Eigenvalues and Eigenvectors

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Overview : Square matrices $\rightarrow$ symmetric matrices $\rightarrow$ positive definite

$Ax_i = \lambda_i x_i$. Let A be an $n \times n$ matrix $(i = 1, 2, \ldots, n)$
Normally, there will be n independent vectors x that have this property (called eigenvectors).

The key property of evectors is seen by looking at A^2

$$A^2 x = A(Ax) = \lambda^2 x$$

$\Rightarrow$ x is an evector of A^2 also with squared eigenvalue.

$$A^n x = \lambda^n x$$

$\Rightarrow$ $A^{-1} x = \frac{1}{\lambda} x$, $\lambda \neq 0$. If $\lambda = 0$, A^{-1} does not exist.

The null space has something non-0, so the matrix is not invertible. $\Leftarrow$ bcoz this means $Ax = 0$ for some x

$\Rightarrow$ $e^{At} x = e^{\lambda t} x$ for eigenvectors x
(look at series expansion of e^t)

Assume A has n indep evectors (a basis)
So, any $v = \sum_i c_i x_i$

$$v_k = A^k v = \sum_i c_i \lambda^k x_i$$

$v_{k+1} = A v_k$ (1-step difference eq^n)

Principal use of eigenvectors.

$$\frac{dv}{dt} = Av \Rightarrow v = c_1 e^{\lambda_1 t} x_1 + \cdots + c_n e^{\lambda_n t} x_n$$

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Idea of Similarity

A matrix B is called similar to A if

$\boxed{B = M^{-1} A M}$, where M is any invertible matrix.

Key fact: Similar matrices have same eigenvalues.

So, to calculate eigenvalues, something like MATLAB will use a seq of matrices $M_1, M_2, \dots$, and bring A to B_i, where $B_i = M_i^{-1} B_{i-1} M_i$ and $B_0 = A$, s.t. the eigenvalues show up on the diagonal of B_i

Proof of the fact:

Note: eigenvalues of similar matrices are same BUT eigenvectors may not be.

$B = M^{-1} A M$

Let y be an evector of B with evalue λ.

$\Rightarrow \quad M^{-1} A M y = \lambda y$

Now we want to show that λ is an evalue of A also

Multiply M on both sides on the left:

$$A(My) = M\lambda y = \lambda (My)$$

Another fact:

$\rightarrow$ if we have 0, B^{-1} does not exist

AB has the same non-0 evalues as BA

Proof:

Find an M s.t. $M(AB)M^{-1} = BA$

Take $M = B$

$$\Rightarrow \quad M(AB)M^{-1} = BA$$

Take $M = B$

$$\Rightarrow \quad M(AB)M^{-1} = BA$$

Note: Although obvious, but still important:

evalues of AB are not evalues of A * evalues of B.
evalues of $A+B$ are not evalues of A + evalues of B.
→ in general.

<u>Fact</u>: add λ's = add diagonals (trace)

(see that roots of the characteristic polynomial are eigenvalues)

\# S is a symmetric matrix.

→ Eigenvalues are real if the matrix is.
(not true for real matrices in general)

e.g. $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (rotation by $\frac{\pi}{2}$), so there's no vector that doesn't change direction.

→ Eigenvectors are orthogonal (look at first page in this section for proof)

→ Even if some eigenvalues are repeated, we may have a full set of eigenvectors (e.g. the identity matrix I)

E.g. $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \lambda = -1, 1$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Let $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ → the eigenvalue matrix.

S and Λ are similar (same evalues)

Now $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = M^{-1} S M \Rightarrow SM = M\Lambda$

$\Rightarrow$ M is the eigenvector matrix!

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

More to see here:

$$S\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} Sx_1 & Sx_2 \end{bmatrix} = \begin{bmatrix} x_1 & -x_2 \end{bmatrix}$$

$$\Rightarrow \begin{array}{ll} Sx_1 = \lambda_1 x_1 & \Rightarrow \lambda_1 = 1 \\ Sx_2 = \lambda_2 x_2 & \Rightarrow \lambda_2 = -1 \end{array}$$

Conclusion: (not only for symmetric matrices)

A : $\lambda_1, \dots, \lambda_n$ then
$x_1, \dots, x_n$

$$A\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\Rightarrow AX = X\Lambda$$
$$A = X\Lambda X^{-1}$$

$$A^2 = X\Lambda X^{-1} X\Lambda X^{-1} = X\Lambda^2 X^{-1}$$

$\Rightarrow$ eigenvectors don't change, eigenvalues squared.

If the matrix is symmetric, $A = S$, then eigenvectors are orthogonal, $X = Q$

$$S = Q\Lambda Q^{-1}$$ (Spectral theorem)

We usually normalize Q, and have

$$S = Q\Lambda Q^T \quad \text{as} \quad Q^{-1} = Q^T$$

This is because orthonormal colums of Q means $QQ^T = I$.