Logic Synthesis & Verification, Fall 2023

National Taiwan University

Reference Solution to Problem Set 4

Due on 2023/12/08 23:59.

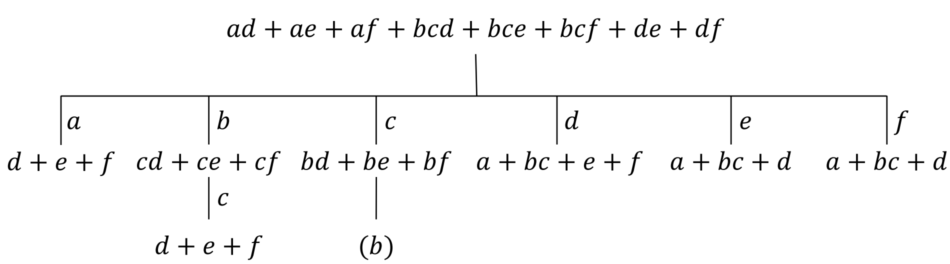
1 [Weak Division]

We first note that weak division is a kind of algebraic division. For an algebraic division $F \mathbin{/\!\!/} G$, once F, G, and H are given, the remainder R is uniquely decided by $R = F \setminus GH$. Therefore, it is sufficient to prove that the quotient H of weak division is unique.

Suppose H_1 and H_2 are both valid quotients obtained from a weak division F/G, i.e, $F = GH_1 + R_1 = GH_2 + R_2$ making R_1 and R_2 have as few cubes as possible. Since weak division is a kind of algebraic division, $GH_1 \subseteq F$ and $GH_2 \subseteq F$ must hold. Therefore, $G(H_1 \cup H_2) = (GH_1 \cup GH_2) \subseteq F$ also holds. Let $H_3 = H_1 \cup H_2$ and $R_3 = F \setminus GH_3$. Then $F = GH_3 + R_3$ is also a valid algebraic division. If $H_1 \neq H_2$, then $H_1 \subsetneq H_3$ and thereby $R_3 \subsetneq R_1$, so R_1 does not have as few cubes as possible, which leads to a contradiction. Therefore, we have $H_1 = H_2$ and thereby $R_1 = R_2$, indicating that the quotient H and the remainder R of weak division F/G are unique.

2 [Kernelling and Factoring]

(a) As shown below.



Kernel	**Co-kernel**
$d + e + f$	$\{a, bc\}$
$a + bc + e + f$	d
$a + bc + d$	$\{e, f\}$
F	1

Table 1

	ad	ae	af	bcd	bce	bcf	de	df
ad	-	$d+e$	$d+f$	$a+bc$	$ad+bce$	$ad+bcf$	$a+e$	$a+f$
ae		-	$e+f$	$ae+bcd$	$a+bc$	$ae+bcf$	$a+d$	$ae+df$
af			-	$af+bcd$	$af+bce$	$a+bc$	$af+de$	$a+d$
bcd				-	$d+e$	$d+f$	$bc+e$	$bc+f$
bce					-	$e+f$	$bc+d$	$bce+df$
bcf						-	$bcf+de$	$bc+d$
de							-	$e+f$
de								-

(b) Two-cube divisors contain:
(1) Directly obtained from Table 1: $\{d+e, d+f, e+f, a+bc, ae+bcd, af+bcd, ad+bcd, af+bce, ad+bcf, ae+bcf, a+e, a+d, af+de, bc+e, bc+d, bcf+de, a+f, ae+df, bc+f, bce+df\}$
(2) Complement of (1): $\{a'b'+a'c', b'e'+c'e', b'd'+c'd', b'f'+c'f'\}$
(3) Complement of (4): $\{a'+d', a'+e', a'+f', b'+c', b'+d', c'+d', b'+e', c'+e', b'+f', c'+f', d'+e', d'+f'\}$
Two-literal cube divisors contain:
(4) Directly obtained from F: $\{ad, ae, af, bc, bd, cd, be, ce, bf, cf, de, df\}$
(2) Complement of (1): $\{d'e', d'f', e'f', a'e', a'd', a'f'\}$
None of the two-cube divisors are kernels because all kernels listed in (a) have more than two cubes.

(c)
Step 1. Do GFACTOR(F) to make $F = D_0Q_0 + R_0$.
We select $D_0 = a+bc+e+f$. Then $Q_0 = F/D_0 = d$. Since $|Q_0| = 1$, we shall return LF(F, Q_0).
Step 2. Do LF(F, Q_0) to make $F = D_1Q_1 + R_1$.
We select $L_1 = d$ and compute $(Q_1, R_1) = F/L_1 = (a+bc+e+f,\ ae+af+bce+bcf)$. We note that Q_1 is cube-free. Therefore, we recursively compute GFACTOR(Q_1) and GFACTOR(R_1). We note that Q_1 has no more non-trivial divisors, so only GFACTOR(R_1) is required. Then we shall return $F = L_1Q_1 +$ GFACTOR(R_1).
Step 3. Do GFACTOR(R_1) to make $R_1 = D_2Q_2 + R_2$.
We select $D_2 = e+f$. Then $Q_2 = R_1/D_2 = a+bc$. Since $|Q_2| \neq 0$ and Q_2 is already cube-free, we overwrite $(D_2, R_2) = R_1/Q_2 = (e+f, 0)$. Then we recursively compute GFACTOR(Q_2), GFACTOR(D_2), and GFACTOR(R_2). We note that Q_2, D_2, and R_2 have no more non-trivial divisors. Therefore, we return $R_1 = D_2Q_2 + R_2$.
Therefore,

$$\begin{aligned} F &= \mathrm{GFACTOR}(F) \\ &= LF(F, Q_0) \\ &= L_1Q_1 + \mathrm{GFACTOR}(R_1) \\ &= d(a+bc+e+f) + (e+f)(a+bc). \end{aligned}$$

(d) Step 1. Do GFACTOR(F) to make $F = D_0Q_0 + R_0$.
We select $D_0 = bc + d$. Then $Q_0 = F/D_0 = e + f$. Since $|Q_0| \neq 0$ and Q_0 is already cube-free, we overwrite $(D_0, R_0) = F/Q_0 = (a+bc+d,\ ad+bcd)$. Then we recursively compute GFACTOR(Q_0), GFACTOR(D_0), and GFACTOR(R_0). We note that Q_0 and D_0 have no more non-trivial divisors, so only GFACTOR(R_0) is required.

Step 2. Do GFACTOR(R_0) to make $R_0 = D_1Q_1 + R_1$.
We select $D_1 = a + bc$. Then $Q_1 = F/D_1 = d$. Since $|Q_1| = 1$, we shall return LF(R_0, Q_1).

Step 3. Do LF(R_0, Q_1) to make $R_0 = D_2Q_2 + R_2$.
We select $L_2 = d$ and compute $(Q_2, R_2) = F/L_2 = (a + bc,\ 0)$. We note that Q_2 is cube-free. Therefore, we recursively compute GFACTOR(Q_2) and GFACTOR(R_2). We note that Q_2 and R_2 have no more non-trivial divisors Therefore, we return $R_0 = L_2Q_2 + R_2$.

Therefore,

$$
\begin{aligned}
F &= \mathrm{GFACTOR}(F) \\
&= Q_0D_0 + \mathrm{GFACTOR}(R_0) \\
&= Q_0D_0 + \mathrm{LF}(R_0) \\
&= (e + f)(a + bc + d) + d(a + bc).
\end{aligned}
$$

3 [Extraction and Rectangle Covering]

The level-0 kernels and co-kernels of F are as follows.

Kernel	**Co-kernel**
$bc + ef$	$\{a, d\}$
$a + d$	$\{bc, ef\}$

The level-0 kernels and co-kernels of G are as follows.

Kernel	**Co-kernel**
$d + e$	ab
$bd + e$	ac
$bc + ef$	ad
$a + c + df$	ae
$ac + f$	bd
$ab + f$	cd
$ae + b + c$	df

Table 2

		a	b	c	d	e	f	ab	ac	ae	bc	bd	cd	df	ef
F	a										$\underline{abc}$				$\underline{aef}$
F	d										$\underline{bcd}$				$\underline{def}$
F	bc	abc			bcd										
F	ef	aef			def										
G	ab					abe							$abcd$		
G	ac					ace						$abcd$			
G	ad										$\underline{abcd}$				$\underline{adef}$
G	ae	abe	ace								$abcd$			$adef$	
G	bd						bdf		$abcd$						
G	cd						cdf	$abcd$							
G	df	bdf	cdf							$adef$					

The co-kernel cube matrix is shown in Table 2, so the simplified F and G are

$$\begin{aligned} h &= bc + ef \\ F &= ah + dh + ce + be \\ G &= adh + abe + ace + bdf + cdf \end{aligned}$$

4 [Functional Dependency]

No, f_1 cannot be re-expressed with a function $h(y_2, y_3)$ for variables y_2 and y_3 being the output variables of f_2 and f_3. The reason comes from the following two facts.

(1) When $(a, b, c, d, e) = (0, 0, 1, 1, 0)$, $f_1 = 1$, $f_2 = 0$, and $f_3 = 1$.
(2) When $(a, b, c, d, e) = (0, 1, 0, 0, 0)$, $f_1 = 0$, $f_2 = 0$, and $f_3 = 1$.

Therefore, given $(y_2, y_3) = (0, 1)$, we cannot decide whether f_1 is 0 or 1. In other words, the value of $h(0, 1)$ cannot be decided, so h cannot exist.

5 [SDC and ODC]

(a)

$$\begin{aligned} \text{SDC} &= (y_1 \oplus f_1) \vee (y_2 \oplus f_2) \vee (y_3 \oplus f_3) \vee (y_4 \oplus f_4) \vee (z_1 \oplus f_5) \vee (z_2 \oplus f_6) \\ &= (y_1 \oplus (x_1 \vee \neg x_2)) \vee (y_2 \oplus \neg x_2 x_3) \vee (y_3 \oplus \neg x_3 \neg x_4) \\ &\quad \vee (y_4 \oplus (\neg y_1 \neg y_2 \vee y_2 \neg y_3 \vee y_1 \neg y_3)) \vee (z_1 \oplus (y_1 \vee y_4)) \vee (z_2 \oplus (y_3 y_4)) \end{aligned}$$

(b) $\text{SDC}_4 = (y_1 \oplus (x_1 \vee \neg x_2)) \vee (y_2 \oplus \neg x_2 x_3) \vee (y_3 \oplus \neg x_3 \neg x_4)$. To make it depend on y_1, y_2, y_3, the resulting formula is

$$\forall x_1, x_2, x_3, x_4.(\text{SDC}_4) = \neg y_1 y_2 \vee y_2 y_3.$$

(c)

$$\begin{aligned} ODC_{41} &= \neg \frac{\partial z_1}{\partial y_4} = \neg(x_1 \vee \neg x_2 \oplus 1) = x_1 \vee \neg x_2 \\ ODC_{42} &= \neg \frac{\partial z_2}{\partial y_4} = \neg(0 \oplus \neg x_3 \neg x_4) = x_3 \vee x_4 \\ ODC_4 &= ODC_{41} \wedge ODC_{42} = (x_1 \vee \neg x_2) \wedge (x_3 \vee x_4) \end{aligned}$$

6 [Don't Cares in Local Variables]

(a) (5%) Compute the don't cares D_4 of Node 4 in terms of its local input variables y_1, y_2, and y_3. (Note that in general the computation of ODC may be affected by XDC especially when there exist different XDCs for different primary outputs.)

$$\begin{aligned} DC_4 &= (ODC_{41} \vee XDC_1) \wedge (ODC_{42} \vee XDC_2) \\ &= (x_1 \vee \neg x_2 \vee \neg x_1 \neg x_2 \neg x_3 \neg x_4) \wedge (x_3 \vee x_4 \vee x_1 x_2 \neg x_3 x_4) \\ &= (x_1 \vee \neg x_2) \wedge (x_3 \vee x_4) \\ D_4 &= \neg(\text{IMG}(\neg DC_4)) \\ &= \forall x_1, x_2, x_3, x_4.[(y_1 \oplus (x_1 \vee \neg x_2)) \vee (y_2 \oplus \neg x_2 x_3) \vee (y_3 \oplus \neg x_3 \neg x_4) \vee DC_4] \\ &= y_1 \neg y_3 \vee y_2 \end{aligned}$$

(b) We can use K-map to minimize f_4 with don't cares D_4, as shown below. The best implementable function for Node 4 is $f_4 = \neg y_1$.

$y_3 \setminus y_1 y_2$	00	01	11	10
0	1	x	x	x
1	1	x	x	0

7 [Complete Flexibility]

(a) (5%)

$$\begin{aligned}
f_5(X) &= f_1 \vee f_4 \\
&= f_1 \vee \neg f_1 \neg f_2 \vee f_2 \neg f_3 \vee f_1 \neg f_3 \\
&= (x_1 \vee \neg x_2) \vee (\neg x_1 x_2)(x_2 \vee \neg x_3) \vee (\neg x_2 x_3)(x_3 \vee x_4) \\
&= x_1 \vee \neg x_2 \vee \neg x_1 x_2 \\
&= 1 \\
f_6(X) &= f_3 f_4 \\
&= f_3(\neg f_1 \neg f_2 \vee f_2 \neg f_3 \vee f_1 \neg f_3) \\
&= f_3(\neg f_1 \neg f_2) \\
&= (\neg x_3 \neg x_4)(\neg x_1 x_2)(x_2 \vee \neg x_3) \\
&= \neg x_1 x_2 \neg x_3 \neg x_4 \\
S(X, Z) &= (\neg x_1 \neg x_2 \neg x_3 x_4 \vee (z_1 \Leftrightarrow f_5(X)))(x_1 x_2 \neg x_3 x_4 \vee (z_2 \Leftrightarrow f_6(X))) \\
&= (\neg x_1 \neg x_2 \neg x_3 x_4 \vee z_1)(x_1 x_2 \neg x_3 x_4 \vee (z_2 \Leftrightarrow \neg x_1 x_2 \neg x_3 \neg x_4))
\end{aligned}$$

(b)

$$I(X, y_4, Z) = (z_1 \Leftrightarrow (x_1 \vee \neg x_2 \vee y_4))(z_2 \Leftrightarrow (\neg x_3 \neg x_4 y_4))$$

(c)

$$E(X, Y) = (y_1 \Leftrightarrow (x_1 \vee \neg x_2))(y_2 \Leftrightarrow (\neg x_2 x_3))(y_3 \Leftrightarrow (\neg x_3 \neg x_4))$$

(d)

$$\begin{aligned}
CF(Y, y_4) &= \forall X, Z.[\neg(E(X, Y) \wedge I(X, y_4, Z) \wedge \neg S(X, Z))] \\
&= y_2 \vee y_1 \neg y_3 \vee y_1 \neg y_4 \vee \neg y_1 y_4 \vee \neg y_3 y_4
\end{aligned}$$

(e) Yes, D_4 is subsumed by CF_4 since $(D_4 \rightarrow CF_4)$ equals 1.

8 [Complete Flexibility for Multi-Nodes]

First, we have

$$\begin{aligned}
y_1 &= \neg x_1 x_2 \vee \neg x_1 x_3 \\
y_2 &= \neg x_1 \neg x_2 \neg x_3 \vee x_2 x_3 \\
z_1 &= \neg(y_1 \vee y_2) \\
&= x_1 \neg x_2 \vee x_1 \neg x_3 \\
z_2 &= \neg(y_1 \wedge y_2) \\
&= x_1 \vee \neg x_2 \vee \neg x_3 \\
S(X, Z) &= (z_1 \Leftrightarrow (x_1 \neg x_2 \vee x_1 \neg x_3))(z_2 \Leftrightarrow (x_1 \vee \neg x_2 \vee \neg x_3)) \\
I(X, y_1, y_2, Z) &= (z_1 \Leftrightarrow \neg y_1 \neg y_2)(z_2 \Leftrightarrow (\neg y_1 \vee \neg y_2))
\end{aligned}$$

Therefore,

$$
\begin{aligned}
R(X, y_1, y_2) &= \forall Z.[I(X, y_1, y_2, Z) \rightarrow S(X, Z)] \\
&= \neg x_1 \neg x_2 \neg x_3 y_1 \neg y_2 \vee \neg x_1 \neg x_2 \neg x_3 \neg y_1 y_2 \vee \neg x_1 \neg x_2 x_3 y_1 \neg y_2 \\
&\vee \neg x_1 \neg x_2 x_3 \neg y_1 y_2 \quad \vee \neg x_1 x_2 \neg x_3 \neg y_1 y_2 \quad \vee \neg x_1 x_2 \neg x_3 y_1 \neg y_2 \\
&\vee \neg x_1 x_2 x_3 y_1 y_2 \qquad \vee x_1 \neg x_2 \neg x_3 \neg y_1 \neg y_2 \vee x_1 \neg x_2 x_3 \neg y_1 \neg y_2 \\
&\vee x_1 x_2 \neg x_3 \neg y_1 \neg y_2 \quad \vee x_1 x_2 x_3 \neg y_1 y_2 \qquad \vee x_1 x_2 x_3 y_1 \neg y_2
\end{aligned}
$$

We note that X are PIs of this circuit. Therefore, $CF(X, y_1, y_2)$ exactly equals $R(X, y_1, y_2)$ (see the proof below).

Then we can use K-map to minimize the circuit, as shown below, where a, b, c, d can be either 0 or 1. For example, $\neg x_1 \neg x_2 \neg x_3 y_1 \neg y_2$ and $\neg x_1 \neg x_2 \neg x_3 \neg y_1 y_2$ are both in $CF(X, y_1, y_2)$, so when $(x_1, x_2, x_3) = (0, 0, 0)$, the value of (y_1, y_2) can be either $(0, 1)$ or $(1, 0)$. Therefore, the $(0, 0, 0)$ entries of y_1 and y_2 are don't cares, but when assigning don't-care values to these two entries, y_1 and y_2 should be assigned with different values, and thereby we can assume $y_1 = a$ and $y_2 = \neg a$.

y_1:

$x_3 \backslash x_1 x_2$	00	01	11	10
0	a	b	0	0
1	c	1	d	0

y_2:

$x_3 \backslash x_1 x_2$	00	01	11	10
0	$\neg a$	$\neg b$	0	0
1	$\neg c$	1	$\neg d$	0

A best choice is $(a, b, c, d) = (1, 1, 1, 0)$, and the resulting simplified functions are $y_1 = \neg x_1$ and $y_2 = x_2 x_3$.

Proof of $\boldsymbol{CF(X, y_1, y_2) = R(X, y_1, y_2)}$

We can imagine that the PIs of the network are $X_p = \{x_{1p}, x_{2p}, x_{3p}\}$, and X is connected to X_p through buffers. Therefore, $E(X, X_p) = (X \Leftrightarrow X_p)$, and $R(X_p, y1, y2) = (y_1 \Leftrightarrow (x_{1p}{}' x_{2p} + x_{1p}{}' x_{3p}))(y_2 \Leftrightarrow (x_{1p}{}' x_{2p}{}' x_{3p}{}' + x_{2p}{}' x_{3p}{}'))$. Then

$$
\begin{aligned}
CF(X, y_1, y_2) &= \forall X_p.[E(X, X_p) \rightarrow R(X_p, y_1, y_2)] \\
&= [(X \Leftrightarrow 000) \rightarrow R(000, y_1, y_2)][(X \Leftrightarrow 001) \rightarrow R(001, y_1, y_2)] \\
&\quad [(X \Leftrightarrow 010) \rightarrow R(010, y_1, y_2)][(X \Leftrightarrow 011) \rightarrow R(011, y_1, y_2)] \\
&\quad [(X \Leftrightarrow 100) \rightarrow R(100, y_1, y_2)][(X \Leftrightarrow 101) \rightarrow R(101, y_1, y_2)] \\
&\quad [(X \Leftrightarrow 110) \rightarrow R(110, y_1, y_2)][(X \Leftrightarrow 111) \rightarrow R(111, y_1, y_2)].
\end{aligned}
$$

We can find that whatever X is, we should replace X_p in $R(X_p, y_1, y_2)$ with the value of X. Therefore, $CF(X, y_1, y_2)$ exactly equals $R(X, y_1, y_2)$.