Homework 5

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CSC 565 - Graph Theory

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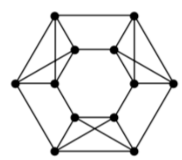
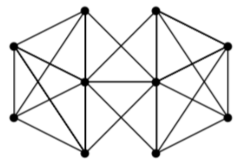
Question 1. Problem 4.1.8, text. For each k, which graphs are k-connected? Which are k-edge connected?



κ: 4
κ':4
δ: 4

k	k-connected	k-edge connected
1	Yes	Yes
2	Yes	Yes
3	Yes	Yes
4	Yes	Yes
$\geq$5	No	No



κ: 2
κ':4
δ: 4

k	k-connected	k-edge connected
1	Yes	Yes
2	Yes	Yes
3	No	Yes
4	No	Yes
≥5	No	No

Question 2. For $k \leq n-1$, prove that every simple n-vertex graph G with $\delta(G) \geq (n+k-2)/2$ is k-connected.

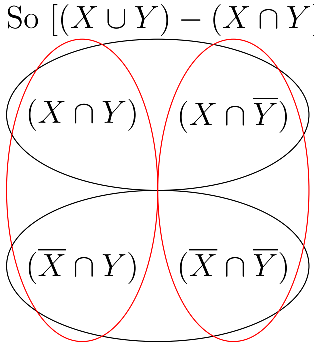
Question 3. (!) Prove that the symmetric difference of two different edge cuts is an edge cut. (Hint: Draw a picture illustrating the two edge cuts and use it to guide the proof.)

Let A be the edge cut of $[X, \bar{X}]$ and B be the edge cut of $[Y, \bar{Y}]$. The symmetric difference of $A\Delta B$ is $(X \cup Y) - (X \cap Y)$.

This can be shown in 4 cases:

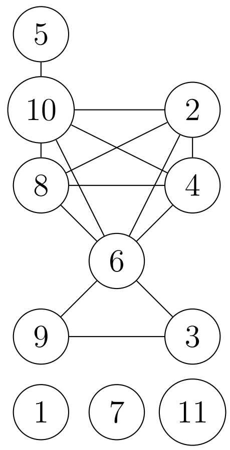
1. An edge in A but not in B.
   In this case $e \in (X \cup Y)$ and $e \notin (X \cap Y)$ Thus $e \in \delta((X \cup Y) - (X \cap Y))$
2. An edge in B but not in A
   In this case $e \in (X \cup Y)$ and $e \notin (X \cap Y)$ Thus $e \in \delta((X \cup Y) - (X \cap Y))$
3. An edge in A and in B
   In this case $e \in (X \cup Y)$ and $e \in (X \cap Y)$ Thus $e \notin \delta((X \cup Y) - (X \cap Y))$
4. An edge not in A or B
   In this case $e \notin (X \cup Y)$ and $e \notin (X \cap Y)$ Thus $e \notin \delta((X \cup Y) - (X \cap Y))$

So $[(X \cup Y) - (X \cap Y), \overline{(X \cup Y) - (X \cap Y)}]$ is the resulting symmetric difference.



Question 4. (−) Let G be the simple graph with vertex set $\{1...11\}$ defined by $i \leftrightarrow j$ if

and only if i, j have a common factor bigger than 1. Determine the blocks of G.

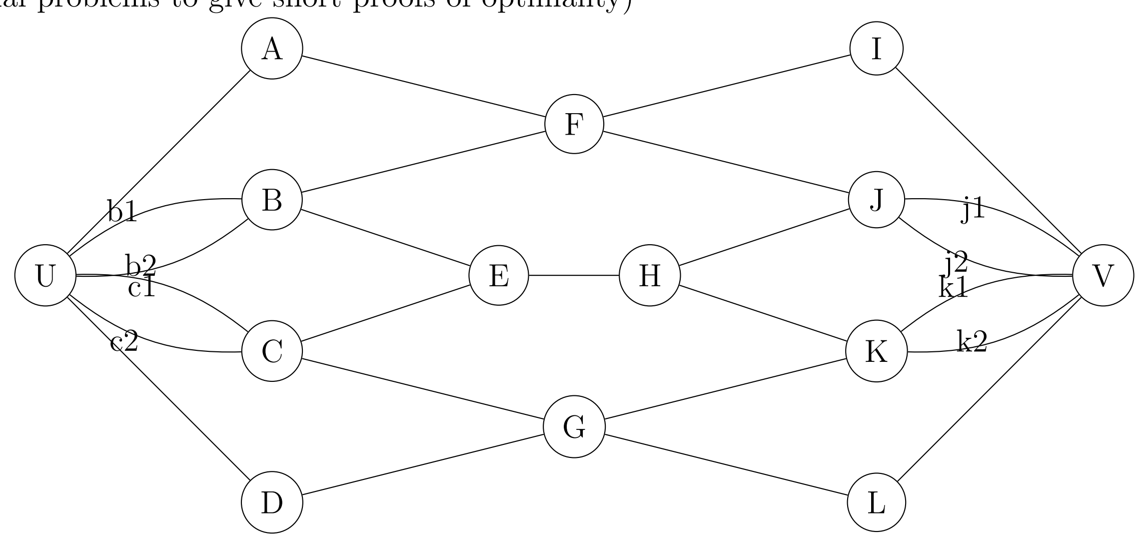


The blocks are: $V(5, 10)$, $V(6, 9, 3)$, $V(1)$, $V(7)$, $V(11)$, $V(10, 2, 4, 6, 8)$

Question 5. Give a formula for the number of spanning trees of G in terms of the number of spanning trees of its blocks.
Let G be a connected graph that can be decomposed into n Blocks B_n. The number of spanning trees in G is the product of the number of spanning trees in all it's blocks:

$\tau(G) = \prod_{i=1}^{n} \tau(B_i)$
This is because:
- The graph can be decomposed into its blocks
- Any combination of spanning trees of all blocks will still leave the resultant graph (now a spanning tree) connected, as long as the original graph is connected.

Question 6. $(-)$ Determine $\kappa(u, v)$ and $\kappa`(u, v)$ in the graph drawn below. (Hint: Use the dual problems to give short proofs of optimality)



For the given graph (with nodes and some edges annotated for convenience):

$\kappa(u, v) = 3$
For proof of optimality, we could use the duality:
$\kappa(u, v) =$ number of vertex-disjoint paths between u and v, which are the following:
$u - A - F - I - v$
$u - B - E - H - J - v$
$u - C - G - K - v$
In other words, all $u - v$ paths must contain at least one of F, E or G.

$\kappa'(u, v) = 5$
For proof of optimality, we could use the duality:
$\kappa'(u, v) =$ number of edge-disjoint paths between u and v, which are the following:
$uA - AF - FI - Iv$
$b1 - BF - FJ - j1$
$b2 - BE - EH - HJ - j2$
$c1 - CG - GK - k1$
$uD - DG - GL - Lv$
In other words, all $u - v$ paths must contain at least one of AF, BF, EH, CG or DG.

Question 7. Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z path through y.

Assume there is a graph G that is two-connected, but does not have a path connecting vertices x and y, creating a triple (x, y, z) that does not have an x, z path through y. To maintain graph connectivity, there must exist a path from x to y. This path cannot contain any continuation that contains z, so y must have a degree of 1, thus contradicting that this graph is 2-connected.

Question 8. Prove that if $\kappa(G) = k$ and diam$(G) = d$, then $n(G) \geq k(d$ - 1$) + 2$ and $\alpha(G) \geq \lceil (1 + d)/2 \rceil$.
Given graph G with diameter d, there exist two vertices a and b with a shortest path between them consisting of at most (d - 1) edges. If $\kappa(G) = k$, then there exist k vertex-disjoint paths between a and b of length $\geq$ (d - 1), since there cannot be a path shorter than (d - 1) between a and b, because that path would be the diameter. These k paths share points a and b, adding 2 to the minimum number of vertices. The end result is $n(G) \geq k(d$ -1$) + 2$.

Given a graph G with diameter d, we can infer that there is a path with $d + 1$ vertices that created the diameter. This path will be isomorphic to P_{d+1}. The maximum independent set will have $\lceil (1 + d)/2 \rceil$ vertices on the diameter alone. Any additional vertices can only increase the size of α.

Question 9. In a network N, with source s and sink t, prove that if there exists no directed (s, t)-path then the value of a maximum flow and the capacity of a minimum cut are both zero. (goal: short, elegant, correct proof)

If we run the Ford-Fulkerson Algorithm on a graph with no directed (s,t)-path from source to sync, the algorithm will not find a f-augmenting path (since it can only reach t by using backwards edges whose value is the 0 flow), it would return a $[S,\bar{S}]$ edge cut with 0 capacity for all the backwards edges from t,(since the algorithm is initialized with every flow(0)). Thus the maximum flow is also 0.

Question 10. (−) A kitchen sink draws water from two tanks according to the network of pipes with capacities per unit time shown below. Find the maximum flow. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.

In every network, the maximum feasible flow is equivalent to the minimum capacity of a source/sink cut by Corollary 4.3.8. The maximum feasible flow in this case is 34, because the edge cut {Tank1, Tank2, B} and {A, C, D, E, F} produces the minimum capacity 34. Every other cut has a greater capacity, and no edges from {Tank1, Tank2, B} to {A, C, D, E, F} have excess capacity, and no edges from {A, C, D, E, F} to {Tank1, Tank2, B} have nonzero flow, this must be the maximum flow possible.