ECN 514: Detection & Estimation Theory
Assignment 1

Due date: Tuesday, 4th February 2024 (by midnight)

Dheeraj Kumar
dheeraj.kumar@ece.iitr.ac.in

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Maximum Marks: 100

1. Consider the problem of estimation of DC level in white noise using N measurements: $x[n] = A + w[n]$, $n = 0, 1, 2, ..., N-1$

   - A is the parameter to be estimated
   - $w[n]$ is the white Gaussian noise with pdf $\mathcal{N}(0, \sigma^2)$

   consider the following four estimators of A

   (a) $\hat{A}_1 = x[0]$

   (b) $\hat{A}_2 = \frac{1}{N}\sum_{n=0}^{N-1} x[n]$

   (c) $\hat{A}_3 = \frac{1}{2\times N}\sum_{n=0}^{N-1} x[n]$

   (d) $\hat{A}_4 = \frac{A^2}{A^2+\frac{\sigma^2}{N}}\frac{1}{N}\sum_{n=0}^{N-1} x[n]$

   Write a Matlab/Python code to simulate these 4 estimators by generating $N = 1000$ realizations of $x[n]$ considering $A = 1$ and $\sigma^2 = 0.1$.

   (a) Write code to plot the histogram bar plot of the four estimators (four plots for four different estimators) and plot a line connecting these bars as an approximation of their PDFs. (Marks: $4 \times 2.5 = 10$)

   (b) Theoretically determine the mean and variance of all four estimators. (Marks: 10)

   (c) Keeping $\sigma^2 = 0.1$, write the code to calculate and plot the MSE of all the four estimators for $A \in \{-25, -24, ..., 23, 24\}$. (Marks: 10)

2. Consider N observations: $\{x[0], x[1], ..., x[N-1]\}$, where $x[n]'s$ are i.i.d. as $\mathcal{N}(0, \sigma^2)$. The objective is to estimate the value of σ^2. Consider an estimator of variance to be $\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2$.

   (a) Theoretically find the mean of $\hat{\sigma^2}$ and comment on its bias. (Marks: 5)

   (b) Theoretically find the variance of $\hat{\sigma^2}$. (Marks: 5)

   (c) Write a Matlab/Python code to simulate this estimator by generating $N = 100$ realizations of $x[n]$ considering $\sigma^2 = 1$. Plot the histogram bar plot of $\hat{\sigma^2}$ and plot a line connecting these bars as an approximation of their PDFs. Also compute and mention the mean and variance of $\hat{\sigma^2}$ based on the simulation. (Marks: 5+2.5)

   (d) Perform the experiment for $N = 10000$ realizations of $x[n]$ and plot the histogram bar plot of $\hat{\sigma^2}$ and plot a line connecting these bars as an approximation of their PDFs. Also compute and mention the mean and variance of $\hat{\sigma^2}$ based on the simulation. (Marks: 5+2.5)

   (e) If $x[n]'s$ are i.i.d. as $\mathcal{N}(1, \sigma^2)$, what would be the theoretical mean and variance of $\hat{\sigma^2} = \frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2$. (Marks: 5+5)

3. Consider a sinusoidal signal added with WGN with mean 0 and variance σ^2: $x[n] = Acos(2\pi fn) + w[n]$, for $n = 0, 1, ..., N-1$. Here, A is known and $0 < f < \frac{1}{2}$ is the parameter to be estimated.

   (a) Theoretically compute the CRLB for the estimator $\hat{f}$ of f. (Marks: 10)

   (b) Write a Matlab/Python code to plot CRLB as a function of f for $A = 1$, $\sigma^2 = 1$ for the following two values of N. (Marks: 5+5)

      i. $N = 10$

      ii. $N = 100$

   (c) Comment on the variations in CRLB for varying f. Can some frequencies be estimated more effectively than others? Why or why not? (Marks: 5)

4. Let $x[n] = Asin(2\pi f_0 n) + w[n]$ for $n = 0, 1, ..., N-1$. Here f_0 is fixed and known, A is the parameter to be estimated and $w[n]$ is WGN with zero mean and σ^2 variance. Theoretically derive the BLUE for A and compute its variance. (Marks: 5+5)