Quantum Computing - Theory Work for Lab #2

Alexandra Semposki
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1). Verify that:

$$\langle \sigma_z \rangle = \frac{1}{N} \cdot \sum_i^{2^N} p_i \langle c_i | \sigma_z | c_i \rangle = \underbrace{\frac{1}{N} \sum_i^{2^N} p_i (n_i^{\uparrow} - n_i^{\downarrow})}_{} = \frac{1}{N} \cdot \sum_i^{2^N} p_i (N - 2n_i^{\downarrow}),$$

is equivalent to measuring:

$$\langle ZIII \ldots I + IZII \ldots I + IIZI \ldots I + \ldots + III \ldots IZ \rangle.$$

For 3 qubits, $\langle ZII + IZI + IIZ \rangle$.

$|c_i\rangle$ can be any of the 2^3 states: $|000\rangle$, $|010\rangle$, $|011\rangle$, $|100\rangle$, $|110\rangle$, $|101\rangle$, $|001\rangle$, $|111\rangle$.

$$\underbrace{\langle 000|ZII|000\rangle}_{\langle 000|000\rangle = 1} + \underbrace{\langle 000|IZI|000\rangle}_{\langle 000|000\rangle = 1} + \underbrace{\langle 000|IIZ|000\rangle}_{\langle 000|000\rangle = 1} = 1 + 1 + 1 = 3.$$

$$\underbrace{\langle 010|ZII|010\rangle}_{\langle 010|010\rangle = 1} + \underbrace{\langle 010|IZI|010\rangle}_{-\langle 010|010\rangle = -1} + \underbrace{\langle 010|IIZ|010\rangle}_{\langle 010|010\rangle = 1} = 1.$$

$$\underbrace{\langle 011|ZII|011\rangle}_{\langle 011|011\rangle = 1} + \underbrace{\langle 011|IZI|011\rangle}_{-\langle 011|011\rangle = 1} + \underbrace{\langle 011|IIZ|011\rangle}_{-\langle 011|011\rangle = -1} = -1.$$

$$\underbrace{\langle 100|ZII|100\rangle}_{-\langle 100|100\rangle = -1} + \underbrace{\langle 100|IZI|100\rangle}_{=1} + \underbrace{\langle 100|IIZ|100\rangle}_{=1} = 1.$$

$$\underbrace{\langle 110|ZII|110\rangle}_{-\langle 110|110\rangle = -1} + \underbrace{\langle 110|IZI|110\rangle}_{-\langle 110|IZI|110\rangle = -1} + \underbrace{\langle 110|IIZ|110\rangle}_{1} = -1.$$

$$\underbrace{\langle 101|ZII|101\rangle}_{-\langle 101|101\rangle = -1} + \underbrace{\langle 101|IZI|101\rangle}_{1} + \underbrace{\langle 101|IIZ|101\rangle}_{-1} = -1.$$

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$$\underbrace{\langle 001|ZII|001\rangle}_{1} + \underbrace{\langle 001|IZI|001\rangle}_{1} + \underbrace{\langle 001|IIZ|001\rangle}_{-1} = 1.$$

$$\underbrace{\langle 111|ZII|111\rangle}_{-1} + \underbrace{\langle 111|IZI|111\rangle}_{-1} + \underbrace{\langle 111|IIZ|111\rangle}_{-1} = -3.$$

$\therefore$ $\boxed{\langle ZII + IZI + IIZ\rangle = 0.}$

Then we check: $\langle \sigma_z\rangle = \frac{1}{N} \cdot \sum_i^{2^N} p_i (n_i^0 - n_i^1)$

and $\langle \sigma_z\rangle = \frac{1}{3} \sum_{i=1}^{8} p_i (n_i^0 - n_i^1)$.

So for each, need p_i and n_i^0, n_i^1:

1. $|000\rangle \Rightarrow p_{000} = |3|^2 = 9.$ $n_{000}^0 = 3;$ $n_{000}^1 = 0.$
2. $|001\rangle \Rightarrow p_{001} = |1|^2 = 1.$ $n_{001}^0 = 2;$ $n_{001}^1 = 1.$
3. $|010\rangle \Rightarrow p_{010} = |1|^2 = 1.$ $n_{010}^0 = 2;$ $n_{010}^1 = 1.$
4. $|011\rangle \Rightarrow p_{011} = |-1|^2 = 1.$ $n_{011}^0 = 1;$ $n_{011}^1 = 2.$
5. $|110\rangle \Rightarrow p_{110} = |-1|^2 = 1.$ $n_{110}^0 = 1;$ $n_{110}^1 = 2.$
6. $|101\rangle \Rightarrow p_{101} = |-1|^2 = 1.$ $n_{101}^0 = 1;$ $n_{101}^1 = 2.$
7. $|100\rangle \Rightarrow p_{100} = |1|^2 = 1.$ $n_{100}^0 = 2;$ $n_{100}^1 = 1.$
8. $|111\rangle \Rightarrow p_{111} = |-3|^2 = 9.$ $n_{111}^0 = 0;$ $n_{111}^3 = 3.$

$$\langle \sigma_z\rangle = \frac{1}{3}\sum_{i=1}^{8} p_i (n_i^0 - n_i^1) = \frac{1}{3}\Big[9(3-0) + 1(2-1) + 1(2-1) + 1(1-2) + 1(1-2) + 1(1-2) + 1(2-1) + 9(0-3)\Big].$$

$$= \frac{1}{3}\left(\cancel{9\cdot 3} + \cancel{1} + \cancel{1} - \cancel{1} - \cancel{1} - \cancel{1} + \cancel{1} - \cancel{9\cdot 3}\right) = 0.$$

$\therefore$ $\boxed{\langle \sigma_z\rangle = 0.}$ ✓. $\therefore$ The methods are equivalent for 3 qubits.

Let's also do this with a 2 qubit configuration:

$\langle ZI + IZ \rangle$ on 4 states total: $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$.

$$\underbrace{\langle 00|ZI|00\rangle}_{1} + \underbrace{\langle 00|IZ|00\rangle}_{1} \equiv 2.$$

$$\underbrace{\langle 01|ZI|01\rangle}_{1} + \underbrace{\langle 01|IZ|01\rangle}_{-1} = 0.$$

$$\underbrace{\langle 10|ZI|10\rangle}_{-1} + \underbrace{\langle 10|IZ|10\rangle}_{1} = 0.$$

$$\underbrace{\langle 11|ZI|11\rangle}_{-1} + \underbrace{\langle 11|IZ|11\rangle}_{-1} = -2.$$

$$\therefore \boxed{\langle ZI + IZ \rangle = 0.}$$

Now we do: $\langle \sigma_z \rangle = \frac{1}{N} \sum_i^{2^N} p_i \langle c_i | \sigma_z | c_i \rangle = \frac{1}{N} \sum_i p_i (n_i^0 - n_i^1)$.

$\therefore |00\rangle \Rightarrow p_{00} = |2|^2 = 4$; $n_{00}^0 = 2$; $n_{00}^1 = 0$.

$|01\rangle \Rightarrow p_{01} = |0|^2 = 0$; $n_{01}^0 = 1$; $n_{01}^1 = 1$.

$|10\rangle \Rightarrow p_{10} = |0|^2 = 0$; $n_{10}^0 = 1$; $n_{10}^1 = 1$.

$|11\rangle \Rightarrow p_{11} = |-2|^2 = 4$; $n_{11}^0 = 0$; $n_{11}^2 = 2$.

$$\therefore \langle \sigma_z \rangle = \frac{1}{2} \sum_{i=1}^{4} p_i (n_i^0 - n_i^1) = \frac{1}{2} \left[4(2-0) + 0(1-1) + 0(1-1) + 4(0-2) \right].$$

$$= \frac{1}{2}\left(4 \cdot 2 + 4(-2)\right) = 0.$$

$$\therefore \boxed{\langle \sigma_z \rangle = 0} \checkmark$$

$\therefore$ By induction, we can conclude that this will also work for n qubits:

$$\langle ZIII\ldots I + IZII\ldots I + IIZI\ldots I + \ldots + III\ldots Z \rangle \equiv \frac{1}{N} \sum_i^{2^N} p_i (n_i^0 - n_i^1)$$

$$\equiv \langle \sigma_z \rangle. \checkmark$$

2). Expand each of the operators in $\mathcal{U}(t)$ and show that they are equivalent to the rotation matrices:

$$ZZ(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 & 0 & 0 \\ 0 & e^{i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}$$

$$X(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}.$$

$$\mathcal{U}(t) = \left[\underbrace{e^{-i\theta_1 Z_0 Z_1}} e^{-i\theta Z_1 Z_2} e^{-i\theta_2 X_0} e^{-i\theta_2 X_1} e^{-i\theta_2 X_2}\right]^{m}.$$

More generally, $\exp(-i\theta Z_0 Z_1)$

$$ZZ(\theta) = e^{-i\theta_1 Z_0 Z_1} = \exp(-i\theta_1 Z_0 \otimes Z_1)$$

$$\therefore ZZ(\theta) = \exp\left(-i\theta \cdot Z_0 \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right).$$

$$\therefore ZZ(\theta) = \exp\left(-i\theta \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right).$$

Call this φ_1.

$$\therefore ZZ(\theta) = \exp\underbrace{\begin{pmatrix} -i\theta & 0 & 0 & 0 \\ 0 & i\theta & 0 & 0 \\ 0 & 0 & i\theta & 0 \\ 0 & 0 & 0 & -i\theta \end{pmatrix}}_{\text{Call this } \xi \text{ for now.}}.$$

Call this φ_2.

$$\therefore ZZ(\theta) = \exp(\xi) \simeq 1 + \xi + \frac{\xi^2}{2!} + \frac{\xi^3}{3!} + \ldots$$

$$\simeq 1 + \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} + \frac{\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}\begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix}}{2!} + \ldots$$

$$ZZ(\theta) \simeq 1 + \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \varphi_1^2 & 0 \\ 0 & \varphi_2^2 \end{pmatrix} + \ldots$$

$$ZZ(\theta) \simeq 1 + \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} + \begin{pmatrix} \varphi_1^2/2! & 0 \\ 0 & \varphi_2^2/2! \end{pmatrix} + \ldots$$

$$\therefore ZZ(\theta) \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \varphi_1 & 0 \\ 0 & \varphi_2 \end{pmatrix} + \begin{pmatrix} \varphi_1^2/2! & 0 \\ 0 & \varphi_2^2/2! \end{pmatrix} + \ldots$$

$$\therefore ZZ(\theta) \simeq \begin{pmatrix} 1+\varphi_1+\varphi_1^2/2!+\ldots & 0 \\ 0 & 1+\varphi_2+\varphi_2^2/2!+\ldots \end{pmatrix}.$$

$$\therefore ZZ(\theta) = \begin{pmatrix} e^{\varphi_1} & 0 \\ 0 & e^{\varphi_2} \end{pmatrix} =$$

e^{φ_1}, from what was just shown, can also be expanded through the same process to get:

$$e^{\varphi_1} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} ; \quad e^{\varphi_2} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

$\therefore$ We can put this all back together to get:

$$\boxed{ZZ(\theta) = \begin{pmatrix} e^{-i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-i\theta} \end{pmatrix}.}$$

(Ignoring the $\theta/2$ condition from the Bloch sphere configuration.)

If we consider the Bloch sphere configuration that indicates $\theta \to \theta/2$ here, we can replace $\theta \to \theta/2$ to get:

$$\boxed{ZZ(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 & 0 & 0 \\ 0 & e^{i\theta/2} & 0 & 0 \\ 0 & 0 & e^{i\theta/2} & 0 \\ 0 & 0 & 0 & e^{-i\theta/2} \end{pmatrix}.}$$

Now we look at $X(\theta)$.

$$X(\theta) = \exp(-i\theta_2 X_0) \equiv \exp(-i\theta X).$$

Expanding:

$$X(\theta) \simeq 1 + (i\theta X) + \frac{(-i\theta X)^2}{2!} + \frac{(-i\theta X)^3}{3} + \frac{(-i\theta X)^4}{4!} + \dots$$

$$\simeq 1 - i\theta X - \frac{1}{2!}\theta^2 X^2 + \frac{i\theta^3 X^3}{3!} + \frac{\theta^4 X^4}{4!} + \dots$$

$$\simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2!}\theta^2 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{X^2 = \mathbb{1}} +$$

$$+ \frac{i\theta^3}{3!} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{X}} + \frac{\theta^4}{4!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots$$

$$\therefore X(\theta) \simeq \begin{pmatrix} 1 - i\frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots & -i\theta + \frac{i\theta^3}{3!} - \dots \\ -i\theta + \frac{i\theta^3}{3!} - \dots & 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \end{pmatrix}$$

$$\therefore X(\theta) \simeq \begin{pmatrix} \cos(\theta) & -i(\theta - \theta^3/3! + \dots) \\ -i(\theta - \theta^3/3! + \dots) & \cos(\theta) \end{pmatrix}$$

$$\therefore X(\theta) = \begin{pmatrix} \cos(\theta) & -i\sin(\theta) \\ -i\sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Including? the Bloch sphere configuration to take $\theta \to \theta/2$ here, we get:

$$\boxed{X(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}}.$$

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