Extension: Existence

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Existence and extension of probability measures.

Members in a σ-field are typically hard to describe
$\Rightarrow$ It is hard to directly describe a probability measure on a σ-field.

Idea: First define it on a smaller / simpler collection of sets then extend it to the σ-field.

Define a class $\mathcal{F}_0$ of subsets of Ω as a field:
(1) $\Omega \in \mathcal{F}_0$
(2) $A \in \mathcal{F}_0 \Rightarrow A^c \in \mathcal{F}_0$
(3) $A, B \in \mathcal{F}_0 \Rightarrow A \cup B \in \mathcal{F}_0$

(this is almost a σ-field, but not quite.)

$\hookrightarrow$ Closed under finite union, not countable union.

Theorem.

Let $\mathcal{F}_0$ be a field on Ω and $\mathbb{P}$ be a set f^n defined on $\mathcal{F}_0$ s.t. it satisfies the probability axioms (on $\mathcal{F}_0$).
Note that $\mathbb{P}$ is not yet a probability measure as it's defined on $\mathcal{F}_0$ which is not a σ-field.
(1) $0 \leq \mathbb{P}(A) \leq 1 \quad \forall A \in \mathcal{F}_0$
(2) $\mathbb{P}(\phi) = 0$, $\mathbb{P}(\Omega) = 1$
(3) $A_1, \ldots$ disjoint $\mathcal{F}_0$-sets:

$$\underbrace{\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}_0}_{\text{this is not reqd if } \mathbb{P} \text{ is a probability measure bcoz then } \mathcal{F}_0 \text{ would be a } \sigma\text{-field and the union will} \in \mathcal{F}_0 \text{ by definition. Here, } \mathcal{F}_0 \text{ is just a field, not a } \sigma\text{-field.}} \Rightarrow \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mathbb{P}(A_k)$$

Then, $\exists$ a unique prob measure $\mathbb{Q}$ on $\sigma(\mathcal{F}_0)$ s.t. $\mathbb{Q}(A) = \mathbb{P}(A) \quad \forall A \in \mathcal{F}_0$.
$\Rightarrow$ The extension exists and is unique.

Proof.

Outer measure: $\boxed{\mathbb{P}^*(A) := \inf \sum_n \mathbb{P}(A_n)}$, where

$\underbrace{A_1, A_2, \dots}$ $\in \mathcal{F}_0$, $A \subseteq \bigcup_n A_n$

So these sets cover A.

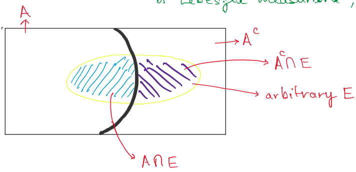
A set $A \subseteq \Omega$ is called $\mathbb{P}^*$-measurable if:

$$\mathbb{P}^*(A \cap E) + \mathbb{P}^*(A^c \cap E) = \mathbb{P}^*(E)$$

for all $E \subseteq \Omega$

[called the Caratheodory's criterion]

used to check if a set is Lebesgue measurable, in real analysis.



We try to cover (and thus approximate) $A \cap E$ and $A^c \cap E$ as well. If those parts add up to parts that cover E, then A is $\mathbb{P}^*$ measurable.

Think about this while looking at the definit^n of the outer measure $\mathbb{P}^*$.

So if $\mathbb{P}^*(A \cap E) + \mathbb{P}^*(A^c \cap E) = \mathbb{P}^*(E)$, then that means we may not estimate E very well but we can estimate A really well.

→ Let $\mathcal{M}$: the class of all $\mathbb{P}^*$-measurable sets.

Proposition: $\mathcal{M}$ is a σ-field and $\mathbb{P}^*$ is countably additive on $\mathcal{M}$. (Proof included in Durrett.)

Other properties for $\mathbb{P}^*$ to be a prob. measure are easy to verify:

(1) $\mathbb{P}^*(A) \geq 0$ by def^n $= \inf \sum_n \mathbb{P}(A_n)$

(2) $\Omega \in \mathcal{F}_0$

$$\Rightarrow \quad \mathbb{P}^*(\Omega) = \mathbb{P}(\Omega) \quad \text{(covers itself as it's } \in \mathcal{L}_0\text{)}$$
$$= 1$$

(3) Countable additivity by the proposition.

Thus, $\boxed{\mathbb{P}^* \text{ is a probability measure on } \sigma\text{-field } \mathcal{M}}$.

→ What we need now is a relationship b/w $\mathcal{M}$ and $\sigma(\mathcal{L}_0)$ in order to prove the **theorem** above.

Proposition: $\mathcal{L}_0 \subseteq \mathcal{M}$

Since $\mathcal{M}$ is a σ-field, we have that $\sigma(\mathcal{L}_0) \subseteq \mathcal{M}$

Proof: Let $A \in \mathcal{L}_0$. Want to show that $A \in \mathcal{M}$ → class of $\mathbb{P}^*$ measurable sets

So we need to show that for any E,

$$\mathbb{P}^*(A \cap E) + \mathbb{P}^*(A^c \cap E) = \mathbb{P}(E)$$

For any $\varepsilon > 0$, let $A_1, A_2, \dots \in \mathcal{L}_0$ s.t.

$E \subseteq \bigcup_n A_n$ (the seq forms a cover of E), and

$$\underbrace{\sum_n \mathbb{P}(A_n)}_{} \leq \mathbb{P}^*(E) + \varepsilon$$

→ Such a cover always exists as $\mathbb{P}^*$ is the inf, so $\exists$ some $A_1, A_2, \dots$ that achieve it.

Now define two seq of events:

→ $B_n = \underbrace{A_n \cap A}_{} \in \mathcal{L}_0$ → Both $A_n, A \in \mathcal{L}_0$ and $\mathcal{L}_0$ is a field.

→ $C_n = A_n \cap A^c \in \mathcal{L}_0$

Moreover, $E \cap A \subseteq \bigcup_n B_n$

$E \cap A^c \subseteq \bigcup_n C_n$

$$\Rightarrow \quad \mathbb{P}^*(E \cap A) \leq \mathbb{P}^*\left(\bigcup_n B_n\right) \leq \sum_n \mathbb{P}(B_n)$$
$$\mathbb{P}^*(E \cap A^c) \leq \sum_n \mathbb{P}(C_n)$$

$$\Rightarrow \quad \mathbb{P}^*(E \cap A) + \mathbb{P}^*(E \cap A^c) \leq \sum_n \mathbb{P}(B_n) + \sum_n \mathbb{P}(C_n)$$

By the choice of A_n, we have that:

$$= \sum_n \mathbb{P}(B_n \cup C_n) \quad \text{as } \mathbb{P} \text{ is additive on } \mathcal{L}_0$$
$$= \sum_n \mathbb{P}(A_n)$$

$\sum_n \mathbb{P}(A_n) \leq \mathbb{P}^*(E) + \varepsilon$

$$\Rightarrow \quad \mathbb{P}^*(E \cap A) + \mathbb{P}^*(E \cap A^c) \leq \mathbb{P}^*(E) + \cancel{\varepsilon}$$

$\because$ this holds for all $\varepsilon > 0$, we can take the limit

On the other hand, we trivially have:

$$\mathbb{P}^*(E \cap A) + \mathbb{P}^*(E \cap A^c) \geq \mathbb{P}^*(E)$$ $\rightarrow$ as this is infimum.

$$\Rightarrow \quad \mathbb{P}^*(E \cap A) + \mathbb{P}^*(E \cap A^c) = \mathbb{P}^*(E)$$

We have this $\forall E$, $\therefore$ A is $\mathbb{P}^*$-measurable. $\therefore$ all $A \in \mathcal{L}_0$ are $\mathbb{P}^*$-measurable. $\Rightarrow \mathcal{L}_0 \subseteq \mathcal{M}$

contains $\mathcal{L}_0$ and is a σ-field.

As a result, $\sigma(\mathcal{L}_0) \subseteq \mathcal{M}$

smallest σ-field containing $\mathcal{L}_0$

$\Rightarrow \mathbb{P}^*$ is also a prob. measure on $\sigma(\mathcal{L}_0)$.

Moreover, $\mathbb{P}^*(A) = \mathbb{P}(A) \quad \forall A \in \mathcal{L}_0$ (can cover itself)

$\Rightarrow \mathbb{P}^*$ is an extension of $\mathbb{P}$ on $\sigma(\mathcal{L}_0)$

This proves the existence of the extension. $\square$

Next we'll look at the uniqueness of this extension.