



PROJECT IN MATHEMATICS

**The Title of the Thesis**

by

*Adrian Swande*

MMA291 — Project in Mathematics

**DIVISION OF MATHEMATICS AND PHYSICS**  
MÄLARDALEN UNIVERSITY  
SE-721 23 VÄSTERÅS, SWEDEN



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*Author:*  
Adrian Swande

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## **Abstract**

Under construction.

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# **Chapter 1**

## **Introduction**

# Chapter 2

## Preliminaries

### 2.1 Groups

**Definition 1** (Group). In mathematics, a *group*  $G$  denotes a set of elements  $\{a, b, c, \dots\}$  and an accompanying binary operator  $*$ , together satisfying the following conditions:

1. *Closure*: For every ordered pair of elements in the group, the binary product thereof (as defined by the binary operator) is also an element of that group;  $\forall a, b \in G (a * b = c \in G)$ .
2. *Associativity*: The order wherein ordered pairs of elements are evaluated has no consequence on the resulting product;  $\forall a, b, c \in G ((a * b) * c = a * (b * c))$ .
3. *Existence of Identity*: There exists in the group an *identity* element, the omission of which from any expression containing any other element or elements has no consequence on the evaluation of that expression;  $\exists e \in G \forall a \in G (e * a = a * e = a)$ .
4. *Existence of Inverse*: Each element in the group has its own *inverse*, whose product therewith evaluates to the identity;  $\forall a \in G \exists b \in G (a * b = b * a = e)$ , where  $e$  is the identity.

**Definition 2** (Commutativity). Let  $G$  be a group. Then it is *commutative* (or *Abelian*) if  $\forall a, b \in G (a * b = b * a)$ .

The integers  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$  together with *addition* constitutes an Abelian group with 0 acting as its identity (since  $\forall a \in \mathbb{Z} (0 + a = a + 0 = a)$ ). The negative numbers exist as the inverses of the positives, and vice versa. We know addition is associative, and since  $\mathbb{Z}$  is infinite in each direction, we will never find a pair of elements therein whose sum is not also an element therein. This group can also be expressed as additively *generated* by the element 1, since 1 and its inverse  $-1$  together with  $+$  can express the whole group. This generative definition is denoted  $G = (\langle\{1\}\rangle, +) = (\mathbb{Z}, +)$ . The group generated by 2 – to give another example – is the group of all *even* integers.

**Definition 3** (Cyclicity). A group which can be generated by a single element is called a *cyclic* group.



When the operator is given implicitly, a cyclic group with the generator  $g$  can be written as  $\langle g \rangle$ .

**Definition 4** (Subgroup). Let  $G$  be a group under a binary operator  $*$ . Then  $H \subset G$  is a *subgroup* of  $G$  if  $H$  is also a group under  $*$ .

## 2.2 Permutations

**Definition 5** (Permutation). A *permutation*  $\pi$  of the set  $A$  is a bijective map  $\pi : A \rightarrow A$ .

We write  $\pi^n$  for a permutation  $\pi$  to denote  $\pi \circ \dots \circ \pi$  where  $\pi$  is composed with itself  $n \in \mathbb{N}$  times.  $\pi^{-n}$  denotes the composition  $\pi^{-1} \circ \dots \circ \pi^{-1}$  where the inverse  $\pi^{-1}$  is composed with itself  $n \in \mathbb{N}$  times. The permutation  $\pi^0$  always equals the identity.

**Definition 6** (Permutation Group). A *permutation group* on a set  $A$  is a group of permutations of  $A$  under composition.

**Definition 7** (Support). The *support*  $\underline{\pi}$  of a permutation  $\pi$  of  $A$  is the set  $\{a \in A \mid \pi(a) \neq a\}$ .

A permutation whose support is the empty set equals the identity. We say that a permutation  $\pi$  of  $A$  *fixes* an element  $a \in A$  if  $a \notin \underline{\pi}$ .

**Definition 8** (Orbit). The *orbit*  $\mathcal{O}^a_\pi$  of an element  $a \in A$  under a permutation  $\pi$  of  $A$  is the set  $\{b \in A \mid \phi(a) = b, \phi \in \langle \pi \rangle\}$ . The orbit  $\mathcal{O}^a_G$  – where  $G$  is a permutation group on  $A$  – is the union  $\bigcup_{\pi \in G} \mathcal{O}^a_\pi$  of the orbits of  $a$  under each permutation in the group – or put differently:  $\mathcal{O}^a_G = \{\pi(a) \mid \pi \in G\}$ .

Herefrom follows that  $b \in \mathcal{O}^a_\pi \Leftrightarrow a \in \mathcal{O}^b_\pi$  and  $b \in \mathcal{O}^a_G \Leftrightarrow a \in \mathcal{O}^b_G$ .

**Definition 9** (Cycle). A *cycle* is a permutation  $\pi$  such that  $\forall a, b \in \underline{\pi} \exists \phi \in \langle \pi \rangle (\phi(a) = b)$ .

Since the support of the identity is the empty set, the identity also a cycle.

**Definition 10** (Transposition). a *transposition* is a cycle  $\pi$  where  $|\underline{\pi}| = 2$ .

**Lemma 1.** *Let  $\pi$  be a permutation of a set  $A$ . Then there exists a family  $(\pi_i)_{i \in I}$  of cycles on  $A$  with pairwise disjoint supports such that  $\forall a \in A \exists i \in I (a \in \underline{\pi_i} \wedge \pi_i(a) = \pi(a) \wedge \pi_j(a) = a \wedge i \neq j)$ .*

*Proof.* Take any element  $a \in A$ . Define the first cycle  $\pi_1$  with  $\forall n \in \mathbb{Z} (\pi_1^n(a) = \pi(a) \wedge b \notin \mathcal{O}^a_{\pi_1} \Rightarrow \pi_1(b) = b)$ . Then  $\underline{\pi_1} = \mathcal{O}^a_{\pi}$ . Let  $a_{k+1} \in A \setminus \bigcup_{i \in \{1, \dots, k\}} \underline{\pi_i}$  where  $k \in \mathbb{N}$  and each  $\pi_i$  is defined in the same way as  $\pi_1$  for each  $a_i$ . Then  $\underline{\pi_{k+1}}$

□

In *cycle notation*, the fact above is used to express permutations in a convenient way. Here, each cycle in the expressed permutation is written as a parenthesized list of elements. For example, the permutation  $(a, b, c)(d, e)$  maps  $a \mapsto b$ ,  $b \mapsto c$  and  $c \mapsto a$ , as well as  $d \mapsto e$  and  $e \mapsto d$ . Note that this permutation is the same as the permutation  $(c, a, b)(e, d)$ , but not the same as  $(a, c, b)(d, e)$ .

In this text, permutations containing symbols or numbers with multiple digits will be written with dividing commas, whilst permutations containing exclusively the numbers 0 through 9 will be written therewithout – for example  $(1234) = (1, 2, 3, 4)$ .

Further, when talking about permutation groups in this text, the operator  $\circ$  will be implicit, and omitted from expressions. As an example, take  $\pi \circ \phi = \pi\phi$ .

**Lemma 2.** *Let  $G$  be a permutation group on  $A$ . Then  $G$  is non-commutative if there are two permutations  $\alpha, \beta \in G$  such that they share an element  $c \in \underline{\alpha} \cap \underline{\beta}$  where  $\alpha_c \neq \beta_c$ ,  $\underline{\alpha}_c = \overleftarrow{\alpha}_c$ ,  $\underline{\beta}_c = \overleftarrow{\beta}_c$  and  $|\alpha_c| \geq 3$ .*

*Proof.* Take the two permutations  $(123)$  and  $(34)$ . Then  $(123)(34) = (1234)$  and  $(34)(123) = (1243)$ . Since  $(123)(34) \neq (34)(123)$ , the two permutations do not commute under composition, wherefrom follows that the permutation group wherein these two are included is non-commutative (or non-Abelian). THIS IS NOT A PROOF!!!  $\square$

# Chapter 3

## Properties of Permutation Groups

Herein, all lowercase geek letters are implied to be elements in a permutation group  $G$  on  $A$ . Elements of  $A$  are represented by lowercase latin characters.

**Definition 11** (Conjugate). The *conjugate*  $\searrow_{\beta}^{\alpha}$  of  $\beta$  by  $\alpha$  is the composition  $\alpha^{-1}\beta\alpha$ .

**Definition 12** (Commutator). The *commutator*  $\times_{\beta}^{\alpha}$  of  $\alpha$  and  $\beta$  is the composition  $\beta^{-1}\alpha^{-1}\beta\alpha$ .

These two constructions – though they can seem arbitrary at first – have interesting and useful properties when working within groups – specifically when the supports of their two permutations share elements.

In the trivial case where they share no element, the permutations commute, so the conjugate of  $\alpha$  and  $\beta$  is  $\beta$ , and the commutator thereof is the identity;  $\alpha \cap \beta = \emptyset$  implies that  $\alpha^{-1}\beta\alpha = \beta$  and  $\beta^{-1}\alpha^{-1}\beta\alpha = e$ .

**Lemma 3.** Let  $\underline{\alpha} \cap \underline{\beta} = \{c\}$ . Then  $\searrow_{\beta}^{\alpha} = (\alpha^{-1}(c), \beta(c), \beta^2(c), \dots, \beta^{-1}(c))$ . TRUE FOR CYCLES!!!!

*Proof.* For each  $b \in \underline{\alpha} \setminus \{\alpha^{-1}(c)\}$ ,  $\alpha(b) \notin \underline{\beta}$ . Since  $\beta\alpha(b) = \alpha(b)$ , the conjugate can be rewritten for  $b$ :  $\alpha^{-1}\beta\alpha(b) = \alpha^{-1}\alpha(b) = b$ . In the case of  $\alpha^{-1}(c)$ ,  $\alpha^{-1}\beta\alpha\alpha^{-1}(c) = \alpha^{-1}\beta(c) = \beta(c)$ , since  $\beta(c) \notin \underline{\alpha} \Rightarrow \alpha^{-1}\beta(c) = \beta(c)$ . For each  $d \in \underline{\beta} \setminus \{\beta^{-1}(c), c\}$ ,  $d, \beta(d) \notin \underline{\alpha}$ , so  $\alpha^{-1}\beta\alpha(d) = \beta(d)$ . Lastly,  $\beta^{-1}(c)$  maps to  $\alpha^{-1}\beta\alpha\beta^{-1}(c) = \alpha^{-1}\beta\beta^{-1}(c) = \alpha^{-1}(c)$ . This gives the function

$$\searrow_{\beta}^{\alpha}(x) = \begin{cases} x & : x \in \underline{\alpha} \setminus \{\alpha^{-1}(c)\} \\ \beta(c) & : x = \alpha^{-1}(c) \\ \beta(x) & : x \in \underline{\beta} \setminus \{\beta^{-1}(c), c\} \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \quad (3.1)$$

which, in cycle notation, is written  $(\alpha^{-1}(c), \beta(c), \beta^2(c), \dots, \beta^{-1}(c))$ . □

Note that if  $\beta$  is a transposition, then  $\beta = (c, \beta(c))$  and  $\beta(c) = \beta^{-1}(c)$ . Thus

$$\begin{aligned} \searrow_{\beta}^{\alpha}(x) &= \begin{cases} x & : x \in \underline{\alpha} \setminus \{\alpha^{-1}(c)\} \\ \beta(c) & : x = \alpha^{-1}(c) \\ \beta(x) & : x \in \emptyset \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \\ &= \begin{cases} x & : x \in \underline{\alpha} \setminus \{\alpha^{-1}(c)\} \\ \beta(c) & : x = \alpha^{-1}(c) \\ \alpha^{-1}(c) & : x = \beta(c) \end{cases} \end{aligned}$$

and the conjugate is always  $(\alpha^{-1}(c), \beta(c))$ .

**Lemma 4.** Let  $\underline{\alpha} \cap \underline{\beta} = \{c\}$ . Then  $\searrow_{\beta}^{\alpha} = (c, \beta^{-1}(c), \alpha^{-1}(c))$ .

*Proof.* Using (3.1) from the previous lemma,

$$\begin{aligned} \searrow_{\beta}^{\alpha}(x) &= \beta^{-1} \searrow_{\beta}^{\alpha}(x) = \begin{cases} \beta^{-1}(x) & : x \in \underline{\alpha} \setminus \{\alpha^{-1}(c)\} \\ \beta^{-1}\beta(c) & : x = \alpha^{-1}(c) \\ \beta^{-1}\beta(x) & : x \in \underline{\beta} \setminus \{\beta^{-1}(c), c\} \\ \beta^{-1}\alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \\ &= \begin{cases} x & : x \in \underline{\alpha} \setminus \{\alpha^{-1}(c), c\} \\ \beta^{-1}(x) & : x = c \\ c & : x = \alpha^{-1}(c) \\ x & : x \in \underline{\beta} \setminus \{\beta^{-1}(c), c\} \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \end{cases} \\ &= \begin{cases} x & : x \in (\underline{\alpha} \cup \underline{\beta}) \setminus \{\alpha^{-1}(c), \beta^{-1}(c), c\} \\ c & : x = \alpha^{-1}(c) \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \\ \beta^{-1}(c) & : x = c \end{cases} \\ &= (c, \beta^{-1}(c), \alpha^{-1}(c)). \end{aligned}$$

□

**Lemma 5.** Let  $\alpha$  and  $\beta$  where  $\underline{\alpha} \cap \underline{\beta} = \{c_1, \dots, c_n\}$ ,  $n \geq 2$  and  $\alpha^k(c_1) = \beta^k(c_1)$  for all  $k \in \{0, \dots, n-1\}$ . Then  $\searrow_{\beta}^{\alpha} = ()$ .

*Proof.* Let  $b \in \underline{\alpha} \setminus \{\alpha^{-1}(c_1), c_1, \dots, c_{n-1}\}$ , then the conjugation fixes  $b$ . Note that if  $\alpha$  and  $\beta$  only share two elements, then  $c_1 = c_{n-1}$ .  $c_{n-1}$  maps to  $\beta(c_n)$  and  $\forall i \in \{1, \dots, n-2\}$  ( $c_i \mapsto c_{i+1}$ ) given that  $n \geq 3$ . Each  $g \in \underline{\beta} \setminus \{\beta^{-1}(c_1), c_1, \dots, c_n\}$  map to  $\beta(g)$ . Lastly,  $\alpha^{-1}(c_1) \mapsto c_1$  and

$\beta^{-1}(c_1) \mapsto \alpha^{-1}(c_1)$ . This yields the function

$$\searrow_{\beta}^{\alpha}(x) = \begin{cases} x & : x \in \underline{\alpha} \setminus \{\alpha^{-1}(c_1), c_1, \dots, c_{n-1}\} \\ c_1 & : x = \alpha^{-1}(c_1) \\ c_{i+1} & : i \in \{1, \dots, n-2\} \wedge n \geq 3 \\ \beta(c_n) & : x = c_{n-1} \\ \beta(x) & : x \in \underline{\beta} \setminus \{\beta^{-1}(c_1), c_1, \dots, c_n\} \\ \alpha^{-1}(c_1) & : x = \beta^{-1}(c_1) \end{cases} \quad (3.2)$$

$$= \left( \alpha^{-1}(c_1), c_1, \dots, c_{n-1}, \beta(c_n), \dots, \beta^{-1}(c_1) \right). \quad (3.3)$$

□

**Lemma 6.** Let  $\alpha$  and  $\beta$  where  $\underline{\alpha} \cap \underline{\beta} = \{c_1, \dots, c_n\}$ ,  $n \geq 2$  and  $\alpha^{k-1}(c_1) = \beta^{k-1}(c_1)$  for all  $k \in \{1, \dots, n\}$ . Then  $\searrow_{\beta}^{\alpha} = (\alpha^{-1}(c), \beta^{-1}(c)) (c_{n-1}, c_n)$ .

*Proof.*

$$\delta(x) = \beta^{-1}\gamma(x) = \begin{cases} x & : x \in (\underline{\alpha} \cup \underline{\beta}) \setminus \{\alpha^{-1}(c), \beta^{-1}(c), c_{n-1}, c_n\} \\ \beta^{-1}(c) & : x = \alpha^{-1}(c) \\ \alpha^{-1}(c) & : x = \beta^{-1}(c) \\ c_n & : x = c_{n-1} \\ c_{n-1} & : x = c_n \end{cases} \quad (3.4)$$

□

# Chapter 4

## The $2 \times 3$ Game

The “ $2 \times 3$  game” is a simple puzzle with six tiles in a grid of numbers and a set of allowed moves which permute them. The goal of the game is to sort the grid into a “solved” state from some scrambled state.

$$\begin{bmatrix} 5 & 4 & 2 \\ 3 & 6 & 1 \end{bmatrix} \xrightarrow{\text{Move}_1} \dots \xrightarrow{\text{Move}_n} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Formally, the game can be and will be hereafter modelled as follows. Let  $G := (\langle \mathfrak{M} \rangle, \circ)$  be a group compositionally generated by the set of moves  $\mathfrak{M} := \{(123), (456), (14)\}$ , named  $U = (123)$  for *upper row*,  $L = (456)$  for *lower row* and  $S = (14)$  for *side flip*. A *problem* is a permutation  $p \in G$ , and a *solution* thereto is a vector  $\mathbf{s} \in \mathfrak{M}^*$  such that

$$\bigcirc_{i=n}^1 \mathbf{s}_i = \mathbf{s}_n \circ \mathbf{s}_{n-1} \circ \dots \circ \mathbf{s}_1 = p$$

where  $n$  denotes the length of  $\mathbf{s}$ . Note that  $\mathbf{s}$  may include the implicitly given inverses of the group generators. Hereafter,  $\bigcirc_{i=n}^1 \mathbf{s}_i$  will be written as  $\acute{\mathbf{s}}$ .

The size of the group  $|G|$  is 720, which shows that  $G = S_6$ , since  $|S_6| = 720$ . This fact can also be proven in the following way. Each transposition between the upper and lower row can be found with the nested conjugate

$$\begin{aligned} T_1(x, y) &= \left( L^{-y+3+1} \right)^{-1} \left( \left( U^{-x+1} \right)^{-1} (S) \left( U^{-x+1} \right) \right) \left( L^{-y+3+1} \right) \\ &= L^{y-4} U^{x-1} S U^{-x+1} L^{-y+4} \\ &= L^{y-1} U^{x-1} S U^{-x+1} L^{-y+1}, \end{aligned}$$

where  $x \in \{1, 2, 3\}$  and  $y \in \{4, 5, 6\}$ . As an example, let  $p = (26)$ . Then

$$\begin{aligned}
T_1(2, 6) &= L^{6-1}U^{2-1}SU^{-2+1}L^{-6+1} \\
&= L^5U^1SU^{-1}L^{-5} \\
&= (456)^5(123)^1(14)(123)^{-1}(456)^{-5} \\
&= (465)(123)(14)(132)(456) \\
&= (465)(24)(456) \\
&= (2, 6).
\end{aligned}$$

This gives – after some reductions – the solution  $\mathbf{s} = (L, U^{-1}, S, U, L^{-1})$  visualized below.

$$\begin{array}{ccccc}
\boxed{\begin{matrix} 1 & 6 & 3 \\ 4 & 5 & 2 \end{matrix}} & \xrightarrow{L} & \boxed{\begin{matrix} 1 & 6 & 3 \\ 2 & 4 & 5 \end{matrix}} & \xrightarrow{U^{-1}} & \boxed{\begin{matrix} 6 & 3 & 1 \\ 2 & 4 & 5 \end{matrix}} \\
\xrightarrow{S} \boxed{\begin{matrix} 2 & 3 & 1 \\ 6 & 4 & 5 \end{matrix}} & \xrightarrow{U} & \boxed{\begin{matrix} 1 & 2 & 3 \\ 6 & 4 & 5 \end{matrix}} & \xrightarrow{L^{-1}} & \boxed{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}}
\end{array}$$

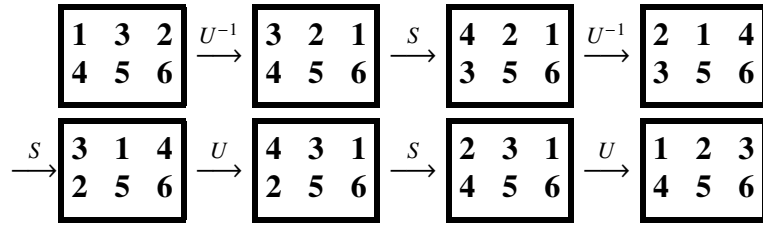
Each transposition between elements on the same row can be found with the nested conjugate

$$\begin{aligned}
T_2(x, y, R) &= \left( (R^{-x+1})^{-1}(S)(R^{-x+1}) \right)^{-1} \left( (R^{-y+1})^{-1}(S)(R^{-y+1}) \right) \left( (R^{-x+1})^{-1}(S)(R^{-x+1}) \right) \\
&= (R^{x-1}SR^{-x+1})^{-1}(R^{y-1}SR^{-y+1})(R^{x-1}SR^{-x+1}) \\
&= (R^{-(x+1)}S^{-1}R^{-(x-1)})(R^{y-1}SR^{-y+1})(R^{x-1}SR^{-x+1}) \\
&= (R^{x-1}SR^{-x+1})(R^{y-1}SR^{-y+1})(R^{x-1}SR^{-x+1}) \\
&= R^{x-1}SR^{-x+1}R^{y-1}SR^{-y+1}R^{x-1}SR^{-x+1} \\
&= R^{x-1}SR^{-x+y}SR^{x-y}SR^{-x+1}
\end{aligned}$$

where  $x, y \in R$  and  $R \in \mathfrak{M} \setminus \{S\}$ . As an example, let  $p = (23)$ . Since 2 and 3 belong to the upper row,  $R$  is set to  $U$ :

$$\begin{aligned}
T_2(2, 3, U) &= U^{2-1}SU^{-2+3}SU^{2-3}SU^{-2+1} \\
&= U^1SU^1SU^{-1}SU^{-1} \\
&= (123)(14)(123)(14)(123)^{-1}(14)(123)^{-1} \\
&= (123)(14)(123)(14)(132)(14)(132) \\
&= (123)(14)(24)(14)(132) \\
&= (123)(12)(132) \\
&= (23).
\end{aligned}$$

This gives the solution  $\mathbf{s} = (U^{-1}, S, U^{-1}, S, U, S, U)$  visualized below.



Since a transposition of any two elements herein can be composed using the generators in  $\mathfrak{M}$  – in other words, since the group  $G$  includes every transposition of the elements 1 through 6 – it follows that the group must include every permutation on the six elements, and thereby equal  $S_6$ .