

# Homework 3

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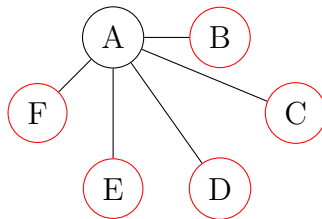
CSC 565 - Graph Theory

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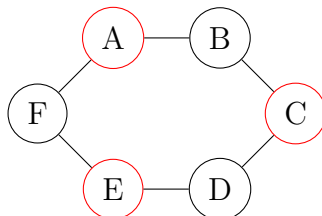
**Question 1.** Show that every simple graph  $G$  with 6 vertices has either a clique of size 3 or an independent set of size 3.

By the definition of independent sets and cliques, any graph  $G$  with a clique of size  $n$  will also have a clique of  $n-1$ ,  $n-2$ , ...  $n-(n-1)$ . The same goes for an independent set. Therefore, using contradiction, if a graph does not have a clique of size 3 and does not have an independent set of size 3, the statement is false.

The graph  $K_{1,5}$  has a maximum clique size of 2, but an independent set of 5. (displayed in red)

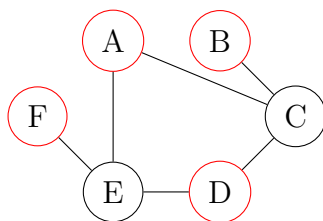


In an effort to reduce the size of the maximum independent set, edges are replaced, creating a cyclic graph:



This graph has a maximum independent set of 3, and a maximum clique of 2. The only way to reduce the size of an independent set, i.e.,  $\{A, E, C\}$  is to connect any of those

vertices with an edge. The connection of any of these vertices results in a clique of size 3. In an effort to maintain our clique constraint, if we remove the edges  $AF$  and  $AB$  to connect the independent set  $\{A, E, C\}$ , we end up with an independent set  $\{A, F, B, D\}$ .



The graph  $C_6$  is the closest we can get to a graph without a clique of size 3 and without an independent set of size 3, but it has an independent set of size three, proving that a graph with a vertex count of 6 has to have an independent set of size 3 or a clique of size 3.

**Question 2.** Prove or disprove: In a simple graph, every closed even trail of length more than 3 contains an even cycle.

**Question 3.** Prove the following by strong induction on the length of the trail: The edge set of every closed trail can be partitioned into zero or more pairwise edge-disjoint cycles.

**Question 4.** Suppose that  $T$  is a maximal trail in a simple graph  $G$  and that  $T$  has at least one edge and is not closed. Prove that the endpoints of  $T$  have odd degree.

**Question 5.** If  $G$  is a graph with vertices  $v_1, v_2, \dots, v_n$  and  $A^k$  denotes the  $k$ th power of the adjacency matrix of  $G$  under matrix multiplication then

(\*\*\*)  $A^k[i, j]$  is the number of  $v_i, v_j$  -walks of length  $k$  in  $G$ .

Show how to use (\*\*\*) to solve the following without multiplying matrices and prove your answer correct: Let  $A$  be the adjacency matrix of  $K_n$ . If  $i = j$ , then  $A^3[i, j] = \text{----}$ . Otherwise  $A^3[i, j] = \text{----}$ .

**Question 6.** Draw a simple, connected graph with the following degree sequence, or prove that no such graph is possible:

- (3, 3, 3, 2, 2, 2)
- (7, 6, 5, 4, 3, 2, 1)
- (3, 3, 2, 2, 1, 1)
- (7, 6, 5, 4, 3, 3, 2)
- (6, 6, 5, 4, 3, 3, 1)

**Question 7.** How many different simple graphs are there with 5 edges and with vertex set  $v_1, v_2, \dots, v_5$ ? (We are counting labeled graphs, not isomorphism classes)

**Question 8.** Prove by contradiction: A graph with every vertex degree even has no cut-edge.