Homework 5

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CSC 565 - Graph Theory

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Question 1. Problem 4.1.8, text. For each k, which graphs are k-connected? Which are k-edge connected?

The Hexagon
κ: 4
κ':4
δ: 4
4-connected? Yes
4-edge-connected? Yes
The Other one
κ: 2
κ':4
δ: 4
4-connected? Yes
4-edge-connected? Yes

Question 2. For $k \leq n-1$, prove that every simple n-vertex graph G with $\delta(G) \geq (n+k-2)/2$ is k-connected.

PROOF BY INDUCTION:
Start with $n = 1$. Since $n = 1, k = 0$. A single node is K_1, and K_1 is 0-connected by virtue of the K-graph rule.

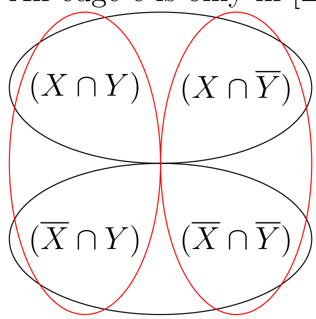
Increment n: As n increases, so will δ. δ also increases. For $n = 2, k = 0, \delta = \frac{2}{2}$. The Harary graph H_21, the graph with the fewest possible edges to achieve $\delta = k$ is 1-connected, therefore 0-connected. δ will increase by 1 for every other increment of k. As n increases, we build a Harary graph $H_{(n)(k)}$, which will be k-connected. This will be the case for every $H_{(n)(k)}$.

Question 3. (!) Prove that the symmetric difference of two different edge cuts is an edge cut. (Hint: Draw a picture illustrating the two edge cuts and use it to guide the proof.)

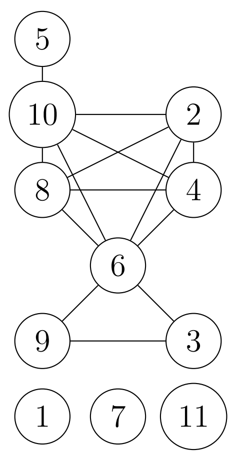
4.1.28 Text

The symmetric difference of $[X, \bar{X}]$ and $[Y, \bar{Y}]$ is the edge cut
$[Z, \overline{Z}] = [(X \cap Y) \cup (\overline{X} \cap \overline{Y}), (X \cap \overline{Y}) \cup (\overline{X} \cap Y)]$

An edge e is only in $[Z, \overline{Z}]$ if it is in exactly one of $[X, \overline{X}]$ or $[Y, \overline{Y}]$



Question 4. (−) Let G be the simple graph with vertex set $\{1...11\}$ defined by $i \leftrightarrow j$ if and only if i, j have a common factor bigger than 1. Determine the blocks of G.



The blocks are: $V(5, 10)$, $V(6, 9, 3)$, $V(1)$, $V(7)$, $V(11)$, $V(10, 2, 4, 6, 8)$

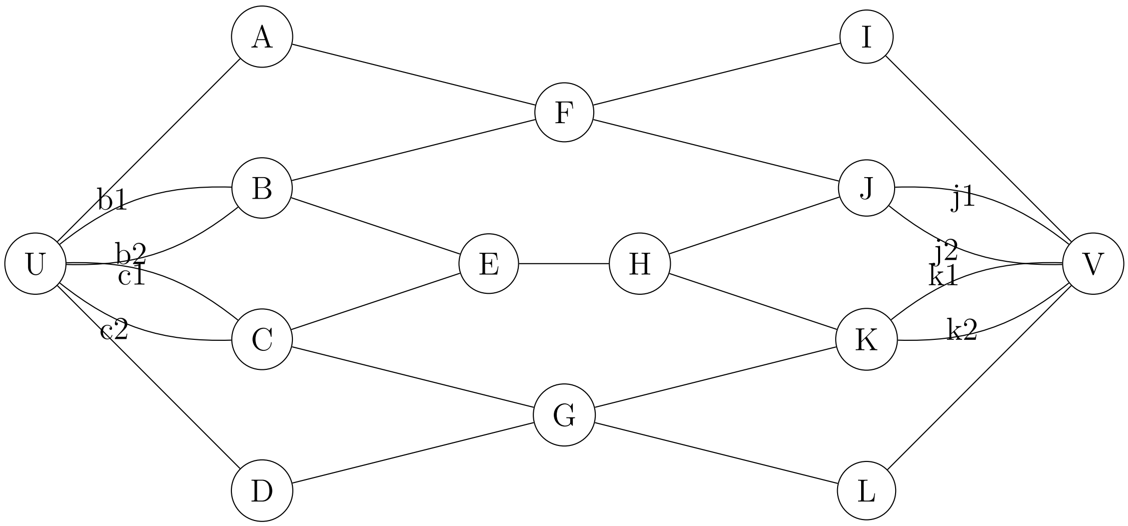
Question 5. Give a formula for the number of spanning trees of G in terms of the number of spanning trees of its blocks.

$\tau(G) = \prod \tau(\text{Blocks})$

This is because:

- The graph can be decomposed into its blocks
- Any combination of spanning trees of all blocks will still leave the resultant graph (now a spanning tree) connected, as long as the original graph is connected.

Question 6. (−) Determine $\kappa(u, v)$ and $\kappa`(u, v)$ in the graph drawn below. (Hint: Use the dual problems to give short proofs of optimality)



For the given graph (with nodes and some edges annotated for convenience):

$\kappa(u,v) = 3$
For proof of optimality, we could use the duality:
$\kappa(u,v) =$ number of vertex-disjoint paths between u and v, which are the following:
$u - A - F - I - v$
$u - B - E - H - J - v$
$u - C - G - K - v$
In other words, all $u - v$ paths must contain at least one of F, E or G.

$\kappa'(u,v) = 5$
For proof of optimality, we could use the duality:
$\kappa'(u,v) =$ number of edge-disjoint paths between u and v, which are the following:
$uA - AF - FI - Iv$
$b1 - BF - FJ - j1$
$b2 - BE - EH - HJ - j2$
$c1 - CG - GK - k1$
$uD - DG - GL - Lv$
In other words, all $u - v$ paths must contain at least one of AF, BF, EH, CG or DG.

Question 7. Prove that a simple graph G is 2-connected if and only if for every triple (x, y, z) of distinct vertices, G has an x, z path through y.

Question 8. Prove that if $\kappa(G) = k$ and $\operatorname{diam}(G) = d$, then $n(G) \geq k(d$ - 1$) + 2$ and $\alpha(G) \geq \lceil (1 + d)/2 \rceil$.
Given graph G with diameter d, there exists two vertices a and b with a shortest path between them consisting of at most (d - 1) edges. If $\kappa(G) = k$, then there exist k vertex-disjoint paths between a and b of length $\geq$ (d - 1), since there cannot be a path shorter than (d - 1) between a and b, because that path would be the diameter. These k paths share points a and b, adding 2 to the minimum number of vertices. The end result is $n(G) \geq k(d$ -1$) + 2$.

Given a graph G with diameter d, we can infer that there is a path with $d+1$ vertices that created the diameter. This path will be isomorphic to P_{d+1}. The maximum independent set will have at least $\lceil (1+d)/2 \rceil$ vertices on the diameter alone. Any additional vertices can only increase the size of α.

Question 9. In a network N, with source s and sink t, prove that if there exists no directed (s,t)-path then the value of a maximum flow and the capacity of a minimum cut are both zero. (goal: short, elegant, correct proof)

If we run the Ford-Fulkerson Algorithm on a graph with no directed (s,t)-path from source to sync, the algorithm will not find a f-augmenting path (since it can only reach t by using backwards edges whose value is the 0 flow), it would return a [S,$\bar{S}$] edge cut with 0 capacity (since the algorithm is initialized with every flow(0)). Thus the maximum flow is also 0.

Question 10. (−) A kitchen sink draws water from two tanks according to the network of pipes with capacities per unit time shown below. Find the maximum flow. Prove that your answer is optimal by using the dual problem, and explain why this proves optimality.