The homogeneous coordinate ring of Grassmannian $(G_{r_{k,n}})$

For $(v_1, v_2, \ldots, v_r)$ in V^r. Let

$$v_j = \sum_{i=1}^{n} x_{ij} e^i, \qquad 1 \leq j \leq r.$$

The x_{ij}'s are coordinate functions on the affine space $V^r \equiv M(n, r)$ and the polynomial k- algebra $k[x_{ij}]$ is the coordinate ring of V^r. The morphism

$$\hat{\pi} : V^r \longrightarrow \Lambda^r V$$

induces a $k-$ algebra homomorphism between the coordinate rings

$$\hat{\pi}^* : k[x_\alpha] \longrightarrow k[x_{ij}]$$

defined by $\hat{\pi}^*(x_\alpha) = p_\alpha$.

where

$$p_\alpha = \pm \det \begin{bmatrix} x_{\alpha_1}1 & x_{\alpha_1}2 & \cdots & x_{\alpha_1}r \\ x_{\alpha_2}1 & x_{\alpha_2}2 & \cdots & x_{\alpha_2}r \\ \vdots & & & \\ x_{\alpha_r}1 & x_{\alpha_r}2 & \cdots & x_{\alpha_r}r \end{bmatrix}$$

the determinant of the α^{th} minor (with its sign) in the $n \times r$ matrix $[x_{ij}]$. Since $\ker \hat{\pi}^*$ is the ideal of im $\hat{\pi}^*$, the cone over $G_{r_{k,n}}$, it is the homogeneous ideal of $G_{r_{k,n}}$ in $k[x_\alpha]$, thus the homogeneous coordinate ring of $G_{r_{k,n}}$ can be identified with the k- subalgebra of $k[x_{ij}]$ generated by $p_\alpha \quad \alpha \in l(r,n)$ and $G_{r_{k,n}}$ has a natural scheme structure defined by its homogeneous coordinate ring $k[p_\alpha]$, namely, proj $\quad k[p_\alpha]$. Indeed $G_{r_{k,n}}$ is a closed integral subscheme of $\mathbb{P}(\Lambda^r V)$.

Remark 1. Grassmannian is describe as an r- dimensional linear subspaces of V endowed with the structure of a variety as follows. Let

$$\begin{aligned} V^r &= \underbrace{V \oplus V \oplus V \oplus \cdots \oplus V}_{r\text{–copies}} \\ &\cong M(n,r), \text{ the set of } r \times n \text{ matrices} \\ \text{and } V^{r,0} &= \{(v_1, v_2, \ldots, v_r) \in V^r | v_1, v_2, \ldots, v_r \text{ are linearly independent } \} \\ &\cong M(n,r)^\circ, \text{ the set of } n \times r \text{ matrices of rank } r. \end{aligned}$$

Clearly the column vectors of a matrix in $M(n,r)^\circ$ generate an element of $G_{r_{k,n}}$.

Grassmannian as a Projective variety

We realize $G_{r_{k,n}}$ as a projective variety by embedding it in the projective space $\mathbb{P}(\Lambda^r V)$.

Let us consider the morphism

$$\hat{\pi}^* : V^r \longrightarrow \Lambda^r$$

defined by $\hat{\lambda}(v_1, v_2, \ldots, v_r) = v_1 \Lambda v_2 \Lambda \cdots \Lambda v_r$. Restricting $\hat{\pi}$ to v^{r0} we have

$$\pi : V^{r,0} \longrightarrow \mathbb{P}(\Lambda^r V)$$

This map induces another morphism

$$\tilde{\pi} : G_{r_{k,n}} \longrightarrow \mathbb{P}(\Lambda^r V)$$

Thus we show that the im $\tilde{\pi}$ is a closed subset of $\mathbb{P}(\Lambda^r V)$. Let $\mathcal{T}_\beta = \{(\omega_\alpha) \in \mathbb{P}(\Lambda^r V) | \omega_\beta = 1\}$ for $\beta \in I(k,n)$

Since $\{\mathcal{T}_\beta; \beta \in I(k,n)\}$ is an open covering of $\mathbb{P}(\Lambda^r V)$, then we demonstrate that for β in $I(k,n)$, $\mathcal{T}_\beta \cap$ im$\tilde{\pi}$ is closed in $\mathcal{T}_\beta$. Suppose β is arbitrary chosen, we see that $\mathcal{T}_\beta \cap \text{im}\tilde{\pi}(\nu_\beta) = \tilde{\pi}(\nu_\beta)$. This implies that $\mathcal{T}_\beta \cap \text{im}\tilde{\pi}$ consists of elements $(\omega_\alpha) \in \mathbb{P}(\Lambda^r V)$ and ω_α is defined as the determinant of the α^{th} minor of an element in ν_β where ν_β consists of $k \times n$ matrices form.

$\binom{I_k}{A}$ with $I_k = k \times k$ identity matrix and $A = [a_{ij}] \in M(n-k, k)$. The element in the intersection above form an equivalent class which has the form

$$(\ldots, a_{ij}, \ldots, f_\lambda(a_{ij}), \ldots)$$

where a_{ij} are arbitrary and f_λ are the polynomial fuction on a_{ij}. Thus im $\tilde{\pi}$ is a projective variety in $\mathbb{P}(\Lambda^r V)$ since the $\mathcal{T}_\beta \cap$ im $\tilde{\pi}$ is closed in the affine space $\mathcal{T}_\beta$.

It also follows that

$$\tilde{\pi} : \nu_\beta \longrightarrow \tilde{\pi}(\nu_\beta)$$

is an isomorphism of affine varieties and $\tilde{\pi}^{-1}(\tilde{\pi}(\nu_\pi)) = \nu_\beta$.

It implies that $\tilde{\pi} : G_{r_{k,n}} \longrightarrow \operatorname{im}\tilde{\pi}$ is an isomorphism, in particular local isomorphism. This $\tilde{\pi}$ embeds $G_{r_{k,n}}$ as a projective variety in $\mathbb{P}(\Lambda^r V)$. This map is called the plücker embedding map and the coordinates of its image are called the Plücker coordinates of $G_{r_{k,n}}$.

Decomposition of $G_{r_{k,n}}$ into cells.

Gransmannian can be decomposed into different cells namely; Matroid cells, Schbert cells and Positroid cells.

Schubert Cell Decomposition

Definition 2 (The Schubert cells.)**.** fix a full flag $\{0\} = v_0 \subset v_1 \subset \cdots \subset v_r = V$ in an n- dimensional vector space V. Define a Schubert cell of $G_{r_{k,n}}$ as follows

$$C(\alpha) = \left\{ \begin{aligned} & W \in G_{r_{k,n}} | \dim W \cap V_j = i \text{ if } a_i \leq j < \alpha_{i+1} \\ & \quad \text{where } 1 \leq j \leq n, \quad 0 \leq i \leq k \text{ and } \alpha_0 = 0 \end{aligned} \right\}$$

The closure of the Schubert cell $C(\alpha)$ in $G_{r_{k,n}}$ is called the Schubert variety corresponding to the index α (or simply the α^{th} schubert variety) denote by $\chi(\alpha)$. As a subscheme of $G_{r_{k,n}}$, $\chi(\alpha)$ is endowed with its canonical reduced subscheme structure with

$$\dim \chi(\alpha) = \sum_{i=1}^{r} \alpha_i - \frac{r(r+1)}{\alpha}$$

Proposition 3. *The homogeneous ideal of a Schubert variety is a Prime ideal.*

Proof. For α in $l(k,n)$, since $C(\alpha)$ is an irreducible variety, its closure $\chi(\alpha)$ is irreducible. This together with the fact that $\chi(\alpha)$ is reduced implies that $\chi(\alpha)$ is an integral scheme, which in turn implies that the homogeneous ideal of $\chi(\alpha)$ is a Prime ideal in $\mathbb{K}[x_\beta, \beta \in I(k,n)]$ the homogeneous coordinate ring of $\mathbb{P}(\Lambda^r V)$ $\square$

Remark 4. We have $\alpha^{\min} = (1,2,\ldots,r)$ and $\alpha^{\max} = (n-r+1, n-r+2,\ldots,n)$. The $C(\alpha^{\min})$ is a point.

$$C(\alpha^{\max}) = \cup_{\alpha \max}$$

and it is the only Schubert cell that is open subset of $G_{r_{k,n}}$ which is called the big cell.

Since Grassmannian is irreducible, $\chi(\alpha^{\max}) = G_{r_{k,n}}$ i.e, $G_{r_{k,n}}$ itself is a Schubert variety.

Positroid cell decomposition

Positroid cells denote as $P(\alpha)$ can be represented by $2 \times n$ matrices $A = [v_1, \ldots, v_r], v_i \in \mathbb{R}^2$ with some possible empty subset of zero columns $v_i = 0$ and some (cyclically). Consecutive columns $v_r, v_{r+1}, \ldots, v_n$ parallel to each other. Suppose $k = 2$, we have the matrix form

$$\begin{pmatrix} r & q & p & t & 0 & 0 & z & x \\ 0 & 0 & 0 & s & 0 & u & y & \end{pmatrix} \tag{1}$$

after the row reduction and deletion of the pivot columns.

We have

$$\begin{pmatrix} * & * & * & * & 0 & 0 & * & * \\ 0 & 0 & 0 & * & 0 & * & * & \end{pmatrix} \tag{2}$$

Thus, a blocked zero is the one with dot over it and everything to the left of a blocked zero is zero. It is not allowed to have a 0 with a dot above and a dot to the left.

One may assume A has no zero columns then the combinatorial structure is given by a decomposition of the set $[n]$ into a disjoint union of cyclically consecutive intervals $[n] = B_1 \cup \ldots \cup B_r$.

Then the Plücker coordinates Δ_{ij} is strictly positive if i and j belong to two different intervals. B_i's and $\Delta_{ij} = 0$ if i and j are in the same interval.

Remark 5. The closure of Positroid cells $P(\alpha)$ is called Positroid variety denote by $Q(x)$.

The classical example of Positroid variety is the Schubert variety $C(\alpha)$ since Grassmannian is a disjoint union of the schubert cells $C(\alpha), \quad \alpha \in I(k,n)$. It will be ideal to say that Grassmannian is a disjoint union of Positroid cells $P(\alpha)$. The positroid varieties are subvarieties of $G_{r_{k,n}}$ indexed by various posets.

Combinatorial Description of Positroid ideals in a totally nonnegative $G_{r_{k,n}}$

Definition 6. There are several posets that can index the Positroid cells namely;

The Grassmann necklace

The Decorative permutation

The plabic network

The Matroids as well as the Schubert varieties.

Definition 7. Let M be a matroid of rank k on $[n]$. Define a sequence of k-element subset $J(M) = (J_1, J_2, \ldots, J_n)$ by letting J_r be the minimal base of the matroid and $J = (J_1, J_2, \ldots, J_r) \in Jugg(k,n)$.

Lemma 8 (3.20 (Pos, Oh)). *Let $J \in Jugg(k,n)$. The collection M_j is a* *matroid.*

The matroid $M_j := \left\{ I \in \binom{[n]}{k} | J \geq I_r \right\}$ *are called positroids.*

Every positroids is a special matroid that can be represented by totally positive matrices.

Definition 9. Given a Grassmann necklace $I = (I_1, \ldots, I_n)$ define the positroid

$$M_I := \begin{Bmatrix} J \in \binom{[n]}{k} | I_i \leq J \\ \\ \text{for all } i \in [n] \end{Bmatrix}$$

Definition 10. A Grassmann necklace is a sequence $I = (I_1, \ldots, I_n, I_{n+1} = I_i)$ of k- element subset, of $[n]$ such that for all $i \in [n]$

$$I_{i+1} = \begin{Bmatrix} I_i | \{i\} \cup \{j\} \text{ for some } j \in [n] \text{ if } i \in I_1 \\ \\ I_i \qquad\qquad\qquad \text{if } i \notin I_i \end{Bmatrix}$$

I is connected if $I_i \neq I_j$ for $i \neq j$

Theorem 11. *The homogeneous ideal* J *of* $G_{r_{k,n}}$ *is generated by the homogeneous polynomials of the form*

$$\sum_{\sigma \in s(r-k,l)} sgn(\sigma) x(\alpha_1, \ldots, \alpha_k, \alpha_{k+1}^\sigma, \ldots, \alpha_r^\sigma) x(\beta_1^\sigma, \ldots, \beta_l^\sigma, \beta_{1+1}, \ldots, \beta_r). \qquad 1 < k < l < r \tag{3}$$

and where k *and* l *are fixed integers with for every* α, β *in* $I(r, n)$*, the above sum runs over all the shufflings of* $\{\alpha_{k+1}, \alpha_{k+2}, \ldots \alpha_r\}$ *and* $\{\beta_1, \beta_2, \ldots, \beta_l\}$

Proposition 12. *The Positroid ideal J of $G_{r_{k,n}} \geq 0$ is generated by the homogeneous polynomial of the form (3) as in the above theorem*

Proof. Consider the homogeneous coordinate rings $S = k[x_\alpha, \alpha \in \binom{[n]}{k}]$ and $R = k[p_\alpha, \alpha \in \binom{[n]}{k}]$ of $\mathbb{P}(\Lambda^r V)$ and $G_{r_{k,n}} \geq 0$ respectively.

There exist a natural homomorphism

$$\phi : S \longrightarrow R$$

$$x_\alpha \longrightarrow P_\alpha$$

whose kernel is the ideal J. If J' is the ideal generated by the polynomials of the form (3), then it implies that $J' \subset \ker \phi = J$. Hence a surjective homomorphism

$$\Phi : S/J' \longrightarrow R$$

$$\bar{x}_\alpha \longrightarrow P_\alpha$$

Thus we show that $J' = J$ by prooving that Φ is injective. Let F e any nonzero element of S/J' and since S/J' is generated by standard monomials, it then seen that F can be written as a Linear combination of distinct standard monomials. Then $\Phi(F)$ is a linear combination of distinct standard monomials on $G_{r_{k,n}} \geq 0$, sincve standard monomials on $G_{r_{k,n}} \geq 0$ are linearly independent, it follows that $\Phi(F) \neq 0$ and hence $J = J'$ □

Remark 13. A standard monomial on $G_{r_{k,n}} \geq 0$ of length m is a formal expression of the form

$$P_\alpha(1)P_\alpha(2)\ldots P_\alpha(m)$$

where $P_\alpha(i)$ are Plücker coordinates and $\alpha(1), \alpha(2), \ldots, \alpha(m)$ is a standard tableau. Thus two standard monomials are distinct if the corresponding standard tableaus are distinct.

Proof. Let J' be the ideal generated by the set $\{P_\alpha | \alpha \geq 1\}$ and let R_β be the homogeneous coordinate ring of $Q(\beta)$. Then having a natural homomorphism

$$\begin{aligned} \bar{\Phi} :& R/J' \longrightarrow R_\beta \\ & P_\alpha \longrightarrow P_\alpha | Q(\beta) \end{aligned}$$

which is surjective. Since R is generated by standard monomials, so is R/J'. Then it follows exactly in the same manner as in above theorem that $\bar{\Phi}$ is injective and hence $J' = J_\beta$. $\square$

Remark 14. For $n \geq k \geq 0$, the Grassmannian $G_{r_{k,n}}$ over R is the space of k dimensional linear subspaces of $\mathbb{R}^n$ which can be identified with the space of $k \times n$ matrices form projective coordinates on the Grassmannian called the Plücker coordinates, that are denoted by Δ_I where $I \in \binom{[n]}{k}$. The totally nonnegative Grassmannian $G_{r_{k,n}} \geq 0$ which is part of $G_{r_{k,n}}$ is identified with the $k \times n$ matrices whose Plücker coordinates are all totally non-negative. The dimension of $G_{r_{k,n}}$ is $k(n-k)$.

Object - to - Object mappings in a $G_{r_{k,n}} \geq 0$

The following objects are important in the study of Positroid ideals in $G_{r_{k,n}} \geq 0$.

- The Grassmann necklace
- The Decorative permutation
- The Plabic graph/reduced Plabic graph

These objects which are in one-to-one correspondence with the positroid ideals help to establish a condition of weak seperation of the positroid ideals.

Definition 15. A decorated permutation $\pi = (\pi, \mathrm{col})$ is a permutation $\pi \in \sigma_n$ together with coloring fuction col from the set of fixed points $\{i|\pi(i) = i\}$ to $\{1, -1\}$.

For $i, j \in [n], \{i, j\}$ forms an alignment in π if $i, \pi(i), \pi(j), j$ are cyclically ordered (and all distict). The number of alignment in π is denoted by $al(\pi)$ and the length $l(\pi)$ is defined to be the $k(n-k) + al(\pi)$.

0.1 *Linking between a Grassmann necklace and a decorative permutation

Given a Grassmann necklace I, denote $\pi_I = (\pi_1, \mathrm{col}I)$ as follows;

- if $I_{i+1} = I_i|\{i\} \cup \{j\}$ for $i \neq j$, then $\pi(i) = j$

  if $I_{i+1} = I_i$ and $i + I_i$ (resp; $i \in I_i$) then $\pi(i) = i$ and $\mathrm{col}(i) = 1$ (resp., $\mathrm{col}(i) = -1$)

Definition 16. A plabic graph (Planar bicoloured graph) is a planar undirected graph G drawn inside a disk with vertices coloured in black or white colour. The vertices on the boundary vertices are labelled in clockwise order by the elements of $[n]$.

Definition 17. A strand in a plabic graph G is a directed path that satisfies the " rules of the road" at every black vertex it makes a sharp right turn, and at every white vertex it makes a sharp left turn.

Definition 18. A plabic graph G is called reduced if the following holds.

- A strand cannot be a closed loop in the interior of G.
- If a strand passes through the same edge twice, then it must be a simple loop that starts and ends at the boundary leaf.
- Guven any two strands, if they have two edges e and e' in common, then one strand should be directed from e to e' while the other strand should be directed from e' to e.

Any strand connects two boundary vertices in a reduced plabic graph G, Linking between a decorative permutation and plabic graph. The associated decorative permutation in a plabic is called a decorated strand permutation denote by $\pi_G = (\pi_G, \mathrm{col}_G)$ with G for which $\pi_G(s) = i$ if the strand that starts at a boundary vertex j ends at a boundary vertex i,we labeled such strand i.

- if $\pi_G(i) = i$, then i must be connected to a boundary leaf v and $\mathrm{col}(i) = +1$ if v is white and $\mathrm{col}(i) = -1$ if v is black.

Connected Components of the Objects

The connected components of these objects namely: the Decorative permutation π, the Grassmann necklace I and the Positroid M_I will have the subsets of $[n]$ that inherit their cyclic order from $[n]$ as their ground set.

Definition 19. Let π be a decorated permutation

Let $[n] = \cup S_i$ be the finest non-crossing partition of $[n]$ such that if $i \in S_j$ then $\pi(i) \in S_j$.

Let $\pi(j)$ be the restriction of π to the set S_j and let $I(j)$ be the associated Grassmann necklace on the ground set S_j, for $j = 1$.We call $\pi(j)$ the connected components of π and $I(j)$ the connected components of I. We say that π and I are connected if they have exactly one connected component.

Note: Each fixed point of π(of either color) form a connected components.

Definition 20. Let $[n] = S_1 \cup S_2 \cup \cdots \cup S_r$ be a partition of $[n]$ into disjoint subsets. We say that $[n]$ is non-crossung if for any circularly ordered (a, b, c, d) we have $\{a, c\} \subseteq S_i$ and $\{b, d\} \subseteq S_j$ then $i = j$

Lemma 21. *The decorative permutation is disconnected if and only if there are two circular intervals $[i, j)$ and $[j, i)$ such that π takes $[i, j)$ and $[j, i)$ to themselves.*

Proof. If such intervals exist, then the pair $[n] = [i, j)$ and $[j, i)$ is a non-crossing partition preserved by π. So there is a non trivial non crossing partition preserved by π and π is not connected.

Conversely, any non trivial non crossing permutation can be coarsened to a pair of intervals of this form so if π is disconnected, then there is a pair of interval of this form $\square$

Lemma 22. *A Grassmann necklace $I = (I_1, \ldots, I_n)$ is connected if and only if the sets $I_1, \ldots, I_n$ are all distinct.*

Proof. If π is disconnected, then let $[i, j)$ and $[j, i)$ be as in above lemma. As we change from I_i to I_{i+1} to I_{i+2} to $\cdots$ to I_j, each element of $[i, j)$ is removed once and is added back in once. So $I_i = I_j$.

Conversely, suppose that $I_i = I_j$. As we change from I_i to I_{i+1} to I_{i+2} and so forth up to I_j, each element of $[i, j)$ is removed Once. In other to have $I_i = I_j$, each elements of $[i, j)$ must be added back in once. So π takes $[i, j)$ to itself. $\square$

Proposition 23. The following conditions are equivalent

(i) Every sequence I of k- element subsets of $[n]$ are finitely generated.

(ii) Every non-empty set of I in $\binom{[n]}{k}$ has a maximal element.

(iii) Every ascending chain (by inclusion) of the set $I = (I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n)$ is stationary.

Proof. $(i) \Rightarrow (ii)$ Let $\sum_i I$ be the family of every set of sequence $I = (I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n)$. Since $\sum_i I$ is non-empty, it has a maximal element say n.

If $I_i \neq I$, consider $I_{i+1}, i \in I, i \notin I$ which is obtain from I_i by deleting $\{i\}$ and adding another element $\{j\}$. This implies is finitely generated , hence a contradiction. Thus, $I_{i+1} = I$, it implies is finitely generated.

$(ii) \Rightarrow (iii)$ Contrarily, if there is a non-empty set I in $\sum_i I$ with no maximal element, then inductively we construct a non-terminating sequence in $\sum_i I$ and thus the set I in $\binom{[n]}{k}$ has a maximal element say I_n.

$(iii) \Rightarrow (i)$ Let $I_1 \subset I_2 \subset \cdots$ be an increasing sequence of every $k-$ element subset in $[n]$, then $I = \bigcup_i^n I_i$, hence $I = (I_i, I_{i+1}, \ldots, I_n)$. This implies that is finitely generated.

If $i \in I$, it shows that $I_{i+1} = I_i$ since I_{i+1} is contained in I_i by deleting $\{i\}$ and adding another element $\{j\}$. Continuing in the same manner, we have that $I_{i+1} = I_i \cdots = I_n$ where n is the maximal element in $\sum_i(I)$. Hence $I_i = I_n$, terminates.

$\square$

Proposition 24. *If a Grassmann necklace I satisfies the above conditions, then the Positroid ideal indexed by I is a Noetherian.*

Proof. Suppose Grassmann necklace I satisfies the condition above, then we define the set of Positroid ideals indexed by the Grassmann necklace as follows;

$$P(I) = \left\{ J \in \binom{[n]}{k} | I \leq J \right\}$$

Since this set is non-empty, it contains the maximal element say n, then for every $i \in I$, there exists $j \in j$ such that if $I_i \neq I$ we have I_{i+1} gotten from I_i by deleting $\{i\}$ once and adding $\{j\}$ at most once if $I_{i+1} \neq I$, continue in that manner until we obtain $I_n = I$. This implies finitely generated. Thus is Noetheriean

Conversely, if $P(I)$ is Noetherian then the set of sequence $I = (I_1 \subseteq I_2 \subseteq \cdots \subseteq I_r)$ with its corresponding positroid $J \in \binom{[n]}{k}$ contains the maximal element such that $I_i \leq J$. If $I_i \neq J$ we obtain I_{i+1} from I_i by deleting an element and adding another element at most once. This implies is finitely generated. Continuity in this manner we have the set of chains $I = I_i \longrightarrow I_{i+1} \longrightarrow \cdots \longrightarrow I_n \longrightarrow I_{n+1} \longrightarrow \cdots J$. Thus $I_n = J$, hence terminates. $\square$

Weak separation of the combinatorial objects of study: Grassmann necklace, Decorative permutation, Plabic graph and ideals of Positroid J.

In this section, we try to equip the condition of weak separation of those objects which have s bijective correspondence with the Positroid ideals of $G_{r_{k,n}} \geq 0$.

We recall a definition of Grassmann necklace $I = (I_1, \ldots, I_n)$, we define the Positroid ideal $P(I)$ as follows

$$P(I) = \begin{bmatrix} J \in \binom{[n]}{k} | I_i \leq J \\ \\ \text{for all } i \in [n] \end{bmatrix}$$

for a Grassmann necklace $I = (I_1, \ldots, I_n)$ corresponding with the $P(I)$, a collection C inside the $P(I)$ is said to be weakly separated and $I \subseteq C \subseteq P(I)$, if C is maximal among the weakly separated collections in P_I, then C is called a maximal weakly separated collection.

Proposition 25. *For any Grassmann necklace I, we have $I \subseteq P(I)$ and I is weakly separated.*

Proof. for every i and j in $[n]$, we must show that $I_i \leq I_j$ and $I_i || I_j$. By definition, I_{k+1} is either obtained from I_k by deleting k and adding another element or else $I_{k+1} = I_k$. As we do the changes

$$I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow \cdots \longrightarrow I_n \longrightarrow I_1$$

we delete each $k \in [n]$ at most once in the transformation $I_k \longrightarrow I_{k+1}$. This implies that we add each k at most once. Let us show that $I_j|I_i \subseteq [j, i)$. Suppose that this is not true and there exists $k \in (I_j|I_i) \cap [i, j)$.
Suppose that this is not true and there exists $k \in (I_j|I_i) \subseteq [j, i)$. Note that $I_{k+1} \neq I_k$ otherwise k belongs to all elements of the Grassmann necklace or k does not belong to all elements of the necklace. Consider the sequence of changes

$$I_i \longrightarrow I_{i+2} \longrightarrow \cdots \longrightarrow I_k \longrightarrow I_{r+1} \longrightarrow \cdots \longrightarrow I_j$$

we should have $k \notin I_i, k \in I_k, k \notin I_{k+1}, k \in I_j$. Thus k should be added twice as we go from I_i to I_k and as we go from I_{k+1} to I_j. We get a contradiction.Thus, $I_j|I_i \subseteq [J, i)$ and similarly $I_i|I_j \subseteq [i, j]$, we conclude that $I_i \leq iI_j$ and $I_i||I_j$ as desired. $\square$

Definition 26. Let π be a decorated permutation. Let $[n] = \cup S_i$ be the finest non crossing partition of $[n]$ such that if $i \in S_j$ then $\pi(i) \in S_j$.

Let $\pi(J)$ be the restricting of π to the set S_j and let $I(j)$ be the associated Grassmann necklace on the ground set S_j, for $j = 1, \ldots, r$. We call $\pi(j)$ the connected components of π and $I(j)$ the connected components of I. Then we say that π and I are connected if they have exactly one connected component.

Remark 27. Each fixed point of π (of either color) form a connected components.

Suppose $I_i = I_j$ for some $i \neq j$, then we let $I' = [i,j) \cap I_i \quad J' = j \cap [i,j) \qquad I^2 = [j,i) \cap I_i \quad J^2 = J \cap [j,i)$ for all j in M.

$|J| = k$ and $|J \cap [i,j)| = k^i$

$|J \cap [j,i)| = k^2$

Proposition 28. *The matroid M is a direct sum of two matroid M' and M^2 supported on the ground set $[i,j)$ and $[j,i)$ having rank k' and k^2. In otherwords, there are matroids M' and M^2 such that J is in M iff $J \cap [i,j)$ is in M' and $J \cap [j,i)$ in M^2*

Proposition 29. *for $k \in [i,j]$ the set i_k is of the form $J \cup I^2$ for some $J \in M$ for $k \in [J,i)$, the set I_k is of the form $I' \cup J$ for some $J \in M^2$.*

Proof. Consider the case that $k \in [i,j]$, the other case is similar. Recall that I_k is the minimal element of M since $M = M' \oplus M^2$. We know that $I_k = J' \cup J^2$ where J^r is the $\leq_k$ minimal element of M'. BUt $[j,i)$ the other $\leq_i$ and $\leq_k$ coincide so J^2 is the minimal element of M^2 namely I^2. □

Lemma 30. *Let $J' = j' \cup j^2 \in M$.*

If I is weakly separated from $I' \cup I^2$, then either $I' = J'$ or $I^2 = J^2$.

Proof. Suppose on the contrary, that $J' \neq I'$ and $J^2 \neq I^2$. Since $I' \leq_i J'$ there are a and $b \in [i,j)$ with $i \leq_i a \leq_i b$ such that $a \in I'|J'$ and $b \in J'|I'$. Similarly there are c and $d \in [j,i)$ with $i \leq_j c \leq_j d$ such that $c \in I^2|J^2$ and

d in $J^2|I^2$.

Then a and c are in $I' \cup I^2|J' \cup J^2$ while b and d are in $J' \cup J^2|I' \cup I^2$. So $I' \cup I^2$ and j are not weakly separated. $\square$

Proposition 31. *If C is a weakly separated collection in M, then there are weakly separated collections c' and c^2 in M' and M^2*
such that

$$C = \{J \cup I^2; J \in C' \cup \{I' \cup J; J \in C^2\}\}$$

Conversely, if c' and c^2 are weakly separated collections in M' and M^2, then the above formula defines a weakly separated collection in M. The collection C is maximal if and only if c' and c^2 are.

Proof. First, suppose that C is a weakly separated collection in M. Since $I \in C$, we have that $I' \cup I^2 \in C$. Since every $j \in C$ is either of the form $J' \cup I^2$ or $I' \cup J^2$. Let C^r be the collection of all sets J^r for which $J^r \cup I^{s-r}$ is in C. The condition that C is weakly separated implies that C^r is the condition that $I \subseteq C \subseteq M$ implies $I^r \subseteq C^r \subseteq M^r$. So C^r is a weakly separated collection in M^r and it is clear that C is brought from C' and C^2 in the indicated manner.

Conversely, it is easy to check that if c' and c^2 are weakly separated collections in M' and M^2, then the above formula gives a weakly separated collection in M.

Finally, if $C \subseteq C'$ with C' a weakly separated collection in M, then either $c' \subseteq (c')$ or $c^2 \subseteq (c')$. So if c is not maximal either c' or c^2 is not. The converse is similar. $\square$