Claim 5.1: We conjecture that this result holds for the general case of $i \neq j$, $i = 1, \ldots, \widehat{k}$, $j = 1, \ldots, k$, not just when $\widehat{k} = k_{\text{eff}} = k$. Consider the case when $k = 1$. For $j > 2$, if $\widehat{\lambda}_j$ is an eigenvalue of $\widehat{X}_n = X_n(I_n + \sigma^2 uu^H)$, then it satisfies $\det(\widehat{\lambda}_j I_n - X_n(I_n + \sigma^2 uu^H)) = \det(\widehat{\lambda}_j I_n - X_n) \det(I_n - (\widehat{\lambda}_j I_n - X_n)^{-1} X_n \sigma^2 uu^H) = 0$. Therefore, if $\widehat{\lambda}_j$ is not an eigenvalue of X_n, the corresponding unit norm eigenvector $\widehat{v}_j$ is in the kernel of $I_n - (\widehat{\lambda}_j I_n - X_n)^{-1} X_n \sigma^2 uu^H$. Therefore

$$|\langle \widehat{v}_j, u \rangle|^2 = \frac{1}{\sigma^4 u^H X_n \left(\widehat{\lambda}_j I_n - X_n \right)^{-2} X_n u}.$$

Recall that Weyl's interlacing lemma for eigenvalues gives $\lambda_j(X_n) \leq \widehat{\lambda}_j \leq \lambda_{j-1}(X_n)$. Letting $X_n = V_n \Lambda_n V_n^H$ and $w = V_n^H u$, we see the importance of the asymptotic spacing of eigenvalues of X_n in

$$u^H X_n (\widehat{\lambda}_j I_n - X_n)^{-2} X_n u = \sum_{\ell=1}^{n} \frac{|w_\ell|^2 \lambda_\ell^2(X_n)}{\left(\widehat{\lambda}_j - \lambda_\ell \right)^2} \geq \frac{\min_j \lambda_j^2(X_n) \min_j |w_j|^2}{\max_j |\lambda_{j-1} - \lambda_j|^2}$$

In [**?**] it is shown that $\min_j \lambda_j^2(X_n) = \lambda_n^2(X_n) \xrightarrow{\text{a.s.}} (1 - \sqrt{c})^4$. The typical spacing between eigenvalues is $O(1/n)$ while the typical magnitude of w_i^2 is $O(1/n)$. Therefore, the above inequality will typically be $O(n)$ and we get the desired result of $|\langle \widehat{v}_j, u \rangle|^2 \xrightarrow{\text{a.s.}} 0$. More generally, it is the behavior of the largest eigenvalue gap and the smallest element of w_i that drives this convergence. Thus, so long as the eigenvector whose elements are w_i are delocalized (having elements of $O(1/\sqrt{n})$ and the smallest gap between k successive eigenvalues is at least as large as $O(1/n + \epsilon)$, we may bound the right hand side of the above inequality. The claim follows after applying a similarity transform as in the proof of Theorem 5.1.