Embedded Bandits for Large-Scale Black-Box Optimization

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Large-Scale Black-Box Optimization

$$\begin{aligned}&\text{minimize} \quad f(\mathbf{x})\\ &\text{subject to } \mathbf{x} \in \mathcal{X}\end{aligned}$$

- $f : \mathcal{X} = [-1, 1]^n \to \mathbb{R}$
- $n \gg 10^2$
- $\min_{\mathbf{x}\in\mathcal{X}} f(\mathbf{x}) = f(\mathbf{x}^*) = f^*$
- High-order information (e.g., derivatives) are unavailable.

Related Work

- Algorithmic work has been based on either *decomposition* or *embedding* techniques.
- Decomposition algorithms break the problem into several subproblems, and solutions for the original problem are recognized in a coordinated manner.
- Embedding algorithms exploit the assumption/empirical observation of *low effective dimensionality*
- Recent works presented *Random Embedding* (RE) techniques based on the random matrix theory and provided probabilistic theoretical guarantees [3, 2, 1].
- Multiple runs are employed for RE to substantiate the probabilistic theoretical performance.

Motivation

Breaking away from the *multiple-run* framework and follow the *optimism in the face of uncertainty* principle via *stochastic hierarchical bandits* over a low-dimensional search space $\mathcal{Y}$.

Notation

- $\mathcal{N}$ denotes the Gaussian distribution with zero mean and $1/n$ variance.
- $\{A_p\}_p \subseteq \mathbb{R}^{n\times d}$, with $d \ll n$, is a sequence of realization matrices of the random matrix $\mathbf{A}$ whose entries are sampled independently from $\mathcal{N}$.
- The Euclidean random projection of the ith coordinate $[\mathbf{y}]_i$ to $[\mathcal{X}]_i$ is defined as follows.

$$[\mathcal{P}_{\mathcal{X}}(A\mathbf{y})]_i = \begin{cases} 1, & \text{if } [\mathbf{y}]_i \geq 1;\\ -1, & \text{if } [\mathbf{y}]_i \leq -1;\\ [A\mathbf{y}]_i & \text{otherwise.}\end{cases}$$

- $g_P(\mathbf{y})$ is a random (stochastic) function such that $g_P(\mathbf{y}) \stackrel{\text{def}}{=} f(\mathcal{P}_{\mathcal{X}}(\mathbf{A}\mathbf{y}))$ and $g_p(\mathbf{y}) = f(\mathcal{P}_{\mathcal{X}}(A_p\mathbf{y}))$ is a realization (deterministic) function, where $\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{R}^d$.

Contribution I

- The mean variation in the objective value for a point y in the low-dimensional space $\mathcal{Y} \subseteq \mathbb{R}^d$ projected randomly into the decision space $\mathcal{X}$ of Lipschitz-continuous problems is *bounded.*
- Mathematically, $\forall y \in \mathcal{Y} \subseteq \mathbb{R}^d$, we have

$$E[|g_p(\mathbf{y}) - g_q(\mathbf{y})|] \leq \sqrt{8}\cdot L \cdot ||\mathbf{y}|| \;.$$

Contribution II

- EmbeddedHunter is a $\mathcal{Y}$-partitioning tree-search algorithm.
- The partitioning is represented by a K-ary tree $\mathcal{T}$, where nodes of the same depth h correspond to a partition of K^h subspaces / cells.
- For each node (h, i), f is evaluated at the center point $\mathbf{y}_{h,i}$ of its cell $\mathcal{Y}_{h,i}$ once or more times with different projections based on $||\mathbf{y}_{h,i}||$.

Algorithm 1 The EmbeddedHunter Algorithm

Input:
stochastic function g_P,
search space $\mathcal{Y} = [-d/\eta, d/\eta]^d$,
evaluation budget v.
Initialization:
$t \leftarrow 1, \mathcal{T}_1 = \{(0,0)\}$, Evaluate $g_P(\mathbf{y}_{0,0})$.
while evaluation budget is not exhausted **do**
  $\nu_{\min} \leftarrow \infty$
  for $l = 0$ **to** $\min\{\text{depth}(\mathcal{T}_t), h_{\max}\}$ **do**
    for $j = 1$ **to** $|\Gamma_{l,t}|$ **do**
      Select $(l, o) = \arg\min_{(h,i)\in\mathcal{L}^j_{l,t}} f^*_{h,i}$
      if $f^*_{l,o} < \nu_{\min}$ **then**
        $\nu_{\min} \leftarrow f^*_{l,o}$
        Expand (l, o) into its child nodes
        Evaluate (l, o)'s child nodes by g_P
        Add (l, o)'s child nodes to $\mathcal{T}_t$
      end if
    end for
    $\mathcal{T}_{t+1} \leftarrow \mathcal{T}_t$
    $t \leftarrow t + 1$
  end for
end while
return $f^*_v = \min_{(h,i)\in\mathcal{T}_t} f^*_{h,i}$

Assumptions

- f is Lipschitz-continuous, i.e., $\forall \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$,

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leq L \cdot ||\mathbf{x}_1 - \mathbf{x}_2|| \;,$$

where $L > 0$ is the Lipschitz constant.
- There exists a decreasing sequence δ in $h \geq 0$ such that for one (or more) optimal cell(s) $\mathcal{Y}_{h,i^*_h}$ at depth h, we have

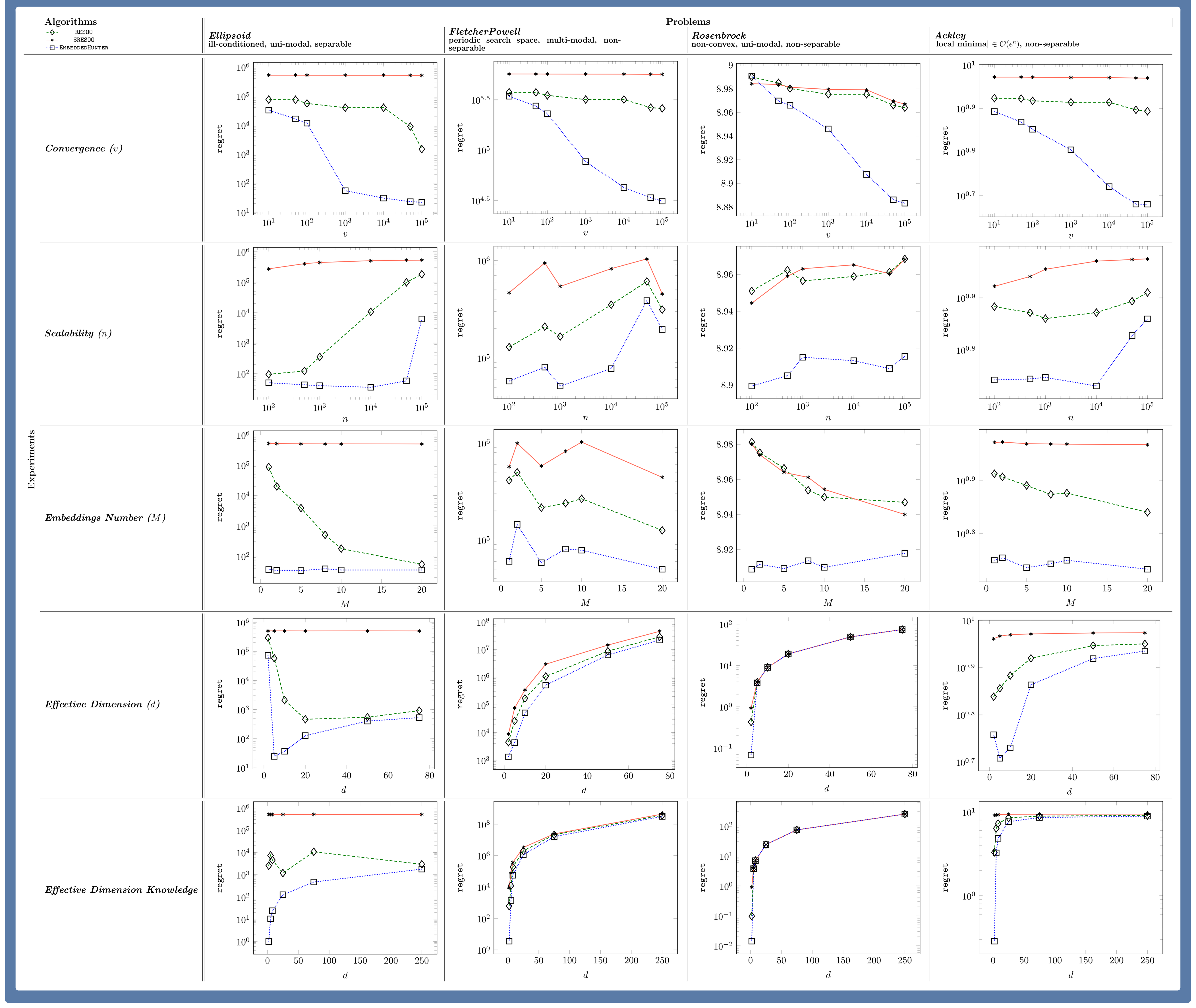
$$0 \leq \sup_{q,\mathbf{y}\in\mathcal{Y}_{h,i^*_p}} |g_q(\mathbf{y}_{h,i}) - g_q(\mathbf{y})| \leq \delta(h) \;.$$

- There exists $m > 0$ such that $\forall (h, i) \in \mathcal{T}, \mathcal{Y}_{h,i}$ contains an ℓ-ball of radius $m\delta(h)$ centered in $\mathbf{y}_{h,i}$.
- There exists two non-decreasing sequences λ and τ in $\mathbf{y}$ and h, respectively, such that for any depth $h \geq 0$, for any optimal cell $\mathcal{Y}_{h,i^*_p}$,

$$0 \leq \sup_{s,t} |g_s(\mathbf{y}_{h,i^*_p}) - g_t(\mathbf{y}_{h,i^*_p})| \leq \lambda(\mathbf{y}_{h,i^*_p}) \;,$$

and $\sup_{i^*_p} \lambda(\mathbf{y}_{h,i^*_p}) \leq \tau(h)$.

Numerical Result



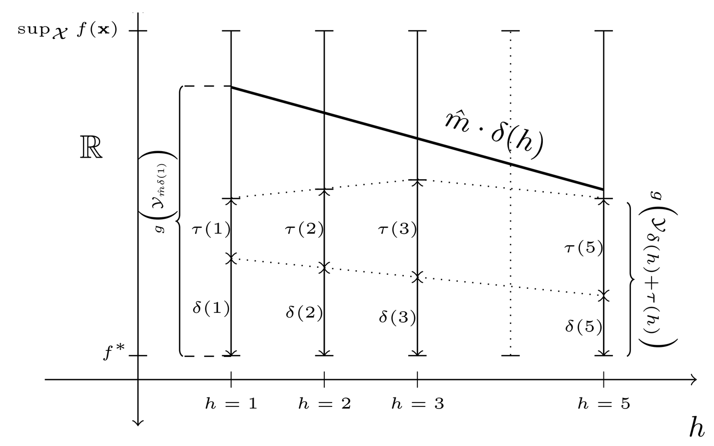
Convergence Analysis

Define $h(t)$ as the smallest $h \geq 0$ such that:

$$Ch_{max}\sum_{l=0}^{h(t)}(\hat{m}\delta(l))^{-\hat{d}} \geq t \;,$$

where t is the number of iterations. Then EmbeddedHunter's regret is bounded as

$$r(t) \leq \min_{h\leq\min(h(t),h_{max}+1)} \tau(h) + \delta(h) \;.$$



Conclusion

- EmbeddedHunter builds a stochastic tree over a low-dimensional search space $\mathcal{Y}$, where stochasticity has shown to be proportional on average with the norm of the nodes' base points.
- Besides its theoretically-proven performance, numerical experiments have validated EmbeddedHunter's in comparison with recent random-embedding methods.

References

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