Orthogonal Matrices

Wednesday, June 10, 2020 17:00

Orthonormal columns in Q, then

$$Q^T Q = I$$

Is $QQ^T = I$? $\longrightarrow$ Yes but only when Q is square

In this square case, Q is called an orthogonal matrix.

$\rightarrow$ E.g. $\Rightarrow$ $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\longrightarrow$ rotation by θ

Claim: for any vector x, $\|Qx\| = \|x\|$ (length doesn't change)

Proof:

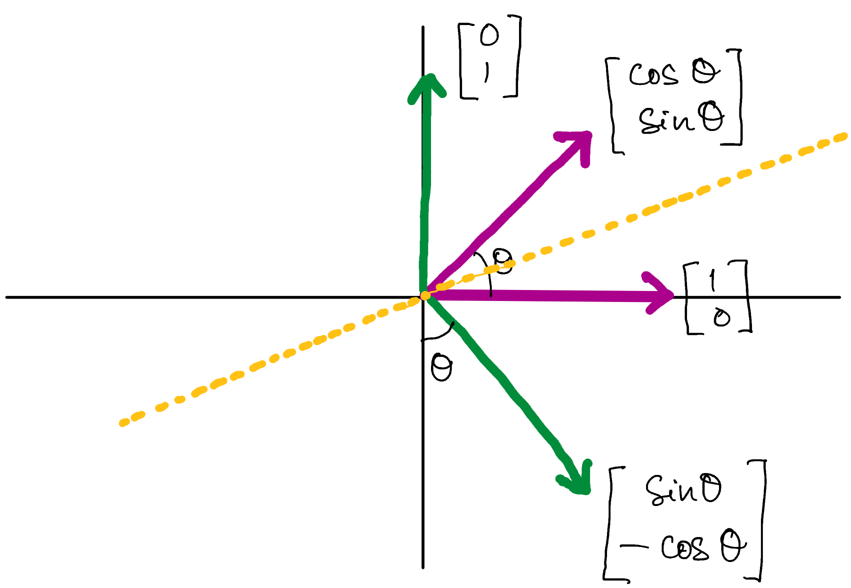
$$\|x\|^2 = x^T x \,, \quad \|Qx\|^2 = (Qx)^T (Qx)$$
$$= x^T Q^T Q x$$
$$= x^T I x$$
$$= x^T x = \|x\|^2$$

Here, Q is an Orthogonal matrix (orthonormal really)

$\Downarrow$

$\boxed{Q^T = Q^{-1}}$ $\longrightarrow$ important property

$\rightarrow$ Another e.g. $Q = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ $\longrightarrow$ NOT a rotation matrix, but a reflection matrix.



These are called "Householder reflections"

→ **Householder reflections**

Start with a unit vector, $u^T u = 1$

$H = I - 2uu^T$

↳ a family of symmetric orthogonal matrices.

To check orthogonal, see that $H^T H = I$

$\Rightarrow \quad H^T H = H^2$ as H is symmetric

$\Rightarrow \quad I - 4uu^T + 4u\underbrace{(u^T u)}_{=1}u^T$

$\Rightarrow \quad I - 4uu^T + 4uu^T = I$

→ **Hadamard**

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

These are not orthogonal matrices yet, we need to have unit vectors, so divide by $\sqrt{2}$ in H_2, 2 in H_4

This generates H_n for $n = 2^k$

Does something like H_{12} exist? Yes

Conjecture

⇒ Always possible if $n/4$ is a whole number

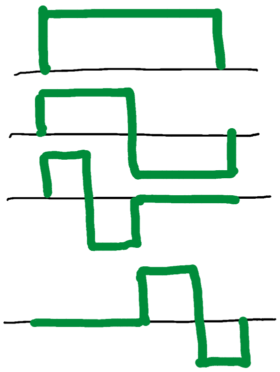
\# Where will we sort of automatically show up with orthogonal vectors? → Eigenvectors of a symmetric matrix.

Wavelet

e.g. for $n = 4$

$\begin{bmatrix} 1 & 1 & 1 & 0 \end{bmatrix}$

e.g. for $n=4$



$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

Wavelets are self-scaling → See pattern of W_8

diff. between Hadamard and Haar wavelet.

name of the guy (1910)

$$W_8 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

Ingrid Daubechies (1988) found a whole lot of families of wavelets, and hence, orthogonal matrices with nice properties.

Eigenvectors of a symmetric matrix ($S^T = S$), and those of an orthogonal matrix ($Q^TQ = I$) are orthogonal.

Eigenvectors of $Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ are 4-discrete Fourier transform (and orthogonal)

evector matrix of Q is F_4

First eigenvector of Q is of course $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & i^2 & i^3 \\ 1 & i^2 & i^4 & i^6 \\ 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & i^3 & i^6 & i^9 \end{bmatrix}$$

Note: To check orthogonality now, we check $\overline{col1} \cdot col2 = 0$ where $\overline{col1}$ is the complex conjugate of col1.

Why? $\Rightarrow$ The usual definition is not sensible for complex vectors. The dot product of a vector with itself could give 0 for non-0 vectors (and other similar problems).

We have $a \cdot b = \overline{b \cdot a}$

Here $a \cdot b = \sum_i a_i \overline{b_i}$ instead of $\sum_i a_i b_i$

Angle b/w two complex vectors, $\cos\theta = \dfrac{Re(a \cdot b)}{\|a\| \, \|b\|}$

So, $\boxed{\begin{array}{l} \text{if } \; Q^T Q = I, \\ \quad Q x = \lambda x \\ \quad Q y = \mu y \\ \\ \text{then } \; \bar{x}^T y = 0 \end{array}}$ (means Q is an orthogonal matrix)

$\Rightarrow$ eigenvectors with diff eigenvalues ($\lambda \neq \mu$)