Homework 3

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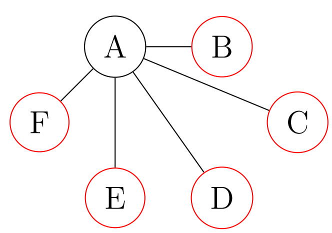
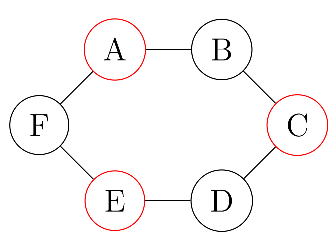
Question 1. Show that every simple graph G with 6 vertices has either a clique of size 3 or an independent set of size 3.

By the definition of independent sets and cliques, any graph G with a clique of size n will also have a clique of n-1, n-2, ... n-(n-1). The same goes for an independent set. Therefore, using contradiction, if a graph does not have a clique of size 3 and does not have an independent set of size 3, the statement is false.

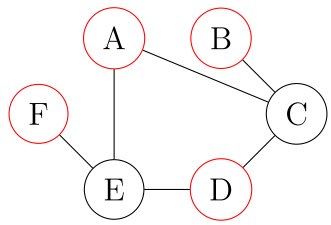
The graph $K_{1,5}$ has a maximum clique size of 2, but an independent set of 5. (displayed in red)



In an effort to reduce the size of the maximum independent set, edges are replaced, creating a cyclic graph:



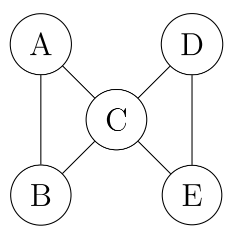
This graph has a a maximum independent set of 3, and a maximum clique of 2. The only way to reduce the size of an independent set, i.e., $\{A, E, C\}$ is to connect any of those vertices with an edge. The connection of any of these vertices results in a clique of size 3. In an effort to maintain our clique constraint, if we remove the edges AF and AB to connect the independent set $\{A, E, C\}$, we end up with an independent set $\{A, F, B, D\}$.



The graph C_6 is the closest we can get to a graph without a clique of size 3 and without an independent set of size 3, but it has an independent set of size three, proving that a graph with a vertex count of 6 has to have an independent set of size 3 or a clique of size 3.

Question 2. Prove or disprove: In a simple graph, every closed even trail of length more than 3 contains an even cycle.

To disprove the given statement, a single counterexample should suffice.
In the following simple graph, the closed even trail A-B-C-E-D-C-A of length 6:



contains no even cycle.

Question 3. Prove the following by strong induction on the length of the trail: The edge set of every closed trail can be partitioned into one or more pairwise edge-disjoint cycles.

Proof: Let G be the graph containing the edge set E.

Let n be the length of the maximum closed trial in G of the form $(v_1,\ v_2)$, ... $(v_n,\ v_1)$.

Let C be the smallest cycle in G of the form $(v_i,\ v_{i+1})...(v_{i+k-1},\ v_{i+k})$, where $v_i = v_{i+k}$ and $v_i...v_{i+k-1}$ are distinct (if the vertexes were not distinct that would mean it would be able to be partitioned into smaller cycles, and we started this exercise assuming C is the smallest cycle).

If $C = G$, the graph can be partitioned into one pairwise edge-disjoint cycle.

Otherwise, Let $G' = G - C$

The length of the closed trial through G' cannot be more than n, and would have the closed trial $(v_1,\, v_2)...(v_n,\, v_1)$.

By induction we can partition the edge set of G' into smaller disjoint cycles. Thus we show that the edge set of G can be partitioned into disjoint cycles consisting of all the cycles in G' and C

Question 4. Suppose that T is a maximal trail in a simple graph G and that T has at least one edge and is not closed. Prove that the endpoints of T have odd degree.

Let v_1 and v_n be the endpoints of T (it must have at least 2 distinct vertexes, since it has at least one edge, and is not closed.).

Since T is not closed, v_1 and v_n cannot be connected.

If v_1 had a neighbors v_0, and their connected edge was not in T, it must be added in T in order for T to be maximal. Thus v_1 can only have one neighbor v_2. And any vertex, on a simple graph, with one neighbor has a degree of 1. The same argument can be applied to v_n showing that it, also, has a degree of 1.

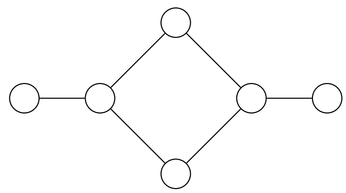
Question 5. If G is a graph with vertices v1, v2, . . . , vn and Ak denotes the kth power of the adjacency matrix of G under matrix multiplication then

(***) $A^k[i, j]$ is the number of vi, vj -walks of length k in G.

Show how to use (***) to solve the following without multipltying matrices and prove your answer correct: Let A be the adjacency matrix of Kn. If i = j, then A3 [i, j] =____ . Otherwise A3 [i, j] = _____.

Question 6. Draw a simple, connected graph with the following degree sequence, or prove that no such graph is possible:
a. (3, 3, 3, 2, 2, 2)
For any graph G, the number of vertices of odd degree should be even. Here, we have 3 vertices of odd degree (3,3,3), hence no such graph is possible.
b. (7, 6, 5, 4, 3, 2, 1)
No such graph is possible as a vertex cannot have degree equal to the total number of vertices in a simple graph (i.e. 7).

c. (3, 3, 2, 2, 1, 1)
A graph is possible



d. (7, 6, 5, 4, 3, 3, 2)
No such graph is possible as a vertex cannot have degree equal to the total number of vertices in a simple graph (i.e. 7).
e. (6, 6, 5, 4, 3, 3, 1)
For the given sequence of seven vertices, two vertices have degree 6. For such a simple graph to exist, all vertices must have a degree of at least 2. Since one vertex has degree 1, such a graph is not possible.

Question 7. How many different simple graphs are there with 5 edges and with vertex set v1, v2, . . . v5? (We are counting labeled graphs, not isomorphism classes)

For an edge, we need to choose 2 vertices out of 5, which can be done in $\binom{5}{2}$ ways. The total number of such edge subsets is $2^{\binom{5}{2}} = 2^{10} = 1024$ graphs.

Question 8. Prove by contradiction: A graph with every vertex degree even has no cut-edge.

Let us assume that there does exist a cut-edge e in graph G (where every vertex has an even degree), joining vertices v_i and v_j.
Removing e would divide G into two the following two graphs:
G_1: containing v_i (now with an odd degree, since e was removed) and zero or more vertices with an even vertex degree.
G_2: containing v_j (now with an odd degree, since e was removed) and zero or more vertices with an even vertex degree.
Since no graph can have an odd number of odd-degree vertices, neither G_1 nor G_2 can exist. This means that our assumption was false.
Hence we have proven that a graph with every vertex degree even has no cut-edge.