CE3007 - Digital Signal Processing

03A: Fourier Analysis Overview

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November 8, 2018

Symbols I

Notations for Fourier Analysis

j	$\sqrt{-1}$
$e^{\pm j\theta}$	$cos(\theta) \pm jsin(\theta)$
ω	$2\pi f$, digital angular frequency, unit = radians/sampl where f is the digital frequency, cycles per sample.
Ω	$2\pi F$, CT angular or radian frequency, unit = radians where F is the frequency in Hz, cycles per seconds.
A_k, B_k, C_k	Continuous Time Fourier Series (CTFS) coefficients
$X(j\Omega)$	CT Fourier Transform of $x(t)$
$X(s)$	Laplace Transform of $x(t)$
c_k	Discrete Time Fourier Series (DTFS) of $x[n]$
$X(e^{j\omega})$	Discrete Time Fourier Transform (DTFT) of $x[n]$
$X[k]$	Discrete Fourier Transform (DFT) of $x[n]$
$X(z)$	Z Transform of $x[n]$

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Part I

Introduction to Fourier Analysis and Synthesis

Who is Joseph Fourier?

- French Mathematician born in the 18^{th} century.
- Introduced the equation (1807) to represent general function by sums of simpler trigonometric functions.
- Used in a vast spectrum of mathematics, sciences and engineering.

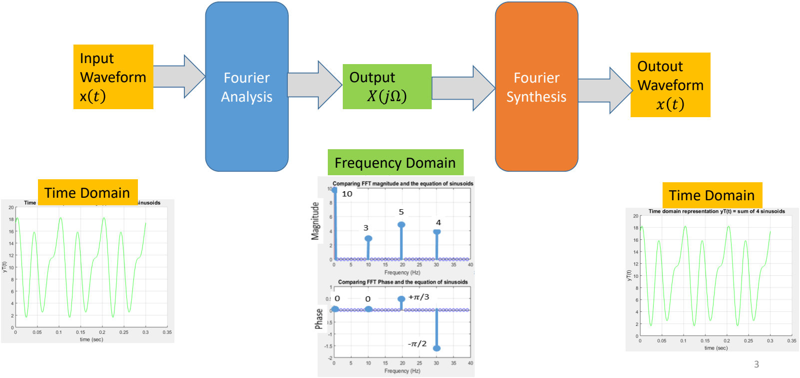
What is Fourier Analysis and Synthesis?

Given a CT input signal $x(t)$ (same for Discrete time $x[n]$)

1. Analysis - decompose $x(t)$'s into terms of cosine waves (complex exponential). Representing x(t) in frequency domain:
   - How many cosines (complex exponential) waves? N.
   - k^{th} cosine term - { Frequency (F_k), Magnitude (M_k), Phase Φ_k}.
2. Synthesis - Given the frequency domain representation, generate (synthesize) the signal $x(t)$ in time domain by combining (adding) all the constituent signal, e.g,

$$x(t) = \sum_{\mathrm{k=1..N}} |M_k| \, cos(2\pi F_k + \Phi_k) \tag{1}$$

Fourier Analysis and Synthesis - Overview



A Simple Example

Example: $x(t)$ and its Frequency Domain $X(j\Omega)$ representation

Input : $x(t) = 10 + 3\cos(2\pi 10\, t) + 5\cos\left(2\pi 20\, t + \frac{\pi}{3}\right) + 4\sin(2\pi 30\, t)$

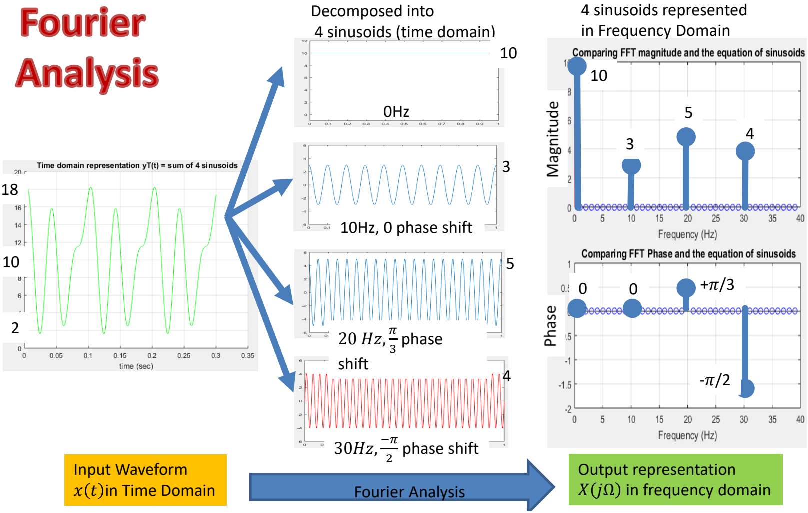
Decomposed into cosine waves with required frequency , amplitude and phase:

$$x(t) = 10\cos(2\pi 0 t + 0) + 3\cos(2\pi 10 t + 0) + 5\cos\left(2\pi 20 t + \frac{\pi}{3}\right) + 4\cos\left(2\pi 30 t - \frac{\pi}{2}\right)$$

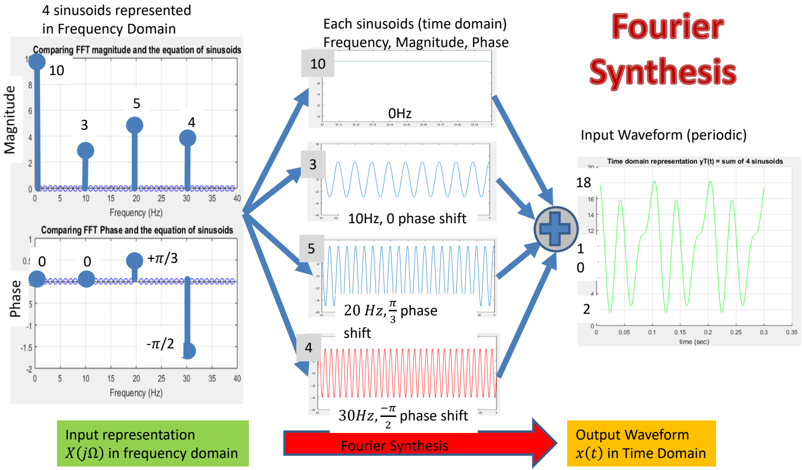
$X(j\Omega)$ = cosine waves at

[{frequency $2\pi 0$, amplitude 10, phase = 0 },
{frequency $2\pi 10$, amplitude 3, phase = 0 },
{frequency $2\pi 20$, amplitude 5, phase = $\frac{\pi}{3}$ },
{frequency $2\pi 30$, amplitude 4, phase = $-\frac{\pi}{2}$ }]

Analysis: Simple Example

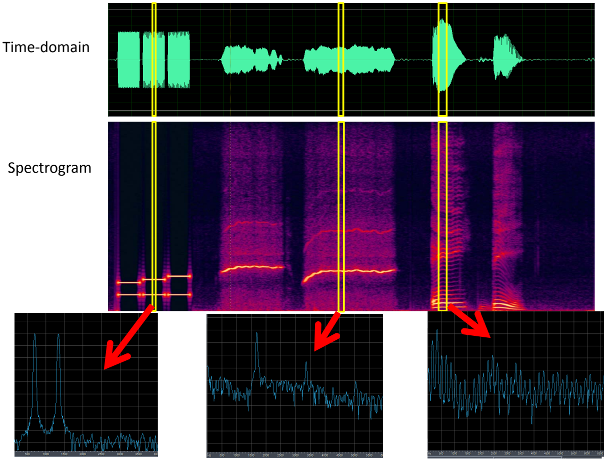


Synthesis: Simple Example



Fourier Analysis of signals

Enables: Feature Generation, compression (mpeg, jpeg,..), locating noise in frequency domain

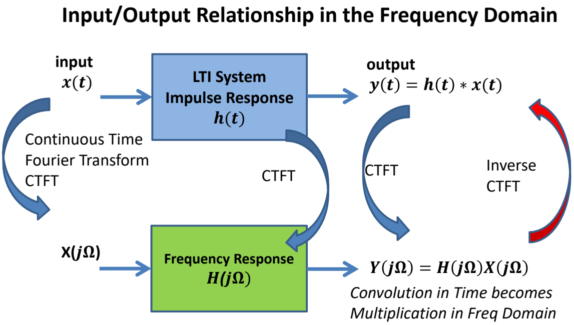


DT Fourier Transform of the marked short segments

FT of LTI's impulse response

Fourier Transform of a Linear Time Invariant (LTI) system's impulse response $h(t)$:

- quantify how system will affect input sinusoids at a particular Ω. This leads to
  a) frequency response $H(j\Omega)$
  b) enables convolution in time to be multiplication in frequency domain.



Frequency Response $H(j\Omega)$

What is $H(j\Omega)$?
Answer: $H(j\Omega)$ = frequency response of LTI system.
$H(j\Omega)$ = CTFT of $h(t)$ (impulse response).

1. Given a single CT cosine signal $Acos(\Omega_1 t)$ as input to LTI, the LTI will output $Gain * Acos(\Omega_1 t + \phi)$
   - $Gain = |H(j\Omega_1)|$ and phase $\phi = \angle H(j\Omega_1)$.
2. Why? Complex Exponentials are eigenfunction of LTI.

Part II

Overview - CT Fourier Analysis

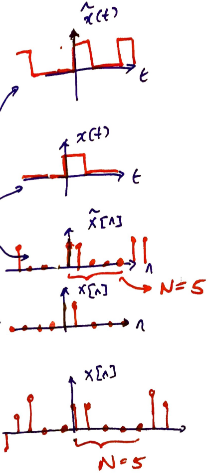
Types of signals for Fourier Analysis

What are examples of Fourier Analysis.

a) depends on
- ↗ CT = continuous time
- ↘ DT = discrete time

| periodic
| not periodic.

Types	Periodicity.	Type of FA.	
CT	periodic	CTFS, CTFT	$\tilde{x}(t)$
CT	NOT periodic	CTFT	$x(t)$
DT	periodic	DTFS, DTFT	$\tilde{x}[n]$
DT	NOT periodic	DTFT.	$x[n]$
DT - analyze a fix window only		DFT.	$x[n]$



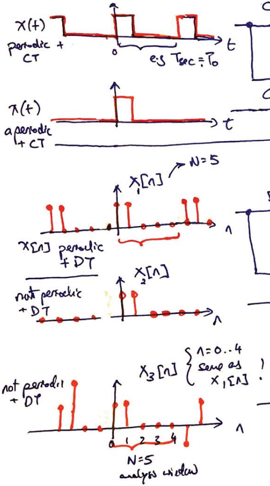
FT = Fourier Transform

FS = Fourier Series

4+1 Types of Fourier Analysis

The (4+1) Fourier Analysis.

Analysis | **Synthesis** FT.



CTFS

$$c_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\Omega_0 t}\, dt \qquad x(t) = \sum_{k=-\infty}^{\infty} c_k\, e^{+jk\Omega_0 t}$$

CTFT.

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\Omega)\, e^{+j\Omega t}\, d\Omega$$

DTFS

$\omega_0 = \frac{2\pi}{N}$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \qquad x[n] = \sum_{k=0}^{N-1} c_k\, e^{+jk\frac{2\pi}{N}n}$$

DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{+j\omega n}\, d\omega$$

DFT

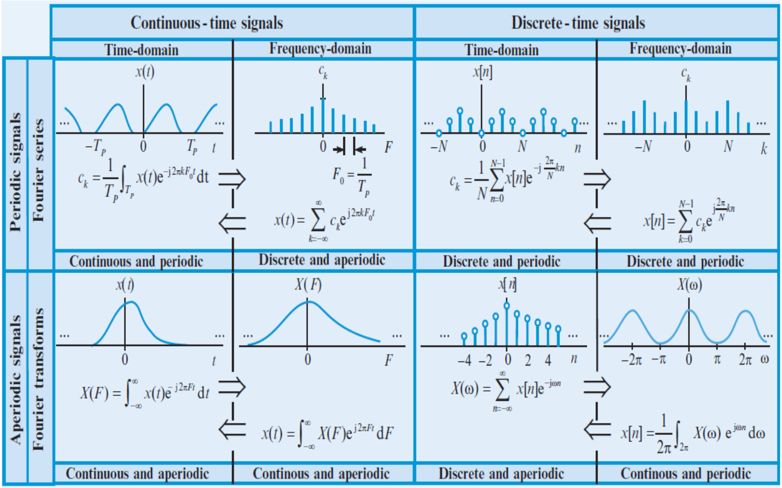
$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-jk\frac{2\pi}{N}n} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+jk\frac{2\pi}{N}n}$$

analyzing $n=0..(N-1)$ samples only!

$n=0..N-1$ only

The Fourier Representations

(taken from Manolakis and Ingle, "Applied DSP" Cambridge University Press)



In Manolakis, he uses c_k to represent the CTFS and DTFS coefficients, we use C_k and $X[k]$. In addition he represents CTFT as $X(F)$ instead of $X(j\Omega)$. Unfortunately representations has never been standardize!

Part III

CT Fourier Series

CT Fourier Series I

Fourier states that, for a given CT periodic signal $x(t)$ with

- period $T_0 \in \mathbb{R}$ seconds, equivalently its frequency $F_0 = 1/T_0$ (Hz), or fundamental angular frequency $\Omega_0 = \frac{2\pi}{T_0} = 2\pi F_0$ (rad/sec).
- that satisfies the Dirichlet condition,

it can be decomposed into an infinite sum of

- complex exponential waveforms at angular frequencies $k(2\pi F_0), \forall k \in \mathbb{Z}$,
- with appropriate complex weights $C_k = |C_k|e^{j\theta_k}$, i.e, magnitude $|C_k|$ and phase $\theta_k = \angle C_k$

The analysis equations is used to find C_k.
The synthesis equation is to reconstruct $x(t)$ from C_k.

Analysis and Synthesis Equations

There are various forms of Fourier Analysis and Synthesis equations:

- Two common analysis form: "Exponential" and "Trigonometric". Here, study only Exponential form.
- Three synthesis forms "Exponential", "Combine Trigo" and "Trigonometric".

The analysis and synthesis equations of CTFS are:

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j(k\Omega_0 t)}dt \qquad (2)$$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\Omega_0 t} \qquad (3)$$

where $k \in \mathbb{Z}$, $C_k = |C_k|e^{j\theta_k}$ is a complex number.

FS: Analysis and 3 Forms of Synthesis Equation

Type	Equation
CTFS Analysis	$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\Omega_0 t}\, dt = \lvert C_k\rvert\, e^{j\theta_k}$, where $k \in Z$, $T_o = Period, \Omega_0 = \frac{2\pi}{T_0} = 2\pi F_0$ (rad/sec)
Synthesis: Complex Exponential Form	$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\Omega_0\, t}$
Synthesis: Combine Trigonometric Form	$x(t) = C_0 + \sum_{k=1}^{\infty} 2\,\lvert C_k\rvert \cos(k\Omega_0 t + \theta_k)$
Synthesis: Trigonometric Form	$x(t) = A_0 + \sum_{k=1}^{\infty} A_k \cos(k\Omega_o t) + B_k \sin(k\Omega_o t)$ where, $2C_k = A_k - jB_k$, $C_0 = A_0$

Synthesis Equation - Complex Exponential Form to Combine Trigo Form

Fourier Series Representation
From complex exponential form $\Rightarrow$ Combine Trigo. form.

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\Omega_0 t} \quad ; \quad C_k = C_{-k}^*$$

$$C_k \text{ usually complex} = \underbrace{|C_k|}_{\text{magnitude}} e^{j\overbrace{\theta_k}^{\text{phase}}}$$

For a given "k"

$$C_{-k} e^{-jk\Omega_0 t} + C_k e^{+jk\Omega_0 t}$$

$$= |C_k| e^{-j\theta_k} e^{-jk\Omega_0 t} + |C_k| e^{+j\theta_k} e^{+jk\Omega_0 t}$$

$$= |C_k| e^{-j(k\Omega_0 t + \theta_k)} + |C_k| e^{+j(k\Omega_0 t + \theta_k)}$$

$$= |C_k| \left[e^{-j(k\Omega_0 t + \theta_k)} + e^{+j(k\Omega_0 t + \theta_k)} \right].$$

Remember: $e^{j\theta} = \cos\theta + j\sin\theta$
$+ \; e^{-j\theta} = \cos\theta - j\sin\theta$

$$= 2\cos\theta$$

$$C_{-k} e^{-jk\Omega_0 t} + C_k e^{+jk\Omega_0 t}$$
$$\therefore = |C_k| \, 2\cos(k\Omega_0 t + \theta_k).$$

$$\therefore \boxed{x(t) = \sum_{k=-\infty}^{\infty} C_k e^{+jk\Omega_0 t}} \quad \text{Complex Exponential Form}$$

$$= C_0 + \sum_{k=1}^{\infty} \left[C_k e^{+jk\Omega_0 t} + C_{-k} e^{-jk\Omega_0 t} \right]$$

$$\boxed{x(t) = C_0 + 2|C_k| \cos(k\Omega_0 t + \theta_k)}$$

combine Trigo Form.

where $C_k = |C_k| e^{j\theta_k}$ *.

Synthesis Equation - Trigo Form to Complex Exponential Form

Fourier Trigo Form → Complex Exponential Form

$$x(t) = A_0 + \sum_{k=1}^{\infty} \left(A_k \cos k\Omega_0 t + B_k \sin k\Omega_0 t\right)$$

DC term   k-th harmonic.   * Trigo Form

$e^{j\theta} = \cos\theta + j\sin\theta$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$\frac{1}{j} = \frac{1}{j}\cdot\frac{j}{j} = \frac{j}{j^2} = -j$ *

$$\therefore \cos\theta = \frac{e^{+j\theta} + e^{-j\theta}}{2}, \quad \sin\theta = \frac{e^{+j\theta} - e^{-j\theta}}{2j}$$

$$\therefore A_k \cos k\Omega_0 t = A_k \left[\frac{e^{+jk\Omega_0 t} + e^{-jk\Omega_0 t}}{2}\right]$$

$$B_k \sin k\Omega_0 t = B_k \left[\frac{e^{+jk\Omega_0 t} - e^{-jk\Omega_0 t}}{2j}\right]$$

Collect $e^{+jk\Omega_0 t}$: $\frac{A_k}{2} e^{+jk\Omega_0 t} + \frac{B_k}{2j} e^{+jk\Omega_0 t}$

$e^{-jk\Omega_0 t}$: $\frac{A_k}{2} e^{-jk\Omega_0 t} - \frac{B_k}{2j} e^{-jk\Omega_0 t}$

$\therefore e^{+jk\Omega_0 t}\left(\frac{A_k}{2} + \frac{B_k}{2j}\right)$ | $e^{-jk\Omega_0 t}\left(\frac{A_k}{2} - \frac{B_k}{2j}\right)$

$= e^{+jk\Omega_0 t}\left(\frac{A_k}{2} - \frac{jB_k}{2}\right)$ | $= e^{-jk\Omega_0 t}\left(\frac{A_k}{2} + \frac{jB_k}{2}\right)$

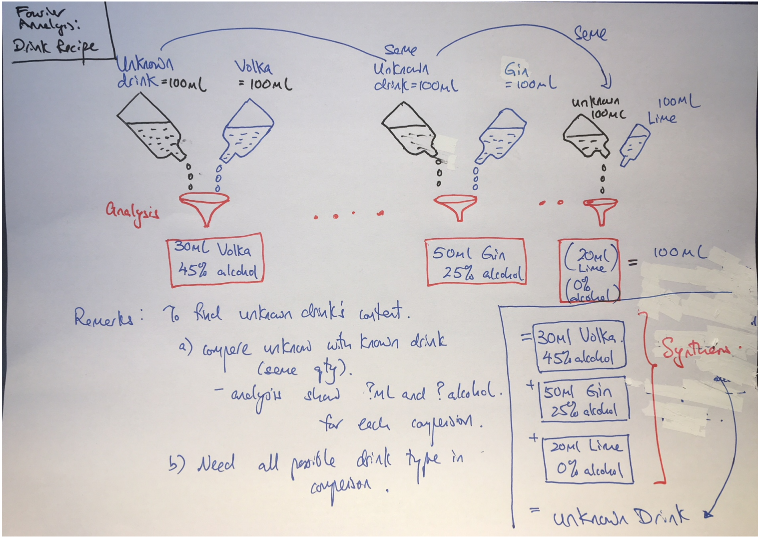
Compare to $\sum_{k=-\infty}^{\infty} C_k e^{+jk\Omega_0 t}$. } complex exponential form.

$$\therefore \boxed{\begin{aligned} C_0 &= A_0, \\ C_k &= \frac{A_k}{2} - j\frac{B_k}{2}, \\ C_{-k} &= \frac{A_k}{2} + j\frac{B_k}{2} \end{aligned}}$$

Complex Exponential Coeff to Trigo Coeff relationships.

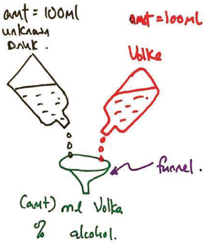
*

Drink Recipe: Explaining Fourier Analysis



Analysis - Corollary

Analysis Eq^n : unknown to 1 type of drink.



Analysis: Given unknown drink
and known (type) drink,
a) equal amount
b) find "mL" amt in unknown
c) find % alcohol in unknown.

Analysis Eq^n : Continuous Time.

CTFS:

amt 100mL

known drink type complex exp with $\Omega = k\Omega_0$

$$\underbrace{c_k}_{|c_k|e^{j\theta}} = \underbrace{\frac{1}{T_0}\int_{T_0}}_{\text{funnel operation}} \underbrace{x(t)}_{\text{unknown drink}} e^{-jk\Omega_0 t}\, dt$$

$|c_k|$: amt ; θ : % alcohol

; T_0 = period. $\Omega = \frac{2\pi}{T_0}$

CTFT:

$$X(j\Omega) = \underbrace{\int_{-\infty}^{\infty}}_{\text{amt = all } \infty} \underbrace{x(t)}_{\text{unknown drink}} \underbrace{e^{-j\Omega t}}_{\text{known drink exp with } \Omega \text{ freq.}}\, dt$$

funnel operation $\int_{-\infty}^{\infty} \cdot \, dt$

Note: the unknown drink is multiplied with known drink type for "funnel" operation. "mix" together.

1. Analysis Eq decomposes $\tilde{x}(t)$ into C_k. Each C_k is associated with a complex exponential at frequency $k\Omega_0$ (corollary - k^{th} Drink Type). $k\Omega_0$ is the k^{th} harmonics, where Ω_0 is the fundamental (angular) frequency.
2. C_k indicates that for $e^{jk\Omega_0 t}$ (k^{th} Drink Type), its
   - $|C_k|$ Magnitude (corollary - amount)
   - θ_k Phase (corollary - alcohol concentration)

Hence, $\tilde{x}(t)$ can be synthesized by:

$$\begin{aligned}\tilde{x}(t) &= \sum_{k=-\infty}^{\infty} C_k e^{j(k\Omega_0 t)} \\ &= \cdots + |C_{-1}|e^{j((-1)\Omega_0 t+\theta_{-1})} + |C_0|e^{j(0\Omega_0 t+\theta_0)} + \\ &\qquad |C_1|e^{j(1\Omega_0 t+\theta_1)} + \cdots \end{aligned} \tag{4}$$

where $k \in \mathbb{Z}$.

Visualizing Harmonics: $cos(k\Omega_0 t)$

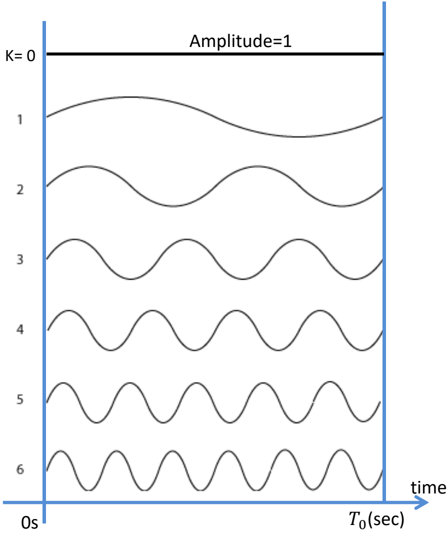
Harmonics of a cosine
Fundamental period = T_0 sec

Fundamental Frequency
1st Harmonic: $\Omega = 1\Omega_0 = \frac{2\pi}{T_0}$

2nd Harmonic: $2\Omega_0$

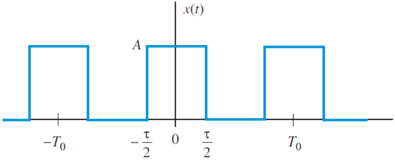
3rd Harmonic: $\Omega = 3\Omega_0$

6th Harmonic: $\Omega = 6\Omega_0$



Work Example: FS periodic rectangle

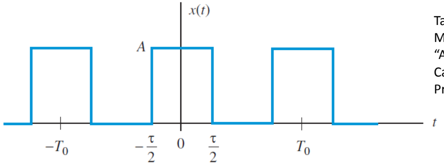
Given a periodic rectangle (even) CT signal with amplitude A, period T_0 and on-period τ, perform Fourier Analysis and hence sketch its magnitude and phase domain representation.



In other words,

1. As signal is CT periodic, we can use CTFS or CTFT for analysis. Its easier to use CTFS, therefore evaluate C_k (Fourier Series Coefficients)
2. Sketch magnitude spectrum $|C_k|$ vs frequencies $(k\Omega_0)$
3. Sketch phase spectrum $\angle C_k$ vs frequencies $(k\Omega_0)$

Ex 1:periodic rectangle I



Taken From Manolakis&Ingle "Applied DSP" Cambridge University Press

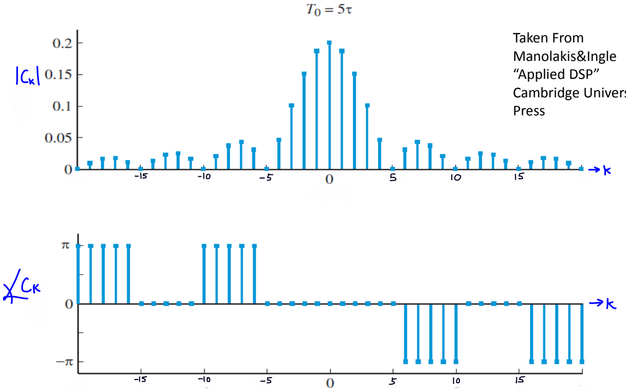
Example 4.1 Rectangular pulse train

Consider the periodic rectangular pulse train in Figure 4.8. The Fourier coefficients are given by

$$
\begin{aligned}
c_k &= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi kF_0 t} dt = \frac{A}{T_0}\left[\frac{e^{-j2\pi kF_0 t}}{-j2\pi kF_0}\right]_{-\tau/2}^{\tau/2} \\
&= \frac{A}{\pi F_0 k T_0}\frac{e^{j\pi kF_0\tau} - e^{-j\pi kF_0\tau}}{2j} \\
&= \frac{A\tau}{T_0}\frac{\sin \pi kF_0\tau}{\pi kF_0\tau}. \quad k = 0, \pm1, \pm2, \ldots
\end{aligned} \tag{4.30}
$$

The values of c_k are obtained by evaluating the function $(A\tau/T_0)\sin(\phi)/\phi$ at equidistant points $\phi = k(\pi F_0\tau)$. Since $\lim_{\phi\to 0}[\sin(\phi)/\phi] = 1$, we have $c_0 = A\tau/T_0$. The function $\sin(\phi)/\phi$ has zero crossings at multiples of π, that is, at $\phi = m\pi$, $m = 0, \pm1, \pm2, \ldots$ The zero crossings occur at $\phi = \pi F\tau = m\pi$ or $F = m/\tau$. The spacing $F = 1/\tau$ between the zero crossings is determined by the width τ of the pulse, whereas the spacing $F_0 = 1/T_0$ between the spectral lines is determined by the fundamental period T_0. ■

Ex 1:periodic rectangle II

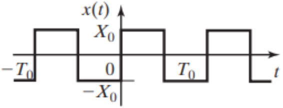
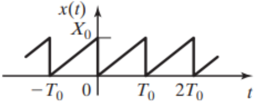
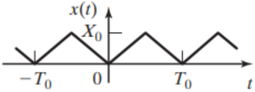
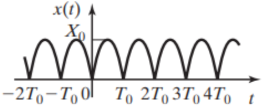


Taken From Manolakis&Ingle "Applied DSP" Cambridge University Press

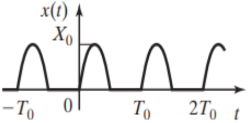
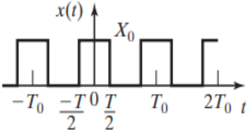
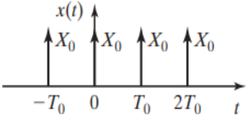
Magnitude and phase spectra of a rectangular pulse train with $A = 1$ and $T_0 = 5\tau$.

- Magnitude spectra are always positive. Hence, negative signs should be absorbed in the phase using the identity: $-A\cos\Omega t = \cos(\Omega t \pm \pi)$. It does not matter whether we take $+\pi$ or $-\pi$ because $\cos(-\pi) = \cos\pi$. However, we use both $+\pi$ and $-\pi$ to bring out the odd symmetry of the phase.

Common FS of periodic CT signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
1. Square wave		0	$-j\,\frac{2X_0}{\pi k}$	$C_k = 0$, k even
2. Sawtooth		$\frac{X_0}{2}$	$j\,\frac{X_0}{2\pi k}$	
3. Triangular wave		$\frac{X_0}{2}$	$\frac{-2X_0}{(\pi k)^2}$	$C_k = 0$, k even
4. Full-wave rectified		$\frac{2X_0}{\pi}$	$\frac{-2X_0}{\pi(4k^2-1)}$	

Common FS of periodic CT signals

Name	Waveform	C_0	$C_k, k \neq 0$	Comments
5. Half-wave rectified		$\frac{X_0}{\pi}$	$\frac{-X_0}{\pi(k^2-1)}$	$C_k = 0$, k odd, except $C_1 = -j\frac{X_0}{4}$ and $C_{-1} = j\frac{X_0}{4}$
6. Rectangular wave		$\frac{TX_0}{T_0}$	$\frac{TX_0}{T_0}\operatorname{sinc}\frac{Tk\omega_0}{2}$	$\frac{Tk\omega_0}{2} = \frac{\pi Tk}{T_0}$
7. Impulse train		$\frac{X_0}{T_0}$	$\frac{X_0}{T_0}$	

Existence Fourier Series I

Sufficient conditions for CTFS of signal to exist:

- Dirichlet conditions:
  1. $x(t)$ is bounded.
  2. $x(t)$ has a finite number of local maxima and minima in one period.
  3. $x(t)$ has finite number of discontinuity in one period.

  If signal satisfies Dirichlet conditions,
  - pointwise convergence between the FS representation and the given signal is guaranteed at all continuous points of the signal.
  - At discontinuity, the FS representation converges to the mid point.
- If Dirichlet conditions not satisfied, but signal is square integrable, i.e, $\frac{1}{T}\int |x(t)|^2 dt < \infty$,
  - then the mean square error (MSE) between the Fourier Series representation to $x(t)$ tends to zero.
  - Very useful - admits a large class of signals.

Part IV

CT Fourier Transform

CT Fourier Transform I

CTFT can analyse both aperiodic and not-periodic CT signal.

- CTFT Analysis and Synthesis equations:

$$X(j\Omega) = \int_{t=-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{5}$$

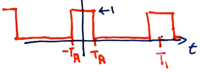
$$x(t) = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty} X(j\Omega)e^{+j\Omega t}d\Omega \tag{6}$$

- Ω is continuous between $-\infty$ to ∞.
- When CTFT and CTFS are used to analyse the same periodic signal, the analysis are slightly different as their synthesis equations are different.
- An overview is given in next page

Relationship - CTFS and CTFT

Relationships of CTFS and CTFT.

$\tilde{x}(t)$ periodic T_1

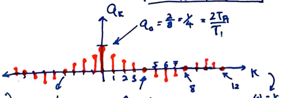


CTFS

$a_0 = \frac{2T_A}{T_1}$

$a_k = \frac{1}{k\pi}\sin(kT_A\Omega_1)$

$|Ex: T_A = 1,\ T_1 = 8|$

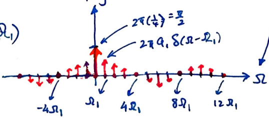


$a_4 = \frac{1}{4\pi}\sin(4\cdot 1\cdot \frac{2\pi}{8})$
$= \frac{1}{4\pi}\sin(\pi) = 0$

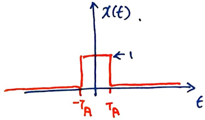
$\omega = k\Omega_1$

CTFT

$X(j\Omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_1)$

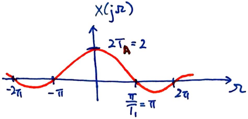


single period of above



$\xrightarrow{\text{CTFT}}$ $X(j0) = 2T_A$

$X(j\Omega) = \frac{2}{\Omega}\sin(\Omega T_A)$



relationship to a_k

$a_k = \frac{1}{T_1} X(j\Omega)\Big|_{\Omega = k\Omega_1}$

$T_1 a_k = X(j\Omega) \longrightarrow$ is the envelope of $T_1 a_k$!

Relating CTFT/CTFS periodic sinusoids I

Lets evaluate $X(j\Omega)$ the $CTFT\{Acos(\pi/4t + \pi/6)\}$.

1. Using the CTFT Synthesis equation and sifting property of $\delta(\Omega)$.

$$
\begin{aligned}
x(t) &= Acos(\pi/4t + \pi/6) \\
&= \frac{A}{2}\left(e^{j(\pi/4t+\pi/6)} + e^{-j(\pi/4t+\pi/6)}\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[2\pi\left(\frac{A}{2}\right)\delta(\Omega - \pi/4)e^{j\pi/6} + \right. \\
&\qquad \left. 2\pi\left(\frac{A}{2}\right)\delta(\Omega + \pi/4)e^{-j\pi/6}\right]e^{j\Omega t}d\Omega
\end{aligned}
$$

Relating CTFT/CTFS periodic sinusoids II

2. Relationship between CTFS representation C_k and CTFT $X(j\Omega)$ of a periodic sinusoid signal with fundamental frequency Ω_1 are :

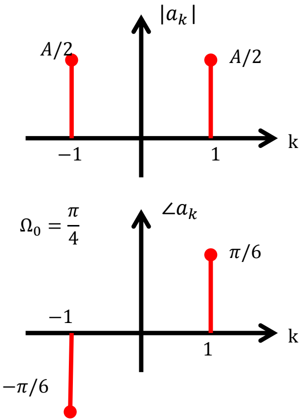
$$C_k = \frac{1}{2\pi} X(j\Omega) \Big|_{\Omega = k\Omega_1} \tag{7}$$

a) the CTFS's C_k coefficients along the x-axis are at discrete values $k \in \mathbb{Z}$.
   - k represents frequency $k\Omega_1$.
   - Relating to CTFT, the C_k coefficients are scaled by $\frac{1}{2\pi}$ of the CTFT values $X(j\Omega)$ at $\Omega = k\Omega_1$.

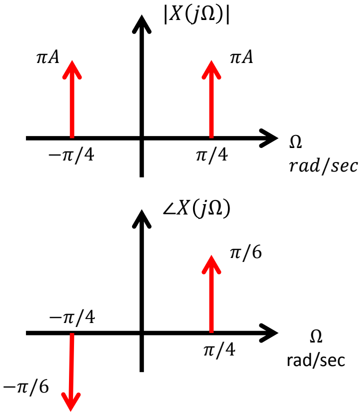
b) In CTFT $X(j\Omega)$, x-axis represents angular frequencies and $\Omega \in R$.
   - Relating to CTFS, $X(j\Omega)$ are $2\pi C_k$ weighted δ at frequencies $k\Omega_1$.

Relating CTFT/CTFS periodic sinusoids III

CTFTS representation of $A(\cos\frac{\pi}{4}t + \frac{\pi}{6})$



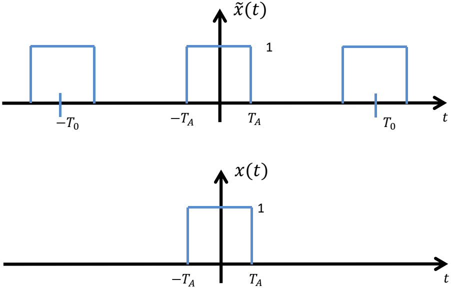
CTFT representation of $A(\cos\frac{\pi}{4}t + \frac{\pi}{6})$



Comparing CTFT of single period of $\tilde{x}(t)$ to CTFS $\tilde{x}(t)$ I

There also exist relationship between

- the CTFS of a periodic signal $\tilde{x}(t)$ to CTFT of a single period $x(t)$.
- Fundamental angular frequency is $\Omega_0 = \frac{2\pi}{T_0} = 2\pi F_0$.
- See Overview Example in pg 37



Comparing CTFT of single period of $\tilde{x}(t)$ to CTFS $\tilde{x}(t)$ II

1. Aperiodic $x(t)$ can be thought of as a periodic signal $\tilde{x}(t)$ with infinite period T_0.
   - Using this idea, the Fourier representation for aperiodic signals using the Fourier series is developed.
   - Example: Begin with periodic square wave and observe its coefficients spacing between C_k in terms of angular frequency as period $T_0 \rightarrow \infty$.
2. The CTFS analysis for the periodic square wave $\tilde{x}(t)$ illustrated in pg 41 is

$$\begin{aligned} C_k &= \frac{1}{T_0}\int_{-T_A/2}^{T_A/2} \tilde{x}(t)e^{-jk\Omega_0 t}dt && (8) \\ &= \frac{2}{k\Omega_0 T_0} sin(k\Omega_0 T_A) && (9) \end{aligned}$$

Comparing CTFT of single period of $\tilde{x}(t)$ to CTFS $\tilde{x}(t)$ III

3. The CTFT analysis of $x(t)$ illustrated in pg 41 is

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt \tag{10}$$

4. To find the relationship between CTFT of a single period of $\tilde{x}(t)$ to the CTFS of $\tilde{x}(t)$.
   a) Compare Eq 10 to Eq 8.
   b) Obvoiusly, C_k is related to $X(j\Omega)$ by

$$C_k = \frac{1}{T_0} X(j\Omega)\Big|_{\Omega = k\Omega_0} \tag{11}$$

   i.e, C_k are scaled versions (by $\frac{1}{T_0}$) and equally spaced samples of $X(j\Omega)$ at $\Omega = k\Omega_0$.

Comparing CTFT of single period of $\tilde{x}(t)$ to CTFS $\tilde{x}(t)$ IV

5. Alternatively, rearranging T_0 to LHS, we have

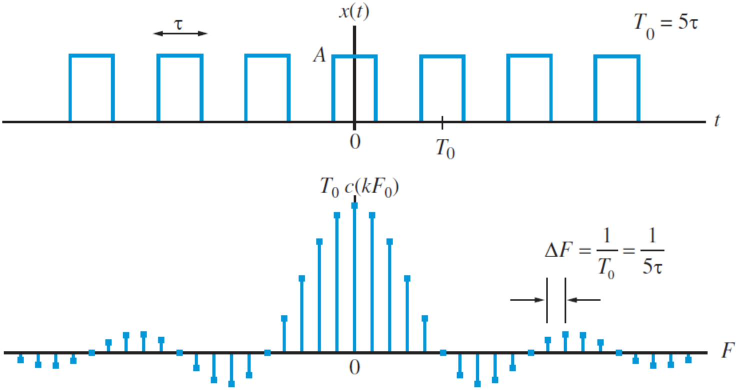
$$T_0 C_k = X(j\Omega)|_{\Omega=k\Omega_0} \tag{12}$$

Interpretation: $X(j\Omega)$ is the envelope of $T_0 C_k$.

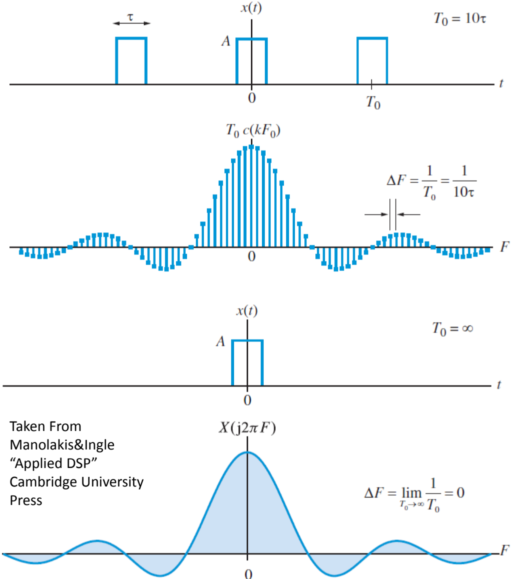
6. Following figures show what happens when period T_0 increases with T_A fixed -
   a) The shape of $X(j\Omega) = T_0 C_k$ remains the same since $x(t)$ does not change.
   b) However, when T_0 increases, the spectral lines C_k becomes closer, e.g, between $k\Omega_0$ and $(k+1)\Omega_0$, since Ω_0 becomes smaller.
   c) When $T_0 \to \infty$, the spectral lines converges to a continum, the continuous spectrum $X(j\Omega)$ for aperiodic signal.

Comparing CTFT of single period of $\tilde{x}(t)$ to CTFS $\tilde{x}(t)$ V

CTFS transform pair of a periodic square wave



Comparing CTFT of single period of $\tilde{x}(t)$ to CTFS $\tilde{x}(t)$ VI



Taken From Manolakis&Ingle “Applied DSP” Cambridge University Press

Part V

Common FS/FT Pairs,Properties and Parseval

Other materials you should know I

1. Various FT pair for CTFT and CTFS:
   - impulse, impulse train, complex exponetial, cosine, symmetric periodic and aperiodic square wave, sinc function.
   - See 1C_Handout_CTFS_CTFT_Examples.pdf for worked examples of CTFS and CTFT Pairs.
2. Properties of CTFT
   - See 1C_Handout_CTFT_Properties_Examples.pdf for worked examples some CTFT Properties.

Common CTFT and CTFS pairs I

Note: this table uses small ω to represent angular frequencies of CTFT

Basic Continuous-Time Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos\omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin\omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1$, $a_k = 0$, $k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \|t\| < T_1 \\ 0, & T_1 < \|t\| \leq \frac{T}{2} \end{cases}$ and $x(t+T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k

Common CTFT and CTFS pairs II

Note: this table uses small ω to represent angular frequencies of CTFT

Basic Continuous-Time Fourier Transform Pairs

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t-t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{a+j\omega}$	—
$te^{-at}u(t), \Re e\{a\} > 0$	$\frac{1}{(a+j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t),$ $\Re e\{a\} > 0$	$\frac{1}{(a+j\omega)^n}$	—

From : http://web.mit.edu/shou/www/6.003/tutorial_tables.pdf

Common CTFT properties I

Note: this table uses small ω to represent angular frequencies of CTFT

Properties of the Continuous-Time Fourier Transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Property	Aperiodic Signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
Time-shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency-shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time-Reversal	$x(-t)$	$X(-j\omega)$

From : http://web.mit.edu/shou/www/6.003/tutorial_tables.pdf

Common CTFT properties II

Properties of the Continuous-Time Fourier Transform

Property	Aperiodic Signal	Fourier transform
Time- and Frequency-Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Note: this table uses small ω to represent angular frequencies of CTFT

Parseval's Relatioship I

Parseval's relationship states

- that the energy or power of a time domain signal is equal to the energy or power of its frequency domain representation.

Proof: For CT aperiodic signal, its energy is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{13}$$

Since $|x(t)| = x(t)x^*(t)$, where $x^*(t)$ is $x(t)$ conjugate if $x(t)$ is complex. Since

$$x^*(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} X^*(j\Omega) e^{-j\Omega t} d\Omega \tag{14}$$

Parseval's Relatioship II

Substitute above into Eq 13 we have

$$E_x = \int_{-\infty}^{\infty} x(t) \int_{-\infty}^{\infty} \frac{1}{2\pi} X^*(j\Omega) e^{-j\Omega t} d\Omega dt \qquad (15)$$

and interchange the order of integration we have

$$\begin{aligned} E_x &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\Omega) \left\{ \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \right\} d\Omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(j\Omega) X(j\Omega) d\Omega \\ \int_{-\infty}^{\infty} |x(t)|^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 \, d\Omega \end{aligned} \qquad (16)$$

Parseval's Relatioship III

- Showed: energy of the signal is equals (normalized by 2π) to its frequency representation.
- $|X(j\Omega)|^2$ is also known as energy spectrum.

Table of Parseval Theorem

Representation	*Parseval Relation*
FT	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(j\omega)\rvert^2 \, d\omega$
FS	$\frac{1}{T}\int_{\langle T\rangle} \lvert x(t)\rvert^2 \, dt = \sum_{k=-\infty}^{\infty} \lvert X[k]\rvert^2$
DTFT	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2 = \frac{1}{2\pi}\int_{\langle 2\pi\rangle} \lvert X(e^{j\Omega})\rvert^2 \, d\Omega$
DTFS	$\frac{1}{N}\sum_{n=\langle N\rangle} \lvert x[n]\rvert^2 = \sum_{k=\langle N\rangle} \lvert X[k]\rvert^2$

Topics not covered: Convergence and the Gibbs Phonemenon

We did not discuss these important topics:

1. How do we know that Fourier Series converges?
2. and what happens at discontinuities!