Logic Synthesis & Verification, Fall 2023

National Taiwan University

Reference Solution to Problem Set 3

Due on 2023/10/28 (Saturday) 23:59.

1 [Cofactor and Generalized Cofactor]

(a) Let $xf_x \oplus (\neg x)f_{\neg x} = h$. Then

$$\begin{aligned} h_x &= (1f_x) \oplus 0 \\ &= f_x, \text{ and} \end{aligned}$$

$$\begin{aligned} h_{\neg x} &= 0 \oplus (1f_{\neg x}) \\ &= f_{\neg x}. \end{aligned}$$

Therefore,

$$\begin{aligned} xf_x \oplus (\neg x)f_{\neg x} &= xh_x + (\neg x)h_{\neg x} \\ &= xf_x + (\neg x)f_{\neg x} \\ &= f. \end{aligned}$$

(b) Let $f_{\neg x} \oplus x(f_{\neg x} \oplus f_x) = h$. Then

$$\begin{aligned} h_x &= f_{\neg x} \oplus 1(f_{\neg x} \oplus f_x) \\ &= f_{\neg x} \oplus f_{\neg x} \oplus f_x \\ &= (f_{\neg x} \oplus f_{\neg x}) \oplus f_x \\ &= 0 \oplus f_x \\ &= f_x, \text{ and} \end{aligned}$$

$$\begin{aligned} h_{\neg x} &= f_{\neg x} \oplus 0(f_{\neg x} \oplus f_x) \\ &= f_x \oplus 0 \\ &= f_{\neg x}. \end{aligned}$$

Therefore,

$$\begin{aligned} xf_x \oplus (\neg x)f_{\neg x} &= xh_x + (\neg x)h_{\neg x} \\ &= xf_x + (\neg x)f_{\neg x} \\ &= f. \end{aligned}$$

(c) Let $f_x \oplus (\neg x)(f_{\neg x} \oplus f_x) = h$. Then

$$\begin{aligned} h_x &= f_x \oplus 0(f_{\neg x} \oplus f_x) \\ &= f_x \oplus 0 \\ &= f_x, \text{ and} \end{aligned}$$

$$\begin{aligned} h_{\neg x} &= f_x \oplus 1(f_{\neg x} \oplus f_x) \\ &= f_x \oplus f_{\neg x} \oplus f_x \\ &= (f_x \oplus f_x) \oplus f_{\neg x} \\ &= 0 \oplus f_{\neg x} \\ &= f_{\neg x}. \end{aligned}$$

Therefore,

$$\begin{aligned} x f_x \oplus (\neg x) f_{\neg x} &= x h_x + (\neg x) h_{\neg x} \\ &= x f_x + (\neg x) f_{\neg x} \\ &= f. \end{aligned}$$

(d)

$$\begin{aligned} & g \wedge co(f, g) \vee \neg g \wedge co(f, \neg g) \\ &= ((g, 0, \neg g) \wedge (fg, \neg g, (\neg f)g)) \vee ((\neg g, 0, g) \wedge (f(\neg g), g, (\neg f)(\neg g))) \\ &= (fg, 0, (\neg f)g \vee (\neg g)) \vee (f(\neg g), 0, g \vee (\neg f)(\neg g)) \\ &= (f, 0, \neg f) \\ &= f. \end{aligned}$$

(e)

$$\begin{aligned} & co(co(f, g), h) \\ &= co((fg, \neg g, (\neg f)g), h) \\ &= ((fg)h, \neg g \vee \neg h, ((\neg f)g)h) \\ &= (f(gh), \neg(gh), (\neg f)(gh)) \\ &= co(f, gh) \end{aligned}$$

(f)

$$\begin{aligned} & co(f, h) + co(g, h) \\ &= (fh, h', f'h) \vee (gh, h', g'h) \\ &= (fh + gh, h', f'g'h) \\ &= ((f+g)h, h', (f+g)'h) \\ &= co(f+g, h) \end{aligned}$$

Note. Operations on incompletely specified functions can be derived from operations on don't cares. Take the $\wedge$ operation for example. Suppose $F = (f_F, d_F, r_F)$ and $G = (f_G, d_G, r_G)$ are incompletely specified functions. Then $H = F \wedge G$ is also an incompletely specified function. Let $H = (f_H, d_H, r_H)$. According to the truth table of 3-valued logic, $x \wedge y = 1$ if and only if $x = y = 1$, so $f_H = f_F \wedge f_G$. Similarly, $x \wedge y = 0$ if and only if $x = 0$ or $y = 0$, so $r_H = r_F \vee r_G$. Finally, the value of $x \wedge y$ cannot be decided for the rest of the conditions, so $d_H = (d_F \wedge f_G) \vee (f_F \wedge d_G) \vee (d_F \wedge d_G)$.
Following the same idea, here are some operations on incompletely specified functions.

$$\begin{aligned}
(f_F, d_F, r_F) \wedge (f_G, d_G, r_G) &= (f_F f_G &&, d_F f_G \vee f_F d_G \vee d_F d_G &&, r_F \vee r_G) \\
(f_F, d_F, r_F) \vee (f_G, d_G, r_G) &= (f_F \vee f_G &&, d_F r_G \vee r_F d_G \vee d_F d_G &&, r_F r_G) \\
\neg(f_F, d_F, r_F) &= (r_F &&, d_F &&, f_F)
\end{aligned}$$

2 [Operation on Cube Lists]

1. Trying to add the cube $(1-0---0)$:
   $(1-0---0)$ is not orthogonal to the second cube $(-0-10-0)$.
   $(1-0---0) - (-0-10-0)$ results in
   $$\{(110---0), (1000--0), (10011-0)\}.$$
2. Trying to add the cube $(110---0)$:
   $(110---0)$ is orthogonal to the cube list.
   Now the cube list becomes
   $$\begin{pmatrix} 0 & - & 0 & - & 1 & 1 & 0 \\ - & 0 & - & 1 & 0 & - & 0 \\ - & 1 & 1 & 0 & - & - & - \\ 1 & 1 & 0 & - & - & - & 0 \end{pmatrix}.$$
3. Trying to add the cube $(1000--0)$:
   $(1000--0)$ is orthogonal to the cube list. Now the cube list becomes
   $$\begin{pmatrix} 0 & - & 0 & - & 1 & 1 & 0 \\ - & 0 & - & 1 & 0 & - & 0 \\ - & 1 & 1 & 0 & - & - & - \\ 1 & 1 & 0 & - & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \end{pmatrix}.$$
4. Trying to add the cube $(10011-0)$:
   $(10011-0)$ is orthogonal to the cube list. The final cube list is
   $$\begin{pmatrix} 0 & - & 0 & - & 1 & 1 & 0 \\ - & 0 & - & 1 & 0 & - & 0 \\ - & 1 & 1 & 0 & - & - & - \\ 1 & 1 & 0 & - & - & - & 0 \\ 1 & 0 & 0 & 0 & - & - & 0 \\ 1 & 0 & 0 & 1 & 1 & - & 0 \end{pmatrix}.$$

3 [Symmetric Functions]

(a) Here we show the process to derive the necessary and sufficient condition of f to be S_1-symmetric on variables x_1 and x_2. By definition, if f is S_1-symmetric on variables x_1 and x_2, then $f(x_1, x_2, x_3, \ldots) = f(x_2, x_1, x_3, \ldots)$. For the left-hand side, we can expand $f(x_1, x_2, x_3)$ as

$$\begin{aligned} f(x_1, x_2, x_3, \ldots) = \quad & \overline{x_1} \cdot \overline{x_2} \cdot f(0, 0, x_3, \ldots) + \overline{x_1} \cdot x_2 \cdot f(0, 1, x_3, \ldots) \\ & + x_1 \cdot \overline{x_2} \cdot f(1, 0, x_3, \ldots) + x_1 \cdot x_2 \cdot f(1, 1, x_3, \ldots). \end{aligned}$$

For the right-hand side, we can expand $f(x_2, x_1, x_3, \ldots)$ as

$$\begin{aligned} f(x_2, x_1, x_3, \ldots) = \quad & \overline{x_1} \cdot \overline{x_2} \cdot f(0, 0, x_3, \ldots) + \overline{x_1} \cdot x_2 \cdot f(1, 0, x_3, \ldots) \\ & + x_1 \cdot \overline{x_2} \cdot f(0, 1, x_3, \ldots) + x_1 \cdot x_2 \cdot f(1, 1, x_3, \ldots). \end{aligned}$$

Note that this form can be obtained by grouping minterms in the minterm canonical form into four groups according to x_1 and x_2, so this form is also canonical. Therefore, $f(x_1, x_2, x_3, \ldots) = f(x_2, x_1, x_3, \ldots)$ if and only if $f(0, 1, x_3, \ldots) = f(1, 0, x_3, \ldots)$. In other words, $f_{x_1 \cdot \overline{x_2}} = f_{\overline{x_1} \cdot x_2}$.

Following the same idea, the necessary and sufficient condition of f to be S_i-symmetric on variables x_1 and x_2 are as follows.
S_1: $f_{x_1 \cdot \overline{x_2}} = f_{\overline{x_1} \cdot x_2}$
S_2: $f_{\overline{x_1} \cdot \overline{x_2}} = f_{x_1 \cdot x_2}$
S_3: f can never be S_3-symmetric
S_4: f can never be S_4-symmetric
S_5: $f_{\overline{x_1} \cdot \overline{x_2}} = f_{\overline{x_i} \cdot x_2} = f_{x_1 \cdot \overline{x_2}} = f_{x_1 \cdot x_2}$ (i.e., f does not depend on x_1 and x_2)
S_6: $f_{\overline{x_1} \cdot \overline{x_2}} = f_{\overline{x_i} \cdot x_2} = f_{x_1 \cdot \overline{x_2}} = f_{x_1 \cdot x_2}$ (i.e., f does not depend on x_1 and x_2)
S_7: $f_{\overline{x_1} \cdot \overline{x_2}} = (\neg f_{\overline{x_1} \cdot x_2}) = (\neg f_{x_1 \cdot \overline{x_2}}) = f_{x_1 \cdot x_2}$
S_8: $f_{\overline{x_1} \cdot \overline{x_2}} = (\neg f_{\overline{x_1} \cdot x_2}) = (\neg f_{x_1 \cdot \overline{x_2}}) = f_{x_1 \cdot x_2}$

(b) Only S_2 does not satisfy transitivity.

- S_1 satisfies transitivity. The proof is as follows.
  Suppose f is S_1-symmetric on (x_1, x_2) and (x_2, x_3), then

$$\begin{aligned} f_{x_1 \cdot \overline{x_3}} &= \overline{x_2} \cdot f_{x_1 \cdot \overline{x_2} \cdot \overline{x_3}} + x_2 \cdot f_{x_1 \cdot x_2 \cdot \overline{x_3}} \\ &= \overline{x_2} \cdot f_{x_1 \cdot \overline{x_2} \cdot \overline{x_3}} + x_2 \cdot f_{x_1 \cdot \overline{x_2} \cdot x_3} && (\because f_{x_2 \cdot \overline{x_3}} = f_{\overline{x_2} \cdot x_3}) \\ &= \overline{x_2} \cdot f_{\overline{x_1} \cdot x_2 \cdot \overline{x_3}} + x_2 \cdot f_{\overline{x_1} \cdot x_2 \cdot x_3} && (\because f_{x_1 \cdot \overline{x_2}} = f_{\overline{x_1} \cdot x_2}) \\ &= \overline{x_2} \cdot f_{\overline{x_1} \cdot \overline{x_2} \cdot x_3} + x_2 \cdot f_{\overline{x_1} \cdot x_2 \cdot x_3} && (\because f_{x_2 \cdot \overline{x_3}} = f_{\overline{x_2} \cdot x_3}) \\ &= f_{\overline{x_1} \cdot x_3}, \end{aligned}$$

  so f is also S_1-symmetric on (x_1, x_3).
- S_2 does not satisfy transitivity. Here is a counterexample.

$$f = x_1 x_2' + x_2' x_3 + x_1 x_3,$$

  where f is S_2-symmetric on (x_1, x_2) and (x_2, x_3), but $f_{\overline{x_1} \cdot \overline{x_3}} = 0 \neq 1 = f_{x_1 \cdot x_3}$, so f is not S_2-symmetric on (x_1, x_3).

- S_5 and S_6 satisfies transitivity. The proof is as follows.
  Suppose f is $S_{5(6)}$-symmetric on (x_1, x_2) and (x_2, x_3), then f does not depend on x_1, x_2 and x_3. Therefore, f is also $S_{5(6)}$-symmetric on (x_1, x_3).
- S_7 and S_8 satisfies transitivity. The proof is as follows.
  Suppose f is $S_{7(8)}$-symmetric on (x_1, x_2) and (x_2, x_3), then

$$\begin{aligned}
f_{\overline{x_1}\cdot\overline{x_3}} &= \overline{x_2}\cdot f_{\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}} + x_2\cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} \\
&= \overline{x_2}\cdot(\neg f_{x_1\cdot\overline{x_2}\cdot\overline{x_3}}) + x_2\cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} && (\because f_{\overline{x_1}\cdot\overline{x_2}} = \neg f_{x_1\cdot\overline{x_2}}) \\
&= \overline{x_2}\cdot(\neg f_{x_1\cdot\overline{x_2}\cdot\overline{x_3}}) + x_2\cdot(\neg f_{x_1\cdot x_2\cdot\overline{x_3}}) && (\because f_{\overline{x_1}\cdot x_2} = \neg f_{x_1\cdot x_2}) \\
&= \neg f_{x_1\cdot\overline{x_3}}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
f_{\overline{x_1}\cdot\overline{x_3}} &= \overline{x_2}\cdot f_{\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}} + x_2\cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2\cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} && (\because f_{\overline{x_2}\cdot\overline{x_3}} = \neg f_{\overline{x_2}\cdot x_3}) \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2\cdot(\neg f_{\overline{x_1}\cdot x_2\cdot x_3}) && (\because f_{x_2\cdot\overline{x_3}} = \neg f_{x_2\cdot x_3}) \\
&= \neg f_{\overline{x_1}\cdot x_3}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
f_{\overline{x_1}\cdot\overline{x_3}} &= \overline{x_2}\cdot f_{\overline{x_1}\cdot\overline{x_2}\cdot\overline{x_3}} + x_2\cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2\cdot f_{\overline{x_1}\cdot x_2\cdot\overline{x_3}} && (\because f_{\overline{x_2}\cdot\overline{x_3}} = \neg f_{\overline{x_2}\cdot x_3}) \\
&= \overline{x_2}\cdot(\neg f_{\overline{x_1}\cdot\overline{x_2}\cdot x_3}) + x_2\cdot(\neg f_{\overline{x_1}\cdot x_2\cdot x_3}) && (\because f_{x_2\cdot\overline{x_3}} = \neg f_{x_2\cdot x_3}) \\
&= \overline{x_2}\cdot(f_{x_1\cdot\overline{x_2}\cdot x_3}) + x_2\cdot(\neg f_{\overline{x_1}\cdot x_2\cdot x_3}) && (\because f_{\overline{x_1}\cdot\overline{x_2}} = \neg f_{x_1\cdot\overline{x_2}}) \\
&= \overline{x_2}\cdot(f_{x_1\cdot\overline{x_2}\cdot x_3}) + x_2\cdot(f_{x_1\cdot x_2\cdot x_3}) && (\because f_{\overline{x_1}\cdot x_2} = \neg f_{x_1\cdot x_2}) \\
&= f_{x_1\cdot x_3},
\end{aligned}$$

  so f is also $S_{7(8)}$-symmetric on (x_1, x_3).

4 [Unate Functions]

(a) True. The proof is as follows.
Let the unate cover without having any single-cube containment be F, and let F be the cover of function f. If F is not a prime, by definition, there exists a literal v in a cube c, where $c \in F$, such that $(F\backslash\{c\}) \cup \{c_v\} = f$.
Without loss of generality, let $c = v\cdot v_2\cdot\ldots\cdot v_k$, $F = c+c_2+...+c_\ell$, and let F depend on variables $\{v, v_2, \ldots, v_k, x_1, \ldots, x_n, y_1, \ldots, y_m\}$, where F is positive unate in $x_1, \ldots, x_n$ and negative unate in $y_1, \ldots, y_n$. For $v, v_2, \ldots, v_k$, we do not make any assumptions, and f can be either positive or negative unate in each of them.
Since $(F\backslash\{c\}) \cup \{c_v\} = (v_2 \ldots v_k) + c_2 + \ldots + c_\ell = f$, f should contain the minterm $m_0 = (\overline{v})(v_2 \ldots v_k)(\overline{x_1} \ldots \overline{x_n})(y_1 \ldots y_m)$. However, if we inspect each cube in F, we will find that c cannot contain m_0 because $v \in c$ and $\overline{v} \in m_0$. As for any other c_i, since there is not any single-cube containment in F, c_i must contain some variables in $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ to avoid containing c. If c_i contains the variable x_j, then $x_j \in c_i$ because F is positive

unate in x_j, so c_i cannot contain m_0. If c_i contains the variable y_j, then $\overline{y_j} \in c_i$ because F is negative unate in y_j, so c_i cannot contain m_0. To sum up, there does not exist any cube in F to contain m_0, so f cannot contain m_0, which leads to a contradiction. Therefore, F must be a prime cover.

(b) True. The proof is as follows.
Let the unate function be f, and the prime cover be F. If F is not unate, by definition, there exist a variable v and two cubes $c_1, c_2 \in F$ such that $v \in c_1$ and $\overline{v} \in c_2$. Consider the two possibilities.
  (1) If f is positive unate in v:
  By definition, $f_{\overline{v}} \subseteq f_v$. Since f contain $c_2 = \overline{v} \cdot c_2 = \overline{v} \cdot (c_2)_{\overline{v}}$, f should also contain $v \cdot (c_2)_{\overline{v}}$. Therefore, f contains $(c_2)_{\overline{v}}$, and the literal $\overline{v}$ in c_2 can be removed without affecting the functionality of F. However, F should be a prime, so $(F \backslash \{c_2\}) \cup \{(c_2)_{\overline{v}}\} \neq f$, which leads to a contradiction.
  (2) If f is negative unate in v:
  By definition, $f_v \subseteq f_{\overline{v}}$. Since f contain $c_1 = v \cdot c_1 = v \cdot (c_1)_v$, f should also contain $\overline{v} \cdot (c_1)_v$. Therefore, f contains $(c_1)_v$, and the literal v in c_1 can be removed without affecting the functionality of F. However, F should be a prime, so $(F \backslash \{c_1\}) \cup \{(c_1)_v\} \neq f$, which leads to a contradiction.
Therefore, F must be a unate cover.

5 [Threshold and Unate Functions]

(a) $x_1 \wedge x_2 \wedge x_3 = f(x_1, x_2, x_3 \mid w_1 = w_2 = w_3 = 1,\, T = 3)$

(b) Without loss of generality, let the threshold function f be defined on variables $\{x_1, \ldots, x_n\}$, where $w_i \geq 0$ for $1 \leq i \leq m$, and $w_i < 0$ for $m < i \leq n$. Let $s(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_i$. Then $f(x_1, \ldots, x_n) = 1$ if and only if $s(x_1, \ldots, x_n) \geq T$.
Consider an arbitrary input assignment $(a_1, \ldots, a_n)$. For any $1 \leq i \leq m$, we have

$$\begin{aligned}
&f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) = 1 \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) \geq T \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) = s(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) + w_i \geq T \quad (\because w_i \geq 0) \\
\Rightarrow &f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) = 1.
\end{aligned}$$

Since the above relation holds for any input assignment, we can conclude

$$f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots x_n) \rightarrow f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots x_n).$$

Therefore, $f_{\overline{x_i}} \subseteq f_{x_i}$, indicating that f is positive unate in x_i $\forall 1 \leq i \leq m$.
Similarly, for any $m < i \leq n$, we have

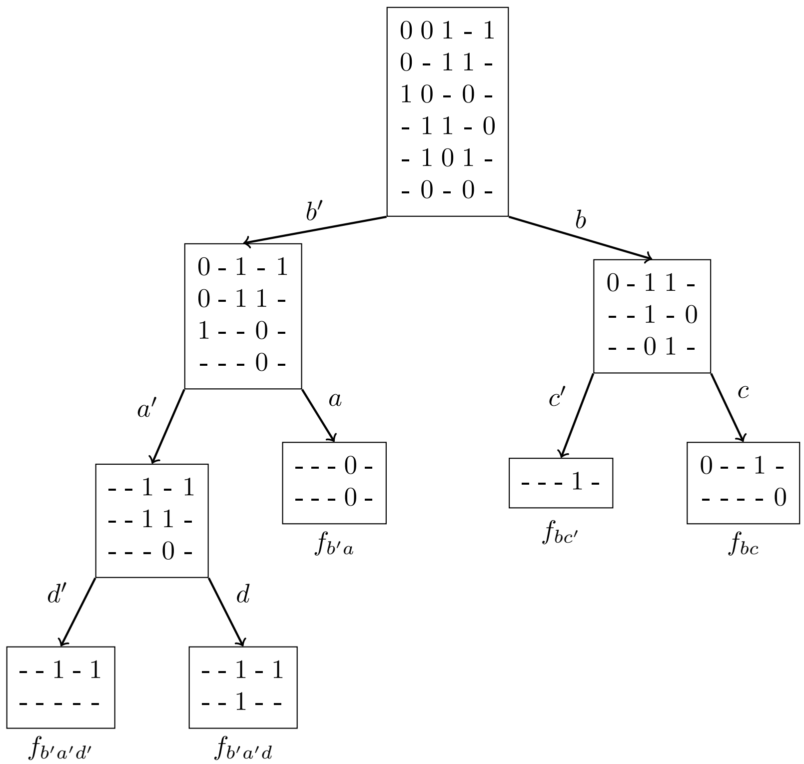
$$\begin{aligned}
&f(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) = 1 \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) \geq T \\
\Rightarrow &s(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) = s(a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots a_n) - w_i \geq T \quad (\because w_i < 0) \\
\Rightarrow &f(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots a_n) = 1.
\end{aligned}$$

Since the above relation holds for any input assignment, we can conclude

$$f(x_1,\ldots,x_{i-1},1,x_{i+1},\ldots x_n) \to f(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots x_n).$$

Therefore, $f_{x_i} \subseteq f_{\overline{x_i}}$, indicating that f is negative unate in x_i $\forall m < i \leq n$. Since f is unate in every variable, f must be a unate function.

6 [Unate Recursive Paradigm: Complementation]



$f_{b'a'd'} = 1$	$\Rightarrow (f_{b'a'd'})' = 0$
Minimal column covers of$f_{b'a'd} = \{\{3\}\}$	$\Rightarrow (f_{b'a'd'})' = c'$
Minimal column covers of$f_{b'a} = \{\{4\}\}$	$\Rightarrow (f_{b'a})' = d$
Minimal column covers of$f_{bc'} = \{\{4\}\}$	$\Rightarrow (f_{bc'})' = d'$
Minimal column covers of$f_{bc} = \{\{1,5\},\{4,5\}\}$	$\Rightarrow (f_{bc})' = ae + d'e$

Therefore,

$$\begin{aligned} f' &= b'a'd' \cdot (f_{b'a'd'})' + b'a'd \cdot (f_{b'a'd})' + b'a \cdot (f_{b'a'})' + bc' \cdot (f_{bc'})' + bc \cdot (f_{bc})' \\ &= a'b'c'd + ab'd + bc'd' + abce + bcd'e. \end{aligned}$$

7 [Quine-McCluskey]

(a) The prime implicants are $\{a'c'd, b'cd', a'bc', bc'd, acd', a'b, bd', ad', bc\}$.

$a\ b\ c\ d$	$f\ g$	
0 0 0 1	1 0	✓
0 0 1 0	1 0	✓
0 1 0 0	1 1	✓
1 0 0 0	0 1	✓
0 1 0 1	1 1	✓
0 1 1 0	0 1	✓
1 0 1 0	1 1	✓
1 1 0 0	0 1	✓
0 1 1 1	0 1	✓
1 1 0 1	1 0	✓
1 1 1 0	1 1	✓
1 1 1 1	0 1	✓

$a\ b\ c\ d$	$f\ g$	
0 - 0 1	1 0	(p_1)
- 0 1 0	1 0	(p_2)
0 1 0 -	1 1	(p_3)
0 1 - 0	0 1	✓
- 1 0 0	0 1	✓
1 0 - 0	0 1	✓
1 - 0 0	0 1	✓
0 1 - 1	0 1	✓
- 1 0 1	1 0	(p_4)
0 1 1 -	0 1	✓
- 1 1 0	0 1	✓
1 - 1 0	1 1	(p_5)
1 1 - 0	0 1	✓
- 1 1 1	0 1	✓
1 1 1 -	0 1	✓

$a\ b\ c\ d$	$f\ g$	
0 1 - -	0 1	(p_6)
- 1 - 0	0 1	(p_7)
1 - - 0	0 1	(p_8)
- 1 1 -	0 1	(p_9)

(b) The Boolean matrix for column covering is as follows, where empty entries mean 0.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9
0 1 0 0 | 1 0			1						
0 1 0 1 | 1 0	1		1	1					
1 0 1 0 | 1 0		1			1				
1 1 1 0 | 1 0					1				
0 1 0 0 | 0 1			1			1	1		
0 1 0 1 | 0 1			1			1			
0 1 1 1 | 0 1						1			1
1 0 0 0 | 0 1								1	
1 0 1 0 | 0 1					1			1	
1 1 0 0 | 0 1							1	1	
1 1 1 1 | 0 1									1

(c) First we list all essential primes. The essential primes are $\{p_3, p_5, p_8, p_9\}$. Then all rows are covered by essential primes and removed. Therefore, the cyclic core is empty.

(d) The minimum column covering is $\{p_3, p_5, p_8, p_9\}$, which corresponds to the minimum multi-output cover $\{a'bc', acd', ad', bc\}$. We verify that $\{a'bc', acd'\}$ is a cover of f, and $\{a'bc', acd', ad', bc\}$ is a cover of g.