CE3007 - Digital Signal Processing

03A1: CTFS Examples

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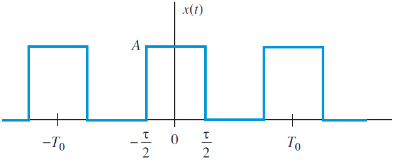
    - Plotting Magnitude and Phase Spectrum

Part I

Analysis of CT Periodic Rectangles

Work Example: FS periodic rectangle

Given a periodic rectangle (even) CT signal with amplitude A, period T_0 and on-period τ, perform Fourier Analysis and hence sketch its magnitude and phase domain representation.

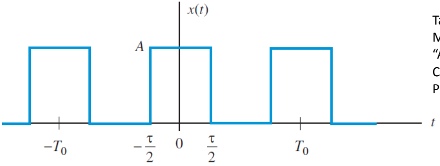


In other words,

1. As signal is CT periodic, we can use CTFS or CTFT for analysis. Its easier to use CTFS, therefore evaluate C_k (Fourier Series Coefficients)
2. Sketch magnitude spectrum $|C_k|$ vs frequencies $(k\Omega_0)$
3. Sketch phase spectrum $\angle C_k$ vs frequencies $(k\Omega_0)$

Type	Equation
CTFS Analysis	$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\Omega_0 t}\, dt = \|C_k\|\, e^{j\theta_k}$, where $k \in Z\,,\ T_o = Period, \Omega_0 = \frac{2\pi}{T_0} = 2\pi F_0$ (rad/sec)
Synthesis: Complex Exponential Form	$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\Omega_0 t}$
Synthesis: Combine Trigonometric Form	$x(t) = C_0 + \sum_{k=1}^{\infty} 2\, \|C_k\| \cos(k\Omega_0 t + \theta_k)$
Synthesis: Trigonometric Form	$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\Omega_o t) + B_k \sin(k\Omega_o t)$ where, $2C_k = A_k - jB_k$, $C_0 = A_0$

Ex 1:periodic rectangle I



Taken From Manolakis&Ingle "Applied DSP" Cambridge University Press

Example 4.1 Rectangular pulse train

Consider the periodic rectangular pulse train in Figure 4.8. The Fourier coefficients are given by

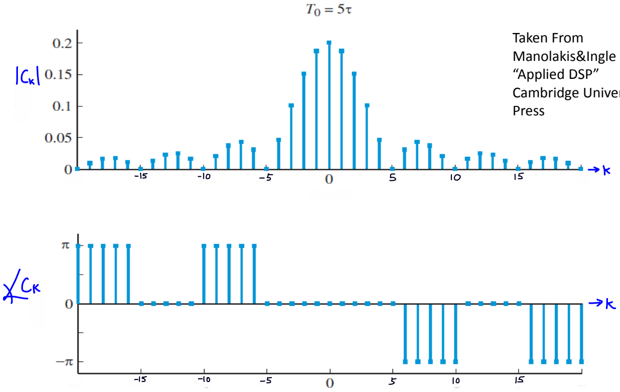
$$c_k = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = \frac{A}{T_0}\left[\frac{e^{-j2\pi k F_0 t}}{-j2\pi k F_0}\right]_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{\pi F_0 k T_0} \frac{e^{j\pi k F_0 \tau} - e^{-j\pi k F_0 \tau}}{2j}$$

$$= \frac{A\tau}{T_0}\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}. \quad k = 0, \pm 1, \pm 2, \ldots \tag{4.30}$$

The values of c_k are obtained by evaluating the function $(A\tau/T_0)\sin(\phi)/\phi$ at equidistant points $\phi = k(\pi F_0 \tau)$. Since $\lim_{\phi\to 0}[\sin(\phi)/\phi] = 1$, we have $c_0 = A\tau/T_0$. The function $\sin(\phi)/\phi$ has zero crossings at multiples of π, that is, at $\phi = m\pi$, $m = 0, \pm 1, \pm 2, \ldots$ The zero crossings occur at $\phi = \pi F \tau = m\pi$ or $F = m/\tau$. The spacing $F = 1/\tau$ between the zero crossings is determined by the width τ of the pulse, whereas the spacing $F_0 = 1/T_0$ between the spectral lines is determined by the fundamental period T_0.

Ex 1:periodic rectangle II



Taken From Manolakis&Ingle "Applied DSP" Cambridge University Press

Magnitude and phase spectra of a rectangular pulse train with $A = 1$ and $T_0 = 5\tau$.

- Magnitude spectra are always positive. Hence, negative signs should be absorbed in the phase using the identity: $-A \cos \Omega t = \cos(\Omega t \pm \pi)$. It does not matter whether we take $+\pi$ or $-\pi$ because $\cos(-\pi) = \cos \pi$. However, we use both $+\pi$ and $-\pi$ to bring out the odd symmetry of the phase.

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03A1 CTFS Examples

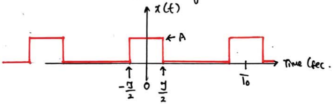
Periodic Rectangle

Example:periodic Square

Detailed Working - 1

Appendix : Example. Detailed Working.

Q) Find the CTFS coefficients of periodic square.



Analysis Eq^n : $C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\Omega_0 t}\, dt$.

$\Omega_0 = 2\pi F_0 = \frac{2\pi}{T_0}$ unit radian/sec.

(angular Fundamental Frequency) — F_0: frequency (cycle/sec) — T_0: period.

$k \in \mathbb{Z}$ integers

Interpretation : Fourier Series tells us, for CT periodic signal with period T_0, we only need sinusoids at $k\Omega_0$ (radian/sample) frequencies for analysis, albeit for all $k \in \mathbb{Z}$ integers.

ie , $\ldots, C_{-2}, C_{-1}, C_0, C_1, C_2, C_3, \ldots$

L1

Working

$$C_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} x(t)\, e^{-jk\Omega_0 t}\, dt$$

$$= \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\, e^{-jk\Omega_0 t}\, dt.$$

$$= \frac{A}{T_0}\left[\frac{1}{-jk\Omega_0} e^{-jk\Omega_0 t}\right]_{t=-\frac{\tau}{2}}^{t=+\frac{\tau}{2}}$$

flip order by -ve sign

$$= \frac{A}{k T_0 \Omega_0}\left[\frac{e^{-jk\Omega_0(-\frac{\tau}{2})} - e^{-jk\Omega_0 \frac{\tau}{2}}}{j}\right]$$

Top Bottom x2

$$= \frac{A}{k \cancel{T_0} \frac{2\pi}{\cancel{T_0}}}\left[\frac{\cancel{2}\left(e^{+jk\Omega_0\frac{\tau}{2}} - e^{-jk\Omega_0\frac{\tau}{2}}\right)}{2j}\right]$$

Euler eq^n.

$$= \frac{A}{k\pi}\left[\sin\left(k\Omega_0\frac{\tau}{2}\right)\right] \quad \#$$

$$= \frac{A}{k\pi}\,\frac{\sin\left(k 2\pi F_0 \frac{\tau}{2}\right)}{\left(2F_0\frac{\tau}{2}\right)}\cdot\left(2F_0\frac{\tau}{2}\right) = \frac{A\tau}{T_0}\,\mathrm{sinc}\left(\pi\frac{\tau k}{T_0}\right) \quad \#$$

Detailed Working - 2

Working.

$$C_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{+\frac{T_0}{2}} x(t)e^{-j\overset{k=0}{0}\Omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} A\,(1)\,dt.$$

$$= \frac{A}{T_0}\left[t\right]_{t=-\tau/2}^{\tau/2} = \frac{A}{T_0}\left[\frac{\tau}{2}-\left(-\frac{\tau}{2}\right)\right]$$

$$= \frac{A\tau}{T_0}$$

interpretation: C_0 ~ at angular frequency $= 0$.

i) value is a real number since A, τ, T_0 are real.

ii) its a ratio, of duty cycle scaled by A.

iii) to represent DC level required to construct $x(t)$.

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Example: $\tau = 0.2$ sec, $A = 1$

$T_0 = 1$sec.

$\therefore$ $k=0$, $C_0 = \frac{1(0.2)}{1} = 0.2$

$k=1$, $C_1 = \frac{A}{k\pi}\left[\sin\left(k2\pi F_0 \frac{\tau}{2}\right)\right]$

$= \frac{1}{1(\pi)}\left[\sin\left(1 2\pi (1) \frac{0.2}{2}\right)\right]$

$= 0.1871$

$k=2$, $C_2 = \frac{1}{2(\pi)}\sin\left((2)2\pi(1)\frac{0.2}{2}\right)$

$= 0.1514.$

$k=3$, $C_3 = 0.1009$

$k=4$, $C_4 = 0.0468$

$k=5$, $C_5 = 0.$

$k=6$, $C_6 = -0.0312$

$\vdots$

$k=10$, $C_{10} = 0.$

at $k=5$, $\left(\frac{5\times\tau}{T_0} = 1\right)$

$C_5 = 0.$

not a coincident. since

$\sin\left(k2\pi F_0 \frac{\tau}{2}\right)$.

$= \sin\left(\pi F_0 k\tau\right)$.

at $1, 2, 3, 4, \dots$

then $= 0$.

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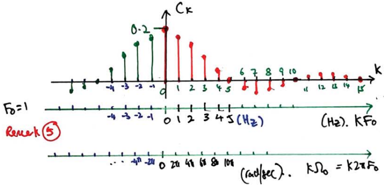
Periodic Rectangle

Example:periodic Square

Detailed Working - 3

$C_{0\to 15} = [0.2, 0.1871, 0.1514, 0.1009, 0.0468, 0, -0.0312, -0.0432, -0.0378, -0.0208, 0, 0.0170, 0.0252, 0.0233, 0.0134, 0, \dots]$.



1) Remark: Evaluate $C_{-15,-14,-13,\dots,0}$

we will find it is <u>complex conjugate</u> of $C_{0..15}$.

2) However in this example, all FS coeff are real, (not so common), then

$C_{-15..0} = C_{0..15}$!!!

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Remark 3: ∴ $|C_k|$ ⇒ plot abs values of C_k.

$\angle C_k$, since C_k is real in this case, for +ve value, $\angle = 0$.

-ve values, $\angle = -\pi$ or π.

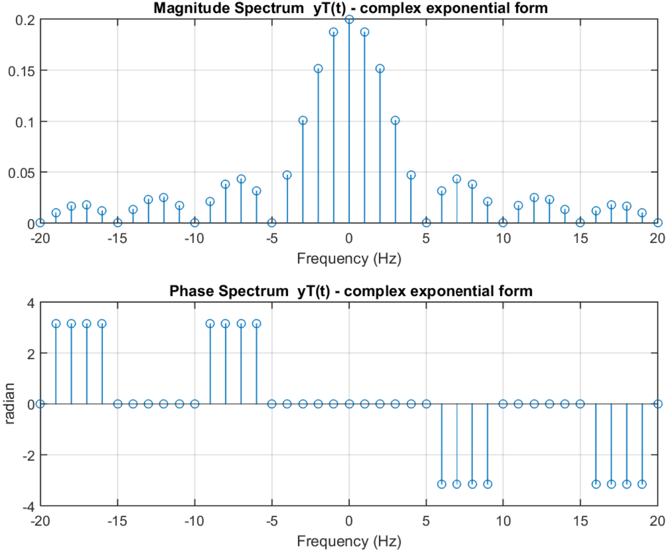
why? let $z = \underbrace{1+j0}_{\text{rectangle}} = 1e^{j0} = 1e^{j2\pi} = 1e^{jk(2\pi)}$.

$z = -1+j0 = 1e^{j\pi} = 1e^{j(-\pi)} = 1e^{j3\pi}, \dots$

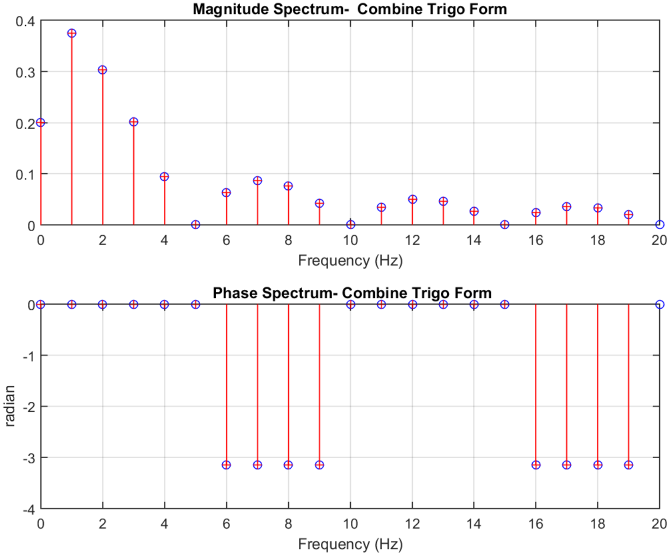
Remark 4: By convention, for square waves, the CTFS phase at +ve freq set to $-\pi$, and -ve freq set to $+\pi$ (its conjugate).

Remark 5: horizontal axis label as "k". Actually its frequency kF_0 (Hz) or $k\Omega_0$ (rad/sec). showing which frequency are required for analysis + synthesis.

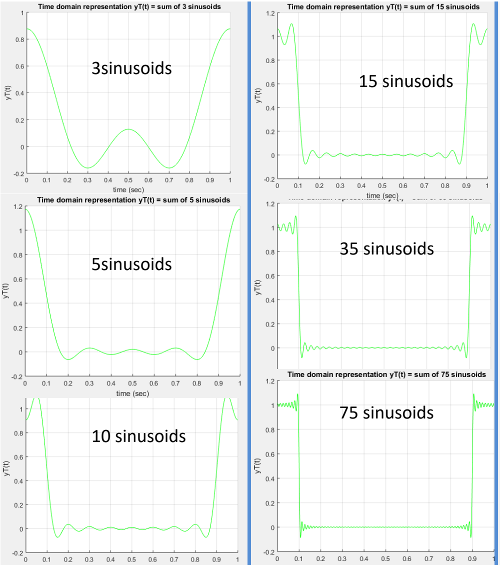
Magnitude and Phase (complex exponential form)



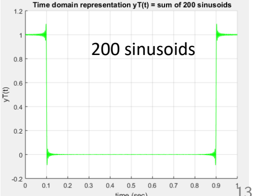
Magnitude and Phase (combine Trigo form)



Synthesised Output using different number sinusoids



Synthesis of rectangular wave using different number of cosine terms.



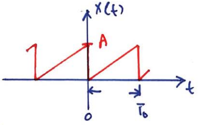
Part II

Analysis of CT Periodic Saw

Work Example: FS periodic Saw

①

Find the CTFS of Saw Tooth $x(t)$.



$x(t) = A\frac{t}{T_0}$ for $t = 0..T_0$. and periodic T_0.

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\Omega_0 t}\, dt.$$

$$\Omega_0 = \frac{2\pi}{T_0} = 2\pi F_0$$

$$C_0 = \frac{1}{T_0}\int_0^{T_0} A\frac{t}{T_0}\underbrace{e^{-j0\Omega_0 t}}_{\text{"1"}}\, dt$$

$$= \frac{A}{T_0^2}\int_0^{T_0} t\, dt = \frac{A}{T_0^2}\left[\frac{t^2}{2}\right]_{t=0}^{T_0}$$

$$= \frac{A}{2} \; \#$$

$$C_k = \frac{1}{T_0}\int_0^{T_0} A\frac{t}{T_0}\, e^{-jk2\pi F_0 t}\, dt.$$

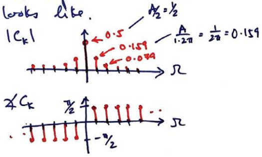
$$= \frac{A}{T_0^2}\int_0^{T_0} t\, e^{-jk2\pi F_0 t}\, dt.$$

integration by parts.

see next page.

$$= \frac{jA}{k2\pi} \quad ; \; k \in \mathbb{Z} \text{ integers.}$$

Remark: ∴ the Magnitude & phase spectrum looks like.



Detailed Working 2

Common Maths. Knowledge for FS.

$$\int u\,dv = uv - \int v\,du.$$

Choose "u" in this order

L - Log
I - Inverse
A - algebraic
T - Trigo
E - exponential.

$$\therefore \int t\, e^{-jk2\pi F_0 t}\, dt$$

let $u = t$

$\frac{du}{dt} = 1$

$\Rightarrow du = dt$

$dv = e^{-jk2\pi F_0 t}$

$\therefore v = \frac{1}{-jk2\pi F_0} e^{-jk2\pi F_0 t}$

(2)

$$\int_0^{T_0} t\, e^{-jk2\pi F_0 t}$$

$$\doteq uv - \int v\,du$$

$$= \left(t \left[\frac{1}{-jk2\pi F_0} e^{-jk2\pi F_0 t} \right] \right)_{t=0}^{T_0}.$$

remember $F_0 = \frac{1}{T_0}$.

$$\therefore T_0 \left[\frac{1}{-jk2\pi F_0} e^{-jk2\pi \cancel{F_0} \cdot \cancel{T_0}} \right] - 0.$$

$$\therefore C_k = \frac{A}{\cancel{T_0^2}} \left[\frac{1}{\cancel{F_0}} \cdot \frac{1}{-jk2\pi \cancel{F_0}} e^{-jk2\pi} \right]$$

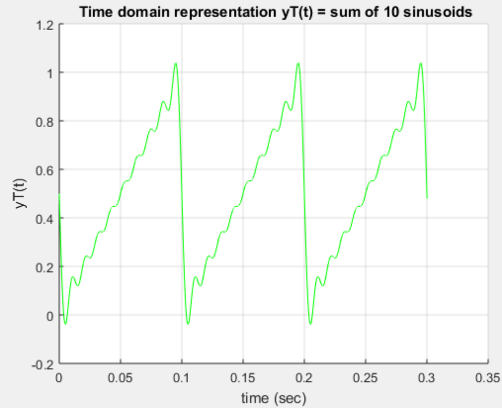
$$= \frac{A}{-jk2\pi} e^{-jk2\pi}; \quad k \in \mathbb{Z}$$

Note $\boxed{\frac{1}{j} = \frac{1}{j} \cdot \frac{j}{j} = \frac{j}{j^2} = \frac{j}{-1}; \quad j = \sqrt{-1}}$

$$\therefore -\frac{A}{-1} j \cdot \frac{1}{k2\pi} (1); \quad \boxed{\text{since } e^{-jk2\pi} = 1}$$

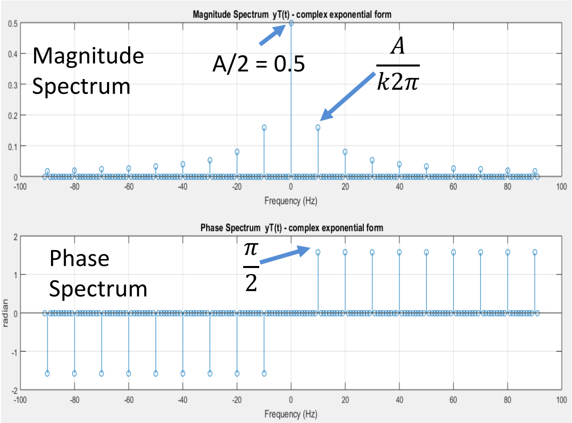
$$\boxed{C_k = \frac{Aj}{k2\pi}}$$

Saw Tooth Reconstructed



$$T_0 = 0.1\,\text{sec} \quad , \qquad F_0 = \frac{1}{T_0} = 10Hz\,, \qquad A = 1$$

Spectrum of Saw Tooth



$$F_0 = \frac{1}{T_0} = 10Hz\,, \qquad A = 1$$