Grade 7

Algebra

Prepared for KAZAKHSTAN

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Class 7
ALGEBRA

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ББК 22.15 я 72
G 37

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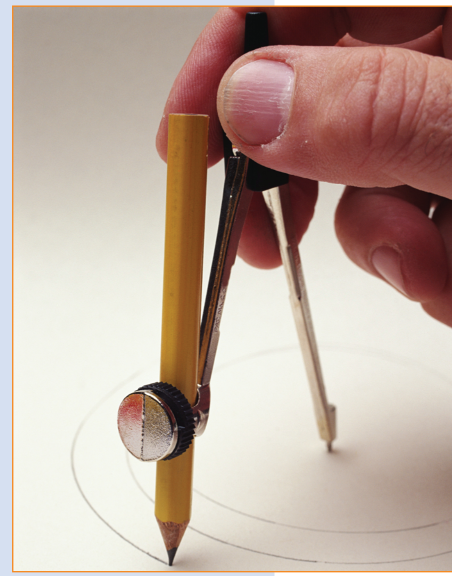
PREFACE

To the Teacher,

Algebra 7 class is designed to provide students to improve and develop their mathematical background needed for further college-level algebra courses.



Mathematics has many branches and makes it easier to solve a wide variety of problems. The goal of this text is to help students develop the skills necessary for solving algebraic problems, and then help students apply these skills. By the end of the book, students will have a good understanding of the analytic approach to solving problems. In addition, we have provided many systematic explanations throughout the text that will help instructors to reach the goals that they have set for their students. As always, we have taken particular care to create a book that students can read, understand, and enjoy, and that will help students gain confidence in their ability to use algebra.

To the Student,

This book consists of five chapters, which cover ***NATURAL EXPONENTS, MONOMIALS AND POLYNOMIALS, FACTORIZATION AND BASIC IDENTITIES, RATIONAL EXPRESSIONS AND APPROXIMATION ERRORS*** respectively. Each chapter begins with basic definitions, theorems, and explanations which are necessary for understanding the subsequent chapter material. In addition, each chapter is divided into subsections so that students can follow the material easily.

Every subsection includes self-test **Check Yourself** problem sections followed by basic examples illustrating the relevant definition, theorem, rule, or property. Teachers should encourage their students to solve Check Yourself problems themselves because these problems are fundemental to understanding and learning the related subjects or sections. The answers to most Check Yourself problems are given directly after the problems, so that students have immediate feedback on their progress. Answers to some Check Yourself problems are not included in the answer key, as they are basic problems which are covered in detail in the preceding text or examples. Giving answers to such problems would effectively make the problems redundant, so we have chosen to omit them, and leave students to find the basic answers themselves.

At the end of every section there are exercises categorized according to the structure and subject matter of the section. **Exercises** are graded in order, from easy (at the beginning) to difficult (at the end). Exercises which involve more ability and effort are denoted by one or two stars. In addition, exercises which deal with more than one subject are included in a separate bank of mixed problems at the end of the section. This organization allows the instructor to deal with only part of a section if necessary and to easily determine which exercises are appropriate to assign.

EXERCISES 1.1

A. Analytic Analysis of Points

1. Plot the following points in the coordinate pla

a. $A(2, 3)$ b. $B(-3, 1)$ c. $C(-3, 2)$
d. $D(5, -3)$ e. $E(0, -4)$ f. $F(-3, 0)$

Every chapter ends with three important sections.

Finally, a **Chapter Review Test** section consists of three tests, each with sixteen carefully-selected problems. The first test covers primitive and basic problems. The second and third tests include more complex problems. These tests help students assess their ability in understanding the coverage of the chapter.

CHAPTER REVIEW TEST 1A

1. What is the length of the median passing throu the vertex A of a triangle ABC with vertices $A(4, 7$ $B(-1, 2)$, and $C(3, 4)$?

A) 5 B) 6 C) 7 D) 8 E)

The answers to the exercises and the tests are given at the end of the book so that students can compare their solution with the correct answer.

Each chapter also includes some subjects which are denoted as **optional**. These subjects complement the topic and give some additional information. However, completion of optional sections is left to the discretion of the teacher, who can take into account regional curriculum requirements.

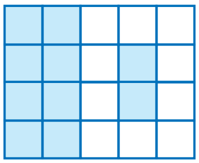
CONTENTS

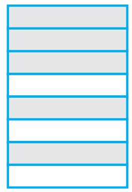
6TH GRADE REVIEW EXERCISES

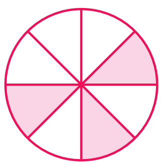
1. Write each shaded area as a fraction.

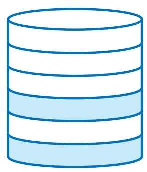
a. 

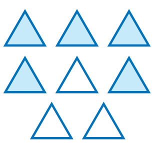
b. 

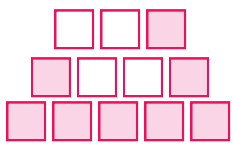
c. 

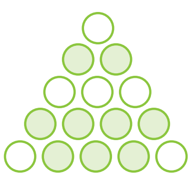
d. 

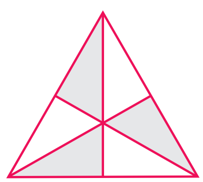
e. 

f. 

g. 

h. 

i. 

j. 

2. Write each mixed number as an improper fraction.

a. $4\frac{14}{5}$ **b.** $100\frac{1}{4}$ **c.** $5\frac{3}{5}$ **d.** $2\frac{27}{6}$

3. Write one word to complete each statement.

a. A ______________ is a part of a whole thing or a part of a group of things.

b. The number above the fraction bar is called the ______________ .

c. The number below the fraction bar is called the ______________ .

4. Order each set of rational numbers.

a. $\frac{9}{8}, \frac{7}{8}, \frac{5}{8}, \frac{4}{8}$ **b.** $\frac{15}{22}, \frac{17}{22}, \frac{13}{22}, \frac{5}{22}$

c. $\frac{11}{5}, \frac{11}{7}, \frac{11}{9}, \frac{11}{4}$ **d.** $\frac{35}{37}, \frac{35}{39}, \frac{35}{30}, \frac{35}{25}$

e. $\frac{16}{16}, \frac{12}{13}, \frac{13}{14}, \frac{17}{12}$ **f.** $\frac{61}{12}, \frac{6}{14}, \frac{2}{4}, \frac{16}{24}$

5. Compare each pair of rational numbers.

a. $\frac{2}{5}$ and $\frac{4}{18}$ **b.** $1\frac{3}{5}$ and $1\frac{4}{7}$

c. $-\frac{6}{9}$ and $-\frac{7}{10}$ **d.** $-1\frac{1}{7}$ and $-1\frac{2}{9}$

6. Add the rational numbers.

a. $\frac{5}{7}+\frac{6}{5}$ **b.** $\frac{1}{2}+\frac{1}{5}+\frac{3}{10}$

c. $\frac{3}{8}+\frac{5}{6}+\frac{11}{12}$ **d.** $2\frac{2}{3}+2\frac{1}{3}$

e. $-\frac{11}{15}+\frac{5}{6}+\frac{3}{5}$ **f.** $\frac{3}{8}+\left(-2\frac{1}{4}\right)+(-1)$

g. $4+\frac{2}{3}+\frac{5}{9}+\left(-\frac{8}{4}\right)$ **h.** $1\frac{2}{3}+3\frac{4}{12}+5\frac{6}{18}$

7. Find the differences.

a. $\frac{5}{7}-\frac{3}{5}$ **b.** $\frac{1}{2}-\frac{1}{5}-\frac{3}{10}$

c. $\frac{3}{8}-\frac{5}{6}-\frac{11}{12}$ **d.** $3\frac{1}{2}-2\frac{1}{3}$

e. $\left(-\frac{11}{15}\right)+\frac{5}{6}-\frac{3}{5}$ **f.** $\frac{3}{8}-2\frac{1}{4}-2$

g. $4-\frac{2}{3}-\frac{5}{9}-\frac{8}{4}$ **h.** $1\frac{2}{3}+3\frac{4}{12}-5\frac{6}{18}$

8. Find x in each equation.

a. $\frac{3}{5}-x=\frac{5}{9}-\frac{1}{4}$ b. $\left(-\frac{5}{12}\right)+x=\frac{6}{12}$

c. $\frac{2}{3}-\frac{7}{8}=x+\frac{2}{3}$

d. $\left(-2\frac{1}{3}\right)-\left(1\frac{1}{5}-\frac{1}{7}\right)=\left(\frac{7}{3}-x\right)$

e. $\left(\frac{1}{5}-x\right)-\frac{7}{6}=\frac{2}{5}-\left(\frac{6}{7}-\frac{1}{5}\right)$

f. $\left(\frac{4}{6}-x\right)-\frac{5}{7}=\frac{3}{5}-\left(\frac{6}{7}-\frac{2}{5}\right)$

9. Find the products and simplify if necessary.

a. $\frac{3}{5}\cdot\frac{7}{11}$ b. $\frac{6}{5}\cdot\frac{10}{12}$ c. $\frac{22}{45}\cdot\frac{5}{11}$

d. $\frac{15}{22}\cdot\frac{7}{9}$ e $3\frac{2}{4}\cdot 5\frac{1}{6}$

f. $\left(-2\frac{3}{4}\right)\cdot\left(-5\frac{2}{7}\right)$ g. $\left(-3\frac{1}{2}\right)\cdot\left(-2\frac{1}{4}\right)$

10. Find x in each equation.

a. $\frac{5}{7}\cdot\frac{9}{19}=\frac{9}{19}\cdot x$ b. $\frac{18}{11}\cdot\left(\frac{3}{4}\cdot\frac{5}{7}\right)=\left(\frac{18}{11}\cdot\frac{3}{4}\right)\cdot x$

c. $\left(\frac{4}{12}\cdot\frac{9}{31}\right)+\left(\frac{4}{12}\cdot\frac{11}{31}\right)=\frac{4}{12}\cdot x$

d. $\left(\frac{4}{9}\cdot\frac{19}{17}\right)+\left(\frac{4}{9}\cdot\frac{15}{17}\right)+\left(\frac{5}{9}\cdot\left(-\frac{31}{47}\right)\right)+\left(\frac{5}{9}\cdot\left(-\frac{16}{47}\right)\right)=x$

11. Perform the operations.

a. $\frac{3}{2}\cdot\frac{4}{3}:\frac{7}{6}$ b. $\frac{2}{5}-\frac{4}{3}\cdot 7\frac{1}{2}+\frac{1}{4}$

c. $\left(-\frac{1}{2}\right)^2\cdot\left(-\frac{1}{8}\right)^3$ d. $\left(1-\frac{1}{2}\right)\cdot\left(1+\frac{1}{2}\right)$

e. $1\frac{3}{2}\cdot\frac{2}{3}-\frac{3}{4}:\frac{5}{6}\cdot\left(\frac{3}{2}\right)$ f. $\left[\left(\frac{12}{4}:\frac{16}{28}\right)+\frac{1}{2}\right]:\frac{23}{12}$

12. Divide the rational numbers.

a. $\frac{8}{15}:\frac{16}{25}$ b. $\frac{8}{9}:\frac{5}{81}$ c. $\frac{3}{25}:\frac{24}{125}$

d. $\left(-3\frac{6}{5}\right):\frac{7}{15}$ e. $\frac{\frac{5}{9}}{10}$ f. $\frac{\frac{5}{8}}{\frac{25}{7}}$

g. $3\frac{1}{5}:\left(-\frac{1}{10}\right)$ h. $\left(-2\frac{2}{3}\right):\left(-3\frac{2}{5}\right)$

i. $\left(-\frac{9}{12}\right):\left[\left(-\frac{1}{4}\right):\frac{15}{2}\right]$

j. $\left[\frac{1}{4}:\left(-\frac{3}{8}\right)\right]:\left(-\frac{5}{6}\right):\left(2\frac{1}{7}\right)$

13. Find the value of x in each equation.

a. $x:\frac{4}{3}=\frac{21}{20}$ b. $x:\frac{2}{5}=\frac{15}{16}$

c. $x:\frac{9}{5}=\frac{20}{63}$ d. $x:\frac{7}{3}=\frac{9}{28}$

14. Find the value of each expression.

a. $2-\cfrac{2}{2+\cfrac{1}{2+\cfrac{1}{2}}}$ b. $2+\cfrac{1+\cfrac{1}{1-\cfrac{1}{2}}}{1+\cfrac{1}{3}}$

c. $\left[\cfrac{2-\cfrac{1}{2-\cfrac{1}{2}}}{1-\cfrac{1}{1-\cfrac{1}{2}}}\right]^2:1\frac{7}{9}$ d. $4+\left[1:\cfrac{1}{1+\cfrac{1}{(-2)-\cfrac{2}{5}}}\right]$

e. $\left[2^{-1}+1+\left(2:\frac{1}{2}\right)\cdot 2:\frac{3}{4}\cdot 2\right]$ f. $\cfrac{2}{\cfrac{5}{3}}-\cfrac{\cfrac{2}{5}}{3}$

15. Perform the operations.

a. $\left(1-\frac{1}{2}\right)\cdot\left(1-\frac{1}{3}\right)\cdot\left(1-\frac{1}{4}\right)\cdot\ldots\cdot\left(1-\frac{1}{50}\right)$

b. $\left(1+\frac{1}{2}\right)\cdot\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{4}\right)\cdot\ldots\cdot\left(1+\frac{1}{50}\right)$

c. $\left(1-\frac{9}{2}\right)\cdot\left(1-\frac{9}{3}\right)\cdot\left(1-\frac{9}{4}\right)\cdot\ldots\cdot\left(1-\frac{9}{71}\right)$

d. $\left(3\frac{1}{5}-2\frac{1}{4}\right)^2:\left(\frac{10}{8}\right)^{-1}+\frac{3}{10}$

e. $\dfrac{\frac{2}{3}+\frac{2}{4}}{\frac{2}{5}+\frac{2}{6}}$

f. $\dfrac{\frac{1}{4}\cdot\left(\frac{1}{3}-\frac{1}{5}\right)}{\left(\frac{6}{5}+\frac{1}{2}\right):5\frac{4}{6}}$

g. $2-\dfrac{\frac{1}{4}-\frac{1}{8}+\frac{1}{16}}{\frac{1}{4}+\frac{1}{8}-\frac{5}{16}}$

h. $\dfrac{\left(3-\frac{4}{2}\right)+\left(2+\frac{3}{4}\right)}{\left(2-\frac{2}{3}\right)-\left(1-\frac{7}{6}\right)}$

i. $2+\dfrac{2+\frac{3}{4}}{2-\frac{3}{4}}$

16. $a=\left(2-\frac{1}{2}\right)\cdot\left(2-\frac{1}{3}\right)\cdot\left(2-\frac{1}{4}\right)\cdot\ldots\cdot\left(2-\frac{1}{10}\right)$, and

$b=\left(1+\frac{1}{3}\right)\cdot\left(1+\frac{1}{5}\right)\cdot\left(1+\frac{\ }{\ }\right)\cdot\ldots\cdot\left(1+\frac{1}{19}\right)$.

Find the product of a and b.

17. $1-\dfrac{1-\dfrac{1-\frac{x}{2}}{2}}{2}=1$. Find x.

18. Evaluate the following.

a. $|7-4|$ **b.** $|3-12|$ **c.** $2\times|5-7|$

d. $\big|3\times|6-4|+|4-7|\big|$

e. $-|-5|+|16-20|\times|12-8|$

f. $|4-6|\times\big|4\times|8-5|\big|$

g. $|5-3|-|2-3|\times|-7-1|$

h. $(|-5|-|-6|)\div(-11-|-12|)$

19. Evaluate the following.

a. $\dfrac{|-2002|-|-2003|}{|-2001|-|-2000|}$

b. $|-1-1|-|-1-(-1)|$

20. Write each comparison as a ratio in its simplest form.

a. 25 to 125 b. 3 to 18

c. 42 kg to 60 kg d. 50 gr to 1 kg

e. 20 cm to 3 m f. 15 cm^2 to 2 dm^2

21. A rope which is 120 m long is divided into three pieces in the ratio 3 : 4 : 5. Find the length of each piece.

22. A father shares \$1000 between his four sons in ratio 3 : 5: 7 : 10. How much money does each receive?

23. The difference of two numbers is 12 and their ratio is 13 to 11. Find the smaller number.

24. The lengths of the sides of a triangle are in the ratio 4 : 7 : 9 and its perimeter is 80 cm. Find the length of each side.

25. Find x in each proportion.

a. $\frac{x+2}{x-2}=\frac{3}{2}$ **b.** $\frac{2x+3}{3x+1}=\frac{4}{5}$

c. $\frac{6x-4}{2x+4}=\frac{11}{4}$ **d.** $\frac{x-3}{x}=\frac{x-1}{x+1}$

26. Find x in each equation. Show your working and check your answer.

a. $x + 5 = 0$
b. $x - 4 = 0$
c. $x + 3 = 7$
d. $y - 4 = -2$
e. $\frac{2}{3} - z = \frac{5}{6}$
f. $-\frac{3}{5} - m = -\frac{2}{5}$
g. $n + \frac{5}{6} = \frac{11}{6}$
h. $x - 4 = -8$
i. $2 - x = -7$
j. $\frac{3+x}{3} = -\frac{1}{2}$
k. $\frac{x}{3} = \frac{7}{12}$
l. $\frac{3x}{2} = \frac{9}{2}$
m. $\frac{x}{2} = -7$

27. Find x in each equation.

a. $6x - 14 = 4$
b. $5 - 4x = 13$
c. $6 + 2x = 28$
d. $\frac{3x+7}{2} = 5$
e. $\frac{8-5x}{3} = 7$
f. $\frac{13+6x}{4} = -2$
g. $2 \cdot (3 + x) = 6$
h. $-3 \cdot (2 - x) = 9$
i. $6 \cdot (5x - 3) = 15$
j. $3x + 2 \cdot (1 + 3x) = 17$
k. $-7x - 2 \cdot (5 - 3x) = 10$
l. $8x - (3x - 5) = 15$
m. $6 \cdot (2 + 4x) + 5 \cdot (4 - 3x) = 7x - 19$
n. $7 \cdot (2x + 1) + 12 = 5 \cdot (2x - 2)$

28. Solve each inequality and graph its solution set on a number line.

a. $x - 3 > 6$
b. $5 + x \leq -2$
c. $6 - x < 4$
d. $3x - 7 > 2$
e. $5x - 7 \leq 8$
f. $2x + 5 \geq -1$
g. $9 - 4x < -3$
h. $3x - 1 \geq 3 + x$
i. $2x - 2 \geq 3+x$
j. $2x - 2 < 4x+2$
k. $3 - 3 \cdot (2 - x) \leq -6$
l. $6 - 4 \cdot (x+2) \leq -2$
m. $12 \cdot (x - 2) > 2x - 4$
n. $2 \cdot (3 + 5x) < 8x + 3$
o. $\frac{x+3}{2} < -3$
p. $\frac{5-4x}{3} \geq 1$
q. $\frac{x-3}{2} > x+1$
r. $\frac{x}{2} \geq 2 - \frac{x}{3}$
s. $\frac{3 \cdot (5x-1)}{7} < 6$
t. $\frac{3 \cdot (6+4x)}{2} \leq -2x+5$
u. $\frac{3-x}{4} - \frac{2x+5}{3} > \frac{1}{6}$
v. $\frac{x+3}{5} - \frac{2x+1}{3} \leq 1 + \frac{x-1}{15}$

29. Graph the solution set of each compound inequality over R on a number line.

a. $-2 < x + 1 < 3$
b. $-7 \leq 3x + 2 < 8$
c. $-6 < 2x - 3 \leq 5$
d. $-5 \leq 4 - 3x < 2$
e. $2 < 1 - \frac{x}{3} < 3$
f. $0 < \frac{3x+2}{3} < \frac{1}{3}$
g. $4x + 3 < 5x + 7 < x + 8$
h. $\frac{1}{2} \leq \frac{x+1}{3} < \frac{3x}{4}$

30. Complete the statements.

a. The symbol > means '________________'.

b. The symbol £ means '________________'.

c. If a < b and b < c, then a < c. This property is called the ____________ property of inequality.

d. If both sides of an inequality are multiplied by a ______________ number, the direction of the inequality remains the same.

e. If both sides of an inequality are multiplied by a ______________ number, the direction of the inequality must be reversed.

f. An ______________ is a statement indicating that two quantities are not necessarily equal.

31. Graph the points in a coordinate plane.

a. A(3, 1) b. B(2, 7)

c. C(–3, 1) d. D(–5, 6)

e. E(–6, 0) f. F(–4, –3)

g. G(–2, –2) h. H(–7, 0)

i. I(3, –2) j. J(4, –4)

k. K(0, 6) l. L(0, 0)

m. $M\left(-2, \frac{5}{2}\right)$ n. $N\left(6, -\frac{3}{2}\right)$

o. $O\left(\frac{1}{2}, \frac{5}{2}\right)$

32. Complete each table of values.

a. $y = x + 3$

x	y	(x, y)
-1		
0		
2		
3		

b. $y = x - 2$

x	y	(x, y)
-3		
0		
1		
5		

d. $y = \frac{x}{3}$

x	y	(x, y)
	1	
	2	
	0	
	-1	
	-3	

e. $x + 2y = 3$

x	y	(x, y)
-5		
0		
-3		
1		
2		

33. Graph each equation by using a table of values.

a. $x = 2$ b. $y = -2$

c. $2x = 6$ d. $x + y = 3$

e. $x + y = -2$ f. $x - y = -2$

g. $y + 3x = -1$ h. $2x + 3y = 6$

i. $3x - 4y + 24 = 0$

34. Use the substitution method to solve each system.

a. $\begin{cases} y - 2x = 0 \\ x + y = 6 \end{cases}$ b. $\begin{cases} x - 3y = 0 \\ 3x + 2y = 11 \end{cases}$

c. $\begin{cases} x + y = 4 \\ x + 3y = 10 \end{cases}$ d. $\begin{cases} 3x - 2y = -14 \\ 2x + 3y = -5 \end{cases}$

e. $\begin{cases} 7x - 2y = 1 \\ 5x - 3y = 7 \end{cases}$ f. $\begin{cases} 6x - 5y = 5 \\ 2x + 3y = 25 \end{cases}$

g. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = \frac{1}{6}, \\ \frac{2}{3}x - \frac{3}{2}y = \frac{13}{6} \end{cases}$ h. $\begin{cases} \frac{1+3y}{4} + \frac{x-1}{4} = \frac{15}{2}, \\ \frac{3x-4}{8} + \frac{4y}{8} = \frac{13}{2} \end{cases}$

35. Two brothers are respectively 12 and 17 years old. In how many years will the ratio of their ages be $\frac{4}{5}$?

36. Three cats can eat three mice in three minutes. In how many minutes can 100 cats eat 100 mice?

37. Eight workers can do a job in twelve days by working ten hours a day. How many days will it take 15 workers to do this job if they work eight hours a day?

38. Nine identical swimming pools can be filled by three identical pipes running five hours a day for nine days. How many pools can be filled by five pipes in six days if they run nine hours a day?

39. Eight workers can do a job in 24 days by working six hours a day. How many workers can do $\frac{1}{4}$ of the same job in four days by working eight hours a day?

40. The ratio of Serik's age to Berik's age is $\frac{2}{3}$. The ratio of Yerik's age to Berik's age is $\frac{1}{2}$. Find the ratio of Yerik's age to Serik's age.

41. A mother is 38 years old and her daughter is 13 years old. In how many years will the mother be twice as old as her daughter?

42. The sum of the ages of two children is 30. Five years ago, one child was six years older than the other child. Find their ages now.

43. The sum of three consecutive integers is 102. Find the smallest number.

44. There are 25 students in a class. The number of boys is three more than the number of girls. Find the number of girls and boys in the class.

CONVERSION TO MATHEMATICAL LANGUAGE

Note

Be careful when translating phrases with the word *less*.

'5 less than x' means x – 5, not 5 – x.

AL-KHWARIZMI
(780-850)

One of the first books about algebra was written in Arabic by a nineteenth-century scientist called Muhammed İbn Musa Al-Khwarizmi. The title of the book was shortened to

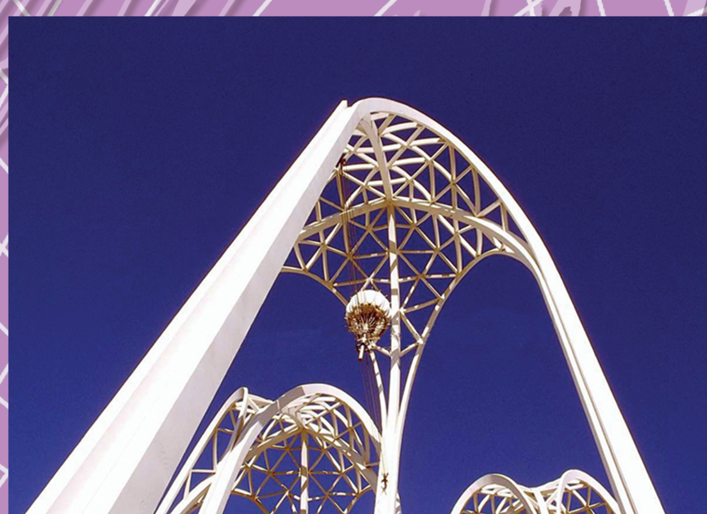
al-jabr, now spelled 'algebra'. The full title meant that equals can be added to both sides of an equation.

Al-Khwarizmi used his *al-jabr* to help him in his scientific work in geography and astronomy.

Operation	Verbal phrase	Algebraic translation
ADDITION	a number plus 4	$x + 4$
	the sum of y and 7	$y + 7$
	a number added to 6	$z + 6$
	3 more than a number	$t + 3$
	a number 5 greater than n	$n + 5$
	k increased by 12	$k + 12$
SUBTRACTION	9 minus a number	$9 - a$
	the difference of x and y	$x - y$
	5 less than a number	$b - 5$
	4 subtracted from t	$t - 4$
	a number decreased by 8	$c - 8$
MULTIPLICATION	6 times a number	$6a$
	the product of m and n	mn
	11 multiplied by a number	$11x$
	twice k	$2k$
	half of ℓ	$\frac{l}{2}$
DIVISION	10 divided by a number	$\frac{10}{x}$
	the quotient of a and b	$\frac{a}{b}$
	the ratio of s to t	$\frac{s}{t}$

CHAPTER 1

Natural Exponents



1 NATURAL EXPONENTS

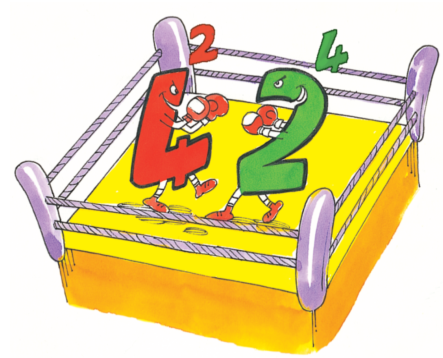
Objectives

After studying this section you will be able to:

1. Understand natural exponents and their properties.
2. Use the properties in solving problems.

A. NATURAL EXPONENTS

Consider the number $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$. We can write this number more simply as 5^8. 5^8 is an **exponential expression** which means 5 multiplied by itself eight times.



Similarly we can write $a \cdot a = a^2$, $a \cdot a \cdot a = a^3$, and $a \cdot a \cdot a \cdot a = a^4$, etc.

In general, $a \cdot a \cdot a \cdot ... \cdot a = a^n$, which means a multiplied by itself n times. Do not confuse a^n with $na = a + a + ... + a$, which is the sum of n terms.

Definition

power of a number, base, exponent

For any rational number a and natural number n,

$$a^n = \underbrace{a \cdot a \cdot a \cdot ... \cdot a}_{n \text{ times}}.$$

In this notation, a^n is called a **power** of a. a is called the **base** and n is called the **exponent**. We read a^n as 'a to the nth power'.

a^2 is read as 'a to the second power' or 'a squared'.

a^3 is read as 'a to the third power' or 'a cubed'.

a^4 is read as 'a to the fourth power'.

Look at some examples of exponential expressions:

$$2^3 = 2 \cdot 2 \cdot 2 = 8, \;\; 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81, \;\; (-2)^4 = (-2) \cdot (-2) \cdot (-2) \cdot (-2) = 16,$$

$$(-4^4) = -(4^4) = -(4 \cdot 4 \cdot 4 \cdot 4) = -256, \;\; (-4)^4 = (-4) \cdot (-4) \cdot (-4) \cdot (-4) = 256,$$

$$5a^3 = 5 \cdot a \cdot a \cdot a, \;\; (5a)^3 = (5a) \cdot (5a) \cdot (5a) = 125a^3.$$

Which expression is greater: $(2^{90} + 2^{90})$ or 2^{100}? Can you explain why?

Remark

Do not multiply the base and the exponent.

$2^6 \neq 2 \cdot 6$

$2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$

$-(a^n)$ and $(-a)^n$ are different: $-a^n = -(\underbrace{a \cdot a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a})$

$$\underbrace{(-a)^n = (-a) \cdot (-a) \cdot (-a)}_{n \text{ factors of } -a} \cdot \ldots \cdot (-a)$$

$a \cdot (x^n)$ and $(a \cdot x)^n$ are different: $a \cdot (x^n) = \underbrace{a \cdot x \cdot x \cdot x \cdot \ldots \cdot x}_{n \text{ factors of } x}$

$$\underbrace{(a \cdot \mathrm{x})^n = (a \cdot x) \cdot (a \cdot x) \cdot (a \cdot x)}_{n \text{ factors of } a \cdot x} \cdot \ldots \cdot (a \cdot x)$$

EXAMPLE 1 Write each expression in exponential form.

a. $x \cdot x \cdot x$ **b.** $(-3) \cdot (-3) \cdot (-3)$ **c.** $(-2) \cdot (-2) \cdot (-2) \cdot (-2)$

d. $a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b$ **e.** $16 \cdot x \cdot x \cdot 27 \cdot y \cdot y \cdot y$ **f.** $(-z) \cdot z \cdot z \cdot z$

Solution

a. $x \cdot x \cdot x = \mathrm{x}^3$

b. $(-3) \cdot (-3) \cdot (-3) = (-3)^3$

c. $(-2) \cdot (-2) \cdot (-2) \cdot (-2) = (-2)^4 = 2^4$

d. $a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b = a^4 b^3$

e. $16 \cdot x \cdot x \cdot 27 \cdot y \cdot y \cdot y = 4^2 \cdot x^2 \cdot 3^3 \cdot y^3 = (4x)^2 \cdot (3y)^3$

f. $(-z) \cdot z \cdot z \cdot z = (-z^4)$

B. PROPERTIES OF EXPONENTS

Property

Multiplication Rule

If a is any rational number and m and n are whole numbers, then

$$a^m \cdot a^n = a^{m+n}.$$

Proof

$$a^m \cdot a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a} = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m+n \text{ factors of } a} = a^{m+n}$$

For example,

$$5^2 \cdot 5^3 = 5^{2+3} = 5^5 = 3125,$$

$$4^5 \cdot 4 = 4^{5+1} = 4^6 = 4096,$$

$$(-3)^2 \cdot (-3)^3 = (-3)^{2+3} = (-3)^5 = -243,$$

$$x^2 \cdot x^3 \cdot x^5 = x^{2+3+5} = x^{10}, \text{ and}$$

$$y \cdot y^4 = y^1 \cdot y^4 = y^{1+4} = y^5.$$

Remark

The multiplication rule only applies to expressions with the same base.

$4^2 \cdot 2^3 \neq 8^{2+3}$, since

$4^2 \cdot 2^3 = 4 \cdot 4 \cdot 2 \cdot 2 \cdot 2 = 128$, and

$8^{2+3} = 8^5 = 8 \cdot 8 \cdot 8 \cdot 8 \cdot 8 = 32\ 768$.

Remark

The multiplication rule applies just to the product, not to the sum of two numbers.

$2^3 + 2^4 \neq 2^{3+4}$, since

$$\begin{aligned} 2^3 + 2^4 &= (2 \cdot 2 \cdot 2) + (2 \cdot 2 \cdot 2 \cdot 2) \\ &= 8 + 16 = 24, \text{ and} \\ 2^{3+4} &= 2^7 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \\ &= 128. \end{aligned}$$

Property

If a, b are any two rational numbers and n is a whole number, then

$$(a \cdot b)^n = a^n \cdot b^n.$$

Proof

$$\begin{aligned} (a \cdot b)^n &= \underbrace{(ab) \cdot (ab) \cdot (ab) \cdot \ldots \cdot (ab)}_{n \text{ factors of } (a \cdot b)} = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ factors of } a} \cdot \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors of } b} \\ &= a^n \cdot b^n \end{aligned}$$

For example,

$$(3 \cdot 5)^3 = 3^3 \cdot 5^3 = 27 \cdot 125 = 3375,$$

$$2^4 \cdot 3^4 = (2 \cdot 3)^4 = 6^4 = 1296,$$

$$(16 \cdot 2)^2 = 16^2 \cdot 2^2 = 256 \cdot 4 = 1024, \text{ and}$$

$$\begin{aligned} (3a)^2 \cdot (-2a)^3 &= 3^2 \cdot a^2 \cdot (-2)^3 \cdot a^3 = 9 \cdot a^2 \cdot (-8) \cdot a^3 \\ &= -72 \cdot a^2 \cdot a^3 \\ &= -72a^5. \end{aligned}$$

Property

If a is any rational number and m and n are whole numbers, then

$$(a^m)^n = a^{m \cdot n}.$$

Proof

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot a^m \cdot \ldots \cdot a^m}_{n \text{ factors of } a^m}$$

$$= \overbrace{\underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors of } a} \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors of } a} \cdot \ldots \cdot \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{m \text{ factors of } a}}^{n \text{ factors of } a^m} = a^{m \times n}$$

For example,

$$(2^3)^2 = 2^{3 \times 2} = 2^6 = 64,$$

$$((-3)^4)^2 = (-3)^{4 \times 2} = (-3)^8 = 3^8,$$

$$(8a^3)^4 = 8^4 \cdot a^{3 \times 4} = 8^4 \cdot a^{12}, \text{ and}$$

$$[(-2 \cdot a \cdot b)^2]^3 = (-2 \cdot a \cdot b)^6 = (-2)^6 \cdot a^6 \cdot b^6 = 64 \cdot a^6 \cdot b^6.$$

Property

If a, b are any two rational numbers and n is a whole number, then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}. \quad (b \neq 0)$$

Proof

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \cdot \ldots \cdot \frac{a}{b}}_{n \text{ factors of } \frac{a}{b}} = \frac{\overbrace{a \cdot a \cdot a \cdot \ldots \cdot a}^{n \text{ factors of } a}}{\underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n \text{ factors of } b}} = \frac{a^n}{b^n}$$

For example,

$$\left(\frac{4}{5}\right)^3 = \frac{4^3}{5^3} = \frac{64}{125},$$

$$\left(\frac{a}{2b}\right)^5 = \frac{a^5}{(2b)^5} = \frac{a^5}{2^5 \cdot b^5} = \frac{a^5}{32b^5}, \text{ and}$$

$$\left(\frac{x}{y^3}\right)^4 = \frac{x^4}{(y^3)^4} = \frac{x^4}{y^{12}}.$$

Definition

Any non-zero number to the power 0 is always equal to 1, i.e. if $a \neq 0$, then $a^0 = 1$.

The expression 0^0 is undefined.

For example,

$$5^0 = 1,\ (-4)^0 = 1, \text{ and } x^0 = 1 \qquad (x \neq 0).$$

$$(x \cdot y \cdot z^3)^0 = 1 \qquad (x, y, z \neq 0)$$

Definition

If $a \neq 0$, then $a^{-1} = \dfrac{1}{a}$.

For example,

$$4^{-1} = \frac{1}{4},\quad 5^{-1} = \frac{1}{5},\quad x^{-1} = \frac{1}{x},\quad (x \cdot y)^{-1} = \frac{1}{x \cdot y},$$

$$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2},\ \left(\frac{1}{20}\right)^{-1} = 20, \text{ and } \left(-\frac{2}{5}\right)^{-1} = -\frac{5}{2}.$$

Definition

If n is an integer and $a \neq 0$, then

$$a^{-n} = \frac{1}{a^n}.$$

For example,

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8},$$

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25},$$

$$(2a)^{-3} = \frac{1}{(2a)^3} = \frac{1}{2^3 \cdot a^3} = \frac{1}{8a^3}, \text{ and}$$

$$(a^{-2})^{-3} = \frac{1}{(a^{-2})^3} = \frac{1}{a^{-6}} = a^6.$$

Property

If a, b are any two non-zero rational numbers and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n} = \frac{b^n}{a^n}. \quad (a \neq 0 \quad \text{and} \quad b \neq 0)$$

For example,

$$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^{2} = \frac{16}{9}, \quad \left(\frac{1}{5}\right)^{-3} = \left(\frac{5}{1}\right)^{3} = 125,$$

$$\left(\frac{a^3}{b^2}\right)^{-4} = \left(\frac{b^2}{a^3}\right)^{4} = \frac{(b^2)^4}{(a^3)^4} = \frac{b^{2\cdot 4}}{a^{3\cdot 4}} = \frac{b^8}{a^{12}}, \text{ and}$$

$$\left(\frac{2a^2b}{3xy^2}\right)^{-3} = \left(\frac{3xy^2}{2a^2b}\right)^{3} = \frac{3^3\cdot x^3\cdot (y^2)^3}{2^3\cdot (a^2)^3\cdot b^3} = \frac{27x^3y^6}{8a^6b^3}.$$

Property

If a is any non-zero rational number and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$

Proof

$$\frac{a^m}{a^n} = a^m \cdot \left(\frac{1}{a^n}\right) = a^m \cdot a^{-n} = a^{m+(-n)} = a^{m-n}.$$

$$\frac{a^m}{a^m} = a^{m-m} = a^0 = 1$$

For example,

$$\frac{a^5}{a^3} = a^{5-3} = a^2, \qquad \frac{3^5}{3^{-11}} = 3^{5-(-11)} = 3^{5+11} = 3^{16},$$

$$\frac{x^2\cdot y^3}{x\cdot y^{-2}} = \frac{x^2}{x}\cdot\frac{y^3}{y^{-2}} = x^{2-1}\cdot y^{3-(-2)} = x^1\cdot y^{3+2} = xy^5, \text{ and}$$

$$\frac{3^{x+1}}{3^{x-1}} = 3^{x+1-(x-1)} = 3^{\cancel{x}+1-\cancel{x}+1} = 3^2 = 9.$$

Now look at the expressions:

$$3x^2 + 4y^2 \qquad \frac{2a^4}{3a^4} \qquad 6p^2 - 7p^3$$

In the expressions, the numbers $3x^2$, $4y^2$, $2a^4$, $3a^4$, $6p^2$, and $7p^3$ are called the *terms* of the expressions.

Property

The terms of an expression which have the same base and the same exponent are called **like terms**. We can add or subtract like terms.

$$(x \cdot a^n) + (y \cdot a^n) + (z \cdot a^n) = (x + y + z) \cdot a^n \qquad (a \neq 0)$$

For example, the terms $2a^3$, $4a^3$, and a^3 are like terms. We can add and subtract them easily:

$2a^3 + 4a^3 - a^3 = (2 + 4 - 1) \cdot a^3 = 5a^3$.

Similarly,

$(3 \cdot 2^x) + (4 \cdot 2^x) + (5 \cdot 2^x) - (11 \cdot 2^x) = (3 + 4 + 5 - 11) \cdot 2^x = 1 \cdot 2^x = 2^x$, and

$$\frac{3^{14} - 3^{13} + 3^{12}}{3^{13} - 3^{12}} = \frac{(3^{12} \cdot 3^2) - (3^{12} \cdot 3^1) + (3^{12} \cdot 1)}{(3^{12} \cdot 3^1) - (3^{12} \cdot 1)}$$

$$= \frac{\cancel{3^{12}} \cdot (3^2 - 3^1 + 1)}{\cancel{3^{12}} \cdot (3^1 - 1)} = \frac{9 - 3 + 1}{3 - 1} = \frac{7}{2}.$$

Property

1. All powers of a positive number a are positive, i.e. for $a > 0$, and $n \in \mathbb{Z}$,
$$a^n > 0.$$
2. The even powers of a negative number a are positive, i.e. for $a \neq 0$ and $n \in \mathbb{Z}$,
$$(-a)^{2n} = a^{2n}.$$
3. The odd powers of a negative number a are negative, i.e. for $a \neq 0$ and $n \in \mathbb{Z}$,
$$(-a)^{2n-1} = -a^{2n-1}.$$

Look at the examples of each rule.

$$\left.\begin{array}{l} 1.\quad 5^3 = 5 \cdot 5 \cdot 5 = 125 \\ \quad (\frac{1}{2})^4 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2^4} = \frac{1}{16} \\ \quad 2^{-3} = (\frac{1}{2})^3 = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8} \end{array}\right\}$$ The powers of the numbers are all positive.

2. $(-2)^2 = (-2) \cdot (-2) = 4$

$(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$

$(-\frac{1}{2})^{-2} = (-2)^2 = (-2) \cdot (-2) = 4$

$(-2)^{-6} = (-\frac{1}{2})^6 = \frac{1}{64}$

$(-1)^{100} = \underbrace{(-1) \cdot (-1) \cdot \ldots \cdot (-1)}_{100 \text{ factors of } -1} = 1$

The even powers of the numbers are all positive.

2. $(-2)^2 = (-2) \cdot (-2) = 4$

$(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$

$(-\frac{1}{2})^{-2} = (-2)^2 = (-2) \cdot (-2) = 4$

$(-2)^{-6} = (-\frac{1}{2})^6 = \frac{1}{64}$

$(-1)^{100} = \underbrace{(-1) \cdot (-1) \cdot \ldots \cdot (-1)}_{100 \text{ factors of } -1} = 1$

The odd powers of the numbers are all negative.

EXAMPLE 2 Simplify each expression.

a. $\dfrac{a^9 \cdot a^{-3}}{(a^2)^3}$ b. $(\dfrac{b^{-3}}{b^{-4}})^{-2}$ c. $\dfrac{x^{10} \cdot x^3}{x^5 \cdot x^{-2}}$

Solution a. $\dfrac{a^9 \cdot a^{-3}}{(a^2)^3} = \dfrac{a^{9+(-3)}}{a^{2\cdot 3}} = \dfrac{a^6}{a^6} = 1$

b. $(\dfrac{b^{-3}}{b^{-4}})^{-2} = (b^{-3-(-4)})^{-2} = (b^{-3+4})^{-2} = (b)^{-2} = \dfrac{1}{b^2}$

c. $\dfrac{x^{10} \cdot x^3}{x^5 \cdot x^{-2}} = \dfrac{x^{10+3}}{x^{5+(-2)}} = \dfrac{x^{13}}{x^3} = x^{13-3} = x^{10}$

EXAMPLE 3 Simplify each expression.

a. $\dfrac{(-1)^4 \cdot (-1)^3 \cdot (-1)^2}{2^{-4} \cdot 4^2 \cdot (-1^4)}$ b. $\left[-[(-1)^{101} - (-1^{100}) - 1]\right]^{101}$ c. $\dfrac{3^8 + 3^8 + 3^8}{9^4 + 9^4 + 9^4}$

d. $\dfrac{5^{x-1} + 5^{x+1}}{5^{x+1} - 5^{x-1}}$ e. $(x^{-3})^{-3} \cdot (-x)^{-6} \cdot (-x)^{-4}$ f. $\dfrac{6^4 \cdot 15^5 \cdot 21^6}{14^6 \cdot 10^5 \cdot 27^5}$

Solution

a. $\dfrac{(-1)^4 \cdot (-1)^3 \cdot (-1)^2}{2^{-4} \cdot 4^2 \cdot (-1^4)} = \dfrac{1 \cdot (-1) \cdot 1}{\dfrac{1}{\cancel{16}} \cdot \cancel{16} \cdot (-1)} = \dfrac{-1}{-1} = 1$

b. $\left[-[(-1)^{101} - (-1^{100}) - 1]\right]^{101} = \left[-[-1 - (-1) - 1]\right]^{101} = \left[-[-1 + 1 - 1]\right]^{101}$

$= \left[-[-1]\right]^{101} = 1^{101} = 1$

c. $\dfrac{3^8 + 3^8 + 3^8}{9^4 + 9^4 + 9^4} = \dfrac{\cancel{3} \cdot 3^8}{\cancel{3} \cdot 9^4} = \dfrac{3^8}{(3^2)^4} = \dfrac{3^8}{3^8} = 3^{8-8} = 3^0 = 1$

d. $\dfrac{5^{x-1} + 5^{x+1}}{5^{x+1} - 5^{x-1}} = \dfrac{5^x \cdot 5^{-1} + 5^x \cdot 5^1}{5^x \cdot 5^1 - 5^x \cdot 5^{-1}} = \dfrac{\cancel{5^x} \cdot (\frac{1}{5} + 5)}{\cancel{5^x} \cdot (5 - \frac{1}{5})} = \dfrac{\frac{26}{5}}{\frac{24}{5}} = \dfrac{26}{24} = \dfrac{13}{12}$

e. $(x^{-3})^{-3} \cdot (-x)^{-6} \cdot (-x)^{-4} = x^9 \cdot x^{-6} \cdot x^{-4} = x^{9-6-4} = x^{-1} = \dfrac{1}{x}$

f. $\dfrac{6^4 \cdot 15^5 \cdot 21^6}{14^6 \cdot 10^5 \cdot 27^5} = \dfrac{(2 \cdot 3)^4 \cdot (3 \cdot 5)^5 \cdot (3 \cdot 7)^6}{(2 \cdot 7)^6 \cdot (2 \cdot 5)^5 \cdot (3^3)^5} = \dfrac{2^4 \cdot 3^4 \cdot 3^5 \cdot 5^5 \cdot 3^6 \cdot 7^6}{2^6 \cdot 7^6 \cdot 2^5 \cdot 5^5 \cdot 3^{15}}$

$= \dfrac{2^4 \cdot \cancel{3^{15}} \cdot \cancel{5^5} \cdot \cancel{7^6}}{2^{11} \cdot \cancel{7^6} \cdot \cancel{5^5} \cdot \cancel{3^{15}}} = \dfrac{2^4}{2^{11}} = 2^{-7} = \dfrac{1}{128}$

Check Yourself 1

1. Evaluate the expressions.

a. 2^5 b. $(-3)^4$ c. -3^{-2} d. $(\frac{1}{2})^{-4}$

e. 5^0 f. -5^0 g. $(-5)^0$ h. 1^{1001}

i. $(-1)^{99}$ j. $(-1)^{100}$ k. -1^{-20} l. $(0.25)^{-4}$

m. $(0.0625)^{-1}$ n. $(-1)^{2n}$, $n \in \mathbb{Z}$ o. $(-1)^{2n+1}$, $n \in \mathbb{Z}$ p. $(-1)^{4n}$, $n \in \mathbb{Z}$

2. Evaluate the expressions.

a. $2^1 \cdot 3^2$ b. $3^2 \cdot 2^4$ c. $5^2 \cdot 2^3$ d. $4^3 \cdot 3^4$

e. $5^1 \cdot 3^5$ f. $2^4 \cdot 3^0$ g. $1^7 \cdot 7^1$ h. $4^4 \cdot 4^4$

3. Simplify the expressions.

a. $x^3 \cdot x^4$ b. $x \cdot x^0 \cdot x^{-1}$ c. $m \cdot n \cdot m^3 \cdot n^5$ d. $3^4 \cdot 2^4 \cdot 6^{-4}$

e. $(3a^2)^3 \cdot (4a^3)^2$ f. $\dfrac{a^{11} \cdot a^{-4}}{(2a \cdot b^{-2})^{-5}}$ g. $\dfrac{(4a^{-1} \cdot b)^4}{(2a \cdot b^{-2})^{-5}}$ h. $\dfrac{a^{2n+1}}{a^{n+2}}$

i. $\dfrac{(x^6 \cdot y^{-4})^2}{(x^3 \cdot y^2)^{-4}}$ j. $(\dfrac{x^2 \cdot y^3 \cdot z}{x^{-1} \cdot y^{-2} \cdot z^{-3}})^{-2}$ k. $\dfrac{a^n \cdot a^{-2}}{a^4}$ l. $\dfrac{\left[(-\frac{1}{a})^5\right]^2}{\left[(-a)^{-2}\right]^5}$

i. $(\dfrac{a^{-1} \cdot b^{-1}}{b^2})^2 (\dfrac{a^2 \cdot b}{a^3})^{-2} (\dfrac{a}{a^2 \cdot b^{-2}})^3$

4. Simplify the expressions.

a. $\dfrac{(-1)^{93} \cdot (-1)^{94} \cdot (-1)^{95}}{(-1)^{96} \cdot (-1)^{97}}$ b. $\dfrac{5^4 + 5^4 + 5^4 + 5^4}{5^5}$ c. $\dfrac{3^{x-2} - 3^{x-3}}{3^x - 3^{x+1}}$

d. $\dfrac{(-2)^{-3} + (-2^3)}{(-3)^{-2} + (-3^2)}$ e. $(-a)^{-3} \cdot (-a^{-2}) \cdot (-\frac{1}{a})^{-5} \cdot (-a^{-3})$ f. $\dfrac{((-4)^3)^4 + ((-2)^8)^3}{(-2^6)^2 - (-2^4)^3}$

5. Evaluate the expressions.

a. $\dfrac{2^8 \cdot 7^9}{14^{10}} \cdot \dfrac{26^5 \cdot 13^{-6}}{2^{-10} \cdot 8^4}$ b. $\dfrac{12^5}{2^8 \cdot 3^4} \cdot \left(\dfrac{10^5}{2^6 \cdot 5^7}\right)^{-1}$ c. $\dfrac{5^9 \cdot 21^3 \cdot 5^4}{5^2 \cdot 35^4 \cdot 15^5}$

d. $\left(\dfrac{3^5 \cdot 5^7}{15^7 \cdot 2^8}\right)^{-1} : \dfrac{22^9 \cdot 3^{12}}{\left(11^2\right)^4 \cdot 9^4}$ e. $\left(\dfrac{8^5 + 2^4}{8^7 - 2^8} \cdot \dfrac{3^9 - 3^{12}}{3^8 + 9^4}\right)^0$

Answers

1. a. 32 b. 81 c. $-\dfrac{1}{9}$ d. 16 e. 1 f. –1 g. 1 h. 1 i. –1 j. 1 k. –1 l. 256 m. 16 n. 1 o. –1 p. 1

2. a. 18 b. 144 c. 200 d. 5184 e. 1215 f. 16 g. 7 h. 64 768   3. a. x^7 b. 1 c. $m^4 \cdot n^6$ d. 1 e. $432a^{12}$ f. $\dfrac{32 \cdot a^{12}}{b^{10}}$ g. $2^{13} \cdot a \cdot b^{-6}$ h. a^{n-1} i. x^{24} j. $x^{-6} \cdot y^{-10} \cdot z^{-8}$ k. a^{n-6} l. 1 m. $a^{-3} \cdot b^{-2}$

4. a. –1 b. $\dfrac{4}{5}$ c. $-\dfrac{1}{27}$ d. $\dfrac{9}{320}$ e. a^{-3} f. 2^{12} 4. a. $\dfrac{2}{91}$ b. 300 c. $\dfrac{5}{63}$ d. $\dfrac{9}{22}$ e. 1

C. THE GRAPHS AND PROPERTIES OF SOME BASIC FUNCTIONS

1. $y = ax^2$

How can we find the graph of the quadratic function $y = ax^2$? If we have the function, we can plot the graph by making a table of values. To find the values, we substitute different values of x into the equation to obtain the corresponding y values. These x and y values provide the coordinates for points which we can plot to form the shape of the graph.

Let us graph the function $y = ax^2$.

x	$-\infty$	–2	–1	0	1	2	$+\infty$
y	$+\infty$	$4a$	a	0	a	$4a$	$+\infty$

If $a > 0$, we get the table of ordered pairs opposite.

Then we plot the points on a graph and draw a parabola through them.

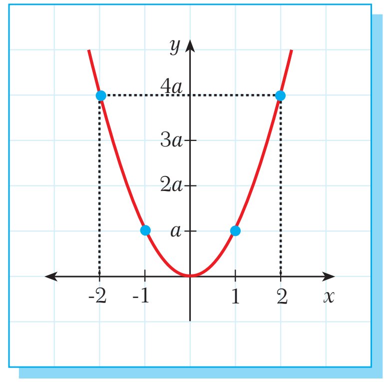


Fig. 1.1

$y = ax^2, \quad a > 0$

We can see that the vertex of the parabola is at the origin (0, 0), and the axis of symmetry lies along the y-axis (the line $x = 0$).

If $a < 0$, we get a different set of ordered pairs.

x	$-\infty$	-2	-1	0	1	2	$+\infty$
y	$-\infty$	$4a$	a	0	a	$4a$	$-\infty$

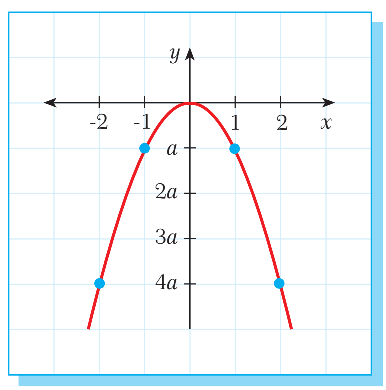


Fig. 1.2

EXAMPLE 4 Sketch the graphs of the functions.

a. $y = x^2$, $y = 2x^2$ and $y = \dfrac{1}{2}x^2$

b. $y = -x^2$, $y = -2x^2$ and $y = -\dfrac{1}{2}x^2$

Solution a.

As $|a|$ increases, the parabola becomes narrower. As $|a|$ decreases, the parabola becomes wider.

x	$-\infty$	-2	-1	0	1	2	∞
$y = x^2$	∞	4	1	0	1	4	∞
$y = 2x^2$	∞	8	2	0	2	8	∞
$y = \frac{1}{2}x^2$	∞	2	$\frac{1}{2}$	0	$\frac{1}{2}$	2	∞

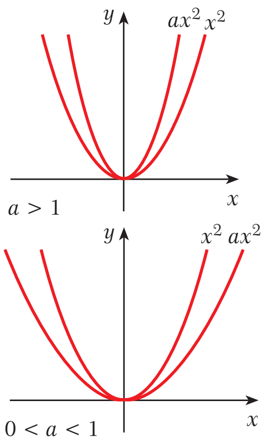
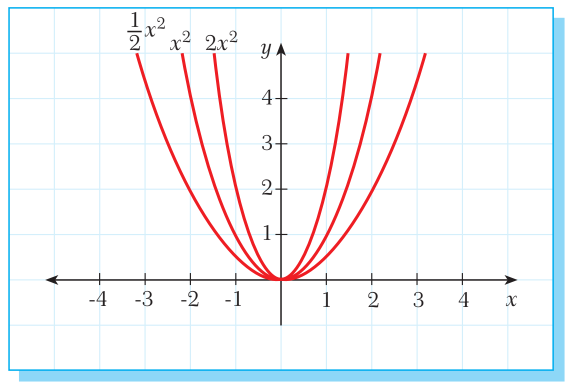
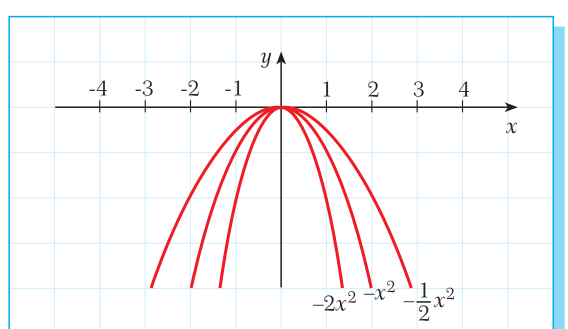


Fig. 1.3



b.

x	$-\infty$	-2	-1	0	1	2	∞
$y = -x^2$	$-\infty$	-4	-1	0	-1	-4	$-\infty$
$y = -2x^2$	$-\infty$	-8	-2	0	-2	-8	$-\infty$
$y = -\frac{1}{2}x^2$	$-\infty$	-2	$-\frac{1}{2}$	0	$-\frac{1}{2}$	-2	$-\infty$



Properties

1. The graph of $y = ax^2$ is a parabola.
2. $y = ax^2$ passes through the origin O(0;0).
3. If $a > 0$, then the graph of $y = ax^2$ lies above x-axis.

   If $a < 0$, then the graph of $y = ax^2$ lies below x-axis.
4. The graph of $y = ax^2$ is symmetric with respect to y-axis.

Check Yourself 2

Graph the functions.

1. $y = 3x^2$ 2. $y = \frac{1}{3}x^2$ 3. $y = -3x^2$ 4. $y = -\frac{1}{3}x^2$

5. Which of the following points pass through $y = -3x^2$?

   a. (1; –3), b. (0; –3), c. (2; 12), d. (–3; –27) e. (–1; 3), f. (0; 0)

6. For which value of 'a' does the point A(–3; a) lie on the graph of these functions?

   a. $y = -x^2$ b. $y = -\frac{1}{3}x^2$ c. $y = 7x^2$ d. $y = 4x^2$ e. $y = -\frac{2}{3}x^2$

7. For which value of 'a' do these points lie on the graph of $y = ax^2$?

   a. A(–1;2) b. B($\frac{1}{4}$, $\frac{1}{2}$) c. C(0; 0) d. D(–3; 9) e. E(2; 1) f. F(0; 5)

Answers

5. A, D and F 6. a. a = –9 b. $a = \frac{9}{2}$ c. a = –63 d. a = 36 e. a = –6

7. a. a=2 b. a = 8 c. a e $\mathbb{R}$ -{0} d. a=1 d. $a = \frac{1}{4}$ d. no solution

2. y = ax³

What about the graph of the cubic function $y = ax^3$? Let's find some points of the graph *by* substituting different x values on the function. If $a > 0$, then we get the following table:

x	$-\infty$	–2	–1	0	1	2	$+\infty$
y	$+\infty$	$-8a$	$-a$	0	a	$8a$	$+\infty$

By using these points we can draw the following graph:

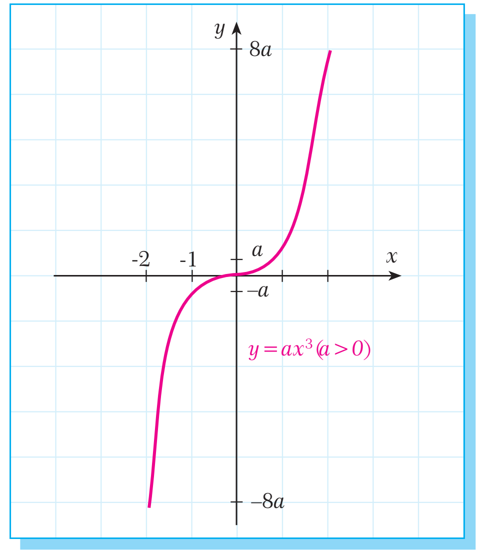


Fig. 1.4

Let's draw the table when $a < 0$:

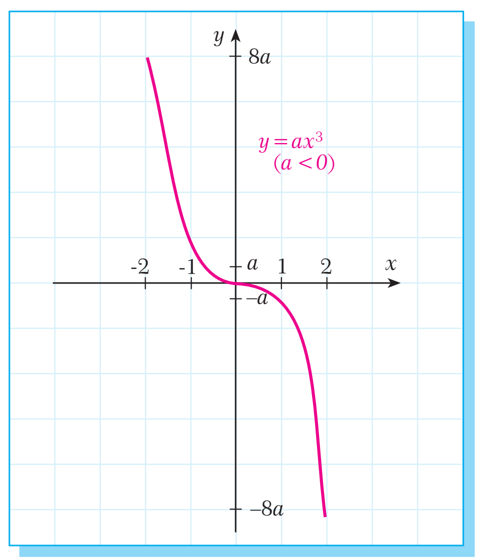


Fig. 1.5

x	$-\infty$	-2	-1	0	1	2	$+\infty$
y	$+\infty$	$-8a$	$-a$	0	a	$8a$	$-\infty$

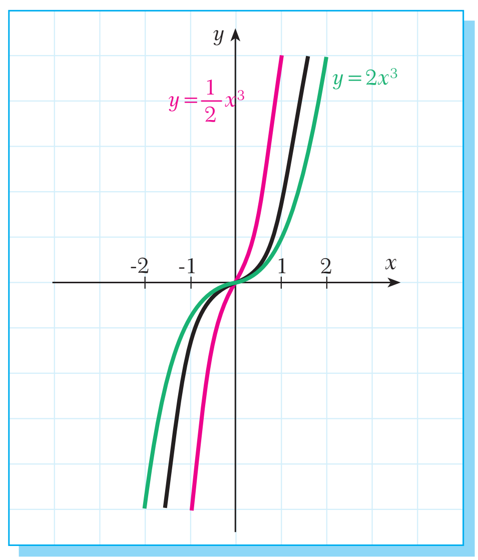
EXAMPLE 5 Sketch the graphs of these functions:

a. $y = x^3$ b. $y = 2x^3$ c. $y = \frac{1}{2}x^3$ d. $y = -x^3$ e. $y = -2x^3$ f. $y = -\frac{1}{2}x^3$

Solution

x	$-\infty$	-2	-1	0	1	2	∞
$y = x^3$	$-\infty$	-8	-1	0	1	8	∞
$y = 2x^3$	$-\infty$	-16	-2	0	2	16	∞
$y = \frac{1}{2}x^3$	$-\infty$	-4	$-\frac{1}{2}$	0	$\frac{1}{2}$	4	∞

By using the table,we are able to draw this graph:



As $|a|$ increases, the graph of $y = ax^3$ becomes narrower. As $|a|$ decreases, the graph becomes wider.

x	$-\infty$	-2	-1	0	1	2	∞
$y = x^3$	$+\infty$	8	1	0	1	8	$-\infty$
$y = -2x^3$	$+\infty$	16	2	0	2	-16	$-\infty$
$y = -\frac{1}{2}x^3$	$+\infty$	4	$\frac{1}{2}$	0	$\frac{1}{2}$	-4	$-\infty$

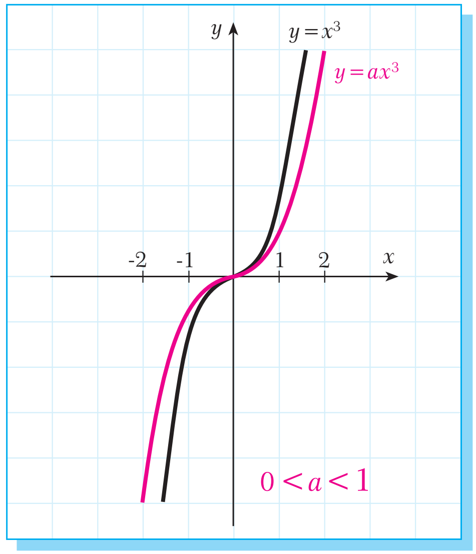


Fig. 1.6

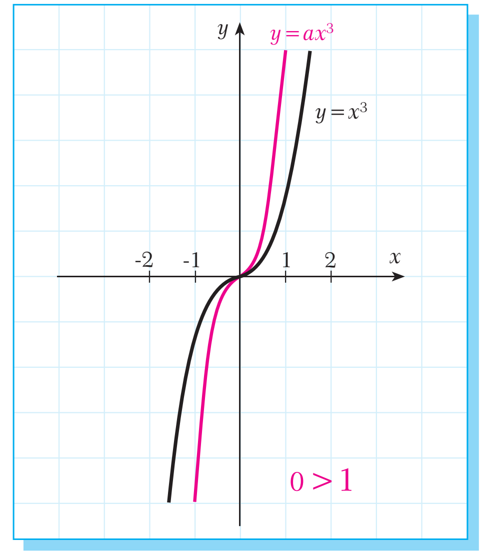


Fig. 1.7

By using the table, we can draw this graph:

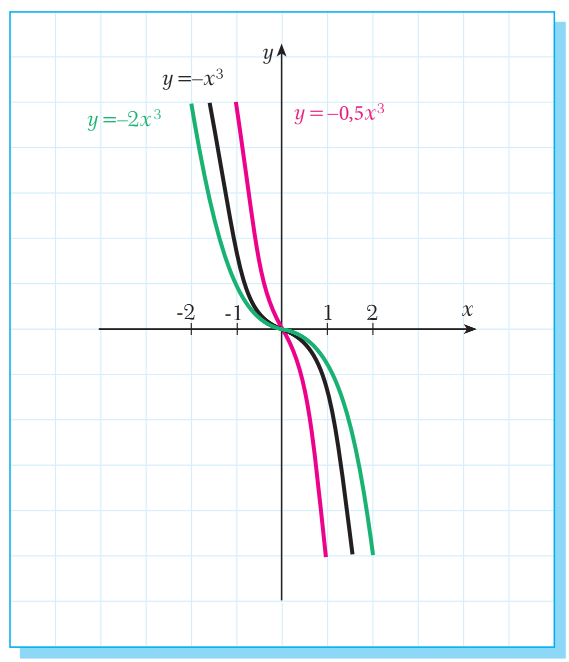


Fig. 1.8

Properties

1. $y = ax^3$ passes through the origin O(0; 0).
3. If $a > 0$, then the graph of $y = ax^3$ lies on the 1st and 3rd quadrant.
   If $a < 0$, then the graph of $y = ax^3$ lies on the 2nd and 4th quadrant.
4. The graph of $y = ax^3$ is symmetric with respect to the origin.

Check Yourself 3

Graph the functions.

1. $y = 3x^3$ 2. $y = \frac{2}{3}x^3$ 3. $y = -4x^3$ 4. $y = -\frac{1}{4}x^3$

5. Which of the following points pass through $y = -5x^3$?

   A(1; 5), B(0; –5), C(2; –40), D$(-\frac{1}{5}; \frac{1}{5})$ E(–1;5), F(0;0)

6. For which value of 'a' does the point A(–2; a) lie on the graph of these functions?

   a. $y = -x^3$ b. $y = \frac{1}{2}x^3$ c. $y = 7x^3$ d. $y = 4x^3$ a. $y = -\frac{2}{3}x^3$

7. For which value of 'a' do these points lie on the graph of $y = ax^3$?

   a. A(–1; 2) b. B$(\frac{1}{2}; \frac{1}{2})$ c. C$(4; \frac{1}{8})$ d. D$(-\frac{2}{5}; 4)$ e. E(–3; –27)

   f. F(0; 0)

Answers

5. C, E and F 6. a. $a = 8$ b. $a = -4$ c. $a = -56$ d. a= –32 e. $a = \frac{16}{3}$

7. a. $a = -2$ e. $a = 4$ e. $\frac{1}{512}$ e. $a = -\frac{125}{2}$ e. a=1 f. $a \in \mathbb{R} - \{0\}$

EXERCISES 1.1

1. Write the expressions in exponential form.

a. $m \cdot m \cdot m \cdot m$ b. $x \cdot x \cdot x \cdot x \cdot x$
c. $x \cdot x \cdot y \cdot y \cdot y$ d. $x \cdot y \cdot y \cdot z \cdot z \cdot z$

2. Simplify the expressions.

a. 2^3 b. 1^4
c. $(-3)^2$ d. $2^3 \cdot 2^4$
e. $5^2 \cdot 3^3$ f. $3^1 \cdot 4^3$
g. $2^3 \cdot 3^2$ h. $1^9 \cdot 9^1$
i. $2^3 \cdot 3^3$ j. $-3^2 \cdot (-3^2) \cdot (-3)^3 \cdot (-3^3)$
k. $x^2 \cdot x^2 \cdot x^6$ l. $(2x)^3 \cdot (3x)^2$
m. $5^x \cdot 5^y \cdot 5$ n. $(-3)^{35} \cdot (-3)^{40}$
o. $4^{2x} \cdot 4^{-3x} \cdot 4^{4x+1}$ p. $3^k + 3^k + 3^k$
q. $(2^{3a-b} \cdot 2^{2b-a}) + (2^{a-b} \cdot 2^{a+2b})$

3. Simplify the expressions.

a. $\dfrac{(-4)^7}{2^7}$ b. $\dfrac{2^2 \cdot 3^2}{4^2}$

c. $\dfrac{3^k \cdot 6^k}{2^k}$ d. $\dfrac{4^x \cdot 4^x \cdot 4^2}{2^{2(x+1)}}$

e. $\dfrac{3^{4k+3}}{3^{2k+1}}$ f. $\dfrac{2^{2x}+2^{2x}}{2^x+2^x}$

4. Perform the operations.

a. $(2^3)^2$ b. $\dfrac{(4^2)^3}{2^2}$ c. $[(-4)^3]^2$

d. $\dfrac{(-3^2)^3}{(-3)^3}$ e. $(5^x)^2 \cdot (5^3)^x$

5. Write the expressions using only positive exponents.

a. $(\dfrac{1}{2^3})^{-2}$ b. $a^{2b} \cdot \dfrac{1}{a^b}$

c. $\dfrac{a^b}{a^{-b}}$ d. $(3^{x+1})^{-1} \cdot (3^{-2})^{-x}$

e. $100^{-3} \cdot 5^8 \cdot 4^5$ f. $(\dfrac{10^5}{3^3})^{-4} \cdot (\dfrac{5^{10}}{6^6})^2$

6. Simplify the expressions.

a. $2^x + 2^x + 2^x + 2^x$ b. $5^6 + 5^7$

c. $7^5 - 7^3 + 7^2$ d. $3^{1-x} + 3^{2-x} + 3^{3-x}$

e. $\dfrac{2^3+2^7+2^9}{2^2+2^6+2^8}$ f. $\dfrac{5^6-5^7+5^{10}}{5^9-5^6+5^5}$

g. $\dfrac{2^k+4^k}{2^k+1}$ h. $\dfrac{3^{3m}+3^{2m}}{3^m+3^{2m}}$

7. Write the expressions.

a. $a = 2^5$ and $b = 3^5$. Write 12^5 in terms of a and b.

b. $a = 2^x$ and $b = 3^x$. Write 6^{2x} in terms of a and b.

c. $a = 2$ and $b = 5$. Write 0.0125 in terms of a and b.

8. Express the relation between m and n.

a. $m = 3^a$, $n = 27^{2a}$
b. $m = 4^{-a}$, $n = 1 + 2^a$
c. $m = 4^{2a}$, $n = 64^a$
d. $m = 3^{k+1}$, $n = 9^{2k}$

10. Simplify $\dfrac{18^3+30^3}{6^3+10^3}$.

11. $m = 3^{2n+1}$ and $27m^2 = 3^{5n}$. Find n.

CHAPTER 2

Monomials and Polynomials

1 MONOMIALS AND POLYNOMIALS

Objectives

After studying this section you will be able to:

1. Describe the concept of polynomial, and find the degree and leading coefficient of a polynomial.
2. Simplify a polynomial by combining like terms.
3. Add, subtract, multiply and divide polynomials.

A. INTRODUCTION TO POLYNOMIALS

In your early studies you began working with algebraic expressions. An algebraic expression may contain several variables at the same time. $4x^3 + 2x - 7$, $5x^7 + 9$, $6xy^4$, $2x^3$, and $4x^0$ are all examples of algebraic expressions. We can define a polynomial as a special type of algebraic expression.

Definition

polynomial

An algebraic expression which comprises a single real number, or the product of a real number and one or more variables raised to whole number powers, is called a **monomial**.

For example, 6, $-2x^3$, $5a^2b^3$, $-\dfrac{7}{2}$, and $3x^4yz^5$ are all monomials.

Each real number preceding the variable(s) in a monomial is called a **coefficient**.

In the examples above 6, -2, 5, $-\dfrac{7}{2}$, and 3 are the coefficients.

A **polynomial** is the sum or difference of a set of monomials. For example, $4x^3 + 5x - 3$, $5xy^2 + 4x + 3y$, and $-7a^3b^3 + 4ab^2$ are all polynomials. Each monomial that forms a polynomial is called a **term** of that polynomial. For example, the terms of polynomial $5xy^2 + 4x + 3y$ are $5xy^2$, $4x$ and $3y$. The term of a polynomial that does not contain a variable is called the **constant term**.

The coefficient of the term containing the variable raised to the highest power is called the **leading coefficient**. For example, consider the polynomial $9x^6 - 5x^4 + 3x^3 - 8x + 2$.

$9x^6$, $-5x^4$, $3x^3$, $-8x$, and 2 are the terms of polynomial. 9, -5, 3, -8, and 2 are the coefficients. 2 is the constant term. 9 is the leading coefficient.

A polynomial is said to be in **standard form** if the terms are written in descending order of degree. $x^3 - 4x^2 + 6$ is a polynomial in standard form. $-4x^2 + 6 + x^3$ is the same polynomial, but it is not in standard form.

EXAMPLE 1 Determine whether each algebraic expression is a polynomial.

a. -8 b. $5x - \dfrac{3}{4}$ c. $4x^2 - 3x^{-2}$ d. $\dfrac{6}{x} + \dfrac{5}{x^2}$ e. $x^{56} - x^{45} + 3x^2 - 3$

Solution

a. –8 is a polynomial containing only a constant term.

b. 5 and $-\dfrac{3}{4}$ are real numbers, so this is a polynomial with two terms.

c. $4x^2 - 3x^{-2}$ is not a polynomial because –2 is not a whole number power.

d. $\dfrac{6}{x}+\dfrac{5}{x^2}=6\cdot x^{-1}+5\cdot x^{-2}$. So this is not a polynomial.

e. The expression is a polynomial.

Remark

1. $x^0 = 1,\ x \neq 0$ 2. $x^1 = x$ 3. $\dfrac{1}{x^n}=x^{-n},\quad x\neq 0$

EXAMPLE 2 The following expressions are polynomials. Find the possible natural number values of m.

a. $5x^{m+2} - 6x^{4-m} + 3$ b. $-2x^{\frac{24}{m}}+4x^{m-6}+x-m$

Solution By the definition of a polynomial, the exponents of the variables must be non-negative integers, i.e. whole numbers. We can use this to calculate the possible natural number values of m.

a. $m + 2 \geq 0$ and $4 - m \geq 0$

$m \geq -2$ $\quad 4 \geq m$

Hence $-2 \leq m \leq 4$, and $m \in \{-2, -1, 0, 1, 2, 3, 4\}$.

b. $\dfrac{24}{m}\in\mathbb{N}$ and $m - 6 \geq 0$

If $m - 6 \geq 0$ then $m \geq 6$.

If $\dfrac{24}{m}\in\mathbb{N}$ then $m \in \{1, 2, 3, 4, 6, 8, 12 ,24\}$.

Combining the possible values gives $m \in \{6, 8, 12 ,24\}$.

Certain polynomials are given special names depending on their number of terms.

A **monomial** is a polynomial containing one term, a **binomial** is a polynomial containing two terms, and a **trinomial** is a polynomial that has three terms.

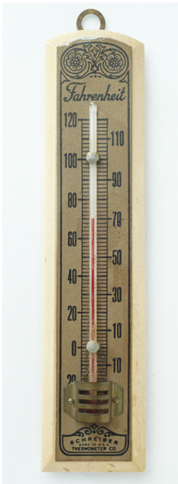
We can also classify polynomials according to their degree. The **degree of a monomial** is the sum of the exponents of the variables.

For example, $5x^2y$ has degree 3 $(5x^2y^1\ ;\ 2 + 1 = 3)$,

$6a^4$ has degree 4,

$-3m^3n^2$ has degree 5 $(-3m^3n^2\ ;\ 3 + 2 = 5)$,

-24 has degree 0 $(-24 = -24 \cdot x^0)$.



Check Yourself 1

Write the degree of each monomial.

1. $63a^2b^5$ 2. $-7x^3$ 3. -12 4. $5x^2y^3zt^6$

Answers

1. 7 2. 3 3. 0 4. 12

The **degree of a polynomial** is the same as the highest degree of the terms in the polynomial.

For example, $-6x^5 + 4x^3 - 7$ has degree 5. $(5 > 3 > 0)$

$9z^7 - 8z^6 + 7z^3 + 3$ has degree 7. $(7 > 6 > 3 > 0)$

$-12x^3yz + 5x^4yz^3 - 6y^4z^3$ has degree 8 because the highest degree of all the monomial terms is 8 (= 4 + 1 + 3). $(-\underbrace{12x^3yz}_{\text{degree 5}} + \underbrace{5x^4yz^3}_{\text{degree 8}} - \underbrace{6y^4z^3}_{\text{degree 7}})$

Note

If the degree of a polynomial is one, then it is called a **linear polynomial**.

A polynomial of degree 2 is called a **quadratic polynomial**.

A polynomial of degree 3 is called a **cubic polynomial**.

EXAMPLE 4 Write the following polynomials in standard form and find their degree and leading coefficient.

a. $1 + 3x$ b. $-7x^2 + 1 + x$ c. $1+\frac{3}{2}x+x^3$ d. $5 - x^5 + 13x^2y^4$

e. $3x^3y^3 + 2xy^2 + 5y^5 + 6$ f. $5+\frac{1}{5}x^2+\frac{1}{25}x$

Solution

	polynomial	standard form	degree of polynomial	leading coefficient
a.	$1 + 3x$	$3x + 1$	1	3
b.	$-7x^2 + 1 + x$	$-7x^2 + x + 1$	2	-7
c.	$1+\frac{3}{2}x+x^3$	$x^3+\frac{3}{2}x+1$	3	1
d.	$5 - x^5 + 13x^2y^4$	$13x^2y^4 - x^5 + 5$	6	13
e.	$3x^3y^3 + 2xy^2 + 5y^5 + 6$	$3x^3y^3+5y^5+2xy^2+6$	6	3
f.	$5+\frac{1}{5}x^2+\frac{1}{25}x$	$\frac{1}{5}x^2+\frac{1}{25}x+5$	2	$\frac{1}{5}$

Check Yourself 2

1. Complete the tables.

a.

Monomial	Degree	Coefficient
7		
$7x$		
$-3x^2$		
$\frac{x^2y}{3}$		
$9x^3y^2$		

b.

Polynomial	Type
-11	
$-3x + 5$	
$x^2 - 7x + 3$	
$x^3 + 4x^2 + 4$	
$\frac{4xy^2z}{3}$	
$x + x^2 + 3x^4 + 5$	

c.

Polynomial	Standard form	Degree	Leading coefficient
$1 - 7x$			
$-x + 4 + 3x^2$			
$2x^4 + 3x + 5x^3 + 1$			
$6x^2 - 4xy + 3x + 1$			
$\frac{1}{2}x + \frac{3}{5}x^2 + \frac{1}{7}x^3 + 1$			
$6x + 7y + 9x^2y^3 + 3x^4$			

Answers

a.

Monomial	Degree	Coefficient
7	0	7
$7x$	1	7
$-3x^2$	2	-3
$\frac{x^2y}{3}$	3	$\frac{1}{3}$
$9x^3y^2$	5	9

b.

Polynomial	Type
-11	monomial
$-3x + 5$	binomial
$x^2 - 7x + 3$	trinomial
$x^3 - 4x^2 + 4$	trinomial
$\frac{4xy^2z}{3}$	monomial
$x + x^2 + 3x^4 + 5$	polynomial

c.

Polynomial	Standard form	Degree	Leading coefficient
$1 - 7x$	$-7x + 1$	1	-7
$-x + 4 + 3x^2$	$3x^2 - x + 4$	2	3
$2x^4 + 3x + 5x^3 + 1$	$2x^4 + 5x^3 + 3x + 1$	4	2
$6x^2 - 4xy + 3x + 1$	$6x^2 - 4xy + 3x + 1$	2	6
$\frac{1}{2}x + \frac{3}{5}x^2 + \frac{1}{7}x^3 + 1$	$\frac{1}{7}x^3 + \frac{3}{5}x^2 + \frac{1}{2}x + 1$	3	$\frac{1}{7}$
$6x + 7y + 9x^2y^3 + 3x^4$	$9x^2y^3 + 3x^4 + 6x + 7y$	5	9

Definition

constant polynomial and zero polynomial

A **constant polynomial** is a polynomial in which the coefficients of all the variables are zero. The value of a constant polynomial is the constant term.

If the constant term in a constant polynomial is zero, then it becomes a special polynomial: a **zero polynomial**. A zero polynomial has no degree. (Can you see why?)

EXAMPLE 5 Find m and n if $(2m-4)x^2 + (5-n)x + 13$ is a constant polynomial.

Solution If the given polynomial is a constant polynomial, then the coefficients of all the variables must be zero.

So $2m-4=0$ and $5-n=0$

$$\begin{aligned} 2m-4 &= 0 & 5-n &= 0 \\ 2m &= 4 & n &= 5. \\ m &= 2 \end{aligned}$$

EXAMPLE 6 Find a if $(b-4)x^3 - (2c+6)x + (a-b+c)$ is a zero polynomial.

Solution In a zero polynomial, every term is equal to zero.

Therefore, $b-4=0$, $2c+6=0$ and $a-b+c=0$. So

$$\begin{aligned} b-4 &= 0 & 2c+6 &= 0 & \text{and} \quad a-b+c &= 0 \\ b &= 4 & 2c &= -6 & a-(4)+(-3) &= 0 \\ & & c &= -3 & a-7 &= 0 \\ & & & & a &= 7. \end{aligned}$$

Check Yourself 3

1. Find a, b, and c if $(8-3a)x^2 + (b^2-9)x + 5x^{\frac{c}{4}}$ is a constant polynomial.
2. $(3b-6)x^7 - (4+a)x$ is a zero polynomial. Find $a+b$.

Answers

1. $a = \frac{8}{3}$, $b = \pm 3$, $c = 0$ 2. $a + b = -2$

B. EQUALITY OF TWO POLYNOMIALS

Definition

like terms

If two or more terms contain the same variable(s) raised to the same power(s) they are called **like terms**.

For example, $5x^2y$ and $-3x^2y$ are like terms. However, $7xy$ and $4x^2y^3$ are not like terms.

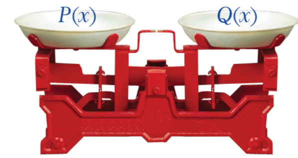
Definition

equal polynomials

Two polynomials are **equal** if and only if they have like terms with the same coefficients.

For example, $P(x, y) = 5x^2y^2 - 3xy + 8x^3y$ and

$Q(x, y) = 8x^3y + 5x^2y^2 - 3xy$ are equal polynomials.



EXAMPLE 7 Find b if $P(x) = (2a - 5)x^5 - (b - a + 2)x^3 + 2x$ and $Q(x) = 5x^5 - 3x^3 + 2x$ are equal polynomials.

Solution The polynomials are equal, so let us match the coefficients of like terms.

$$(2a - 5) \cdot x^5 - (b - a + 2) \cdot x^3 + 2x = 5x^5 - 3x^3 + 2x$$

So $\quad 2a - 5 = 5, \quad -(b - a + 2) = -3$ and $2 = 2$.

$2a = 10 \qquad b - a = 1$

$a = 5 \qquad b - 5 = 1$

$b = 6$

EXAMPLE 8 Find a, b, c, d, and e if $P(x) = (a + 4) \cdot x^6 - 4x^3 + (d - 2)x^2 + (2c - 8)x + e - 2$ and $Q(x) = (3b + 2)x^3 + 5x^2 + 2x$ are equal polynomials.

Solution $$(a + 4)x^6 - 4x^3 + (d - 2)x^2 + (2c - 8)x + e - 2 = 0 \cdot x^6 + (3b + 2)x^3 + 5x^2 + 2x + 0$$

So $\quad a + 4 = 0, \quad -4 = 3b + 2, \quad d - 2 = 5, \quad 2c - 8 = 2, \quad e - 2 = 0$

$a = -4 \qquad -6 = 3b \qquad d = 7 \qquad 2c = 10 \qquad e = 2.$

$-2 = b \qquad c = 5$

Check Yourself 4

Find a, b, and c if $P(x) = 7x^4 - (2a - 3)x^3 + 5x - (c - 3)$ and $Q(x) = (3b + 4)x^4 + 2x^3 + 5x$ are equal polynomials.

Answers

$a = \dfrac{1}{2}$, $b = 1$, $c = 3$

C. OPERATIONS ON POLYNOMIALS

To combine like terms:
1. Add or subtract the coefficients of the like terms.
2. Attach the common variables.

EXAMPLE 9 Simplify the each polynomial by combining like terms.

a. $6x + 4x$ b. $-7y + 6y + 3y$ c. $-4a - (-3a) + 5a$

Solution a. $6x + 4x = (6 + 4) \times x$ (add the coefficients of the like terms)

$= 10x$ (keep the variable x)

b. $-7y + 6y + 3y = (-7 + 6 + 3)y = 2y$

c. $-4a - (-3a) + 5a = -4a + 3a + 5a = (-4 + 3 + 5)a = 4a$

1. Adding Polynomials

Recall that the degree of a polynomial is the same as the highest degree of the terms in the polynomial. We write **deg[$P(x)$]** to mean the degree of a polynomial function $P(x)$.

For example, if $P(x) = x^3 - x^2 + 1$, then $\deg[P(x)] = 3$.

Definition

sum of two polynomials

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ and $Q(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_2x^2 + b_1x + b_0$ be two polynomials such that $\deg[Q(x)] \geq \deg[P(x)]$. Then the sum of these polynomials is defined as

$$P(x) + Q(x) = b_m x^m + \ldots + (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_2 + b_2)x^2 + (a_1 + b_1)x + a_0 + b_0.$$

To add polynomials, we add the like terms of the polynomials.

For example, let us add $3x^2 + 5x + 1$ and $7x^2 + 2x + 3$.

$(3x^2 + 5x + 1) + (7x^2 + 2x + 3)$ (write the sum)

$= (3x^2 + 7x^2) + (5x + 2x) + (1 + 3)$ (group the like terms)

$= (3 + 7)x^2 + (5 + 2)x + (1 + 3)$ (add the coefficients of the like terms)

$= 10x^2 + 7x + 4$ (simplify)

EXAMPLE 10 Let $P(x) = -6x^4 + 5x^3 - 2x + 5$, $Q(x) = x^5 + x^3 + x$, and $R(x) = 2x^5 + x^4 - x^2$. Find each sum.

a. $P(x) + Q(x)$ b. $P(x) + R(x)$ c. $R(x) + Q(x)$

Solution a. $P(x) + Q(x) = x^5 - 6x^4 + 6x^3 - x + 5$

b. $P(x) + R(x) = 2x^5 - 5x^4 + 5x^3 - x^2 - 2x + 5$

c. $R(x) + Q(x) = 3x^5 + x^4 + x^3 - x^2 + x$

2. Subtracting Polynomials

Definition

difference of two polynomials

The difference of two polynomials $P(x)$ and $Q(x)$ is defined as $P(x) - Q(x) = P(x) + [-Q(x)]$.

In other words, to subtract polynomials we subtract like terms.

For example, let $P(x) = 7x^3 - 4x^2 + 5$ and $Q(x) = 4x^3 + 5x - 2$.

Then $P(x) - Q(x) = (7x^3 - 4x^2 + 5) - (4x^3 + 5x - 2)$

$= (7 - 4)x^3 - 4x^2 - 5x + (5 - (-2))$

$= 3x^3 - 4x^2 - 5x + 7.$

EXAMPLE 11 Let $P(x) = 8x^4 - 3x^3 + 5x - 4$ and $Q(x) = 6x^3 + 2x^2 - 10x + 6$. Perform the calculations.

a. $P(x) - Q(x)$ b. $Q(x) - P(x)$

Solution a. $P(x) - Q(x) = (8x^4 - 3x^3 + 5x - 4) - (6x^3 + 2x^2 - 10x + 6)$

$= 8x^4 + (-3 - 6)x^3 - 2x^2 + (5 - (-10))x + (-4 - 6)$

$= 8x^4 - 9x^3 - 2x^2 + 15x - 10$

b. $Q(x) - P(x) = (6x^3 + 2x^2 - 10x + 6) - (8x^4 - 3x^3 + 5x - 4)$

$= -8x^4 + (6 - (-3))x^3 + 2x^2 + (-10 - 5)x + 6 - (-4)$

$= -8x^4 + 9x^3 + 2x^2 - 15x + 10$

EXAMPLE 12 Subtract $7x^3y + 5x^2y^2 - 3xy^3 - 4x$ from $5x^3y + 7x^2y^2 + 3xy^2 + 5x + 1$.

Solution Subtracting a polynomial is like adding the negative (opposite) of the polynomial.

$$\begin{array}{rl} & 5x^3y + 7x^2y^2 + 3xy^2 + 5x + 1 \\ - & 7x^3y + 5x^2y^2 - 3xy^3 - 4x \\ \hline & \end{array} \xleftrightarrow{\text{opposites}} \begin{array}{rl} & 5x^3y + 7x^2y^2 + 3xy^3 + 5x + 1 \\ + & -7x^3y - 5x^2y^2 - 3xy^3 - 4x \\ \hline & -2x^3y + 2x^2y^2 - 3xy^3 + 3xy^2 + 9x + 1 \end{array}$$

Note

Let $\deg[P(x)] = m$ and $\deg[Q(x)] = n$.

1. If $m > n$, then $\deg[P(x) \pm Q(x)] = m$.
2. If $m = n$, then $\deg[P(x) \pm Q(x)] \leq m$.

EXAMPLE 13 Add the polynomials $3x^3 + 5x^2 + x + 1$ and $6x^3 - 3x + 5$.

Solution We can use either horizontal or vertical addition to combine the like terms.

With horizontal addition:

$(3x^3 + 5x^2 + x + 1) + (6x^3 - 3x + 5)$

$= (3x^3 + 6x^3) + 5x^2(x - 3x) + (1 + 5)$

$= (3 + 6)x^3 + 5x^2 + (1 - 3)x + 6$

$= 9x^3 + 5x^2 - 2x + 6.$

With vertical addition:

$$\begin{array}{rl} & 3x^3 + 5x^2 + x + 1 \\ + & 6x^3 \qquad\quad - 3x + 5 \\ \hline & 9x^3 + 5x^2 - 2x + 6 \end{array} \quad \text{or} \quad \begin{array}{rl} & 3x^3 + 5x^2 + x + 1 \\ + & 6x^3 + 0x^2 - 3x + 5 \\ \hline & 9x^3 + 5x^2 - 2x + 6 \end{array}.$$

Notice that the coefficient of x^2 in the polynomial $6x^3 - 3x + 5$ is 0.

To subtract one polynomial from another polynomial, we subtract like terms.

EXAMPLE 14 Subtract $2x^3 + 7x^2 - 3x + 9$ from $7x^3 - 3x^2 + 5x + 8$.

Solution $(7x^3 - 3x^2 + 5x + 8) - (2x^3 + 7x^2 - 3x + 9)$

$= 7x^3 - 3x^2 + 5x + 8 - 2x^3 - 7x^2 + 3x - 9$ (remove parentheses)

$= (7x^3 - 2x^3) + (-3x^2 - 7x^2) + (5x + 3x) + (8 - 9)$ (combine like terms)

$= 5x^3 - 10x^2 + 8x - 1$ (simplify)

Check Yourself 5

1. Add the polynomials.

   a. $(3x + 1) + (7x + 9)$
   b. $(2x + y) + (5x + 3y)$
   c. $(x^3 + 3) + (x^3 + 3x^2 + 7)$
   d. $(3xy + y) + (5xy - y + y^2)$
   e. $(12x^3 + 3x^2 + 5) + (7x^2 + 3x^3 + 6)$
   f. $(6x^2 - 5x) + (6 + 6x + x^2)$
   g. $(2x^2 + y^2) + (x^2 - 2y^2 + xy)$
   h. $(2x^2 + 1) + (2x + x^2) + (7x - 3x^3 + 7x^2)$

2. Subtract the polynomials.

   a. $(7x + 3) - (3x + 1)$
   b. $(3x + y) - (2x + y)$
   c. $(7a + 3b + 9c) - (6a + 4b + 7c)$
   d. $(x^3 + x^2 + x + 1) - (x^3 - 3x + 1)$
   e. $(10x^3 + 15) - (12x^3 - 3x + 12)$
   f. $(a^2 + 2b) - (-2a^2 + 3b - 4)$
   g. $(5x^2 - 3) - (7x + 1) - (3x^2 - 2x - 2)$
   h. $(5x^3 - 7x + 1) - (3x^4 + 2x^2)$

3. Let $P(x) = 9x^5 - 6x^3 + 2x^2$, $Q(x) = 7x^4 - 3x^3 - x^2 + 5$, and $R(x) = 2x^5 + 5x^4 - x + 6$.

   Perform the calculations.

a. $P(x) + Q(x)$ b. $Q(x) - R(x)$ c. $P(x) - R(x)$ d. $R(x) - Q(x)$

Answers

1. a. $10x + 10$ b. $7x + 4y$ c. $2x^3 + 3x^2 + 10$ d. $y^2 + 8xy$ e. $15x^3 + 10x^2 + 11$
f. $7x^2 + x + 6$ g. $3x^2 + xy - y^2$ h. $-3x^3 + 10x^2 + 9x + 1$

2. a. $4x + 2$ b. x c. $a - b + 2c$ d. $x^2 + 4x$ e. $-2x^3 + 3x + 3$
f. $3a^2 - b + 4$ g. $2x^2 - 5x - 2$ h. $-3x^4 + 5x^3 - 2x^2 - 7x + 1$

3. a. $9x^5 + 7x^4 - 9x^3 + x^2 + 5$ b. $-2x^5 + 2x^4 - 3x^3 - x^2 + x - 1$
c. $7x^5 - 5x^4 - 6x^3 + 2x^2 + x - 6$ d. $2x^5 - 2x^4 + 3x^3 + x^2 - x + 1$

3. Multiplying Polynomials

Definition

product of two polynomials

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0$ and $Q(x) = b_m x^m + b_{m-1}x^{m-1} + \dots + b_1 x + b_0$. Then the product of $P(x)$ and $Q(x)$ is defined as

$$P(x) \cdot Q(x) = (a_n x^n + \dots + a_1 x + a_0) \cdot (b_m x^m + \dots + b_1 x + b_0).$$

In other words, to multiply polynomials we first multiply each term of the first polynomial by each term of the second polynomial, and then combine the like terms.

For example,

1. Multiply $3 \cdot (5x + 1)$.

$$3 \cdot (5x + 1) = 3 \cdot 5x + 3 \cdot 1 = 15x + 3$$

2. Multiply $7x \cdot (2x^3 + 3x)$.

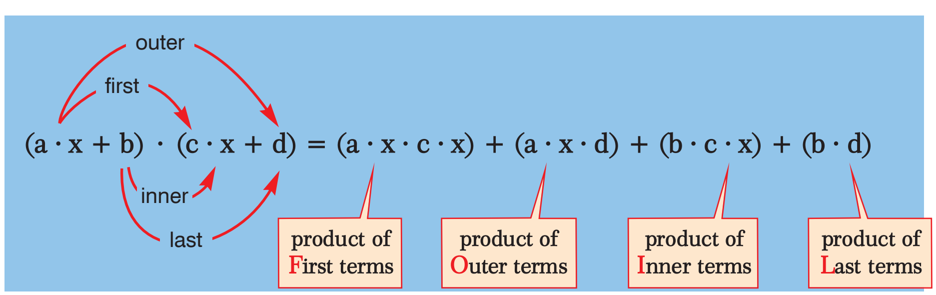
$$\begin{aligned} 7x \cdot (2x^3 + 3x) &= 7x \cdot 2x^3 + 7x \cdot 3x \\ &= (7 \cdot 2) \cdot x^1 \cdot x^3 + (7 \cdot 3) \cdot x^1 \cdot x^1 \\ &= 14x^4 + 21x^2 \end{aligned}$$

3. Multiply $(2x + 1) \cdot (x + 3)$.

To multiply two binomials, we apply the distributive property twice.

$$\begin{aligned} (2x + 1) \cdot \underbrace{(x + 3)} &= 2x \cdot (x + 3) + 1 \cdot (x + 3) && \text{(distributive property)} \\ &= 2x \cdot x + 2x \cdot 3 + 1 \cdot x + 1 \cdot 3 && \text{(distributive property)} \\ &= 2x^2 + 6x + x + 3 && \text{(combine like terms)} \\ &= 2x^2 + 7x + 3 \end{aligned}$$

We can see that the product of two binomials is the sum of the products of the first terms, the outer terms, the inner terms and the last terms. This is called the **FOIL** pattern.



4. Multiply $(3x + 2) \cdot (2x + 4)$.

$$\begin{aligned}(3x + 2) \cdot (2x + 4) &= 3x \cdot 2x + 3x \cdot 4 + 2 \cdot 2x + 2 \cdot 4 \\ &= \overbrace{6x^2}^{F} + \overbrace{12x}^{O} + \overbrace{4x}^{I} + \overbrace{8}^{L} \\ &= 6x^2 + 16x + 8\end{aligned}$$

5. Multiply $(x + 2) \cdot (x^2 + 3x + 1)$.

$$\begin{aligned}(x + 2) \cdot (x^2 + 3x + 1) &= x \cdot (x^2 + 3x + 1) + 2 \cdot (x^2 + 3x + 1) \\ &= x \cdot x^2 + x \cdot 3 \cdot x + x \cdot 1 + 2 \cdot x^2 + 2 \cdot 3x + 2 \cdot 1 \\ &= x^3 + \underline{3x^2} + \underline{\underline{x}} + \underline{2x^2} + \underline{\underline{6x}} + 2 \\ &= x^3 + 5x^2 + 7x + 2\end{aligned}$$

EXAMPLE 15 Let $P(x) = x^3 + 2x$, $Q(x) = 2x^2 - x + 1$, and $R(x) = -x^2 + 5$. Find each product.

a. $P(x) \cdot Q(x)$ b. $P(x) \cdot R(x)$ c. $Q(x) \cdot R(x)$

Solution a. $$\begin{aligned}P(x) \cdot Q(x) &= (x^3 + 2x) \cdot (2x^2 - x + 1) \\ &= x^3 \cdot 2x^2 + x^3 \cdot (-x) + x^3 \cdot 1 + 2x \cdot 2x^2 + 2x \cdot (-x) + 2x \cdot 1 \\ &= 2x^5 - x^4 + x^3 + 4x^3 - 2x^2 + 2x \\ &= 2x^5 - x^4 + 5x^3 - 2x^2 + 2x\end{aligned}$$

b. $$\begin{aligned}P(x) \cdot R(x) &= (x^3 + 2x) \cdot (-x^2 + 5) \\ &= -x^5 + 5x^3 - 2x^3 + 10x \\ &= -x^5 + 3x^3 + 10x\end{aligned}$$

c. $$\begin{aligned}Q(x) \cdot R(x) &= (2x^2 - x + 1) \cdot (-x^2 + 5) \\ &= -2x^4 + 10x^2 + x^3 - 5x - x^2 + 5 \\ &= -2x^4 + x^3 + 9x^2 - 5x + 5\end{aligned}$$

EXAMPLE 16 In the picture, each house has height $a + b$. The houses have width x, y, and z respectively. Write a polynomial to represent the surface area of the fronts of the houses.

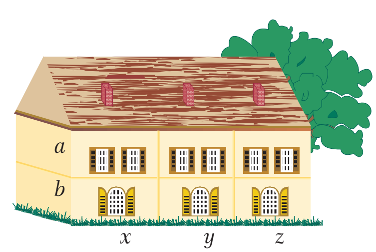


Fig. 2.1

Solution 1 $$\begin{aligned}(a+b)(x+y+z) &= a(x+y+z) + b(x+y+z) \\ &= ax + ay + az + bx + by + bz\end{aligned}$$

Note

1. In order to multiply a polynomial by a real number, multiply each coefficient by that real number.
2. The degree of the product of two polynomials is the sum of the degrees of the polynomials: $\deg[P(x) \cdot Q(x)] = \deg[P(x)] + \deg[Q(x)]$ (can you see why?).

EXAMPLE 17 Let $P(x) = 5x^7 + 4x^3 - 3x$, $Q(x) = 6x^6 + 8x^5$. Find

a. $4 \cdot P(x)$. b. $-5 \cdot Q(x)$. c. $\deg[P(x) \cdot Q(x)]$.

Solution

a. $4 \cdot P(x) = 4 \cdot (5x^7 + 4x^3 - 3x) = 20x^7 + 16x^3 - 12x$

b. $-5 \cdot Q(x) = -5 \cdot (6x^6 + 8x^5) = -30x^6 - 40x^5$

c. $\deg[P(x)] = 7$ and $\deg[Q(x)] = 6$.

So $\deg[P(x) \cdot Q(x)] = \deg[P(x)] + \deg[Q(x)] = 7 + 6 = 13$.

EXAMPLE 18 Let $P(x, y) = -3x^3y^2 + 5x^2y$ and $Q(x, y) = 2xy^3 - 4x^2y$. Find

a. $P(x, y) \cdot Q(x, y)$. b. $2 \cdot P(x, y) - 3 \cdot Q(x, y)$.

Solution

a.
$$\begin{aligned} P(x, y) \cdot Q(x, y) &= (-3x^3y^2 + 5x^2y) \cdot (2xy^3 - 4x^2y) \\ &= -3x^3y^2(2xy^3 - 4x^2y) + 5x^2y(2xy^3 - 4x^2y) \\ &= -6x^4y^5 + 12x^5y^3 + 10x^3y^4 - 20x^4y^2 \end{aligned}$$

b.
$$\begin{aligned} 2P(x) - 3Q(x, y) &= 2(-3x^3y^2 + 5x^2y) - 3(2xy^3 - 4x^2y) \\ &= -6x^3y^2 + 10x^2y - 6xy^3 + 12x^2y \\ &= -6x^3y^2 + 22x^2y - 6xy^3 \end{aligned}$$

EXAMPLE 19 We have a rectangular piece of metal with dimensions 30 cm and 42 cm. We want to make a box or tray by cutting out squares with dimension x from each corner. Write down a polynomial formula for the volume of the box, $V(x)$, in terms of x.

Solution

$$\begin{aligned} V(x) &= x \cdot (42 - 2x) \cdot (30 - 2x) \\ &= x \cdot (1260 - 84x - 60x + 4x^2) \\ &= 4x^3 - 114x^2 + 1260x \end{aligned}$$

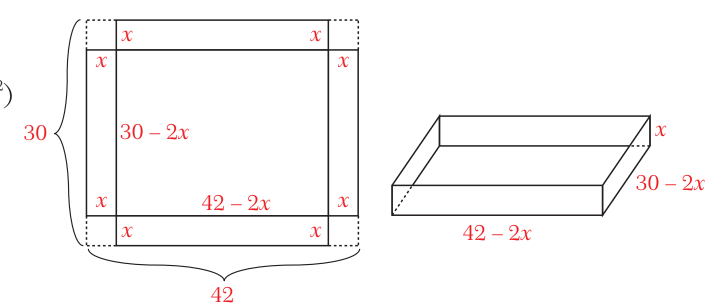


Fig. 2.2

We can show polynomial products geometrically. Look at the examples.

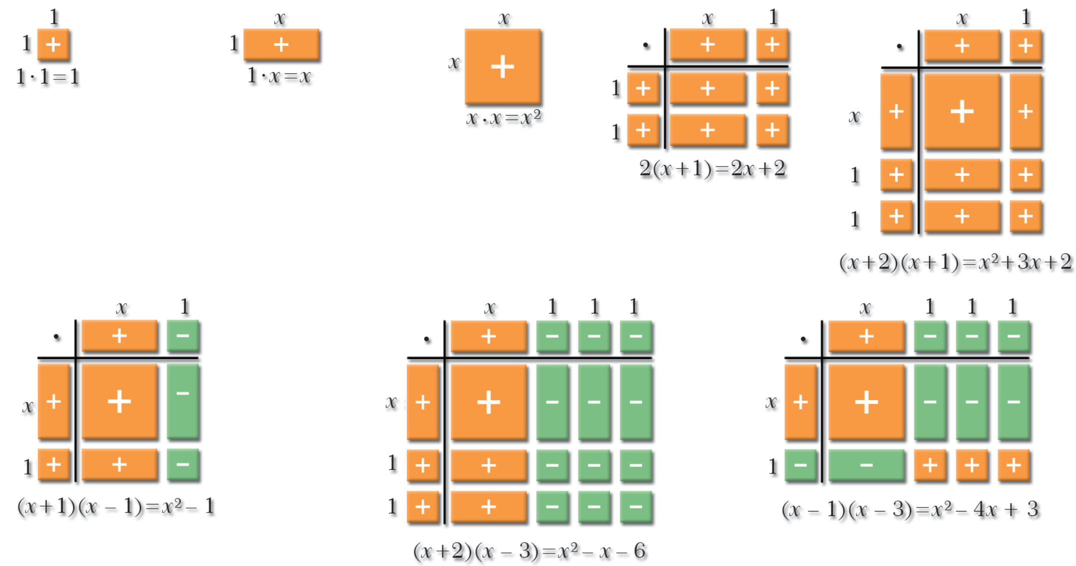


Fig. 2.3

Check Yourself 6

1. Multiply the polynomials.

a. $x \cdot (3x + 5)$

b. $2x \cdot (5 - 4x)$

c. $-5 \cdot (x^3 + 2x)$

d. $(x + 1) \cdot (2x + 3)$

e. $(2x + 1) \cdot (5x - 4)$

f. $(3x - 4) \cdot (x - \frac{1}{2})$

g. $(2x + y) \cdot (x - y)$

h. $(x + y) \cdot (3x + 2y)$

i. $(2x + 3y) \cdot (3x - 5y)$

j. $3x \cdot (x^2 + 2x + 1)$

k. $-4xy \cdot (x^2y^2 + 3xy + 2)$

l. $(2x + 1) \cdot (x^2 + 3x - 1)$

m. $(4x - 1) \cdot (x^3 + 3x - 3)$

n. $(x + y) \cdot (x^2 - xy + y^2)$

o. $(x - y) \cdot (x^2 + xy + y^2)$

2. Let $P(x) = 4x^7 - 3x^6 + 2$, $Q(x) = x^8 + 8x$, $R(x, y) = 7x^3y^4 - 5x^6y$, and $T(x, y) = -2x^5y^4 + x^4y^7$. Find

a. $P(x) \cdot Q(x)$.

b. $4 \cdot Q(x) - 5 \cdot P(x)$.

c. $R(x, y) \cdot T(x, y)$.

d. $\deg[P(x) \cdot Q(x)]$.

e. $\deg[R(x, y) \cdot T(x, y)]$.

Answers

1. **a.** $3x^2 + 5x$ **b.** $-8x^2 + 10x$ **c.** $-5x^3 - 10x$ **d.** $2x^2 + 5x + 3$ **e.** $10x^2 - 3x - 4$ **f.** $\dfrac{6x^2 - 11x + 4}{2}$

g. $2x^2 - xy - y^2$ **h.** $3x^2 + 5xy + 2y^2$ **i.** $6x^2 - xy - 15y^2$ **j.** $3x^3 + 6x^2 + 3x$ **k.** $-4x^3y^3 - 12x^2y^2 - 8xy$

l. $2x^3 + 7x^2 + x - 1$ **m.** $4x^4 - x^3 + 12x^2 - 15x + 3$ **n.** $x^3 + y^3$ **o.** $x^3 - y^3$

2. **a.** $4x^{15} - 3x^{14} + 34x^8 - 24x^7 + 16x$ **b.** $4x^8 - 20x^7 + 15x^6 + 32x - 10$
c. $-14x^8y^8 + 7x^7y^{11} - 5x^{10}y^8 + 10x^{11}y^5$ **d.** 15 **e.** 18

4. Dividing Polynomials

a. Dividing a Polynomial by a Monomial

When we divide a polynomial by a monomial of lower degree, we divide each term of the polynomial by that monomial. For example,

$$\frac{4x^2 - 6x + 5}{2} = \frac{4x^2}{2} - \frac{6x}{2} + \frac{5}{2} = 2x^2 - 3x + \frac{5}{2},$$

$$\frac{12x^3}{x} = 12x^2, \quad \frac{6x^4 - 8x^2}{2x^2} = \frac{6x^4}{2x^2} - \frac{8x^2}{2x^2} \quad 3x^2 - 4, \text{ and}$$

$$\frac{5x^3 + 3x^2 - 4x}{x} = \frac{5x^3}{x} + \frac{3x^2}{x} - \frac{4x}{x} = 5x^2 + 3x - 4.$$

Check Yourself 7

1. Divide the polynomials.

a. $\dfrac{3x^3}{x}$ **b.** $\dfrac{7x^2y}{xy}$ **c.** $\dfrac{3x^3y^2z}{xyz}$ **d.** $\dfrac{-12x^4y^2z^3}{6x^3y^2z^2}$

e. $\dfrac{27x^3y^2 + 9x^2y^2 - 3x^2y}{6x^2y}$ **f.** $\dfrac{3xy(x - 2y) + 4x(y - 2xy)}{xy}$

Answers

1. **a.** $3x^2$ **b.** $7x$ **c.** $3x^2y$ **d.** $-2xz$ **e.** $\dfrac{9xy + 3y - 1}{2}$ **f.** $-5x - 6y + 4$

b. Dividing Two Polynomials

Definition

quotient of two polynomials

Let $P(x)$ and $D(x)$ be two polynomials such that $\deg [P(x)] \geq \deg [D(x)] \geq 1$. If there exist $Q(x)$ and $R(x)$ such that $P(x) = D(x) \cdot Q(x) + R(x)$ where $\deg[R(x)] < \deg [D(x)]$, then

$P(x)$ is called the **dividend**,

$D(x)$ is called the **divisor**,

$Q(x)$ is called the **quotient**, and

$R(x)$ is called the **remainder**.

$$\begin{array}{r|l} P(x) & D(x) \\ \cline{2-2} & Q(x) \\ \hline - \quad R(x) & \end{array}$$

Note

If $R(x) = 0$, then we say that $P(x)$ is **divisible** by $D(x)$ and write $P(x) = D(x) \cdot Q(x)$.

Note

When you are dividing polynomials, make sure that the terms of the dividend and the divisor are in standard form.

EXAMPLE 20 Divide $2x^2 - x - 6$ by $x - 2$.

Solution

$$\begin{array}{r|l} 2x^2 - x - 6 & x - 2 \\ \cline{2-2} & 2x \end{array}$$

Step 1 Divide the first term of $2x^2 - x - 6$ by the first term of $x - 2$ and write the result: $\dfrac{2x^2}{x} = 2x$.

$$\begin{array}{r|l} 2x^2 - x - 6 & x - 2 \\ \cline{2-2} -/\,2x^2 - 4x & 2x \\ \hline 3x - 6 & \end{array}$$

Step 2 Multiply $(x - 2)$ by $2x$ and subtract the result from $2x^2 - x - 6$.

$$\begin{array}{r|l} 2x^2 - x - 6 & x - 2 \\ \cline{2-2} -/\,2x^2 - 4x & 2x + 3 \\ \hline 3x - 6 & \end{array}$$

Step 3 Divide $3x - 6$ by x. The result is the second term of the quotient: $\dfrac{3x}{x} = 3$.

$$\begin{array}{r|l} 2x^2 - x - 6 & x - 2 \\ \cline{2-2} -/\,2x^2 - 4x & 2x + 3 \\ \hline 3x - 6 & \\ -/\,3x - 6 & \\ \hline 0 & \end{array}$$

Step 4 Multiply $(x - 2)$ by 3 and subtract from $(3x - 6)$.

Here, $(2x + 3)$ is the quotient and $2x^2 - x - 6 = (x - 2) \cdot (2x + 3)$. The remainder is zero, so $2x^2 - x - 6$ is divisible by $x - 2$.

EXAMPLE 21 Divide $6x^3 - 9x^2 + 12x - 7$ by $2x - 3$.

Solution

$$\begin{array}{rl|l}
 & 6x^3 - 9x^2 + 12x - 7 & 2x - 3 \\ \cline{3-3}
-/ & 6x^3 - 9x^2 & 3x^2 + 6 \\ \cline{1-2}
 & 12x - 7 & \\
-/ & 12x - 18 & \\ \cline{2-2}
 & 11 &
\end{array}$$

Step 1 $\dfrac{6x^3}{2x} = 3x^2$

Step 2 $3x^2(2x - 3) = 6x^3 - 9x^2$

Step 3 $\dfrac{12x}{2x} = 6$

So

$$\underbrace{6x^3 - 9x^2 + 12x - 7}_{\text{dividend}} = \underbrace{(2x-3)}_{\text{divisor}}\underbrace{(3x^2+6)}_{\text{quotient}} + \underbrace{11.}_{\text{remainder}}$$

EXAMPLE 22 Divide $2x^2 + 4x^4 - 3 + 5x$ by $2x^2 + x$.

Solution First let us rearrange the dividend so that it is in standard form: $4x^4 + 2x^2 + 5x - 3$. Now we can perform the division:

$$\begin{array}{rl|l}
 & 4x^4 + 2x^2 + 5x - 3 & 2x^2 + x \\ \cline{3-3}
-/ & 4x^4 + 2x^3 & 2x^2 - x + \dfrac{3}{2} \\ \cline{1-2}
 & -2x^3 + 2x^2 + 5x - 3 & \\
-/ & -2x^3 - x^2 & \\ \cline{2-2}
 & 3x^2 + 5x - 3 & \\
-/ & 3x^2 + \dfrac{3}{2}x & \\ \cline{2-2}
 & \dfrac{7}{2}x - 3. &
\end{array}$$

Check Yourself 8

Find the quotients.

1. $(5x^3 + 2x^2 + 4x) \div (2x)$
2. $(12x^5 + 7x^3 + 3x) \div (x - 2)$
3. $(8x^4 + 5x^3 + 6x - 3) \div (2x^2 - 3)$

Answers

1. $2{,}5x^2 + x + 2$ 2. $12x^4 + 24x^3 + 55x^2 + 110x + 223$ 3. $4x^2 + 2{,}5x + 6$

C. Finding the Remainder

Theorem

remainder theorem

When $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

Proof Consider the following division.

$$\left.\begin{array}{r|l} P(x) & x-a \\ \cline{2-2} - & Q(x) \\ \hline R(x) & \end{array}\right\} P(x) = (x-a)\cdot Q(x) + R(x)$$

Since deg$[(x-a)] >$ deg$[R(x)]$ and deg$[(x-a)] = 1$, we can see deg$[R(x)] = 0$, and so $R(x)$ is a constant polynomial. So we can say that $R(x) = r$.

Substituting a for x gives us $P(a) = (a-a)\cdot Q(a) + r$

$P(a) = r.$

EXAMPLE 23 Find the remainder when $P(x) = x^4 - 2x^3 + 3x$ is divided by $Q(x) = x - 2$.

Solution

$$\begin{aligned} P(x) &= (x-2)\cdot R(x) + r \\ P(2) &= r \\ 2^4 - 2\cdot 2^3 + 3\cdot 2 &= r \\ 6 &= r \end{aligned}$$

$$\begin{array}{r|l} x^4 - 2x^3 + 3x & x-2 \\ \cline{2-2} -\ x^4 - 2x^3 & x^3 + 3 \\ \cline{1-1} 3x & \\ -\ 3x - 6 & \\ \cline{1-1} 6 & \end{array}$$

Theorem

factor theorem

Let $P(x)$ be a polynomial. Then the following statements are true.

1. If $P(x)$ has a factor $(x-a)$ then $P(a) = 0$.
2. If $P(a) = 0$ then $(x-a)$ is a factor of $P(x)$.

Conclusion

1. If a polynomial $P(x)$ is divisible by $x - a$ then $P(a) = 0$, i.e if $P(x) = (x-a)\cdot Q(x)$ then $P(a) = (a-a)\cdot Q(a) = 0$.
2. Conversely, if $P(a) = 0$, then $P(x) = (x-a)\cdot Q(x)$.

EXAMPLE 24 Find m if $x + 2$ is a factor of $P(x) = x^5 - 3x^3 + mx - 2$.

Solution

$$\left.\begin{aligned} x + 2 = x - (-2) &= x - a \\ a &= -2 \end{aligned}\right\}$$

$$\begin{aligned} P(a) &= 0 \\ P(-2) &= 0 \\ (-2)^5 - 3\cdot(-2)^3 + m(-2) - 2 &= 0 \\ -32 + 24 - 2m - 2 &= 0 \\ -2m - 10 &= 0 \\ m &= -5 \end{aligned}$$

EXAMPLE 25 When $P(x) = 2x^4 + ax^3 - bx^2 + 3x + 4$ is divided by $x - 1$ and $x + 2$ the remainders are both 10. Find the remainder when $P(x)$ is divided by $x + 1$.

Solution $P(1) = 10$ and $P(-2) = 10$.

$P(1) = 2 \cdot (1)^4 + a \cdot (1)^3 - b \cdot (1)^2 + 3 \cdot 1 + 4 = 2 + a - b + 3 + 4 = 10$

$a - b = 1$ ... (1)

$P(-2) = 2 \cdot (-2)^4 + a \cdot (-2)^3 - b \cdot (-2)^2 + 3 \cdot (-2) + 4$

$32 - 8a - 4b - 6 + 4 = 10$

$-8a - 4b = -20$ ...(2)

We can solve the system of equations (1) and (2) by using the elimination method:

$$\begin{array}{rl} 4\,/ & a - b = 1 \\ & 8a + 4b = 20 \\ \hline \end{array} \quad \Rightarrow \quad \begin{array}{rl} & 4a - 4b = 4 \\ + & 8a + 4b = 20 \\ \hline \end{array} \qquad \text{so } \begin{array}{r} 12a = 24 \\ a = 2. \end{array}$$

Substituting $a = 2$ into one of the equations above gives us $b = 1$.

So $P(x) = 2x^4 + 2x^3 - x^2 + 3x + 4$.

The remainder when $P(x)$ is divided by $x + 1$ is the same as $P(-1)$.

We can calculate this easily:

$$\begin{aligned} P(-1) &= 2 \cdot (-1)^4 + 2 \cdot (-1)^3 - (-1)^2 + 3 \cdot (-1) + 4 \\ &= 2 - 2 - 1 - 3 + 4 \\ &= 0. \end{aligned}$$

Check Yourself 9

1. Find the remainder when $P(x) = 3x^2 - 4x + 8$ is divided by $2x + 1$.
2. When $Q(x) = 7x^3 + 4x^2 + mx + 3$ is divided by $x + 2$ the remainder is 2. Find m.
3. When $P(x) = 4x^4 - ax^3 + bx + 6$ is divided by $x - 2$ and $x + 1$ the remainders are 56 and 9 respectively. Find the remainder when $P(x)$ is divided by $x + 2$.
4. When $Q(x)$ is divided by $x + 3$ and $x + 2$ the remainders are 5 and 8 respectively. Find the remainder when $Q(x)$ is divided by $(x + 3) \cdot (x + 2)$.

Answers

1. $\frac{43}{4}$ 2. $m = -\frac{39}{2}$ 3. 84 4. $3x + 14$

d. Finding the Remainder of Division by $x^n + a$

In order to find the remainder when $P(x)$ is divided by $x^n - a$, we substitute a for x^n. Let us look at some examples.

EXAMPLE 26 Find the remainder when $P(x) = x^6 - 2x^5 + 5x^2 + 4$ is divided by $x^2 - 2$.

Solution Let $R(x)$ be the remainder.

$$P(x) = (x^2 - 2) \cdot Q(x) + R(x)$$

$$\left.\begin{aligned} x^2 - 2 &= 0 \\ x^2 &= 2 \end{aligned}\right\}$$

$$P(x) = (x^2)^3 - 2 \cdot (x^2)^2 \cdot x + 5 \cdot x^2 + 4 \quad \text{(substitute 2 for } x^2 \text{ in } P(x))$$

$$\begin{aligned} R(x) &= 2^3 - 2 \cdot 2^2 \cdot x + 5 \cdot 2 + 4 \\ &= 8 - 8x + 10 + 4 \\ &= 8 - 8x + 10 + 4 \\ &= -8x + 22 \end{aligned}$$

EXAMPLE 27 Find the remainder when $P(x) = 4x^8 + 5x^6 + 3x$ is divided by $x^3 + 4$.

Solution Let $R(x)$ be the remainder.

$$P(x) = (x^3 + 4) \cdot Q(x) + R(x)$$

$$\left.\begin{aligned} x^3 + 4 &= 0 \\ x^3 &= -4 \end{aligned}\right\}$$

$$P(x) = 4 \cdot (x^3)^2 \cdot x^2 + 5 \cdot (x^3)^2 + 3x$$

$$R(x) = 4 \cdot (-4)^2 \cdot x^2 + 5 \cdot (-4)^2 + 3x$$

$$R(x) = 64x^2 + 3x + 80$$

Check Yourself 10

1. Find the remainder when $P(x) = 6x^5 + 4x^3 + 2x + 1$ is divided by $x^2 + 4$.
2. Find the remainder when $Q(x) = -2x^{12} + x^{10} + 6x^5$ is divided by $x^5 + 3$.

Answers

1. $82x + 1$ 2. $-18x^2 - 9$

EXERCISES 2.1

A. Introduction to Polynomials

1. Determine whether each expression is a polynomial.

   a. $x^3 + \frac{5}{2}x - 7$ b. $\sqrt{3}x^5 - \frac{6}{7}x^2 + \frac{3}{4}$

   c. $3x^4 - \frac{4}{x^3} - 2$ d. $7x^6 - 5x^5 + 3x^{-2}$

2. State the terms, coefficients, leading coefficient, constant term, and degree of each polynomial.

   a. $3x^2 + 5x - 7$ b. $-7x^6 + 12x^3 + 5x$

   c. $2xy^2 - 5x^3y + 4$ d. $6a^5 - b^2c - 4a^3b^4 + 3^2b^2c^2$

3. Find the possible value(s) of m if the expression is a polynomial.

   a. $5x^{7-m} + 4x^{2m-8} + 3$

   b. $6x^{3m-9} - 5x^{\frac{6}{m}} + 4m$

   c. $8x^{3-\frac{8}{m}} + 2x^{m+2}$

   d. $11x^{\sqrt{m}} - 9x^{\frac{36}{m}} - 3$

4. Find $a + b$ if $(3a - 6)x^3 + (5 - b)x + 15$ is a constant polynomial.

5. Find k if $(2k - 5)x^5 + 9x^2 - 3$ is a quadratic polynomial.

6. Find m if $(2n - 8)x^4 - (n - m + 2)x^2$ is a zero polynomial.

7. Find the constant term of

   a. $P(x) = 5x^3 - 11x^2 + 5$.

   b. $Q(x) = (5x^2 + 2)^5 - 3x$.

   c. $R(x, y) = (2x + 3y^2x - 1)^{12}$.

   d. $P(x + 2)$ if $P(x) = x^2 + 4x - 3$.

   e. $P(x^2 - 3)$ if $P(x + 4) = 2x^2 - 9x + 7$.

8. Find the sum of the coefficients of

   a. $P(x) = (5x^3 - 4)^6$.

   b. $Q(x, y) = (2x^2y - xy^3 + 2)^4$.

   c. $P(2x + 1)$ if $P(x) = x^3 - 2x + 1$.

   d. $R(x^2 + 1)$ if $R(x) = 3x^4 - 5x^2 + 6$.

9. Let $P(x) = (2x^3 - 1)^4$. Find the sum of the coefficients of

   a. the terms with even powers.

   b. the terms with odd powers.

B. Equality of Two Polynomials

10. The leading coefficient of a quadratic polynomial is 3 and its constant term is –3. Find $P(2) + P(-1)$ if $P(3) = -3$.

11. $P(x)$ is a cubic polynomial such that $P_E = -5$, $P_0 = 2$, $P(2) = -1$ and the constant term is –3. Find $P(3)$.

12. The constant term of $P(x + 2)$ is 5. Find the sum of the coefficients of $P(3x - 1)$.

13. $P(3 - x^2) = 2x^6 - 5x^4 + mx^2 + 6$ is given. The constant term of $P(x)$ is 18. Find m.

14. Find the unknowns if the two given polynomials are equal.

    a. $P(x) = 5x^2 - (a - 3)x + 4$ and
    $Q(x) = (b - 3)x^2 - 2x + 4$

    b. $P(x) = 6x^3 + (2a - 4)x^2 + (c - b)x$ and
    $Q(x) = 6x^3 + 2x^2 + 3x + b - 2$

EXERCISES 2.2

1. Write each polynomial in standard form.

a. $x + 2x^3 - 3$

b. $13 - 3x + 2x^2$

c. $3x^4 - 8 + 4x^2 + x^3 - 2x$

d. $x^2y + 4x^3y^2 + xy + x^4 + 4$

e. $3+\frac{1}{3}x^3+\frac{1}{9}x+\frac{1}{27}x^2$

2. Determine whether each expression is a monomial, a binomial or a polynomial.

a. x b. $2x + 1$

c. $x^2 + 4x + 1$ d. $3x^4 + 3x^2 + x$

e. $x^4 + x^3$

3. Write the degree of each polynomial.

a. $125 - x^2$ b. $x^3 + 2x^2 + 1$

c. $4x - x^4 + 3$ d. $2x^3+3x^2+\frac{1}{5}x^5$

4. Add the polynomials.

a. $(3x + 5) + (7x - 4)$

b. $(y^3 + 3) + (y^2 + 3y + 1)$

c. $(x^3 + 3x^2 + 2) + (x^2 + 2x + 1)$

d. $(4y^2 - y + 6) + (3y^2 - 6)$

e. $(3x^2 - 2x + 10) + (x^2 - 4x + 3)$

f. $(2x^2 + 5x) + (x^3 + 2x^2 + 4x + 1)$

g. $(x^5 + x^3 + x + 1) + (x^4 + x^2 + 3)$

5. Subtract the polynomials.

a. $(x^3 + x^2 + 7) - (x^2 + x)$

b. $(y^2 - 4y + 5) - (3y^2 - 4)$

c. $(w^3 + w + 2) - (w^2 + 2w + 3)$

d. $(1 - x^2 - x^4) - (3 - x - x^3)$

e. $(x^3 - x^2 + x) - (x - 8x^3)$

6. Use the distributive property to find each product.

a. $4(x + 2)$ b. $6(2x + 3)$

c. $x(x + 10)$ d. $-x(x + 3)$

e. $2x(-x + 3)$ f. $4x(3 - 2x)$

g. $8(4x^2 + 2x)$ h. $3x(4x^2 + 2x + 1)$

i. $-3x^2(x^3 + x)$

7. Use the FOIL method to find each product.

a. $(x + 1) \times (x + 2)$ b. $(x + 2) \times (x - 3)$

c. $(2x + 1) \times (x + 1)$ d. $(x - 5) \times (x - 1)$

e. $(x + 5) \times (2x + 7)$ f. $(x + 6) \times (x - 6)$

g. $(3x - 4) \times (3x - 4)$ h. $(x^2 + 1) \times (x^2 - 1)$

i. $(2a^2 - 3) \times (2a^2 + 3)$ j. $\left(y+\frac{1}{2}\right)\cdot\left(y+\frac{3}{2}\right)$

k. $(x + 2y) \times (2x + y)$

8. Divide the polynomials.

a. $\frac{15x^3}{3x^2}$ b. $\frac{12x^2y}{3xy}$

c. $\frac{10x^4y^2a}{5x^3ya}$ d. $\frac{3x^2y+2xy}{xy}$

EXERCISES 2.3

Adding, Subtracting, and Multiplying Polynomials

1. Let $P(x) = 2x^3 + 3x^2 - x + 1$,
$Q(x) = -4x^3 + x^2 + 5x$, and
$R(x) = 7x^3 - 8x^2 + x + 6$. Find the following.

   a. $P(x) + Q(x)$ b. $2R(x) + 3Q(x)$
   c. $4Q(x) - 5P(x)$ d. $P(x) - 2Q(x) + 3R(x)$

2. Let $P(x) = 3x^3 + x^2 - 1$, $Q(x) = 2x^2 + x$, and $R(x) = x^3 + 2x$. Find the following.

   a. $P(x) \cdot Q(x)$
   b. $R(x) \cdot Q(x) - 4P(x)$
   c. $P(2x) \cdot Q(x^2)$
   d. $2x^2 \cdot P(x) - x \cdot Q(x)$
   e. $x^2 \cdot P(2x) - x \cdot Q(x^2)$

3. Find $\deg[P(x) + R(x)]$ given $\deg[P(x) \cdot Q(x)] = 9$, $\deg[Q(x) \cdot R(x)] = 6$, and $\deg[P(x) \cdot R(x)] = 7$.

4. ✪ Find a and b if $4x^4 + ax^3 + bx^2 + 6x + 1 = [P(x)]^2$.

5. ✪ Find $P(5) - P(3)$ if $P(2x) + P(4x) + P(6x) = 24x - 6$.

6. Find $P(4)$ if
$P(3x - 2) = (x^2 - 1) \cdot P(x + 2) + x^2 - x + 1$.

7. ✪ Find the polynomials which satisfy $2P(x) = P(2x)$.

8. ✪ Find $P(x)$ if $P(x - 2) + P(x + 1) = 6x + 5$.

9. ✪ Find $Q(x)$ if $Q(x) \cdot Q(2x) = 8x^2 + 30x + 25$.

10. Let $\deg[P(x) + Q(x)] = 1$ and $\deg[P(x) \cdot Q(x)] = 4$. What is the degree of the quotient when $P(x)$ is divided by $(x + 5)$?

11. Find the sum of the coefficients of $P(x)$ if
$(x^2 - 3x) \cdot P(x - 1) = x^5 - 3x^4 + 2x^2 - 5x + 1$.

Dividing Polynomials

12. Find the quotients.

   a. $(3x^3 - 2x^2 + 5x) \div (x + 2)$
   b. $(6x^6 - 10x^4 + 7x^3 - 2x) \div (2x - 3)$

13. a. Find $P(x)$ if $(2x + 1) \cdot P(x) = 6x^3 + 7x^2 + 2x$.
    b. Find $Q(x)$ if $(3x + 1) \cdot Q(x) = 27x^3 + 1$.

14. Find m if $P(x) = x^4 + 5x^3 + mx + 1$ is divisible by $x + 2$.

15. Let $P(x) = (4x^2 - 7x + m)^5$. The constant term of $P(x)$ is -32. Find the remainder when $P(x)$ is divided by $x - 2$.

16. When $P(x)$ is divided by $(x + 2)$ and $(x - 1)$ the remainders are 4 and 1 respectively. What is the remainder when $P(x)$ is divided by $(x - 1) \cdot (x + 2)$?

17. What is the remainder when
$P(x) = 3x^8 + 5x^6 + 4x^3 + 4x^2 - 5$ is divided by $x^2 + 2$?

18. What is the remainder when
$P(x) = 5x^7 + 6x^6 + 3x^4 + 2x - 3$ is divided by $x^2 - x + 1$?

19. Find $a + b$ if $P(x) = 2x^5 + ax^3 + bx^2 + x - 3$ is divisible by $x^2 - 3$.

20. $P(x + 2) = (x - 3)^4 - 4(x - 3)^2 + 2x - 5$ is given. What is the remainder when $P(2x)$ is divided by $x - 1$?

21. $P(x + 3) = (x^2 + 2) \cdot Q(x + 4)$ is given. When $Q(x)$ is divided by $x - 2$ the remainder is 3. What is the sum of the coefficients of $P(x)$?

CHAPTER REVIEW TEST 2A

1. Which one of the following is not a polynomial?

A) $5x^3 - \frac{13}{2}x - 7$ B) $\sqrt{5}x^7 - 6x^2$

C) $-\frac{43}{2}$ D) $16x^2 + 5x + 3x^{-2}$

E) $4x^2 - 2$

2. How many possible integer values of n exist if $P(x) = -2x^{9-n} + 5x^{n-4} + x - 5$ is a polynomial function?

A) 7 B) 6 C) 5 D) 4 E) 3

3. What is the constant term of the polynomial $P(x, y) = (3x^2y - 5xy^4 + 2)^4 + 2x - 7$?

A) 16 B) 12 C) 9 D) 1 E) –7

4. What is the sum of the coefficients of $P(x)$ if $P(2x - 3) = 4x^3 - 5x + 6$?

A) 28 B) 18 C) 10 D) 6 E) 5

5. What is the sum of the coefficients of $P(x, y) = (6x^4y^5 - 3xy^2)^3 - 5$?

A) –5 B) 5 C) 16 D) 20 E) 22

6. What is the coefficient of the term with degree 4 in the product $(5x^3 + 2x^2 + 4x - 3) \cdot (2x^2 - 4x + 7)$?

A) –20 B) –18 C) –16 D) 4 E) 6

7. What is $\deg[P(x^2) \cdot [Q(x)]^3]$ if $P(x) = 4x^3 + 5x$ and $Q(x) = (x - 3)^4$?

A) 20 B) 18 C) 12 D) 10 E) 6

8. When $P(x) = 3x^4 + 6x^3 + 5x^2 + 8x - 4$ is divided by $(x + 2)$ the quotient is $Q(x)$. What is the sum of the coefficients of $Q(3x - 1)$?

A) –4 B) 12 C) 28 D) 32 E) 44

9. When $P(x - 2)$ is divided by $x^2 - 2x + 5$ the quotient is $2x - 3$ and the remainder is 7. What is the constant term of $P(3x - 1)$?

A) 7 B) 6 C) 5 D) 4 E) 3

10. What is the remainder when $P(x) = -6x^4 + 5x^3 - 2x + 1$ is divided by $x - 2$?

A) –59 B) –36 C) 1 D) 8 E) 12

11. What is the remainder when $P(x)$ is divided by $x - 5$ if $P(2x + 3) = (29x^3 - 17x^2 - 10x)^6$?

A) 70 B) 64 C) 32 D) 16 E) 12

12. What is the remainder when $P(3x + 4)$ is divided by $x + 2$ if $P(2x + 4) = 5x^2 + 6x - 3$?

A) 16 B) 18 C) 22 D) 24 E) 30

13. When a polynomial is divided by $x^3 - 2x^2 - 15x$ the remainder is $2x^2 + 5x - 3$. What is the remainder when the same polynomial is divided by $x + 3$?

A) –2 B) 0 C) 2 D) 4 E) 6

14. What is the value of m if $P(x) = x^2 + 5x + 11$ and $Q(x) = (x + 3)(x + k) + m$ are equal polynomials?

A) 5 B) 6 C) 7 D) 8 E) 9

15. What is the remainder when $P(3x - 2)$ is divided by $x - 2$ if $(x - 3) \cdot P(x + 2) = x^3 - 4x^2 + mx + 3$?

A) 5 B) 4 C) 3 D) 2 E) 1

16. When $P(x)$ is divided by $x + 2$ the quotient is $Q(x)$ and the remainder is 3. When $Q(x)$ is divided by $x + 3$ the remainder is –4. What is the remainder when $P(x)$ is divided by $x^2 + 5x + 6$?

A) $-4x - 11$ B) $-4x - 8$ C) $-4x - 5$

D) $4x + 3$ E) $4x + 8$

17. When $P(x) = 2x^4 - 5x^2 + mx + 3$ is divided by $x - 2$ the remainder is –5. Find m.

A) –12 B) –10 C) –6 D) –4 E) –2

18. What is the value of a if $P(x) = 4x^6 - ax^5 + 2$ is divisible by $x - 1$?

A) 6 B) 5 C) 4 D) 3 E) 2

19. What is the remainder when

$P(x) = -3x^4 + 4x^3 + 2x^2 - 1$ is divided by $x^2 - x + 1$?

A) $6x + 3$ B) $9x - 5$ C) $9x - 11$

D) $5x - 7$ E) $10x + 13$

20. When $P(x)$ is divided by $x - 3$ and $x + 3$ the remainders are 2 and 8 respectively. What is the remainder when $P(x)$ is divided by $x^2 - 9$?

A) $5x - 1$ B) $5x + 1$ C) $x - 1$

D) $x - 5$ E) $-x + 5$

CHAPTER REVIEW TEST 2B

1. What is the sum of the integer values of n that make the expression $6x^{\frac{24}{n}} - 5x^{9-n} + 2^{\frac{n}{4}}$ a polynomial?

A) 4 B) 8 C) 12 D) 24 E) 36

2. $P(x)$ and $Q(x)$ are cubic and quadratic polynomials respectively. What is the degree of the polynomial $Q(P(x)) \cdot [P(x^4)]^3$?

A) 12 B) 18 C) 24 D) 42 E) 48

3. $x + 2$ is a factor of $P(x) = 3x^2 + ax + 6$. What is the remainder when $P(x - 2)$ is divided by $x - 1$?

A) –2 B) 0 C) 1 D) 2 E) 3

4. When $P(x) = 5x^3 + 2x^2 - mx + n$ is divided by $x - 1$ and $x + 2$ the remainders are 12 and –36 respectively. What is the value of $m + n$?

A) –3 B) –2 C) –1 D) 0 E) 1

5. $P(4x - 5) = 4 \cdot x^{2003} - 32x^{2000} + 4$ is given. What is the remainder when $P(x)$ is divided *by* $x - 3$?

A) 8 B) 7 C) 6 D) 5 E) 4

6. When $P(x)$ is divided by $5x^2 - x - 4$ the remainder is $3x - 8$. What is the sum of the coefficients of $P(x)$?

A) –5 B) –3 C) –1 D) 0 E) 1

7. When $P(x + 4)$ is divided by $x + 7$ the remainder is 3. Which one of the following is divisible by $x - 1$?

A) $P(x) - 2$ B) $P(3x - 6) - 1$

C) $P(2x + 5) + 7$ D) $P(2x - 5) - 3$

E) $P(x^2) + 12$

8. What is the constant term of $P(x)$ if $P(x - 2) + P(x + 3) = 10x - 9$?

A) –10 B) –9 C) –7 D) –3 E) 2

9. Which one of the following is $P(x)$ if $P(x + 3) + P(x - 4) = 4x^2 - 18x + 79$?

A) $2x^2 + 7x + 11$ B) $2x^2 - 7x + 11$

C) $2x^2 - 7x - 11$ D) $x^2 - 7x - 11$

E) $x^2 - 7x + 11$

10. What is the sum of the coefficients of $P(4x - 7)$ if $4 \cdot P(x) + 2 \cdot P(-x) = 3x^2 + 2x - 3$?

A) 5 B) 4 C) 3 D) 2 E) 1

11. $\dfrac{P(x+5)}{Q(x-3)} = x^2 - 5x$ is given. When $Q(x)$ is divided by $(x - 1)$ the remainder is 3. What is the remainder when $P(x)$ is divided by $x - 9$?

A) –15 B) –12 C) –7 D) –6 E) –4

12. $P(x) = 2x^2 - 7x + 8$ and $Q(x) = -x + 4$ are given. What is the remainder when $P(Q(x + 3))$ is divided by $x + 2$?

A) 8 B) 7 C) 6 D) 5 E) 4

13. When $P(x)$ and $Q(x)$ are divided by $x + 5$ the remainders are 4 and –3 respectively. What is the remainder when $3P(x) - 2Q(x)$ is divided by $x + 5$?

A) 18 B) 14 C) 10 D) 6 E) 4

14. When $P(x)$ is divided by $x^2 - x + 3$ the remainder is $x - 4$. What is the remainder when $[P(x)]^2$ is divided by $x^2 - x + 3$?

A) $-x + 7$ B) $-x - 7$ C) $-7x + 13$ D) $7x - 13$ E) $7x + 13$

15. When $P(x)$ and $Q(x)$ are divided by $x^2 + 3x - 2$ the remainders are $x + 2$ and $x - 2$ respectively. What is the remainder when $x \cdot P(x) - [Q(x)]^2$ is divided by $x^2 + 3x - 2$?

A) $-6x - 4$ B) $4x + 6$ C) $4x - 6$ D) $6x - 4$ E) $6x + 4$

16. $2 \cdot P(x + 5) - 3 \cdot P(x - 2) = 4$ is given. When $P(x)$ is divided by $x + 3$ the remainder is 2. What is the remainder when $P(x)$ is divided by $x - 4$?

A) 6 B) 5 C) 4 D) 3 E) 2

17. When $P(x)$ is divided by $x^2 - 3x$ the quotient is $Q(x)$ and the remainder is $2x - 11$. When $Q(x)$ is divided by $x^2 + 3x$ the remainder is 2. What is the remainder when $P(x)$ is divided by $x^2 - 9$?

A) $-4x - 7$ B) $4x - 7$ C) $-4x + 7$ D) $4x + 7$ E) $-4x - 11$

18. What is the value of a if $P(x) = x^4 + 3x^3 + 2x^2 + ax$ is divisible by $x^2 + 2$?

A) –6 B) 1 C) 2 D) 6 E) 7

19. When $P(x) = x^5 - x^3 + x^2 + x$ is divided by $Q(x)$ the quotient is $x^2 - 2$. What is the remainder in this operation?

A) $3x + 2$ B) $2x + 2$ C) $3x - 2$ D) $2x + 3$ E) $7x + 2$

20. $P(x) + P(x + 2) = 2x^2 + 12x + 14$ is given. What is the remainder when $P(2x - 5)$ is divided by $x - 3$?

A) 2 B) 3 C) 4 D) 5 E) 6

CHAPTER 3

Factorization and Basic Identities

1 FACTORIZATION

Objectives

After studying this section you will be able to:

1. Describe the main ways of factoring polynomials and use them to solve problems.
2. Describe and apply the different methods for factoring trinomials.

In this section we will look at strategies for factoring (or factorizing) polynomial expressions. These strategies will ultimately help us to solve polynomial equations. Before we begin, let us recall the basic concepts of factor, common factor and greatest common factor.

Factoring is the reverse operation of multiplying. When we factor a number such as 12, we break it into smaller parts. $12 = 2 \cdot 6 = 2 \cdot 2 \cdot 3$ so 2 and 3 are **factors** of 12. However, they are not its only factors: 1, 4, 6, and 12 are other factors of 12.

A **common factor** of two or more integers is a number which is a factor of each integer. For example, 3 is a common factor of 12 and 15:

$$12 = 3 \cdot 2 \cdot 2$$

$$15 = 3 \cdot 5.$$

The **greatest common factor** (GCF) of two or more numbers is the greatest number that is a factor of each number.

For example, the greatest common factor of 18 and 48 is 6:

$$18 = 2 \cdot 3 \cdot 3 \qquad 48 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$$

$$\text{GCF}(18, 48) = 2 \cdot 3 = 6.$$

We can also apply the concept of greatest common factor to monomials and polynomials. For example, we can write $3x^2 + 15x$ as $(3x \cdot x) + (3x \cdot 5)$. We can see that the two terms have 3x as their greatest common factor.

EXAMPLE 1 Find the greatest common factor of $3xy^2z^3$ and $6xy^3$.

Solution

$3xy^2z^3 = 3 \cdot x \cdot y \cdot y \cdot z \cdot z \cdot z$

$6xy^3 = 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$

So $3xy^2$ is the greatest common factor of the two terms.

A. FACTORIZATION METHODS

1. Taking Out a Common Factor

If the same monomial factor occurs in each term of *a* polynomial, we can rewrite the polynomial as *a* product of this factor.

This strategy uses the distributive property:

$$3x^2 + 15x = (3x) \cdot x + (3x) \cdot 5 = \underbrace{3x}_{\text{common monomial factor}} \cdot (x + 5).$$

This method of factoring a polynomial is called 'taking out a common factor', because we take *a* common factor and put it outside the parentheses.

EXAMPLE 2 Factor the polynomials.

a. $3x^2 - 12x$ b. $4xy^3 - 6x^2y^2$ c. $8a^2b^3 + 12ab^2 - 6a^2b$

Solution a. $3x^2 - 12x = (3x \cdot x) - (3x \cdot 4)$. The greatest common factor of the terms is $3x$.

$3x \cdot (x - 4)$

b. $4xy^3 - 6x^2y^2 = (2 \cdot 2 \cdot x \cdot y \cdot y \cdot y) - (2 \cdot 3 \cdot x \cdot x \cdot y \cdot y)$ $2xy^2$ is the GCF

$= 2 \cdot x \cdot y^2 \cdot (2y) - 2 \cdot x \cdot y^2 \cdot (3x)$

$= 2xy^2(2y - 3x)$

c. $8a^2b^3 + 12ab^2 - 6a^2b = 2 \cdot a \cdot b \cdot (4ab^2) + 2 \cdot a \cdot b \cdot (6b) - 2 \cdot a \cdot b \cdot (3a)$

$= 2ab \cdot (4ab^2 + 6b - 3a).$

2. Factoring by Grouping

Sometimes it is impossible to find a common factor of all the terms in a polynomial easily. In this case, we can first try grouping the terms with a common factor. Sometimes this gives us an expression which is easier to factorize. This procedure is called factoring by grouping.

For example, let us factorize $ac + bc + ad + bd$ *by* grouping.

Since c is common factor in the first two terms we can group them.

Since d is common factor in the last two terms we can group them.

$$ac + bc + ad + bd = c \cdot (a + b) + d \cdot (a + b) = (a + b) \cdot (c + d)$$

or $$ac + bc + ad + bd = a \cdot (c + d) + b \cdot (c + d) = (a + b) \cdot (c + d)$$

EXAMPLE 3 Factor $ax + by + ay + bx$.

Solution

$$ax + by + ay + bx = ax + bx + ay + by$$
$$= x(a + b) + y(a + b)$$
$$= (a + b) \cdot (x + y)$$

EXAMPLE 4 Factor the polynomials.

a. $x^3 + x^2 + 4x + 4$ b. $ax - 3a - bx + 3b$ c. $xy + 2x - y - 2$

d. $x^2 + yz + xy + xz$ e. $a^3 + a^2 + a + 1$

Solution

a. $x^3 + x^2 + 4x + 4 = x^2(x + 1) + 4(x + 1) = (x + 1) \cdot (x^2 + 4)$

b. $ax - 3a - bx + 3b = a \cdot (x - 3) - b(x - 3) = (x - 3) \cdot (a - b)$

c. $xy + 2x - y - 2 = x \cdot (y + 2) - 1 \cdot (y + 2) = (y + 2) \cdot (x - 1)$

d. $x^2 + yz + xy + xz = x^2 + xy + yz + xz = x \cdot (x + y) + z \cdot (x + y)$
$= (x + y) \cdot (x + z)$

e. $a^3 + a^2 + a + 1 = a^2(a + 1) + 1 \cdot (a + 1) = (a + 1) \cdot (a^2 + 1)$

EXAMPLE 5 Factor $x^2 + 2xy + y^2 - z^2$.

Solution

$\underbrace{x^2 + 2xy + y^2} - z^2$ $\quad ((a + b)^2 = a^2 + 2ab + b^2))$

$= (x + y)^2 - z^2$ $\quad (a^2 - b^2 = (a + b) \cdot (a - b))$

$= (x + y + z) \cdot (x + y - z)$

EXAMPLE 6 Factor $4 - 4xy - x^2 - 4y^2$.

Solution
$$\begin{aligned}4 - 4xy - x^2 - 4y^2 &= 4 - (x^2 + 4xy + 4y^2)\\ &= 2^2 - (x^2 + 2 \cdot (x) \cdot (2y) + (2y)^2)\\ &= 2^2 - (x + 2y)^2\\ &= [2 + (x + 2y)] \cdot [2 - (x + 2y)]\\ &= (2 + x + 2y) \cdot (2 - \mathrm{x} - 2y)\end{aligned}$$

B. FACTORING TRINOMIALS

1. Factoring Perfect Square Trinomials

Recall the expansions of $(a + b)^2$ and $(a - b)^2$:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2.$$

$(a + b)^2$ and $(a - b)^2$ are perfect squares.

Their equivalent trinomials $a^2 + 2ab + b^2$ and $a^2 - 2ab + b^2$ are called **perfect square trinomials**.

If we can recognize that a trinomial is a perfect square trinomial then we can write it easily as a perfect square (in the form $(a + b)^2$ or $(a - b)^2$). Therefore, recognizing perfect square trinomials is another useful technique for factoring polynomials.

For example,

$$a^2 + 6a + 9 = a^2 + (2 \cdot a \cdot 3) + 3^2 = (a + 3)^2$$
$$x^2 - 12x + 36 = x^2 - (2 \cdot x \cdot 6) + 6^2 = (x - 6)^2.$$

EXAMPLE 7 Factor the polynomials.

a. $x^2 + 2x + 1$ **b.** $x^2 + 4x + 4$ **c.** $x^2 - 10x + 25$

d. $16x^2 + 24x + 9$ **e.** $4y^2 - 20y + 25$

Solution

a. $x^2 + 2x + 1 = x^2 + 2 \cdot x \cdot 1 + 1^2 = (x + 1)^2$

b. $x^2 + 4x + 4 = x^2 + 2 \cdot x \cdot 2 + 2^2 = (x + 2)^2$

c. $x^2 - 10x + 25 = x^2 - 2 \cdot x \cdot 5 + 5^2 = (x - 5)^2$

d. $16x^2 + 24x + 9 = 16x^2 + 24x + 9 = (4x)^2 + 2 \cdot (4x) \cdot (3) + 3^2$
$= (4x + 3)^2$

e. $4y^2 - 20y + 25 = (2y)^2 - 2 \cdot (2y) \cdot 5 + 5^2 = (2y - 5)^2$

2. Completing the Square

Consider a quadratic expression $x^2 + mx + n$. When we complete the square, we add and subtract the square of half of the coefficient of the second term of the quadratic to get a perfect square. Then we factorize the expression as the difference of two squares.

$$\begin{aligned} x^2+mx+n &= x^2+mx+(\frac{m}{2})^2-(\frac{m}{2})^2+n \\ &= (x+\frac{m}{2})^2-\frac{m^2+4n}{4} \\ &= (x+\frac{m}{2})^2-\left(\sqrt{\frac{m^2+4n}{4}}\right)^2 \\ &= \left(x+\frac{m}{2}-\sqrt{\frac{m^2+4n}{4}}\right)\cdot\left(x+\frac{m}{2}+\sqrt{\frac{m^2+4n}{2}}\right). \end{aligned}$$

Let us look at some examples of completing the square to factorize polynomials.

EXAMPLE 8 Factorize $x^2 + 5x + 4$.

Solution $x^2+5x+4 = x^2+5x+(\frac{5}{2})^2-(\frac{5}{2})^2+4$ (the square of half the coefficient of the second term is $(\frac{5}{2})^2$

$= x^2+5x+(\frac{5}{2})^2-\frac{9}{4}$

$= (x+\frac{5}{2})^2-(\frac{3}{2})^2$ (write as the difference of two squares)

$= (x+\frac{5}{2}+\frac{3}{2})(x+\frac{5}{2}-\frac{3}{2})$ (factorize)

$= (x+4)(x-1)$ (simplify)

EXAMPLE 9 Factorize $x^2 - x - 1$.

Solution

$$\begin{aligned} x^2-x+(\frac{1}{2})^2-(\frac{1}{2})^2-1 &= (x-\frac{1}{2})^2-\frac{5}{4} \\ &= (x-\frac{1}{2})^2-(\frac{\sqrt{5}}{4})^2 \\ &= (x-\frac{1}{2}-\frac{\sqrt{5}}{2})\cdot(x-\frac{1}{2}+\frac{\sqrt{5}}{2}). \end{aligned}$$

EXAMPLE 7 Factor the polynomials.

a. $x^2 + 2x + 1$ b. $x^2 + 4x + 4$ c. $x^2 - 10x + 25$

d. $16x^2 + 24x + 9$ e. $4y^2 - 20y + 25$

Solution

a. $x^2 + 2x + 1 = x^2 + 2 \cdot x \cdot 1 + 1^2 = (x + 1)^2$

b. $x^2 + 4x + 4 = x^2 + 2 \cdot x \cdot 2 + 2^2 = (x + 2)^2$

c. $x^2 - 10x + 25 = x^2 - 2 \cdot x \cdot 5 + 5^2 = (x - 5)^2$

d. $16x^2 + 24x + 9 = 16x^2 + 24x + 9 = (4x)^2 + 2 \cdot (4x) \cdot (3) + 3^2$

$= (4x + 3)^2$

e. $4y^2 - 20y + 25 = (2y)^2 - 2 \cdot (2y) \cdot 5 + 5^2 = (2y - 5)^2$

3. Other Factoring Methods

a. Factoring trinomials of the form $x^2 + bx + c$

Definition

monic polynomial

If the leading coefficient of a polynomial is 1 then the polynomial is called a **monic polynomial**.

Now let $x^2 + bx + c$ be a monic polynomial.

We can factorize this polynomial into two binomials by using the following procedure:

First identify two factors m and n of c whose sum is equal to b, so $c = m \cdot n$ and $b = m + n$.

Then $x^2 + bx + c = x^2 + (m + n)x + m \cdot n$

$= x^2 + mx + nx + m \cdot n$

$= x \cdot (x + m) + n \cdot (x + m)$

$= (x + m) \cdot (x + n)$.

In short, $x^2 + \underbrace{b}_{m+n}x + \underbrace{c}_{m \cdot n} = (x + m) \cdot (x + n)$.

For example, let us factorize $x^2 + 5x - 6$ using this method:

$x^2 + \underbrace{5}_{6+(-1)}x - \underbrace{6}_{6 \cdot (-1)} = (x + 6) \cdot (x - 1)$

EXAMPLE 10 Factorize the polynomials.

a. $x^2 + 8x + 7$ b. $x^2 + 7x + 10$ c. $x^2 - 4x - 12$

d. $x^2 - x - 2$ e. $x^2 + 7x - 18$ f. $x^2 - 7x + 10$

Solution a. $x^2 + \underbrace{8x}_{7+1} + \underbrace{7}_{7\cdot 1} = (x + 7)\cdot(x + 1)$

b. $x^2 + \underbrace{7x}_{5+2} + \underbrace{10}_{5\cdot 2} = (x + 5)\cdot(x + 2)$

c. $x^2 - \underbrace{4x}_{-6+2} - \underbrace{12}_{-6\cdot 2} = (x - 6)\cdot(x + 2)$

d. $x^2 - \underbrace{x}_{-2+1} - \underbrace{2}_{-2\cdot 1} = (x - 2)\cdot(x + 1)$

e. $x^2 + \underbrace{7x}_{9-2} - \underbrace{18}_{9\cdot(-2)} = (x + 9)\cdot(x - 2)$

f. $x^2 - \underbrace{7x}_{(-5)+(-2)} + \underbrace{10}_{(-5)\cdot(-2)} = (x - 5)\cdot(x - 2)$

Check Yourself 1

1. Factorize the polynomials.

a. $x^2 - 2x - 3$ b. $x^2 - 8x - 9$ c. $x^2 - x - 56$ d. $x^2 + 5x - 36$

e. $x^2 - 7x - 18$ f. $x^2 - 7x + 12$ g. $y^2 - 5y + 6$ h. $t^2 + t - 110$

i. $m^2 - m\cdot a - 6a^2$ j. $y^2 - y - 20$

Answers

1. a. $(x - 3)(x + 1)$ b. $(x - 9)(x + 1)$ c. $(x - 8)(x + 7)$ d. $(x + 9)(x - 4)$

e. $(x - 9)(x + 2)$ f. $(x - 4)(x - 3)$ g. $(y - 3)(y - 2)$ h. $(t + 11)(t - 10)$

i. $(m - 3a)(m + 2a)$ j. $(y - 5)(y + 4)$

b. Factoring trinomials of the form $ax^2 + bx + c$

We can factor trinomials of the form $ax^2 + bx + c$ in a similar way to monic trinomials.

Look at the following example:

$$3x^2 + 10x + 3$$

$$\begin{array}{ll} 3x & 3 \\ x & 1 \\ \hline \end{array}$$

$$3x \cdot 1 + 3 \cdot x = 6x$$

$6x \neq 10x$: doesn't satisfy the second term of the trinomial.

$$3x^2 + 10x + 3$$

$$\begin{array}{ll} 3x & 1 \\ x & 3 \\ \hline \end{array}$$

$$3x \cdot 3 + x \cdot 1 = 10x$$

This satisfies the second term of the trinomial.

So $3x^2 + 10x + 3 = (3x + 1) \cdot (x + 3)$.

More generally,

Rule

$$ax^2 + bx + c = mnx^2 + (mq + np)x + p \cdot q = (mx + p) \cdot (nx + q)$$

$$\begin{array}{ll} mx & p \\ nx & q \\ \hline \end{array}$$

$$mqx + npx = (mq + np)x$$

where m and n are factors of a and p and q are factors of c.

EXAMPLE 11 Factorize $2x^2 + 13x + 21$.

Solution

$$2x^2 + 13x + 21$$

$$\begin{array}{ll} 2x & 7 \\ x & 3 \\ \hline \end{array}$$

$$3 \cdot 2x + 7 \cdot x = 13x$$

So $2x^2 + 13x + 21 = (2x + 7) \cdot (x + 3)$.

EXAMPLE 12 Factorize $4x^2 - 7x - 2$.

Solution

$4x^2 - 7x - 2$

$$\begin{array}{ll} 4x & 1 \\ x & -2 \end{array}$$

$4x \cdot (-2) + x \cdot 1 = -8x + x = -7x$

So $4x^2 - 7x - 2 = (4x + 1) \cdot (x - 2)$.

Check Yourself 2

1. Factor the polynomials.

a. $6x^2 + 5x - 4$

b. $3x^2 + 15x + 12$

c. $6x^2 + 13x + 6$

d. $4x^2 + 11x - 3$

e. $2x^2 - 5x + 3$

f. $4y^2 - 13y + 10$

g. $2(x + 3)^2 - 7 \cdot (x + 3) - 4$

h. $4x^3 - 5x^2 + x$

i. $36x^2 + 30x - 50$

j. $2(x - y)^2 + 5(x - y) + 3$

k. $x^4 + 12x^2 + 32$

l. $x^{2y} - 5x^y - 24$

Answers

1. a. $(2x - 1)(3x + 4)$ b. $(3x + 3)(x + 4)$ c. $(3x + 2)(2x + 3)$ d. $(4x - 1)(x + 3)$

e. $(2x - 3)(x - 1)$ f. $(4y - 5)(y - 2)$ g. $(2x + 7)(x - 1)$ h. $x \, . \, (x - 1)(4x - 1)$

i. $2(3x + 5)(6x - 5)$ j. $(x - y + 1)(2x - 2y + 3)$ k. $(x^2 + 8)(x^2 + 4)$ l. $(x^y - 8)(x^y + 3)$

EXERCISES 3.1

1. Find all the factors of each number.

   a. 18 b. 48 c. 60 d. 96 e. 128

2. Find the greatest common factor of each pair of numbers.

   a. 12, 27 b. 24, 56
   c. 60, 72 d. 36, 48, 60

3. Find the greatest common factor of the given monomials.

   a. $3x$, $6xy$ b. $8xy^2$, $12x^2y^2$
   c. $15x^2yz^2$, $20x^3y^2z$ d. $3x^2yz, 9xy^2z, 27xyz^2$

4. Factor the polynomials.

   a. $ab + ac$ b. $xy + 3x^2y - 2xy^2$
   c. $35x^2y^3 + 21xy^4$
   d. $13a^2b^2c^2 + 26a^3bc^2 - 52a^3bc$

5. Factor by grouping.

   a. $ax + by + ay + bx$
   b. $ma - na + mb - nb$
   c. $a^3 - a^2 + ab - b$
   d. $z^2 + 2z + 3zy + 6y$
   e. $x^2 - 3x - xy + 3y$
   f. $a^3 + a^2b - a - b$
   g. $6a^2 + 2a - 3ab - b$
   h. $x^2 - y^2 - x - y$
   i. $a - 2b + a^2 - 4b^2$

6. Factor the trinomials.

   a. $x^2 + 3x - 4$ b. $x^2 + 2x - 15$
   c. $x^2 - 9x + 20$ d. $x^2 - 4x - 21$
   e. $x^2 - 6x - 27$ f. $x^2 - 10x + 9$

7. Factor the trinomials.

   a. $2x^2 + 5x - 3$ b. $3x^2 + 10x + 3$
   c. $3x^2 - 7x - 6$ d. $6x^2 + 7x - 3$
   e. $8x^2 + 42x + 49$ f. $14x^2 + 47x - 27$

8. Factor each expression and simplify it.

   a. $4 \cdot (x - 1)^2 - 6 \cdot (x - 1) - 10$
   b. $6x^{2y} - 7x^y - 5$

9. $8x^2 + (3a - 1)x + 45 = (4x - 5) \cdot (2x - 9)$ is given. What is the value of a?

2 BINOMIAL EXPANSIONS

Objectives

After studying this section you will be able to:

1. Understand the concept of binomial expansion.
2. Describe Pascal's triangle and explain how it is useful when expanding binomials.
3. Use basic identities to expand some common types of binomial.

A. BINOMIAL EXPANSION

1. Binomial Expansion Using Pascal's Triangle

In the previous section we learned how to use the FOIL method to find the product of two binomials. For example,

$$(x + 2)^2 = (x + 2) \cdot (x + 2) = x^2 + 2x + 2x + 4 = x^2 + 4x + 4.$$

In this example, $x + 2$ is a binomial and $(x + 2)^2$ is a binomial product. $x^2 + 4x + 4$ is called the **expansion** of $(x + 2)^2$. Similarly, the expansion of $(x - 6)^2$ is $x^2 - 12x + 36$. The process of expanding an expression of the form $(a + b)^n$ $(n \in W)$ is called **binomial expansion**.

Look at the first few expansions of $(a + b)^n$ for $n = 0$ to 5:

$(a + b)^0 = 1$

$(a + b)^1 = 1 \cdot a + 1 \cdot b$

$(a + b)^2 = 1 \cdot a^2 + 2 \cdot ab + 1 \cdot b^2$

$(a + b)^3 = 1 \cdot a^3 + 3 \cdot a^2b + 3 \cdot ab^2 + 1 \cdot b^3$

$(a + b)^4 = 1 \cdot a^4 + 4 \cdot a^3b + 6 \cdot a^2b^2 + 4 \cdot ab^3 + 1 \cdot b^4$

$(a + b)^5 = 1 \cdot a^5 + 5 \cdot a^4b + 10 \cdot a^3b^2 + 10 \cdot a^2b^3 + 5 \cdot ab^4 + 1 \cdot b^5.$

Can you see a pattern in the coefficients (in red) of each expansion?

If we look closely, we can see that they form a symmetrical triangle.

```
          1
        1   1
      1   2   1
    1   3   3   1
  1   4   6   4   1
1   5   10  10  5   1
```

```
             1
           1   1
         1   2   1
       1   3   3   1
     1   4   6   4   1
   1   5   10  10  5   1
 1   6   15  20  15  6   1
```

Each row in the triangle begins and ends with 1.

To find the other numbers in each row, we add the two numbers in the row directly above. We can

BLAISE PASCAL (1623-1662)

Blaise Pascal was a French scientist, philosopher and mathematician. He was born on June 19, 1623.

Pascal was interested in geometry at school. By the age of twelve he had studied Euclid's famous book, *Elements*. In 1642, Pascal developed a mechanical calculator to speed up arithmetic for his father. The calculator had metal wheels for entering numbers and showed the solution in small windows at the top.

Pascal's triangle has many different number patterns. It is used in many branches of mathematics, including algebra and probability. Pascal died on August 19, 1662.

use this rule to extend the triangle by any number of rows.

This special triangle is called **Pascal's triangle**. We can use Pascal's triangle to write the expansion of a binomial.

EXAMPLE 13 Expand $(x + 3)^3$.

Solution Use the binomial expansion given for $(a + b)^3$:

$(x + 3)^3 = x^3 + (3 \times x^2 \times 3^1) + (3 \times x^1 \times 3^2) + 3^3 = x^3 + 9x^2 + 27x + 27.$

EXAMPLE 14 Expand $(2a + b)^2$.

Solution $(2a + b)^2 = (2a)^2 + [2 \times (2a) \times b] + b^2 = 4a^2 + 4ab + b^2$

EXAMPLE 15 Expand $(2x + 3y)^4$.

Solution $(2x + 3y)^4 = (2x)^4 + [4\times(2x)^3\times(3y)] + [6\times(2x)^2\times(3y)^2] + [4\times(2x)\times(3y)^3] + (3y)^4$

$= 16x^4 + [4 \times 8x^3 \times 3y] + [6 \times 4x^2 \times 9y^2] + [4 \times 2x \times 27y^3] + 81y^4 = 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$

EXAMPLE 16 Expand $(x - y)^2$.

Solution $(x - y)^2 = (x + (-y))^2 = x^2 + [2 \times x(-y)] + (-y)^2 = x^2 - 2xy + y^2$

EXAMPLE 17 Expand $(3a - b)^3$.

Solution $(3a - b)^3 = (3a + (-b))^3 = (3a)^3 + [3 \times (3a)^2 \times (-b)] + [3 \times (3a) \times (-b)^2] + (-b)^3$

$= 27a^3 + [3 \times 9a^2 \times (-b)] + [3 \times (3a) \times b^2] - b^3 = 27a^3 - 27a^2b + 9ab^2 - b^3$

EXAMPLE 18 Expand $(2x - 3)^4$.

Solution $(2x - 3)^4 = (2x + (-3))^4$

$= (2x)^4 + [4\times(2x)^3\times(-3)] + [6\times(2x)^2\times(-3)^2] + [4\times(2x)^1\times(-3)^3] + (-3)^4 = 16x^2 - 96x^3 + 216x^2 - 216x + 81$

EXAMPLE 19 Expand $(x + y)^6$.

Solution In the expansion of $(x + y)^n$, the degree of each term is n. In each term the sum of the powers of x and y is n.

$(x+y)^6 = (1\times x^6y^0)+(6\times x^5y^1)+(15\times x^4y^2)+(20\times x^3y^3)+(15\times x^2y^4)+(6\times x^1y^5)+(1\times x^0y^6)$

$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

3 BASIC IDENTITIES

Recall that an **identity** is an equation which is true for all the numbers in a set.

For example, $x \cdot (y + 2) = xy + 2x$ is an identity in the set of real numbers.

Pascal's triangle helps us to find the identities for binomial expansions. In this section we will revise the common binomial expansions and look at some other identities which are useful in algebra.

A. SQUARE OF A BINOMIAL

The square of *a* binomial is the simplest binomial expansion.

Rule

$(a + b)^2 = a^2 + 2ab + b^2$

Proof $(a + b)^2 = (a + b) \cdot (a + b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2$

Alternatively, we can find this expansion using geometry.

The area of the square opposite is $(a + b)^2$.

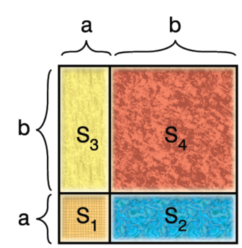


Fig. 3.1

The total area is $S_1 + S_2 + S_3 + S_4$.

$S_1 = a \cdot a = a^2$

$S_2 = a \cdot b = ab$

$S_3 = b \cdot a = ab$

$S_4 = b \cdot b = b^2$

$(a + b)^2 = S_1 + S_2 + S_3 + S_4 = a^2 + ab + ab + b^2$

$(a + b)^2 = a^2 + 2ab + b^2$

EXAMPLE 20 Expand $(x + 2)^2$.

Solution $(x + 2)^2 = x^2 + 2 \cdot x \cdot 2 + 2^2 = x^2 + 4x + 4$.

EXAMPLE 21 Expand $(3x + 2y)^2$.

Solution $(3x + 2y)^2 = (3x)^2 + 2 \cdot (3x) \cdot (2y) + (2y)^2 = 9x^2 + 12xy + 4y^2$.

EXAMPLE 22 Expand $(x+\frac{y}{2})^2$.

Solution $(x+\frac{y}{2})^2 = x^2 + 2 \cdot x \cdot \frac{y}{2} + (\frac{y}{2})^2 = x^2 + xy + \frac{y^2}{4}$.

Rule

$(a-b)^2 = a^2 - 2ab + b^2$

Proof $(a-b)^2 = (a-b) \cdot (a-b) = a^2 - ab - ba + b^2 = a^2 - 2ab + b^2$

Alternatively, we can find the expansion using geometry.

The area of the square opposite is a^2 and we want to find the area of $S = (a-b)^2$.

$S_1 = (a-b) \cdot b = ab - b^2$

$S_2 = b^2$

$S_3 = b \cdot (a-b) = ab - b^2$

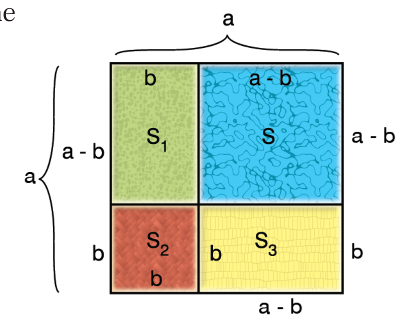


Fig. 3.2

Now $a^2 = S + S_1 + S_2 + S_3$, which we can

rewrite as $S = a^2 - (S_1 + S_2 + S_3)$

$= a^2 - (ab - b^2 + b^2 + ab - b^2)$

$= a^2 - (2ab - b^2)$

$= a^2 - 2ab + b^2$.

So $(a-b)^2 = a^2 - 2ab + b^2$.

EXAMPLE 23 Expand $(x-2)^2$.

Solution $(x-2)^2 = x^2 - 2 \cdot (x) \cdot (2) + 2^2 = x^2 - 4x + 4$.

EXAMPLE 24 Expand $(x^2-1)^2$.

Solution $(x^2-1)^2 = (x^2)^2 - 2 \cdot (x^2) \cdot (1) + 1^2 = x^4 - 2x^2 + 1$

EXAMPLE 25 Expand $(2x - 3y)^2$.

Solution $(2x - 3y)^2 = (2x)^2 - 2 \cdot (2x) \cdot (3y) + (3y)^2 = 4x^2 - 12xy + 9y^2$

EXAMPLE 26 Find 98^2 using binomial expansion.

Solution 98^2 is a difficult number to calculate quickly. However, we know that 100^2 is 10 000. Therefore, can write 98 as 100 – 2 and use binomial expansion.

$98^2 = (100 - 2)^2 = 100^2 - 2 \cdot 100 \cdot 2 + 2^2 = 10000 - 400 + 4 = 9604$

B. SQUARE OF A TRINOMIAL

Rule

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

Proof 1
$$\begin{aligned}(a + b + c)^2 &= (a + b + c) \cdot (a + b + c)\\ &= a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2\\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\end{aligned}$$

Proof 2 We can obtain the same result using geometry. Look at the rectangle.

$$\begin{array}{rl} & S_1 = a^2\\ & S_2 = a \cdot b\\ & S_3 = a \cdot c\\ & S_4 = b \cdot a\\ & S_5 = b^2\\ & S_6 = b \cdot c\\ & S_7 = c \cdot a\\ & S_8 = c \cdot b\\ + & S_9 = c^2\\ \hline \end{array}$$

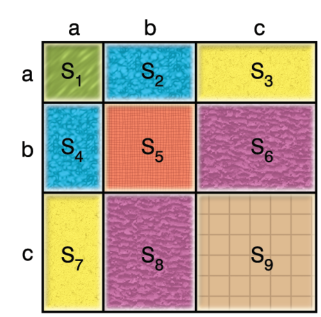


Fig. 3.3

$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$

or $(a + b + c)^2 = a^2 + b^2 + c^2 + 2 \cdot (ab + ac + bc)$, which is the required result.

Notice that rewriting this result gives us another identity:

$a^2 + b^2 + c^2 = (a + b + c)^2 - 2 \cdot (ab + ac + bc)$.

EXAMPLE 27 Expand $(2x - 3y - z)^2$.

Solution $(2x - 3y - z)^2 = (2x + (-3y) + (-z))^2$

$$= (2x)^2 + (-3y)^2 + (-z)^2 + 2 \cdot (2x) \cdot (-3y) + 2 \cdot (2x) \cdot (-z) + 2 \cdot (-3y) \cdot (-z)$$

$$= 4x^2 + 9y^2 + z^2 - 12xy - 4xz + 6yz$$

EXAMPLE 28 Expand $(4a + b - 5c)^2$.

Solution $(4a + b - 5c)^2 = (4a + b + (-5c))^2$

$$= (4a)^2 + (b)^2 + (-5c)^2 + 2 \cdot (4a) \cdot b + 2 \cdot (4a) \cdot (-5c) + 2 \cdot (b) \cdot (-5c)$$

$$= 16a^2 + b^2 + 25c^2 + 8ab - 40ac - 10bc$$

EXAMPLE 29 $x^2 + y^2 = 34$ and $x - y = 8$ are given. What is $x + y$?

Solution Let us begin by finding the relation between $(x + y)^2$ and $(x - y)^2$.

$$\begin{array}{rl} & (x + y)^2 = x^2 + 2xy + y^2 \\ - & (x - y)^2 = x^2 - 2xy + y^2 \\ \hline & (x + y)^2 - (x - y)^2 = 4xy \end{array}$$

$$(x - y)^2 = x^2 - 2xy + y^2$$

$$2xy = x^2 + y^2 - (x - y)^2$$

$$2xy = x^2 + y^2 - (x - y)^2$$

$$2xy = 34 - 64$$

$$2xy = -30$$

$$4xy = -60$$

$$(x + y)^2 - (x - y)^2 = 4xy$$

$$(x + y)^2 = -60 + 64$$

$$(x + y)^2 = 4$$

$$x + y = 2 \text{ or } x + y = -2$$

Check Yourself 3

1. Expand the expressions.

a. $(x+3)^2$ b. $(x+\frac{1}{2})^2$ c. $(x^2+2)^2$

d. $(-3x+4)^2$ e. $(2x+y^2)^2$ f. $(xy+3)^2$

g. $(x^2+\frac{2}{3}y^2)^2$ h. $(-x+(-y))^2$ i. 53^3

j. 104^2 k. 97^2 l. 48^2

m. $(2x-1)^2$ n. $(2x-3y)^2$ o. $(3m-n)^2$

p. $(2xy-3)^2$ q. $(p^2-q^2)^2$ r. $(x-\frac{1}{2x})^2$

s. $(2x+3y-z)^2$ t. $(a-2b+3c)^2$

Answers

1. a. x^2+6x+9 b. $x^2+x+\frac{1}{4}$ c. x^4+4x^2+4 d. $9x^2-24x+16$ e. $4x^2+4xy^2+y^4$

f. $x^2y^2+6xy+9$ g. $x^4+\frac{4}{3}x^2y^2+\frac{4}{9}y^4$ h. $x^2+2xy+y^2$ i. $(50+3)^2=2809$

j. $(100+4)^2=10816$ k. $(100-3)^2=9409$ l. $(50-2)^2=2304$ m. $4x^2-4x+1$

n. $4x^2-12xy+9y^2$ o. $9m^2-6mn+n^2$ p. $4x^2y^2-12xy+9$ q. $p^4-2p^2q^2+q^4$

r. $x^2-1+\frac{1}{4x^2}$ s. $4x^2+9y^2+z^2+12xy-4xz-6yz$ t. $a^2+4b^2+9c^2-4ab+6ac-12bc$

C. CUBE OF A BINOMINAL

Rule

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$

Proof 1 $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

$$\begin{aligned}(a+b)^3 &= (a+b)\cdot(a+b)\cdot(a+b)\\ &= (a+b)\cdot(a^2+2ab+b^2)\\ &= a^3+2a^2\cdot b+ab^2+a^2\cdot b+2a\cdot b^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

Proof 2 We can also think of the expansion geometrically. Look at the cube below.

If we divide each side of the cube into two parts, a and b, then the length of each side is $a + b$.

The volume of the whole cube is $(a + b)^3$. It consists of the sum of the volumes of eight rectangular solids:

one cube with volume a^3, one cube with volume b^3, three blocks each with volume a^2b, and three blocks with volume ab^2.

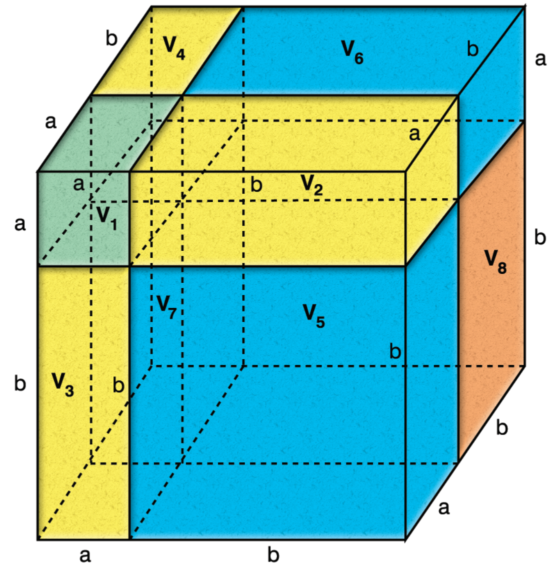


Fig. 3.4

$$
\begin{array}{ll}
 & V_1 = a^3 \\
 & V_2 = a^2 \cdot b \\
 & V_3 = a^2 \cdot b \\
 & V_4 = a^2 \cdot b \\
 & V_5 = a \cdot b^2 \\
 & V_6 = a \cdot b^2 \\
 & V_7 = a \cdot b^2 \\
+ & V_8 = b^3 \\
\hline
 & V_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7 + V_8 = a^3 + 3a^2b + 3ab^2 + b^3.
\end{array}
$$

So the volume of the whole cube is $a^3 + 3a^2b + 3ab^2 + b^3$.

EXAMPLE 30 Expand the expressions.

a. $(a + 1)^3$ b. $(x + 2)^3$ c. $(\frac{x}{2}+2y)^3$

Solution

a. $(a + 1)^3 = a^3 + 3a^2 \cdot 1 + 3 \cdot a \cdot 1^2 + 1^3$

$= a^3 + 3a^2 + 3a + 1.$

b. $(x + 2)^3 = x^3 + 3x^2 \cdot 2 + 3 \cdot x \cdot 2^2 + 2^3$

$= x^3 + 3x^2 + 12x + 8.$

c. $(\frac{x}{2}+2y)^3 = (\frac{x}{2})^3 + 3 \cdot (\frac{x}{2})^2 \cdot (2y) + 3 \cdot (\frac{x}{2}) \cdot (2y)^2 + (2y)^3$

$= \frac{x^3}{8} + \frac{3}{2}x^2y + 6xy^2 + 8y^3.$

Rule

$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

Proof

$$\begin{aligned}(a-b)^3 &= (a-b)\cdot(a-b)\cdot(a-b)\\ &= (a-b)\cdot(a^2-2ab+b^2)\\ &= a^3-2a^2b+ab^2-a^2b+2ab^2-b^3\\ &= a^3-3a^2b+3ab^2-b^3\end{aligned}$$

EXAMPLE 31 Expand the expressions.

a. $(x-1)^3$ b. $(\frac{x}{2}-\frac{y}{3})^3$ c. 19^3

Solution

a. $(x-1)^3 = x^3-3x^2+3x-1$

b. $$\begin{aligned}(\frac{x}{2}-\frac{y}{3})^3 &= (\frac{x}{2})^3-3\cdot(\frac{x}{2})^2\cdot(\frac{y}{3})+3\cdot(\frac{x}{2})\cdot(\frac{y}{3})^2-(\frac{y}{3})^3\\ &= \frac{x^3}{8}-\frac{x^2y}{4}+\frac{xy^2}{6}-\frac{y^3}{27}\end{aligned}$$

c. $$\begin{aligned}19^3 = (20-1)^3 &= 20^3-3\cdot20^2\cdot1+3\cdot20\cdot1^2-1^2\\ &= 8000-1200+60-1 = 6859\end{aligned}$$

Check Yourself 4

1. Expand the expressions.

a. $(x+1)^3$ b. $(\frac{x}{2}+3)^3$ c. $(x+2y)^3$ d. $(2x+\frac{1}{2})^3$

e. $(a+3b)^3$ f. $(x^2+1)^3$ g. $(x^3+y)^3$ h. $(5a+3b)^3$

i. $(x-3)^3$ j. $(\frac{x}{3}-4)^3$ k. $(2y-x)^3$ l. $(-x+3)^3$

m. $(-x-y)^3$ n. $(x-\frac{1}{x})^3$ o. $(x^2-1)^3$ p. $(t^3-1)^3$

Answers

1. a. x^3+3x^2+3x+1 b. $\frac{1}{8}.x^3+\frac{9}{4}.x^2+\frac{27}{2}.x+27$ c. $x^3+6x^2y+12xy^2+8y^3$

d. $8x^3+6x^2+\frac{3}{2}.x+\frac{1}{8}$ e. $a^3+9a^2b+27ab^2+27b^3$ f. $x^6+3x^4+3x^2+1$

g. $x^9 + 3x^6y + 3x^3y^2 + y^3$ h. $125a^3 + 225a^2b + 135ab^2 + 27b^3$ i. $x^3 - 9x^2 + 27x - 27$

j. $\frac{1}{27}x^3 - \frac{4}{3}x^2 + 16x - 64$ k. $8y^3 - 12y^2x + 6yx^2 - x^3$ l. $-x^3 + 9x^2 - 27x + 27$

m. $-x^3 - 3x^2y - 3xy^2 - y^3$ n. $x^3 - 3x + 3 \cdot \frac{1}{x} - \frac{1}{x^3}$ o. $x^6 - 3x^4 + 3x^2 - 1$ p. $t^9 - 3t^6 + 3t^3 - 1$

D. DIFFERENCE OF TWO SQUARES

Rule

$(a + b) \cdot (a - b) = a^2 - b^2$

The product of the sum and the difference of two terms is equal to the difference of their squares.

Proof 1 $(a + b) \cdot (a - b) = a^2 - ab + ab - b^2 = a^2 - b^2$.

Proof 2 We can prove the identity geometrically.

The area of the square opposite is $a^2 - b^2$.

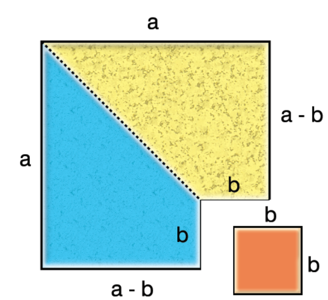


Fig. 3.5

If we draw a diagonal and cut along the diagonal, we can rearrange the pieces to form a rectangle.

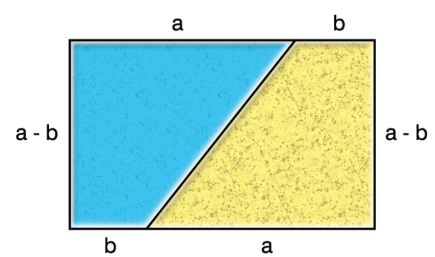


Fig. 3.6

The area of the rectangle is $(a - b) \cdot (a + b)$.

So $a^2 - b^2 = (a - b) \cdot (a + b)$.

Alternatively, we can use a different area model:

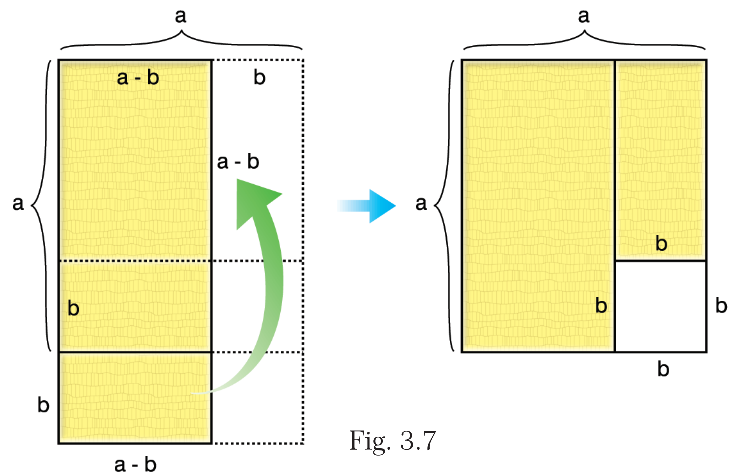


Fig. 3.7

$$(a + b) \cdot (a - b) = a^2 - b^2.$$

EXAMPLE 32 Find the products.

a. $(x + 4) \cdot (x - 4)$ b. $(x^2 + 1) \cdot (x^2 - 1)$ c. $(2a - b) \cdot (2a + b)$ d. $12 \cdot 8$

Solution a. $(x + 4) \cdot (x - 4) = x^2 - 4^2 = x^2 - 16$

b. $(x^2 + 1) \cdot (x^2 - 1) = (x^2)^2 - 1^2 = x^4 - 1$

c. $(2a - b) \cdot (2a + b) = (2a)^2 - b^2 = 4a^2 - b^2$

d. $12 \cdot 8 = (10 + 2)(10 - 2) = 10^2 - 2^2 = 100 - 4 = 96$

E. SUM OF TWO CUBES

Rule

$$a^3 + b^3 = (a + b) \cdot (a^2 - ab + b^2)$$

Proof $(a + b) \cdot (a^2 - ab + b^2) = a^3 - a^2b + ab^2 + a^2b - ab^2 + b^3 = a^3 + b^3$

EXAMPLE 33 Find the products.

a. $(x + 3) \cdot (x^2 - 3x + 9)$ b. $(x + 1) \cdot (x^2 - x + 1)$

Solution a. $(x + 3) \cdot (x^2 - 3x + 9) = (x + 3) \cdot (x^2 - 3x + 3^2) = x^3 + 3^3 = x^3 + 27$

b. $(x + 1) \cdot (x^2 - x + 1) = x^3 + 1^3 = x^3 + 1$

F. DIFFERENCE OF TWO CUBES

Rule

$a^3 - b^3 = (a - b) \cdot (a^2 + ab + b^2)$

Proof

$(a - b) \cdot (a^2 + ab + b^2) = a^3 + \cancel{a^2b} + \cancel{ab^2} - \cancel{ba^2} - \cancel{ab^2} - b^3 = a^3 - b^3$

EXAMPLE 34 Find the products.

a. $(x - 2y) \cdot (x^2 + 2xy + 4y^2)$ b. $(x - 1) \cdot (x^2 + x + 1)$

Solution

a. $(x - 2y) \cdot (x^2 + 2xy + 4y^2) = (x - 2y) \cdot (x^2 + x \cdot (2y) + (2y)^2)$
$= x^3 - (2y)^3 = x^3 - 8y^3$

b. $(x - 1) \cdot (x^2 + x + 1) = x^3 - 1^3 = x^3 - 1$

Check Yourself 5

1. Find the products.

a. $(x + 1) \cdot (x - 1)$ b. $(x + 2y) \cdot (x - 2y)$ c. $(\frac{a}{2} + b) \cdot (\frac{a}{2} - b)$

d. $(3x - y) \cdot (3x + y)$ e. $(x^2 - 2) \cdot (x^2 + 2)$ f. $(xy + 3) \cdot (xy - 3)$

2. Find the products.

a. $(x + y) \cdot (x^2 - xy + y^2)$ b. $(x + 2y) \cdot (x^2 - 2xy + 4y^2)$

c. $(x - 2) \cdot (x^2 + 2x + 4)$ d. $(x - \frac{y}{2}) \cdot (x^2 + \frac{xy}{2} + \frac{y^2}{4})$

3. Calculate $2001^2 - 1999^2$ using an identity. Show your working.

4. Calculate $1998 \cdot 2002$ using an identity. Show your working.

Answers

1. a. $x^2 - 1$ b. $x^2 - 4y^2$ c. $\frac{a^2}{4} - b^2$ d. $9x^2 - y^2$ e. $x^4 - 4$ f. $x^2y^2 - 9$

2. a. $x^3 + y^3$ b. $x^3 + 8y^3$ c. $x^3 - 8$ d. $x^3 - \frac{y^3}{8}$ 3. 8000 4. 3 999 996

EXERCISES 3.3

1. Expand the expressions.

a. $(x+2)^2$ b. $(x-1)^2$ c. $(2x+3)^3$
d. $(x-2y)^3$ e. $(x+1)^4$ f. $(x-a)^5$
g. $(1+x)^5$ h. $(a^2-b^2)^3$ i. $(3x-1)^6$

2. Expand the expressions.

a. $(x+\frac{1}{3})^2$ b. $(x^3+1)^2$ c. $(x^3-y^3)^2$
d. 98^2 e. 105^2 f. $(x-2y)^2$
g. $(3x^2+2y)^2$ h. $(2xy+1)^2$ i. $(x^2-y^2)^2$
j. $(a+b+1)^2$ k. $(a+2b-c)^2$ l. $(2a-b-3c)^2$

3. Expand the expressions.

a. $(x+1)^3$ b. $(x+2y)^3$
c. $(x-3)^3$ d. $(x-\frac{1}{x})^3$
e. $(x^2+1)^3$ f. $(x^3-1)^3$
g. 29^3 h. $(-x+y)^3$

4. Find the products.

a. $(x+1)\cdot(x+1)$
b. $(x+1)\cdot(x-1)$
c. $(\frac{a}{b}-c)\cdot(\frac{a}{b}+c)$
d. $(x^3+y^3)\cdot(x^3-y^3)$
e. $(a+b)\cdot(a^2-ab+b^2)$
f. $(x-y)\cdot(x^2+xy+y^2)$
g. $(x-2y)\cdot(x^2+2xy+4y^2)$
h. $(x-1)\cdot(x^2+x+1)$
i. $(x+3y)\cdot(x^2-3xy+9y^2)$

5. Calculate 2003^2-1997^2 using an identity.

6. Calculate $2003\cdot 1997$ using an identity.

7. Calculate the missing term in each sentence.

a. If $x-y=3$ and $x\cdot y=2$ then $x^2+y^2=?$
b. If $x+y=5$ and $x\cdot y=3$ then $x^2+y^2=?$
c. If $x-y=7$ and $x\cdot y=8$ then $x^2-y^2=?$
d. If $x^2+y^2=8$ and $x+y=3$ then $x\cdot y=?$
e. If $x-\frac{1}{x}=5$ then $x^2+\frac{1}{x^2}=?$
f. If $x-y=5$ and $x\cdot y=3$ then $x^3-y^3=?$
g. If $x^2+y^2=34$ and $x\cdot y=-15$ then $x^3+y^3=?$

8. Factor by using the difference of two squares.

a. $9a^2-64$ b. $81x^4-16y^2$
c. $64x^4-144y^6$ d. $(a+b)^2-c^2$
e. $a^2-(b-c)^2$ f. $7x^2-112$
g. $5x^3-180x$ h. $a^{2x}-b^{2y}$
i. $a^{6m}-b^{6n}$

9. Factor by using the difference of two cubes.

a. $(2x)^3-8$ b. a^3-64b^3
c. $3x^3-81y^3z^6$ d. $16a^3b^3-54c^3d^3$

10. Factor.

a. $x^5-(2y)^5$ b. a^5+b^5
c. $1+m^5$ d. $125-x^6$

Chapter 4

RATIONAL EXPRESSIONS

RATIONAL EXPRESSIONS

Objectives

After studying this section you will be able to:
1. Understand the concept of rational expression.
2. Simplify (reduce) rational expressions.
3. Add, subtract, multiply and divide rational expressions.

Definition

rational expression

A **rational expression** is an expression written as the quotient of two polynomials.

In other words a rational expression is an expression of the form $\dfrac{P(x)}{Q(x)}$ where $Q(x) \neq 0$.

For example, the expressions $\dfrac{x^2-7}{x^2+3x+1}$ and $\dfrac{2x^5+5}{5x-1}$ are rational expressions.

We can perform the same operations on rational expressions that we perform on rational numbers.

A. SIMPLIFYING (Reducing) RATIONAL EXPRESSIONS

Dividing the terms of rational expression by a common factor is commonly called 'cancelling'.

A rational expression is **simplified** (or reduced to lowest terms) if its numerator and denominator have no common factor except 1.

In order to simplify a rational expression we use the following procedure:

1. Factorize the numerator and denominator completely.
2. Divide both the numerator and the denominator by each common factor.

Note

1. If two expressions are the same then their quotient is 1.
2. If two expressions are additive inverses of each other then their quotient is –1.

For example, $\dfrac{8a^2b}{8a^2b}=1$, $\dfrac{5a-3b}{3b-5a}=-1$, and $\dfrac{\cancel{(a-b)}\,\cancel{(c-d)}}{\cancel{(b-a)}\,\cancel{(d-c)}}=1$.

EXAMPLE 1 Simplify $\dfrac{5x-10}{2x-4}$.

Solution $\dfrac{5\cancel{(x-2)}}{2\cancel{(x-2)}}=\dfrac{5}{2}$

Note

When we are working with rational expressions we assume that no divisor is equal to zero, otherwise the rational expression is undefined.

EXAMPLE 2 Simplify $\dfrac{x^2 - x - 6}{x^2 + 5x + 6}$.

Solution $\dfrac{(x-3)\cancel{(x+2)}}{(x+3)\cancel{(x+2)}} = \dfrac{x-3}{x+3}$ (Here we assume that $x + 2 \neq 0$.)

Look at some more examples:

1. $\dfrac{20x^2y^3z}{45x^3y \cdot z^2} = \dfrac{\cancel{4} \cdot \cancel{5} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{y} \cdot y \cdot y \cdot \cancel{z}}{9 \cdot \cancel{5} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot \cancel{y} \cdot \cancel{z} \cdot z} = \dfrac{4 \cdot y^2}{9 \cdot x \cdot z}$

2. $\dfrac{-9a^3b^4}{3a^2b^3} = \dfrac{-3 \cdot \cancel{3} \cdot \cancel{a^2} \cdot a \cdot \cancel{b^3} \cdot b}{\cancel{3} \cdot \cancel{a^2} \cdot \cancel{b^3}} = -3ab$

Common mistake:

$\dfrac{a + \cancel{b}}{c + \cancel{b}} \neq \dfrac{a}{c}$

$\dfrac{\cancel{x} + 1}{\cancel{x} + 2} \neq \dfrac{1}{2}$

$\dfrac{\cancel{2x} - 2}{\cancel{2x} + 5} \neq -\dfrac{2}{5}$

3. $\dfrac{2x^2 - 3y}{15y - 10x^2} = \dfrac{\overset{(-1)}{\cancel{2x^2 - 3y}}}{5\cancel{(3y - 2x^2)}} = \dfrac{-1}{5}$

4. $\dfrac{x^2 + 2x - 8}{x^2 - 9x + 14} = \dfrac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}(x+7)} = \dfrac{(x+4)}{(x+7)}$

5. $\dfrac{a^3b + 3a^2b + 9ab}{a^3 - 27} = \dfrac{ab\cancel{(a^2 + 3a + 9)}}{(a-3)\cancel{(a^2 + 3a + 9)}} = \dfrac{ab}{a-3}$

6. $\dfrac{ac + bc + ad + bd}{abc + abd} = \dfrac{c(a+b) + d(a+b)}{ab(c+d)} = \dfrac{(a+b)\cancel{(c+d)}}{ab\cancel{(c+d)}} = \dfrac{a+b}{ab}$.

EXAMPLE 3 Find the possible integer values of a if $\dfrac{x^2 - 2x - 15}{x^2 + ax + 30}$ is reducible.

Solution $\dfrac{P(x)}{Q(x)} = \dfrac{x^2 + 2x - 15}{x^2 + ax + 30} = \dfrac{(x+5)(x-3)}{x^2 + ax + 30}$.

If $\dfrac{P(x)}{Q(x)}$ is reducible then we must be able to cancel by a common factor, so $Q(x)$ must be divisible by $x + 5$ or $x - 3$.

So $Q(-5) = 0$ or $Q(3) = 0$.

If $Q(-5) = 0$ then
$$\begin{aligned} Q(-5) &= (-5)^2 + a \cdot (-5) + 30 \\ &= 25 - 5a + 30 \\ 5a &= 55 \\ a &= 11. \end{aligned}$$

If $Q(3) = 0$ then
$$\begin{aligned} Q(3) &= 3^2 + a \cdot 3 + 30 \\ 0 &= 9 + 3a + 30 \\ -3a &= 39 \\ a &= -13. \end{aligned}$$

So the possible integer values of a are 11 and –13.

Check Yourself 1

Simplify the rational expressions.

1. $\dfrac{27a^3b}{18a^2b^4}$
2. $\dfrac{ab+b^2}{4b+4a}$
3. $\dfrac{x^2-1}{x^2+2x+1}$
4. $\dfrac{x^3-3x^2y+3xy^2-y^3}{x^3-y^3}$
5. $\dfrac{2x^2-x-6}{2x^2+5x+3}$

Answers

1. $\dfrac{3a}{2b^3}$ 2. $\dfrac{b}{4}$ 3. $\dfrac{x-1}{x+1}$ 4. $\dfrac{x^2-2xy+y^2}{x^2+xy+y^2}$ 5. $\dfrac{x-2}{x+1}$

B. OPERATIONS ON RATIONAL EXPRESSIONS

1. Multiplying Rational Expressions

Definition

product of two rational expressions

The product of two rational expressions $\dfrac{P(x)}{Q(x)}$ and $\dfrac{R(x)}{T(x)}$ is defined as $\dfrac{P(x)}{Q(x)} \cdot \dfrac{R(x)}{T(x)} = \dfrac{P(x) \cdot R(x)}{Q(x) \cdot T(x)}$.

For example, $\dfrac{x+2}{x-2} \cdot \dfrac{x+5}{x-1} = \dfrac{(x+2)(x+5)}{(x-2)(x-1)}$

When multiplying rational expressions we always simplify the product if possible.

Here are some more examples of multiplying rational expressions:

1. $\dfrac{10x^2y}{8ab^3}\cdot\dfrac{4a^2}{2xy^3}=\dfrac{10x^2y\cdot 4a^2}{8ab^3\cdot 2xy^3}=\dfrac{\overset{5}{\cancel{10}}\cdot\cancel{4}\cdot\cancel{x}\cdot x\cdot\cancel{y}\cdot\cancel{a}\cdot a}{\cancel{8}\cdot 2\cdot\cancel{a}\cdot b^3\cdot\cancel{x}\cdot y^2\cdot\cancel{y}}=\dfrac{5ax}{2b^3y^2}$

2. $\dfrac{x^2-1}{x+xy}\cdot\dfrac{y^2+2y+1}{x^2-2x+1}=\dfrac{(x^2-1)\cdot(y^2+2y+1)}{(x+xy)\cdot(x^2-2x+1)}=\dfrac{\cancel{(x-1)}\cdot(x+1)\cdot(y+1)^{\cancel{2}}}{x\cancel{(y+1)}\cdot(x-1)^{\cancel{2}}}=\dfrac{(x+1)\cdot(y+1)}{x(x-1)}$

3. $\dfrac{x^2-4x+3}{x^2+x-12}\cdot\dfrac{x^2-x-6}{x^2+x-2}=\dfrac{(x-3)\cancel{(x-1)}}{\cancel{(x-3)}(x+4)}\cdot\dfrac{\cancel{(x-3)}\cancel{(x+2)}}{\cancel{(x+2)}\cancel{(x-1)}}=\dfrac{x-3}{x+4}$

4. $\dfrac{xy-1}{xy+y}\cdot\dfrac{2yx^2+4xy+2y}{x^2y+xy-x-1}=\dfrac{(xy-1)2y(x^2+2x+1)}{y(x+1)[xy(x+1)-(x+1)]}=\dfrac{2y(xy-1)(x+1)^2}{y(x+1)(xy-1)(x+1)}=2.$

2. Dividing Rational Expressions

Definition

quotient of two rational expressions

The quotient of two rational expressions $\dfrac{P(x)}{Q(x)}$ and $\dfrac{R(x)}{T(x)}$ is defined as $\dfrac{P(x)}{Q(x)}\div\dfrac{R(x)}{T(x)}=\dfrac{P(x)}{Q(x)}\cdot\dfrac{T(x)}{R(x)}$ where $Q(x)\neq 0$ and $T(x)\neq 0$.

For example, $\dfrac{5x^2}{4y^3}\div\dfrac{4x}{3y^2}=\dfrac{5x^2}{4y^3}\cdot\dfrac{3y^2}{4x}=\dfrac{15\cancel{x^2}\,\cancel{y^2}}{16\cancel{x}\,\cancel{y^3}}=\dfrac{15x}{16y}$.

Look at some more examples:

1. $\dfrac{6a^2b}{10xy^3}\div\dfrac{12ab^3}{4x^2}=\dfrac{6a^2b}{10xy^3}\cdot\dfrac{4x^2}{12ab^3}=\dfrac{\cancel{6}\cdot\cancel{4}\cdot a\cdot\cancel{a}\cdot\cancel{b}\cdot x\cdot\cancel{x}}{\underset{5}{\cancel{10}}\cdot\cancel{12}\cdot\cancel{x}\cdot y^3\cdot\cancel{a}\cdot b^2\cdot\cancel{b}}=\dfrac{ax}{5y^3b^2}$

2. $\dfrac{x^2-100}{a^2-b^2}\div\dfrac{3x+30}{a+b}=\dfrac{x^2-100}{a^2-b^2}\cdot\dfrac{a+b}{3x+30}=\dfrac{(x-10)\cancel{(x+10)}\,\cancel{(a+b)}}{(a-b)\cancel{(a+b)}\cdot 3\cancel{(x+10)}}=\dfrac{x-10}{3(a-b)}$.

Check Yourself 2

1. Simplify the rational expressions.

a. $\dfrac{12x^2}{-18x^3}$ b. $\dfrac{18xy^3}{60x^2y^2}$ c. $\dfrac{-15x^5y^4}{20x^3y^2}$ d. $\dfrac{3x^2\cdot(y+z)}{6x\cdot(y^2-z^2)}$

e. $\dfrac{3x^2y^2z}{15x^2y^2z^2+21xyz}$ f. $\dfrac{(a-b)\cdot(a-c)\cdot(b-c)}{(c-b)\cdot(a-c)\cdot(b-a)}$

2. Find the products.

a. $\frac{x}{2}\cdot\frac{4xy}{x^2y^2}$ b. $\frac{3ab}{x^2y}\cdot\frac{3xy^2}{27a^2b}$ c. $\frac{24xy}{3a^2}\cdot\frac{2a^3b}{16y}$ d. $\frac{-3m^2n^3}{4pq}\cdot\frac{12p^2q}{-mn^2}$

e. $\frac{3x+6}{5y-15}\cdot\frac{2y-6}{4x+8}$ f. $\frac{x^2-3x+2}{x-1}\cdot\frac{x+2}{x^2-4}$

3. Perform the operations and simplify.

a. $\frac{6a^5x^4}{3y^3b^2}\cdot\frac{8b^3y^2}{4a^4x^5}$ b. $\frac{3a^5b^4}{5x}\div\frac{4a^3b^3}{10x^2}$

c. $\frac{x^2+3x+2}{x^2-x-6}\cdot\frac{x^2-5x+6}{x^2-x-2}$ d. $\frac{x^2-4x-21}{x^2-5x-24}\div\frac{x^2-49}{x^2+2x-35}$

Answers

1. a. $-\frac{2}{3x}$ b. $\frac{3y}{5x}$ c. $-\frac{3}{4}x^2y^2$ d. $\frac{x}{2(y-z)}$ e. $\frac{xy}{5xyz+7}$ f. 1

2. a. $\frac{2}{y}$ b. $\frac{y}{3ax}$ c. xab d. 9mnp e. $\frac{3}{10}$ f. 1

3. a. $\frac{4ab}{xy}$ b. $\frac{3}{2}a^2bx$ c. 1 d. $\frac{x-5}{x-8}$

3. Adding and Subtracting Rational Expressions

Definition

sum or difference of two rational expressions

The sum or difference of two rational expressions $\frac{P(x)}{Q(x)}$ and $\frac{R(x)}{T(x)}$ is defined as

$$\frac{P(x)}{Q(x)}\pm\frac{R(x)}{T(x)}=\frac{P(x)\cdot T(x)}{Q(x)\cdot T(x)}\pm\frac{R(x)\cdot Q(x)}{T(x)\cdot Q(x)}=\frac{P(x)\cdot T(x)\pm R(x)\cdot Q(x)}{Q(x)\cdot T(x)}.$$

For example,

$$\frac{2x}{x-3}\pm\frac{x-2}{x+4}=\frac{2x}{x-3}\cdot\frac{(x+4)}{(x+4)}\pm\frac{x-2}{x+4}\cdot\frac{(x-3)}{(x-3)}=\frac{2x(x+4)\pm(x-2)\cdot(x-3)}{(x+4)(x-3)}.$$

When we add or subtract rational expressions we use the following procedure:

1. Find the least common multiple of the denominators and change each fraction to an equivalent fraction with this denominator.
2. Combine the fractions and simplify if necessary.

The least common multiple of two or more denominators is also called the **lowest common denominator** of the rational expressions.

Look at some examples of adding and subtracting rational expressions:

1. $\dfrac{3}{x}+\dfrac{4}{x^2}=\dfrac{3\cdot x}{x\cdot x}+\dfrac{4}{x^2}=\dfrac{3x+4}{x^2}$

2. $\dfrac{4}{a^3}-\dfrac{2}{a^2}+\dfrac{1}{a}=\dfrac{4}{a^3}-\dfrac{2}{a^2}\cdot\dfrac{a}{a}+\dfrac{1}{a}\cdot\dfrac{a^2}{a^2}=\dfrac{4}{a^3}-\dfrac{2a}{a^3}+\dfrac{a^2}{a^3}=\dfrac{4-2a+a^2}{a^3}$

3. $\dfrac{2}{x+2}+\dfrac{3}{x-2}-\dfrac{4x}{x^2-4}=\dfrac{2}{x+2}\cdot\dfrac{x-2}{x-2}+\dfrac{3}{x-2}\cdot\dfrac{x+2}{x+2}-\dfrac{4x}{x^2-4}=\dfrac{2x-4}{x^2-4}+\dfrac{3x+6}{x^2-4}-\dfrac{4x}{x^2-4}$

$$=\frac{2x-4+3x+6-4x}{x^2-4}=\frac{x+2}{x^2-4}=\frac{\cancel{(x+2)}}{(x-2)\cancel{(x+2)}}=\frac{1}{x-2}$$

4. $\dfrac{x-y}{x+y}-\dfrac{x+y}{x-y}=\dfrac{(x-y)}{(x+y)}\cdot\dfrac{(x-y)}{(x-y)}-\dfrac{(x+y)}{(x-y)}\cdot\dfrac{(x+y)}{(x+y)}=\dfrac{x^2-2xy+y^2}{(x+y)(x-y)}-\dfrac{x^2+2xy+y^2}{(x-y)(x+y)}$

$$=\frac{x^2-2xy+y^2-(x^2+2xy+y^2)}{(x+y)(x-y)}=\frac{\cancel{x^2}-2xy+\cancel{y^2}-\cancel{x^2}-2xy-\cancel{y^2}}{x^2-y^2}=\frac{-4xy}{x^2-y^2}$$

5. $\dfrac{3}{3-\dfrac{3}{1+\dfrac{1}{x+1}}}=\dfrac{3}{3-\dfrac{3}{\dfrac{x+2}{x+1}}}=\dfrac{3}{3-3\cdot\dfrac{x+1}{x+2}}=\dfrac{3}{3-\dfrac{3x+3}{x+2}}=\dfrac{3}{\dfrac{3\cdot(x+2)-(3x+3)}{x+2}}$

$$=\frac{3}{\dfrac{3x+6-3x-3}{x+2}}=\frac{3}{\dfrac{3}{x+2}}=3\cdot\frac{x+2}{3}=x+2.$$

Check Yourself 3

1. Add or subtract the rational expressions.

a. $\dfrac{3}{x+4}-\dfrac{1-x}{x+4}$ b. $\dfrac{2x+1}{x+1}+\dfrac{x+2}{x+1}$ c. $\dfrac{2x+1}{x+1}+\dfrac{x+2}{x+1}$ d. $\dfrac{3y}{y-3}-\dfrac{9}{y-3}$

e. $\dfrac{3}{x+4}-\dfrac{1-x}{x+4}+\dfrac{2}{x+4}$ f. $\dfrac{x\cdot(x+2)}{x^2-2x+3}-\dfrac{4x-3}{x^2-2x+3}$

2. Add or subtract the rational expressions.

a. $\dfrac{1}{3x^2}+\dfrac{2}{6x}-\dfrac{3}{x}$ b. $\dfrac{x}{x-3}+\dfrac{3}{x+5}$ c. $1+\dfrac{1}{x+1}+\dfrac{2}{x}$ d. $\dfrac{x+1}{x+3}-\dfrac{x-1}{3-x}$

e. $\dfrac{x}{x^2+3x+2}+\dfrac{x}{x^2-4}$ f. $\dfrac{1}{x-1}+\dfrac{x}{x^2-1}+\dfrac{2x}{x+1}$ g. $\dfrac{1}{x^2+x}-\dfrac{1}{x^2+x}+\dfrac{2x}{x^2-1}$

3. Perform the operations and simplify.

a. $\dfrac{3x+1}{x+2}+\dfrac{5}{x+2}$ b. $\dfrac{5a^2+4}{a-1}-\dfrac{9}{a-1}$ c. $\dfrac{3}{a-2}+\dfrac{3}{a+2}$

d. $\dfrac{6}{x^2-1}+\dfrac{3}{x+1}-\dfrac{2}{x-1}$ e. $(1+\dfrac{xy}{x^2-xy+y^2})\div(\dfrac{x^3-x^2y}{x^3+y^3}-1)$

Answers

1. a. $\dfrac{a-1}{a+b}$ b. $\dfrac{x+4}{x+3}$ c. 3 d. 3 e. 1 f. 1

2. a. $\dfrac{1-8x}{3x^2}$ b. $\dfrac{x^2+8x-9}{x^2+2x-15}$ c. $\dfrac{x^2+4x+2}{x(x+1)}$ d. $\dfrac{2x^2-6}{x^2-9}$ e. $\dfrac{2x^2-1}{(x^2-4)(x+1)}$

f. $\dfrac{2x^2+1}{x^2-1}$

3. a. 3 b. 5a + 5 c. $\dfrac{6a}{a^2-4}$ d. $\dfrac{1}{x-1}$ e. $\dfrac{x+y}{-y}$

EXERCISES 4.1

1. Simplify the rational expressions.

a. $\dfrac{56a^3b^4}{63a^2b^2}$ b. $\dfrac{45x^4y^4z^2}{36x^2y^7z}$

c. $\dfrac{x-3}{4x-12}$ d. $\dfrac{3x+12}{5x^2+20x}$

e. $\dfrac{12a-18b}{16a^2-36b^2}$ f. $\dfrac{x^2-x}{x^2-2x+1}$

g. $\dfrac{a^2+2ab+b^2}{-ax-bx}$ h. $\dfrac{x^2+5x-24}{x^2+15x+56}$

i. $\dfrac{2x^2-3x-20}{x^2-16}$ j. $\dfrac{(a+b)^2-c^2}{a^2-(b+c)^2}$

k. $\dfrac{x^2+4xy+4y^2-z^2}{x+2y+z}$ l. $\dfrac{9\cdot(x-2y)^2+5x-10y-4}{x^2-4xy+4y^2-1}$

2. Perform the operations.

a. $\dfrac{2x}{9}+\dfrac{x}{6}$ b. $\dfrac{4}{a}-\dfrac{3}{b}$

c. $\dfrac{4}{3x}-\dfrac{7}{9x^2}$ d. $\dfrac{5}{7a-7b}-\dfrac{7}{5b-5a}$

e. $\dfrac{1}{x-3}+\dfrac{1}{x+3}+\dfrac{6}{x^2-9}$ f. $\dfrac{4a^2b}{5xy}\cdot\dfrac{9x^2}{8ab}$

g. $\dfrac{x^2-4}{x^3-8}-\dfrac{6}{3x^2+6x+12}$

h. $\dfrac{x^2-12x+32}{x^2-5x-24}\cdot\dfrac{x^2+5x+6}{x^2-2x-8}$

i. $\dfrac{2x^2+x-10}{3x^2-2x-8}\cdot\dfrac{3x^2+10x+8}{x^2+6x+8}$

j. $\dfrac{x^2+3x-4}{x^2+x-12}\div\dfrac{x^2+4x-5}{x^2+4x-21}$

k. $\dfrac{9a^2-4}{a^2+9a+20}\cdot\dfrac{a^2+4a}{3a^2-4a-4}\div\dfrac{12a-8}{a^2+3a-10}$

3. Simplify the expressions.

a. $(\dfrac{a}{b}-2)\div(\dfrac{a}{b}+2)$

b. $(a-\dfrac{1}{a})\div(a+2+\dfrac{1}{a})$

c. $(1+\dfrac{xy}{x^2-xy+y^2})\div(\dfrac{x^3-x^2y}{x^3+y^3}-1)$

d. $\dfrac{\dfrac{3}{a}-\dfrac{4}{b}}{\dfrac{9}{a^2}-\dfrac{16}{b^2}}$ e. $\dfrac{\dfrac{1}{3x}-1}{1-2x^2-\dfrac{7x}{3}}$

f. $5-\dfrac{5}{1-\dfrac{5}{5-\dfrac{5}{a^2}}}$ g. $1-\dfrac{1}{2-\dfrac{1}{3-\dfrac{2a-1}{2a+1}}}$

4. Simplify each complex fraction.

a. $\dfrac{\dfrac{5x}{y}}{\dfrac{15xy^2}{y^3}}$ b. $\dfrac{\dfrac{a-b}{ab}}{\dfrac{a+b}{ab}}$

c. $\dfrac{\dfrac{x^2+3x+2}{x+1}}{\dfrac{x^2-4}{x-2}}$ d. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{y}}$

e. $\dfrac{\dfrac{1}{x}}{\dfrac{1}{x}-\dfrac{1}{y}}$ f. $\dfrac{\dfrac{1}{x}-\dfrac{1}{y}}{\dfrac{1}{x}+\dfrac{1}{y}}$

g. $\dfrac{\dfrac{x}{y}-\dfrac{y}{x}}{\dfrac{1}{x}-\dfrac{1}{y}}$ h. $\dfrac{1+\dfrac{3}{x}+\dfrac{2}{x^2}}{\dfrac{6}{x^2}+\dfrac{5}{x}+1}$

ý. $\dfrac{1+\dfrac{2}{1+\dfrac{x}{y}}}{3+\dfrac{x}{y}}$

5. Simplify the rational expressions.

a. $\dfrac{36x^2y^3}{12xy^2}$ b. $\dfrac{x^2+3x+2}{x^2-x-6}$

c. $\dfrac{x-3}{9-6x+x^2}$ d. $\dfrac{(x^2-4x-5)(3x+15)}{(x-5)(x^2+4x-5)}$

6. Multiply the rational expressions.

a. $\dfrac{4x^2y^3}{x^2-9}\cdot\dfrac{3-x}{6x^3y^2}$

b. $\dfrac{x^2-3x+2}{x^2-2x+1}\cdot\dfrac{x^2+4x-5}{x^2+3x-10}$

c. $\dfrac{x^2+xy+y^2}{x+y}\cdot\dfrac{x^2-y^2}{x^3-y^3}$

d. $\dfrac{2x^2+7x-4}{10x-5}\cdot\dfrac{8-2x}{x^3-16x}$

7. Find the quotients.

a. $\dfrac{x+1}{x}\div\dfrac{-2x-2}{4x^2}$

b. $\dfrac{x^2-9}{4x^2+12x}\div\dfrac{5x-15}{8x}$

c. $\dfrac{7x^2-49x-56}{x^2-4x-5}\div\dfrac{7x^2-56x}{4x^3-20x^2}$

d. $\dfrac{x^2+7x+12}{x^2-9}\div\dfrac{x^2+5x+4}{x^2-3x}$

8. $\dfrac{7x-6}{x^2-x-6}=\dfrac{A}{x-3}+\dfrac{B}{x+2}$ is given. What is A · B?

9. Add or subtract the rational expressions.

a. $\dfrac{3x}{x+5}+\dfrac{15}{x+5}$ b. $\dfrac{7x+1}{x+1}+\dfrac{x+7}{x+1}$

c. $\dfrac{x+3}{x-2}+\dfrac{x+2}{x-3}$ d. $\dfrac{4}{x+6}-\dfrac{8}{x^2+2x-24}$

e. $\dfrac{-x-1}{x^2-7x+10}+\dfrac{x+3}{x-5}-\dfrac{1}{x-2}$

f. $\dfrac{x^2}{x-5}+\dfrac{5\cdot(2x-5)}{5-x}$

10. Write each expression in its simplest form.

a. $\dfrac{x^2-2x-3}{x^2-x-6}+\dfrac{x^2+x-6}{x^2-4}$

b. $\left(\dfrac{x}{x^2-4}-\dfrac{8}{x^2+2x}\right)\div\dfrac{4-x}{x^2-2x}$

11. Simplify each complex fraction.

a. $\dfrac{x}{x+1}+\dfrac{x}{\dfrac{1}{x}+1}$ b. $a+\dfrac{1+\dfrac{1+\dfrac{1}{a}}{a+1}}{\dfrac{a}{\dfrac{1}{a}}}$

c. $\dfrac{3xy\left(\dfrac{1}{4x^2}-\dfrac{9}{y^2}\right)}{y+6x}$

CHAPTER REVIEW TEST 4

1. What is the simplest form of $\dfrac{6x^2+11x+3}{2x+3}-3$?

A) $2x + 3$ B) $2x - 3$ C) $3x - 2$ D) $3x + 2$ E) 1

2. What is the simplest form of $\dfrac{x^3-x^2y-3x+3y}{3-x^2}$?

A) 1 B) $-x - y$ C) $x + y$ D) $y - x$ E) $x - y$

3. What is the simplest form of

$$\frac{x^2+3x-10}{x^2+4x-5} \div \frac{x^2+2x-8}{x^2+3x-4}?$$

A) 1 B) $\dfrac{x-2}{x+5}$ C) $\dfrac{x-1}{x-2}$ D) $x - 4$ E) $\dfrac{x+5}{x-1}$

4. What is the simplest form of

$$\left(\frac{10}{x^2-9}+\frac{5}{x-3}\right) \div \left(\frac{2}{x+3}+1\right)?$$

A) $\dfrac{5}{x+3}$ B) $\dfrac{5}{x-3}$ C) $\dfrac{1}{x-3}$ D) $\dfrac{2}{x+3}$ E) 1

5. What is the simplest form of $\dfrac{a+1}{a-\dfrac{1}{a}}+\dfrac{1}{1-a}$?

A) $a^2 + 1$ B) a C) 1 D) $a - 1$ E) $a + 1$

6. What is the value of $\dfrac{x^3-1}{x^2-1} \div \dfrac{x^3+x^2+x}{x^2+3x+2}$ if $x=\dfrac{1}{10}$?

A) 21 B) 11 C) 5 D) 2 E) $\dfrac{6}{5}$

7. Which one of the following is a possible value of

$$\left(\frac{x^3-y^3}{xy}+3y-3x\right)\cdot\frac{x+y}{\dfrac{y}{x}+\dfrac{x}{y}-2}+y^2$$

if x is an integer?

A) 12 B) 16 C) 18 D) 21 E) 23

8. What is the simplest form of

$$\frac{2x}{2xy-y^2}+\frac{2}{y-2x}-\frac{y}{2xy-4x^2}?$$

A) 1 B) $\dfrac{x^2-y^2}{xy}$ C) $\dfrac{xy}{x+y}$ D) $\dfrac{2x-y}{2xy}$ E) $\dfrac{x-y}{xy}$

9. What is the value of $\left(1+\frac{a^2-b^2-c^2}{2bc}\right)\div\frac{a+b-c}{2bc}$ if $b = a + c$?

A) 4 B) 3 C) 2 D) 1 E) 0

10. Find the simplest form of $\frac{a^2-b^2}{b-a}\div\frac{a+b}{a}$.

A) 1 B) a C) –a D) $-\frac{1}{a}$

11. Find the simplest form of
$\frac{x^3+x^2+x+1}{x^2+1}\cdot\frac{x}{x^2+x}$.

A) 1 B) $x^2 + 1$ C) $\frac{x+1}{x^2+1}$ D) $\frac{x^2+1}{x+1}$

12. Find the simplest form of $\frac{x^2-1}{x+1}\div\frac{x^3-1}{x^2+x+1}$.

A) x B) x – 1 C) x + 1 D) 1

13. Find the simplest form of
$\left(1-\frac{x^2+1}{x^3+1}\right)\div\frac{x-1}{x^2-x+1}$

A) $\frac{x}{x+1}$ B) $\frac{x^2}{x+1}$ C) $\frac{x^2}{x-1}$ D) $\frac{x^2+1}{x+1}$

14. Find the simplest form of $\frac{x-\frac{1}{x}}{1-\frac{1}{x}}\cdot\frac{x}{x^2+x}$.

A) 0 B) 1 C) 2 D) $\frac{x}{x+1}$

15. If a ≠ 0 and a ≠ 1 find $\frac{\frac{1}{1-a}}{1-\frac{1}{1-\frac{1}{a}}}$

A) 3 B) 2 C) 1 D) 0

16. Find the simplest form of $\frac{\frac{x}{x-y}-\frac{y}{x+y}}{\frac{x}{x+y}+\frac{y}{x-y}}$.

A) x(x – y) B) $x^2 + y^2$ C) 0 D) 1

Chapter 5

APPROXIMATION ERRORS

APPROXIMATION

Objectives

After studying this section you will be able to:
1. Round off numbers and measeures to a specified degree of accuracy.
2. Make estimates of numbers and measures.
3. Calculate absolute and relative errors.

A. ESTIMATION AND ROUNDING

In our daily life, we often need to use estimation when getting a precise answer is impossible, unnecessary, or inconvinient.

Estimation often involves rounding. In Mathematics we round off a number to the nearest. For example we round off 5.264 to 5.26 and round of 9.568 to 9.57.

Can you estimate the number of oranges in the picture shown? Many time we have to make estimation and approximation. A methodical way of making estimations is a far better method than making a wild guesses.

B. APPROXIMATIONS IN MEASUREMENTS AND ACCURACY

In most of our daily situations we do not need to use highl sensitive measuring devices. How accurate our measurements are depends on what we need the information for. For example if we use a compass to guide us from one end of the school to the other, it would not be a serious error if we are 1 off course. However, 1 off course on a journey from the earth to the moon will mean an error of 644000km!

Besides the errors arising from the use of different instruments, human error is another source of error. In an athletics meet, the time for the first placing of a 100-m race given by two timekeepers may be slightly different. That is why in a school athletics meet there are usually two or more timekeepers for the first few placings.

In fact, all physical measurements such as mass, length, time, area, and volume can never be absolutely accurate. They are only approximations. The accuracy of a measurement depends on the measuring instruments and the person taking the measurement. Both of them can never be absolutely accurate.

In rounding, we may round up, round down, or round off to the nearest. For example, we round up 80 to 1000 when we budget for a trip that will cost at least $850. At the car park, the car park charges are often rounded up. For example, you park for 1 hour and 50 minutes, you pay $2 at a parking rate of 50 cents for every half hour. At some supermarkets, the bill is rounded down to the nearest 5 cents. For exaamle, if your bill is $12.03, you pay $12.00 and if your bill is 12.09, you pay $12.05

EXAMPLE 1 During a sale, one kilogram of fish was sold for $4.95. Estimate how many kilograms of fish you could buy with $20.

Solution

Firstly, let's round price of fish $4.95 to $5.

Estimated amount of fish can be calculated by dividing the amount of money by rounded price of fish per kilogram.

So, $20 / $5 = 4 kilograms.

EXAMPLE 2 Estimate the cost of 8 copies of a textbook at $10.49 each.

Solution

To estimate the cost of 8 copies of the textbook,we may choose to round $10.49 to the nearest 10 cents and obtain the result as shown below. The actual cost is also provided for comparison.

Estimation cost: $10.50 x 8 = $84

Actual cost: $10.49 x 8 = $83.92

For a quick estimate, which can be done mentally, round $10.49 to the nearest dollar and the estimate is $10 x 8 = $80.

C. ABSOLUTE AND RELATIVE ERRORS

Many a times it happens that there will approximately some error in the instruments due to negligence in measuring precisely. These approximation values with errors when used in calculations may lead to larger errors in the values. There are two ways to measure errors commonly - absolute error and relative error.

The absolute error tells about how much the approximate measured value varies from true value whereas the relative error decides how incorrect a quantity is from the true value.

Suppose the measurement has some errors compared to true values. Relative error decides how incorrect a quantity is from a number considered to be true. Unlike absolute error where the error decides how much the measured value deviates from the true value. the relative error is expressed as a percentage ratio of absolute error to the true value. tells what's the error percentage?

To calculate the relative error use the following way:

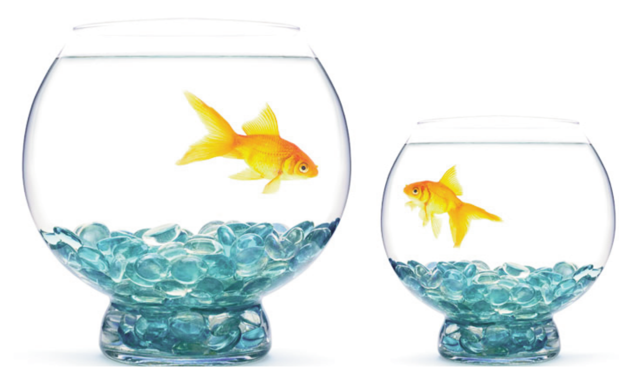
Observe the true value (x) and approximate measured value (x_o). Then find the absolute error using formula

Absolute error $\Delta x = |\text{True value - measured value}| = |x - x_o|$

Relative error =Absolute error / Value of thing to be measured = $\Delta x/x$.

In terms of percentage it is expressed as

Relative error = $\Delta x/x \times 100\ \%$



EXAMPLE 3 Darkhan measures the size of metal ball as 3.97 cm but the actual size of it is 4 cm. Calculate the absolute error and relative error.

Solution The measured value of metal ball $x_o = 3.97$ cm

The true value of ball $x = 4$ cm

Absolute error Δx = True value – Measured value =

$= x - x_o = 4\text{cm} - 3.97\text{cm} = 0.03$ cm

Relative error $= \Delta x/x = 0.03/4 = 0.0075$.

EXAMPLE 4 If the approximate value of π is 3.14. Calculate the absolute and relative errors?

Solution

The measured value π is $x_o = 3.14$

The true value of π is $x = 3.142$

Absolute error Δx = True value - Measured value $= x - x_o = 3.142 - 3.14 = 0.002$

Relative error $= \Delta x/x = 0.002/3.142 = 0.0006$.

EXERCISES 5.1

1. Estimate each ofthe following mentally:
   a. 3.14 x 80.5
   b. 4831.9 x 229.78

2. Write the following correct to 3 decimal places:
   a. 712.8923
   b. 0.00272

3. During a sale, one car was sold for 4980230 tenges. Estimate how much money should you have to buy 30 cars.

4. One million seconds is about
   a. 3 days b. 12 days
   c. 3 months d. 1 year

5. Write the following correct to second decimal places and find absolute error which was made:
   a. 4.525 b. 12.972

6. Write the following correct to third decimal places and find absolute error which was made:
   a. 0.8274 b. 7.02458

7. Write the following correct to second decimal places and find absolute and relative errors:
   a. 3:7 b. 5:9

8. The temperature in Astana in summer generally is between 22C and 23C. What is relative error if we take average temperature in Astana in summer as the arithmetic mean of 22C and 23C?

ANSWERS TO EXERCISES

Review Exercises

2. a. $\frac{34}{5}$ b. $\frac{401}{4}$ c. $\frac{28}{5}$ d. $\frac{39}{6}$ 3. a. fraction b. numerator c. denominator

4. a. $\frac{4}{8}<\frac{5}{8}<\frac{7}{8}<\frac{9}{8}$ c. $\frac{11}{9}<\frac{11}{7}<\frac{11}{5}<\frac{11}{4}$ e. $\frac{12}{13}<\frac{13}{14}<\frac{16}{16}<\frac{17}{12}$ 5. a. $\frac{2}{5}>\frac{4}{18}$ b. $\frac{8}{5}>\frac{11}{7}$ c. $-\frac{6}{9}>-\frac{7}{10}$ d. $-\frac{8}{7}>-\frac{11}{9}$ 6. a. $\frac{67}{35}$ c. $\frac{51}{24}$ e. $\frac{7}{10}$ g. $\frac{29}{9}$ 7. a. $\frac{4}{35}$ c. $-\frac{11}{8}$ e. $-\frac{1}{2}$ g. $\frac{7}{9}$

8. a. $\frac{53}{180}$ b. $\frac{11}{12}$ c. $-\frac{7}{8}$ d. $\frac{601}{105}$ e. $-\frac{149}{210}$ f. $-\frac{4}{21}$ 9. a. $\frac{21}{55}$ b. 1 c. $\frac{2}{9}$ d. $\frac{35}{66}$ e. $\frac{217}{12}$ f. $\frac{407}{28}$ g. $\frac{63}{8}$

10. a. $\frac{5}{7}$ b. $\frac{5}{7}$ c. $\frac{20}{31}$ d. $\frac{1}{3}$ 11. a. $\frac{12}{7}$ b. $-\frac{187}{20}$ c. 2^{18} d. $\frac{3}{4}$ e. $\frac{16}{15}$ f. 3 12. a. $\frac{5}{6}$ b. $\frac{72}{5}$ c. $\frac{5}{8}$ d. ð9 e. $\frac{1}{18}$ f. $\frac{7}{40}$ g. ð32 h. $\frac{40}{51}$ i. $\frac{45}{2}$ j. $\frac{28}{75}$ 13. a. $\frac{7}{5}$ b. $\frac{3}{8}$ c. $\frac{4}{7}$ d. $\frac{3}{4}$ 14. a. $\frac{7}{6}$ b. $\frac{17}{4}$ c. 1 d. $\frac{55}{12}$ e. $\frac{137}{6}$ f. $\frac{16}{15}$

15. a. $\frac{1}{50}$ b. $\frac{51}{2}$ c. 0 d. $\frac{457}{320}$ e. $\frac{35}{22}$ f. $\frac{1}{9}$ g. –1 h. $\frac{5}{2}$ i. $\frac{21}{5}$ 16. 512 17. –2

18. a. 3 b. 9 c. 4 d. 9 e. 11 f. 24 g. –6 h. $-\frac{1}{23}$ 19. a. –1 b. 2 21. 30, 40, 50 22. 120, 200, 280, 400 23. 66

24. 16, 28, 36 25. a. no solution b. 5,5 c. 30 d. –3

Exercises 1.1

1. a. m^4 b. x^5 c. x^2y^3 d. xy^2z^3 2. a. 8 b. 1 c. 9 d. 128 e. 675 f. 192 g. 72 h. 9 i. 216 j. -3^{10} k. x^{10} l. $72x^5$ m. 5^{x+y+1} n. -3^{75} o. 4^{3x+1} p. 3^{k+1} q. 2^{2a+b+1} 3. a. -2^7 b. $\frac{9}{4}$ c. 3^{2k} d. 2^{2x+2} e. 3^{2k+2} f. 2^x 4. a. 2^6 b. 2^{10} c. 2^{12} d. 3^3 e. 5^{5x} 5. a. 2^6 b. a^b c. a^{2b} d. 3^{x-1} e. 400 f. 2^{-32} 6. a. 2^{x+2} b. 6×5^6 c. 16513 d. 39×3^{-x} e. 2 f. 5 g. 2^k h. 3^m 7. a. a^2b b. $(ab)^2$ c. $a^{-4} \times b^{-1}$ 8. a. $n = m^6$ b. $m = \frac{1}{(n-1)^2}$ c. $m = n^{\frac{2}{3}}$ d. $(\frac{m}{3})^4$ 9. a. 5 b. 6 c. 3 d. –3 e. 2 f. 2 10. 3^3 11. $n = 5$

Exercises 2.1

1. a. polynomial b. polynomial c. not a polynomial d. not a polynomial 3. a. {4, 5, 6, 7} b. {3, 6} c. {–1, –2, 4, 8} d. {1, 4, 9, 36} 4. 7 5. $\frac{5}{2}$ 6. 6 7. a. 5 b. 32 c. 1 d. 9 e. 168 8. a. 1 b. 81 c. 22 d. 34 9. a. 41 b. –40 10. 0 11. 9 12. 5 13. 1 14. a. $a = 5, b = 8$ b. $a = 3, b = 2, c = 5$

Exercises 2.2

1. a. $2x^3 + x - 3$ b. $2x^2 - 3x + 13$ c. $3x^4 + x^3 + 4x^2 - 2x - 8$ d. $4x^3y^2 + x^4 + x^2y + xy + 4$ e. $\frac{9x^3 + x^2 + 3x + 81}{27}$
2. a. monomial b. binomial c. trinomial d. trinomial e. binomial 3. a. 2 b. 3 c. 4 d. 5 4. a. $10x + 1$ b. $y^3 + y^2 + 3y + 4$ c. $x^3 + 4x^2 + 2x + 3$ d. $7y^2 - y$ e. $4x^2 - 6x + 13$ f. $x^3 + 4x^2 + 9x + 1$ g. $x^5 + x^4 + x^3 + x^2 + x + 4$ 5. a. $x^3 - x + 7$ b. $-2y^2 - 4y + 9$ c. $w^3 - w^2 - w - 1$ d. $-x^4 + x^3 - x^2 + x - 2$ e. $9x^3 - x^2$ 6. a. $4x + 8$ b. $12x + 18$ c. $x^2 + 10x$ d. $-x^2 - 3x$ e. $-2x^2 + 6x$ f. $-8x^2 + 12x$ g. $32x^2 + 16x$ h. $12x^3 + 6x^2 + 3x$ i. $-3x^5 - 3x^3$ 7. a. $x^2 + 3x + 2$ b. $x^2 - x - 6$ c. $2x^2 + 3x + 1$ d. $x^2 - 6x + 5$ e. $2x^2 + 17x + 35$ f. $x^2 - 36$ g. $9x^2 - 24x + 16$ h. $x^4 - 1$ i. $4a^4 - 9$ j. $y^2 + 2y + \frac{3}{4}$ k. $2x^2 + 5xy + 2y^2$ 8. a. $5x$ b. $4x$ c. $2xy$ d. $3x + 2$

Exercises 2.3

1. a. $-2x^3 + 4x^2 + 4x + 1$ b. $2x^3 - 13x^2 + 17x + 12$ c. $-26x^3 - 11x^2 + 25x - 5$ d. $31x^3 - 23x^2 - 8x + 19$

2. a. $6x^5 + 5x^4 + x^3 - 2x^2 - x$ b. $2x^5 + x^4 - 8x^3 - 2x^2 + 4$ c. $48x^7 + 8x^6 + 24x^5 + 2x^4 - x^2$ d. $6x^5 + 2x^4 - 2x^3 - 3x^2$ e. $22x^5 + 4x^4 - x^3 - x^2$ 3. 5 4. $a = 12, b = 13$ 5. 4 6. $-\frac{3}{2}$ 7. $P(x) = ax, a \in \mathbb{R}$ 8. $3x + 4$ 9. $2x + 5$ 10. 1 11. $\frac{17}{2}$ 12. a. $3x^2 - 8x + 11$ b. $3x^5 + \frac{9}{2}x^4 + \frac{7}{4}x^3 + \frac{49}{8}x^2 + \frac{147}{16}x + \frac{409}{32}$ 13. a. $3x^2 + 2x$ b. $9x^2 - 3x + 1$ 14. $-\frac{23}{2}$ 15. 0 16. $-x + 2$ 17. $-8x - 5$ 18. $4x + 3$ 19. $-\frac{16}{3}$ 20. 40 21. 18

Exercises 3.1

1. a. {1, 2, 3, 6, 9, 18} b. {1, 2, 3, 4, 6, 8, 12, 16, 24, 48} c. {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60} d. {1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96} e. {1, 2, 4, 8, 16, 32, 64, 128} 2. a. 3 b. 8 c. 12 d. 12 3. a. $3x$ b. $4xy^2$ c. $5x^2yz$ d. $3xyz$ 4. a. $a(b + c)$ b. $xy(3x - 2y + 1)$ c. $7xy^3(5x + 3y)$ d. $13a^2bc(bc + 2ac - 4a)$ 5. a. $(x + y)(a + b)$ b. $(a + b)(m - n)$ c. $(a - 1)(a^2 + b)$ d. $(z + 2)(z + 3y)$ e. $(x - y)(x - 3)$ f. $(a + b)(a - 1)(a + 1)$ g. $(2a - b)(3a + 1)$ h. $(x + y)(x - y - 1)$ i. $(a - 2b)(a + 2b + 1)$ 6. a. $(x + 4)(x - 1)$ b. $(x + 5)(x - 3)$ c. $(x - 4)(x - 5)$ d. $(x - 7)(x + 3)$ e. $(x - 9)(x + 3)$ f. $(x - 9)(x - 1)$ 7. a. $(2x - 1)(x + 3)$ b. $(3x + 1)(x + 3)$ c. $(3x + 2)(x - 3)$ d. $(3x - 1)(2x + 3)$ e. $(4x + 7)(2x + 7)$ f. $(7x + 27)(2x - 1)$ 8. a. $2x \cdot (2x - 7)$ b. $(3x^y - 5)(2x^y + 1)$ 9. –15

Exercises 3.3

1. a. $x^2 + 4x + 4$ **b.** $x^2 - 2x + 1$ **c.** $8x^3 + 36x^2 + 54x + 27$ **d.** $x^3 - 6x^2y + 12xy^2 + 8y^3$ **e.** $x^4 + 4x^3 + 6x^3 + 4x + 1$ **f.** $x^5 - 5x^4a + 10x^3a^2 - 10x^2a^3 + 5xa^4 - a^5$ **g.** $1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$ **h.** $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ **i.** $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$ **2. a.** $x^2 + \frac{2}{3}x + \frac{1}{9}$ **b.** $x + 2x + 1$ **c.** $x - 2xy + y$ **d.** $(100 - 2)^2 = 9604$ **e.** $(100 + 5)^2 = 11025$ **f.** $x^2 - 4xy + 4y^2$ **g.** $9x^4 + 12x^2y + 4y^2$ **h.** $4x^2y^2 + 4xy + 1$ **i.** $x^4 - 2x^2y^2 + y^4$ **j.** $a^2 + b^2 + 1 + 2ab + 2a + 2b$ **k.** $a^2 + 4b^2 + c^2 + 4ab - 2ac - 4bc$ **l.** $4a^2 + b^2 + 9c^2 - 4ab - 12ac + 6bc$ **3. a.** $x^3 + 3x^2 + 3x + 1$ **b.** $x^3 + 6x^2y + 12xy^2 + 8y^3$ **c.** $x^3 - 9x^2 + 27x - 27$ **d.** $x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}$ **e.** $x^6 + 3x^4 + 3x^2 + 1$ **f.** $x^9 - 3x^6 + 3x^3 - 1$ **g.** $(30 - 1)^3 = 24389$ **h.** $-x^3 + 3x^2y - 3xy^2 + y^3$ **4. a.** $x^2 + 2x + 1$ **b.** $x^2 - 1$ **c.** $\frac{a^2}{b^2} - c^2$ **d.** $x - y$ **e.** $a^3 + b^3$ **f.** $x^3 - y^3$ **g.** $x^3 - 8y^3$ **h.** $x^3 - 1$ **i.** $x^3 + 27y^3$ **5.** 24000 **6.** 3999991 **7. a.** 13 **b.** 19 **c.** 63 **d.** $\frac{1}{2}$ **e.** 27 **f.** 170 **g.** 98 **8. a.** $(3a - 8)(3a + 8)$ **b.** $(9x^2 - 4y)(9x^2 + 4y)$ **c.** $(8x^2 - 12y^3)(8x^2 + 12y^3)$ **d.** $(a + b - c)(a + b + c)$ **e.** $(a - b + c)(a + b - c)$ **f.** $7(x - 4)(x + 4)$ **g.** $5x(x - 6)(x + 6)$ **h.** $(a^x - b^y)(a^x + b^y)$ **i.** $(a^{3m} - b^{3n})(a^{3m} + b^{3n})$ **9. a.** $8 . (x - 1)(x^2 + x + 1)$ **b.** $(a - 4b)(a^2 + 4ab + 16b^2)$ **c.** $3(x - 3yz^2)(x^2 + 3xyz^2 + 9y^2z^4)$ **d.** $2(2ab - 3cd)(4a^2b^2 + 6abcd + 9c^2d^2)$ **10. a.** $(x - 2y) . (x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4)$ **b.** $(a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$ **c.** $(m + 1)(m^4 - m^3 + m^2 - m + 1)$ **d.** $(5 - x^2)(25 + 5x^2 + x^4)$

Exercises 4.1

1. a. $\frac{8ab^2}{9}$ **b.** $\frac{5x^2z}{4y^3}$ **c.** $\frac{1}{4}$ **d.** $\frac{3}{5x}$ **e.** $\frac{3}{4a+6b}$ **f.** $\frac{x}{x-1}$ **g.** $-\frac{a+b}{x}$ **h.** $\frac{x-3}{x+7}$ **i.** $\frac{2x+5}{x+4}$ **j.** $\frac{a+b-c}{a-b-c}$ **k.** $x + 2y - z$ **l.** $\frac{9(x-2y)-4}{x-2y-1}$ **2. a.** $\frac{7x}{18}$ **b.** $\frac{4b-3a}{ab}$ **c.** $\frac{12x-7}{9x^2}$ **d.** $\frac{74}{35(a-b)}$ **e.** $\frac{2}{x-3}$ **f.** $\frac{9ax}{10y}$ **g.** $\frac{x}{x^2+2x+4}$ **h.** 1 **i.** $\frac{2x+5}{x+4}$ **j.** $\frac{x+7}{x+5}$ **k.** $\frac{a}{4}$

3. a. $\frac{a-2b}{a+2b}$ **b.** $\frac{a-1}{a+1}$ **c.** $\frac{x+y}{-y}$ **d.** $\frac{ab}{3b+4a}$ **e.** $\frac{1}{x(2x+3)}$ **f.** $5a^2$ **g.** $\frac{2a+3}{6a+7}$

4. a. $\frac{1}{3}$ **b.** $\frac{a-b}{a+b}$ **c.** 1 **d.** $\frac{x+y}{x}$ **e.** $\frac{y}{y-x}$ **f.** $\frac{y-x}{y+x}$ **g.** $-x - y$ **h.** $\frac{x+1}{x+3}$ **i.** $\frac{y}{x+y}$

5. a. $3xy$ **b.** $\frac{x+1}{x-3}$ **c.** $\frac{1}{x-3}$ **d.** $\frac{3x+3}{x-1}$ **6. a.** $\frac{-2y}{3x^2+9x}$ **b.** 1 **c.** $x - y$ **d.** $-\frac{2}{5x}$ **7. a.** $-2x$ **b.** $\frac{2}{5}$ **c.** $4x$ **d.** $\frac{x}{x+1}$ **8.** $\frac{216}{25}$

9. a. 3 **b.** 8 **c.** $\frac{2x^2-13}{x^2-5x+6}$ **d.** $\frac{4x-24}{x^2+2x-24}$ **e.** $\frac{x+1}{x-5}$ **f.** $x - 5$ **10. a.** 2 **b.** $\frac{x^2-8x+16}{x^3-4x}$ **11. a.** x **b.** $3a$ **c.** $\frac{3y-18x}{4x}$

Exercises 5.1

1. a. 243 **b.** 1000000 **2. a.** 712.892 **b.** 0.003 **3.** 150 000 000 tg **4.** 12 days

ANSWERS TO TESTS

TEST 2A

1.	D	11.	B
2.	B	12.	D
3.	C	13.	B
4.	A	14.	A
5.	E	15.	E
6.	C	16.	C
7.	B	17.	B
8.	D	18.	A
9.	E	19.	D
10.	A	20.	E

TEST 4

1.	C	9.	E
2.	D	10.	C
3.	A	11.	A
4.	B	12.	D
5.	C	13.	B
6.	A	14.	B
7.	B	15.	C
8.	D	16.	D

TEST 2B

1.	C	11.	B
2.	D	12.	D
3.	B	13.	A
4.	C	14.	C
5.	E	15.	D
6.	A	16.	B
7.	D	17.	C
8.	C	18.	D
9.	B	19.	A
10.	E	20.	E