# Exponential Distribution

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

**Problem:**

Let’s say the average lifetime of a bulb from a manufacturer is 5 years. If you buy the bulb from this manufacturer what is the probability that the

a) bulb lasts **between 6 to 9 years**

b) bulb lasts **exactly 5 years.**

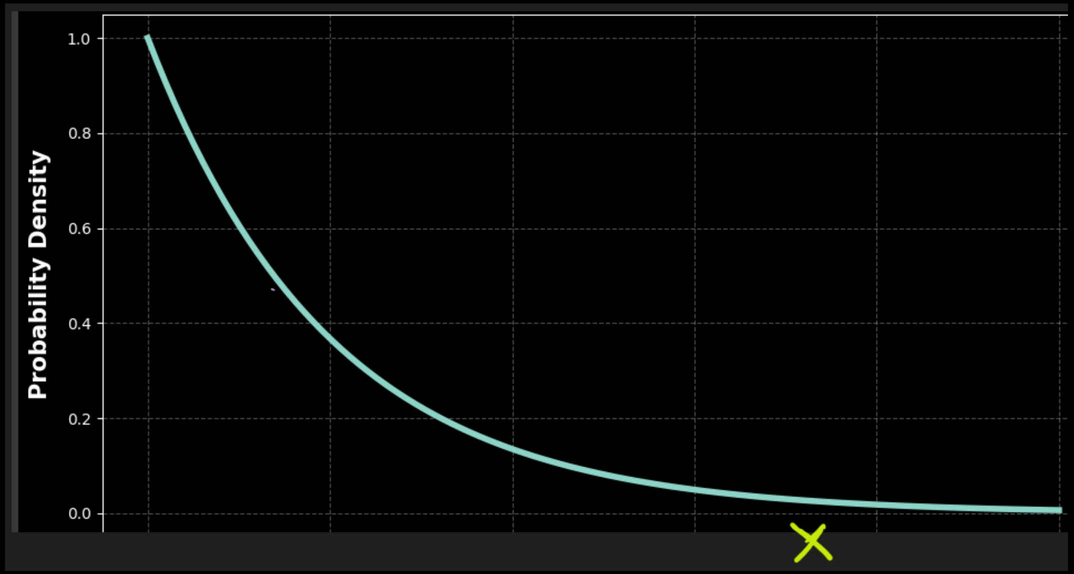
# Exponential Distribution: Theory

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The **exponential distribution** is a **continuous probability distribution.**
It models the **time between events** in a process where events occur

**a) continuously and independently,**
**b) at a constant average rate**.

Events could be **waiting times** for next customer at store or **lifetimes** of bulb, etc.




The PDF is given by:

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Here,
$X$ is random variable that follows an exponential distribution,
$\lambda > 0$ is the **rate parameter** (events per unit time).

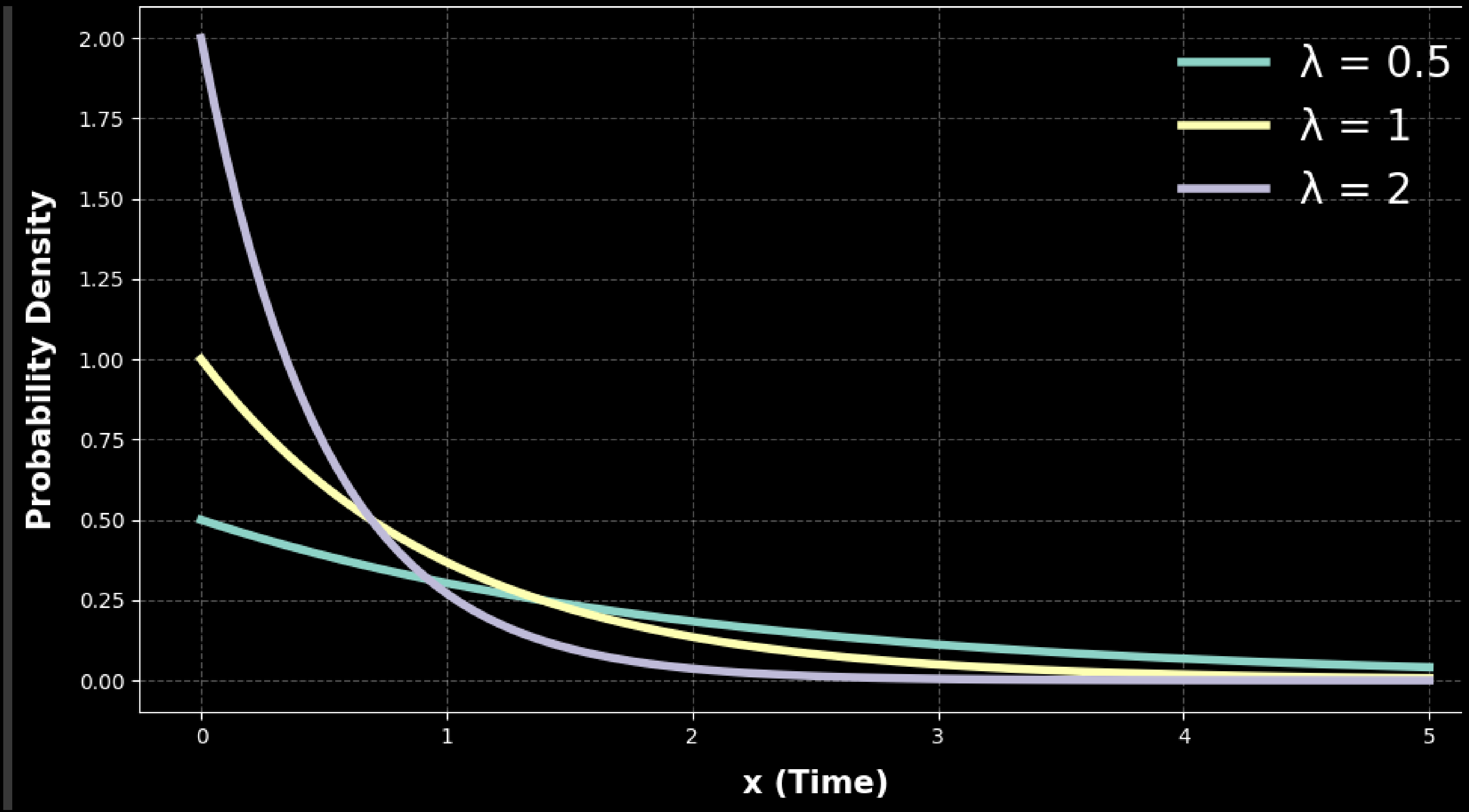
These distributions are used to determine

- lifetime of an electronic component (until it fails),
- time between customer arrivals at a store , etc.

# Exponential Distribution: Plots

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

The **exponential distribution** curve for different values of $\lambda > 0$

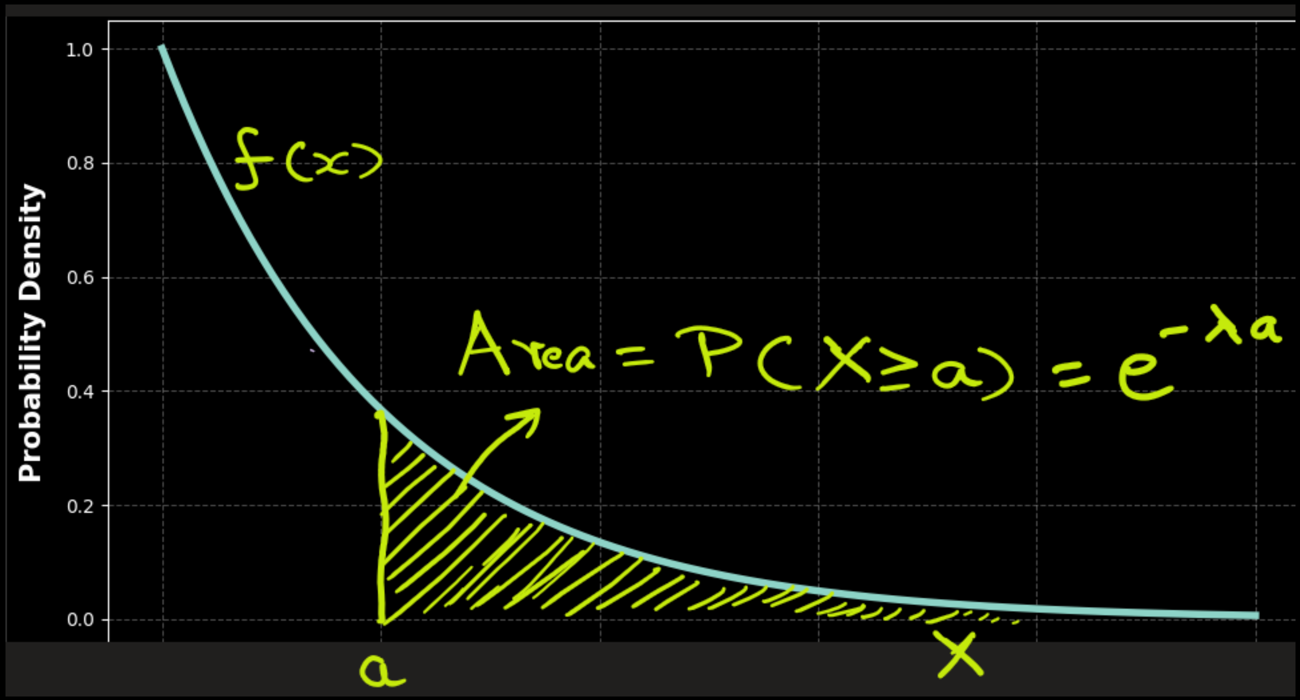


# Exponential Distribution: Probability

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

The probability that random variable X > a, is given by area under the curve:

$$P(X > a) = \int_a^\infty f(x)\, dx$$




Substituting, $f(x) = \lambda e^{-\lambda x}, \quad x \ge 0$

$$P(X > a) = \int_a^\infty \lambda e^{-\lambda x}\, dx$$

After integrating we get,

$$P(X > a) = e^{-\lambda a}$$

Note:

- P(X=a) = 0
- P(X > 0) = 1, i.e. the total area under the curve is 1.
  This means, $P(X < a) = 1 - e^{-\lambda a}$

# Exponential Distribution: Applications

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

**Problem:**

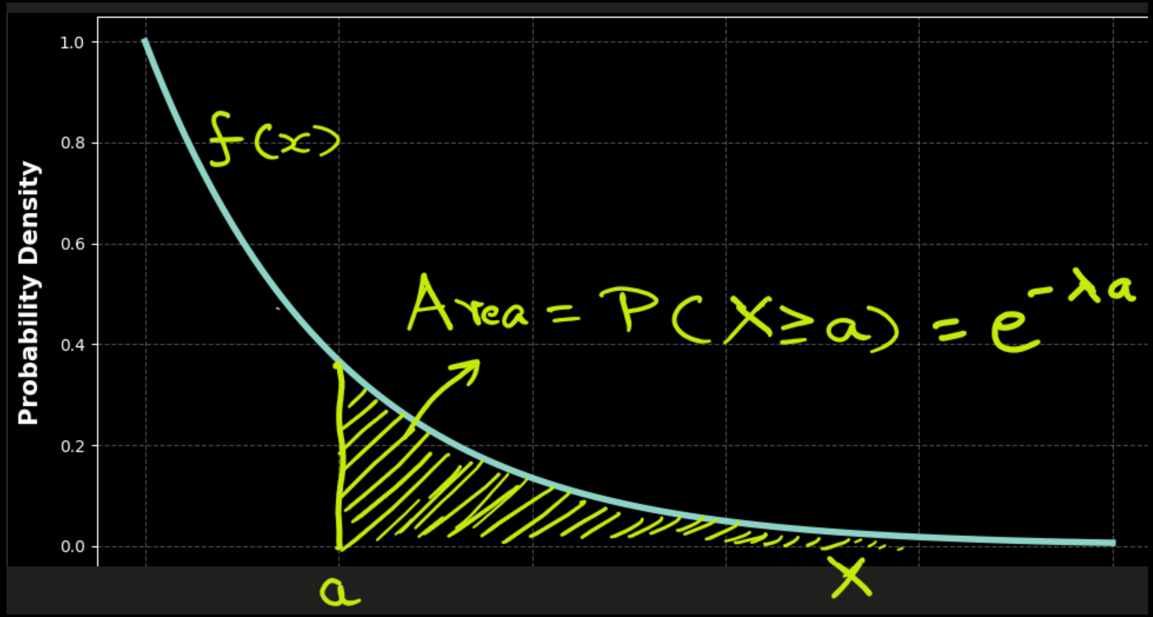
Suppose customers arrive at a store **on average every 5 minutes**. Find the probability that

a) the next customer arrives **after** 3 minutes.

b) the next customer arrives **within** 3 minutes.


Ans:

That means the rate $\lambda = 1/5 = 0.2$ customers per minute.




a) P (next customer arrives **after** 3 minutes)

= $P(X > 3) = e^{-\lambda x} = e^{-0.2\times 3} = e^{-0.6} = 0.5488$

b) P (next customer arrives **within** 3 minutes)

= 1 - P (next customer arrives **after** 3 minutes)

= 1 – 0.5488 = 0.4512

# Exponential Distribution: Applications

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

**Problem:** Let's say the average lifetime of a bulb is 5 years. Find the probability

a) the bulb lasts **more than 8 years,**

b) the bulb lasts **less than 8 years**

c) the bulb lasts **between 6 to 9 years**

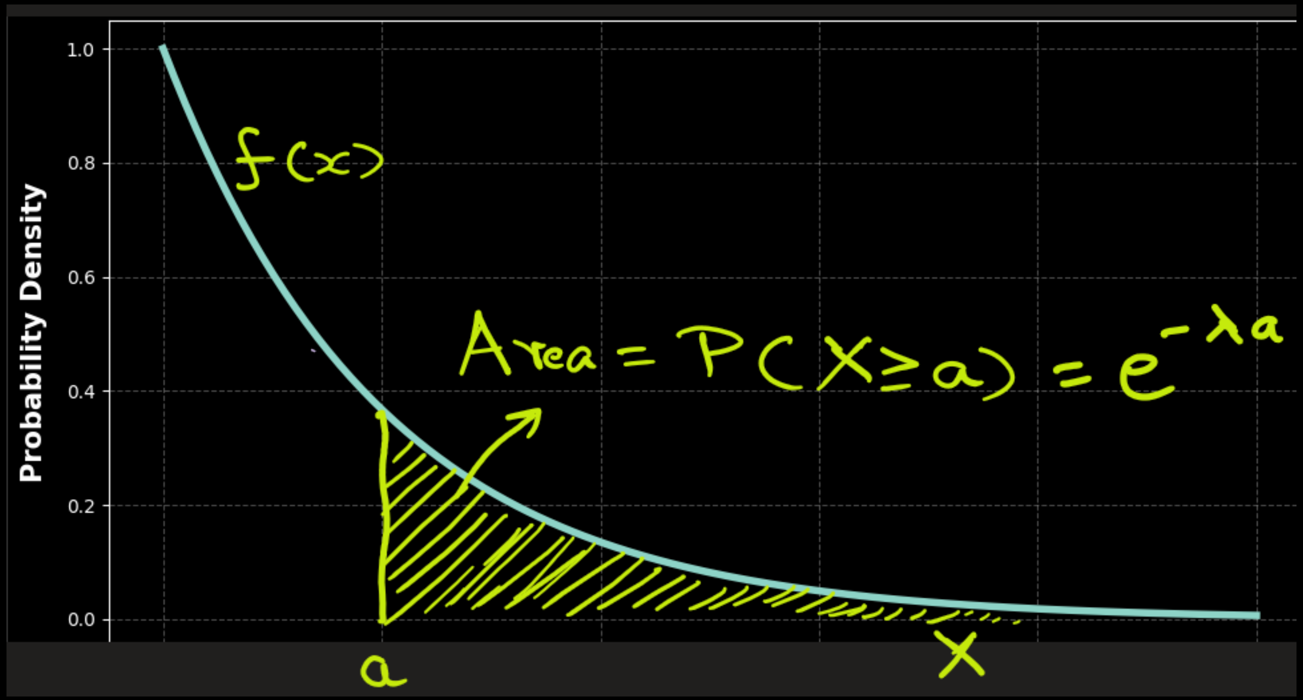
d) the bulb lasts **exactly 5 years**

**Ans:** Here $\lambda = \frac{1}{5} = 0.2$

a) P(bulb lasts **more than 8 years**)

= Area to under the curve to the **right** of X = 8

= $P(X > 8) = e^{-0.2\times 8} = e^{-1.6} = 0.2019$




b) P(bulb lasts **less than 8 years**)

= Area to the **left** of X = 8

= P( X < 8 )

= 1 – P( X > 8 )

= 1 – 0.2019 = 0.7981 = 79.81%

# Exponential Distribution: Applications

$$f(x;\lambda) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$$

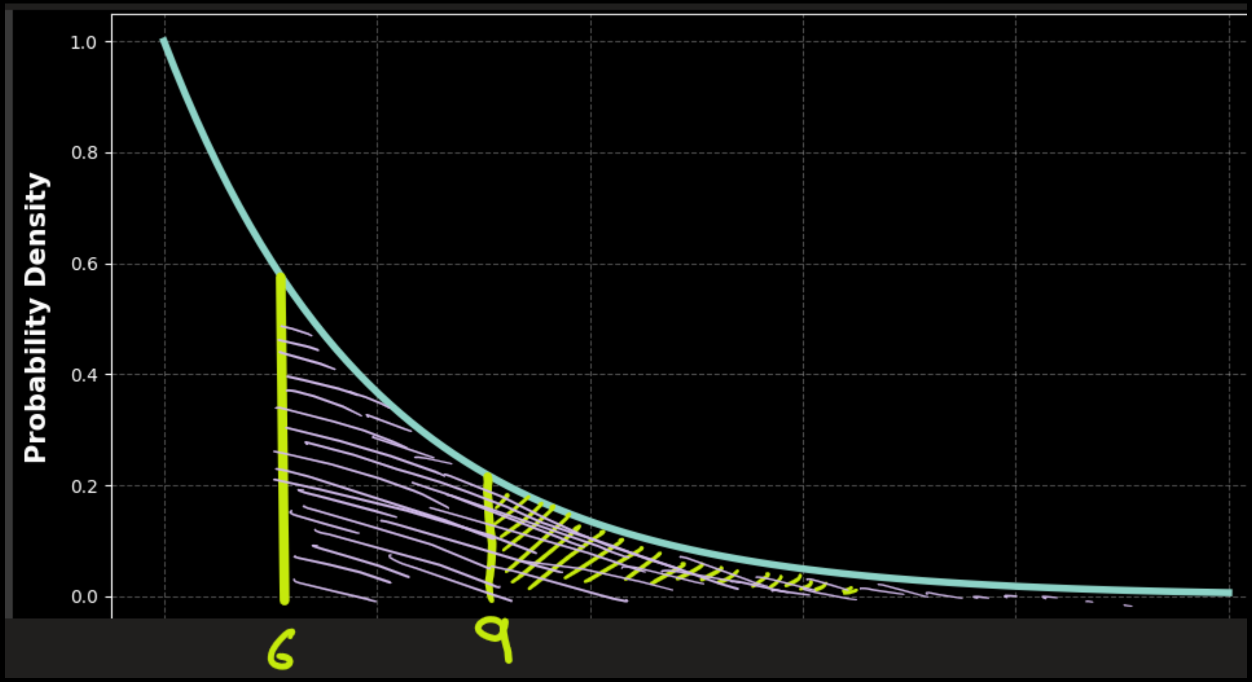
c) P(bulb lasts **between 6 to 9 years**)

= Area between X = 6 and X = 9

= P(X > **6**) - P(X > **9**)

= 0.3010 – 0.1653

= 0.1357 = 13.57 %

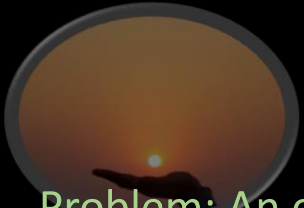



d) P(bulb lasts **exactly 5 years**)

= Area under X = 5

= 0

# EXTRA



Problem: An online recommendation model triggers an automatic **retrain** whenever a sudden drop in validation performance occurs. The **time between retrain triggers** is modeled as an **Exponential** random variable with **mean 30 days** (so the rate $\lambda = 1/30$per day).

Answer:

1.What is the probability the next retrain happens **within 10 days**?

2.What is the probability the model **survives** (i.e., no retrain) for **60 days**?

3.Given the model has already survived **20 days** without a retrain, what is the probability it survives an **additional 30 days**? (Illustrate the memoryless property.)

4.What is the **expected number of retrains** in one year (365 days)?

Ans:

λ=1/30≈0.0333333 per day.

**1)** $P(X \leq 10)$

$$P(X \leq 10) = 1 - e^{-\lambda \cdot 10} = 1 - e^{-10/30} = 1 - e^{-1/3} \approx 0.283469$$

= 28.35 %

**2)** $P(X > 60)$

$$P(X > 60) = e^{-\lambda \cdot 60} = e^{-60/30} = e^{-2} \approx 0.135335$$

= 13.53%

**3) Memoryless:** $P(X > 20 + 30 \mid X > 20) = P(X > 30)$

$$P(\textit{survive additional 30 days} \mid \textit{survived 20}) = e^{-\lambda \cdot 30} = e^{-1} \approx 0.367879$$

**= 36. 79 %**

(So, the conditional probability depends only on the additional waiting time, not on the fact that it already lasted 20 days.)

**4) Expected retrains in 365 days**

$$\mathbb{E}[\textit{retrain count in 365 days}] = \lambda \cdot 365 = \frac{365}{30} \approx 12.1667$$

**About 12.17 retrains per year (on average).**

# Exponential Distribution

The exponential distribution is commonly used for:

**Modeling waiting times:**

Time between customer arrivals at a store.

Time between phone calls at a call center.

**Reliability and lifetime modeling:**

Lifetime of an electronic component (until it fails).

**Queueing systems:**

Time between arrivals in a queue (Poisson process).

Essentially, when events occur **randomly but at a constant rate**, the exponential distribution fits perfectly.

# Exponential Distribution vs Poisson Distribution

| Concept | **Poisson Distribution** | **Exponential Distribution** |
|---|---|---|
| **Type** | Discrete | Continuous |
| **Describes** | Number of events in a fixed interval | Time (or distance) between two consecutive events |
| **Example Question** | “How many customers arrive in 10 minutes?” | “How long until the next customer arrives?” |
| **Parameter** | $\lambda$: average number of events per interval | $\lambda$: rate of events per unit time |

**Example**
Suppose customers arrive at a rate of **2 per minute** ( $\lambda = 2$).

**a) Poisson:**
Probability that **3 customers arrive in 1 minute**:

$$P(X=3) = e^{-2}\frac{2^3}{3!} = 0.180$$

**b) Exponential:**
Probability that **next customer arrives within 30 seconds (0.5 minutes)**:

$$P(T \leq 0.5) = 1 - e^{-2(0.5)} = 1 - e^{-1} = 0.632$$

# Exponential Distribution vs Poisson Distribution

**Example**
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# Extra

The **mean (expected value)** is $\frac{1}{\lambda}$.

The **variance** is $\frac{1}{\lambda^2}$.

The PDF:

The CDF