# Strong Duality - Why do We Need Slater's Condition?

## Conic Programming Scenario

An Euclidean space $\mathbb{E}$ is a finite-dimensional vector space over $\mathbb{R}$ endowed with an inner product $\langle \cdot, \cdot \rangle$. If $\mathbb{E}$ and $\mathbb{Y}$ are Euclidean spaces, the *direct sum* of $\mathbb{E}$ and $\mathbb{Y}$ is the Euclidean space

$$\mathbb{E} \oplus \mathbb{Y} := \{x \oplus y : x \in \mathbb{E} \text{ and } y \in \mathbb{Y}\},$$

where we consider for each $x_1 \oplus y_1$, $x_2 \oplus y_2 \in \mathbb{E} \oplus \mathbb{Y}$ and $\alpha \in \mathbb{R}$:

(i) $(x_1 \oplus y_1) + (x_2 \oplus y_2)_{\mathbb{E}\oplus\mathbb{Y}} = (x_1 +_{\mathbb{E}} x_2) \oplus (y_1 +_{\mathbb{Y}} y_2)$;

(ii) $\alpha(x_1 \oplus y_1) = \alpha x_1 \oplus \alpha y_1$;

(iii) $\langle x_1 \oplus y_1, x_2 \oplus y_2 \rangle_{\mathbb{E}\oplus\mathbb{Y}} = \langle x_1, x_2 \rangle_{\mathbb{E}} +_{\mathbb{R}} \langle y_1, y_2 \rangle_{\mathbb{Y}}$.

**Definition.** Let $\mathbb{E}$ be an Euclidean space. A *cone* is a set $K \subseteq \mathbb{E}$ such that $\alpha x \in K$ for each $x \in K$ and $\alpha \in \mathbb{R}_{++}$. A *hyperplane* is a set of the form $\{x \in \mathbb{E} : \langle x, a \rangle = \beta\}$ for some $0 \neq a \in \mathbb{E}$ and $\beta \in \mathbb{R}$. Similarly, a (closed) *halfspace* is a set of the form $\{x \in \mathbb{E} : \langle a, b \rangle \leq \beta\}$ for some $0 \neq a \in \mathbb{E}$ and $\beta \in \mathbb{R}$. A *polyhedron* is the intersection of finitely many halfspaces.

**Definition.** Let $\mathbb{E}$ be an Euclidean space. A cone $K \subseteq \mathbb{E}$ is *pointed* if $K \cap -K = \{0\}$. We also say that $K$ is proper if $K$ is *convex*, *closed*, pointed, and $\operatorname{int}(K) \neq \varnothing$. Moreover, the cone $K$ is *polyhedral* if it is a polyhedron.

**Definition.** Let $\mathbb{E}$ and let $K \subseteq \mathbb{E}$ be a cone. The *dual cone* of $K$ is the set

$$K^* := \{x \in \mathbb{E} : \langle x, k \rangle \geq 0 \text{ for each } k \in K\}.$$

## Cone Partial Order

**Definition.** Let $\mathbb{E}$ be an Euclidean space, $K \subseteq \mathbb{E}$ be a proper cone and $x, y \in \mathbb{E}$. Then, the cone $K$ induces an order in $\mathbb{E}$ as follows:

$$x \succeq_K y \text{ if } x - y \in K.$$

Moreover,

$$x \succ_K y \text{ if } x - y \in \operatorname{int}(K).$$

**Proposition 1.** Let $\mathbb{E}$ be an Euclidean space and let $K \subseteq \mathbb{E}$ be a proper cone. Then, $\succeq_K$ is a partial order on $\mathbb{E}$.

## Overview of Duality Theory

**Definition.** Let $\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3$ and $\mathbb{Y}_1, \mathbb{Y}_2, \mathbb{Y}_3$ be Euclidean spaces. Let $K \subseteq \mathbb{E}_1$ and $L \subseteq \mathbb{Y}_1$ both be proper cones. Let $K_p \subseteq \mathbb{E}_2$ and $L_p \subseteq \mathbb{Y}_2$ both be polyhedral cones. Consider $c_1 \oplus c_2 \oplus c_3 \in \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3$ and $b_1 \oplus b_2 \oplus b_3 \in \mathbb{Y}_1 \oplus \mathbb{Y}_2 \oplus \mathbb{Y}_3$. Let $A \colon \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \mathbb{E}_3 \to \mathbb{Y}_1 \oplus \mathbb{Y}_2 \oplus \mathbb{Y}_3$ be a linear function.

A *conic optimization problem* is an optimization problem of the form:

$$\begin{aligned} \text{minimize} \quad & \langle x_1 \oplus x_2 \oplus x_3, c_1 \oplus c_2 \oplus c_3 \rangle \\ \text{subject to} \quad & A(x_1 \oplus x_2 \oplus x_3) \succeq_{L^* \oplus L_p^* \oplus \{0\}} b_1 \oplus b_2 \oplus b_3, \\ & x_1 \oplus x_2 \oplus x_3 \in K \oplus K_p \oplus \mathbb{E}_3. \end{aligned} \tag{1}$$

The set $G := \{x_1 \oplus x_2 \oplus x_3 \in K \oplus K_p \oplus \mathbb{E}_3 \colon A(x_1 \oplus x_2 \oplus x_3) \succeq_{L^* \oplus L_p^* \oplus \{0\}} b_1 \oplus b_2 \oplus b_3\}$ is the feasible set of (1). According to the notation presented in the Preliminaries, a conic optimization problem can be represented simply as $(G, \langle \cdot, c_1 \oplus c_2 \oplus c_3 \rangle)$.

We now define the dual problem of (1).

**Definition.** Consider the conic optimization problem (1). The *dual problem* of (1) is the conic optimization problem

$$\begin{aligned} \text{maximize} \quad & \langle b_1 \oplus b_2 \oplus b_3, y_1 \oplus y_2 \oplus y_3 \rangle \\ \text{subject to} \quad & A^*(y_1 \oplus y_2 \oplus y_3) \preceq_{K^* \oplus K_p^* \oplus \{0\}} c_1 \oplus c_2 \oplus c_3, \\ & y_1 \oplus y_2 \oplus y_3 \in L \oplus L_p \oplus \mathbb{Y}_3. \end{aligned} \tag{2}$$

To simplify the notation on the following propositions, we will denote $x \coloneqq x_1 \oplus x_2 \oplus x_3$, $y \coloneqq y_1 \oplus y_2 \oplus y_3$, $c \coloneqq c_1 \oplus c_2 \oplus c_3$, and $b \coloneqq b_1 \oplus b_2 \oplus b_3$ on (1) and (2) until the end of this section.

**Theorem 2** (Weak duality)**.** Let $\alpha$ be the optimal value of (1) and let $\beta$ be the optimal value of (2). If $x$ is feasible in (1) and $y$ is feasible in (2), then $\langle b, y\rangle \leq \langle c, x\rangle$. In particular, $\alpha \geq \beta$. Moreover, if $\langle x, c\rangle = \langle b, y\rangle$ then $x$ and $y$ are optimal solutions for their respective problems and $\alpha = \beta$.

**Corollary 3** (Complementary Slackness)**.** Let $\alpha$ be the optimal value of (1) and let $\beta$ be the optimal value of (2). Let $x$ be a feasible solution in (1) and $y$ be a feasible solution in (2). Then $x$ and $y$ are optimal in their respective problems and $\alpha = \beta$ if, and only if,

$$\langle x, c - A^*(y)\rangle = \langle A(x) - b, y\rangle = 0.$$

**Proposition 4.** Let $\mathbb{E}$ be an Euclidean space, let $K \subseteq \mathbb{E}$ be a proper cone, let $K_p \subseteq \mathbb{E}$ be a polyhedral cone and $S \subseteq \mathbb{E}$ a linear subspace. If $\operatorname{int}(K) \cap K_p \cap S \neq \varnothing$, then

$$(K \cap K_p \cap S)^* = (K^* + K_p^* + S^\perp).$$

**Theorem 5** (Strong duality)**.** Consider the optimization problem (1). If (1) is bounded below and has a restricted Slater point, then the optimal values of (1) and its dual (2) are equal and (2) has an optimal solution.

## Race for Closedness

Our proof of strong duality relies heavily on Proposition 4, which in turn needs the set $K^* + K_p^* + S^\perp$ (in that context) to be closed. As we shall discuss, this fact is closely related with the commutativity between closures and linear images of convex sets and their duals.

Let $\mathbb{E}$ be an Euclidean space and let $\varnothing \neq C_1, C_2 \subseteq \mathbb{E}$ both be convex sets. Then $C_1 + C_2$ can be seen as $A(C_1 \oplus C_2)$ where $A\colon \mathbb{E} \oplus \mathbb{E} \to \mathbb{E}$ is given by $A(x_1 \oplus x_2) = x_1 + x_2$.

If we imagine an ideal scenario where we had $\overline{A(C_1 \oplus C_2)} = A(\overline{C_1 \oplus C_2})$, we could apply $A$ to $K^* + K_p^* + S^\perp$ and nothing we will see next would be required.

**Remark.** In the case $C_1$ and $C_2$ are polyhedra, the ideal scenario is actually real. That's why there is no need for constraint qualifications in linear programming.

Unfortunately, the ideal scenario does not hold in general. In fact, we have that

**Proposition 6.** Let $\mathbb{E}$ and $\mathbb{Y}$ be Euclidean spaces, let $C \subseteq \mathbb{E}$ be a convex set, and let $A\colon \mathbb{E} \to \mathbb{Y}$ be a linear function. Then:

(i) $A(\overline{C}) \subseteq \overline{A(C)}$;

(ii) $\operatorname{ri}(A(C)) = A(\operatorname{ri}(C))$.

The first question one asks is: When does item (ii) of Proposition 6 hold with equality? In order to have an satisfactory answer, we need one additional tool.

**Definition.** Let $\mathbb{E}$ be an Euclidean space and let $\varnothing \neq C \subseteq \mathbb{E}$ be a convex set. The *recession cone* of $C$ is the set

$$0^+C \coloneqq \{y \in \mathbb{E} : x + \alpha y \in C \text{ for each } x \in C \text{ and } \alpha \in \mathbb{R}_{++}\}.$$

The *lineality space* of $C$ is the set $\operatorname{lin}(C) \coloneqq 0^+C \cap (-0^+C)$.

**Proposition 7.** Let $\mathbb{E}$ and $\mathbb{Y}$ be Euclidean spaces, let $A\colon \mathbb{E} \to \mathbb{Y}$ be a linear function, and let $\varnothing \neq C \subseteq \mathbb{E}$ be a closed convex set. If $0^+C \cap \operatorname{Null}(A) \subseteq \operatorname{lin}(C)$, then $A(C)$ is closed.

**Corollary 8.** Let $\mathbb{E}$ be an Euclidean space, let $\{C_i\}_{i\in I} \subseteq \mathbb{E}$ be a finite family of nonempty convex sets. If $x_i \in C_i$ for each $i \in I$ and $\sum_{i\in I} x_i = 0$ implies that $x_i \in \operatorname{lin}(C_i)$ for each $i \in I$, then $\overline{\sum_{i\in I} C_i} = \sum_{i\in I} \overline{C_i}$.

Finally, we need to relate our previous achievements with duality theory and the next result is the key to it.

**Proposition 9.** Let $\mathbb{E}$ be an Euclidean space, let $S \subseteq \mathbb{E}$ be a subspace of $\mathbb{E}$, and let $K \subseteq \mathbb{E}$ be a convex cone. Then exactly one of the following two statements is true:

(i) There is no hyperplane separating $S$ and $K$ properly;

(ii) There exists $x$ such that $x \in S^{\perp}$, $x \in -K^*$, and $x \notin K^*$.

Now things start to connect:

**Corollary 10.** Let $\mathbb{E}$ be an Euclidean space and let $\{K_i\}_{i\in I} \subseteq \mathbb{E}$ be a finite family of convex cones. Then exactly one of the following two statements is true:

(i) There exists $y \in \bigcap_{i\in I} \mathrm{ri}(K_i)$;

(ii) There exists a family $\{x_i\}_{i\in I}$ such that $\sum_{i\in I} x_i = 0$, $x_i \in -K_i^*$ for each $i \in I$, and $x_i \notin K_i^*$ for some $i \in I$.

**Corollary 11.** Let $\mathbb{E}$ be an Euclidean space and let $\{K_i\}_{i\in I} \subseteq \mathbb{E}$ be a finite family of convex cones. If $\bigcap_{i\in I} \mathrm{ri}(K_i) \neq \varnothing$ then $\sum_{i\in I} K_i^*$ is closed.

Proposition 9 and Corollaries 10 and 11 can all trivially be adapted to consider a finite family of polyhedral cones. In this case, similar statements can be proved if we replace proper separation for strong separation and weaken the relative interior requirements. The following theorem is a refinement of these results.

**Theorem 12.** Let $\mathbb{E}$ be an Euclidean space, Let $\{K_i\}_{i\in I} \subseteq \mathbb{E}$ be a finite family of convex cones. Assume that there exists $I_0 \subseteq I$ such that $K_i$ is polyhedral for each $i \in I_0$. If $\bigcap_{i\in I_0} K_i \cap \bigcap_{i\in I\setminus I_0} \mathrm{ri}(K_i) \neq \varnothing$, then $\sum_{i\in I} K_i^*$ is closed.

## Homomorphisms - What Does Equivalence Mean in Optimization?

**Definition.** An *optimization problem* is an ordered pair $P = (X, f)$, where $X$ is a set and $f\colon X \to \overline{\mathbb{R}}$ is an extended real-valued function. The problem $P$ is more commonly denoted as

$$\begin{aligned} \text{minimize} \quad & f(x) \\ \text{subject to} \quad & x \in X. \end{aligned}$$

The set $X$ is the *feasible region* of $P$ and the function $f$ is the *objective function* of $P$. The elements of $X$ are the *feasible points* or *feasible solutions* of $P$; everything else is *infeasible*. The optimization problem $P$ is *feasible* if $X \neq \varnothing$. Otherwise it is *infeasible*. The *objective value* of $x \in X$ is $f(x)$. The *optimal value* of $P$ is $\inf_{x\in X} f(x) \in \overline{\mathbb{R}}$. A feasible solution $\bar{x}$ is *optimal* if $f(\bar{x})$ is the optimal value of the problem. If the optimal value of $P$ is $-\infty$, the problem is *unbounded*.

When we write

$$\begin{aligned} \text{maximize} \quad & f(x) \\ \text{subject to} \quad & x \in X \end{aligned}$$

we are referring to the optimization problem $(X, -f)$ and, besides the definition of optimal value which becomes $-\inf_{x\in X} -f(x)$, we use the same terminology as above.

**Definition.** Let $P = (X, f)$ be an optimization problem. A feasible point $\bar{x}$ is *locally optimal* if there exists $V \subseteq X$ such that $\bar{x}$ is the optimal solution of the problem $(V, f)$.

**Definition.** Let $P = (X, f)$ and $Q = (Y, g)$ be optimization problems. A *homomorphism* from $P$ to $Q$ is a function $\varphi\colon X \to Y$ such that $g(\varphi(x)) \leq f(x)$ for each $x \in X$.

**Proposition 13.** Let $P = (X, f)$, $Q = (Y, g)$, and $R = (Z, h)$ be optimization problems. If $\varphi\colon X \to Y$ be a homomorphism from $P$ to $Q$ and $\psi\colon Y \to Z$ be a homomorphism from $Q$ to $R$, then $\psi \circ \varphi\colon X \to Z$ is a homomorphism from $P$ to $R$.

**Proposition 14.** Let $P = (X, f)$ and $Q = (Y, g)$ be optimization problems. If $\varphi\colon X \to Y$ is a bijective function such that $g(\varphi(x)) = f(x)$ for each $x \in X$, then $\varphi^{-1}\colon Y \to X$ is a homomorphism from $Q$ to $P$.

**Corollary 15.** Let $P = (X, f)$ and $Q = (Y, g)$ be optimization problems. If there exists a bijective function $\varphi\colon X \to Y$ such that $g(\varphi(x)) = f(x)$ for each $x \in X$ then $P$ and $Q$ are equivalent.

**Proposition 16.** Consider, for every optimization problems $A$ and $B$,

$$A \sim B \text{ if, and only if there exists a homomorphism from} A \text{ to } B \text{ and vice-versa.}$$

Then $\sim$ is an equivalence relation in $\mathcal{C}$.

## First Results

**Proposition 17.** Let $P = (X, f)$ and $Q = (Y, g)$ be equivalent optimization problems. If either $P$ or $Q$ has finite optimal value $\alpha \in \mathbb{R}$, then $\alpha$ is the optimal value of both problems.

**Proposition 18.** Let $P = (X, f)$ and $Q = (Y, g)$ be equivalent optimization problems. Then $P$ has an optimal solution if, and only if $Q$ has an optimal solution.

**Proposition 19.** Let $P = (X, f)$ and $Q = (Y, g)$ be equivalent optimization problems. Then $P$ and $Q$ have the same outcome. That is:

(i) $P$ is infeasible if and only if $Q$ is infeasible.

(ii) $P$ is unbounded if and only if $Q$ is unbounded.

(iii) $P$ has finite optimal value and does not have optimal solution if and only if $Q$ has finite optimal value and does not have optimal solution.

(iv) $P$ has finite optimal value and an optimal solution if and only if $Q$ has finite optimal value and an optimal solution.

## Further Developments