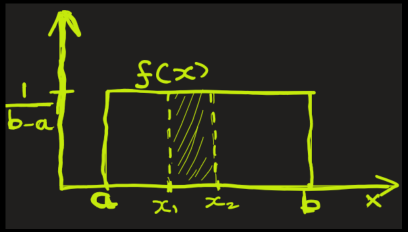
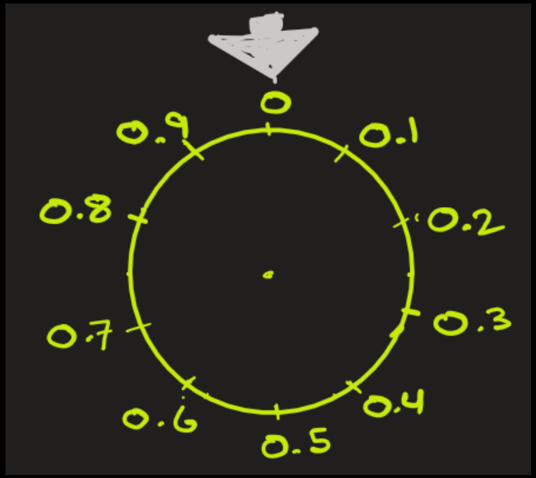
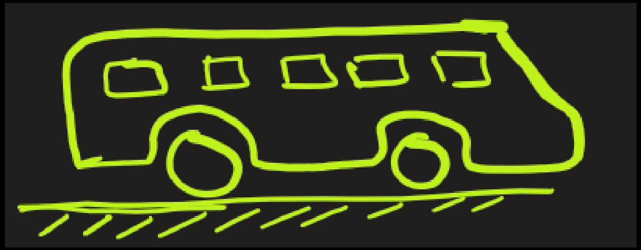
Uniform Distribution



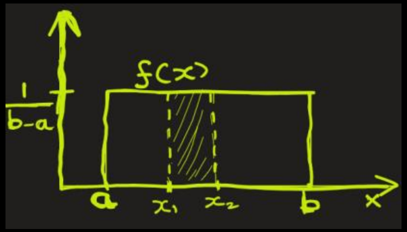
Problem1: If you spin a fair wheel that can stop **anywhere** between 0 and 1 then what is the probability that it stops **exactly** at 0.5 ?





Problem2: You arrive at a bus stop where buses come every 10 minutes. What's the probability that you'll wait **exactly 5 minutes** for the next bus?

Uniform Distribution: Theory

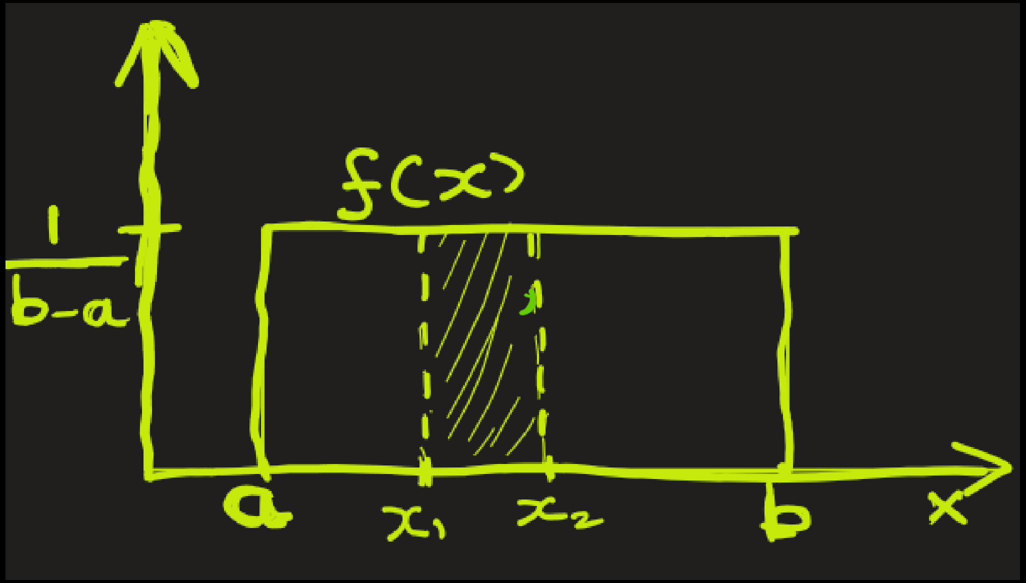


It generally refers to continuous distribution.

Uniform Distribution is a **probability distribution** in which **all outcomes are equally likely** within a given range.

In other words, all intervals of equal length within [a, b] are **equally likely**.

1) The total area under curve = 1   (True for all PDF)



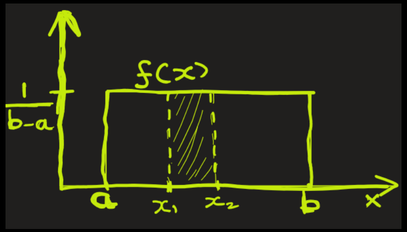
2) The PDF is given by,

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

3) The probability of x1 < X < x2 is the area between curve and x1 and x2:

$$P(x_1 \leq X \leq x_2) = \frac{x_2 - x_1}{b-a}$$

Uniform Distribution: Theory

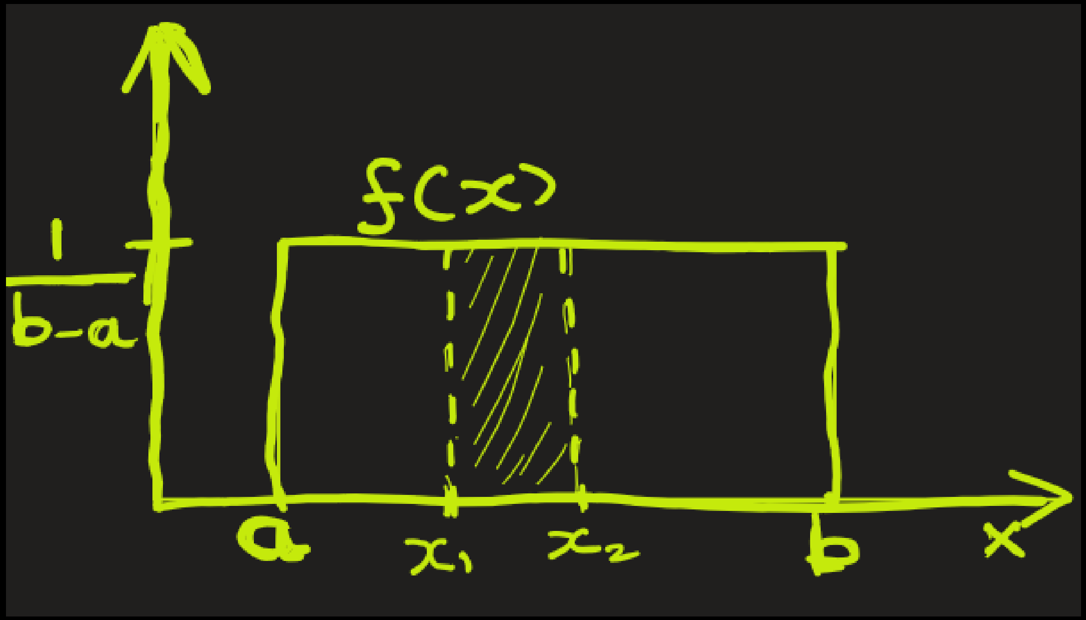


A common notation that is used for a random variable X that follows a **uniform distribution** between two values a and b, is

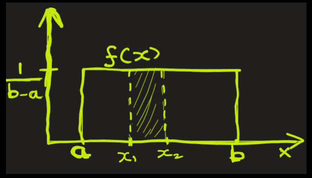
$$X \sim \mathbf{U\ (a,\ b)}$$

$$X \sim \mathbf{Uniform(a,\ b)}$$

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

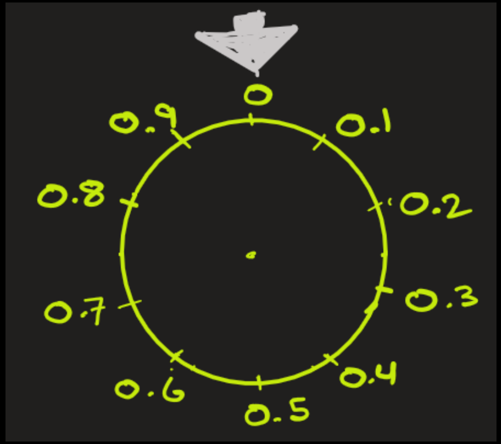


Uniform Distribution: Applications



Problem: If you spin a fair wheel that can stop anywhere between 0 and 1 with uniform probability distribution **Uniform(0, 1)**, then what is the probability that it stops

a) Between 0.1 and 0.3
b) Between 0.4 and 0.6
c) Exactly at 0.5

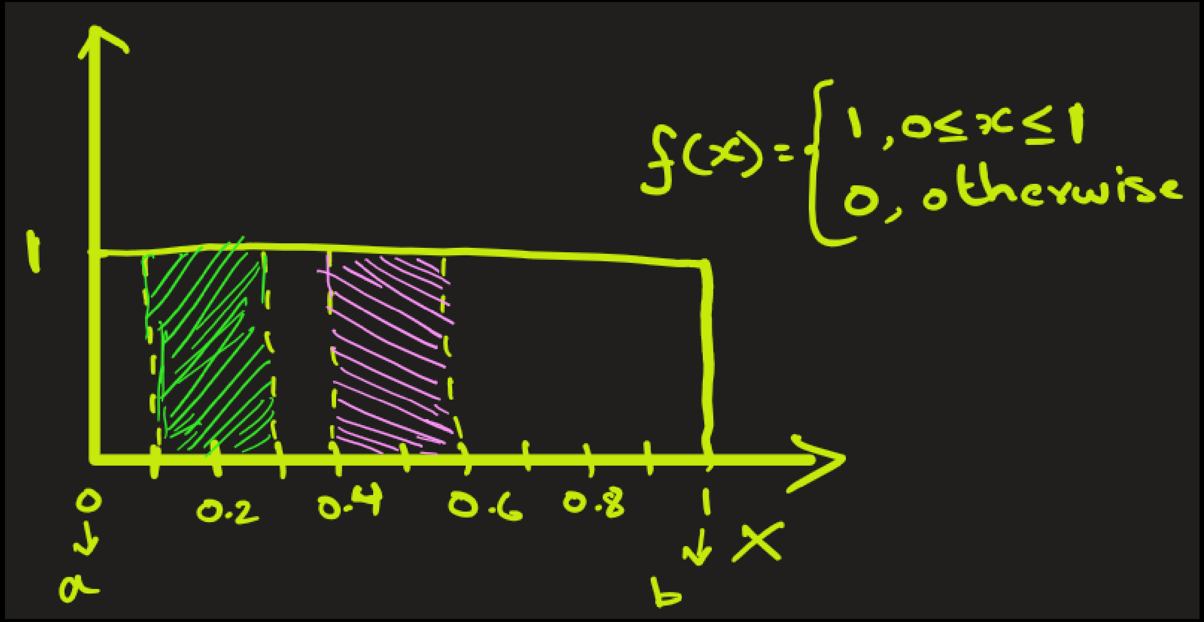


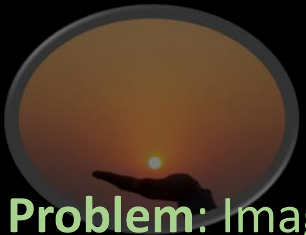
Ans: First we find PDF f(x). That's derived easily using formula.

a) P(0.1 < X < 0.3)
= Area between x = 0.1 and 0.3 under the PDF curve
= (0.3 – 0.1) x (1)
= 0.20 = 20 %

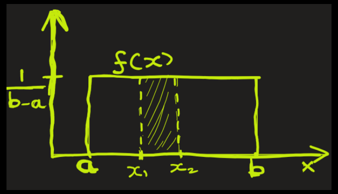
b) P(0.4 < X < 0.6)
= Area between x = 0.4 and 0.6 under the PDF curve
= (0.6 – 0.4) x (1)
= 0.20 = 20 % (same as above)

c) P(X = 0.5)
= Area between x = 0.5 and 0.5 under the PDF curve
= (0.5 – 0.5) x (1) = 0

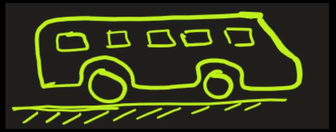




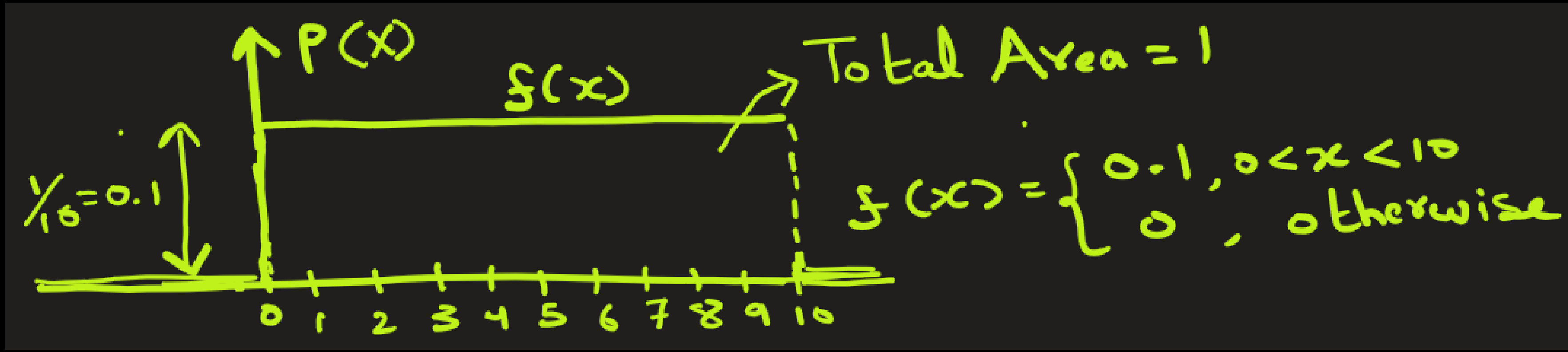
Uniform Distribution: Applications



Problem: Imagine a bus that runs every 10 minutes, and you arrive randomly at the bus stop. Your waiting time X follows a Uniform(0, 10) distribution, i.e. you could wait anywhere between 0 and 10 minutes with equal probability. After arriving at bus stop, what is the probability that you wait



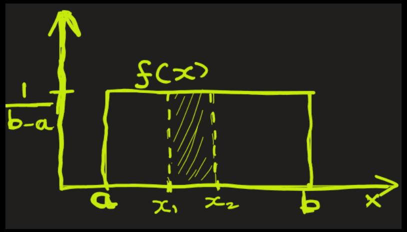
a) Less than 2 minutes
b) More than 4 minutes
c) Between 7 and 9 minutes
d) Between 5 and 12 minutes
e) Between 11 and 12 minutes
f) Exactly 5 minutes



Ans: First we find PDF f(x).
a) P(0 < X < 2) = Area between 0 and 2 = 0.1 x 2 = 0.2
b) P(4 < X < 10) = Area between 4 and 10 = 0.1 x 6 = 0.6
c) P(7 < X < 9) = Area between 7 and 9 = 0.1 x 2 = 0.2
d) P(5 < X < 12) = Area between 5 and 12 = 0.1 x 5 = 0.5
c) P(11 < X < 12) = Area between 11 and 12 = 0 x 1 = 0
d) P(X = 5) = Area between 5 and 5 = 0.1 x 0 = 0

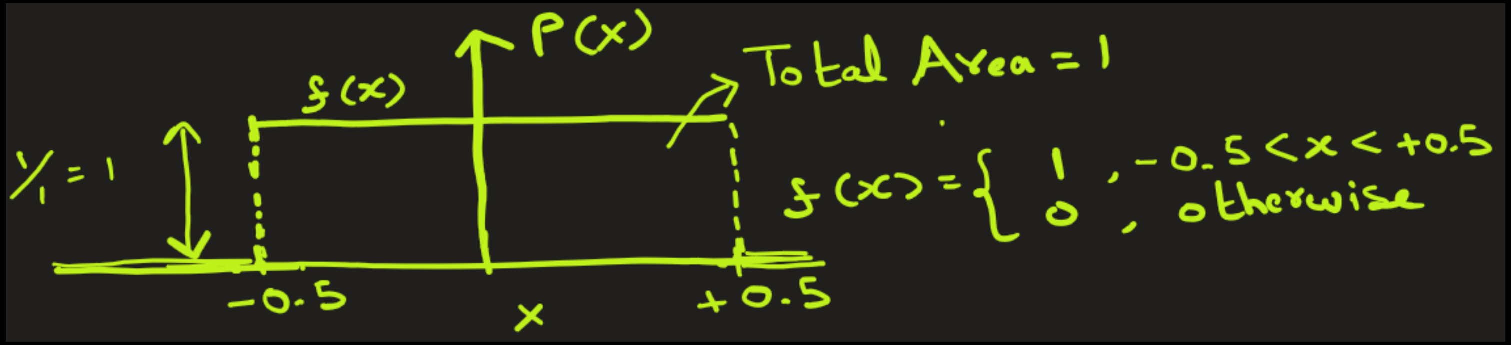
EXTRA

Uniform Distribution: Applications



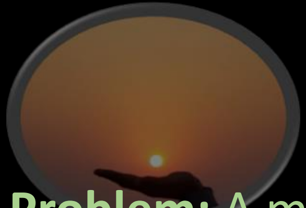
Problem: A voltage sensor reads values with random noise, X, that can vary between **–0.5 V** and **+0.5 V**.
There is **no bias** — every noise value within this range is **equally likely**.
What is the probability that a reading has noise

a) That is below -0.25 V
b) That is above + 0.3 V
c) That is between –0.2 V and + 0.2 V
d) That is exactly 0.4 V

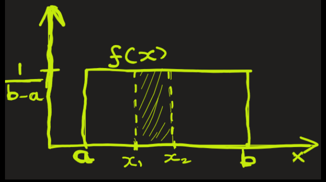


Ans: First we find PDF f(x).
Since every noise value within the range is **equally likely,** so the noise is uniformly distributed.

a) P(-0.5 < X < -0.25) = Area between -0.50 and -0.25 = 1 x 0.25 = 0.25
b) P(0.3 < X < 0.5) = Area between 0.3 and 0.5 = 1 x 0.20 = 0.20
c) P(-0.2 < X < +0.2) = Area between -0.2 and +0.2 = 1 x 0.40 = 0.40
d) P(X = 0.4) = Area between 0.4 and 0.4 = 1 x 0 = 0



Uniform Distribution: Applications



Problem: A municipal water lab collects **one random sample per day**. Due to staffing, samples are taken **uniformly at random** during two separate windows:
Either between 6:00 AM to 8:00 AM or between 10:00 AM to 12:00AM.
Let X= time (in hours) that the sample is taken. If you are given a sample then what is the probability the sample was taken **between 6:30 AM and 10:30 AM**?

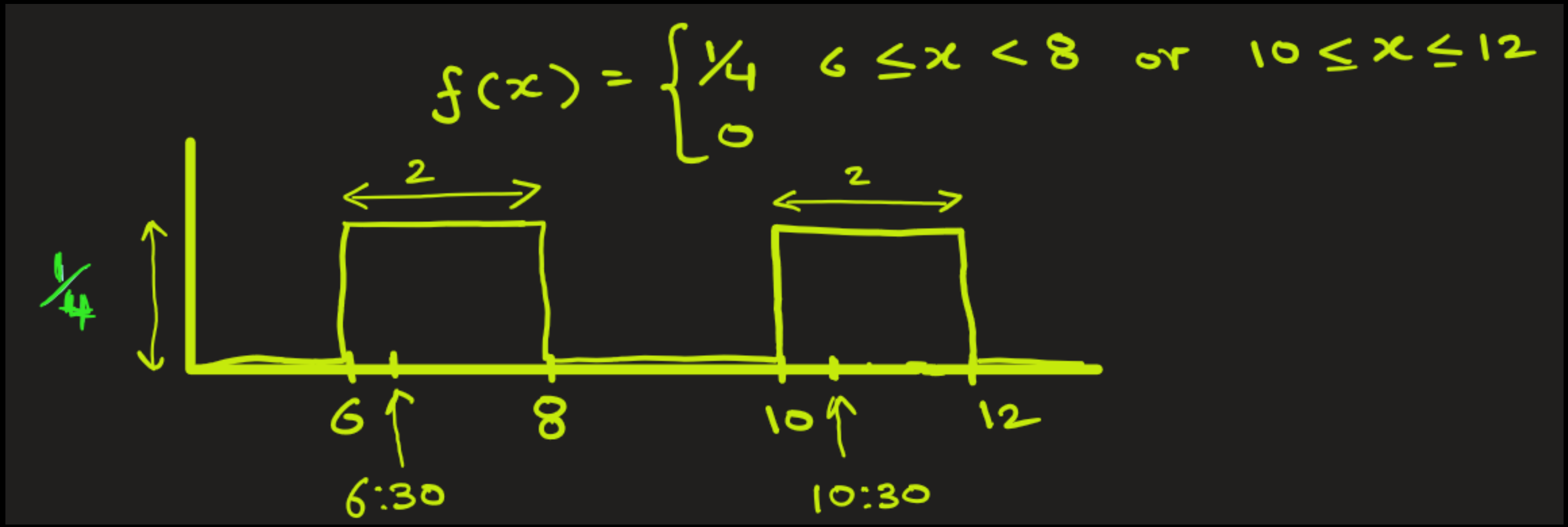
Ans: First we find PDF f(x). Since we know the total area under PDF is 1 and the intervals are [6, 8] U [10, 12], using simple geometry we find the equation for f(x).

P(6:30AM < X < 10:30 AM)

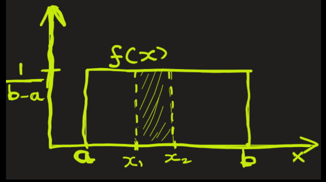
= Area under the curve between
6:30AM and 10:30 AM

= 1.5 x ¼ + 0.5 x ¼

= 0.5 = 50%



Uniform Distribution: Applications



ADVANCED: A municipal water lab collects **one random sample per day**. Due to staffing, samples are taken **uniformly at random** during two separate windows:

Either between 6:00 AM to 8:00 AM with 90% chance or between 10:00 AM to 12:00AM with 10% chance.

Let X= time (in hours) that the sample is taken. If you are given a sample then what is the probability

a) the sample was taken **between 6:30 AM and 8:00 AM ?**

b) the sample was taken **between 6:30 AM and 10:30 AM**?

c) the sample was taken **between 5:00 AM and 1:00 PM**?

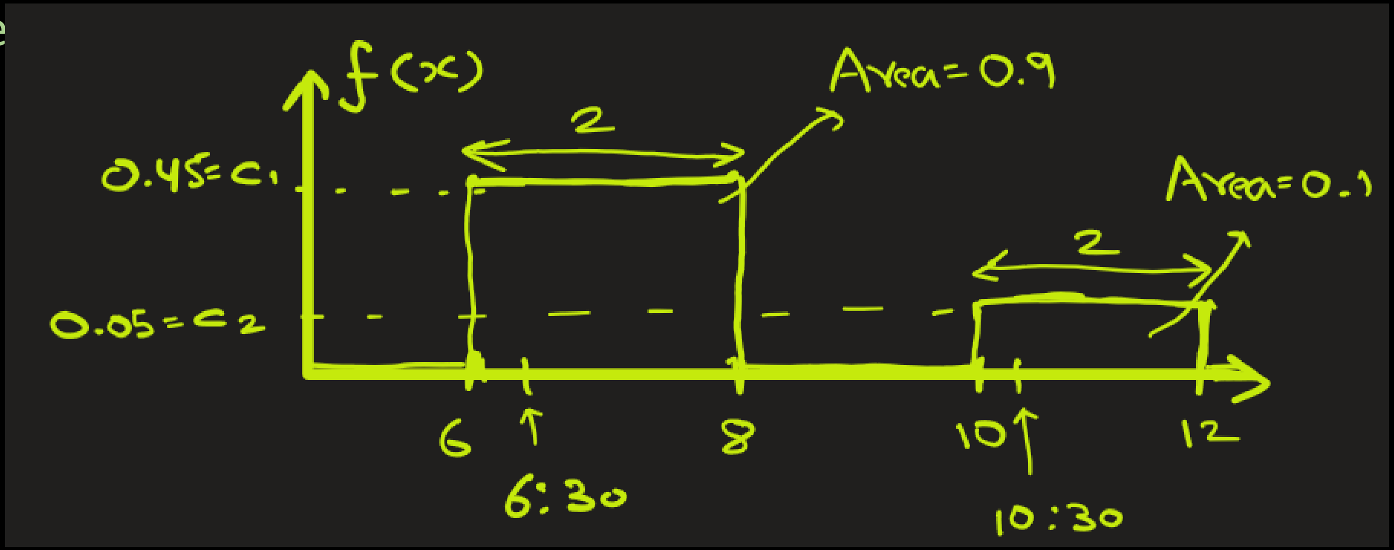
Ans:

First, we calculate the height of f(x) for the 2 intervals [6, 8] and [10, 12]. Since we know the total area under PDF is 1, the area under interval [6, 8] is 0.9 and the

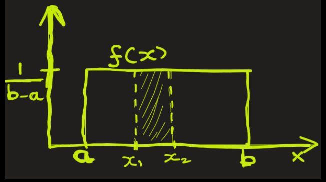
we find c1 = 0.45 and c2=0.05

$[6,8]$ $\quad 2c_1 = 0.9 \Rightarrow c_1 = 0.45$

$[10,12]$ $\quad 2c_2 = 0.1$, $\quad c_2 = 0.05$



Uniform Distribution: Applications



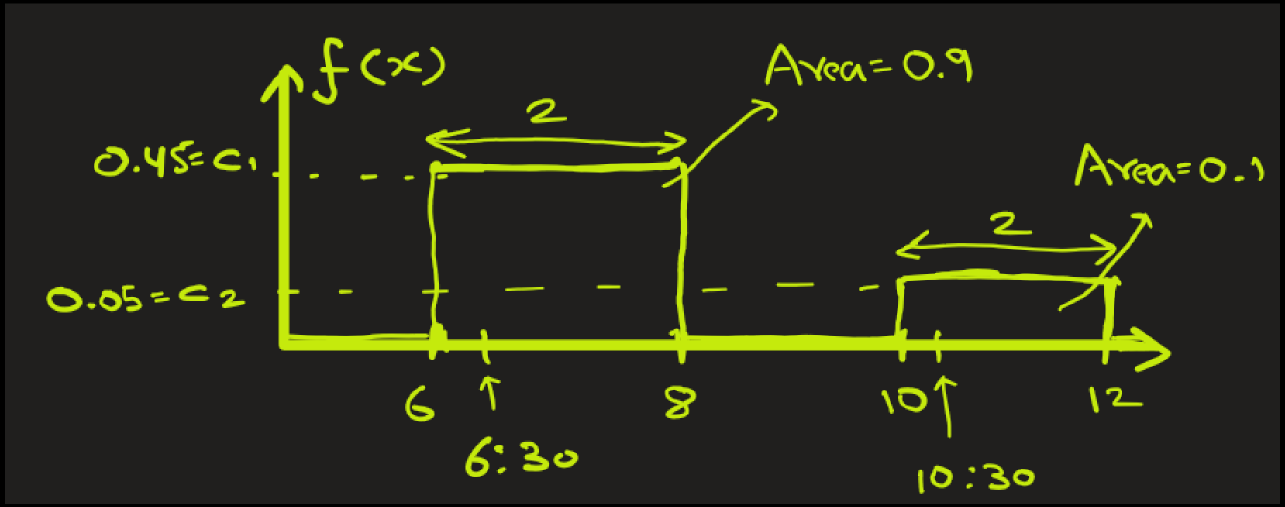
Continued…..

a) P(6:30AM < X < 8:00 AM)

= Area under the curve between 6:30 AM and 8:00 AM

= 1.5 x 0.45

= 0.675 = 67.5 %



b) P(6:30AM < X < 10:30 AM)

= Area under the curve between 6:30 and 8:00 AM

+ Area under the curve between 10:00 and 10:30 AM

= 1.5 x 0.45 + 0.5 x 0.05

= 0.7 = 70%

c) P(5:00 AM < X < 1:00 PM)

= Area under the curve between 5:00 AM and 1:00 PM

= (2 x 0.45) + (2 x 0.05)

= 1.00 = 100 %

Uniform Distribution

Use case:

- **Simulation and Random Sampling**
  - Many random number generators produce numbers using a **Uniform(0,1)** distribution.
  - Used as a base to generate other distributions (like normal, exponential, etc.).

- **Modeling Equal-Likelihood Scenarios**
  - When every value within a range is equally possible.
    Example:
    - Randomly picking a number between 1 and 10.
    - A bus that can arrive anytime within a 10-minute window.

- **Monte Carlo Methods**
  - Uniformly distributed random numbers are used in simulations and numerical integrations.

- **Quality Control**
  - If a machine produces items uniformly within tolerance limits, the thickness or weight may be modeled by a uniform distribution.