Reference: some basic distributions

An excellent reference for probability distributions is: Evans, Merran, Nicholas Hastings, and Brian Peacock. 2000. Statistical Distributions, 3rd Edition. John Wiley & Sons. New York.

Continuous

Expected, or mean

$$E[Z] = \int z\, f(z)\, dz$$

Variance

$$\mathrm{VAR}(Z) = E[(Z - E[Z])^2] = \int (z - E[z])^2\, f(z)\, dz$$

For continuous variables, the **probability density function** is the probability of the value *z* given the parameters

Uniform

Uniform Distribution:
a uniform distribution on the interval [a,b] where a < b
probability density function:

$f(z) = \frac{1}{b-a}$, where a ≤ z ≤ b

mean: $\frac{b+a}{2}$

variance: $\frac{(b-a)^2}{12}$

r

```
dunif(z, min = a, max = b)

# generate 1 uniform random number over the range 1.4 to 2.4
runif(1, 1.4, 2.4)
```

mathematica

```
f[z_, a_, b_] := 1/(b - a)

(* random number over range 0,1 *)
Random[]

0.0407265

(* random number of specified range *)
```

```
Random[Real, {3, 6}]
```

```
4.87348
```

■ Normal

Normal Distribution:
has two parameters: m is the location parameter, in this case the mean; σ is the scale parameter, in this case the standard deviation
probability density function:

$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-m)^2}{2\sigma^2}\right)$$

mean: m
variance: σ^2
note: plotting $f(z)$ will give you the familiar bell curve

■ r

probability density function

```
dnorm (z, mean, sd)
# plot the pdf
curve (dnorm (x, 4, 1), 0, 7)
```

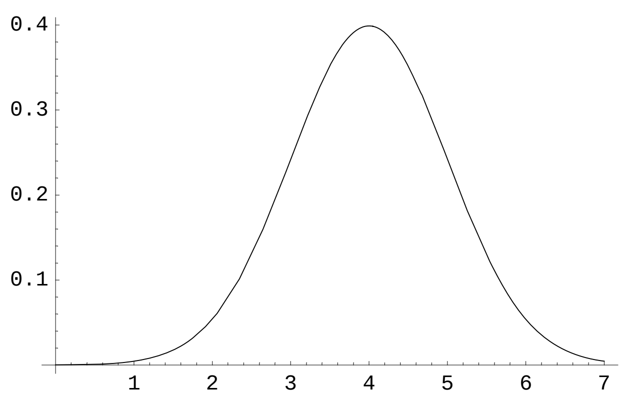
random number generation, where n is the number random numbers to generation. In other words, n = 1 will create 1 random number.

```
rnorm (n, mean, sd)
```

■ mathematica

```
f[z_, m_, σ_] := 1/Sqrt[2 π σ^2] Exp[-(z - m)^2/(2 σ^2)]
```

```
Plot[f[z, 4, 1], {z, 0, 7}];
```



Show that pdf integrates to 1

```
Integrate[f[z, 1, 1], {z, -∞, ∞}]
```

```
1
```

```
<< Statistics`ContinuousDistributions`
(* normal random number with mean of 3 and standard deviation 2 *)
Random[NormalDistribution[3, 2]]
```

```
1.87451
```

■ Lognormal

Lognormal Distribution:
has two parameters: *m* is the location parameter; σ is the shape parameter
probability density function:

$$f(z) = \frac{1}{z\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(\ln(z)-\log(m))^2}{2\sigma^2}\right), \text{ where } z > 0,\ m > 0, \text{ and } \sigma > 0$$

mean: $m\exp\left(\frac{\sigma^2}{2}\right)$
median: m
σ^2 is the variance of the ln(z)
variance: $m^2\exp(\sigma^2)(\exp(\sigma^2)-1)$
note: This describes a variable in which the logarithm of the variable is a normal distribution. Whereas the normal distribution is additive, the lognormal distribution is multiplicative.

■ r

```
dlnorm (z, meanlog = m, sdlog = σ)
```

R uses a slightly different form of the lognormal model. Use the following form so that *m* is defined in the same way as described in the *f*(*z*) equation above.

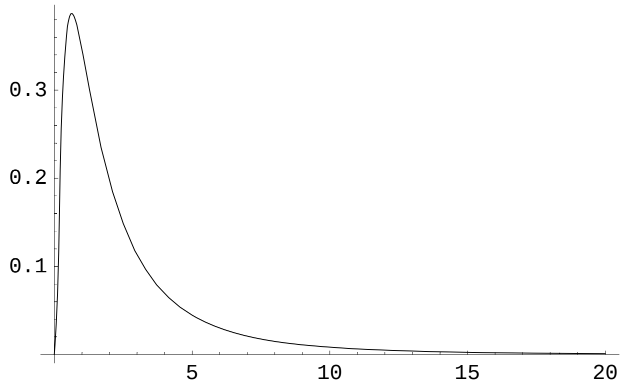
```
dlnorm (z, log ( m), σ)
```

```
# generate random number
rlnorm (1, meanlog = 1, sdlog = 1.2)
```

■ mathematica

$$\mathtt{f[z\_, m\_, \sigma\_] :=} \frac{1}{z\sqrt{2\pi\sigma^2}}\,\mathtt{Exp}\left[-\frac{(\mathtt{Log[z]} - \mathtt{Log[m]})^2}{2\sigma^2}\right]$$

```
Plot[f[z, 1.7, 1], {z, 0, 20}];
```



Show that pdf integrates to 1

```
Integrate[f[z, 1, 1], {z, 0, ∞}]
```

1

Derive expected value

```
Integrate[z f[z, m, σ], {z, 0, ∞}, Assumptions → {σ > 0, m > 0, σ ∈ Reals, m ∈ Reals}]
```

$e^{\frac{\sigma^2}{2}}\, m$

Derive variance

```
Integrate[
 (z - Integrate[z f[z, m, σ], {z, 0, ∞}, Assumptions → {σ > 0, m > 0, σ ∈ Reals, m ∈ Reals}])^2
  f[z, m, σ], {z, 0, ∞}, Assumptions → {σ > 0, m > 0, σ ∈ Reals, m ∈ Reals}]
```

$e^{\sigma^2} \left(-1 + e^{\sigma^2}\right) m^2$

```
<< Statistics`ContinuousDistributions`
(* lognormal random number with the first number being the Log[m]
  and the second number being σ.  In the case below, m = 3 and σ = 2 *)
Random[LogNormalDistribution[Log[3], 2]]
```

6.94065

■ Exponential

Exponential Distribution:
has three parameters: *m* is the location parameter; σ is the shape parameter
probability density function:

$f(z) = a \, \mathrm{Exp}[-a\, z]$, where z > 0

mean: $\frac{1}{a}$
variance: $\frac{1}{a^2}$
note: Continuous distribution analog of Poisson. Can be used for dispersal distance. Widely used in Baysian Analysis

■ r

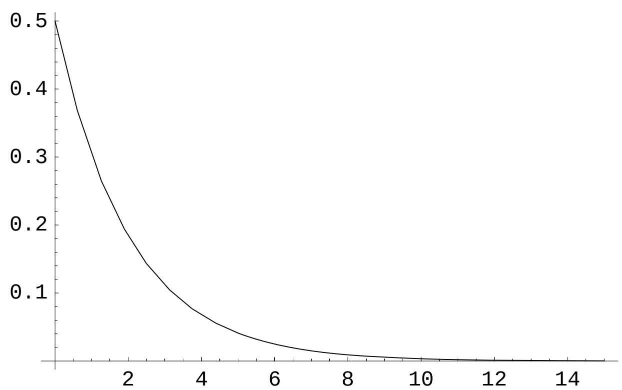
```
dexp (z, rate = a)
```

```
# generate random number
rexp (1, rate = a)
```

■ mathematica

```
f[z_, a_] := a Exp[-a z]
```

```
Plot[f[z, 0.5], {z, 0, 15}, PlotRange → All];
```




```
<< Statistics`ContinuousDistributions`
(* exponential random number with a = 0.5 *)
Random[ExponentialDistribution[0.5]]
```

```
2.1514
```

Gamma

Gamma Distribution:
has two parameters: *a* is the rate parameter; *n* is the shape parameter
probability density function:

$$f(z) = \frac{a^n}{\Gamma(n)} \text{Exp}[-a\, z]\, z^{n-1}, \text{ where z} > 0$$

mean: $\frac{n}{a}$
variance: $\frac{n}{a^2}$
note: can have a long tail. When $n = 1$, the gamma distribution becomes the exponential distribution. When n = degrees of freedom / 2 and $a = 2$, the gamma distribution becomes the chi-square distribution.

r

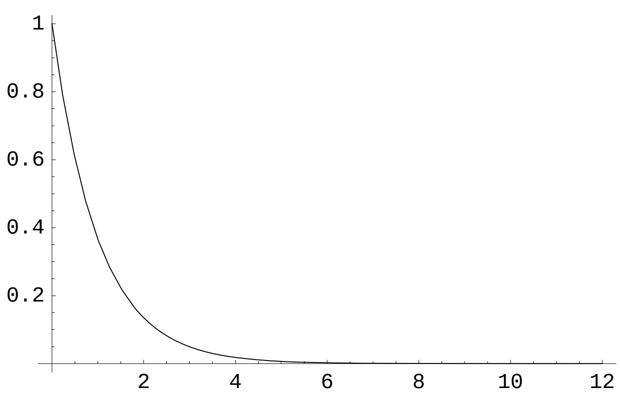
```
dgamma (z, shape = n , rate = a)
```

```
# generate random number
rgamma (1, shape = n, rate = a)
```

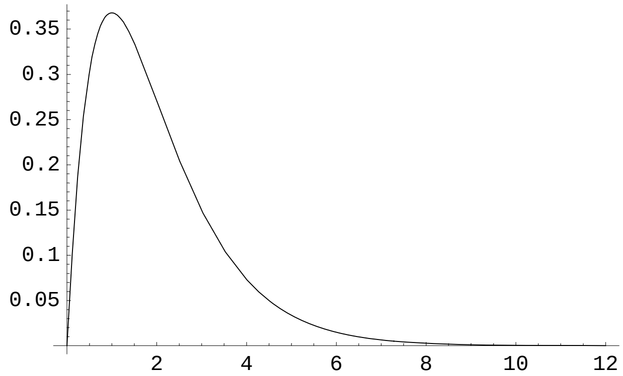
mathematica

```
f[z_, a_, n_] := a^n / Gamma[n] Exp[-a z] z^(n-1)
```

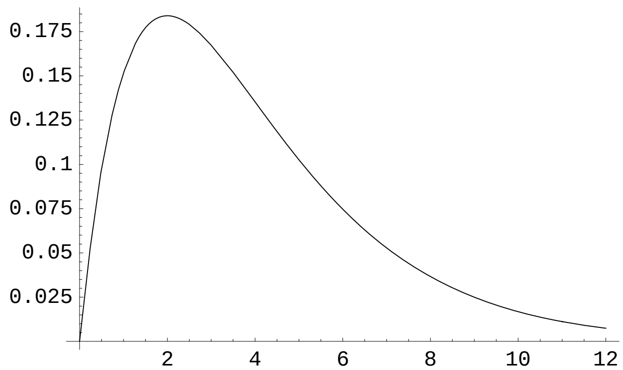
```
Plot[f[z, 1, 1], {z, 0, 12}, PlotRange → All];
```



```
Plot[f[z, 1, 2], {z, 0, 12}, PlotRange → All];
```




```
Plot[f[z, 0.5, 2], {z, 0, 12}, PlotRange → All];
```



Derive expected value

```
Integrate[z f[z, a, n], {z, 0, ∞}, Assumptions → {a > 0, n > 0, a ∈ Reals, n ∈ Reals}]
```

$$\frac{n}{a}$$

Derive variance

```
Integrate[
 (z - Integrate[z f[z, a, n], {z, 0, ∞}, Assumptions → {a > 0, n > 0, a ∈ Reals, n ∈ Reals}])^2
  f[z, a, n], {z, 0, ∞}, Assumptions → {a > 0, n > 0, a ∈ Reals, n ∈ Reals}]
```

$$\frac{n}{a^2}$$

```
<< Statistics`ContinuousDistributions`
(* gamma random number with n = 2 and a = 0.5 *)
Random[GammaDistribution[2.0 , 1 / 0.5]]
```

```
2.82494
```

■ Chi-Square

Note: this section is a bit incomplete

Chi-Square Distribution: A squared normal distribution is a chi-square distribution.

probability density function for 1 degree of freedom:

$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2 z}} \exp\left(-\frac{z}{2\sigma^2}\right)$$

probability density function for n degrees of freedom:

$$f(z) = \frac{1}{\sigma^n 2^{(n/2)} \Gamma(\frac{n}{2})} z^{(n/2)-1} \exp\left(-\frac{z}{2\sigma^2}\right), \text{ for } z \geq 0$$

mean: $n\,\sigma^2$
variance: $2\,n\,\sigma^4$

notes:
- n can also be thought of as a shape parameter in the context of a general distribution
- often assume unit variance, $\sigma = 1$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

■ r

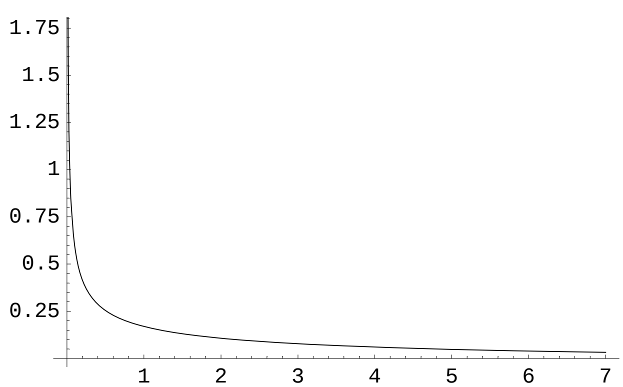
```
dchisq ()
```

■ mathematica

```
f[z_, σ_] := 1/Sqrt[2 π σ^2 z] Exp[-z/(2 σ^2)]
```

$$f[z\_, \sigma\_] := \frac{1}{\sqrt{2\pi\sigma^2 z}} \text{Exp}\left[-\frac{z}{2\sigma^2}\right]$$

```
Plot[f[z, 2.1], {z, 0, 7}];
```



Show that pdf integrates to 1

```
Integrate[f[z, 2.1], {z, 0, ∞}]
```

1.

```
<< Statistics`ContinuousDistributions`
(* Chi-Square distributed random number with 1 degree of freedom *)
Random[ChiSquareDistribution[1]]
```

0.111756

Add non-centrality, or location, parameter with a single degree of freedom, although no scale parameter
mean: $1 + \mu$

Two forms are given below

$$f[z\_, \mu\_] := \text{Exp}\left[-\frac{z+\mu}{2}\right] \frac{1}{2} \left(\frac{z}{\mu}\right)^{\frac{-1}{4}} \text{BesselI}\left[-1/2, \sqrt{\mu z}\right]$$

$$f[z\_, \mu\_] := \text{Hypergeometric0F1}\left[\frac{1}{2}, \frac{1}{4} z \mu\right] \frac{1}{\sqrt{2 \pi z}} \text{Exp}\left[-\frac{z+\mu}{2}\right]$$

```
NIntegrate[ f[z, 2.0], {z, 0, 100000}]
```

1.

```
NIntegrate[z f[z, 2.0], {z, 0, 100000}]
```

3.

```
Integrate[z f[z, μ], {z, 0, ∞}, Assumptions → {μ > 0, μ ∈ Reals}]
```

$1 + \mu$

$$\text{Hypergeometric0F1}\left[\frac{1}{2}, \frac{1}{4} z \mu\right]$$

$$\text{Cosh}\left[\sqrt{z \mu}\right]$$

$$\frac{1}{2} \left(\text{Exp}\left[\sqrt{z \mu}\right] + \text{Exp}\left[-\sqrt{z \mu}\right]\right) \frac{1}{\sqrt{2 \pi z}} \text{Exp}\left[-\frac{z+\mu}{2}\right]$$

$$\left(\text{Exp}\left[\sqrt{z \mu}\right] + \text{Exp}\left[-\sqrt{z \mu}\right]\right) \frac{1}{2\sqrt{2 \pi z}} \text{Exp}\left[-\frac{z+\mu}{2}\right]$$

$$\frac{1}{2\sqrt{2 \pi z}} \text{Exp}\left[\sqrt{z \mu} - \frac{z+\mu}{2}\right] + \frac{1}{2\sqrt{2 \pi z}} \text{Exp}\left[-\sqrt{z \mu} - \frac{z+\mu}{2}\right]$$

■ Bimodal Normal

Normal Distribution:
has two parameters: m is the location parameter, in this case the mean; σ is the scale parameter, in this case the standard deviation
probability density function:

$$f(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(z-m)^2}{2\sigma^2}\right)$$

mean: m
variance: σ^2
note: plotting $f(z)$ will give you the familiar bell curve

Here I extend that to a bimodal normal distribution. I am assuming that I can just divide each unimodal pdf by 2, but it would be nice to confirm this

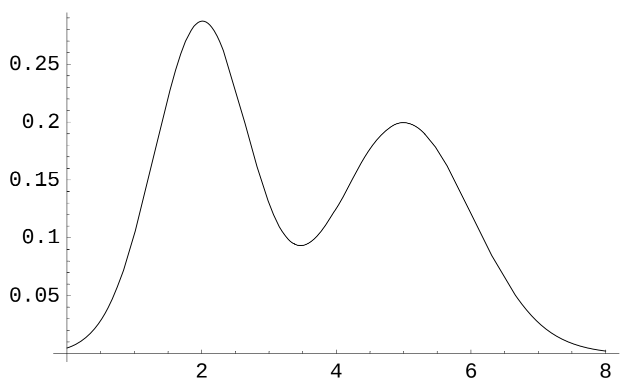
■ r

```
# assuming equal weighting to each mode
(dnorm (z, m1, s1) + dnorm (z, m2, s2)) / 2

# weighting c for modes
(1 / (1 + exp (-c))) * dnorm (z, m1, s1) + (1 / (1 + exp (c))) dnorm (z, m2, s2)
```

■ mathematica

$$\texttt{f[z\_, m1\_, }\sigma\texttt{1\_, m2\_, }\sigma\texttt{2\_] :=} \frac{1}{2\sqrt{2\pi\sigma 1^2}} \,\mathrm{Exp}\left[-\frac{(z-m1)^2}{2\sigma 1^2}\right] + \frac{1}{2\sqrt{2\pi\sigma 2^2}} \,\mathrm{Exp}\left[-\frac{(z-m2)^2}{2\sigma 2^2}\right]$$

```
Plot[f[z, 5, 1, 2, 0.7], {z, 0, 8}];
```



Show that pdf integrates to 1

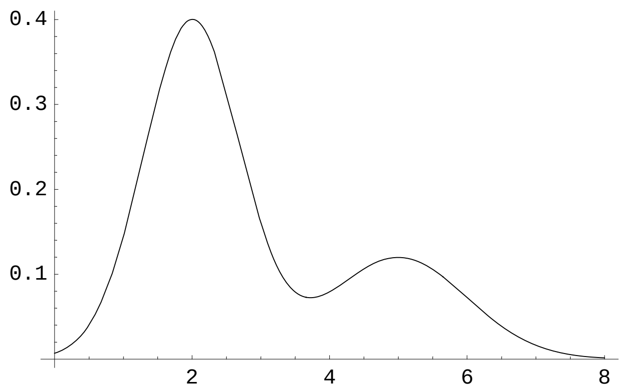
```
Integrate[f[z, 5, 1, 2, 0.7], {z, -∞, ∞}]

1.
```

Therefore the support function is

```
(* bimodal normal distribution *)
sf[z_, m1_, σ1_, m2_, σ2_] :=
  Log[2 √(2 π)] - Log[1/√(σ1^2) Exp[-(z - m1)^2/(2 σ1^2)] + 1/√(σ2^2) Exp[-(z - m2)^2/(2 σ2^2)]];

bimodalRandom[m1_, σ1_, m2_, σ2_] := Block[{indef, x},
  indef[x_Real] := NIntegrate[bimodalPDF[z, m1, σ1, m2, σ2], {z, -∞, x}];
  x /. FindRoot[Random[] == indef[x], {x, (m1 + m2)/2 * 0.5, (m1 + m2)/2 * 1.0}]]
```

Perhaps I should not assume an equal distribution of each mode, in which case I add the parameter c which is between [0,1] and gives the percent of each mode in generating the overall distribution

```
f[z_, m1_, σ1_, m2_, σ2_, c_] := c/√(2 π σ1^2) Exp[-(z - m1)^2/(2 σ1^2)] + (1 - c)/√(2 π σ2^2) Exp[-(z - m2)^2/(2 σ2^2)]

Plot[f[z, 5, 1, 2, 0.7, 0.3], {z, 0, 8}];
```



Show that pdf integrates to 1

```
Integrate[f[z, 5, 1, 2, 0.7, 0.3], {z, -∞, ∞}]

1.

(* bimodal normal distribution *)
sf[z_, m1_, σ1_, m2_, σ2_, c_] :=
  Log[√(2 π)] - Log[c/√(σ1^2) Exp[-(z - m1)^2/(2 σ1^2)] + (1 - c)/√(σ2^2) Exp[-(z - m2)^2/(2 σ2^2)]];
```

■ Beta

Beta Distribution:
This distribution is good for modeling fractions because it is constrained to the domain between 0 and 1. It has two parameters: both *p* and *q* are shape parameters with *p*>0 and *q*>0, but *z* is restrict to the range (0,1)

probability density function:

$f(z) = \frac{z^{p-1}(1-z)^{q-1}}{B(p,q)}$, *B* is the beta function, $B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,du$)

$f(z) = \frac{z^{p-1}(1-z)^{q-1}}{\int_0^1 u^{p-1}(1-u)^{q-1}\,du}$

mean: $\frac{p}{p+q}$

variance: $\frac{p\,q}{(p+q)^2\,(p+q+1)}$

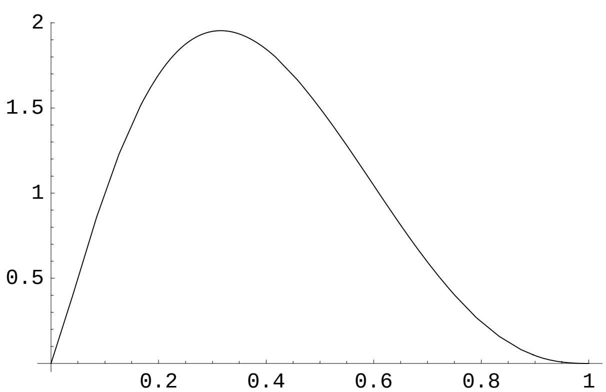
■ r

```
dbeta (z, shape1 = p, shape2 = q)

# random number generation
rbeta (1, shape1 = p, shape2 = q)
```

■ mathematica

$$f[z\_, p\_, q\_] := \frac{z^{p-1}\,(1-z)^{q-1}}{\mathrm{Beta}[p, q]}$$

```
Plot[f[z, 2.2, 3.6], {z, 0, 1}];
```



Therefore the support function is

```
(* beta distribution *)
sf[z_, p_, q_] := Log[Beta[p, q]] - (p - 1) Log[z] - (q - 1) Log[(1 - z)]
```

The Beta distribution can be transformed to be in terms of a mean parameter and variance parameter instead of *p* and *q*. This can be more useful because one is often interested in modeling the mean.

```
substiution = {q → μ/σ^2 (μ (1 - μ) - σ^2) (1 - μ)/μ, p -> μ/σ^2 (μ (1 - μ) - σ^2)};
```

$$f[z\_, \mu\_, \sigma\_] := \frac{(1-z)^{-1+\frac{(1-\mu)\left((1-\mu)\mu-\sigma^2\right)}{\sigma^2}}\, z^{-1+\frac{\mu\left((1-\mu)\mu-\sigma^2\right)}{\sigma^2}}}{\mathrm{Beta}\left[\frac{\mu((1-\mu)\mu-\sigma^2)}{\sigma^2}, \frac{(1-\mu)((1-\mu)\mu-\sigma^2)}{\sigma^2}\right]};$$

$$sf[z\_, \mu\_, \sigma\_] := \left(1 - \frac{(-1+\mu)(-\mu+\mu^2+\sigma^2)}{\sigma^2}\right) \mathrm{Log}[(1-z)] + \left(\frac{-\mu^2+\mu^3+\sigma^2+\mu\,\sigma^2}{\sigma^2}\right) \mathrm{Log}[z] + \mathrm{Log}\left[\mathrm{Beta}\left[-\frac{\mu(-\mu+\mu^2+\sigma^2)}{\sigma^2}, \frac{(-1+\mu)(-\mu+\mu^2+\sigma^2)}{\sigma^2}\right]\right];$$

where $\mu \in [0,1]$
and $\sigma^2 < \mu(1-\mu)$

mean: μ
variance: σ^2

The Beta distribution can be transformed to be in terms of a mean parameter and dispersion parameter, $p = s(1-m)$ and $q = s\, m$. This just scales p and q without the specific constraints on the variance version. Here $m \in (0,1)$, $s \in (0,\infty)$.

■ Bimodal Beta

extend the Beta Distribution by adding 2 pdf's together and dividing by 2
probability density function:

$$f(z) = \frac{z^{p-1}(1-z)^{q-1}}{2\,B(p,q)} + \frac{z^{r-1}(1-z)^{s-1}}{2\,B(r,s)},\ B \text{ is the beta function, } B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,du)$$

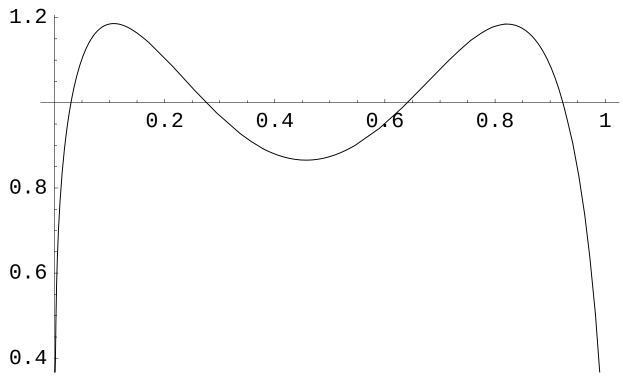
mean:
variance:

■ mathematica

$$f[z\_, v\_, w\_, x\_, y\_] := \frac{z^{v-1}(1-z)^{w-1}}{2\,\mathrm{Beta}[v, w]} + \frac{z^{x-1}(1-z)^{y-1}}{2\,\mathrm{Beta}[x, y]}$$

```
f[0.4, 2.2, 3.6, 2.2, 3.6]
1.8448
```

```
Plot[f[z, 1.3, 3.6, 4.1, 1.6], {z, 0, 1}];
```




```
Integrate[f[z, 1.3, 3.6, 4.1, 1.6], {z, 0, 1}]
```

```
1.
```

Therefore the support function is

```
(* bimodal beta distribution *)
```

$$\texttt{sf[z\_, v\_, w\_, x\_, y\_] := -Log}\left[\frac{z^{v-1}\,(1-z)^{w-1}}{2\,\texttt{Beta[v, w]}}+\frac{z^{x-1}\,(1-z)^{y-1}}{2\,\texttt{Beta[x, y]}}\right]$$

Discrete

Expected, or mean

$E[Z] = \sum_z z\, f_z$

Variance

$\mathrm{VAR}(Z) = E[(Z - E[Z])^2] = \sum_z (z - E[z])^2\, f_z$

where Z is a random variable, z is a particular instance of that variable, f_z is the frequency of occurance of z.

For discrete variables, the **probability distribution function** is the probability of the value z given the parameters

Binomial

Binomial Distribution:
has two parameters: N is the number of times the trial was conducted in which one of two outcomes can occur. p is the probability of one outcome, and z is the total number of times that outcome occurs. For example, flip a coin 10 times and count the number of heads. The result would be a binomial distribution, and if the coin is fair, $p = 0.5$. Whereas the other discrete distributions are used to model discrete observations of continuous processes, the binomial models discrete processes.
probability distribution function:

$$\Pr\{Z = z\} = \left(\frac{N!}{(N-z)!\, z!}\right) p^z (1-p)^{N-z}$$

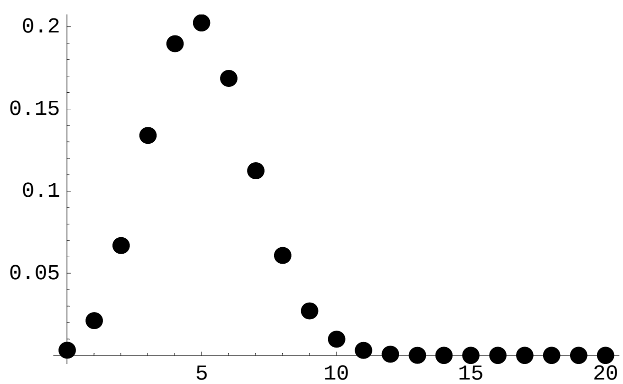
mean: $N\,p$
variance: $N\; p\,(1-p)$

r

```
dbinom (z, n, p)
plot (dbinom (seq (1 : 20), 20, 0.3))

# random number generation
rbinom (1, n, p)

# see function description below
curve (1 / (1 + exp (-x)), -10, 10)
```

mathematica

```
f[z_, n_, p_] := n! / ((n - z)! z!) p^z (1 - p)^(n-z)

ListPlot[Table[{z, f[z, 20, 0.25]}, {z, 0, 20}], PlotStyle → PointSize[0.03]];
```



Sums to 1

```
Plus @@ Table[f[z, 20, 0.3], {z, 0, 20}]

1.
```

```
<< Statistics`DiscreteDistributions`
(* binomial random number with n = 3 and p = 0.5 *)
Random[BinomialDistribution[3 , 0.5]]
```

```
1
```

Here is a useful function for modeling a probability. The funciton is constrained to a range between 0 and 1 for all real values of $f(x)$, both positive and negative. This function is derived by the following logic:

1) Probabilities $\in [0,1]$. An odds ratio is a ratio of the probability of sucess over the probability of failure, and odds ratios $\in [0,\infty)$. Therefore
ln(odds ratios) $\in (-\infty,\infty)$.

2) For a binary model, only two outcomes are examined, therefore the Pr{failure} = 1 - Pr{success}.

$\ln\left(\frac{\Pr\{\text{success}\}}{\Pr\{\text{failure}\}}\right) = f(x)$

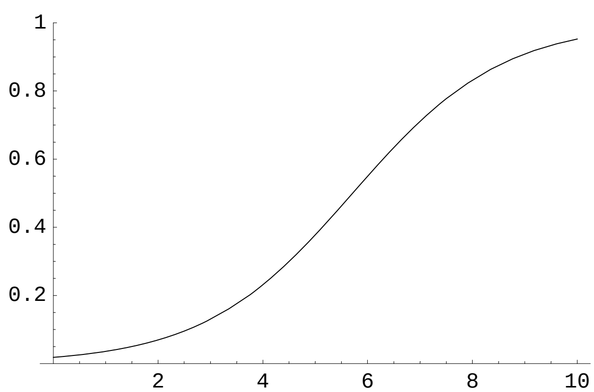
$\frac{\Pr\{\text{success}\}}{\Pr\{\text{failure}\}} = \exp(f(x))$

$\frac{\Pr\{\text{success}\}}{1-\Pr\{\text{success}\}} = \exp(f(x))$

$\Pr\{\text{success}\} = \frac{\exp(f(x))}{\exp(f(x))+1}$

$\Pr\{\text{success}\} = \frac{1}{1+\exp(-f(x))}$

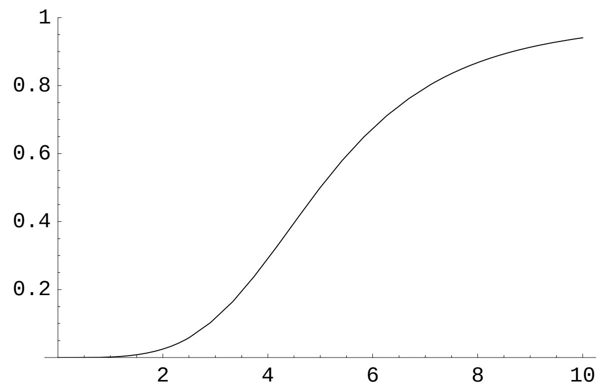
```
parameters = {a → -4, b → 0.7};
f[x_] := a + b x /. parameters;
Plot[1/(1 + Exp[-f[x]]) /. parameters, {x, 0, 10}, PlotRange → {0, 1}];
```



The Hill function is also useful for modeling probabilities as it contains interpretable parameters: *a* is the point at which the probability equals 0.50 and *b* is the "slope" at the 0.50 point.

```
parameters = {a → 5, b → 4};
```

$$\texttt{Plot}\left[\frac{1}{1+\left(\frac{a}{x}\right)^{b}}\ \texttt{/. parameters, \{x, 0, 10\}, PlotRange} \to \texttt{\{0, 1\}}\right];$$



■ Multinomial

Multinomial Distribution:
This is an extension of the binomial to m possible outcomes each with a probability p_i.
probability distribution function:

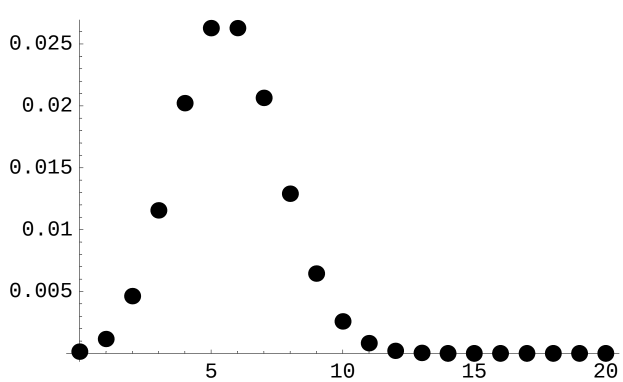
$$\Pr\{Z_m = z_m\} = \left(\frac{N!}{z_1!\, z_2! \ldots z_m!}\right) p_1{}^{z_1}\; p_2{}^{z_2} \ldots p_m{}^{z_m}$$

mean: $N\ p_m$
variance: $N\ p_m\ (1 - p_m)$

■ mathematica

$$\texttt{f[z1\_, z2\_, n\_, p1\_, p2\_] :=} \frac{n!}{z1!\ z2!\ (n - z1 - z2)!}\ p1^{z1}\ p2^{z2}\ (1 - p1 - p2)^{n-z1-z2}$$

```
ListPlot[Table[{z, f[z, 3, 20, 0.25, 0.25]}, {z, 0, 20}], PlotStyle → PointSize[0.03]];
```



Sums to 1

```
Plus @@ Flatten[Table[f[z, i, 5, 0.25, 0.25], {i, 0, 5}, {z, 0, 5}]]
```

```
1.
```

```
<< Statistics`MultiDiscreteDistributions`
```

```
(* multinomial random vector for an event that occurs 10 times;
 the probability vector follows *)
Random[MultinomialDistribution[10, {0.2, 0.2, 0.3, 0.3}]]
```

```
{1, 3, 2, 4}
```

r

```
dmultinom ()
```

```
# random number generation
```

Poisson

Poisson Distribution:
has one parameter that can really be broken into two parameters: r is the rate constant and t is the time.
One description of a poisson distribution: Imagine a process with a constant probability of occurance r. The event can occur at any time, and repeatedly. However, if we measure at discrete time intervals, and count the number of times the event has occurred for discrete entities, the result will be a Poisson Distribution. Imagine rain falling onto floor tiles. If you count the number of drops that hit each tile in one minute, the result would be distribute as a Poisson distribution.
probability distribution function:

$$\Pr\{Z = z\} = \frac{e^{-r\,t}(r\,t)^z}{z!}$$

mean: $r\,t$
variance: $r\,t$
note: if evenly spaced time intervals, set t equal to 1

r

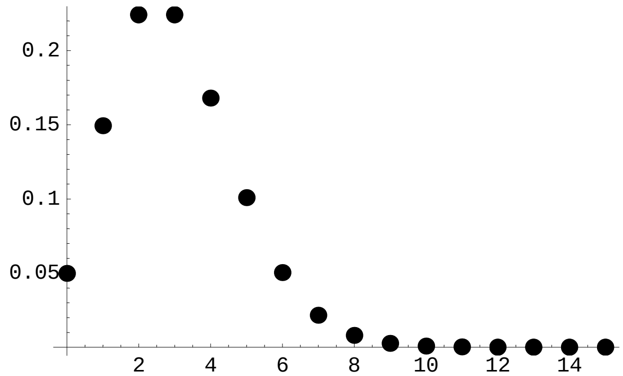
```
dpois (z, lambda)
```

```
# random number generation
rpois (1, lambda)
```

mathematica

$$f[z\_,\ \lambda\_] := \frac{\mathrm{Exp}[-\lambda]\ \lambda^z}{z\,!}$$

```
ListPlot[Table[{z, f[z, 3]}, {z, 0, 15}], PlotStyle → PointSize[0.03]];
```



Sums to 1

```
Plus @@ Table[f[z, 20, 0.3], {z, 0, 100}]
```

```
1.
```

```
<< Statistics`DiscreteDistributions`
(* poisson random number with λ = 3.26 *)
Random[PoissonDistribution[3.26]]
```

```
2
```

■ Negative binomial

Negative Binomial Distribution:
The negative binomial is similar to the poisson except that the data are overdispersed. As a consequence, the variance will always be larger than the mean. The negative binomial can be derived assuming the rate constant in the poisson is no longer constant but is gamma distributed. As $k \to \infty$, the distribution approaches a Poisson. k is the shape parameter.
probability distribution function:

$$\Pr\{Z = z\} = \frac{\Gamma(k+z)}{\Gamma(k)\, z!} \left(\frac{k}{k+m}\right)^k \left(\frac{m}{k+m}\right)^z$$

where Γ is the gamma function, $\Gamma(n) = \int_0^\infty \mathrm{Exp}(-t)\, t^{n-1}\, dt$

mean: m
variance: $m + \frac{m^2}{k}$

Alternative form: used to describe the number the number of failures that occur before n sucesses have occured, where the probability of success is p.

■ r

```
# note, "mu =" is needed to specify this particular parameterization
dnbinom (z, size = k, mu = m)
```

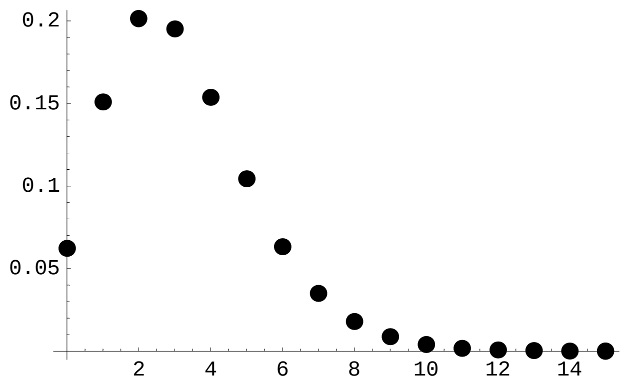
```
rnbinom (1, size = k, mu = m)
```

■ mathematica

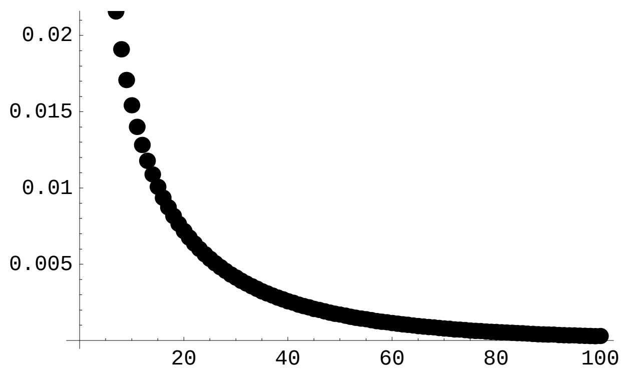
```
f[z_, k_, m_] := Gamma[k + z]/(Gamma[k] z!) (k/(k + m))^k (m/(k + m))^z
```

$$f[z\_, k\_, m\_] := \frac{\text{Gamma}[k+z]}{\text{Gamma}[k]\, z!} \left(\frac{k}{k+m}\right)^k \left(\frac{m}{k+m}\right)^z$$

```
ListPlot[Table[{z, f[z, 10, 3.2]}, {z, 0, 15}], PlotStyle → PointSize[0.03]];
```




```
ListPlot[Table[{z, f[z, 0.26, 10]}, {z, 0, 100}], PlotStyle → PointSize[0.03]];
```



Sums to 1

```
Plus @@ Table[f[z, 10, 3], {z, 0, 100}] // N
```

```
1.
```

Derive expected value

```
Sum[z f[z, k, m], {z, 0, ∞}]
```

```
m
```

Derive variance

```
Sum[(z - Sum[z f[z, k, m], {z, 0, ∞}])^2 f[z, k, m], {z, 0, ∞}] // Simplify
```

$$\frac{m\,(k+m)}{k}$$

Mathematica does not contain a built-in random number generator for this form of the negative binomial. While not very elegant, the function below can be used to randomly generate numbers from a negative binomial distribution.

```
NegBiRandom[m_, k_] := {
   funtable = Table[Gamma[k + x]/(Gamma[k] x!) (k/(k + m))^k (m/(k + m))^x, {x, 0, 250}];
   CDF = Table[Sum[funtable[[i]], {i, endsum}], {endsum, 1, Length[funtable]}];
   x = Random[];
   i = 1;
   While[x > CDF[[i]],
    i++;
   ];
   i - 1}[[1]]
```

```
NegBiRandom[10, 3]
```

```
5
```

```
(* negative binomial distribution *)
Clear[sf];
sf[z_, m_?NumberQ, k_?NumberQ] := If[m > 0 && k > 0, -Log[Gamma[k + z]] + Log[z!] +
    Log[Gamma[k]] - k Log[k] + k Log[k + m] - z Log[m] + z Log[k + m], 10000.0];
SetAttributes[sf, Listable];
```

■ Generalized Poisson

The generalized Poisson distribution (Consul and Jain. 1973. Technometrics. 15: 791-799) can be used to represent a Poisson process in which the data are over-dispersed. It is also known as Lagrangian Poisson Distribution (Johnson, Kotz, and Kemp. 1992. Univariate Discrete Distributions). Despite the claims of the original authors, this distribution cannot be used for under-dispersed data because the sum of the probabilities do not add to 1.

This distribution explicity models an increase in the rate of moving to subsequent categories and can represent data with very long right-tails as compared to the negative binomial. This distribution worked very well for the distribution of pollen deposited on flowers (Castellanos et al. 2003. Evolution 57:2742-2752.) When $\lambda_2 = 0$ this distribution becomes the Poisson distribution.

$\Pr\{Z = z\} = \frac{e^{-(\lambda_1 + z\lambda_2)}\lambda_1(\lambda_1 + z\lambda_2)^{z-1}}{z!}$, where $0 < \lambda_1$ and $0 \le \lambda_2 < 1$
mean: $\frac{\lambda_1}{1-\lambda_2}$
variance: $\frac{\lambda_1}{(1-\lambda_2)^3}$

```
f[z_, λ1_, λ2_] := λ1 (λ1 + z λ2)^(z-1) Exp[-(λ1 + z λ2)] / z!
```

Consul and Jain (1973) list the following method for estimating $\lambda_1$and $\lambda_2$from the mean and variance, but I don't trust these as the the best estimates, just a close estimate.

$\lambda_2 = 1 - \sqrt{\frac{\text{mean}}{\text{variance}}}$

$\lambda_1 = \text{mean}\,(1 - \lambda_2)$

▪ Beta-Binomial

Beta Binomial Distribution:
This distribution assumes that *p* in the binomial distribution has a beta distribution with parameters *v* and w (see beta distribution above).

probability distribution function:

$\Pr\{Z = z\} = \left(\frac{N!}{(N-z)!\,z!}\right) \frac{B(v+z, N+w-z)}{B(v,w)}$, where *B(v,w)* is the beta function

mean:
variance:

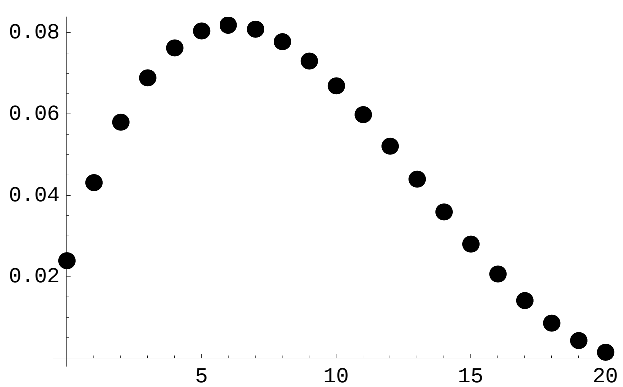
The Beta distribution can be transformed to be in terms of a mean parameter and dispersion parameter, $p = s(1\text{-}m)$ and $q = s\,m$. This just scales *p* and *q* without the specific constraints on the variance version. Here $m \in (0,1)$, $s \in (0,\infty)$. The *m*, *s* version is the one used for sfBetaBinomial in *Mathematica*. This form has been used to describe correlated instances of the Binomial Distribution, although more mechanistic distributions for these cases exist.

▪ r

```
library (rmutil)
# this is based on a parameterization so that m is the mean and s is
# an overdispersion parameters. As s approaches infintiy,
# the Beta - Binomial collapses to a Biomial Distribution
dbetabinom (z, N, m, s)
```

▪ mathematica

```
f[z_, n_, m_, s_] := n! / ((n - z)! z!) Beta[s m + z, n + s (1 - m) - z] / Beta[s m, s (1 - m)]

f[z_, n_, v_, w_] := n! / ((n - z)! z!) Beta[v + z, n + w - z] / Beta[v, w]

ListPlot[Table[{z, f[z, 20, 2.0, 3.2]}, {z, 0, 20}], PlotStyle → PointSize[0.03]];
```



Sums to 1

```
Plus @@ Table[f[z, 20, 2.0, 3.2], {z, 0, 20}]
```

```
1.
```

```
(* Beta binomial distribution *)
sf[z_, n_, v_, w_] :=
 Log[(n - z) !] + Log[z !] - Log[n !] + Log[Beta[v, w]] - Log[Beta[v + z, n + w - z]]
```

The beta binomial is derived by multiply the pdf of the beta distribution by the pdf of the binomial, and then integrating from across the the full range of z for the beta binomial, in this case [0,1]

```
(* pdf for beta distribution *)
```

$$f[z\_, p\_, q\_] := \frac{z^{p-1} (1-z)^{q-1}}{\mathrm{Beta}[p, q]}$$

```
(* pdf for binomial distribution *)
```

$$f[z\_, n\_, p\_] := \frac{n!}{(n-z)!\, z!} p^z (1-p)^{n-z}$$

$$\mathrm{Integrate}\left[\frac{n!}{(n-z)!\, z!} p^z (1-p)^{n-z} \frac{p^{v-1} (1-p)^{w-1}}{\mathrm{Beta}[v, w]},\ \{p, 0, 1\}, \mathrm{Assumptions} \to \{v > 0, w > 0, z \geq 0, n \geq 0\}\right]$$

$$\frac{1}{\mathrm{Beta}[v, w]\ (n-z)!\, z!} \left(n!\ \mathrm{If}\left[z < n + w, \frac{\mathrm{Gamma}[n + w - z]\ \mathrm{Gamma}[v + z]}{\mathrm{Gamma}[n + v + w]}, \mathrm{Integrate}\left[(1-p)^{-1+n+w-z} p^{-1+v+z},\ \{p, 0, 1\}, \mathrm{Assumptions} \to v > 0 \,\&\&\, w > 0 \,\&\&\, n \geq 0 \,\&\&\, z \geq 0 \,\&\&\, z \geq n + w\right]\right]\right)$$

Mathematica doesn't do a good job of pulling out the beta function, but this simplifies to

$$\frac{n!}{(n-z)!\, z!} \frac{\mathrm{Beta}[z + v]\ \mathrm{Beta}[w + n - z]}{\mathrm{Beta}[v, w]}$$

■ Bimodal Beta-Binomial

Using the same assumptions from the derivation of the unimodal beta binomial distribution, multiply the bimodal beta distribition by the binomial distribution and integrate p from 0 to 1

```
(* pdf for binomial distribution *)
```

$$f[z\_, n\_, p\_] := \frac{n!}{(n-z)!\, z!} p^z (1-p)^{n-z}$$

```
(* pdf for bimodal beta *)
```

$$f[z\_, v\_, w\_, x\_, y\_] := \frac{z^{v-1} (1-z)^{w-1}}{2\,\mathrm{Beta}[v, w]} + \frac{z^{x-1} (1-z)^{y-1}}{2\,\mathrm{Beta}[x, y]}$$

```
(* multiply both distributions by each other *)
```

$$\frac{n!}{(n-z)!\,z!}\,p^{z}\,(1-p)^{n-z}\left(\frac{p^{v-1}\,(1-p)^{w-1}}{2\,\mathrm{Beta}[v,w]}+\frac{p^{x-1}\,(1-p)^{y-1}}{2\,\mathrm{Beta}[x,y]}\right)$$

$$\frac{n!}{(n-z)!\,z!}\,p^{z}\,(1-p)^{n-z}\,\frac{p^{v-1}\,(1-p)^{w-1}}{2\,\mathrm{Beta}[v,w]}+\frac{n!}{(n-z)!\,z!}\,p^{z}\,(1-p)^{n-z}\,\frac{p^{x-1}\,(1-p)^{y-1}}{2\,\mathrm{Beta}[x,y]}$$

$$\frac{n!}{(n-z)!\,z!}\,\frac{p^{z+v-1}\,(1-p)^{w+n-z-1}}{2\,\mathrm{Beta}[v,w]}+\frac{n!}{(n-z)!\,z!}\,\frac{p^{z+x-1}\,(1-p)^{y+n-z-1}}{2\,\mathrm{Beta}[x,y]}$$

$$\frac{n!}{(n-z)!\,z!}\left(\frac{p^{z+v-1}\,(1-p)^{w+n-z-1}}{2\,\mathrm{Beta}[v,w]}+\frac{p^{z+x-1}\,(1-p)^{y+n-z-1}}{2\,\mathrm{Beta}[x,y]}\right)$$

$$\int_0^1\frac{n!}{(n-z)!\,z!}\left(\frac{p^{z+v-1}\,(1-p)^{w+n-z-1}}{2\,\mathrm{Beta}[v,w]}+\frac{p^{z+x-1}\,(1-p)^{y+n-z-1}}{2\,\mathrm{Beta}[x,y]}\right)dp$$

$$\frac{n!}{(n-z)!\,z!}\int_0^1\frac{p^{z+v-1}\,(1-p)^{w+n-z-1}}{2\,\mathrm{Beta}[v,w]}\,dp+\frac{n!}{(n-z)!\,z!}\int_0^1\frac{p^{z+x-1}\,(1-p)^{y+n-z-1}}{2\,\mathrm{Beta}[x,y]}\,dp$$

$$\frac{n!}{(n-z)!\,z!}\,\frac{\mathrm{Beta}[z+v,w+n-z]}{2\,\mathrm{Beta}[v,w]}+\frac{n!}{(n-z)!\,z!}\,\frac{\mathrm{Beta}[z+x,y+n-z]}{2\,\mathrm{Beta}[x,y]}$$

$$\frac{n!}{(n-z)!\,z!}\left(\frac{\mathrm{Beta}[z+v,w+n-z]}{2\,\mathrm{Beta}[v,w]}+\frac{\mathrm{Beta}[z+x,y+n-z]}{2\,\mathrm{Beta}[x,y]}\right)$$

```
(* pdf for bimodal beta-binomal distribution *)
```

$$\mathrm{f}[z\_,n\_,v\_,w\_,x\_,y\_]:=\frac{n!}{(n-z)!\,z!}\left(\frac{\mathrm{Beta}[z+v,w+n-z]}{2\,\mathrm{Beta}[v,w]}+\frac{\mathrm{Beta}[z+x,y+n-z]}{2\,\mathrm{Beta}[x,y]}\right)$$

Bimodal Beta-Binomial Distribution:
has 5 parameters: shape parameters for one node are *v* and *w*; shape parameters for the other node are *x* and *y*; *n* is the total number of individuals in the set and *z* is the number of success in that set.

Note: this distribution and the unimodal beta-binomial assume a different draw from beta for the *p* of each set, but it assumes the same *p* for all the individuals within a set. This statement is based on Skellam. 1948. Journal of the Royal Statistical Society. Series B, 10: 257-261.

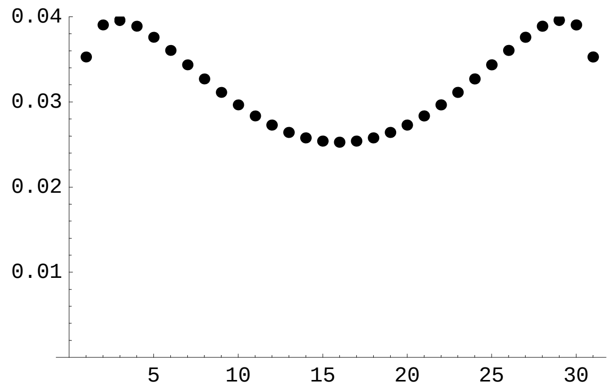
probability density function:

$f(z) = \frac{z^{p-1}(1-z)^{q-1}}{B(p,q)}$, B is the beta function, $B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}\,du$)

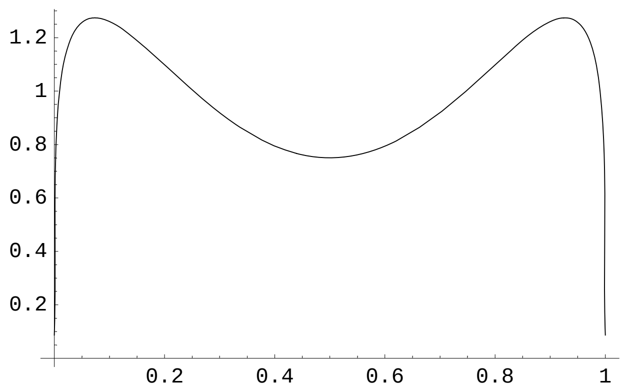
$$f(z) = \frac{z^{p-1}(1-z)^{q-1}}{\int_0^1 u^{p-1}(1-u)^{q-1}\,du}$$

mean:
variance:

```
ListPlot[Table[f[z, 30, 1.2, 3.6, 3.6, 1.2], {z, 0, 30}],
  PlotStyle → {PointSize[0.02]}, PlotRange → {0, 0.04}];
```



$$\texttt{Plot}\left[\frac{z^{v-1}\,(1-z)^{w-1}}{2\,\texttt{Beta}[v,\,w]}+\frac{z^{x-1}\,(1-z)^{y-1}}{2\,\texttt{Beta}[x,\,y]}\ \texttt{/.}\ \{v\to 1.2,\ w\to 3.6,\ x\to 3.6,\ y\to 1.2\},\ \{z,\,0,\,1\}\right];$$




```
Plus @@ Table[f[z, 30, 1.2, 3.6, 3.6, 1.2], {z, 0, 30}]

1.
```

Therefore the support function is

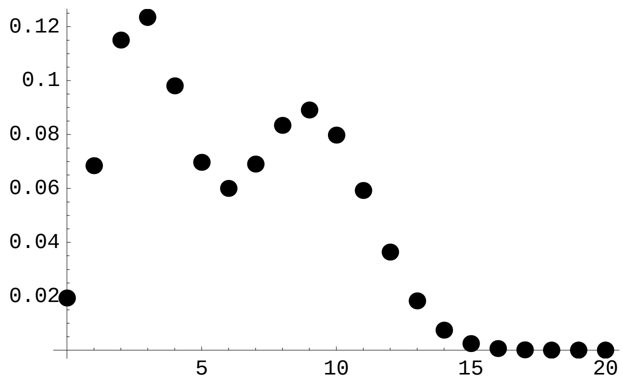
```
(* bimodal beta-binomial distribution *)
sf[z_, n_, v_, w_, x_, y_] :=
```

$$\texttt{Log}[(n-z)\,!]+\texttt{Log}[z\,!]-\texttt{Log}[n\,!]-\texttt{Log}\left[\frac{\texttt{Beta}[z+v,\ w+n-z]}{2\,\texttt{Beta}[v,\,w]}+\frac{\texttt{Beta}[z+x,\ y+n-z]}{2\,\texttt{Beta}[x,\,y]}\right]$$

■ Bimodal Binomial

$$\texttt{f}[z\_,\,n\_,\,p\_,\,q\_] := \frac{n\,!}{2\,(n-z)\,!\,z\,!}\,p^{z}\,(1-p)^{n-z}+\frac{n\,!}{2\,(n-z)\,!\,z\,!}\,q^{z}\,(1-q)^{n-z}$$

```
ListPlot[Table[{z, f[z, 20, 0.15, 0.45]}, {z, 0, 20}], PlotStyle → PointSize[0.03]];
```



Sums to 1

```
Plus @@ Table[f[z, 20, 0.3], {z, 0, 20}]
```

```
1.
```

```
(* bimodal binomial *)
sf[z_, n_, p_, q_] := Log[2 (n - z) !] + Log[z !] - Log[n !] - Log[p^z (1 - p)^(n-z) + q^z (1 - q)^(n-z)]
```