Indian Institute of Technology Ropar
Department of Mathematics
MAL424/MA205/MA204 - Numerical Analysis
Second Semester of Academic Year 2020-2021

Lab Tutorial sheet 4 March 4, 2021

Instructions:

- Write and save the codes in matlab script files (·m file) along with maintaining the lab report.

Exercise

1. Using LU decomposition, solve the system $A\vec{x} = \vec{b}$, where A and $\vec{b}$ are given as

   (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 4 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 1 & 2 & 3 & 4 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 0.001 & 1 \\ 2 & 3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$
   What do you observe in (c), (d)?

2. Solve the following system of equations using Jacobi's iterative method until two successive iterations are identical when rounded to correct to four significant digits.

   (a) $27x + 6y - z = 85, 6x + 15y + 2z = 72, x + y + 54z = 110$

   (b) $20x + y - 2z = 17, 3x + 20y - z = -18, 2x - 3y + 20z = 25$

   (c) $10x - y + 2z = 6, -x + 11y - z + 3w = 25, 2x - y + 10z - w = -11, 3y - z + 8w = 15$ with initial approximation (0,0,0,0)

3. Solve the following system of equations using Gauss–Seidel method accurate to four significant digits.

   (a) $2x_1 - x_2 + 2x_3 = 3, x_1 + 3x_2 + 3x_3 = -1, x_1 + 2x_2 + 5x_3 = 1$ with initial approximation (0.3, -0.8, 0.3).

   (b) $10x + y + 2z = 44, 2x + 10y + z = 51, x + 2y + 10z = 61$

4. Apply Gauss Siedel method,start with initial approximation (0,0,0) and implement two iterations on the system

   (a) $6x + y + z = 107$

   (b) $x + 9y - 2z = 36$

   (c) $2x - y + 8z = 121$ next consider equations in order (b),(c),(a) and apply same procedure. Compare progress of method in two cases.

5. Find the number of iterations required for the convergence of the solution to the following system of equations using Gauss- Seidel method as well as Jacobi's method and determine which method converges faster.

$$\begin{aligned} 10x - 2y - z - w &= 3 \\ -2x + 10y - z - w &= 15 \\ -x - y + 10z - 2w &= 27 \\ -x - y - 2z + 10w &= -9 \end{aligned}$$

6. Solve the following system of equations using a) Jacobi's iterative method b) Gauss Seidel iterative method:

$$\begin{aligned} x - 5y &= -4 \\ 7x - y &= 6 \end{aligned}$$

   and explain why both methods diverges faster.

7. The linear system

$$\begin{aligned}2x_1 - x_2 + x_3 &= -1\\ 2x_1 + 2x_2 + 2x_3 &= 4\\ -x_1 - x_2 + 2x_3 &= -5\end{aligned}$$

has the solution $(1, 2, -1)^t$.
a. Show that $\rho(T_j) = \frac{\sqrt{5}}{2} > 1$.
b. Show that the Jacobi method with $\mathbf{x}^{(0)} = \mathbf{0}$ fails to give a good approximation after 25 iterations.
c. Show that $\rho(T_g) = \frac{1}{2}$.
d. Use the Gauss-Seidel method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within 10^{-5} in the l_∞ norm.

Note that $T_j = D^{-1}(L + U)$ and $T_g = (D - L)^{-1}U$.

8. The linear system

$$\begin{aligned}x_1 + 2x_2 - 2x_3 &= 7\\ x_1 + x_2 + x_3 &= 2\\ 2x_1 + 2x_2 + x_3 &= 5\end{aligned}$$

has the solution $(1, 2, -1)^t$. a. Show that $\rho(T_j) = 0$.
b. Use the Jacobi method with $\mathbf{x}^{(0)} = \mathbf{0}$ to approximate the solution to the linear system to within

10^{-5}in the l_∞ norm.

c. Show that $\rho(T_g) = 2$.
d. Show that the Gauss-Seidel method applied as in part (b) fails to give a good approximation in 25 iterations.

LU-Decomposition

The goal of this tutorial is to demonstrate that the process of Gaussian Elimination applied to a $n \times n$ matrix A is equivalent to factoring A as the product of a unit lower triangular (say L) and upper triangular matrix (say U), i.e.

$$A = LU.$$

Thus, in order to solve a given system $A\vec{x} = \vec{b}$, where $\vec{x}, \vec{b} \in \mathbb{R}^n$. We first solve $L\vec{y} = \vec{b}$ for $\vec{y} \in \mathbb{R}^n$ and then $U\vec{x} = \vec{y}$. Since L and U are triangular, this is easy.

Algorithm for LU decomposition

STEP 1: Input the matrix A and check that it is a square matrix or not. If not, input a square matrix.

STEP 2: If $A(j, j) = 0$, for any $j = 1, 2, \ldots, n$, pivoting is needed.

STEP 3: Initialize two matrices, namely L as an identity matrix and U as a the given matrix A.

STEP 4: # a. for $k = 1 : n - 1$
b. $L(k + 1 : n, k) = U(k + 1 : n, k)/U(k, k)$; these are Gauss multipliers
c. for $j = k + 1 : n$
d. $U(j, k : n) = U(j, k : n) - L(j, k) * U(k, k : n)$;
e. end
f. end

STEP 5: Now solve the system $L\vec{y} = \vec{b}$ by **forward substitution** for given $\vec{b}$.

STEP 6: Use the value of $\vec{y}$ from STEP 5: to solve $U\vec{x} = \vec{y}$ by **backward substitution**.

Observations

(i). Note that unlike Gauss elimination, we don't have to make an augmented matrix $\left[A|\vec{b}\right]$.

(ii). Note the changes in STEP 4:, if you replace the line numbers # c. to # e. by

$$U(k + 1 : n, k : n) = U(k + 1 : n, k : n) - L(k + 1 : n, k) * U(k, k : n);$$

PA=LU decomposition

It is essentially Gauss elimination with partial pivoting.

Algorithm

STEP 1 - 3: as in LU-decomposition, additionally initialize the matrix P as an identity matrix.

STEP 4: # a. for $i = 1, 2, \ldots,$ col-1
b. for $j = i+1, i+2, \ldots,$ row
c. if $|U(j,i)| \geq |U(i,i)|$
d. $R_j \leftrightarrow R_i$; $P_j \leftrightarrow P_i$
e. end
f. if $U(i,i) \neq 0$
. Set $\lambda = U(j,i)/U(i,i)$; $R_j \rightarrow R_j - \lambda R_i$; $L(j,i) = \lambda$
. else Stop.
. end
. end
. end

Algorithm for Jacobi's Method

STEP 1: Input the matrix A, vector b and initial condition x.

STEP 2: if A is a square matrix
do STEP 3- STEP5
else Error, A must be a square matrix.

STEP 3: Decompose the matrix A as $L + D + U$, where
L is lower triangular matrix such that $L(i,j) = A(i,j)$, $i > j$
m D is diagonal matrix such that $D(i,i) = A(i,i)$ for $i = 1, 2, ..., n$.
U is upper triangular matrix with zero diagonal entries and $U(i,j) = A(i,j)$ for $i < j$.

STEP 4: Define error = Some non zero number
Define epsilon= Tolerance

STEP 5: Let X and x be two successive approximations to the root.
While error > epsilon
do $X = (D)^{-1} * (b - (L+U) * x)$
error $= \dfrac{norm(X - x)}{norm(X)}$
$x = X$
end while

********* End *********