Factorization

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Matrix factorizations

→ $A = \underbrace{L\ U}$ lower & upper triangular (for square matrices)
elimination, solving linear systems

→ $A = QR$ Q: orthonormal columns, Gram-Schmidt

→ $S = Q \Lambda Q^T$ S: symmetric matrices

Eigenvectors
Diagonal eigenvalue matrix

$$Q = \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} q_1^T \\ \vdots \\ q_n^T \end{pmatrix}$$

Fact: Eigenvectors are orthogonal for a symmetric matrix.

$(Q\Lambda)Q^T$

$\Rightarrow \sum (\text{col of } Q\Lambda) \times (\text{row of } Q^T)$

$= \sum [\ ] [\qquad]$ a matrix like that is rank 1

$\Rightarrow$ Sum of rank 1 matrices

$Q\Lambda$'s 1st column $= \lambda_1 q_1$

$\Rightarrow$ Sum of $\lambda_1 q_1 q_1^T \cdots$

$= \sum_i \lambda_i q_i q_i^T$ (spectral theorem)

Break a symmetric matrix S in rank 1 pieces

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots \lambda_n q_n q_n^T$$

Verify: $Sq_1 = \lambda_1 q_1$ why? → Because eigenvectors are orthonormal
$Sq_2 = \lambda_2 q_2$

→ $A = U \Sigma V^T$ — singular value decomposition (SVD) (works for every matrix)
Σ: Diagonal
U, V: Orthogonal

→ $A = X \Lambda X^{-1}$