Fundamental Limits of Singular Value Based Signal Detection from Randomly Compressed Signal-plus-Noise Matrices

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Signal-plus-noise Matrix

$$\widetilde{X}_n = \sum_{i=1}^{r} \theta_i u_i v_i^T + X_n.$$

Parameters

- $u_i \in \mathbb{C}^{n\times 1}$, $u_i^H u_j = \delta_{i=j}$
- $v_i \in \mathbb{C}^{N\times 1}$, $v_i^H v_j = \delta_{i=j}$
- $\theta_i > 0$
- $X_n \in \mathbb{C}^{n\times N}$ random matrix with singular values $\sigma_1, \ldots, \sigma_{\min(n,N)}$
- Empirical singular value distribution:

$$\mu_{X_n} = \frac{1}{\min(n,N)} \sum_{i=1}^{\min(n,N)} \delta_{\sigma_i}.$$

Goal

Detect the presence of the r signals

Standard Operating Procedure

- Take SVD of $\widetilde{X}_n$ to get $\widehat{\sigma}_i$
- Compare singular values to a threshold to determine significance
- The corresponding singular vectors, $\widehat{u}_i$, estimate signals

Challenges in High Dimensions

- As $n, N \to \infty$, computing the SVD of $\widetilde{X}_n$ becomes prohibitive

Idea

- Project $\widetilde{X}_n$ to a lower dimensional space, $Y_n = P_n^H \widetilde{X}_n$
- $P_n \in \mathbb{C}^{n \times m}$
- Take SVD of Y_n to detect signals

Paper Goals

Goals

- Quantify how m, n, N, θ affect the behavior of the singular values of Y_n
- Uncover fundamental signal detection limits
- Compare the performance of Gaussian and unitary projection matrices

Gaussian Projection Matrix

- P_n has i.i.d. $\mathcal{CN}(0,1)$ entries
- Gaussian-like matrix, $P_n = G_n$

$$G_{ij} = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$

Unitary Projection Matrix

- $P_n = Q_n$ s.t. $Q_n^H Q_n = I_m$
- Discrete Fourier matrix, $Q_n = F$

$$F_{kj} = \frac{1}{\sqrt{n}} \exp\left\{ \frac{-2\pi i(k-1)(j-1)}{n} \right\}$$

Main Theoretical Results

Theorem (Almost sure limit of singular values of Y_n)

The largest r singular values of Y_n exhibit the following behavior as $n, m, N \to \infty$ with $n/N \to c_1$ and $m/n \to c_2$. For each fixed $1 \leq i \leq r$, $\sigma_i(Y_n)$ solves

$$\sigma_i^2 \varphi_F(\sigma_i)\varphi_H(\sigma_i) = \frac{1}{\theta_i^2},$$

where

$$\varphi_F(\sigma_i) \xrightarrow{a.s.} -\mathbb{E}\left[x m_{\mu_{RS|R}}\left(\sigma_i^2, x\right)\right]_{\mu_R}$$

$$\varphi_H(\sigma_i) \xrightarrow{a.s.} -\frac{n}{N} m_{M_3}(\sigma_i^2) - \frac{1}{\sigma_i^2}\frac{n-N}{N}$$

where

- m_{μ_M} *is the Stieltjes transform of a matrix M defined as* $m_{\mu_M}(z) = \int \frac{1}{x-z}\mu_M(x)$
- μ_R *is the limiting eigenvalue density of either* $G_n G_n^H$ *or* $Q_n Q_n^H$
- μ_S *is the limiting eigenvalue density of* $X_n X_n^H$
- $m_{\mu_{RS|S}}$ *is the Stieltjes transform of the limiting conditional density*
- $m_{\mu_{M_3}}$ *is the Stieltjes transform of* $G_n G_n^H X_n X_n^H$ *or* $Q_n Q_n^H X_n X_n^H$.

Corollaries

Corollary (Fundamental detection limit)

Define the critical SNR threshold as

$$\theta_{crit} = \frac{1}{b\sqrt{\varphi_F(b)\varphi_H(b)}}.$$

When $\theta_i < \theta_{crit}$*, then*

$$\sigma_i \xrightarrow{a.s.} b,$$

where b *is the almost sure limit of the largest singular value of* $P_n^H X_n$*.*

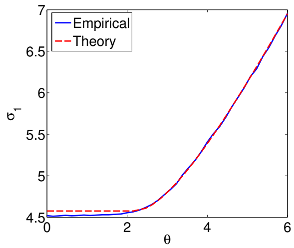
Corollary (Closed form solution for unitary P)

When Y_n *is a generated using a unitary matrix* Q_n*, we have that for each fixed* $1 \leq i \leq r$*,*

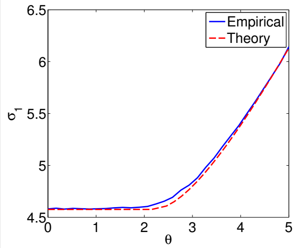
$$\sigma_i \xrightarrow{a.s.} \begin{cases} \sqrt{\frac{c_1}{\theta_i^2} + c_2\theta_i^2 + 1 + c_1 c_2} & \text{if } \theta_i \geq \left(\frac{c_1}{c_2}\right)^{1/4} \\ \sqrt{c_1 c_2} + 1 & \text{if } \theta_i < \left(\frac{c_1}{c_2}\right)^{1/4} \end{cases}.$$

Empirical Results - Singular Value Prediction

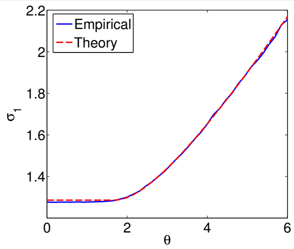
- Parameters: $r = 1$, $n = 1000$, $N = 1220$ and $m = 100$, 500 trials



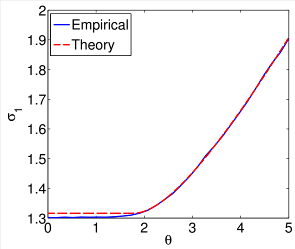
(a) Gaussian G



(b) Gaussian-like G



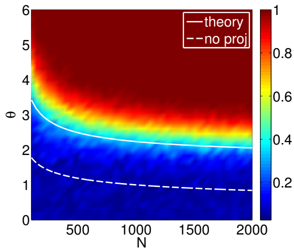
(c) Unitary Q



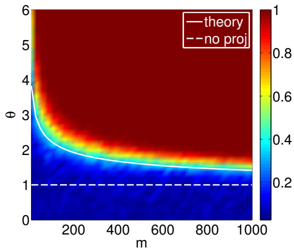
(d) Fourier Q

Empirical Results - Phase Transition Prediction

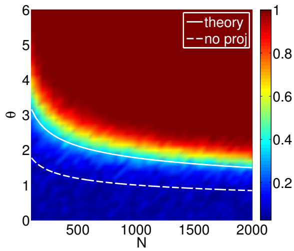
- KS Statistic: $r = 1$, $n = 1000$, 500 trials



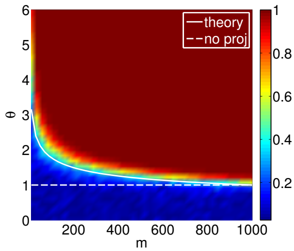
(e) Gaussian $m = 100$



(f) Gaussian $N = 1000$



(g) Unitary $m = 100$



(h) Unitary $N = 1000$

Discussion and Conclusion

Takeaways

- Accurately predict top singular values
- Accurately predict detection limit
- Unitary projection outperforms Gaussian projection
- Theory allows practitioners to set system parameters to achieve desired performance
- No closed form for Gaussian - relies on numerical techniques

Generating projection matrices

- Easy - Gaussian, Gaussian-like (i.i.d. entries)
- Hard - Unitary
- Easy - Fourier