Homework 5

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Question 1. Problem 4.1.8, text. For each k, which graphs are k-connected? Which are k-edge connected?

The Hexagon
κ: 4
κ':4
δ: 4
4-connected? Yes
4-edge-connected? Yes
The Other one
κ: 2
κ':4
δ: 4
4-connected? Yes
4-edge-connected? Yes

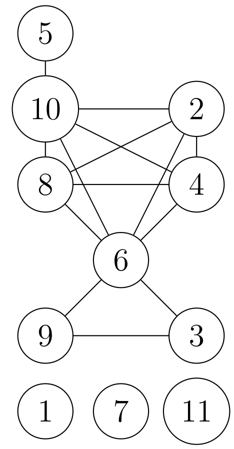
Question 2. For $k \leq n-1$, prove that every simple n-vertex graph G with $\delta(G) \geq (n+k-2)/2$ is k-connected.

PROOF BY INDUCTION:
Start with $n = 1$. Since $n = 1, k = 0$. A single node is K_1, and K_1 is 0-connected by virtue of the K-graph rule.

Increment n: As n increases, so will δ. δ also increases. For $n = 2, k = 0, \delta = \frac{2}{2}$. The Harary graph $H_2 1$, the graph with the fewest possible edges to achieve $\delta = k$ is 1-connected, therefore 0-connected. δ will increase by 1 for every other increment of k. As n increases, we build a Horary graph $H_{(n)(k)}$, which will be k-connected. This will be the case for every $H_{(n)(k)}$.

Question 3. (!) Prove that the symmetric difference of two different edge cuts is an edge cut. (Hint: Draw a picture illustrating the two edge cuts and use it to guide the proof.)

Question 4. (−) Let G be the simple graph with vertex set $\{1...11\}$ defined by $i \leftrightarrow j$ if and only if i, j have a common factor bigger than 1. Determine the blocks of G.



The blocks are: $V(5, 10)$, $V(6, 9, 3)$, $V(1)$, $V(7)$, $V(11)$, $V(10, 2, 4, 6, 8)$

Question 5. Give a formula for the number of spanning trees of G in terms of the number of spanning trees of its blocks.
$\tau(G) = \sum \tau(\text{Blocks})$

Question 6. (−) Determine$\kappa(u, v)$ and $\kappa`(u, v)$ in the graph drawn below. LOL I'M NOT DRAWING THAT. (Hint: Use the dual problemas to give short proofs of optimality)

Question 7. Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z path through y.

Question 8. Prove that if $\kappa(G) = k$ and $\text{diam}(G) = d$, then $n(G) \geq k(d$ - 1$) + 2$ and $\alpha(G) \geq \lceil (1 + d)/2 \rceil$.
Given graph G with diameter d, there exsist two vertices a and b with a shortest path between them consisting of at most (d - 1) edges. If $\kappa(G) = k$, then there exist k vertex-disjoint paths between a and b of length $\geq$ (d - 1), since there cannot be a path shorter than (d - 1) between a and b, because that path would be the diameter. These k paths share points a and b, adding 2 to the minimum number of vertices. The end result is $n(G) \geq k(d$ -1$) + 2$.

Given a graph G with diameter d, we can infer that there is a path with $d + 1$ vertices that created the diameter. This path will be isomorphic to P_{d+1}. The maximum independent set will have at least $\lceil (1 + d)/2 \rceil$ vertices on the diameter alone. Any additional vertices can only increase the size of α.

Question 9. In a network N, with source s and sink t, prove that if there exists no directed (s, t)-path then the value of a maximum flow and the capacity of a minimum cut are both zero. (goal: short, elegant, correct proof)

Question 10. (−) A kitchen sink draws water from two tanks according to the network of pipes with papacities per unit time shown below. Find the maximum flow. Prove that your answer is optimal by using the dual problem, nad explain why this proves optimality.