Positive Definite Matrices

Thursday, June 11, 2020 15:34

Symmetric Positive Definite S

In green: modification for +ve SEMI-definite.

1. All $\lambda_i \geq 0$
2. Energy $x^T S x \geq 0$ (all $x \neq 0$)
3. $S = A^T A$ (independent cols in A) (could be dep)
4. All leading determinants ≥ 0
5. All pivots in elimination > 0

Essentially, PD matrices are symmetric matrices with +ve eigenvalues. If a matrix has any of the above 5 properties, it is PD and has all 5 properties.

→ Recall that $\sum_i \lambda_i = \text{Trace}$ and $\underbrace{\prod_i \lambda_i = \text{determinant}}_{\text{connects (1) and (4)}}$

→ What does all leading determinants mean?
It means determinant of all square sub-matrices with top left corner fixed. Total n such determinants if the matrix is $n \times n$.

→ Pivots. E.g. for $\begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$, pivots are 3, 2/3

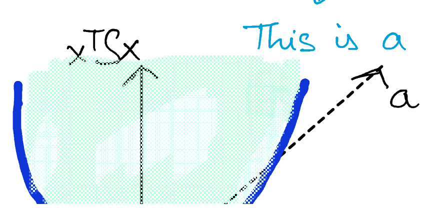
$\begin{bmatrix} 3 & 4 \\ 0 & 2/3 \end{bmatrix}$ → All pivots +ve. Actually, 2^{nd} pivot $= \dfrac{2\times 2 \text{ det}}{1 \times 1 \text{ det}} = \dfrac{2}{3}$

→ Define Energy in a vector x for matrix S $= \underbrace{x^T S x}_{\text{This is a scalar}}$

e.g. $x = \begin{bmatrix} a \\ b \end{bmatrix}$, $S = \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}$

$f = x^T S x = 3a^2 + 6b^2 + 8ab$

If we plot this for (a, b):



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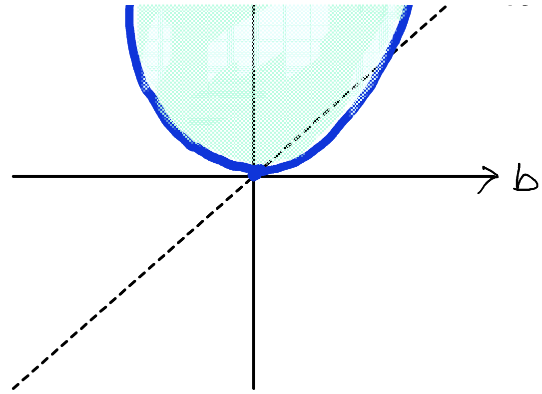


So energy > 0

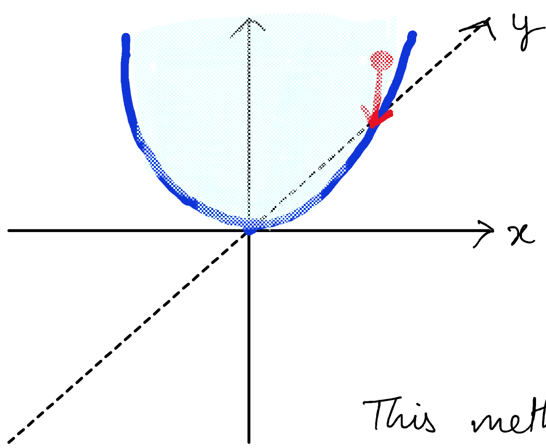
This is what deep learning is about, where something like this energy could be a loss function.

What if we have $x^T S x + \overbrace{x^T C}^{\text{A linear term.}} = f$

The plot would still be a bowl, but shifted.

In both the cases, f is convex, but that may not be true always.

→ <u>How to find the min?</u>



→ Start at some point.

Find:

→ $\frac{df}{dx}, \frac{df}{dy}$ (gradients of f) ↳ ∇f

→ Follow the gradient

→ Recalculate gradients at new point and follow it.

This method is called gradient descent.

Q: Good question to ask: **Does this work well? How well?**

A: Not always (we see more of this later).

E.g. exercise: If S, T are PD, is $S+T$ also PD?

→ We don't want to use the +ve λ test (i) as evalues of S & T don't say much about evalues of $S+T$.

→ Look at the energy:

$$\underbrace{x^T(S+T)x}_{+ve} = \underbrace{x^T S x}_{+ve} + \underbrace{x^T T x}_{+ve} \Rightarrow \underline{\text{YES}}$$

If S is PD, is S^{-1} PD?
YES, as eigenvalues of S^{-1} are $\frac{1}{\lambda_i}$ for λ_i eigenvalues of S.

What about $Q^T S Q$?
Here Q is orthogonal and $Q^T S Q$ is symmetric.
$\hookrightarrow Q^T = Q^{-1}$
S is PD.

$\therefore \ M = Q^{-1} S Q$ is similar to S

$\Rightarrow$ M and S have same eigenvalues $\Rightarrow$ M is PD

Another way: energy.

$x^T Q^T S Q x$. Let $Qx = y$

$\Rightarrow$ $y^T S y$ which is +ve. So, <u>YES</u>

→ **What is Positive SEMI-definite?**

e.g. $\begin{bmatrix} 3 & 4 \\ 4 & 16/3 \end{bmatrix}$ det $= 0$, $\lambda_i \geq 0$

→ Consider matrix $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

rank$(M) = 1$ $\Rightarrow$ there is only 1 non-0 eigenvalue

$\Rightarrow$ There are total 3, 2 of them are 0

$\Rightarrow$ Trace tells us 3rd one is 3.

$\lambda_i = 3, 0, 0.$

Express M as $A^T A$. Since M is rank-1, A^T will be just a vector. $A = [1 \ 1 \ 1]$.

We can also see $M = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \lambda_3 q_3 q_3^T$

$\lambda_1 = 3, \ \lambda_2 = \lambda_3 = 0$

$M = 3 q_1 q_1^T \qquad q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ (normalized eigenvector).