Homework 3

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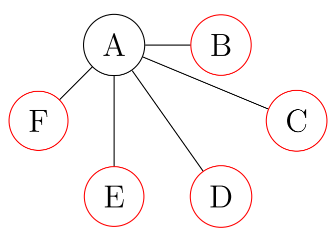
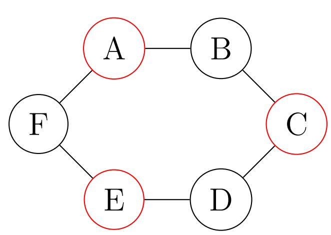
Question 1. Show that every simple graph G with 6 vertices has either a clique of size 3 or an independent set of size 3.

By the definition of inependent sets and cliques, any graph G with a clique of size n will also have a clique of n-1, n-2, ... n-(n-1). The same goes for an independent set. Therefore, using contradiction, if a graph does not have a clique of size 3 and does not have an independent set of size 3, the statement is false.

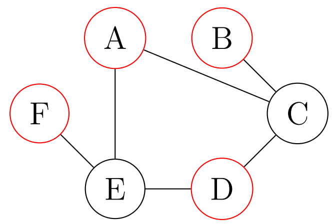
The graph $K_{1,5}$ has a maximum clique size of 2, but an independent set of 5. (displayed in red)



In an effort to reduce the size of the maximum independent set, edges are replaced, creating a cyclic graph:



This graph has a a maximum independent set of 3, and a maximum clique of 2. The only way to reduce the size of an independent set, i.e., $\{A, E, C\}$ is to connect any of those vertices with an edge. The connection of any of these vertices results in a clique of size 3. In an effort to maintain our clique constraint, if we remove the edges AF and AB to connect the independent set $\{A, E, C\}$, we end up with an independent set $\{A, F, B, D\}$.



The graph C_6 is the closest we can get to a graph without a clique of size 3 and without an independent set of size 3, but it has an independent set of size three, proving that a graph with a vertex count of 6 has to have an independent set of size 3 or a clique of size 3.

Question 2. Prove or disprove: In a simple graph, every closed even trail of length more than 3 contains an even cycle.

Question 3. Prove the following by strong induction on the length of the trail: The edge set of every closed trail can be partitioned into zero or more pairwise edge-disjoint cycles.

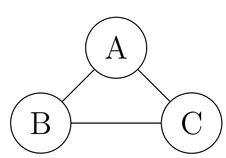
Question 4. Suppose that T is a maximal trail in a simple graph G and that T has at least one edge and is not closed. Prove that the endpoints of T have odd degree.

Question 5. If G is a graph with vertices $v_1, v_2, ..., v_n$ and A^k denotes the kth power of the adjacency matrix of G under matrix multiplication then

(***) $A^k[i, j]$ is the number of v_i, v_j -walks of length k in G.

Show how to use (***) to solve the following without multipltying matrices and prove your answer correct: Let A be the adjacency matrix of K_n. If $i = j$, then $A^3[i, j] =$____ . Otherwise $A^3[i, j] =$ ____.

Consider the graph K_3, whose vertex count is n.



These two branches can be calculated as follows:

TO CALCULATE (***) FOR $i \neq j$

There are 3 possible paths from A to C.

{AC, CA, AC}
{AC, CB, BC}
{AB, BA, AC}

To calculate the number of walks from A to C, we need to break down the number of moves needed from two branches: going to C, and not going to C

{AC, CA, AC}
{AC, CB, BC}
{AB, BA, AC}

BEGINNING MOVE TO C (THE END POINT)

MOVE TO	OPTIONS	REASONING
C	1	You can only move to C, as per definition
A or B	$n-1$	Any option but C works for this step
C	1	The walk's third step has to end on C

BEGINNING MOVE TO ANY POINT (THE END POINT)

MOVE TO	OPTIONS	REASONING
B	$n-2$	You can only move to any vertex that isn't C
A or C	$n-2$	C cannot be the choice in this step
C	1	The walk's third step has to end on C

If we add these two options together, we get $(1 \times (n-1) \times 1) + ((n-2) \times (n-2) \times 1)$, which simplifies to $n^2 - 3n + 3$.

The formula for calculating (***) $A^3[i, j]$ where $i \neq j$ therefore is $n^2 - 3n + 3$.

TO CALCULATE (***) FOR $i = j$

There are 2 possible paths from A to A.

{AB, BC, CA}
{AC, CB, BA}

To calculate the number of paths, we have the following options:

STARTING AND ENDING AT THE SAME POINT

MOVE TO	OPTIONS	REASONING
B or A	$n-1$	Any option will work on this step
A or B	$n-2$	Any option will work, except for C
C	1	The walk's third step has to end on C

When we combine these, we get $(n-1) \times (n-2) \times 1$, which simplifies to $n^2 - 3n + 2$. The formula for calculating (***) $A^3[i, j]$ where $i = j$ therefore is $n^2 - 3n + 2$.

IN CONCLUSION

$$A^3[i,j] = \begin{cases} n^2 - 3n + 3 & \text{if } i \neq j \\ n^2 - 3n + 2 & \text{if } i = j \end{cases}$$

where n is the number of vertices in G.

Question 6. Draw a simple, connected graph with the following degree sequence, or prove that no such graph is possible:
a. (3, 3, 3, 2, 2, 2)
b. (7, 6, 5, 4, 3, 2, 1)
c. (3, 3, 2, 2, 1, 1)
d. (7, 6, 5, 4, 3, 3, 2)
e. (6, 6, 5, 4, 3, 3, 1)

Question 7. How many different simple graphs are there with 5 edges and with vertex set v1, v2, . . . v5? (We are counting labeled graphs, not isomorphism classes)

Question 8. Prove by contradiction: A graph with every vertex degree even has no cut-edge.