Homework 3

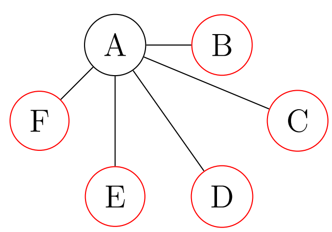
Dustin Lambright - dalambri
Aseem Raina - araina
Bihan Zhang - bzhang28
CSC 565 - Graph Theory

February 5, 2018

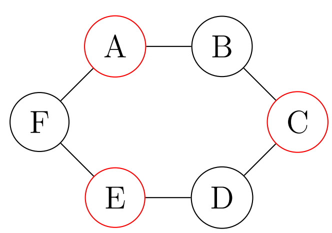
Question 1. Show that every simple graph G with 6 vertices has either a clique of size 3 or an independent set of size 3.

By the definition of inependent sets and cliques, any graph G with a clique of size n will also have a clique of n-1, n-2, ... n-(n-1). The same goes for an independent set. Therefore, using contradiction, if a graph does not have a clique of size 3 and does not have an independent set of size 3, the statement is false.

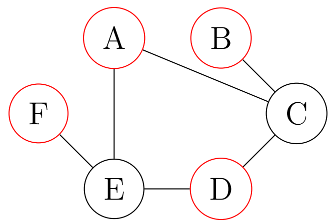
The graph $K_{1,5}$ has a maximum clique size of 2, but an independent set of 5. (displayed in red)



In an effort to reduce the size of the maximum independent set, edges are replaced, creating a cyclic graph:



This graph has a a maximum independent set of 3, and a maximum clique of 2. The only way to reduce the size of an independent set, i.e., $\{A, E, C\}$ is to connect any of those vertices with an edge. The connection of any of these vertices results in a clique of size 3. In an effort to maintain our clique constraint, if we remove the edges AF and AB to connect the independent set $\{A, E, C\}$, we end up with an independent set $\{A, F, B, D\}$.



The graph C_6 is the closest we can get to a graph without a clique of size 3 and without an independent set of size 3, but it has an independent set of size three, proving that a graph with a vertex count of 6 has to have an independent set of size 3 or a clique of size 3.

Question 2. Prove or disprove: In a simple graph, every closed even trail of length more than 3 contains an even cycle.

Question 3. Prove the following by strong induction on the length of the trail: The edge set of every closed trail can be partitioned into one or more pairwise edge-disjoint cycles.

i. The edge set E of every closed trail can be partitioned into one or more pairwise edge-disjoint cycles

Proof: Let G be the graph containing the edge set E.

Let n be the length of the maximum closed trial in G of the form $(v_1,\, v_2)$, ... $(v_n,\, v_1)$.

Let C be the smallest cycle in G of the form $(v_i,\, v_{i+1})...(v_{i+k-1},\, v_{i+k})$, where $v_i = v_{i+k}$ and $v_i...v_{i+k-1}$ are distinct (if the vertexes were not distinct that would mean it would be able to be partitioned into smaller cycles, and we started this exercise assuming C is the smallest cycle).

If $C = G$, the graph can be partitioned into one pairwise edge-disjoint cycle.

Otherwise, Let $G' = G - C$

The length of the closed trial through G' cannot be more than n, and would have the closed trial $(v_1,\, v_2)...(v_n,\, v_1)$.

By induction we can partition the edge set of G' into smaller disjoint cycles. Thus we show that the edge set of G can be partitioned into disjoint cycles consisting of all the cycles in G' and C

ii. If an edge set E can be partitioned into one or more pairwise edge-disjoint cycles, it is a closed trial.

Proof: Let G be the graph containing the edge set E.

If the edge set of G can be partitioned into pairwise edge-disjoint cycles, every vertex must have an even degree (for v to be on a cycle it must have 2 degrees). Since every vertex is even G must be Eulerian by Theorem 1.2.26. Every Eulerian graph must have a closed trial. Thus G must have a closed trial.

Question 4. Suppose that T is a maximal trail in a simple graph G and that T has at least one edge and is not closed. Prove that the endpoints of T have odd degree.

Let v_1 and v_n be the endpoints of T (it must have at least 2 distinct vertexes, since it has at least one edge, and is not closed.).

Since T is not closed, v_1 and v_n cannot be connected.

If v_1 had a neighbors v_0, and their connected edge was not in T, it must be added in T in order for T to be maximal. Thus v_1 can only have one neighbor v_2. And any vertex, on a simple graph, with one neighbor has a degree of 1. The same argument can be applied to v_n showing that it, also, has a degree of 1.

Question 5. If G is a graph with vertices v1, v2, . . . , vn and Ak denotes the kth power of the adjacency matrix of G under matrix multiplication then

(***) $A^k[i, j]$ is the number of vi, vj -walks of length k in G.

Show how to use (***) to solve the following without multipltying matrices and prove your answer correct: Let A be the adjacency matrix of Kn. If i = j, then A3 [i, j] =____ . Otherwise A3 [i, j] = ____.

Question 6. Draw a simple, connected graph with the following degree sequence, or prove that no such graph is possible:
a. (3, 3, 3, 2, 2, 2)
b. (7, 6, 5, 4, 3, 2, 1)
c. (3, 3, 2, 2, 1, 1)
d. (7, 6, 5, 4, 3, 3, 2)
e. (6, 6, 5, 4, 3, 3, 1)

Question 7. How many different simple graphs are there with 5 edges and with vertex set v1, v2, . . . v5? (We are counting labeled graphs, not isomorphism classes)

Question 8. Prove by contradiction: A graph with every vertex degree even has no cut-edge.