From last week: Integration of $\underbrace{\text{rational fractions}}_{\frac{P(x)}{Q(x)}}$

- method:
  partial fraction decomposition

$$\frac{P(x)}{Q(x)} = \sum \left(\underbrace{\text{rat. func.}} \text{ in basic types} \right)$$

$$\int \frac{P}{Q} = \sum \int \text{b. r. f.}$$

b, $\displaystyle\int \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx$

1. $\deg(P)=2 < 3=\deg(Q)$

2. $Q(x)=x^3+2x^2-3x=$

$$=x(x^2+2x-3)=\cancel{x}(x+3)(x-1)$$

3. $\displaystyle\frac{P(x)}{Q(x)}=\frac{A}{x}+\frac{B}{x+3}+\frac{C}{x-1}$ $\quad A, B, C=?$

$$\frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1} = \frac{A(x+3)(x-1) + Bx(x-1) + Cx\cdot(x+3)}{x(x+3)(x-1)} =$$

$$= \frac{A(x^2+2x-3) + B(x^2-x) + C\cdot(x^2+3x)}{x(x+3)(x-1)}$$

$$= \frac{(A+B+C)x^2 + (2A-B+3C)\cdot x - 3A}{x(x+3)(x-1)} =$$

$$= \frac{4x^2+13x-9}{x(x+3)(x-1)} \iff (A+B+C)x^2 + (2A-B+3C)x - 3A$$

$$\begin{array}{ccccc} \| & & \| & & \| \\ 4x^2 & + & 13x & & -9 \end{array}$$

$$\begin{cases} A + B + C = 4 \\ 2A - B + 3C = 13 \\ \quad -3A = -9 \end{cases} \longrightarrow \boxed{A = 3}$$

$$3 + B + C = 4 \rightarrow B + C = 1$$

$$B = 1 - C$$

$$\underbrace{2 \cdot 3}_{6} - (1 - C) + 3C = 13 \longrightarrow \boxed{\begin{matrix} C = 2 \\ B = -1 \end{matrix}}$$

$$4C = 8$$

$$\frac{P(x)}{Q(x)} = 3 \cdot \frac{1}{x} - \frac{1}{x+3} + 2 \cdot \frac{1}{x-1}$$

$$\int \frac{P}{Q} = 3 \cdot \int \frac{1}{x}\,dx - \int \frac{1}{x+3}\,dx + 2 \cdot \int \frac{1}{x-1}\,dx =$$

$$= 3 \cdot \ln|x| - \ln|x+3| + 2 \cdot \ln|x-1| + C$$

$$\int \frac{P}{Q}$$

$$\deg(P) \geq \deg(Q)$$

$$\Rightarrow P(x) = T(x) \cdot Q(x) + P^{*}(x) \quad \text{s.t.} \quad \deg(P^{*}) < \deg(Q)$$

$$\int \frac{P(x)}{Q(x)} = \int \frac{T(x) \cdot \cancel{Q(x)}}{\cancel{Q(x)}} + \frac{P^{*}(x)}{Q(x)} =$$

$$= \int T(x) + \frac{P^{*}(x)}{Q(x)} \, dx \quad)$$

c) $\displaystyle\int \frac{1}{x^3+2x^2+x}\,dx$

① $\deg(P) < \deg(Q)$

② $Q(x) = x\left(x^2+2x+1\right) = \overbrace{x}\,\overbrace{(x+1)^2}$

③ $$\frac{P(x)}{Q(x)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + Bx\cdot(x+1) + C\cdot x}{x(x+1)^2} =$$

$$= \frac{(A+B)x^2 + (2A+B+C)x + A}{x(x+1)^2} = \frac{1 = 0\cdot x^2 + 0\cdot x + 1}{x(x+1)^2}$$

$$\Updownarrow$$

$$\begin{cases} A+B=0 \\ 2A+B+C=0 \\ \boxed{A=1} \end{cases} \to \boxed{B=-1}$$

$$2-1+C=0 \to \boxed{C=-1}$$

$$\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}$$

$$\| \quad \frac{P}{Q}$$

$$\left(\int \frac{1}{(x+1)^2}dx = \int (x+1)^{-2}dx\right)$$

$$\int f^{\alpha}\cdot f' = \frac{f^{\alpha+1}}{\alpha+1} + C$$

$$\to \int \frac{1}{x(x+1)^2}dx = \ln|x| - \ln|x+1| - \frac{(x+1)^{-1}}{-1} + C$$

Integration by parts

Th: $\int f \cdot g' = f \cdot g - \int f' \cdot g$

$$\left(\rightsquigarrow \text{product rule w.r.t differentiation} \right.$$

$$\left. \int \underbrace{f \cdot g' + f' \cdot g}_{(f \cdot g)'} = f \cdot g + C \right)$$

1. a, $\int x \cdot e^{2x}\, dx$

type I: $P(x) \cdot T(ax+b)$, where $P(x)$ is a polynom,

$T \in \{\exp, \cos, \sin, \sinh, \cosh\}$

$a, b \in \mathbb{R}$

$$\int x \cdot e^{2x}\, dx = x \cdot \frac{1}{2} \cdot e^{2x} - \int 1 \cdot \frac{e^{2x}}{2}\, dx =$$

$f(x) = x \quad f'(x) = 1$

$g'(x) = e^{2x} \quad g(x) = \frac{e^{2x}}{2} \quad \left(\int e^{2x}\, dx\right)$

$$= \frac{x}{2}e^{2x} - \frac{1}{2}\underbrace{\int e^{2x}\,dx}_{\frac{e^{2x}}{2} + C} =$$

$$= \left(\frac{x}{2} - \frac{1}{4}\right)e^{2x} + C$$

б) HW

type II. integrand = $P(x) \cdot G(ax)$, $a \in \mathbb{R}$

$P(x)$ is a polynomial, $G \in \{\ln, \text{arcus}\ldots, \text{area}\ldots\}$

$f(x) := G(ax)$ ($\rightarrow f'$ is a rational func.)

$g'(x) = P(x)$

d, $\int 1 \cdot \ln(x)\,dx = x \cdot \ln(x) - \int \frac{1}{x} \cdot x\,dx = x \cdot \ln(x) - x + C$

$f(x) := \ln(x)$ $\quad f'(x) = \frac{1}{x}$

$g'(x) := 1$ $\quad g(x) := x$

$$\underline{\underline{= x \cdot \ln(x) - x + C}}$$

e) $\int (x^3 + 2x + 2) \cdot \ln(x) \, dx =$

$f(x) = \ln(x)$ $\qquad$ $f'(x) = \frac{1}{x}$

$g'(x) = x^3 + 2x + 2$ $\qquad$ $g(x) := \frac{x^4}{4} + x^2 + 2x$

$$= \left(\frac{x^4}{4} + x^2 + 2x\right) \ln(x) - \int \frac{x^3}{4} + x + 2 \, dx =$$

$$= \left(\frac{x^4}{4} + x^2 + 2x\right) \cdot \ln(x) - \frac{x^4}{16} - \frac{x^2}{2} + 2x + C$$

$$\int x^2 \cdot \ln(x)\, dx =$$

$f(x) = x^2$

$g'(x) = \ln(x)$ $\qquad g(x)$

$\int \ln(x)\, dx$

$$\int \ln(x) \cdot e^x \, dx = e^x \cdot \ln(x) - \underbrace{\int \frac{e^x}{x}\, dx}_{= \dots \text{ HK}}$$

$f(x) = \ln(x)$ $\qquad f'(x) = \frac{1}{x}$

$g'(x) = e^x$ $\qquad g(x) = e^x$

Substitution

FORM I: $\int f(g(x)) \cdot g'(x)\,dx = F(g(x)) + C$

$F' = f$

FORM II: $\int f(x)\,dx = \int f(g(t)) \cdot g'(t)\,dt$

$x = g(t)$

$t = g^{-1}(x)$

From Lecture Notes, Section 7.3

2/a $$\int \underbrace{\frac{e^x}{\sqrt{1-e^{2x}}}}_{f(e^x)} \, dx$$

$$t = e^x > 0$$

$$x = \ln(t) = g(t) \quad \left(t \in (0,+\infty) \right)$$

$$g'(t) = \frac{1}{t} > 0$$

$$\int \frac{e^x}{\sqrt{1-e^{2x}}} \, dx = \int \frac{\cancel{t}}{\sqrt{1-t^2}} \cdot \frac{1}{\cancel{t}} \, dt =$$

$$= \int \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin(t) + C \Big|_{t=e^x} =$$

b, HW*

$$= \underline{\underline{\arcsin(e^x) + C}}$$

c, $\int \frac{x}{\sqrt{3x+5}}\,dx$

$t = \sqrt{3x+5}$ $\left(x \geq -\frac{5}{3},\ t \geq 0\right)$

$t^2 = 3x+5$

$x = \frac{1}{3}(t^2-5) = g(t),$ $g'(t) = \frac{2}{3}t > 0$

$(0, +\infty)$

$$\int \frac{x}{\sqrt{3x+5}}\,dx = \int \frac{\frac{1}{3}(t^2-5)}{\cancel{t}} \cdot \frac{2}{3} \cdot \cancel{t}\,dt =$$

$$= \int \frac{2}{9} \cdot (t^2 - 5)\,dt =$$

$$= \frac{2}{9} \cdot \left(\frac{t^3}{3} - 5 \cdot t\right) + C =$$

$$t = \sqrt{3x+5}$$

$$= \frac{2}{9} \cdot \left(\frac{1}{3}(3x+5)^{\frac{3}{2}} - 5 \cdot \sqrt{3x+5}\right) + C$$

Definite Integrals

Newton-Leibniz formula:

$$\int_a^b f(x)\,dx = \underbrace{F(b) - F(a)}_{\in \mathbb{R}}, \quad F' = f$$

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2 a, $\int_0^{\frac{\pi}{3}} \sin(8x)\,dx =$

$F = ?$

$$\int \underbrace{\sin(8x)}_{\text{linear subs.}}\, dx = \frac{-\cos(8x)}{8} + C$$

$$\int f(ax+b)\, dx = \frac{F(ax+b)}{a} + C$$

$$\boxed{\begin{aligned} &\cos\left(\frac{8\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2} \\ &\frac{8\pi}{3} = \frac{6\pi + 2\pi}{3} = 2\pi + \frac{2\pi}{3} \end{aligned}}$$

$$\Rightarrow F(x) := -\frac{\cos(8x)}{8}$$

$$\rightarrow \int_0^{\frac{\pi}{3}} \sin(8x)\, dx = F\left(\frac{\pi}{3}\right) - F(0) =$$

$$-\frac{\cos\left(\frac{8\pi}{3}\right)}{8} - \left(-\frac{\cos(0)}{8}\right) =$$

$$= \frac{1}{8}\left(+\frac{1}{2} + 1\right) = \underline{\underline{\frac{3}{2\cdot 8}}} = \frac{3}{16}$$