Multiplication Rule

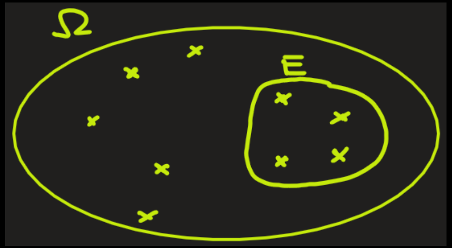
Problem: You roll a die. Let's define two events :
A: "The number is even." → {2, 4, 6}
B: "The number is greater than 3." → {4, 5, 6}

Are these independent ?
In other words, are they related ?

Probability: Definition (Review)

Probability of an event E:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$



Or,

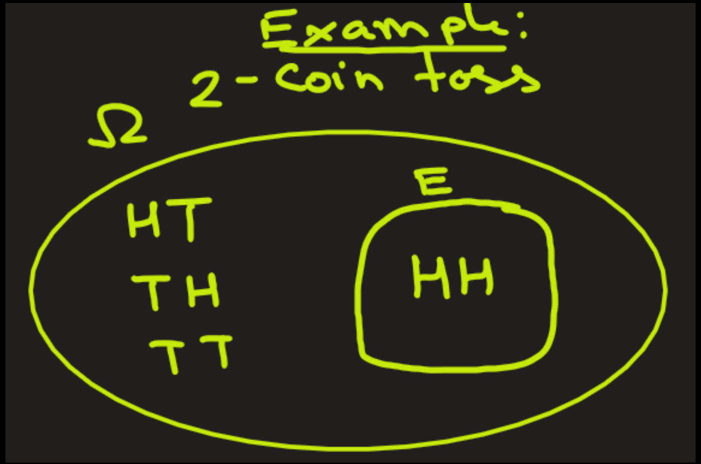
$$P(E) = \frac{|E|}{|\Omega|}$$

Here,

| E | = number (size) of elements in event E,

| Ω | = total number (size) of elements in sample space

coin1 coin2



Example: In 2-coin toss, what is the probability of getting **both heads**?

Ans: Here sample space is Ω = {HH, HT, TH, TT}. Size of Ω is | Ω | = 4

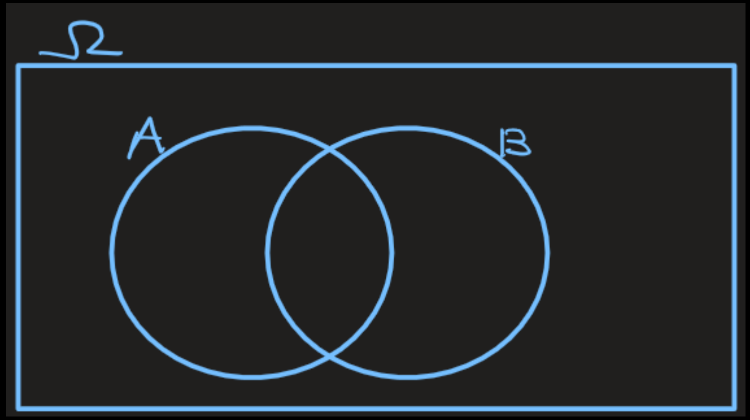
The event E is both toss are heads: E = {HH}. Size of E is |E|= 1

P(E) = 1/4.

Multiplication Rule

The **multiplication rule** is used to find the probability that **two (or more) events happen together** — that is, the intersection of events.
Mathematically, for **two events A and B**:

$$\mathbf{P(A \cap B) = P(A) \times P(B|A)}$$



$P(A \cap B)$: Probability that both A **and** B occur.
$P(A)$: Probability that event A occurs.
$P(B|A)$: Probability that event B occurs given that A has already occurred.

Special Case: If A and B are independent, then $P(B|A) = P(B)$. In this case,

$$P(A \cap B) = P(A) \times P(B)$$

We can use above equation to define independence of two events:

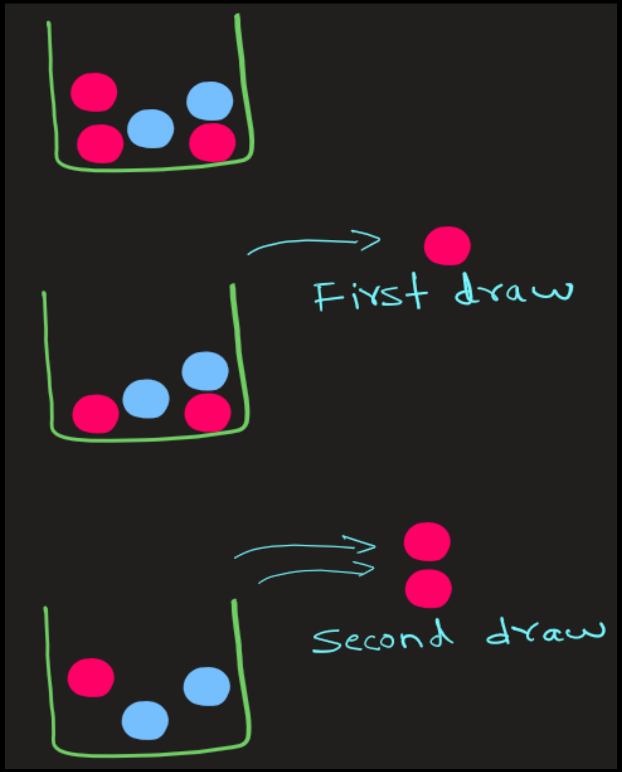
- Two events A and B are **independent** if and only if

$$P(A \cap B) = P(A) \times P(B)$$

Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: A box contains 5 balls — 3 red and 2 blue.
Two balls are drawn one after another **without replacement**. What is the probability that both balls are red ?



Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: A box contains 5 balls — 3 red and 2 blue.
Two balls are drawn one after another **without replacement**. What is the probability that both balls are red ?

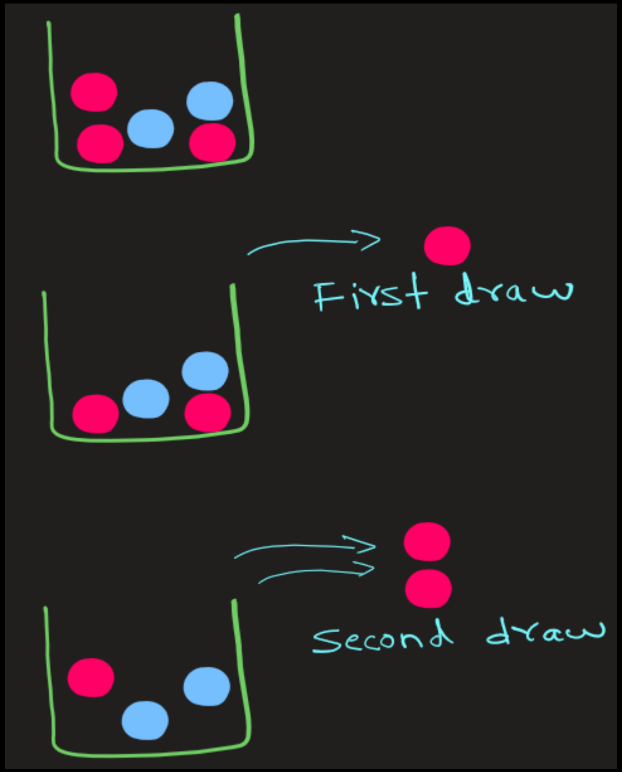
Ans: Let
A: event that "the first ball is red."
B: event that "the second ball is red."
$A \cap B$ means that first draw is red ball and second draw is red ball.
Here we have to find $P(A \cap B)$.

A: First draw is red ball

$$P(A) = \frac{3}{5}$$

B|A: Second draw gives red ball, given that 1st draw was red:
If the first ball was red, only 2 red and 2 blue remain, so

$$P(B \mid A) = \frac{2}{4} = \frac{1}{2}$$



Then, by the **multiplication rule**,

$$P(A \cap B) = P(A) \times P(B \mid A) = \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$$

Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: Suppose you roll two fair dice.
What is the probability that the first die shows a 3 and the second die shows an even number.

3

Even

Multiplication Rule

P(A∩B) = P(A) × P(B|A)

Problem: Suppose you roll two fair dice.
What is the probability that the first die shows a 3 and the second die shows an even number.

3

Even

Ans: Let
A : event that "the first die shows a 3." A = {3}
B : event that "the second die shows an even number." B = {2, 4, 6}
$A \cap B$ means that the first die shows a 3 and the second die shows an even number.
Here we have to find $P(A \cap B)$.

A: First die shows 3

$$P(A) = \frac{1}{6}$$

B|A: The second die shows even number, given that first die shows 3.
Because the two dice are **independent**,

P(B|A) = $P(B) = \frac{3}{6}$

$P(A \cap B) = P(A) \times P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$. This can be verified from the figure

Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: You roll a die. Let's define two events :

A: "The number is even." → {2, 4, 6}

B: "The number is greater than 3." → {4, 5, 6}

Are these independent ?

Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: You roll a die. Let's define two events :

A: "The number is even." → {2, 4, 6}

B: "The number is greater than 3." → {4, 5, 6}

Are these independent ?

Ans:

These appear to look dependent — because both involve overlapping conditions on the same die outcome (evenness and size).

Let's check mathematically: Two events A and $B$are **independent** if and only if $P(A \cap B) = P(A) \times P(B)$

Step1) Compute the probabilities:

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}$$

$$A \cap B = \{4, 6\} \Rightarrow P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

Step2) Test for independence: If $A$and B were independent, we would have

$$P(A \cap B) = P(A) \times P(B)$$

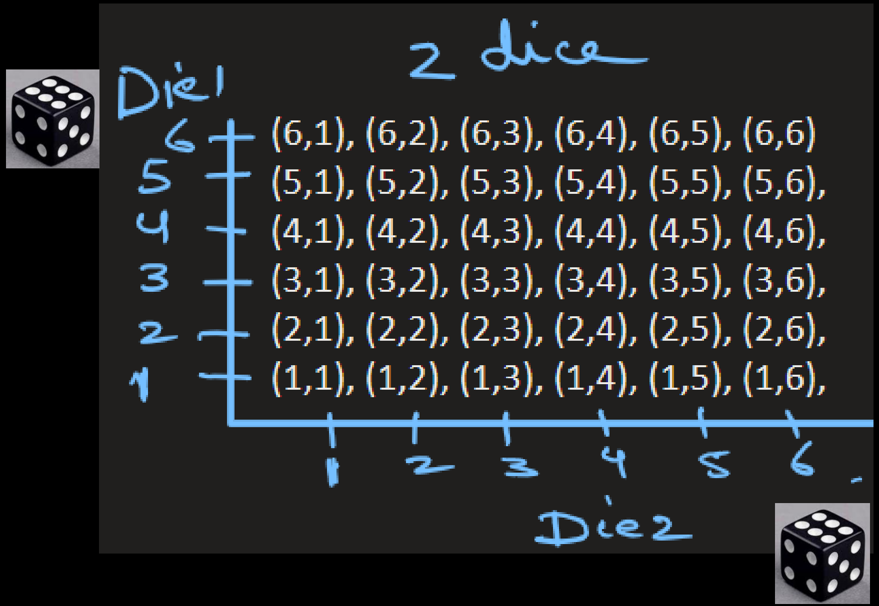
$$\frac{1}{3} \neq \frac{1}{2} \times \frac{1}{2}$$

So, A and B are not independent

Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: You roll a pair of dice. What is the probability that the sum of two dice is greater than 8 and the first die shows a 5 ?



Multiplication Rule

P(A∩B) = P(A) × P(B|A)

Problem: You roll a pair of dice. What is the probability that the sum of two dice is greater than 8 and the first die shows a 5 ?

Ans: Let
A: event that “the first die shows a 5.”
B: event that “the sum of two dice is greater than 8.”
$A \cap B$ means that the first die shows a 5 and the sum of two dice is greater than 8.
Here we have to find $P(A \cap B)$.

A: First die shows 5. Here

$$P(A) = \frac{1}{6}$$

B|A: The sum of two dice is greater than 8, given that the first die shows a 5. The possibles outcomes of B|A are {4, 5, 6}

$$P(B|A) = \frac{3}{6} = \frac{1}{2}$$

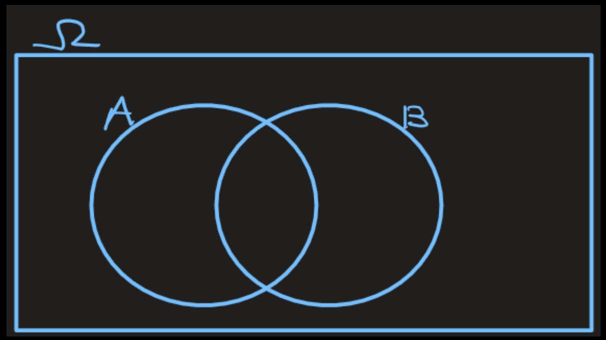
So, P(A∩B) = P(A) × P(B/A) = $\frac{1}{6} \times \frac{\mathbf{1}}{\mathbf{2}} = \frac{1}{12}$

Verify: Here |A∩B| = |{ (5,4), (5,5), (5,6)}| = 3 and size of sample space = 36 → P(A∩B) = 3/36 = 1/12

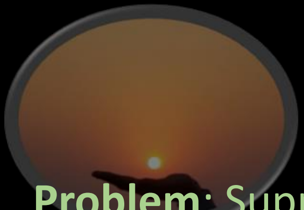
Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: Suppose an **AI spam filter** classifies emails.
Based on past data we know that
30% of all emails contain 'discount'
80% of emails containing 'discount' are spam
How many emails contain the word “discount” and are spam.



Multiplication Rule



$P(A \cap B) = P(A) \times P(B|A)$

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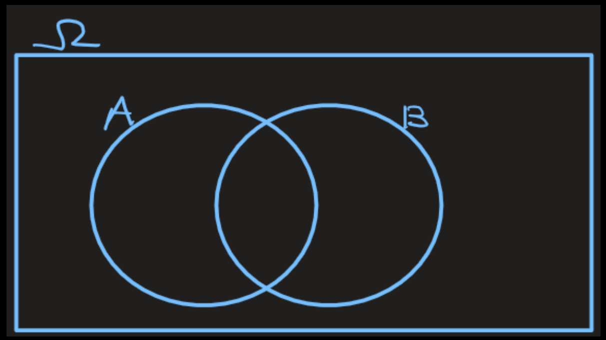


Ans: Let
A = “email contains the word ‘discount’”
B = “email is spam”
$A \cap B$ means that emails contain the word “discount” and are spam
Here we have to find $P(A \cap B)$. This would give you the percentage.

A: 30% of all emails contain 'discount’ → $P(A) = 0.3$

B|A: 80% of emails are spam, given that they contain the word 'discount’ → $P(B/A) = 0.8$

$P(A \cap B) = P(A) \times P(B/A)$
$= 0.3 \times 0.8$
$= 0.24$ So, 24% of all emails are both spam and contain the word “discount”

Multiplication Rule

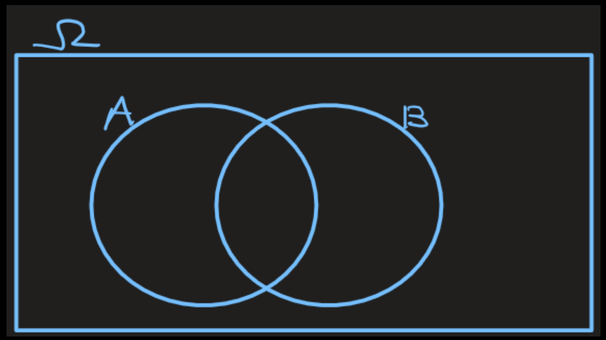
P(A∩B) = P(A) × P(B|A)

Problem: Suppose

1% of people have the disease → $P(D) = 0.01$

Test is 99% sensitive → $P(T^+ \mid D) = 0.99$

Test is 95% specific → $P(T^- \mid No\ Disease) = 0.95$



What is the probability that a person has disease and test is positive ?

Multiplication Rule

P(A∩B) = P(A) × P(B|A)

Problem: Suppose
1% of people have the disease → $P(D) = 0.01$
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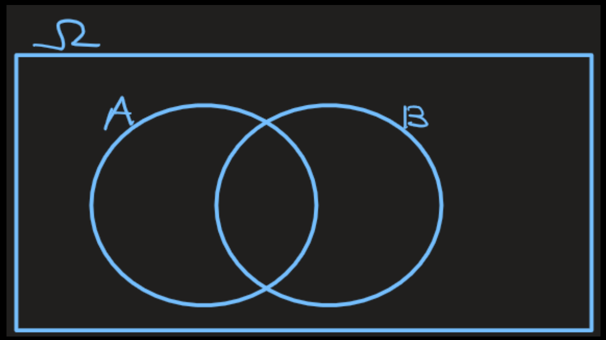


What is the probability that a person has disease and test is positive ?

Ans:

P(Disease and Test Positive) = P(Disease) × P(Test Positive | Disease)
= $P(D)$ x $P(T^+ \mid D)$
= 0.01 × 0.99
= 0.0099

EXTRA

$P(A \cap B) = P(A) \times P(B|A)$

Extra problems

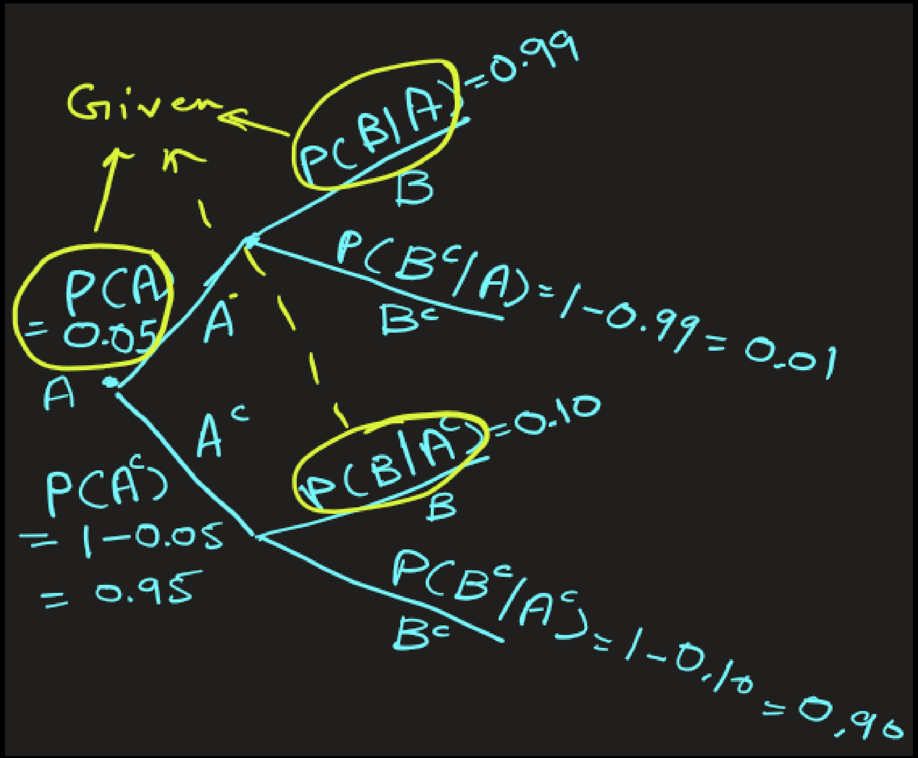
Multiplication Rule

$P(A \cap B) = P(A) \times P(B|A)$

Problem: (Radar Detection) If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present. the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05.
What is the probability of aircraft presence and no detection?

Ans: Let
A: event that "an aircraft is present". Then, A^C means that aircraft is not present..
B: event that "the radar generates an alarm". Then, B^C means that radar does not generate an alarm, i.e. no detection.
Given to us:

Here $P(A) = 0.05$,
$P(B|A) = 0.99$,
$P(B|A^C) = 0.10$



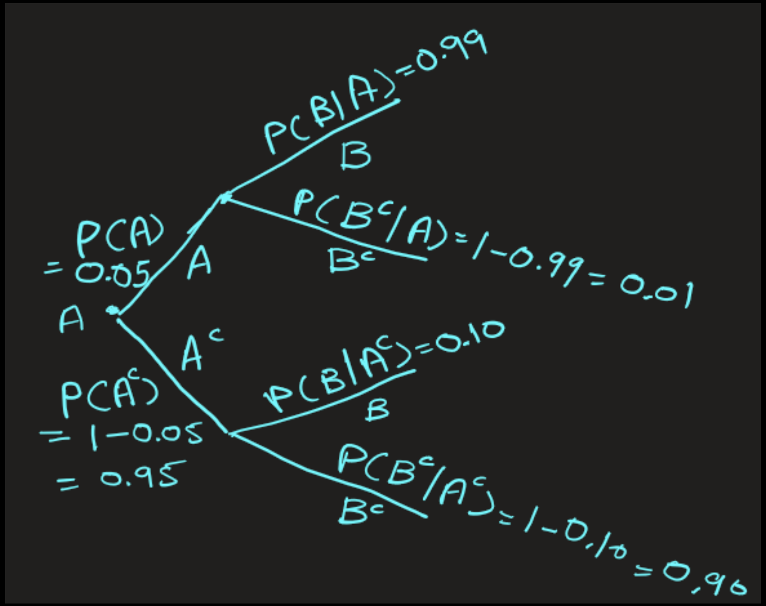
$A \cap B^C$ means that aircraft presence and no detection by radar.
Here we have to find $P(A \cap B^C)$.

Multiplication Rule

P(A∩B) = P(A) × P(B|A)

$\boldsymbol{B^C|A}$ **:** The radar shows no detection, given that an aircraft is present.
$P(B^C|A) = 0.01$

So, $P(A \cap B^C) = P(A) \times P(B^C|A)$
= 0.05 x 0.01 = 0.0005



Multiplication Rule: Generalization

$P(A \cap B) = P(A) \times P(B|A)$

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2|A_1)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2 \cap A_1)$$
$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_2 \cap A_1) \cdot P(A_4|A_3 \cap A_2 \cap A_1)$$