Homework 3

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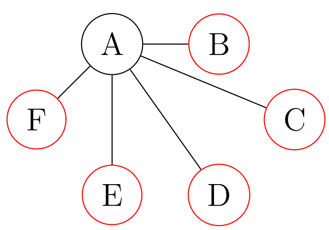
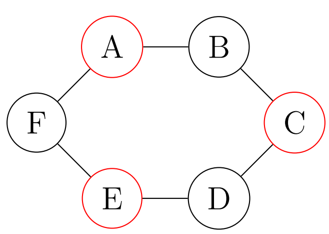
Question 1. Show that every simple graph G with 6 vertices has either a clique of size 3 or an independent set of size 3.

By the definition of independent sets and cliques, any graph G with a clique of size n will also have a clique of n-1, n-2, ... n-(n-1). The same goes for an independent set. Therefore, using contradiction, if a graph does not have a clique of size 3 and does not have an independent set of size 3, the statement is false.

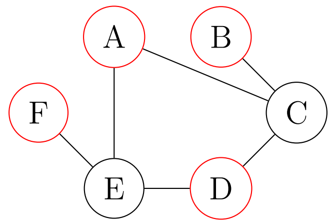
The graph $K_{1,5}$ has a maximum clique size of 2, but an independent set of 5. (displayed in red)



In an effort to reduce the size of the maximum independent set, edges are replaced, creating a cyclic graph:



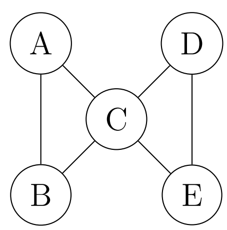
This graph has a a maximum independent set of 3, and a maximum clique of 2. The only way to reduce the size of an independent set, i.e., $\{A, E, C\}$ is to connect any of those vertices with an edge. The connection of any of these vertices results in a clique of size 3. In an effort to maintain our clique constraint, if we remove the edges AF and AB to connect the independent set $\{A, E, C\}$, we end up with an independent set $\{A, F, B, D\}$.



The graph C_6 is the closest we can get to a graph without a clique of size 3 and without an independent set of size 3, but it has an independent set of size three, proving that a graph with a vertex count of 6 has to have an independent set of size 3 or a clique of size 3.

Question 2. Prove or disprove: In a simple graph, every closed even trail of length more than 3 contains an even cycle.

To disprove the given statement, a single example should suffice.
In the following simple graph, the closed even trail A-B-C-E-D-C-A of length 6:



contains no even cycle.

Question 3. Prove the following by strong induction on the length of the trail: The edge set of every closed trail can be partitioned into one or more pairwise edge-disjoint cycles.

Proof: Let G be the graph containing the edge set E.

Let n be the length of the maximum closed trial in G of the form $(v_1,\, v_2)$, ... $(v_n,\, v_1)$.

Let C be the smallest cycle in G of the form $(v_i,\, v_{i+1})$...$(v_{i+k-1},\, v_{i+k})$, where $v_i = v_{i+k}$ and v_i...v_{i+k-1} are distinct (if the vertexes were not distinct that would mean it would be able to be partitioned into smaller cycles, and we started this exercise assuming C is the smallest cycle).

If $C = G$, the graph can be partitioned into one pairwise edge-disjoint cycle.

Otherwise, Let $G' = G - C$

The length of the closed trial through G' cannot be more than n, and would have the closed trial $(v_1, v_2)...(v_n, v_1)$.

By induction we can partition the edge set of G' into smaller disjoint cycles. Thus we show that the edge set of G can be partitioned into disjoint cycles consisting of all the cycles in G' and C

Question 4. Suppose that T is a maximal trail in a simple graph G and that T has at least one edge and is not closed. Prove that the endpoints of T have odd degree.

Let v_1 and v_n be the endpoints of T (it must have at least 2 distinct vertexes, since it has at least one edge, and is not closed.).

Since T is not closed, v_1 and v_n cannot be connected.

If v_1 had a neighbors v_0, and their connected edge was not in T, it must be added in T in order for T to be maximal. Thus v_1 can only have one neighbor v_2. And any vertex, on a simple graph, with one neighbor has a degree of 1. The same argument can be applied to v_n showing that it, also, has a degree of 1.

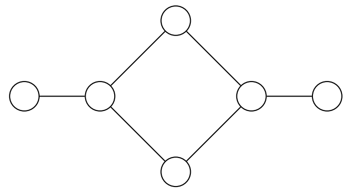
Question 5. If G is a graph with vertices v1, v2, . . . , vn and Ak denotes the kth power of the adjacency matrix of G under matrix multiplication then

(***) $A^k[i, j]$ is the number of vi, vj -walks of length k in G.

Show how to use (***) to solve the following without multipltying matrices and prove your answer correct: Let A be the adjacency matrix of Kn. If i = j, then A3 [i, j] =____ . Otherwise A3 [i, j] = _____.

Question 6. Draw a simple, connected graph with the following degree sequence, or prove that no such graph is possible:
a. (3, 3, 3, 2, 2, 2)
For any graph G, the number of vertices of odd degree should be even. Here, we have 3 vertices of odd degree (3,3,3), hence no such graph is possible.
b. (7, 6, 5, 4, 3, 2, 1)
No such graph is possible as a vertex cannot have degree equal to the total number of vertices in the graph (i.e. 7).

c. (3, 3, 2, 2, 1, 1)
A graph is possible



d. (7, 6, 5, 4, 3, 3, 2)
No such graph is possible as a vertex cannot have degree equal to the total number of vertices in the graph (i.e. 7).
e. (6, 6, 5, 4, 3, 3, 1) No such graph is possible as two vertices cannot have degree 6 for the following sequence.

Question 7. How many different simple graphs are there with 5 edges and with vertex set v1, v2, . . . v5? (We are counting labeled graphs, not isomorphism classes) For an edge, we need to choose 2 vertices out of 5, which can be done in $\binom{5}{2}$ ways. The total number of such edge subsets is $2^{\binom{5}{2}} = 2^{10} = 1024$ graphs.

Question 8. Prove by contradiction: A graph with every vertex degree even has no cut-edge.

Let us assume that there does exist a cut-edge e in graph G (where every vertex has an even degree), joining vertices v_i and v_j.
Removing e would divide G into two the following two graphs:
G_1: containing v_i (now with an odd degree, since e was removed) and zero or more vertices with an even vertex degree.
G_2: containing v_j (now with an odd degree, since e was removed) and zero or more vertices with an even vertex degree.
Since no graph can have an odd number of odd-degree vertices, neither G_1 nor G_2 can exist. This means that our assumption was false.
Hence we have proven that a graph with every vertex degree even has no cut-edge.