Logic Synthesis & Verification, Fall 2023

National Taiwan University

Problem Set 1

Due on 2023/09/28 23:59 on NTU Cool.

1 [Characteristic Function]

current state	input	next state	value
0 0	0	0 0	1
0 0	0	0 1	1
0 0	0	1 0	0
0 0	0	1 1	0
0 0	1	0 0	0
0 0	1	0 1	0
0 0	1	1 0	1
0 0	1	1 1	0
0 1	0	0 0	0
0 1	0	0 1	1
0 1	0	1 0	0
0 1	0	1 1	0
0 1	1	0 0	1
0 1	1	0 1	0
0 1	1	1 0	0
0 1	1	1 1	0

current state	input	next state	value
1 0	0	0 0	0
1 0	0	0 1	0
1 0	0	1 0	0
1 0	0	1 1	1
1 0	1	0 0	0
1 0	1	0 1	1
1 0	1	1 0	0
1 0	1	1 1	1
1 1	0	0 0	0
1 1	0	0 1	1
1 1	0	1 0	0
1 1	0	1 1	0
1 1	1	0 0	0
1 1	1	0 1	1
1 1	1	1 0	0
1 1	1	1 1	0

2 [Boolean Algebra Definition]

No, $(D_{90}, \mathrm{lcm}, \mathrm{gcd}, 1, 90)$ does not form a Boolean algebra because it does not satisfy the postulate of complements. For example, for $x = 3 \in D_{90}$, there does not exist an element $y \in D_{90} = \{1, 2, 3, 6, 9, 10, 12, 15, 18, 30, 45, 90\}$ satisfying both $\mathrm{lcm}(x, y) = 90$ and $\mathrm{gcd}(x, y) = 1$.

3 [Uniqueness of Complement]

We prove it by contradiction. If a' is not unique, we can find two different elements $a'_1, a'_2 \in \mathbb{B}$, $a'_1 \neq a'_2$ that are both complements of a. By the definition of complements, we have

$$
\begin{aligned}
a + a'_0 &= \underline{1} \\
a \ \cdot \ a'_0 &= \underline{0} \\
a + a'_1 &= \underline{1} \\
a \ \cdot \ a'_1 &= \underline{0}
\end{aligned}
$$

Therefore,

$$\begin{aligned}
a_0' &= \underline{1} \cdot a_0' && \text{(Identities)} \\
&= (a + a_1') \cdot a_0' && \text{(Complements)} \\
&= a_0' \cdot (a + a_1') && \text{(Commutative)} \\
&= (a_0' \cdot a) + (a_0' \cdot a_1') && \text{(Distributive)} \\
&= (a \cdot a_0') + (a_0' \cdot a_1') && \text{(Commutative)} \\
&= \underline{0} + (a_0' \cdot a_1') && \text{(Complements)} \\
&= \underline{0} + (a_1' \cdot a_0') && \text{(Commutative)} \\
&= (a \cdot a_1') + (a_1' \cdot a_0') && \text{(Complements)} \\
&= (a_1' \cdot a) + (a_1' \cdot a_0') && \text{(Commutative)} \\
&= a_1' \cdot (a + a_0') && \text{(Distributive)} \\
&= (a + a_0') \cdot a_1' && \text{(Commutative)} \\
&= \underline{1} \cdot a_1' && \text{(Complements)} \\
&= a_1' && \text{(Identities).}
\end{aligned}$$

This result violates $a_0' \neq a_1'$. Therefore, the assumption that a' is not unique is incorrect, so a' must be unique.

4 [Properties of Boolean Algebra]

(a)

$$\begin{aligned}
(a \cdot b) + c &= c + (a \cdot b) && \text{(Commutative)} \\
&= (c + a) \cdot (c + b) && \text{(Distributive)} \\
&= (a + c) \cdot (c + b) && \text{(Commutative)} \\
&= (a + c) \cdot (b + c) && \text{(Commutative)}
\end{aligned}$$

(b)

$$\begin{aligned}
(a + b) \cdot a &= a \cdot (a + b) && \text{(Commutative)} \\
&= (a \cdot a) + (a \cdot b) && \text{(Distributive)} \\
&= (\underline{0} + (a \cdot a)) + (a \cdot b) && \text{(Identities)} \\
&= ((a \cdot a') + (a \cdot a)) + (a \cdot b) && \text{(Complements)} \\
&= (a \cdot (a' + a)) + (a \cdot b) && \text{(Distributive)} \\
&= (a \cdot (a + a')) + (a \cdot b) && \text{(Commutative)} \\
&= (a \cdot \underline{1}) + (a \cdot b) && \text{(Complements)} \\
&= a \cdot (\underline{1} + b) && \text{(Distributive)} \\
&= a \cdot (\underline{1} \cdot (\underline{1} + b)) && \text{(Identities)} \\
&= a \cdot ((b + b') \cdot (\underline{1} + b)) && \text{(Complements)}
\end{aligned}$$

$$
\begin{aligned}
&= a \cdot ((b + b') \cdot (b + \underline{1})) && \text{(Commutative)}\\
&= a \cdot (b + (b' \cdot \underline{1})) && \text{(Distributive)}\\
&= a \cdot (b + (\underline{1} \cdot b')) && \text{(Commutative)}\\
&= a \cdot (b + b') && \text{(Identities)}\\
&= a \cdot \underline{1} && \text{(Complements)}\\
&= \underline{1} \cdot a && \text{(Commutative)}\\
&= a && \text{(Identities)}
\end{aligned}
$$

(c)

$$
\begin{aligned}
(x')' &= \underline{1} \cdot (x')' && \text{(Identities)}\\
&= (x + x') \cdot (x')' && \text{(Complements)}\\
&= (x')' \cdot (x + x') && \text{(Commutative)}\\
&= ((x')' \cdot x) + ((x')' \cdot x') && \text{(Distributive)}\\
&= ((x')' \cdot x) + (x' \cdot (x')') && \text{(Commutative)}\\
&= ((x')' \cdot x) + \underline{0} && \text{(Complements)}\\
&= \underline{0} + ((x')' \cdot x) && \text{(Commutative)}\\
&= (x \cdot x') + ((x')' \cdot x) && \text{(Complements)}\\
&= (x \cdot x') + (x \cdot (x')') && \text{(Commutative)}\\
&= x + (x' \cdot (x')') && \text{(Distributive)}\\
&= x + \underline{0} && \text{(Complements)}\\
&= \underline{0} + x && \text{(Commutative)}\\
&= x && \text{(Identities)}
\end{aligned}
$$

5 [Minterm Canonical Form]

Theorem 2 (Minterm Canonical Form). *A function $f : \mathbb{B}^n \to \mathbb{B}$ is Boolean if and only if it can be expressed in the minterm canonical form*

$$f(X) = \sum_{A \in \{\underline{0}, \underline{1}\}^n} f(A) \cdot X^A$$

where $X = (x_1, ..., x_n) \in \mathbb{B}^n, A = (a_1, ..., a_n) \in \{\underline{0}, \underline{1}\}^n$, and $X^A \equiv {x_1}^{a_1} \cdot {x_2}^{a_2} \cdot ... \cdot {x_n}^{a_n}$ (with $x_{\underline{0}} \equiv x'$ and $x_{\underline{1}} \equiv x$).

(1) The "if" direction:

Observe that each $f(A) \in \mathbb{B}$ is a valid Boolean formula. Also, each ${x_i}^{a_i}$ is either a variable or its negation, which are both valid Boolean formulae. Therefore, each X^A is the product of n Boolean formulae, which is also a valid Boolean formula. Thus, $f(X) = \sum_{A \in \{\underline{0},\underline{1}\}^n} f(A) \cdot X^A$ is a valid Boolean formula and corresponds to a Boolean function.

(2) The "only if" direction:

When $n = 1$, by Boole's expansion theorem, the Boolean function $f(x_1) = x_1' f(\underline{0}) + x_1 f(\underline{1})$ is in minterm canonical form, so the statement holds.

Suppose the statement holds for $n = k$. Then for $n = k+1$, by Boole's expansion theorem, we have $f(x_1, ..., x_{k+1}) = x_{k+1}' f(x_1, ..., x_k, \underline{0}) + x_{k+1} f(x_1, ..., x_k, \underline{1})$, where both $f(x_1, ..., x_k, \underline{0})$ and $f(x_1, ..., x_k, \underline{1})$ are Boolean functions of k variables. By induction hypothesis, we can represent them as $f(x_1, ..., x_k, \underline{0}) = f_0(x_1, ..., x_k) = \sum_{A \in \{\underline{0},\underline{1}\}^k} f_0(A) \cdot X^A$ and $f(x_1, ..., x_k, \underline{1}) = f_1(x_1, ..., x_k) = \sum_{A \in \{\underline{0},\underline{1}\}^k} f_1(A) \cdot X^A$. Therefore,

$$\begin{aligned} f(x_1, ..., x_{k+1}) &= x_{k+1}' f(x_1, ..., x_k, \underline{0}) + x_{k+1} f(x_1, ..., x_k, \underline{1}) \\ &= x_{k+1}' \sum_{A \in \{\underline{0},\underline{1}\}^k} f_0(A) \cdot X^A + x_{k+1} \sum_{A \in \{\underline{0},\underline{1}\}^k} f_1(A) \cdot X^A \\ &= \sum_{A \in \{\underline{0},\underline{1}\}^k} f(A, \underline{0}) \cdot X^A \cdot x_{k+1}^{\underline{0}} + \sum_{A \in \{\underline{0},\underline{1}\}^k} f(A, \underline{1}) \cdot X^A \cdot x_{k+1}^{\underline{1}} \\ &= \sum_{A \in \{\underline{0},\underline{1}\}^{k+1}} f(A) \cdot X^A. \end{aligned}$$

In other words, f can also be represented in minterm canonical form when $n = k + 1$. By induction, the statement holds for every $n \in \mathbb{N}$.

6 [Number of Boolean Functions]

(a) A function is defined by assigning an output to each input combination. Since there are m^n input combinations, there are $m^{(m^n)}$ functions.

(b) By minterm canonical form, let $f(X) = \sum_{A \in \{\underline{0},\underline{1}\}^n} f(A) \cdot X^A$. Once all $f(A)$'s are assigned, the functionality of f is decided. Since there are 2^n $f(A)$'s, there are $m^{(2^n)}$ Boolean functions.

7 [Boolean Functions]

There are not any Boolean functions consistent with the function table. By minterm canonical form, we can represent

$$\begin{aligned} f(a, a) &= a'a' f(0, 0) + a'a f(0, 1) + aa' f(1, 0) + aa f(1, 1) \\ &= a'a'a + a'a1 + aa'0 + aa f(1, 1) \\ &= 0 + 0 + 0 + a \cdot f(1, 1) \\ &= a \cdot f(1, 1). \end{aligned}$$

We found that whether $f(1, 1)$ is 0, 1, a or a' cannot make $a \cdot f(1, 1)$ equal to $f(a, a) = a'$. Therefore, the function table cannot be a Boolean function.