Mälardalen University

PROJECT IN MATHEMATICS

The Title of the Thesis

by

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MMA291 — Project in Mathematics

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MMA291 — Project in Mathematics

Date of presentation:
XX June 2026

Project name:
The Title of the Thesis

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Version: 1
19th January 2026

Supervisor:
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Examiner:
To be determined

Comprising:
7.5 ECTS credits

Abstract

Under construction.

Acknowledgements

Under construction.

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2.1 Groups

Definition 1 (Group)**.** In mathematics, a *group* G denotes a set of elements $\{a, b, c, \cdots\}$ and an accompanying binary operator $*$, together satisfying the following conditions:

1. *Closure*: For every ordered pair of elements in the group, the binary product thereof (as defined by the binary operator) is also an element of that group; $\forall a, b \in G \ (a * b = c \in G)$.

2. *Associativity*: The order wherein ordered pairs of elements are evaluated has no consequence on the restulting product; $\forall a, b, c \in G \ ((a * b) * c = a * (b * c))$.

3. *Existence of Identity*: There exists in the group an *identity* element, the omission of which from any expression containing any other element or elements has no consequence on the evaluation of that expression; $\exists e \in G \, \forall a \in G \ (e * a = a * e = a)$.

4. *Existence of Inverse*: Each element in the group has its own *inverse*, who's product therewith evalutes to the identity; $\forall a \in G \, \exists b \in G \ (a * b = b * a = e)$, where e is the identity.

Definition 2 (Commutativity)**.** Let G be a group. Then it is *commutative* (or *Abelian*) if $\forall a, b \in \mathbb{G} \ (a * b = b * a)$.

The integers $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ together with *addition* constitutes an Abelian group with 0 acting as its identity (since $\forall a \in \mathbb{Z} \ (0 + a = a + 0 = a)$). The negative numbers exist as the inverses of the positives, and vice versa. We know addition is associative, and since $\mathbb{Z}$ is infinite in each direction, we will never find a pair of elements therein whose sum is not also an element therein. This group can also be expressed as additively *generated* by the element 1, since 1 and its inverse -1 together with + can express the whole group. This generative definition is denoted $G = (\langle\{1\}\rangle, +) = (\mathbb{Z}, +)$. The group generated by 2 – to give another example – is the group of all *even* integers.

Definition 3 (Cyclicity)**.** A group which can be generated by a single element is called a *cyclic* group.

When the operator is given implicitly, a cyclic group with the generator g can be written as $\langle g \rangle$.

Definition 4 (Subgroup)**.** Let G be a group under a binary operator $*$. Then $H \subset G$ is a *subgroup* of G if H is also a group under $*$.

2.2 Permutations

Definition 5 (Permutation)**.** A *permutation* π of the set A is a bijective map $\pi : A \to A$.

We write π^n for a permutation π to denote $\pi \circ \cdots \circ \pi$ where π is composed with itself $n \in \mathbb{N}$ times. π^{-n} denotes the composition $\pi^{-1} \circ \cdots \circ \pi^{-1}$ where the inverse π^{-1} is composed with itself $n \in \mathbb{N}$ times. The permutation π^0 always equals the identity.

Definition 6 (Permutation Group)**.** A *permutation group* on a set A is a group of permutations of A under composition.

Definition 7 (Support)**.** The *support* $\underline{\pi}$ of a permutation π of A is the set $\{a \in A \mid \pi(a) \neq a\}$.

A permutation whoose support is the empty set equals the identity. We say that a permutation π of A *fixes* an element $a \in A$ if $a \notin \underline{\pi}$.

Definition 8 (Orbit)**.** The *orbit* $\overset{a}{\circlearrowleft\pi}$ of an element $a \in A$ under a permutation π of A is the set $\{b \in A \mid \phi(a) = b,\ \phi \in \langle \pi \rangle\}$. The orbit $\overset{a}{\circlearrowleft G}$ – where G is a permutation group on A – is the union $\bigcup_{\pi \in G} \overset{a}{\circlearrowleft\pi}$ of the orbits of a under each permutation in the group – or put differently: $\overset{a}{\circlearrowleft G} = \{\pi(a) \mid \pi \in G\}$.

Herefrom follows that $b \in \overset{a}{\circlearrowleft\pi} \Leftrightarrow a \in \overset{b}{\circlearrowleft\pi}$ and $b \in \overset{a}{\circlearrowleft G} \Leftrightarrow a \in \overset{b}{\circlearrowleft G}$.

Definition 9 (Cycle)**.** A *cycle* is a permutation π such that $\forall a, b \in \underline{\pi}\ \exists\, \phi \in \langle \pi \rangle\ (\phi(a) = b)$.

Since the support of the identity is the empty set, the identity also a cycle.

Definition 10 (Transposition)**.** a *transposition* is a cycle π where $|\underline{\pi}| = 2$.

Lemma 1. *Let π be a permutation of a set A. Then there exists a family $(\pi_i)_{i \in I}$ of cycles on A with pairwise disjoint supports such that $\forall a \in A\ \exists i \in I\ \big(a \in \underline{\pi_i} \wedge \pi_i(a) = \pi(a) \wedge \pi_j(a) = a \wedge i \neq j\big)$.*

Proof. Take any element $a \in A$. Define the first cycle π_1 with $\forall n \in \mathbb{Z}\ \Big(\pi_1^n(a) = \pi(a) \wedge b \notin \overset{a}{\circlearrowleft\pi_1} \Rightarrow \pi_1(b)$
Then $\underline{\pi}_1 = \overset{a}{\circlearrowleft\pi}$. Let $a_{k+1} \in A \setminus \bigcup_{i \in \{1, \cdots, k\}} \underline{\pi}_i$ where $k \in \mathbb{N}$ and each π_i is defined in the same way as π_1 for each a_i. Then $\underline{\pi}_{k+1}$

□

In *cycle notation*, the fact above is used to express permutations in a convenient way. Here, each cycle in the expressed permutation is written as a parenthesized list of elements. For example, the permutation $(a, b, c)(d, e)$ maps $a \mapsto b$, $b \mapsto c$ and $c \mapsto a$, as well as $d \mapsto e$ and $e \mapsto d$. Note that this permutation is the same as the permutation $(c, a, b)(e, d)$, but not the same as $(a, c, b)(d, e)$.

In this text, permutations containing symbols or numbers with multiple digits will be written with dividing commas, whilst permutations containing exclusively the numbers 0 through 9 will be written therewithout – for example $(1234) = (1, 2, 3, 4)$.

Further, when talking about permutation groups in this text, the operator $\circ$ will be implicit, and omitted from expressions. As an example, take $\pi \circ \phi = \pi\phi$.

Lemma 2. *Let α and β be permutations of A. Then α and β do not commute if and only if there is an element $c \in \underline{\alpha} \cap \underline{\beta}$ where the cycles of α and β wherein c is contained are neither equal nor each other's inverses.*

Proof. Let α and β be permutations of A and $\underline{\alpha} \cap \underline{\beta} = \emptyset$. Then $\forall b \in A \;\; (\alpha(b) = b \vee \beta(b) = b)$. Thus for each $b \in A$ both $(\alpha \circ \beta)(b)$ and $(\beta \circ \alpha)(b)$ evaluate to the same expression, which will be either $\alpha(b)$, $\beta(b)$ or b. If $\underline{\alpha}$ and $\underline{\beta}$ share an element c and $\alpha_c = \beta_c$ where α_c and β_c are the cycles of α and β wherein c occurs, then $\alpha_c \circ \beta_c = \beta_c \circ \alpha_c$ whilst the other cycles of α and β commute (as noted above). If $\alpha_c = \beta_c^{-1}$, then the cycles neutralize each other $\alpha_c \circ \beta_c = \beta_c \circ \alpha_c = e$. If, however, there is an element $c \in \alpha \cap \beta$ such that $\alpha(c) \neq \beta(c)$, then $\alpha_c \neq \beta_c$ and $(\alpha \circ \beta)(c) = \beta(c) \neq (\beta \circ \alpha)(c) = \alpha(c)$, NOT TRUE!!!!! since $\beta(c) \notin \underline{\alpha}$ and $\alpha(c) \notin \underline{\beta}$. $\square$

Chapter 3

The Conjugates and Commutators of Permutations

Herein, all lowercase geek letters are implied to be elements in a permutation group G on A. Elements of A are represented by lowercase latin characters.

Definition 11 (Conjugate)**.** The *conjugate* $\searrow^{\alpha}_{\beta}$ of β by α is the composition $\alpha^{-1}\beta\alpha$.

Definition 12 (Commutator)**.** The *commutator* $\nwarrow\!\!\!\!\searrow^{\alpha}_{\beta}$ of α and β is the composition $\beta^{-1}\alpha^{-1}\beta\alpha$.

These two constructions – though they can seem arbitrary at first – have interesting and useful properties when working within groups – specifically when the supports of their two permutations share elements.

In the trivial case where they share no element, the permutations commute, so the conjugate of α and β is β, and the commutator thereof is the identity; $\alpha \cap \beta = \emptyset$ implies that $\alpha^{-1}\beta\alpha = \beta$ and $\beta^{-1}\alpha^{-1}\beta\alpha = e$.

Lemma 3. *Let $\underline{\alpha} \cap \underline{\beta} = \{c\}$. Then the cycle of $\searrow^{\alpha}_{\beta}$ wherein c occurs has the form $\left(\alpha^{-1}(c), \beta(c), \beta^2(c), \cdots, \beta^{-1}(c)\right)$.*

Proof. For each $b \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c)\right\}$, $\alpha(b) \notin \underline{\beta}$. Since $\beta\alpha(b) = \alpha(b)$, the conjugate can be rewritten for b: $\left(\alpha^{-1}\beta\alpha\right)(b) = \left(\alpha^{-1}\alpha\right)(b) = b$. In the case of $\alpha^{-1}(c)$, $\left(\alpha^{-1}\beta\alpha\alpha^{-1}\right)(c) = \left(\alpha^{-1}\beta\right)(c) = \beta(c)$, since $\beta(c) \notin \underline{\alpha} \Rightarrow \left(\alpha^{-1}\beta\right)(c) = \beta(c)$. For each $d \in \underline{\beta} \setminus \left\{\beta^{-1}(c), c\right\}$, $d, \beta(d) \notin \underline{\alpha}$, so $\left(\alpha^{-1}\beta\alpha\right)(d) = \beta(d)$. Lastly, $\beta^{-1}(c)$ maps to $\left(\alpha^{-1}\beta\alpha\beta^{-1}\right)(c) = \left(\alpha^{-1}\beta\beta^{-1}\right)(c) = \alpha^{-1}(c)$. This gives the function

$$\searrow^{\alpha}_{\beta}(x) = \begin{cases} x & : x \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c)\right\} \\ \beta(c) & : x = \alpha^{-1}(c) \\ \beta(x) & : x \in \underline{\beta} \setminus \left\{\beta^{-1}(c), c\right\} \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \tag{3.1}$$

which, in cycle notation, is written $\left(\alpha^{-1}(c), \beta(c), \beta^2(c), \cdots, \beta^{-1}(c)\right)\cdots$. $\square$

Note that if β is a transposition, then $\beta = (c, \beta(c))$ and $\beta(c) = \beta^{-1}(c)$. Thus

$$
\begin{aligned}
\searrow^{\alpha}_{\beta}(x) &= \begin{cases} x & : x \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c)\right\} \\ \beta(c) & : x = \alpha^{-1}(c) \\ \beta(x) & : x \in \emptyset \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \\
&= \begin{cases} x & : x \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c)\right\} \\ \beta(c) & : x = \alpha^{-1}(c) \\ \alpha^{-1}(c) & : x = \beta(c) \end{cases} \\
&= \left(\alpha^{-1}(c), \beta(c)\right).
\end{aligned}
$$

Lemma 4. *Let* $\underline{\alpha} \cap \underline{\beta} = \{c\}$. *Then* $\diagup\!\!\!\!\searrow^{\alpha}_{\beta} = \left(c, \beta^{-1}(c), \alpha^{-1}(c)\right)$.

Proof. Using (3.1) from the previous lemma,

$$
\begin{aligned}
\diagup\!\!\!\!\searrow^{\alpha}_{\beta}(x) = \beta^{-1} \searrow^{\alpha}_{\beta}(x) &= \begin{cases} \beta^{-1}(x) & : x \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c)\right\} \\ \beta^{-1}\beta(c) & : x = \alpha^{-1}(c) \\ \beta^{-1}\beta(x) & : x \in \underline{\beta} \setminus \left\{\beta^{-1}(c), c\right\} \\ \beta^{-1}\alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \\
&= \begin{cases} x & : x \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c), c\right\} \\ \beta^{-1}(x) & : x = c \\ c & : x = \alpha^{-1}(c) \\ x & : x \in \underline{\beta} \setminus \left\{\beta^{-1}(c), c\right\} \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \\
&= \begin{cases} x & : x \in (\underline{\alpha} \cup \underline{\beta}) \setminus \left\{\alpha^{-1}(c), \beta^{-1}(c), c\right\} \\ c & : x = \alpha^{-1}(c) \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \\ \beta^{-1}(c) & : x = c \end{cases} \\
&= \left(c, \beta^{-1}(c), \alpha^{-1}(c)\right).
\end{aligned}
$$

□

Lemma 5. *Let* α *and* β *where* $\underline{\alpha} \cap \underline{\beta} = \{c_1, \cdots, c_n\}$, $n \geq 2$ *and* $\alpha^k(c_1) = \beta^k(c_1)$ *for all* $k \in \{0, \cdots, n-1\}$. *Then* $\searrow^{\alpha}_{\beta} = ()$.

Proof. Let $b \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c_1), c_1, \cdots, c_{n-1}\right\}$, then the conjugation fixes b. Note that if α and β only share two elements, then $c_1 = c_{n-1}$. c_{n-1} maps to $\beta(c_n)$ and $\forall i \in \{1, \cdots, n-2\}$ $(c_i \mapsto c_{i+1})$ given that $n \geq 3$. Each $g \in \underline{\beta} \setminus \left\{\beta^{-1}(c_1), c_1, \cdots, c_n\right\}$ map to $\beta(g)$. Lastly, $\alpha^{-1}(c_1) \mapsto c_1$ and

$\beta^{-1}(c_1) \mapsto \alpha^{-1}(c_1)$. This yields the function

$$\searrow^{\alpha}_{\beta}(x) = \begin{cases} x & : \ x \in \underline{\alpha} \setminus \left\{\alpha^{-1}(c_1), c_1, \cdots, c_{n-1}\right\} \\ c_1 & : \ x = \alpha^{-1}(c_1) \\ c_{i+1} & : \ i \in \{1, \cdots, n-2\} \wedge n \geq 3 \\ \beta(c_n) & : \ x = c_{n-1} \\ \beta(x) & : \ x \in \underline{\beta} \setminus \left\{\beta^{-1}(c_1), c_1, \cdots, c_n\right\} \\ \alpha^{-1}(c_1) & : \ x = \beta^{-1}(c_1) \end{cases} \tag{3.2}$$

$$= \left(\alpha^{-1}(c_1), c_1, \cdots, c_{n-1}, \beta(c_n), \cdots, \beta^{-1}(c_1)\right). \tag{3.3}$$

$\square$

Lemma 6. *Let α and β where $\underline{\alpha} \cap \underline{\beta} = \{c_1, \cdots, c_n\}, n \geq 2$ and $\alpha^{k-1}(c_1) = \beta^{k-1}(c_1)$ for all $k \in \{1, \cdots, n\}$. Then ${\nearrow\!\!\!\!\searrow}^{\alpha}_{\beta} = \left(\alpha^{-1}(c), \beta^{-1}(c)\right)(c_{n-1}, c_n)$.*

Proof.

$$\delta(x) = \beta^{-1}\gamma(x) = \begin{cases} x & : \ x \in (\underline{\alpha} \cup \underline{\beta}) \setminus \left\{\alpha^{-1}(c), \beta^{-1}(c), c_{n-1}, c_n\right\} \\ \beta^{-1}(c) & : \ x = \alpha^{-1}(c) \\ \alpha^{-1}(c) & : \ x = \beta^{-1}(c) \\ c_n & : \ x = c_{n-1} \\ c_{n-1} & : \ x = c_n \end{cases} \tag{3.4}$$

$\square$

Chapter 4

The 2×3 Game

The "2×3 game" is a simple puzzle with six tiles in a grid of numbers and a set of allowed moves which permute them. The goal of the game is to sort the grid into a "solved" state from some scrambled state.

$$\boxed{\begin{array}{ccc}\mathbf{5} & \mathbf{4} & \mathbf{2}\\ \mathbf{3} & \mathbf{6} & \mathbf{1}\end{array}} \xrightarrow{\text{Move}_1} \cdots \xrightarrow{\text{Move}_n} \boxed{\begin{array}{ccc}\mathbf{1} & \mathbf{2} & \mathbf{3}\\ \mathbf{4} & \mathbf{5} & \mathbf{6}\end{array}}$$

Formally, the game can be and will be hereafter modelled as follows. Let $G := (\langle \mathfrak{M} \rangle, \circ)$ be a group compositionally generated by the set of moves $\mathfrak{M} := \{(123), (456), (14)\}$, named $U = (123)$ for *upper row*, $L = (456)$ for *lower row* and $S = (14)$ for *side flip*. A *problem* is a permutation $p \in G$, and a *solution* thereto is a vector $\mathbf{s} \in \mathfrak{M}^*$ such that

$$\bigcirc_{i=n}^{1} \mathbf{s}_i = \mathbf{s}_n \circ \mathbf{s}_{n-1} \circ \cdots \circ \mathbf{s}_1 = p$$

where n denotes the length of $\mathbf{s}$. Note that $\mathbf{s}$ may include the implicitly given inverses of the group generators. Hereafter, $\bigcirc_{i=n}^{1} \mathbf{s}_i$ will be written as $\acute{\mathbf{s}}$.

The size of the group $|G|$ is 720, which shows that $G = S_6$, since $|S_6| = 720$. This fact can also be proven in the following way. Each transposition beteen the upper and lower row can be found with the nested conjugate

$$\begin{aligned} T_1(x, y) &= \left(L^{-y+3+1}\right)^{-1} \left(\left(U^{-x+1}\right)^{-1} (S) \left(U^{-x+1}\right)\right) \left(L^{-y+3+1}\right) \\ &= L^{y-4} U^{x-1} S U^{-x+1} L^{-y+4} \\ &= L^{y-1} U^{x-1} S U^{-x+1} L^{-y+1}, \end{aligned}$$

where $x \in \{1, 2, 3\}$ and $y \in \{4, 5, 6\}$. As an example, let $p = (26)$. Then

$$\begin{aligned}
T_1(2,6) &= L^{6-1}U^{2-1}SU^{-2+1}L^{-6+1} \\
&= L^5U^1SU^{-1}L^{-5} \\
&= (456)^5(123)^1(14)(123)^{-1}(456)^{-5} \\
&= (465)(123)(14)(132)(456) \\
&= (465)(24)(456) \\
&= (2,6).
\end{aligned}$$

This gives – after some reductions – the solution $\mathbf{s} = \left(L, U^{-1}, S, U, L^{-1}\right)$ visualized below.

$$\boxed{\begin{matrix}\mathbf{1} & \mathbf{6} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{2}\end{matrix}} \xrightarrow{L} \boxed{\begin{matrix}\mathbf{1} & \mathbf{6} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} & \mathbf{5}\end{matrix}} \xrightarrow{U^{-1}} \boxed{\begin{matrix}\mathbf{6} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{4} & \mathbf{5}\end{matrix}}$$

$$\xrightarrow{S} \boxed{\begin{matrix}\mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{6} & \mathbf{4} & \mathbf{5}\end{matrix}} \xrightarrow{U} \boxed{\begin{matrix}\mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{6} & \mathbf{4} & \mathbf{5}\end{matrix}} \xrightarrow{L^{-1}} \boxed{\begin{matrix}\mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6}\end{matrix}}$$

Each transposition beteen elements on the same row can be found with the nested conjugate

$$\begin{aligned}
T_2(x,y,R) &= \Big((R^{-x+1})^{-1}(S)(R^{-x+1})\Big)^{-1}\Big((R^{-y+1})^{-1}(S)(R^{-y+1})\Big)\Big((R^{-x+1})^{-1}(S)(R^{-x+1})\Big) \\
&= (R^{x-1}SR^{-x+1})^{-1}(R^{y-1}SR^{-y+1})(R^{x-1}SR^{-x+1}) \\
&= (R^{-(-x+1)}S^{-1}R^{-(x-1)})(R^{y-1}SR^{-y+1})(R^{x-1}SR^{-x+1}) \\
&= (R^{x-1}SR^{-x+1})(R^{y-1}SR^{-y+1})(R^{x-1}SR^{-x+1}) \\
&= R^{x-1}SR^{-x+1}R^{y-1}SR^{-y+1}R^{x-1}SR^{-x+1} \\
&= R^{x-1}SR^{-x+y}SR^{x-y}SR^{-x+1}
\end{aligned}$$

where $x, y \in R$ and $R \in \mathfrak{M} \setminus \{S\}$. As an example, let $p = (23)$. Since 2 and 3 belong to the upper row, R is set to U:

$$\begin{aligned}
T_2(2,3,U) &= U^{2-1}SU^{-2+3}SU^{2-3}SU^{-2+1} \\
&= U^1SU^1SU^{-1}SU^{-1} \\
&= (123)(14)(123)(14)(123)^{-1}(14)(123)^{-1} \\
&= (123)(14)(123)(14)(132)(14)(132) \\
&= (123)(14)(24)(14)(132) \\
&= (123)(12)(132) \\
&= (23).
\end{aligned}$$

This gives the solution $\mathbf{s} = \left(U^{-1}, S, U^{-1}, S, U, S, U\right)$ visualized below.

$$\boxed{\begin{array}{ccc} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array}} \xrightarrow{U^{-1}} \boxed{\begin{array}{ccc} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array}} \xrightarrow{S} \boxed{\begin{array}{ccc} \mathbf{4} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} \end{array}} \xrightarrow{U^{-1}} \boxed{\begin{array}{ccc} \mathbf{2} & \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{5} & \mathbf{6} \end{array}}$$

$$\xrightarrow{S} \boxed{\begin{array}{ccc} \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{5} & \mathbf{6} \end{array}} \xrightarrow{U} \boxed{\begin{array}{ccc} \mathbf{4} & \mathbf{3} & \mathbf{1} \\ \mathbf{2} & \mathbf{5} & \mathbf{6} \end{array}} \xrightarrow{S} \boxed{\begin{array}{ccc} \mathbf{2} & \mathbf{3} & \mathbf{1} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array}} \xrightarrow{U} \boxed{\begin{array}{ccc} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{array}}$$

Since a transposition of any two elements herein can be composed using the generators in $\mathfrak{M}$ – in other words, since the group G includes every transposition of the elements 1 through 6 – it follows that the group must include every permutation on the six elements, and thereby equal S_6.