ECN 514: Detection & Estimation Theory
Assignment 2

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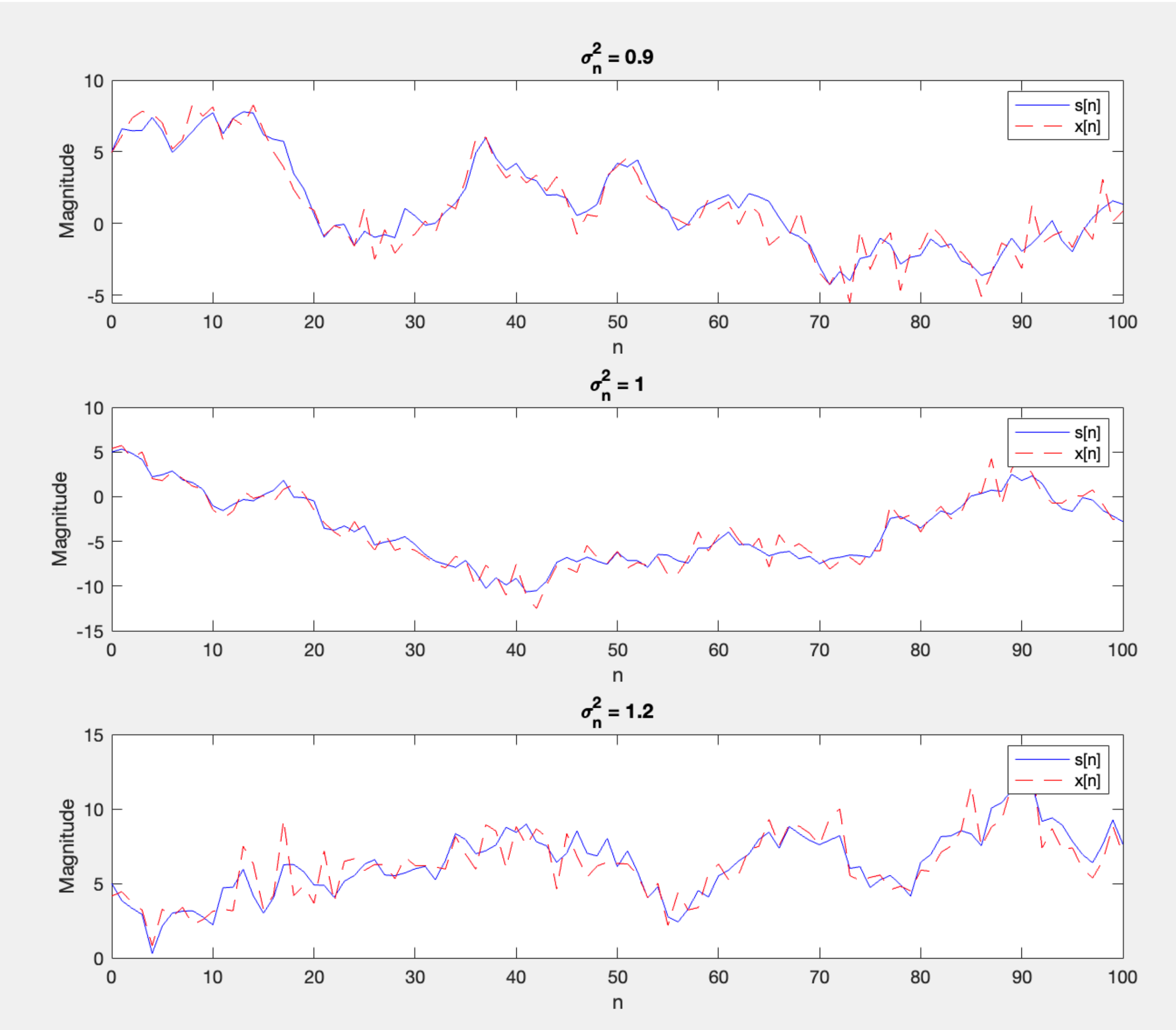
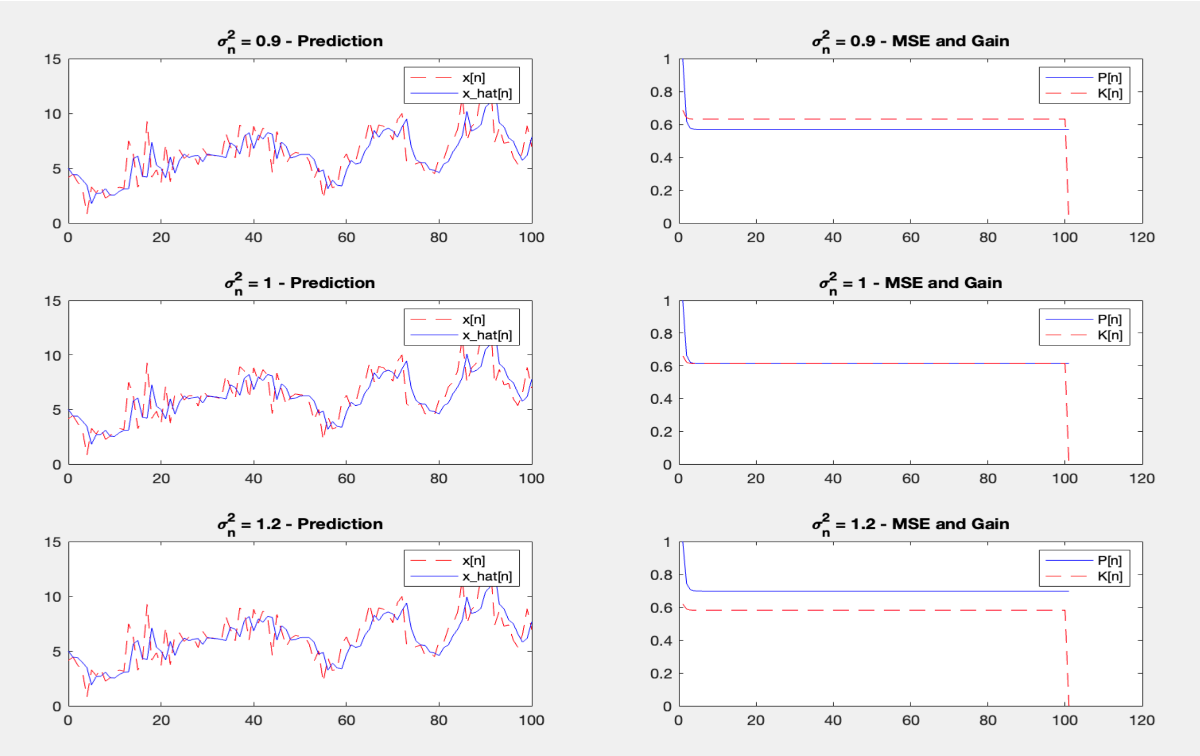
Q1:
Output:

Predicted state: 4.8391
Predicted state covariance: 1.5895
Kalman gain: 0.6138
State estimate: 4.9379
State estimate variance: 0.6138

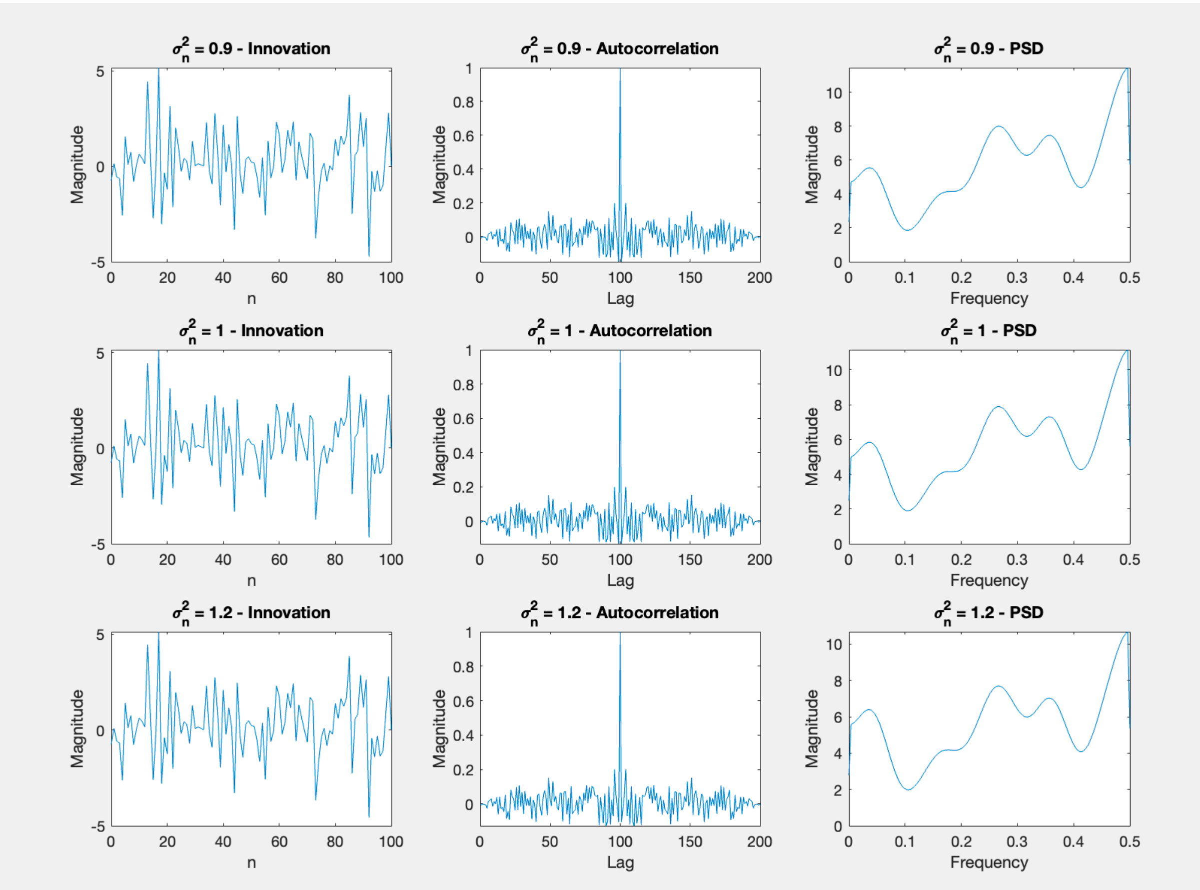
Q1.a)



Q1.b)


σ²n = 0.9 - Prediction
σ²n = 0.9 - MSE and Gain
σ²n = 1 - Prediction
σ²n = 1 - MSE and Gain
σ²n = 1.2 - Prediction
σ²n = 1.2 - MSE and Gain
x[n]
x_hat[n]
P[n]
K[n]

Q1.c)

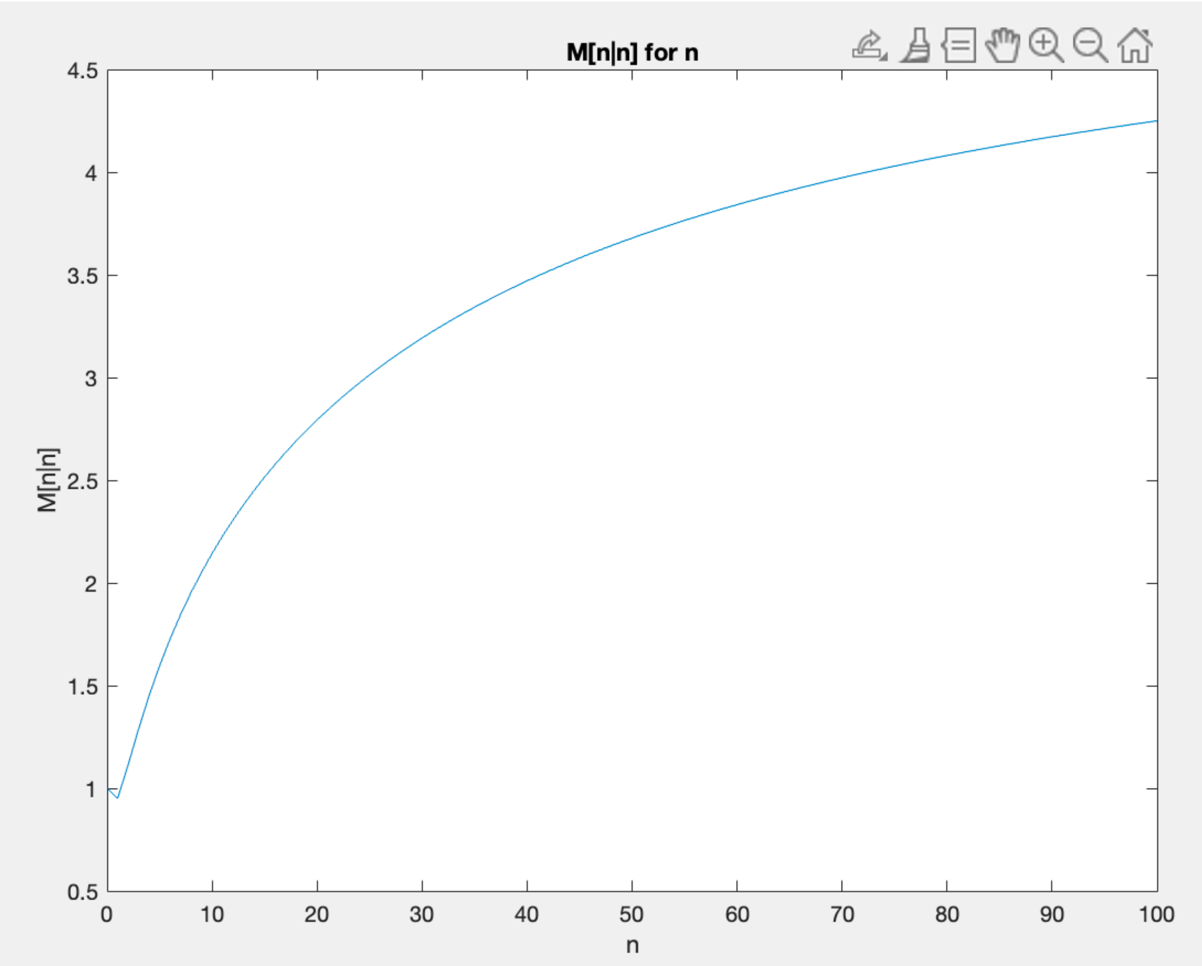

σ²n = 0.9 - Innovation
σ²n = 0.9 - Autocorrelation
σ²n = 0.9 - PSD
σ²n = 1 - Innovation
σ²n = 1 - Autocorrelation
σ²n = 1 - PSD
σ²n = 1.2 - Innovation
σ²n = 1.2 - Autocorrelation
σ²n = 1.2 - PSD
Magnitude
n
Lag
Frequency

Q1d):
To assess the whiteness of the innovation sequence, we would typically inspect both the ACF and the PSD. If both indicate minimal correlation or structure in the residuals, we can conclude that the innovation sequence is close to being white, indicating a successful filtering process. However, if either the ACF or the PSD shows significant deviations from the expected patterns of white noise, further investigation may be necessary to understand and address any underlying issues in the filtering process.

In part c however, we can see that ACF is an impulse at lag 100 and zero at all other lags. This shows that there is no correlation between different samples of innovation sequence.

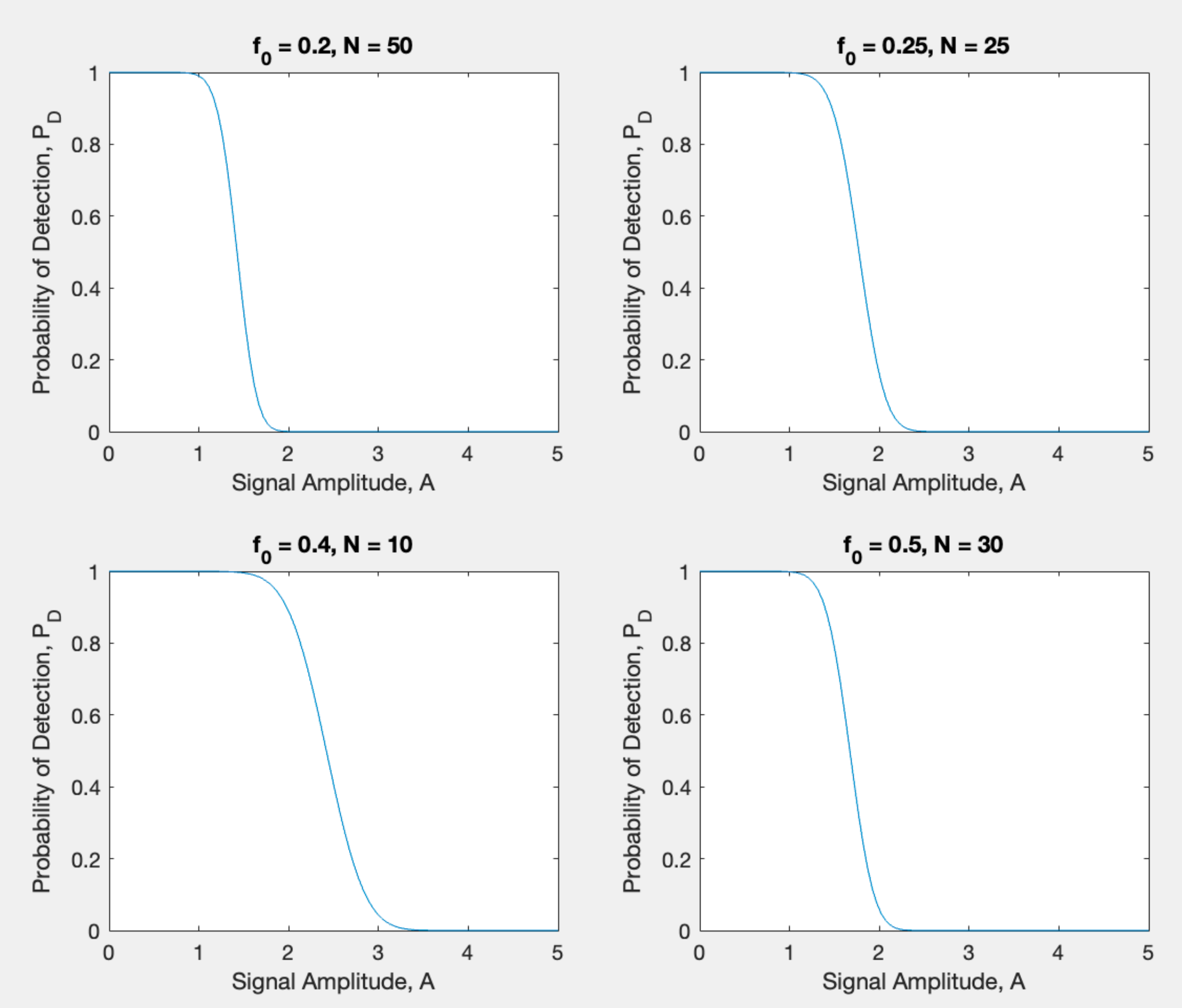
But PSD shows some peaks at some specific frequencies which is not expected from a white sequence. Hence we can conclude that innovation sequence is non-white in nature.

Q2:
Output:



As n increases, M[n|n] and starts saturating at higher values of n.

Q3:
Output:



f0 = 0.2, N = 50
f0 = 0.25, N = 25
f0 = 0.4, N = 10
f0 = 0.5, N = 30
Probability of Detection, PD
Signal Amplitude, A
0
0.2
0.4
0.6
0.8
1
0
1
2
3
4
5

Q2.
Ans

$$M[n|n] = (1 - k[n])\, M[n|n-1]$$

where

$$k[n] = \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

So

$$M[n|n] = \left(1 - \frac{M[n|n-1]}{\sigma_n^2 + M[n|n-1]}\right) M[n|n-1]$$

$$M[n|n] = \frac{\sigma_n^2\, M[n|n-1]}{\sigma_n^2 + M[n|n-1]}$$

Since,

$$M[n|n-1] = E\left[(s[n] - \hat{s}[n|n-1])^2\right]$$

$$= E\left[(a\, s[n-1] + u[n] - a\,\hat{s}[n-1|n-1])^2\right]$$

$$= E\left[(a(s[n-1] - \hat{s}[n-1|n-1]) + u[n])^2\right]$$

$$= E\left[a^2 (s[n-1] - \hat{s}[n-1|n-1])^2\right] + E(u^2[n])$$

$$+ E\left[2a(s[n-1] - \hat{s}[n-1|n-1])\, u[n]\right]$$

$$= a^2 M[n-1|n-1] + \sigma_u^2$$

where,

$s[n-1]$ and $\hat{s}[n-1|n-1]$ are independent of $u[n]$

$$\therefore M[n|n] = \frac{\sigma_n^2 \left[a^2 M[n-1|n-1] + \sigma_u^2\right]}{\sigma_n^2 + a^2 M[n-1|n-1] + \sigma_u^2}$$

Q3.

$$x[n] = \begin{cases} \mathcal{N}(0, \sigma_0^2) & \text{under } H_0 \\ \mathcal{N}(0, \sigma_1^2) & \text{under } H_1 \quad (\sigma_1^2 > \sigma_0^2) \end{cases}$$

NP test decides H_1 if

$$\frac{\dfrac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma_1^2} \sum_{n=0}^{N-1} x^2[n]}}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma_0^2} \sum_{n=0}^{N-1} x^2[n]}} > \gamma$$

Taking log on both sides, we have

$$-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) \sum_{n=0}^{N-1} x^2[n] > \ln\gamma + \frac{N}{2} \ln\left(\frac{\sigma_1^2}{\sigma_0^2}\right)$$

Since $\sigma_1^2 > \sigma_0^2$

we have $\quad \frac{1}{N} \sum_{n=0}^{N-1} x^2[n] > \gamma'$

where $\gamma' = \dfrac{\frac{2}{N}\ln\gamma + \ln\left(\frac{\sigma_1^2}{\sigma_0^2}\right)}{\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)}$

Here $N = 2$.

So, decides H_1 if $\quad \frac{1}{2} \sum_{n=0}^{1} x^2[n] > \gamma'$

$$\frac{1}{2}\left(x^2[0] + x^2[1]\right) > \gamma'$$

Under H_0, $\sigma_*^2 = \sigma_0^2$ so,

$$P_{FA} = P_r \left\{ \frac{x^2[0] + x^2[1]}{\sigma_0^2} > \frac{2\gamma'}{\sigma_0^2} : H_0 \right\}$$

$$= P_r \left\{ \chi_2^2 > \frac{2\gamma'}{\sigma_0^2} \right\} = \int_{\frac{2\gamma'}{\sigma_0^2}}^{\infty} \frac{1}{2} e^{-\frac{1}{2}t} \, dt$$

$$= -e^{-\frac{1}{2}t} \Big|_{\frac{2\gamma'}{\sigma_0^2}}^{\infty}$$

Here $N = 2$

$$Q_{\chi_\nu^2}(x) = e^{-\frac{1}{2}x} \sum_{k=0}^{\nu/2 - 1} \frac{\left(\frac{x}{2}\right)^k}{k!}$$

$$= e^{-\frac{1}{2}x}$$

$$\left[P_{FA} = e^{-\frac{\gamma'}{\sigma_0^2}} \right]$$

$$P_D = P_r \left\{ \frac{x^2[0] + x^2[1]}{\sigma_1^2} > \frac{2\gamma'}{\sigma_1^2} : H_1 \right\}$$

$$= P_r \left\{ \chi_2^2 > \frac{2\gamma'}{\sigma_1^2} \right\}$$

$$\boxed{P_D = e^{-\frac{\gamma'}{\sigma_1^2}}}$$

$$\therefore \ln P_{FA} = -\frac{\gamma'}{\cancel{2}\sigma_0^2} \quad \& \quad \ln P_D = \frac{-\gamma'}{\sigma_1^2}$$

$$\sigma_0^2 \ln P_{FA} = -\gamma' \quad \& \quad \sigma_1^2 \ln P_D = -\gamma'$$

$$\Rightarrow \sigma_0^2 \ln P_{FA} = \sigma_1^2 \ln P_D$$

$$\ln P_{FA}^{\sigma_0^2} = \ln P_D^{\sigma_1^2}$$

$$\boxed{P_{FA} = P_D^{\sigma_1^2/\sigma_0^2}}$$

Q5.

$s[n]$ with zero mean and covariance matrix

$$C_s = \text{diag}\left(\sigma_{s_0}^2, \sigma_{s_1}^2, \ldots, \sigma_{s_{N-1}}^2 \right)$$

$w[n]$ is WGN with variance σ^2

$$\text{So} \quad x \sim \begin{cases} \mathcal{N}(0, \sigma^2 I) & : H_0 \\ \mathcal{N}(0, C_s + \sigma^2 I) & : H_1 \end{cases}$$

NP detector decides H_1 if :

$$L(x) = \frac{\dfrac{1}{(2\pi)^{N/2}\sqrt{\det(C_s+\sigma^2 I)}} e^{-\frac{1}{2} x^T (C_s + \sigma^2 I)^{-1} x}}{\dfrac{1}{(2\pi\sigma^2)^{N/2}} e^{-\frac{1}{2\sigma^2} x^T x}} > \gamma$$

taking log on both sides,

$$-\frac{1}{2} x^T (C_s + \sigma^2 I)^{-1} x + \frac{1}{2\sigma^2} x^T x > \gamma'$$

$$-\frac{1}{2} x^T \left[(C_s + \sigma^2 I)^{-1} - \frac{1}{\sigma^2} I \right] x > \gamma'$$

$$\Rightarrow \quad T(x) = \sigma^2 x^T \left[\frac{1}{\sigma^2} I - (C_s + \sigma^2 I)^{-1} \right] > 2\gamma' \sigma^2$$

From matrix inversion lemma :

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B\left(DA^{-1}B + C^{-1}\right)^{-1} DA^{-1}$$

Putting $A = \sigma^2 I$, $B = D = I$, $C = C_s$.

$$(\sigma^2 I + C_s)^{-1} = \frac{I}{\sigma^2} - \frac{1}{\sigma^4}\left(\frac{1}{\sigma^2} I + C_s^{-1}\right)^{-1}$$

Hence,

$$T(x) = \sigma^2 x^T \left[\frac{I}{\sigma^2} - \left\{\frac{I}{\sigma^2} - \frac{1}{\sigma^4}\left(\frac{1}{\sigma^2} I + C_s^{-1}\right)^{-1}\right\}\right] x > \gamma''$$

$$= x^T \left[\frac{1}{\sigma^2}\left(\frac{I}{\sigma^2} + C_s^{-1}\right)^{-1}\right] x > \gamma''$$

$$T(x) = x^T \left[\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2}\left(C_s + \sigma^2 I\right) C_s^{-1}\right)^{-1}\right] x$$

$$T(x) = x^T (C_s + \sigma^2 I)^{-1} C_s x$$

Let $\hat{x} = C_s (C_s + \sigma^2 I)^{-1} x$

Since,

$$C_s = \begin{bmatrix} \sigma_{s_0}^2 & \cdots & 0 & 0 \\ \vdots & \sigma_{s_1}^2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \sigma_{s_{N-1}}^2 \end{bmatrix}$$

$$\left[C_s + \sigma^2 I\right]^{-1} = \begin{bmatrix} \sigma_{s_0}^2 + \sigma^2 & & \cdots & 0 \\ & \sigma_{s_1}^2 + \sigma^2 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & \sigma_{s_{N-1}}^2 + \sigma^2 \end{bmatrix}$$

$$C_s (C_s + \sigma^2 I)^{-1} = \begin{bmatrix} \frac{\sigma_{s_0}^2}{\sigma_{s_0}^2 + \sigma^2} & \cdots & 0 & 0 \\ \vdots & \frac{\sigma_{s_1}^2}{\sigma_{s_1}^2 + \sigma^2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \frac{\sigma_{s_{N-1}}^2}{\sigma_{s_{N-1}}^2 + \sigma^2} \end{bmatrix}$$

So, NP detector decides H_1 if

$$T(x) = x^T \hat{s} = \sum_{n=0}^{N-1} \frac{\sigma_{s_n}^2}{\sigma_{s_n}^2 + \sigma^2} x^2[n]$$