

First-Order System: Transient Response of a Thermocouple to a Step Temperature Change

Last Modified 9/6/06

Many mechanical, thermal, and electrical systems can be modeled using first-order differential equations. In this experiment we will study the behavior of a first-order system, the transient response of a thermocouple (TC).

First-Order Systems

A one-degree-of-freedom first-order system is governed by the first-order ordinary differential equation

$$a_1 \frac{dy}{dt} + a_0 y = F(t), \quad (1)$$

where $y(t)$ is the response of the system (the output) to some forcing function $F(t)$ (the input). Eq. (1) may be rewritten as

$$\tau \frac{dy}{dt} + y = kF(t), \quad (2)$$

where $\tau = a_1/a_0$ has the dimension of time and is the **time constant** for the system and $k = 1/a_0$ is the **gain**.

Response of a First-order System to a Step Input

Consider a first-order system subjected to a constant force applied instantaneously at the initial time $t = 0$

$$F(t) = \begin{cases} 0, & t < 0 \\ A, & t > 0 \end{cases} \quad (3)$$

The initial condition is $y(0) = 0$. The solution to Eq. (2) with the step input Eq. (3) is then

$$y(t) = kA(1 - e^{-t/\tau}). \quad (4)$$

The response approaches the final value $y_\infty = kA$ exponentially. By using the boundary conditions equation (4) then may be rewritten as

$$\frac{y(t) - y_\infty}{y_0 - y_\infty} = e^{-t/\tau} = \frac{y(t) - kA}{-kA}. \quad (5)$$

The rate at which the response approaches the final value is determined by the time constant. When $t = \tau$, y has reached 63.2% of its final value as illustrated in Figure 1.

When $t = 5\tau$, y has reached 99.3% of its final value.

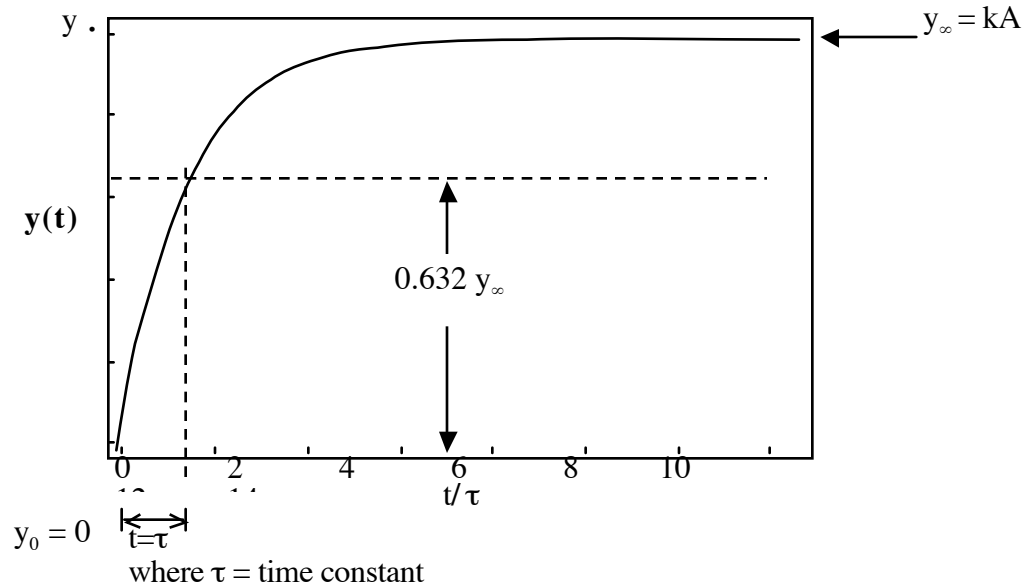


Figure 1: Response of a first-order system to a step input.

The time constant of a system can be determined from the measured response using a linear regression. Taking the natural log Eq. (5) yields

$$-\frac{t}{\tau} = \ln\left(\frac{y - y_{\infty}}{y_0 - y_{\infty}}\right) = \ln\left(\frac{y - kA}{-kA}\right) \quad (6)$$

The slope s of the natural log term plotted against t gives the time constant τ through the relation $s = -1/\tau$.

Transient Response of a Thermocouple

The dynamic response of a sensor is often an important consideration in designing a measurement system. The response of a temperature sensor known as a thermocouple (TC) may be modeled as a first-order system. When the TC is subjected to a rapid temperature change, it will take some time to respond. If the response time is slow in comparison with the rate of change of the temperature that you are measuring, then the TC will not be able to faithfully represent the dynamic response to the temperature fluctuations.

A model of the response of a TC is based on a simple heat transfer analysis. The rate at which the sensor exchanges heat with its environment must equal the rate of change of the internal energy of the sensor. If the dominant mechanism of heat exchange is convection

(neglecting conduction and radiation), as it is for a TC in a fluid, then this energy balance is

$$hA(T_{\infty} - T) = mc \frac{dT}{dt}, \quad (7)$$

h is the convection coefficient, A is the surface area of the sensor, T is the temperature, m is the TC mass, and c is the heat capacity. Writing Eq. (7) in the form of Eq. (2)

$$\tau \frac{dT}{dt} + T = T_{\infty}, \quad (8)$$

where the time constant is

$$\tau = \frac{mc}{hA}. \quad (9)$$

As a first-order system, the transient response of the thermocouple to an instantaneous change in temperature is given by Eq. (5).

References

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