Optimal control

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1 Introduction

According to Kallrath (2004) in an optimization problem, the objective is to minimize or maximize a quantity associated with a decision process such as elapsed time or cost by exploiting available degrees of freedom under a set of restrictions (constraints). There are many different problems of optimization a wide variety of industries or fields such as transportation, logistics, production, amongst others. This industries have great impact in the world's revenues for companies but also have to do with the use of the resources necessary in the development of such activities.

The fishing industry is growing everyday around the world due to the increasing population and the need of more resources to supply the alimentary needs. Also, it leaves good profits for the companies that are dedicated to that business, as stated by Haas et al. (2019) the fishing sector in the United States alone was valued at 10.05 billion U.S. dollars in 2018. This kind of revenue only encourages the fishing industry to grow constantly but this is done harming the environment and causing the fish population to decrease or get extincted because the rate of fishing is higher than the recovery rate of fish population.

In this project, a fishing example was solved using a software but also the mathematical equations of the problem are presented the, in order to provide deeper understanding of the solution and the theoretical background behind it. The objective is to understand through the development of the problem the implications, not only in the economic aspect but also from an ecological point of view due to the impact in the fish population. By analyzing the results and variables carefully, conclusions are presented using the optimal intervals obtained by solving the example with optimal control optimization.

2 Problem statement

The optimal control in the commercial fishing industry can be described as the integral of profit that takes into account the revenue of sales and the operational cost. To make this operations correctly the population of fishes denoted by the letter x in this problem is affected by the amount of fishes removed each year, this variable depends on u.

The time horizon of this project is 10 years, and the main objective is to maximize the income or revenue during this period of time. Now, there are some conditions to consider in order to achieve this objective which are that if there is overfishing (high u) then the returns in the following years will be diminished and also that the fish population will not recover.
The function that represents the revenue over the study time period is

$$max_{u(t)} \int_0^{10} (E - \frac{c}{x}) u * Umax * dt \tag{1}$$

The restrictions associated to the problem are

$$\frac{dx}{dt} = r * x(t)(1 - \frac{x(t)}{k}) - u * Umax \tag{2}$$

$$x(0) = 70 \tag{3}$$

$$0 \le u(x) \le 1 \tag{4}$$

$$E = 1, c = 17.5, r = 0.71, k = 80.5, Umax = 20 \tag{5}$$

The previous equations can be re-written as

$$J = \int_0^{10} (E - \frac{c}{x}) u * Umax * dt \tag{6}$$

That is $max_{u(t)} J(t_f)$ and its restrictions are

$$\frac{dx}{dt} = r * x(t)(1 - \frac{x(t)}{k}) - u * Umax \tag{7}$$

$$\frac{dJ}{dt} = (E - \frac{c}{x}) u * Umax \tag{8}$$

$$x(0) = 70 \tag{9}$$

$$J(0) = 0 \tag{10}$$

$$0 \le u(t) \le 1 \tag{11}$$

$$t_f = 10, E = 1, c = 17.5, r = 0.71, k = 80.5, Umax = 20 \tag{12}$$

3 Theoretical and computational basis

The general discrete time optimization problem can be described by the equation

$$x_{k+1} = f^k(x_k, u_k) \tag{13}$$

where the initial condition is x_0, the state x_k a vector of size n and the control input u_k is a vector of size m.
The scalar performance index is expressed in the general form as:

$$J_i = \phi(N, x_N) + \sum_{k=1}^{N-1} L^k(x_k, u_k) \tag{14}$$

where [i,N] is the time interval, $\phi(N, x_N)$ is a function of the final time N and the state at the final time and $L^k(x_k, u_k)$ is a generally time-varying function of the state and control input at each intermediate time k in [i,N].

Now that the general form of the optimization problem has been explained, the optimal control equations will be presented. The objective of the optimal control problem is to find the u_K^* on the interval [i,N] that allows the optimization of the performance index presented before, this problems can be solved for minimization or maximization.

Lagrange multipliers are used to find the optimal control sequence $u_i^*, u_{i+1}^*, ..., u_{N-1}^*$ to minimize or maximize J. As mentioned before, there is a constraint function but this function is different for each time k in the interval [i,N], and that is the reason why there is a Lagrange Multiplier for each time, therefore is one for each constraint.

The next step is to define the augmented performance index J', this can be done by appending the equation (1) to the performance index in equation (2) along with $\lambda_k \in R^n$. The resulting equation is:

$$J' = \phi(N, x_N) + \sum_{k=i}^{N-1} [L^k(x_k, u_k) + \lambda_{k+1}^T (f^k(x_k, u_k) - x_{k+1})] \tag{15}$$

To simplify J' a Hamiltonian function can be used expressed in this form

$$H^k(x_k, u_k) = L^k(x_k, u_k) + \lambda_{k+1}^T (f^k(x_k, u_k) - x_{k+1}) \tag{16}$$

The simplified J' function is

$$J' = \phi(N, x_N) - \lambda_N^T * x_N + H^i(x_i, u_i) + \sum_{k=i+1}^{N-1} [H^k(x_k, u_k) - \lambda_k^T x_k] \tag{17}$$

There is a Hamiltonian function for each time k, just as with the Lagrange multipliers.

In the continuous case, the equations are similar but there are some other things to consider. The Pontriagyn Maximum Principle guarantees the existence of the optimum. The first step is to change the constrained dynamic optimization problem into a unconstrained problem, and express that as the Hamiltonian function H

$$H(t, y, u, \lambda) = F(t, y, u) + \lambda(t) f(t, y, u) \tag{18}$$

λ is associated to the Lagrange multipliers.

The conditions of the Maximum Principle or Pontryagin conditions are

$$max_u H(t, y, u, \lambda) \forall t \in [0, T] \tag{19}$$

$$\dot{y} = \frac{\partial H}{\partial \lambda} \tag{20}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial y} \tag{21}$$

$$0 = \frac{\partial H}{\partial u} \tag{22}$$

$$\lambda(T) = 0 \tag{23}$$

And the main condition is

$$H(t, y, u^*, \lambda) \geq H(t, y, u, \lambda) \forall t \in [0, T] \tag{24}$$

The optimal control equation for the continuous case taking into account the Lagrange multipliers and the Hamiltonian is

$$max(J) = \int_0^T F(t, y, u) dt \tag{25}$$

and the restrictions associated to it are as follows

$$\dot{y} = f(t, y, u) \tag{26}$$

$y(0) = y_0$ where this value is given by the problem, and $u(t) \in U \forall t \in [0, T]$

3.1 Computational basis

The development of the project was done using the programming language Python with the library Gekko. As stated by Beal et al. (2018) Gekko specializes in dynamic optimization problems for mixed-integer, nonlinear, and differential algebraic equations problems. By blending the approaches of typical algebraic modeling languages (AML) and optimal control packages, GEKKO greatly facilitates the development and application of tools such as nonlinear model predicative control (NMPC), real-time optimization (RTO), moving horizon estimation (MHE), and dynamic simulation.

For the resolution and optimization of the model, Gekko discretizes the differential equations through the use of orthogonal finite element placement, using the implicit Runge-Kutta method Rafau II-A. Optimization problems are solved using different specific algorithms for non-linear problems.

In order to develop the simulation and dynamic optimization processes, the system of differential and algebraic equations is totally discretized by means of the orthogonal placement method of finite elements. The problem is solved in finite elements of equal length, allowing the use of the implicit Runge-Kutta method Radau II-A of the fifth order. The Radau II-A method describes the differentials of the state variables u in each of the finite elements as polynomials, whose coefficients are determined through Lagrange interpolation (E. 28) and evaluated on the different points of temporary placement (E. 27)

$$t_{i,k} = t_{i-1} + h_f \tau_k \qquad 0 \leq t \leq 1 \tag{27}$$

$$\frac{du_t}{dt} = \sum_{j=1}^{n} l_j(t) \left(\frac{du}{dt} \right)_{t=tj} \tag{28}$$

where $l_j(t)$ are permutations over the different points of interpolation (E. 29)

$$l_j(t) = \prod_{j=1, i \neq j}^{n} \frac{\tau - \tau_j}{\tau_i - \tau_j} \tag{29}$$

4 Problem development

Link to the project's code: `https://tinyurl.com/h2aywrkm`

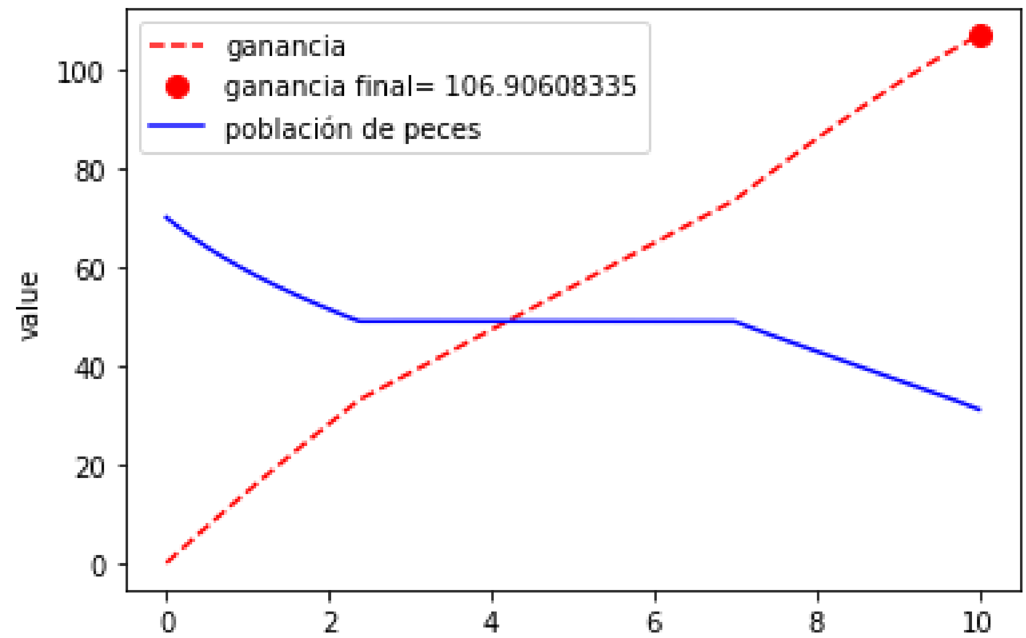

Figure 1: Fishing rate vs revenue

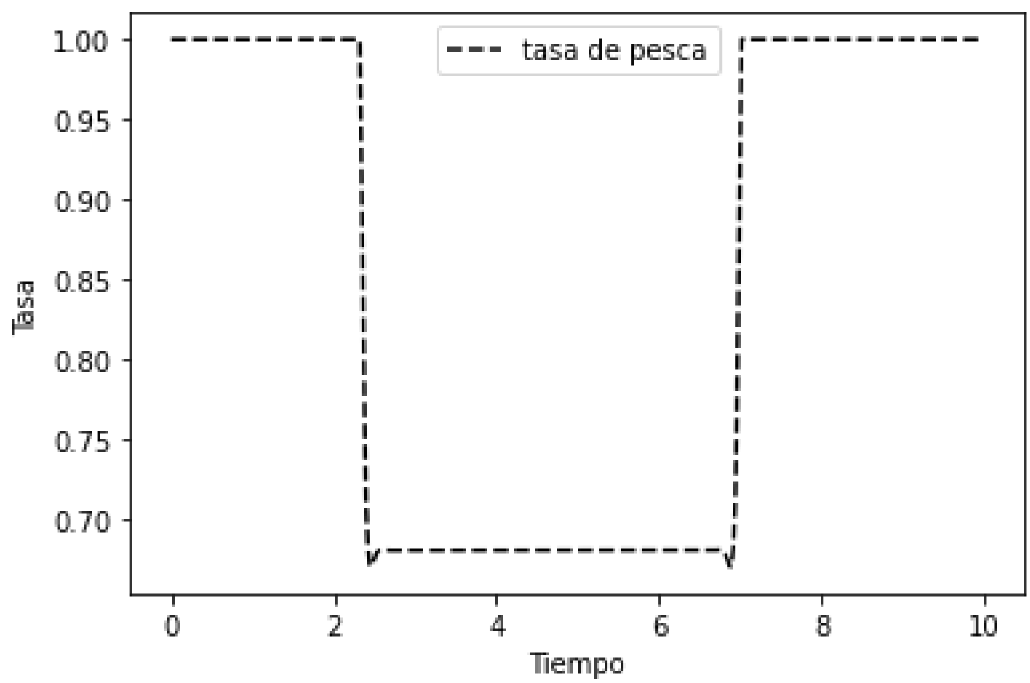

Figure 2: Control variable

What is interesting about these results is that one can see the optimal trajectories of the profits, the state variable (Fish population) and the control variable, which is the fishing rate.

From these optimal trajectories one can see that, the company changes the fishing rate depending on the dynamics of the fish population. See, for example, that the company reduces the fishing rate when the population of fishes start to rapidly decrease, and increases back again when the fish population restarts growing. The results -optimal trajectories- allows one to see the relationship between the dynamics of the state variable, the controls and the value or objective function.

5 Conclusions

After developing this problem using optimal control optimization some important conclusions were drawn. From a theoretical point of view it is certain that a good approach to solve optimal control problems is by using finite elements because this outlook enables the solution to be discretized.

Another important point to highlight is that the previous example was a deterministic problem, that is to say that there are non stochastic elements included because the profits, fishing, fish population recovery, fish growth amongst others were determined by specific functions in a closed time interval.
Finally, it becomes clear that in the problem analyzed the company is aware of the fish recovery rate in every instant of the time interval analyzed, this is why they fish according to the fish population dynamic in order to allow fish population to recover and allow them to continue their business.

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