Improved Estimation of Canonical Vectors in Canonical Correlation Analysis

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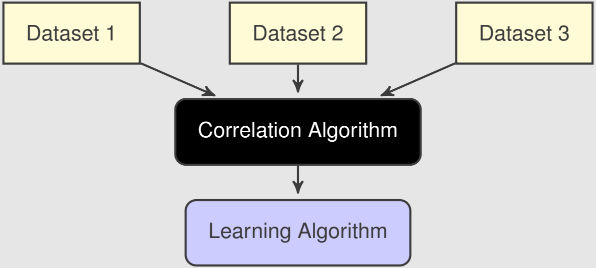
Motivation



Goal

Develop theoretically justified, robust correlation algorithms for multi-dataset fusion

Multiple Datasets

- Audio-Video
- Audio-Audio

Machine Learning

- emotion identification
- shopping predictions
- music genre classification

Medical Signal Processing

- MRI, fMRi, EEG, MEG, etc.

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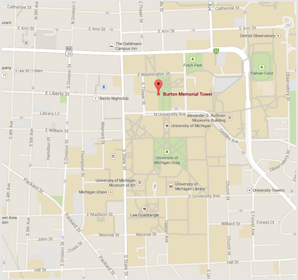

A Myriad of Applications

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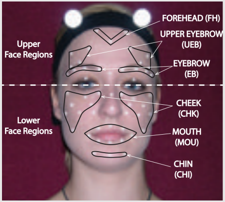

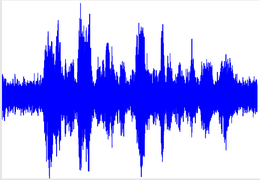

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- disco influences
- danceable grooves
- repetitive melodic phrasing
- extensive vamping
- minor key tonality

Canonical Correlation Analysis

What is it?

- Dimensionality reduction algorithm for exactly 2 datasets, X, Y
- Correlation coefficients, linear transformations

What is it not?

- Data fusion algorithm

Covariance matrices

- $R_{xx} = \mathbb{E}\left[xx^H\right]$
- $R_{yy} = \mathbb{E}\left[yy^H\right]$
- $R_{xy} = \mathbb{E}\left[xy^H\right]$

Optimization problem

$$\begin{aligned} \underset{w_x, w_y}{\operatorname{argmax}} \quad & \rho = w_x^H R_{xy} w_y \\ \text{subject to} \quad & w_x^H R_{xx} w_x = 1 \\ & w_y^H R_{yy} w_y = 1 \end{aligned}$$

Variable Transformation

- $f = R_{xx}^{1/2} w_x$
- $g = R_{yy}^{1/2} w_y$

What is it?

- Dimensionality reduction algorithm for exactly 2 datasets
- Correlation coefficients, linear transformations

What is it not?

- Data fusion algorithm

Optimization problem

$$\underset{f,g}{\operatorname{argmax}} \quad \rho = f^H \underbrace{R_{xx}^{-1/2} R_{xy} R_{yy}^{-1/2}}_{C_{\text{cca}}} g$$

$$\text{subject to} \quad \|f\|_2 = 1\,,\ \|g\|_2 = 1$$

Canonical Vectors

- $w_x = R_{xx}^{-1/2} f$
- $w_y = R_{yy}^{-1/2} g$

Insight

$\#$ correlated signals $= k = \text{rank}(C_{\text{cca}})$

Empirical CCA

Training Datasets

- $X = [x_1, \ldots, x_n]$
- $Y = [y_1, \ldots, y_n]$

Sample Covariance Matrices

- $\widehat{R}_{xx} = \frac{1}{n} XX^H$
- $\widehat{R}_{yy} = \frac{1}{n} YY^H$
- $\widehat{R}_{xy} = \frac{1}{n} XY^H$

Estimate

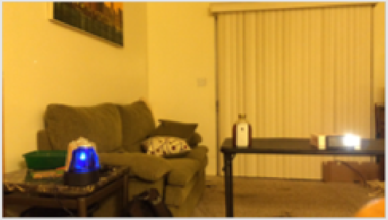
$$\boxed{\begin{aligned} \widehat{C}_{\text{cca}} &= \widehat{R}_{xx}^{-1/2} \widehat{R}_{xy} \widehat{R}_{yy}^{-1/2} \\ &= \widehat{F} \widehat{K} \widehat{G}^H \end{aligned}}$$

Data SVDs

- $X = \widehat{U}_x \widehat{\Sigma}_x \widehat{V}_x^H$
- $Y = \widehat{U}_y \widehat{\Sigma}_y \widehat{V}_y^H$
- $\sigma\left(\widehat{C}_{\text{cca}}\right) = \sigma\left(\widehat{V}_x^H \widehat{V}_y\right)$

Motivational Example - Flashing Light Video

Left Camera



Right Camera

System parameters

- Vectorize 135×240 image $\Rightarrow p = q = 32400$ pixels
- 30 fps @ 30 seconds $\Rightarrow n = 900$ frames

Goal

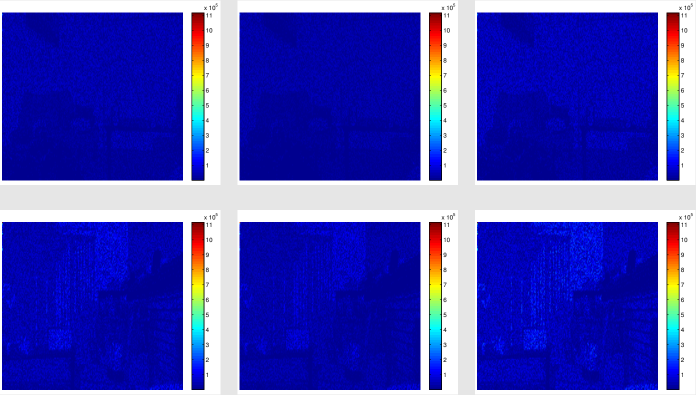
Identify correlated pixels between camera views

- After 900 frames = 30 seconds of video

First **Second** **Third**

Left

Right



Two-Dataset Model

Linear Subspace Model

$$x_i = U_x s_{x,i} + z_{x,i}$$
$$y_i = U_y s_{y,i} + z_{y,i}$$

Parameters

- $U_x^H U_x = I_{k_x}$, $U_y^H U_y = I_{k_y}$
- $z_{x,i} \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, I_p)$, $\;z_{y,i} \overset{\text{i.i.d.}}{\sim} \mathcal{CN}(0, I_q)$
- $\mathbb{E}\left[\begin{bmatrix} s_{x,i} \\ s_{y,i} \end{bmatrix}\begin{bmatrix} s_{x,i}^H & s_{y,i}^H \end{bmatrix}\right] = \begin{bmatrix} \Theta_x & K_{xy} \\ K_{xy}^H & \Theta_y \end{bmatrix}$
- $K_{xy} = \Theta_x^{1/2} P_{xy} \Theta_y^{1/2}$
- $\Theta_x = \textbf{diag}\left(\left(\theta_1^{(x)}\right)^2, \ldots, \left(\theta_{k_x}^{(x)}\right)^2\right)$, $\;\Theta_y = \textbf{diag}\left(\left(\theta_1^{(y)}\right)^2, \ldots, \left(\theta_{k_y}^{(y)}\right)^2\right)$
- P_{xy} contains correlations ρ_{kj} between signals of x_i and y_i
- $\widetilde{K}_{xy} = \left(\Theta_x + I_{k_x}\right)^{-1/2} K_{xy} \left(\Theta_y + I_{k_y}\right)^{-1/2}$, with singular values $\kappa_1, \ldots, \kappa_{\min(k_x, k_y)}$

Informative CCA (ICCA)

Not all singular vectors are informative! (Nadakuditi, 2011)

- $\sigma_i\left(\widehat{C}_{\text{cca}}\right) = \sigma_i\left(\widehat{V}_x^H \widehat{V}_y\right)$
- Trim data SVD's to only use informative components

1. Trim data SVD's: $X = \widehat{U}_x \widehat{\Sigma}_x \widehat{V}_x^H$ and $Y = \widehat{U}_y \widehat{\Sigma}_y \widehat{V}_y^H$
   - $\mathring{U}_x = \widehat{U}_x\left(:, 1 : \widehat{k}_x\right)$, $\mathring{U}_y = \widehat{U}_y\left(:, 1 : \widehat{k}_y\right)$
   - $\mathring{V}_x = \widehat{V}_x\left(:, 1 : \widehat{k}_x\right)$, $\mathring{V}_y = \widehat{V}_y\left(:, 1 : \widehat{k}_y\right)$
2. Form $\widehat{C}_{\text{icca}} = \mathring{U}_x \mathring{V}_x^H \mathring{V}_y \mathring{U}_y$
3. Take SVD: $\widehat{C}_{\text{icca}} = \widetilde{F}\widetilde{K}\widetilde{G}^H$
4. $\widehat{\rho}_{\text{icca}}^{(i)} = \widetilde{k}_i$
5. $\widetilde{w}_x^{(i)} = \widehat{R}_{xx}^{-1/2} \widetilde{f}_i$
6. $\widetilde{w}_y^{(i)} = \widehat{R}_{yy}^{-1/2} \widetilde{g}_i$

New Algorithm: ICCA+

Motivation

- We expect ICCA to outperform CCA
- However, we expect ICCA to be suboptimal because we substitute $\widehat{U}_x$ and $\widehat{U}_y$ without considering accuracy

Proposed Estimate

$$\widehat{w}_{x,i}^{\text{icca+}} = \mathring{U}_x \,\textbf{diag}\left(\lambda_{x,i}^{\text{opt}}\right) \widehat{U}_{\widetilde{K}}\,(:,i)$$

$$\widehat{w}_{y,i}^{\text{icca+}} = \mathring{U}_y \,\textbf{diag}\left(\lambda_{y,i}^{\text{opt}}\right) \widehat{V}_{\widetilde{K}}\,(:,i)\,,$$

Optimization Problem

$$\lambda_{x,i}^{\text{opt}} = \underset{\lambda_x}{\operatorname{argmin}} \left\| w_x^{(i)} - \widehat{U}_x \,\textbf{diag}(\lambda_x) \widehat{U}_{\widetilde{K}}\,(:,i) \right\|_F$$

$$\lambda_{y,i}^{\text{opt}} = \underset{\lambda_y}{\operatorname{argmin}} \left\| w_y^{(i)} - \widehat{U}_y \,\textbf{diag}(\lambda_x) \widehat{V}_{\widetilde{K}}\,(:,i) \right\|_F .$$

Closed Form Soluation

Proposition

The solutions to the previous optimization problem is given by

$$\lambda_{x,i}^{opt} = \textbf{diag}\left(\mathring{U}_x^H U_x \left(\Theta_x + I_{k_x}\right)^{-1/2}\right)$$

$$\lambda_{y,i}^{opt} = \textbf{diag}\left(\mathring{U}_y^H U_y \left(\Theta_y + I_{k_y}\right)^{-1/2}\right).$$

Results from random matrix theory: Eigenvector accuracy

$$\boxed{|\langle \widehat{u}_i, u_i \rangle|^2 \xrightarrow{\text{a.s.}} \begin{cases} \dfrac{-2\varphi_{\mu_Z}(\rho)}{\theta_i^2 D'_{\mu_Z}(\rho)} & \theta_i^2 > 1/D_{\mu_Z}\left(b^+\right) \\ 0 & \text{otherwise} \end{cases}}$$

$$D_\mu(z) =: \left[\int \frac{z}{z^2 - t^2} d_\mu(t)\right] \times \left[c_x \int \frac{z}{z^2 - t^2} d_{\mu(t)} + \frac{1 - c_x}{z}\right]$$

$$\varphi_\mu(z) =: \int \frac{z}{z^2 - t^2} d\mu(t)$$

Main Theorem

Theorem

The solution to the optimal ICCA+weights exhibits the following behavior in the asymptotic regime where $p, q, n \to \infty$ with $p/n \to c_x$ and $q/n \to c_y$.
For $i = 1, \ldots, k_x$,

$$\lambda_{x,opt}^{(i)} \xrightarrow{a.s.} \begin{cases} D_{\mu_{Z_x}}\left(\sigma_x^{(i)}\right)\sqrt{\dfrac{-2\varphi_{\mu_{Z_x}}\left(\sigma_x^{(i)}\right)}{D'_{\mu_{Z_x}}\left(\sigma_x^{(i)}\right)\left(1+D_{\mu_{Z_x}}\left(\sigma_x^{(i)}\right)\right)}} & \text{if } \left(\theta_i^{(x)}\right)^2 > 1/D_{\mu_{Z_x}}(b_x^+) \\ 0 & \text{otherwise} \end{cases}$$

where $\sigma_x^{(i)} = D_{\mu_{Z_x}}^{-1}\left(1/\left(\theta_i^{(x)}\right)^2\right)$ and the D-transform. A similar expression for $\lambda_{y,opt}$ exists by replacing subscripts of x with y.

Estimation using training data

- Using data X, Y, we can estimate $\widehat{D}$, $\widehat{D}'$, $\widehat{\varphi}_\mu$, and $\widehat{\varphi}_{\widetilde{\mu}}$

Canonical Vector Estimation

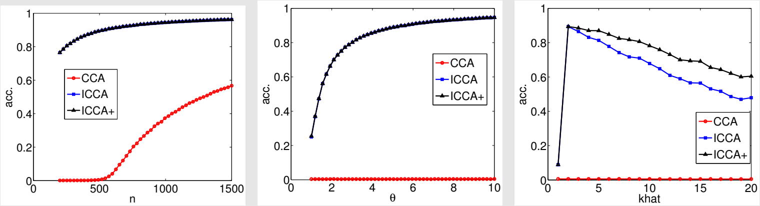
Data SVDs

- $X = \widehat{U}_x \widehat{\Sigma}_x \widehat{V}_x^H$ with trimmed versions $\mathring{U}_x$, $\mathring{\Sigma}_x$, $\mathring{V}_x$
- $U_{\widetilde{K}}$ left singular vectors of $\widetilde{K}_{xy}$
- $\widehat{U}_{\widetilde{K}}$ left singular vectors of $\widehat{C}_{\text{cca}}$
- $\mathring{U}_{\widetilde{K}}$ left singular vectors of $\widehat{C}_{\text{icca}}$

Estimate	
Population	$W_x = U_x \left(\Theta_x + I_{k_x}\right)^{-1/2} U_{\widetilde{K}}$
Empirical CCA	$\widehat{W}_x^{\text{cca}} = \widehat{U}_x \left(\widehat{\Sigma}_x\right)^{-1} \widehat{U}_{\widetilde{K}}$
ICCA	$\widehat{W}_x^{\text{icca}} = \mathring{U}_x \mathring{\Sigma}_x^{-1} \mathring{U}_{\widetilde{K}}$
ICCA+	$\widehat{W}_x^{\text{icca+}} = \mathring{U}_x \Lambda_x^{\text{opt}} \mathring{U}_{\widetilde{K}}$

Numerical Simulations

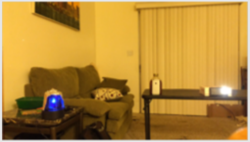
- Rank-2 setting $k_x = k_y = 2$, $p = 200$, $q = 250$,
- $\Theta_x = \Theta_y = \textbf{diag}(16, 4)$, $P_{xy} = \textbf{diag}(0.9, 0.5)$
- Plot the accuracy of the first canonical vector for empirical CCA, ICCA, and ICCA+



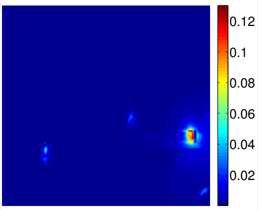
(a) n sweep　(b) θ sweep　(c) $\widehat{k}$ sweep

Canonical Vector Accuracy Experiment

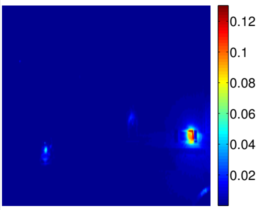
Original Scene



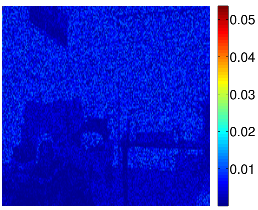
ICCA



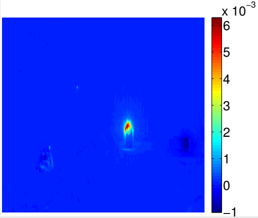
ICCA+



Empirical CCA



ICCA - ICCA+



Conclusion

- Improved canonical vector accuracy in low-sample, low-SNR regime in Canonical Correlation Analysis (CCA)
- Proposed a new algorithm: ICCA+
- ICCA+ uses insights from random matrix theory to optimally scale sample eigenvectors setting
- ICCA+ is more robust to over estimation of number of signals