Problem 1

(a)

Suppose $|A_k\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$ for some $a, b, c, d \in \mathbb{C}$

$a^2+b^2=1,\ c^2+d^2=1$

$\rightarrow$ $ac = \frac{1}{2}$, $ad = \frac{1}{2}$, $bc = \frac{1}{2}$, $bd = \frac{(-1)^k}{2}$

$\rightarrow$ if $k=1$, $\frac{-1}{4} = \frac{1}{2} \cdot \frac{-1}{2} = abcd = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$, $\rightarrow\leftarrow$.

$\searrow$ if $k=0$, $(a, b, c, d) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is a solution

Ans:

$|A_0\rangle$: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$, unentangled.

$|A_1\rangle$: entangled, proved by the contradiction of hypothesis.

(b)

Ans:

For $|A_0\rangle$: $U \triangleq \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$ $\begin{matrix} |0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\ |1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{matrix}$, $|0\rangle|0\rangle \xrightarrow{U \otimes U} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |A_0\rangle$

$|A_0\rangle$ can be!!

$|A_1\rangle$: Suppose it can be, $|0\rangle|0\rangle \xrightarrow{U_1 \otimes U_2} (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) = |A_1\rangle$. $\rightarrow\leftarrow$

$|A_1\rangle$ cannot be!! By the contradiction of entangle.

(c)

Suppose

$\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$

$\rightarrow$ $ac = \alpha$, $ad = \beta$, $bc = \gamma$, $bd = \delta$

$\rightarrow$ if $\alpha\delta \neq \beta\gamma$, then $\alpha\delta = abcd \neq \beta\gamma = abcd$, $\rightarrow\leftarrow$

(entangle!!)

$\searrow$ if $\alpha\delta = \beta\gamma$, $\frac{\gamma}{\alpha} = \frac{\delta}{\beta}$, then $(a, b, c, d) = \left(\sqrt{\alpha^2+\beta^2}, \frac{\gamma}{\alpha}\sqrt{\alpha^2+\beta^2}, \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}, \frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right)$

then $\cdot\ c^2+d^2=1$

$\cdot\ (a^2+b^2)(c^2+d^2) = (\alpha^2+\beta^2)\left(1^2 + \frac{\gamma^2}{\alpha^2}\right) = \alpha^2+\beta^2+\gamma^2+\delta^2 \overset{\text{given}}{=} 1$.

$\Rightarrow$ $(a|0\rangle + b|1\rangle)$, $(c|0\rangle + d|1\rangle)$ are two states. #

Problem 2.

Consider $|\psi\rangle$, $X|\psi\rangle$, $Y|\psi\rangle$, $Z|\psi\rangle$

Given any projection matrix $P_k \triangleq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with outcome k, we have:

- $$Pr(k) = \frac{1}{4}\Big(\langle\psi|P_k|\psi\rangle + \langle\psi|X^{+}P_kX|\psi\rangle + \langle\psi|Y^{+}P_kY|\psi\rangle + \langle\psi|Z^{+}P_kZ|\psi\rangle \Big)$$

$$= \frac{1}{4}\langle\psi|P'|\psi\rangle,$$

- with $P' = P_k + X^{+}P_kX + Y^{+}P_kY + Z^{+}P_kZ$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} d & c \\ b & a \end{pmatrix} + \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} + \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

$$= (2a+2d)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = 2\,\mathrm{trace}(P_k)\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} = 2I$$

- $$Pr(k) = \frac{1}{4}\langle\psi|2I|\psi\rangle = \frac{1}{2}\langle\psi|\psi\rangle = \frac{1}{2}\|\psi\|^2 = \underline{\underline{\frac{1}{2}}}\ \#$$

Problem 3

(a)

Schmidt bases: $\{|a\rangle, |a^\perp\rangle\}$ of A, $\{|b\rangle, |b^\perp\rangle\}$ of B

$$|a\rangle|b\rangle = 1|a\rangle|b\rangle + 0\,|a^\perp\rangle|b^\perp\rangle, \quad \text{rank} = 1.$$

Schmidt bases: $\{|0\rangle, |1\rangle\}$ of A, $\{\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\}$ of B

$$|A_1\rangle = \frac{1}{\sqrt{2}}|0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle+\frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}|1\rangle\left(\frac{1}{\sqrt{2}}|0\rangle-\frac{1}{\sqrt{2}}|1\rangle\right), \quad \text{rank} = 2.$$

Notice: rank > 1 is proved by "entangle".

(b)

Consider a Isomophism: $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \longleftrightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

$$|\psi\rangle \longleftrightarrow [\psi]$$

In this case, we apply singular value decomposition thm:

$\exists\, U, V$ orthogonal matrix, Σ is singular matrix s.t.

$$[\psi] = U\Sigma V^* = \sum_{i=1}^{2} \sigma_i u_i v_i^*$$

$$\longleftrightarrow |\psi\rangle = \sum_{i=1}^{2} \sigma_i |u_i\rangle|v_i\rangle \quad \#$$

(c)

Part 1: Claim $U \triangleq \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a unitary matrix

Proof:

$\because \{a|0\rangle+b|1\rangle, c|0\rangle+d|1\rangle\}$ is a orthonormal basis

$\therefore \left\{\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right\}$ is a orthnormal basis

$\therefore \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is a unitary matrix. #

So, $|0\rangle \xrightarrow{U} \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$

$|1\rangle \xrightarrow{U} \begin{pmatrix} c \\ d \end{pmatrix} = c|0\rangle + d|1\rangle$. #

Problem 3 (c)

Given any target 2-qubit state $|\psi\rangle \overset{\triangle}{=} \sum_{i=1}^{2} \lambda_i |\alpha_i\rangle|\beta_i\rangle$ ← by Schmidt form

for some $\lambda_i, \alpha_i, \beta_i$, $i=0,1$.

Consider $U_1 = \begin{pmatrix} \lambda_0 & \lambda_1 \\ \lambda_1 & -\lambda_0 \end{pmatrix}$, $U_2 = \left(\alpha_0 \middle| \alpha_1\right)$, $U_3 = \left(\beta_0 \middle| \beta_1\right)$ are unitary matrices.

Notice: since $|\psi\rangle$ is a state, $\lambda_0^2 + \lambda_1^2 = 1$, so U_1 is a unitary matrix.

In this case, we have:

$|0\rangle|0\rangle$

$\xrightarrow{U_1} (\lambda_0|0\rangle + \lambda_1|1\rangle)|0\rangle = \lambda_0|00\rangle + \lambda_1|10\rangle$

$\xrightarrow{CNOT} \lambda_0|00\rangle + \lambda_1|11\rangle$

$\xrightarrow{U_2 \otimes U_3} \lambda_0|\alpha_0\rangle|\beta_0\rangle + \lambda_1|\alpha_1\rangle|\beta_1\rangle = |\psi\rangle$ #

If $|\psi\rangle$ is unentangle, $\exists\, |\alpha\rangle, |\beta\rangle$ states s.t. $|\psi\rangle = |\alpha\rangle|\beta\rangle$.

Then, $U_1 \overset{\triangle}{=} (\alpha | \alpha^{\perp})$, $U_2 \overset{\triangle}{=} (\beta | \beta^{\perp})$; $|0\rangle|0\rangle \xrightarrow{U_1 \otimes U_2} |\alpha\rangle|\beta\rangle = |\psi\rangle$

⇒ <u>No need CNOT</u>

If $|\psi\rangle$ is entangle, without CNOT, we only can get the form like $|\alpha\rangle|\beta\rangle$.

So it is impossible to arrive $|\psi\rangle$. ⇒ <u>Need CNOT</u>

Ans: <u>Need CNOT</u> iff <u>target 2-qubit state is entangle.</u> #

Problem 4

(a)

- If such deleting operation is unitary $\Rightarrow$

$$\langle \psi_0 | \psi_1 \rangle \langle \psi_0 | \psi_1 \rangle \langle M | M \rangle = \langle \psi_0 | \psi_1 \rangle \langle 0 | 0 \rangle \langle M_0 | M_1 \rangle$$

$$\xrightarrow{\text{by } \langle \psi_0 | \psi_1 \rangle \in (0,1)} \quad \langle \psi_0 | \psi_1 \rangle = \langle M_0 | M_1 \rangle$$

$$\longrightarrow \quad \langle \psi_0 | \psi_1 \rangle \langle N | N \rangle = \langle 0 | 0 \rangle \langle M_0 | M_1 \rangle \quad \text{... condition } *$$

- Claim: For any states $a_0, a_1, b_0, b_1 \in \mathbb{C}^n$,
  if $\langle a_0 | a_1 \rangle = \langle b_0 | b_1 \rangle$, then $\exists$ unitary V s.t. $a_0 \xrightarrow{V} b_0$, $a_1 \xrightarrow{V} b_1$.

- As a result, by condition $*$, $\exists V$ s.t. $|0\rangle |M_i\rangle \longrightarrow |\psi_i\rangle |N\rangle$. #

- Prove claim:

consider $s \triangleq \langle a_0 | a_1 \rangle = \langle b_0 | b_1 \rangle$

$$t \triangleq \| a_1 - s a_0 \| = \|a_1\|^2 - s\langle a_1, a_0 \rangle - s \langle a_0, a_1 \rangle + s^2 \|a_0\|^2$$

$$= 1 - s^2$$

$$= \| b_1 - s b_0 \|$$

$\because \left\{ a_0, a_2 \triangleq \frac{a_1 - s a_0}{t} \right\}, \left\{ b_0, b_2 \triangleq \frac{b_1 - s b_0}{t} \right\}$ are orthonormal linear indep set

$\therefore \exists \{a_0, a_2, a_3, \dots, a_{n+1}\}, \{b_0, b_2, b_3, \dots, b_{n+1}\}$ are orthonormal bases

$\therefore \exists V: a_i \to b_i, \; i = 0, 2, 3, \dots, n+1$

In this case

$$a_0 \xrightarrow{V} b_0$$

$$s a_0 \longrightarrow s b_0$$

$$t a_2 \longrightarrow t b_2$$

$$a_1 = t a_2 + s a_0 \longrightarrow t b_2 + s b_0 = b_1 \quad \#$$

(So, $\exists V$ unitary s.t. $a_0 \to b_0$, $a_1 \to b_1$, the claim is proved!!)

Problem 4

(a)

On the other hand, if the info. can not be retrieved, that is:

$$\exists U \text{ unitary s.t. } |\psi_i\rangle|\psi_i\rangle|M\rangle \rightarrow |\psi_i\rangle|0\rangle|M'\rangle$$

then, we have:

$$\langle\psi_0|\psi_1\rangle\langle\psi_0|\psi_1\rangle\langle M|M\rangle = \langle\psi_0|\psi_1\rangle\langle 0|0\rangle\langle M'|M'\rangle$$

$$\rightarrow \langle\psi_0|\psi_1\rangle^2 = \langle\psi_0|\psi_1\rangle$$

$\rightarrow \langle\psi_0|\psi_1\rangle = 0$ or 1 if it is real number, contradict to $\langle\psi_0|\psi_1\rangle \in (0,1)$. #

(b) Consider $|\psi_0\rangle = a|0\rangle + b|1\rangle$, if measurements is allowed, then $|\psi_1\rangle = c|0\rangle + d|1\rangle$

$\exists U$ unitary s.t. $|\psi_0\rangle \rightarrow |0\rangle$, $|\psi_1\rangle \rightarrow |1\rangle$.

In this case, we have:

$$|\psi_i\rangle|\psi_i\rangle|M\rangle \xrightarrow{U\otimes U\otimes I} |i\rangle|i\rangle|M\rangle \xrightarrow{CNOT} |i\rangle|0\rangle|M\rangle \xrightarrow{U^{\dagger}\otimes I\otimes I} |\psi_i\rangle|0\rangle|M\rangle. \ \#$$

(c) <u>It can be!!</u>

- We have: $|i\rangle|i\rangle|M\rangle \xrightarrow{CNOT} |i\rangle|0\rangle|M\rangle$
- CNOT is reversible boolean operation:

CNOT: $00 \rightarrow 00$, $01 \rightarrow 01$, $10 \rightarrow 11$, $11 \rightarrow 10$

$CNOT^2$: $00 \rightarrow 00 \rightarrow 00$, $01 \rightarrow 01 \rightarrow 01$, $10 \rightarrow 11 \rightarrow 10$, $11 \rightarrow 10 \rightarrow 11$

$\Rightarrow CNOT^2 = \text{Identity}$.

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Problem 5

(a)

- Suppose not, $\{a_i\}$ are linear dependent set, $\exists k$ s.t. $|\alpha_k\rangle = \sum_i a_i |\alpha_i\rangle$.

- Note: $a_k = 0$ in this case.

- Define process: $|\alpha\rangle \xrightarrow{U} U|\alpha\rangle|A\rangle \xrightarrow{\text{measurement}} \langle A|\langle\alpha|U^+\Pi_t U|\alpha\rangle|A\rangle$
$= p_t \delta_t(\alpha)$

, where $\Pi_1, \Pi_2, \cdots, \Pi_{N+1}$ are projection; $p_1, p_2, \cdots, p_{N+1}$ are nonzero prob. of occurring;

$$\delta_t(\alpha) = \begin{cases} 0, & \text{if } \alpha \neq \alpha_t \\ 1, & \text{if } \alpha = \alpha_t. \end{cases}$$

- Consider Π_k in the process, we have:

$|\alpha_k\rangle \longrightarrow p_k \delta_k(\alpha) = p_k$

$\alpha \triangleq \sum_i a_i|\alpha_i\rangle \longrightarrow \langle A|\langle\alpha|U^+\Pi_k U|\alpha\rangle|A\rangle$

$= \sum_{i,j} a_i a_j \langle A|\langle\alpha_i|U^+\Pi_k U|\alpha_j\rangle|A\rangle$

- $p_k^2 \leq \sum_{i,j} \left(a_i a_j \langle A|\langle\alpha_i|U^+\Pi_k U|\alpha_j\rangle|A\rangle\right)^2$

$\overset{\text{科西不等式}}{\leq} \sum_{i,j} a_i^2 a_j^2 \left(\langle A|\langle\alpha_i|U^+\Pi_k U|\alpha_i\rangle|A\rangle + \langle A|\langle\alpha_j|U^+\Pi_k U|\alpha_j\rangle|A\rangle\right)$

$= 2\sum_i a_i^2 \left(\sum_j a_j^2\right) \langle A|\langle\alpha_i|U^+\Pi_k U|\alpha_i\rangle|A\rangle$

$\overset{S \triangleq \sum_j a_j^2}{=\!=} 2\sum_i a_i^2 \cdot S \cdot p_k \delta_k(\alpha_i)$

$= 2a_k^2 \cdot S \cdot p_k = 0 \cdot S \cdot p_k = 0.$

As a result, $p_k^2 \leq 0$, contradict to $p_k \in (0,1]$. #.

[Hence, $\{a_i\}$ are linear independent set]

Problem 6

(a)

$$P(k|k) = \langle A| \langle \alpha_k | \Pi_k | \alpha_k \rangle |A\rangle$$

$$\underline{\underline{P_S = \frac{\sum_i P(i|i)}{N} = \frac{1}{N} \sum_k \langle A| \langle \alpha_k | \Pi_k | \alpha_k \rangle |A\rangle}}$$

(b).

Property : If X is positive semi-definite, then $\langle \psi | X | \psi \rangle \le \mathrm{Tr}(X)$, $\forall$ state $|\psi\rangle$.

Property: Projection matrix is symmetric, and so it is positive semi-definite.

As a result, we have:

$$\langle A| \langle \alpha_1 | \Pi_1 | \alpha_1 \rangle |A\rangle + \dots + \langle A| \langle \alpha_N | \Pi_N | \alpha_N \rangle |A\rangle$$

$$\le \mathrm{trace}(\Pi_1) + \dots + \mathrm{trace}(\Pi_N)$$

$$= \mathrm{trace}(\Pi_1 + \Pi_2 + \dots + \Pi_N) = \mathrm{trace}(I) = d.$$

Hence, $\underline{P_S \le \frac{1}{N} \cdot d}$. #

(c)

- When $\frac{d}{N} \ge 1$, $P_S \le 1 \le \frac{d}{N}$ must hold. #
- When $\frac{d}{N} < 1$, $d < N$, we have:

consider $|\alpha_i\rangle = \begin{cases} \text{if } i \le d, & e_i \\ \text{if } i > d, & \text{any} \end{cases}$, $\Pi_i = \begin{cases} \text{if } i \le d, & E_{ii} \triangleq \begin{array}{c} \scriptstyle (i) \\ \begin{pmatrix} & & \\ & 1 & \\ & & \end{pmatrix} \end{array} \\ \text{if } i > d, & \text{any} \end{cases}$.

$$P_S = \frac{1}{N}\left(\langle A| \langle \alpha_1 | \Pi_1 | \alpha_1 \rangle |A\rangle + \dots + \langle A| \langle \alpha_N | \Pi_N | \alpha_N \rangle |A\rangle \right)$$

$$\ge \frac{1}{N}\left(\underbrace{1 + 1 + \dots + 1}_{d} \underbrace{+ 0 + \dots + 0}_{N-d} \right) = \frac{d}{N}.$$

As a result, $P_S = \frac{d}{N}$ in this case. It is a tight bound. #.