

# AI Applications Lecture 19

Learning Theory and Generalization Gap

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# Outline

Introduction

Preparation: Restatement of Mathematical Notations

Generalization Loss and Generalization Gap

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Performance Limits of Machine Learning by Fano's Inequality

Summary and Next Lecture Preview

# Introduction

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## Introduction (1)

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Machine learning using parametric functions, including generative AI, is the problem of determining good parameters to solve a given task based on training data. For this to be possible, information about the task must be sufficiently provided through the training data sequence. A practical problem, then, is specifically how much training data should be provided.

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## Introduction (2)

In principle, the greater the expressive power of the parametric function, the larger the number of training data required tends to be. Conversely, if the number of parameters to be changed is small and the expressive power of the parametric function is low, as in PEFT, the required number of training data tends to be small. In this lecture, we will theoretically examine the relationship between the expressive power of parametric functions and the number of training data required for learning.

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## Learning Outcomes

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- Explain that when the set of candidate hypotheses in machine learning is large, there exist fundamental performance limits independent of any algorithm or model.

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- Explain that when the set of candidate hypotheses in machine learning is large, there exist fundamental performance limits independent of any algorithm or model.

In the following, we restate the necessary mathematical preparation to achieve these, formulate generalization loss and generalization gap, and describe their quantitative evaluation and limits.

## **Preparation: Restatement of Mathematical Notations**

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## Mathematical Notations (1): Definition

- **Definition:**
  - $(\text{LHS}) := (\text{RHS})$ : Indicates that the left-hand side is defined by the right-hand side. For example,  $a := b$  indicates that  $a$  is defined by  $b$ .

## Mathematical Notations (2): Set

- **Set:**

- Sets are often denoted by uppercase calligraphic letters. Example:  $\mathcal{A}$ .
- $x \in \mathcal{A}$ : Indicates that element  $x$  belongs to set  $\mathcal{A}$ .
- $\{\}$ : Empty set.
- $\{a, b, c\}$ : Set consisting of elements  $a, b, c$  (extensional notation of a set).
- $\{x \in \mathcal{A} \mid P(x)\}$ : Set consisting of elements of set  $\mathcal{A}$  for which proposition  $P(x)$  is true (intensional notation of a set).
- $\mathbb{R}$ : Set of all real numbers.
- $\mathbb{R}_{>0}$ : Set of all positive real numbers.
- $\mathbb{R}_{\geq 0}$ : Set of all non-negative real numbers.
- $\mathbb{Z}$ : Set of all integers.
- $\mathbb{Z}_{>0}$ : Set of all positive integers.
- $\mathbb{Z}_{\geq 0}$ : Set of all non-negative integers.
- $[1, k]_{\mathbb{Z}} := \{1, 2, \dots, k\}$ : Set of integers from 1 to  $k$  for a positive integer  $k$ .

## Mathematical Notations (3): Function

- **Function:**

- $f : \mathcal{X} \rightarrow \mathcal{Y}$ : Indicates that function  $f$  is a map that takes an element of set  $\mathcal{X}$  as input and outputs an element of set  $\mathcal{Y}$ .
- $y = f(x)$ : Indicates that the output is  $y \in \mathcal{Y}$  when  $x \in \mathcal{X}$  is input to function  $f$ .

## Mathematical Notations (4): Vector

- **Vector:**

- In this course, a vector refers to a column of numbers.
- Vectors are denoted by bold italic lowercase letters. Example:  $v$ .
- $v \in \mathbb{R}^n$ : Indicates that vector  $v$  is an  $n$ -dimensional real number vector.
- The  $i$ -th element of vector  $v$  is denoted as  $v_i$ .

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$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}. \quad (1)$$

## Mathematical Notations (5): Matrix

- **Matrix:**

- Matrices are denoted by bold italic uppercase letters. Example:  $A$ .
- $A \in \mathbb{R}^{m,n}$ : Indicates that matrix  $A$  is a real matrix with  $m$  rows and  $n$  columns.
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$$\mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}. \quad (2)$$

## Mathematical Notations (6): Transpose

- **Matrix (transpose):**
  - The transpose of matrix  $A$  is denoted as  $A^\top$ . If  $A \in \mathbb{R}^{m,n}$ , then  $A^\top \in \mathbb{R}^{n,m}$ , and

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$$v^\top = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{1,n}. \quad (4)$$

## Mathematical Notations (7): Tensor

- **Tensor:**
  - In this lecture, the term tensor simply refers to a multidimensional array. A vector can be regarded as a 1st-order tensor, and a matrix as a 2nd-order tensor. Tensors of 3rd order or higher are denoted by underlined bold italic uppercase letters, such as *A*.

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  - Students who have already learned about abstract tensors in mathematics or physics may find it resistant to call simple multidimensional arrays tensors. There is (some) consistency in terminology if one assumes that the basis is always fixed to the standard basis and the tensor in the mathematical sense is identified with its component representation (which becomes a multidimensional array).

# **Generalization Loss and Generalization Gap**

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## Generalization Loss and Generalization Gap

In this section, we formulate the learning problem abstractly and rigorously define expected risk, empirical risk, and the generalization gap.

## Abstract Formulation of Learning Problems

First, we provide a general framework for learning problems consisting of three elements: data, function space, and loss function.

# General Learning Problem Framework

## Definition (General Learning Problem Framework)

- Let the data space be  $\mathcal{Z}$ . An element  $z \in \mathcal{Z}$  of  $\mathcal{Z}$  is a single data point.
- Let the function space (hypothesis set) be  $\mathcal{F}$ .  $\mathcal{F}$  is a set of collected models (hypotheses)  $f$  that are candidates for execution.
- Let the loss function be

$$\ell : \mathcal{F} \times \mathcal{Z} \rightarrow \mathbb{R}. \quad (5)$$

For  $f \in \mathcal{F}$  and  $z \in \mathcal{Z}$ ,  $\ell(f, z)$  is a scalar measuring "how bad  $f$  is for  $z$ ".

## Remark

In Definition 1, the first argument of  $\ell$  is the function  $f$  itself, and the second argument is the data point  $z$ . In practice,  $z$  is often an input-output pair  $(x, y)$ , and  $\ell(f, z)$  is a function measuring the distance between  $f(x)$  and  $y$ . However, in order to treat it more abstractly without limiting it to such a structure, we first formulate it as a function for elements of  $\mathcal{Z}$ .

# Prediction Problem Framework

## Example (Prediction Problem Framework)

Let the data space be  $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ , where the input space is  $\mathcal{X}$  and the output space is  $\mathcal{Y}$ . A single data point is  $z = (x, y) \in \mathcal{X} \times \mathcal{Y}$ .

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Let the predictive function space be  $\mathcal{F} \subset \mathcal{Y}^{\mathcal{X}}$ . That is,  $\mathcal{F}$  is a set of functions  $f$  that take an input  $x \in \mathcal{X}$  and return a prediction  $\hat{y} = f(x) \in \mathcal{Y}$ .

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Write the loss function regarding prediction  $\ell_{\text{pred}} : \mathcal{Y} \times \mathcal{Y} \rightarrow \mathbb{R}$  as

$$\ell_{\text{pred}}(\hat{y}, y). \tag{6}$$

For example, in the case of regression, it is the squared loss

$\ell_{\text{pred}}(\hat{y}, y) = (\hat{y} - y)^2$ , and in the case of classification, it is the 0 – 1 loss

$\ell_{\text{pred}}(\hat{y}, y) = \mathbf{1}\{\hat{y} \neq y\}$ , etc. Here,  $\mathbf{1}\{\cdot\}$  is an indicator function that returns 1 if the event is true and 0 if it is false.

## Prediction Problem Framework (2)

### Example (Prediction Problem Framework, continued)

At this time, if we define the overall loss function  $\ell : \mathcal{F} \times \mathcal{Z} \rightarrow \mathbb{R}$  as

$$\ell(f, (x, y)) := \ell_{\text{pred}}(f(x), y), \quad (7)$$

it becomes a special case within the framework of Definition 1.

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it becomes a special case within the framework of Definition 1.

On a computer, the function space  $\mathcal{F}$  is often represented as a **parametric family**. That is, introducing a parameter space  $\Theta \subset \mathbb{R}^d$ , we prepare a map

$$f_{(\cdot)} : \Theta \rightarrow \mathcal{Y}^{\mathcal{X}}, \quad \theta \mapsto f_{\theta}. \quad (8)$$

At this time, the function space can be written as

$$\mathcal{F} = f_{\Theta} := \{f_{\theta} \mid \theta \in \Theta\}. \quad (9)$$

## Expected and Empirical Risk

Next, we define the expected risk, which is the performance indicator in the real environment, and the empirical risk, which is calculable on a finite number of training data. Hereinafter, the notation of probability distributions and expected values follows standard learning theory textbooks [5].

## Definition (Expected Risk)

Let the probability distribution on the data space  $\mathcal{Z}$  be  $P$ . That is,  $Z \sim P$  is a random data point in the environment. For loss function  $\ell$  and function  $f \in \mathcal{F}$ , we define the **expected risk** as

$$\text{ExpRisk}_{\ell, P}(f) := \mathbb{E}_{Z \sim P} [\ell(f, Z)]. \quad (10)$$

# Expected Risk Meaning

## Remark

$P$  is interpreted as a probability distribution representing the data-generating process in the real environment. Therefore,  $\text{ExpRisk}_{\ell,P}(f)$  is "the expected loss when using  $f$  in the real environment", and is the quantity we ultimately want to minimize. Formally written, ideally we want to solve

$$\underset{f \in \mathcal{F}}{\text{Minimize}} \text{ExpRisk}_{\ell,P}(f). \quad (11)$$

When described by a parametric family  $f_\theta$ , this can be written as

$$\underset{\theta \in \Theta}{\text{Minimize}} \text{ExpRisk}_{\ell,P}(f_\theta). \quad (12)$$

# Empirical Risk

## Definition (Empirical Risk)

Let the data sequence (training dataset) be

$$\mathbf{z} = (z_1, z_2, \dots, z_m) \in \mathcal{Z}^m. \quad (13)$$

At this time, we define the **empirical risk** as

$$\text{EmpRisk}_{\ell, \mathbf{z}}(f) := \frac{1}{m} \sum_{i=1}^m \ell(f, z_i). \quad (14)$$

# Empirical Risk Minimization

## Remark

The empirical risk  $\text{EmpRisk}_{\ell,z}(f)$  is calculable on the training data, and its gradient

$$\nabla_{\theta} \text{EmpRisk}_{\ell,z}(f_{\theta}) \quad (15)$$

can also be efficiently calculated by backpropagation if it is a neural network. Therefore, actual learning algorithms often solve the minimization problem of empirical risk instead of expected risk:

$$\underset{\theta \in \Theta}{\text{Minimize}} \text{EmpRisk}_{\ell,z}(f_{\theta}). \quad (16)$$

This framework is called **Empirical Risk Minimization** [5].

## Generalization Gap and Algorithms

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# Learning Algorithm

## Definition (Learning Algorithm)

Let  $\mathcal{Z}^* := \bigcup_{m=0}^{\infty} \mathcal{Z}^m$  be the set of data sequences of all lengths. We define the **learning algorithm** as

$$\mathfrak{A} : \mathcal{Z}^* \rightarrow \mathcal{F}. \quad (17)$$

$\mathfrak{A}(z)$  is a map that takes a data sequence  $z$  as input and returns a single function  $f$  from the hypothesis set  $\mathcal{F}$ .

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On the other hand, the algorithm  $\mathfrak{A}$  is usually designed to minimize the empirical risk:

$$\text{EmpRisk}_{\ell, z}(\mathfrak{A}(z)). \quad (19)$$

# Generalization Gap

## Definition (Generalization Gap)

Assume that data generating distribution  $P$ , loss function  $\ell$ , data sequence  $z$ , and learning algorithm  $\mathfrak{A}$  are given. At this time, we define the **generalization gap** as

$$\text{GenGap}_{\ell,P,z}(\mathfrak{A}) := \text{ExpRisk}_{\ell,P}(\mathfrak{A}(z)) - \text{EmpRisk}_{\ell,z}(\mathfrak{A}(z)). \quad (20)$$

# Decomposition via Generalization Gap

## Remark

Using equation (20), we can decompose as follows:

$$\text{ExpRisk}_{\ell,P}(\mathfrak{A}(z)) = \text{EmpRisk}_{\ell,z}(\mathfrak{A}(z)) + \left( \text{ExpRisk}_{\ell,P}(\mathfrak{A}(z)) - \text{EmpRisk}_{\ell,z}(\mathfrak{A}(z)) \right) \quad (21)$$

$$= \text{EmpRisk}_{\ell,z}(\mathfrak{A}(z)) + \text{GenGap}_{\ell,P,z}(\mathfrak{A}). \quad (22)$$

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$$= \text{EmpRisk}_{\ell,z}(\mathfrak{A}(z)) + \text{GenGap}_{\ell,P,z}(\mathfrak{A}). \quad (22)$$

The first term on the right-hand side is observable, and the second term is the generalization gap. If the generalization gap is small, it can be said that it "generalizes well" in the sense that the performance on the training data is close to the performance in the real environment as it is.

## Qualitative Discussion of Generalization Gap

Qualitatively, the following intuition holds:

- If there is a sufficient amount of training data to approximate the data generating distribution  $P$ , the empirical risk  $\text{EmpRisk}_{\ell,z}(f)$  becomes a good approximation of the expected risk  $\text{ExpRisk}_{\ell,P}(f)$ , and the generalization gap tends to be small.

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- Even with the same amount of data, if the function space  $\mathcal{F}$  is large (high expressive power), it means the search range of the optimization algorithm  $\mathfrak{A}$  is wide, so  $\mathfrak{A}(z)$  selected dependent only on training data is likely to deviate from the original purpose of expected risk minimization. As a result, the generalization gap tends to be large.

## From Qualitative Discussion to Quantitative Discussion

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In the next section, we introduce Rademacher complexity and covering number for this purpose, and provide an upper bound evaluation of the generalization gap.

## **Quantitative Evaluation of Generalization Gap**

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## Quantitative Evaluation of Generalization Gap

In this section, we quantitatively evaluate the generalization gap when the function space  $\mathcal{F}$  is given, using **Rademacher complexity** and **covering number**.

## Rademacher Complexity: Overview

First, we define Rademacher complexity. This is a measure that gauges how "flexibly" a function space can adapt to a random sign sequence [5].

# Rademacher Random Variables

## Definition (Rademacher Random Variables)

Fix  $m \in \mathbb{Z}_{>0}$ . **Rademacher random variables** are a sequence of independent and identically distributed  $\{-1, +1\}$ -valued random variables

$$\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_m) \tag{23}$$

where each satisfies

$$\mathbb{P}(\sigma_i = 1) = \mathbb{P}(\sigma_i = -1) = \frac{1}{2}. \tag{24}$$

# Empirical Rademacher Complexity

## Definition (Empirical Rademacher Complexity of Function Class)

Assume a data sequence  $z = (z_1, \dots, z_m) \in \mathcal{Z}^m$  and a real-valued function class

$$\mathcal{G} \subset \mathbb{R}^{\mathcal{Z}} \tag{25}$$

are given. At this time, we define the **empirical Rademacher complexity** as

$$\widehat{\mathfrak{R}}_z(\mathcal{G}) := \mathbb{E}_{\sigma} \left[ \sup_{g \in \mathcal{G}} \frac{1}{m} \sum_{i=1}^m \sigma_i g(z_i) \right]. \tag{26}$$

Here, the expectation is with respect to the Rademacher random variable sequence  $\sigma$  in Definition 8.

## Expected Rademacher Complexity

### Definition (Expected Rademacher Complexity)

Let the data generating distribution be  $P$ , and sample  $m$  data sequences  $\mathbf{Z} = (Z_1, \dots, Z_m) \sim P^m$  independently and identically. At this time, we define the **expected Rademacher complexity** as

$$\mathfrak{R}_{P,m}(\mathcal{G}) := \mathbb{E}_{\mathbf{Z} \sim P^m} [\widehat{\mathfrak{R}}_{\mathbf{Z}}(\mathcal{G})]. \quad (27)$$

## Loss-Composed Function Class

We are interested in the Rademacher complexity for the loss function  $\ell$  and the hypothesis class  $\mathcal{F}$ . Therefore, we consider the function class composed of the loss function and the hypothesis class:

$$\ell \circ \mathcal{F} := \{ g_f : \mathcal{Z} \rightarrow \mathbb{R} \mid f \in \mathcal{F}, g_f(z) := \ell(f, z) \}. \quad (28)$$

## Rademacher Complexity with Loss

### Definition (Rademacher Complexity of Hypothesis Class with Loss)

For loss function  $\ell$  and hypothesis class  $\mathcal{F}$ , we define the **empirical Rademacher complexity** on data sequence  $z$  as

$$\text{EmpRad}_{\ell,z}(\mathcal{F}) := \widehat{\mathfrak{R}}_z(\ell \circ \mathcal{F}), \quad (29)$$

and the **expected Rademacher complexity** on data length  $m$  as

$$\text{ExpRad}_{\ell,P,m}(\mathcal{F}) := \mathfrak{R}_{P,m}(\ell \circ \mathcal{F}). \quad (30)$$

# Rademacher Complexity Intuition

## Remark

Rademacher complexity measures the ability to select a function  $g$  from the function class  $\mathcal{G}$  that maximizes the inner product  $\frac{1}{m} \sum_{i=1}^m \sigma_i g(z_i)$  with the random label, when regarding the random sequence  $\sigma$  as a "random label". In other words, it is an indicator of "how well random noise can be fitted", representing the flexibility (complexity) of the function class.

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Also, for the inclusion relationship of function classes  $\mathcal{G} \subset \mathcal{G}'$ ,

$$\mathfrak{R}_{P,m}(\mathcal{G}) \leq \mathfrak{R}_{P,m}(\mathcal{G}') \tag{31}$$

holds [5, Proposition 26.9]. Therefore,  $\text{ExpRad}_{\ell,P,m}(\mathcal{F})$  is, as the name suggests, a measure of the "size" or **complexity** of the function space.

## Rademacher Complexity and Generalization Gap

Using Rademacher complexity, the generalization gap can be bounded from above with high probability. The following theorem is based on Shalev-Shwartz and Ben-David's textbook [5, Theorem 26.5].

## Generalization Gap Upper Bound

### Theorem (Generalization Gap Upper Bound by Rademacher Complexity)

Let the data generating distribution be  $P$ , and sample  $Z = (Z_1, \dots, Z_m) \sim P^m$  independently and identically. For loss function  $\ell$  and hypothesis class  $\mathcal{F}$ , assume that

$$\ell(f, z) \in [0, 1] \tag{32}$$

holds for all  $f \in \mathcal{F}$  and  $z \in \mathcal{Z}$  (i.e., the loss is bounded in the interval  $[0, 1]$ ).

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holds for all  $f \in \mathcal{F}$  and  $z \in \mathcal{Z}$  (i.e., the loss is bounded in the interval  $[0, 1]$ ).

At this time, for any  $\delta \in (0, 1)$ , with probability at least  $1 - \delta$ ,

$$\sup_{f \in \mathcal{F}} \left( \text{ExpRisk}_{\ell, P}(f) - \text{EmpRisk}_{\ell, Z}(f) \right) \leq 2 \text{EmpRad}_{\ell, Z}(\mathcal{F}) + 3 \sqrt{\frac{\log(2/\delta)}{2m}} \quad (33)$$

holds.

## Rademacher Bound: Interpretation

### Remark

Theorem 12 indicates the following points:

- The left-hand side is the maximum value of the generalization gap

$$\text{ExpRisk}_{\ell,P}(f) - \text{EmpRisk}_{\ell,Z}(f) \quad (34)$$

for all functions  $f \in \mathcal{F}$ . Therefore, whatever  $f = \mathfrak{A}(Z)$  any learning algorithm  $\mathfrak{A}$  chooses, its generalization gap is bounded from above by the right-hand side.

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- The first term on the right-hand side  $\text{EmpRad}_{\ell, Z}(\mathcal{F})$  is the complexity of the function class, and the second term is a term dependent only on the sample size  $m$  and confidence  $\delta$ . Therefore, **the larger the function class** (the larger the Rademacher complexity), the larger the upper bound of the generalization gap tends to be.

## Rademaher Bound: Interpretation 2

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- On the other hand, to reduce the empirical risk  $\text{EmpRisk}_{\ell, \mathcal{Z}}(f)$ , it is often advantageous to allow a function class with higher "expressive power". That is,
  - If the function class  $\mathcal{F}$  is enlarged, the minimum empirical risk decreases, but the upper bound of the generalization gap increases.
  - If the function class  $\mathcal{F}$  is made smaller, the upper bound of the generalization gap decreases, but a function close to the optimal value of the expected risk might be outside  $\mathcal{F}$  in the first place.

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  - If the function class  $\mathcal{F}$  is made smaller, the upper bound of the generalization gap decreases, but a function close to the optimal value of the expected risk might be outside  $\mathcal{F}$  in the first place.
- Therefore, to **reduce both empirical risk and generalization gap**, it is necessary to appropriately adjust the size of the function space (model capacity), and there exists a trade-off here. This provides the theoretical background when comparing large-scale models with many parameters and models with limited effective degrees of freedom like PEFT.

## Covering Number and Dudley Integral

Directly evaluating Rademacher complexity itself is often difficult.

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Directly evaluating Rademacher complexity itself is often difficult. Therefore, we introduce the **covering number** as another measure of the "size" of the function class, and use this to evaluate Rademacher complexity [3, 1].

# Covering Number

## Definition (Covering Number)

Let the universal set be  $\mathcal{S}$ , and its subset be  $\mathcal{A} \subset \mathcal{S}$ . Let the distance (or pseudometric) on  $\mathcal{S}$  be

$$d : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}. \quad (35)$$

For  $\epsilon \geq 0$ , an  $\epsilon$ -cover of  $\mathcal{A}$  is a finite set  $C \subset \mathcal{S}$  such that

$$\forall a \in \mathcal{A} \exists c \in C \text{ such that } d(a, c) \leq \epsilon \quad (36)$$

is satisfied. That is, each point in  $\mathcal{A}$  is within distance  $\epsilon$  from some point in  $C$ .

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is satisfied. That is, each point in  $\mathcal{A}$  is within distance  $\epsilon$  from some point in  $C$ .

We define the  $\epsilon$ -covering number of  $\mathcal{A}$  as

$$\text{CN}_{\mathcal{A}, d}(\epsilon) := \min\{ |C| \mid C \subset \mathcal{S} \text{ is an } \epsilon\text{-cover of } \mathcal{A} \}. \quad (37)$$

# Covering Number as Size Measure

## Remark

The covering number  $\text{CN}_{\mathcal{A},d}(\epsilon)$  represents the "size" of set  $\mathcal{A}$  at distance scale  $\epsilon$ . Intuitively, it counts "how many balls of radius  $\epsilon$  are needed to cover  $\mathcal{A}$ ".

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In particular, if  $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{S}$ , then for any  $\epsilon \geq 0$ ,

$$\text{CN}_{\mathcal{A},d}(\epsilon) \leq \text{CN}_{\mathcal{A}',d}(\epsilon) \tag{38}$$

holds. This is obvious from the fact that the number of balls needed to cover  $\mathcal{A}'$  must be greater than or equal to the number needed to cover its subset  $\mathcal{A}$ . Therefore, the covering number is also a measure of the complexity (size) of the set.

## Dudley Integral Bound

Next, we describe the **Dudley entropy integral** type evaluation which evaluates Rademacher complexity from the covering number. The following theorem is an example of a Dudley-type evaluation for Rademacher processes, based on textbooks on concentration inequalities such as Boucheron, Lugosi, Massart [1, Theorem 13.2] (constants may vary slightly depending on the literature).

# Dudley Integral for Rademacher Complexity

## Theorem (Evaluation of Rademacher Complexity by Dudley Integral)

Consider a data sequence  $z = (z_1, \dots, z_m) \in \mathcal{Z}^m$  and a real-valued function class  $\mathcal{G} \subset \mathbb{R}^{\mathcal{Z}}$ . Define the pseudometric  $d_{2,z}$  on  $\mathcal{G}$  as

$$d_{2,z}(g, g') := \sqrt{\frac{1}{m} \sum_{i=1}^m (g(z_i) - g'(z_i))^2}. \quad (39)$$

Also, let the diameter of  $\mathcal{G}$  be  $\text{diam}_{d_{2,z}}(\mathcal{G}) := \sup_{g,g' \in \mathcal{G}} d_{2,z}(g, g')$ .

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At this time, there exists an absolute constant  $C > 0$  such that

$$\widehat{\mathfrak{R}}_z(\mathcal{G}) \leq \frac{C}{\sqrt{m}} \int_0^{\text{diam}_{d_{2,z}}(\mathcal{G})} \sqrt{\log \text{CN}_{\mathcal{G}, d_{2,z}}(\epsilon)} d\epsilon \quad (40)$$

holds.

## Dudley Bound: Interpretation

### Remark

In the right-hand side of Theorem 14, the covering number  $\text{CN}_{\mathcal{G}, d_2, z}(\epsilon)$  at each scale  $\epsilon$  of the function class  $\mathcal{G}$  appears. If the function class is very large and many balls are needed at every scale, the integral value becomes large, and the Rademacher complexity also becomes large. Conversely, if the function class is relatively small (or smooth), the covering number becomes small, and the Rademacher complexity is also kept small.

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By combining this theorem with Theorem 12, we can mathematically support the qualitative trend that:

- If the covering number of the function space is small (i.e., the function space is "small" in a certain sense), the Rademacher complexity becomes small, and as a result, the generalization gap also becomes small.

# **Performance Limits of Machine Learning by Fano's Inequality**

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## Performance Limits via Fano's Inequality

In the previous chapter, we gave an evaluation method for the **upper bound** of the generalization gap when a function set included in parametric functions is given. This can be regarded as a **sufficient condition** for generalization in the sense that "if certain conditions (e.g., upper bound of Rademacher complexity) are met, generalization works well".

## Performance Limits via Fano's Inequality

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This can be regarded as a **sufficient condition** for generalization in the sense that "if certain conditions (e.g., upper bound of Rademacher complexity) are met, generalization works well".

In this chapter, conversely, we provide a **necessary condition** that "better performance cannot be achieved" no matter what algorithm or model is used.

Specifically, we show the performance limits of machine learning using **Fano's inequality** in information theory [2, 4].

## Information-Theoretic Preliminaries

First, we define the basic concepts of information theory necessary to state Fano's inequality. The notation hereinafter follows the textbook by Cover and Thomas [2].

# Shannon Entropy

## Definition (Shannon Entropy)

Let a random variable  $V$  on a finite set  $\mathcal{V}$  have a probability mass function

$$p_V(v) := \mathbb{P}(V = v), \quad v \in \mathcal{V}. \quad (41)$$

We define **Shannon entropy** as

$$H(V) := - \sum_{v \in \mathcal{V}} p_V(v) \log p_V(v). \quad (42)$$

Here, the base of the logarithm is 2, and the unit is bits.

# Conditional Entropy and Mutual Information

## Definition (Conditional Entropy and Mutual Information)

For random variables  $V, Y$  on finite sets  $\mathcal{V}, \mathcal{Y}$ , we define **conditional entropy** as

$$H(V | Y) := \mathbb{E}_Y [H(V | Y = y)] = - \sum_{y \in \mathcal{Y}} p_Y(y) \sum_{v \in \mathcal{V}} p_{V|Y}(v | y) \log p_{V|Y}(v | y). \quad (43)$$

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Also, we define **mutual information** as

$$I(V; Y) := H(V) - H(V | Y). \quad (44)$$

# Kullback–Leibler Divergence

## Definition (Kullback–Leibler Divergence)

For two probability distributions  $P$  and  $Q$  on the same finite set  $\mathcal{Y}$ , we define **Kullback–Leibler divergence** as

$$D(P\|Q) := \sum_{y \in \mathcal{Y}} P(y) \log \frac{P(y)}{Q(y)}. \quad (45)$$

# Binary Entropy Function

## Definition (Binary Entropy Function)

For  $p \in [0, 1]$ , we define the **binary entropy function** as

$$h_2(p) := -p \log p - (1 - p) \log(1 - p) \quad (46)$$

(in the case of  $0 \log 0$ , it is considered 0).

## Fano's Inequality: Setup

Fano's inequality is an inequality that bounds the error probability from below by mutual information in the problem of estimating a discrete parameter  $V$  from observation  $Y$  [2, Theorem 2.10.1].

# Fano's Inequality

## Theorem (Fano's Inequality)

Consider a random variable  $V$  on a finite set  $\mathcal{V}$  and an observation  $Y$ . For any estimator  $\hat{V} = \hat{V}(Y)$  that estimates the value of  $V$  from  $Y$ , consider the error probability

$$P_e := \mathbb{P}(\hat{V} \neq V). \quad (47)$$

At this time, the following holds:

$$H(V \mid Y) \leq h_2(P_e) + P_e \log(|\mathcal{V}| - 1). \quad (48)$$

Especially when  $|\mathcal{V}| \geq 2$ , as a coarser upper bound,

$$H(V \mid Y) \leq 1 + P_e \log |\mathcal{V}| \quad (49)$$

holds.

## Fano's Inequality (continued)

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Furthermore, when  $V$  follows a uniform distribution

$$\mathbb{P}(V = v) = \frac{1}{|\mathcal{V}|}, \quad v \in \mathcal{V}, \tag{50}$$

the error probability satisfies

$$P_e \geq 1 - \frac{I(V; Y) + \log 2}{\log |\mathcal{V}|}. \tag{51}$$

## Fano's Inequality: Interpretation

### Remark

Equation (48) shows that "the uncertainty of  $V$ ,  $H(V | Y)$ , is constrained from below by the error probability  $P_e$  and the number of states  $|\mathcal{V}|$ ". Intuitively,

- If  $P_e$  becomes very small,  $H(V | Y)$  must also become small (that is, the uncertainty of  $V$  is small when  $Y$  is known).
- Conversely, if  $H(V | Y)$  is still large ( $V$  is not well known even looking at  $Y$ ),  $P_e$  must be large to some extent.

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Equation (51) for the special case of uniform distribution shows that when the number of possible states of  $V$ ,  $|\mathcal{V}|$ , is large, **unless the mutual information  $I(V; Y)$  is sufficiently large, the error probability  $P_e$  cannot be made small.** This mathematically expresses the basic intuition in machine learning that

- "The larger the number of hypotheses to distinguish (number of

## Fano and Minimax Risk

Fano's inequality is used to give a lower bound for **minimax risk** in statistical estimation problems. The following explanation follows [4].

## Minimax Risk

### Definition (Minimax Risk)

Let the parameter space be  $\Theta$ , and assume that for each  $\theta \in \Theta$ , a distribution  $P_\theta^n$  of observation vector  $Y^n$  is given (e.g., product distribution of  $n$  independent observations). Let the loss function

$$\ell : \Theta \times \Theta \rightarrow \mathbb{R}_{\geq 0} \tag{52}$$

be the loss between the true parameter  $\theta$  and the estimated value  $\hat{\theta}$ .

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Let the estimator be

$$\hat{\theta} : \mathcal{Y}^n \rightarrow \Theta, \tag{53}$$

and define the risk under  $\theta$  as

$$R_n(\theta, \hat{\theta}) := \mathbb{E}_{Y^n \sim P_\theta^n} [\ell(\theta, \hat{\theta}(Y^n))]. \tag{54}$$

## Remark

If we regard  $\Theta$  as "a set of candidate generative models",  $P_\theta^n$  as "data distribution observed under parameter  $\theta$ ", and  $\ell(\theta, \hat{\theta})$  as "distance between true model and learned model", then  $M_n(\Theta, \ell)$  represents the performance limit in the sense that "the risk cannot be reduced below this for the worst true model no matter what learning algorithm is used".

## Local Minimax Risk Lower Bound

Using Fano's inequality, we can provide a lower bound for this minimax risk [4, Corollary 1].

## Local Minimax Risk Lower Bound (Theorem)

**Theorem (Local Minimax Risk Lower Bound Based on Fano's Inequality [4, Corollary 1])**

Assume the setting of Definition 21. Fix a finite set of parameters

$$\{\theta_1, \dots, \theta_M\} \subset \Theta, \quad (56)$$

and let the distance (or pseudometric) between  $\theta$  be  $d : \Theta \times \Theta \rightarrow \mathbb{R}_{\geq 0}$ . Also, assume there exists a non-decreasing function

$$\Phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \quad (57)$$

representing the relationship between loss function  $\ell$  and distance  $d$  such that

$$\forall \theta, \theta' \in \Theta, \ell(\theta, \theta') \geq \Phi(d(\theta, \theta')) \quad (58)$$

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## Local Minimax Risk: Interpretation

### Remark

Theorem 22 corresponds to the following intuition:

- The set  $\{\theta_1, \dots, \theta_M\}$  is sufficiently separated from each other in the sense of distance  $d$  (packing condition (59)), and much information is needed to distinguish them.
- On the other hand, if the data distribution  $P_{\theta_v}^n$  under each  $\theta_v$  is not very far from a certain auxiliary distribution  $Q_n$  (Kullback–Leibler divergence is small), it is difficult to identify the index  $v$  from the observation  $Y^n$ .
- By Fano's inequality, the error probability of index identification is bounded from below, and as a result, a lower bound is derived that the estimation error in the sense of distance  $d$  or loss  $\ell$  cannot be reduced below a certain level.

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## **Summary and Next Lecture Preview**

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## Summary: Learning Outcomes Revisited

Finally, we briefly review the main points corresponding to each of the Learning Outcomes stated in this lecture.

- **Difference between empirical risk and expected risk**

Expected risk is the expected value of loss with respect to the data generating distribution and represents the average performance in the real environment.

On the other hand, empirical risk is the average loss on a finite number of training data and is a quantity that can be actually calculated and optimized.

The difference between the two is the generalization gap, and the smaller this is, the more the performance on the training data generalizes to the real environment.

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## Next Lecture Preview

Next time, we will learn about the basic framework of **learning from rewards**, that is, **Reinforcement Learning**.

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