 1{,}250{,}000 - 1{,}450{,}000 = 450{,}000$

6. **a)** In this case the union is just A_3, so the answer is $|A_3| = 10{,}000$.
b) The cardinality of the union is the sum of the cardinalities in this case, so the answer is $100+1000+10000 = 11{,}100$.
c) $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| = 100 + 1000 + 10000 - 2 - 2 - 2 + 1 = 11{,}095$

8. $270 - 64 - 94 - 58 + 26 + 28 + 22 - 14 = 116$

10. $100 - \lfloor 100/5 \rfloor - \lfloor 100/7 \rfloor + \lfloor 100/(5\cdot 7) \rfloor = 100 - 20 - 14 + 2 = 68$

12. There are $\lfloor \sqrt{1000} \rfloor = 31$ squares and $\lfloor \sqrt[3]{1000} \rfloor = 10$ cubes. Furthermore there are $\lfloor \sqrt[6]{1000} \rfloor = 3$ numbers that are both squares and cubes, i.e., sixth powers. Therefore the answer is $31 + 10 - 3 = 38$.

14. There are $26!$ strings in all. To count the strings that contain *fish*, we glue these four letters together as one and permute it and the 22 other letters, so there are $23!$ such strings. Similarly there are $24!$ strings that contain *rat* and $23!$ strings that contain *bird*. Furthermore, there are $21!$ strings that contain both *fish* and *rat* (glue each of these sets of letters together), but there are no strings that contain both *bird* and another of these strings. Therefore the answer is $26! - 23! - 24! - 23! + 21! \approx 4.0 \times 10^{26}$.

16. $4 \cdot 100 - 6 \cdot 50 + 4 \cdot 25 - 5 = 195$

18. There are $C(10,1) + C(10,2) + \cdots + C(10,10) = 2^{10} - C(10,0) = 1023$ terms on the right-hand side of the equation.

20. $5 \cdot 10000 - 10 \cdot 1000 + 10 \cdot 100 - 5 \cdot 10 + 1 = 40{,}951$

22. The base case is $n = 2$, for which we already know the formula to be valid. Assume that the formula is true for n sets. Look at a situation with $n+1$ sets, and temporarily consider $A_n \cup A_{n+1}$ as one set. Then by the inductive hypothesis we have

$$\begin{aligned}|A_1 \cup \cdots \cup A_{n+1}| = \sum_{i<n} |A_i| + |A_n \cup A_{n+1}| - \sum_{i<j<n} |A_i \cap A_j| \\ - \sum_{i<n} |A_i \cap (A_n \cup A_{n+1})| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_{n-1} \cap (A_n \cup A_{n+1})| \,.\end{aligned}$$

Next we apply the distributive law to each term on the right involving $A_n \cup A_{n+1}$, giving us

$$\sum |(A_{i_1} \cap \cdots \cap A_{i_m}) \cap (A_n \cup A_{n+1})| = \sum |(A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n) \cup (A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1})| \,.$$

Now we apply the basis step to rewrite each of these terms as

$$\sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n| + \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_{n+1}| - \sum |A_{i_1} \cap \cdots \cap A_{i_m} \cap A_n \cap A_{n+1}| \,,$$

which gives us precisely the summation we want.

24. Let E_1, E_2, and E_3 be these three events, in the order given. Then $p(E_1) = C(5,3)/2^5 = 10/32$; $p(E_2) = 2^3/2^5 = 8/32$; and $p(E_3) = 2^3/2^5 = 8/32$. Furthermore $p(E_1 \cap E_2) = C(3,1)/2^5 = 3/32$; $p(E_1 \cap E_3) = 1/32$; and $p(E_2 \cap E_3) = 2/32$. Finally $p(E_1 \cap E_2 \cap E_3) = 1/32$. Therefore the probability that at least one of these events occurs is $(10 + 8 + 8 - 3 - 1 - 2 + 1)/32 = 21/32$.

26. We only need to list the terms that have one or two events in them. Thus we have

$$p(E_1 \cup E_2 \cup E_3 \cup E_4) = \sum_{1 \le i \le 4} p(E_i) - \sum_{1 \le i < j \le 4} p(E_i \cap E_j) \,,$$

or, explicitly, $p(E_1 \cup E_2 \cup E_3 \cup E_4) = p(E_1) + p(E_2) + p(E_3) + p(E_4) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_3 \cap E_4)$.

28. The probability of the union, in this case, is the sum of the probabilities of the events:

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{i=1}^{n} p(E_i) = p(E_1) + p(E_2) + \cdots + p(E_n)$$

SECTION 8.6 Applications of Inclusion–Exclusion

2. $1000 - 450 - 622 - 30 + 111 + 14 + 18 - 9 = 32$

4. $C(4+17-1,17) - C(4+13-1,13) - C(4+12-1,12) - C(4+11-1,11) - C(4+8-1,8) + C(4+8-1,8) + C(4+7-1,7) + C(4+4-1,4) + C(4+6-1,6) + C(4+3-1,3) + C(4+2-1,2) - C(4+2-1,2) = 20$

6. Square-free numbers are those not divisible by the square of a prime. We count them as follows: $99 - \lfloor 99/2^2 \rfloor - \lfloor 99/3^2 \rfloor - \lfloor 99/5^2 \rfloor - \lfloor 99/7^2 \rfloor + \lfloor 99/(2^2 3^2) \rfloor = 61$.

8. $5^7 - C(5,1)4^7 + C(5,2)3^7 - C(5,3)2^7 + C(5,4)1^7 = 16{,}800$

10. This problem is asking for the number of onto functions from a set with 8 elements (the balls) to a set with 3 elements (the urns). Therefore the answer is $3^8 - C(3,1)2^8 + C(3,2)1^8 = 5796$.

12. 2143, 2341, 2413, 3142, 3412, 3421, 4123, 4312, 4321

14. We use Theorem 2 with $n = 10$, which gives us

$$\frac{D_{10}}{10!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{10!} = \frac{1334961}{3628800} = \frac{16481}{44800} \approx 0.3678794643\,,$$

which is almost exactly $e^{-1} \approx 0.3678794412\ldots$.

16. There are $n!$ ways to make the first assignment. We can think of this first seating as assigning student n to a chair we will label n. Then the next seating must be a derangement with respect to this numbering, so there are D_n second seatings possible. Therefore the answer is $n!D_n$.

18. In a derangement of the numbers from 1 to n, the number 1 cannot go first, so let $k \neq 1$ be the number that goes first. There are $n-1$ choices for k. Now there are two ways to get a derangement with k first. One way is to have 1 in the k^{th} position. If we do this, then there are exactly D_{n-2} ways to derange the rest of the numbers. On the other hand, if 1 does not go into the k^{th} position, then think of the number 1 as being temporarily relabeled k. A derangement is completed in this case by finding a derangement of the numbers 2 through n in positions 2 through n, so there are D_{n-1} of them. Combining all this, by the product rule and the sum rule, we obtain the desired recurrence relation. The initial conditions are $D_0 = 1$ and $D_1 = 0$.

20. We apply iteration to the formula $D_n = nD_{n-1} + (-1)^n$, obtaining

$$\begin{aligned}
D_n &= n\big((n-1)D_{n-2} + (-1)^{n-1}\big) + (-1)^n \\
&= n(n-1)D_{n-2} + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)\big((n-2)D_{n-3} + (-1)^{n-2}\big) + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)(n-2)D_{n-3} + n(n-1)(-1)^{n-2} + n(-1)^{n-1} + (-1)^n \\
&\ \ \vdots \\
&= n(n-1)\cdots 2D_1 + n(n-1)\cdots 3 - n(n-1)\cdots 4 + \cdots + n(-1)^{n-1} + (-1)^n \\
&= n(n-1)\cdots 3 - n(n-1)\cdots 4 + \cdots + n(-1)^{n-1} + (-1)^n\,,
\end{aligned}$$

which yields the formula in Theorem 2 after factoring out $n!$.

22. The numbers not relatively prime to pq are the ones that have p and/or q as a factor. Thus we have

$$\phi(pq) = pq - \frac{pq}{p} - \frac{pq}{q} + \frac{pq}{pq} = pq - q - p + 1 = (p-1)(q-1)\,.$$

24. The left-hand side of course counts the number of permutations of the set of integers from 1 to n. The right-hand side counts it, too, by a two-step process: first decide how many and which elements are to be fixed (this can be done in $C(n,k)$ ways, for each of $k = 0, 1, \ldots, n$), and in each case derange the remaining elements (which can be done in D_{n-k} ways).

26. This permutation starts with $4, 5, 6$ in some order ($3! = 6$ ways to choose this), followed by $1, 2, 3$ in some order ($3! = 6$ ways to decide this). Therefore the answer is $6 \cdot 6 = 36$.

SUPPLEMENTARY EXERCISES FOR CHAPTER 8

2. **a)** Let a_n be the amount that remains after n hours. Then $a_n = 0.99a_{n-1}$.
b) By iteration we find the solution $a_n = (0.99)^n a_0$, where a_0 is the original amount of the isotope.

4. **a)** Let B_n be the number of bacteria after n hours. The initial conditions are $B_0 = 100$ and $B_1 = 300$. Thereafter, $B_n = B_{n-1} + 2B_{n-1} - B_{n-2} = 3B_{n-1} - B_{n-2}$.
b) The characteristic equation is $r^2 - 3r + 1 = 0$, which has roots $(3 \pm \sqrt{5})/2$. Therefore the general solution is $B_n = \alpha_1((3+\sqrt{5})/2)^n + \alpha_2((3-\sqrt{5})/2)^n$. Plugging in the initial conditions we determine that $\alpha_1 = 50 + 30\sqrt{5}$ and $\alpha_2 = 50 - 30\sqrt{5}$. Therefore the solution is $B_n = (50 + 30\sqrt{5})((3+\sqrt{5})/2)^n + (50 - 30\sqrt{5})((3-\sqrt{5})/2)^n$.
c) Plugging in small values of n, we find that $B_9 = 676{,}500$ and $B_{10} = 1{,}771{,}100$. Therefore the colony will contain more than one million bacteria after 10 hours.

6. We can put any of the stamps on first, leaving a problem with a smaller number of cents to solve. Thus the recurrence relation is $a_n = a_{n-4} + a_{n-6} + a_{n-10}$. We need 10 initial conditions, and it is easy to see that $a_0 = 1$, $a_1 = a_2 = a_3 = a_5 = a_7 = a_9 = 0$, and $a_4 = a_6 = a_8 = 1$.

8. If we add the equations, we obtain $a_n + b_n = 2a_{n-1}$, which means that $b_n = 2a_{n-1} - a_n$. If we now substitute this back into the first equation, we have $a_n = a_{n-1} + (2a_{n-2} - a_{n-1}) = 2a_{n-2}$. The initial conditions are $a_0 = 1$ (given) and $a_1 = 3$ (follows from the first recurrence relation and the given initial conditions). We can solve this using the characteristic equation $r^2 - 2 = 0$, but a simpler approach, that avoids irrational numbers, is as follows. It is clear that $a_{2n} = 2^n a_0 = 2^n$, and $a_{2n+1} = 2^n a_1 = 3 \cdot 2^n$. This is a nice explicit formula, which is all that "solution" really means. We also need a formula for b_n, of course. From $b_n = 2a_{n-1} - a_n$ (obtained above), we have $b_{2n} = 3 \cdot 2^n - 2^n = 2^{n+1}$, and $b_{2n+1} = 2 \cdot 2^n - 3 \cdot 2^n = -2^n$.

10. Following the hint, we let $b_n = \log a_n$. Then the recurrence relation becomes $b_n = 3b_{n-1} + 2b_{n-2}$, with initial conditions $b_0 = b_1 = 1$. This is solved in the usual manner. The characteristic equation is $r^2 - 3r - 2 = 0$, which gives roots $(3 \pm \sqrt{17})/2$. Plugging the initial conditions into the general solution and doing some messy algebra gives

$$b_n = \frac{17 - \sqrt{17}}{34}\left(\frac{3+\sqrt{17}}{2}\right)^n + \frac{17 + \sqrt{17}}{34}\left(\frac{3-\sqrt{17}}{2}\right)^n.$$

The solution to the original problem is then $a_n = 2^{b_n}$.

12. The characteristic equation is $r^3 - 3r^2 + 3r - 1 = 0$. This factors as $(r-1)^3 = 0$, so there is only one root, 1, and its multiplicity is 3. Therefore the general solution is $a_n = \alpha_1 + \alpha_2 n + \alpha_3 n^2$. Plugging in the initial conditions gives us $2 = \alpha_1$, $2 = \alpha_1 + \alpha_2 + \alpha_3$, and $4 = \alpha_1 + 2\alpha_2 + 4\alpha_3$. Solving yields $\alpha_1 = 2$, $\alpha_2 = -1$, and $\alpha_3 = 1$. Therefore the solution is $a_n = 2 - n + n^2$.

14. The success of this algorithm relies heavily on the fact that the weights are integers. The time complexity is nW. If the weights are real numbers (or, what effectively amounts to the same thing, W is prohibitively large), then no efficient algorithm is known for solving the knapsack problem. Indeed, the problem is NP-complete.

a) In this case the weight of item j by itself exceeds w, so no subset of the first j items whose total weight does not exceed w can contain item j. Therefore the maximum total weight not exceeding w among the first j items is achieved by a subset of the first $j-1$ items, and $M(j-1,w)$ is that maximum.

b) The maximum total weight not exceeding w among the first j items either is achieved by using item j or is achieved without using item j. In the latter case, that maximum is the same as the maximum total weight not exceeding w among the first $j-1$ items, namely $M(j-1,w)$. In the latter case, the maximum weight that a subset of the first $j-1$ items can contribute is $M(j-1,w-w_j)$, so $M(j,w)=w_j+M(j-1,w-w_j)$ in this case.

c) Without loss of generality, we can assume that each $w_j \le W$; overweight items cannot contribute to the desired subset, so they can be discarded before we start. We need to compute $M(j,w)$ for all $1 \le j \le n$ and all $0 \le w \le W$. To initialize, we set $M(1,w)=w_1$ for $w_1 \le w \le W$, set $M(1,w)=0$ for $0 \le w < w_1$, and set $M(j,0)=0$ for $1 \le j \le n$. We then loop through $j=2,3,\ldots,n$, and for each j loop through $w=1,2,\ldots,W$, computing the values of $M(j,w)$ according to the rules given in parts **(a)** and **(b)**.

d) The maximum total weight is given by $M(n,W)$. By the way the algorithm works, that value is either $M(n-1,W)$ or it is $w_n+M(n-1,W-w_n)$. By computing those two quantities, we can determine which it is; in the former case we know that item n is not in the optimal subset, and we can proceed with this same calculation by looking at $M(n-1,W)$, whereas in the latter case we know that item n is in the optimal subset and we can proceed with this same calculation by looking at $M(n-1,W-w_n)$.

16. The initial conditions $L(i,0)=L(0,j)=0$ are trivial. That $L(i,j)=L(i-1,j-1)+1$ when the last symbols match follows immediately from Exercise 15a. That $L(i,j)=\max(L(i,j-1),L(i-1,j))$ when the last symbols do not match follows immediately from Exercise 15b.

18. The length of the longest common subsequence is given by $L(m,n)$. If $a_m=b_n$ then we know that the longest common subsequence ends with that symbol, and the first $L(m,n)-1$ symbols can then be found by proceeding with this same calculation by looking at $L(m-1,n-1)$. Otherwise we compare $L(m,n-1)$ and $L(m-1,n)$ and proceed with this same calculation at the location in the table at which the larger value is located (that value will be the same as $L(m,n)$).

20. We use the result of Exercise 31 in Section 8.3, with $a=3$, $b=5$, $c=2$, and $d=4$. Thus the solution is $f(n)=625n^4/311-314n^{\log_5 3}/311$.

22. The algorithm compares the largest elements of the two halves (this is one comparison), and then it compares the smaller largest element with the second largest element of the other half (one more comparison). This is sufficient to determine the largest and second largest elements of the list. (If the list has only one element in it, then the second largest element is declared to be $-\infty$.) Let $f(n)$ be the number of comparisons used by this algorithm on a list of size n. The list is split into two lists, of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$, respectively. Thus our recurrence relation is $f(n)=f(\lfloor n/2 \rfloor)+f(\lceil n/2 \rceil)+2$, with initial condition $f(1)=0$. (This algorithm could be made slightly more efficient by having the base cases be $n=2$ and $n=3$, rather than $n=1$.)

24. **a)** That a_m is greater than a_{m-1} and greater than a_{m+1} follows immediately from the definition given. Note that it might happen that $a_m=a_1$ or $a_m=a_n$, in which case half of the condition is satisfied vacuously. Furthermore, because the terms strictly increase up to a_m and strictly decrease afterwards, there cannot be two terms satisfying this condition.

b) If m were less than or equal to i, then the condition $a_i < a_{i+1}$ would violate the fact that the terms in the sequence must decrease once a_m is encountered.

c) If m were greater than i, then the condition $a_i > a_{i+1}$ would violate the fact that the terms in the sequence must increase until a_m is encountered.

d) The algorithm is similar to binary search. Suppose we have narrowed the search down to $a_i, a_{i+1}, \ldots, a_j$, where initially $i = 1$ and $j = n$. If $j - i = 1$, then $a_m = a_i$; and if $j - i = 2$, then a_m is the larger of a_i and a_j. Otherwise, we look at the middle term in that sequence, a_k, where $k = \lfloor (i+j)/2 \rfloor$. By part **(b)**, if $a_{k-1} < a_k$, then we know that a_m must be in $a_k, a_{k+1}, \ldots, a_j$, so we can replace i by k and iterate. By part **(c)**, if $a_k > a_{k+1}$, then we know that a_m must be in $a_i, a_{i+1}, \ldots, a_k$, so we can replace j by k and iterate. (And if we wish, we could declare that $a_m = a_k$ if both of these conditions are met.) The algorithm could also be written recursively.

26. **a)** $\Delta a_n = 3 - 3 = 0$ **b)** $\Delta a_n = 4(n+1) + 7 - (4n+7) = 4$
c) $\Delta a_n = \big((n+1)^2 + (n+1) + 1\big) - (n^2 + n + 1) = 2n + 2$

28. We prove something a bit stronger. If $a_n = P(n)$ is a polynomial of degree at most d, then Δa_n is a polynomial of degree at most $d - 1$. To see this, let $P(n) = c_d n^d + (\text{lower order terms})$. Then

$$\begin{aligned}\Delta P(n) &= c_d(n+1)^d + (\text{lower order terms}) - c_d n^d + (\text{lower order terms})\\ &= c_d n^d + (\text{lower order terms}) - c_d n^d + (\text{lower order terms})\\ &= (\text{lower order terms}).\end{aligned}$$

If we apply this result $d+1$ times, then we get that $\Delta^{d+1} a_n$ has degree at most -1, i.e., is identically 0.

30. Since it is valid to use the commutative, associative, and distributive laws for absolutely convergent infinite series, we simply write

$$(cF + dG)(x) = cF(x) + dG(x) = c\sum_{k=0}^{\infty} a_k x^k + d\sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty}(ca_k + db_k)x^k\,.$$

32. $14 + 18 - 22 = 10$

34. If the queries are correct, then by inclusion–exclusion the number of students who are freshmen and have not taken courses in either subject must equal $2175 - 1675 - 1074 - 444 + 607 + 350 + 201 - 143 = -3$. Since a negative number here is not possible, we conclude that the responses cannot all be accurate.

36. There will be $C(7,i)$ terms involving combinations of i of the sets at a time. Therefore the answer is $C(7,1) + C(7,2) + C(7,3) + C(7,4) + C(7,5) = 119$.

38. For a more compact notation, let us write 1,000,000 as M.
a) $\lfloor M/2 \rfloor + \lfloor M/3 \rfloor + \lfloor M/5 \rfloor - \lfloor M/(2\cdot 3) \rfloor - \lfloor M/(2\cdot 5) \rfloor - \lfloor M/(3\cdot 5) \rfloor + \lfloor M/(2\cdot 3\cdot 5) \rfloor = 733{,}334$
b) $M - \lfloor M/7 \rfloor - \lfloor M/11 \rfloor - \lfloor M/13 \rfloor + \lfloor M/(7\cdot 11) \rfloor + \lfloor M/(7\cdot 13) \rfloor + \lfloor M/(11\cdot 13) \rfloor - \lfloor M/(7\cdot 11\cdot 13) \rfloor = 719{,}281$
c) This is asking for numbers divisible by 3 but not by 21. Since the set of numbers divisible by 21 is a subset of the set of numbers divisible by 3, this is simply $\lfloor M/3 \rfloor - \lfloor M/21 \rfloor = 285{,}714$.

40. After the assignments of the hardest and easiest job have been made, there are 4 different jobs to assign to 3 different employees. No restrictions are stated, so we assume that there are none. Therefore we are just looking for the number of functions from a set with 4 elements to a set with 3 elements, and there are $3^4 = 81$ such functions. (If we impose the restriction that every employee must get at least one job, then it is a little harder. In particular, we must rule out all the assignments in which the jobs go only to the two employees that already have jobs. There are $2^4 = 16$ such assignments, so the answer would be $81 - 16 = 65$ in this case.)

42. We will count the number of bit strings that do contain four consecutive 1's. Bits 1 through 4 could be 1's, or bits 2 through 5, or bits 3 through 6, and in each case there are 4 strings meeting those conditions (since the other two bits are free). This gives a total of 12. However we overcounted, since there are ways in which more than one of these can happen. There are 2 strings in which bits 1 through 4 and bits 2 through 5 are 1's, 2 strings in which bits 2 through 5 and bits 3 through 6 are 1's, and 1 string in which bits 1 through 4 and bits 3 through 6 are 1's. Finally, there is 1 string in which all three substrings are 1's. Thus the number of bit strings with 4 consecutive 1's is $12 - 2 - 2 - 1 + 1 = 8$. Therefore the answer to the exercise is $2^6 - 8 = 56$.