Do both approaches yield the same dynamics?

$$\textbf{Substitution} \xrightarrow{\textbf{followed by}} \textbf{Variation}$$

- $L = L_t(-A_r(q_r)\dot{q}_r) + L_r(q_r, \dot{q}_r)$

- $L = L_t(\dot{q}_t) + L_r(q_r, \dot{q}_r)$

$$\rightarrow \left(\frac{d}{dt} \frac{\partial L_r}{\partial \dot{q}_r} - \frac{\partial L_r}{\partial q_r} \right) = A_r(q_r)^T \frac{d}{dt} \frac{\partial L_t}{\dot{q}_t}$$

$$\rightarrow \frac{d}{dt}\left(-A_r(q_r)^T \frac{\partial L_t}{\partial \dot{q}_t} + \frac{\partial L_r}{\dot{q}_r} \right)$$

$$\rightarrow -\left(\frac{\partial(A_r(q_r)\dot{q}_r)}{q_r}\right)^T \frac{\partial L_t}{\partial \dot{q}_t} + \frac{\partial L_r}{q_r}$$

$$\textbf{Variation} \xrightarrow{\textbf{followed by}} \textbf{Substitution}$$

$$\rightarrow \left(\frac{d}{dt} \frac{\partial L_r}{\partial \dot{q}_r} - \frac{\partial L_r}{\partial q_r} \right) = A_r(q_r)^T \frac{d}{dt} \frac{\partial L_t}{\dot{q}_t} + \left(\dot{A}_r - \frac{\partial \left(A_r(q_r) \dot{q}_r \right)}{\partial q_r} \right)^T \frac{\partial L_t}{\partial \dot{q}_t}$$

$$\rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = 0$$

A. Lewis and R. Murray Variational Principles for Constrained Systems : Theory and Experiment
S. Ray and J. Shamanna On Virtual Displacements and Virtual Work in Lagrangian Systems

Vakonomic Dynamics

Nonholonomic Dynamics

Both are mathematically valid approaches to impose constraints
However, experimentally mechanical systems have been found to obey Nonholonomic Dynamics

$$\rightarrow \left(\frac{d}{dt} \frac{\partial L_r}{\partial \dot{q}_r} - \frac{\partial L_r}{\partial q_r} \right) = A_r(q_r)^T \frac{d}{dt} \frac{\partial L_t}{\dot{q}_t}$$

$$+ \left(\dot{A}_r - \frac{\partial \left(A_r(q_r) \dot{q}_r \right)}{\partial q_r} \right)^T \frac{\partial L_t}{\partial \dot{q}_t}$$

Do both approaches yield the same dynamics?

Substitution $\xrightarrow{\text{followed by}}$ Variation

- $L = L_t(-A_r(q_r)\dot{q}_r) + L_r(q_r, \dot{q}_r)$

$$\rightarrow \left(\frac{d}{dt}\frac{\partial L_r}{\partial \dot{q}_r} - \frac{\partial L_r}{\partial q_r} \right) = A_r(q_r)^T \frac{d}{dt}\frac{\partial L_t}{\dot{q}_t} + \left(\dot{A}_r - \frac{\partial \left(A_r(q_r)\dot{q}_r \right)}{\partial q_r} \right)^T \frac{\partial L_t}{\partial \dot{q}_t}$$

Vakonomic Dynamics

Variation $\xrightarrow{\text{followed by}}$ Substitution

- $L = L_t(\dot{q}_t) + L_r(q_r, \dot{q}_r)$

$$\rightarrow \left(\frac{d}{dt}\frac{\partial L_r}{\partial \dot{q}_r} - \frac{\partial L_r}{\partial q_r} \right) = A_r(q_r)^T \frac{d}{dt}\frac{\partial L_t}{\dot{q}_t}$$

Nonholonomic Dynamics

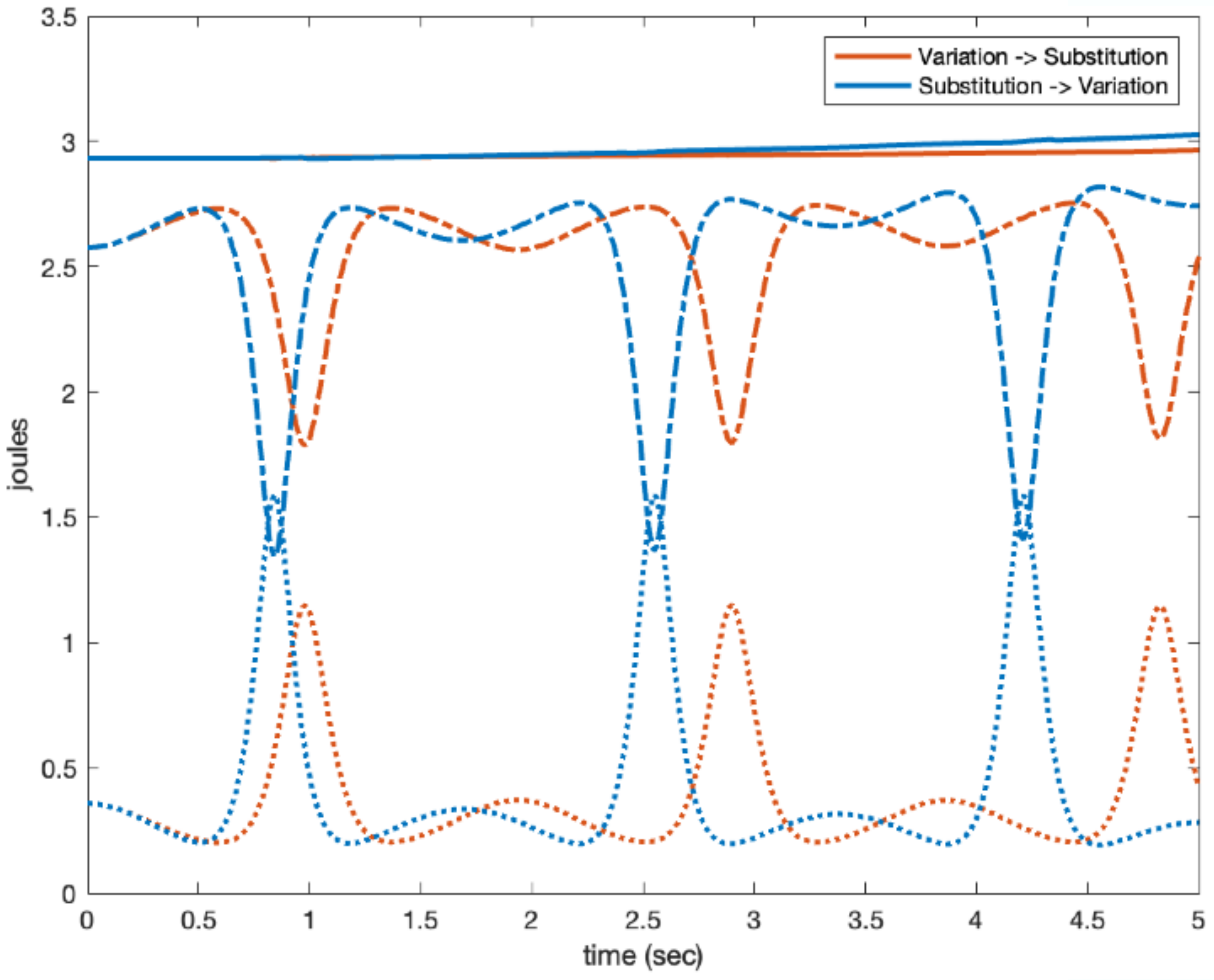
Both are mathematically valid approaches to impose constraints

However, experimentally mechanical systems have been found to obey Nonholonomic Dynamics

A. Lewis and R. Murray Variational Principles for Constrained Systems : Theory and Experiment

S. Ray and J. Shamanna On Virtual Displacements and Virtual Work in Lagrangian Systems

Rolling disk



Both satisfy conservation of energy!