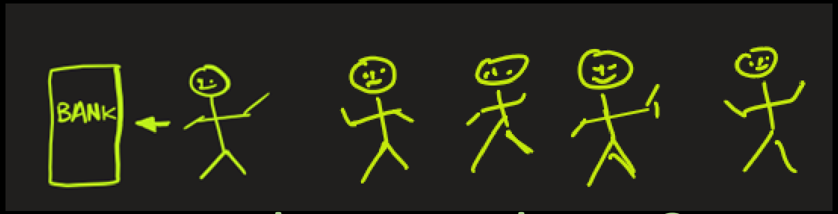
Poisson Distribution

Problem1:



A bank receives an average of **5 customers per hour**.
What is the probability that exactly **3 customers** will arrive in the next hour?

Problem2:

Scientists have observed that, on average, **2 meteorites** strike Earth **per year**.

Find the probability that **3 or more meteorites** strike the Earth in a year.

Poisson Distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

The **Poisson distribution** is a **discrete probability distribution** that gives the probability of a certain number of events happening in a **fixed interval of time, distance, area, or volume.** We assume that these events occur

- **Independently** of each other, and
- At a **constant average rate (λ or "lambda")**.

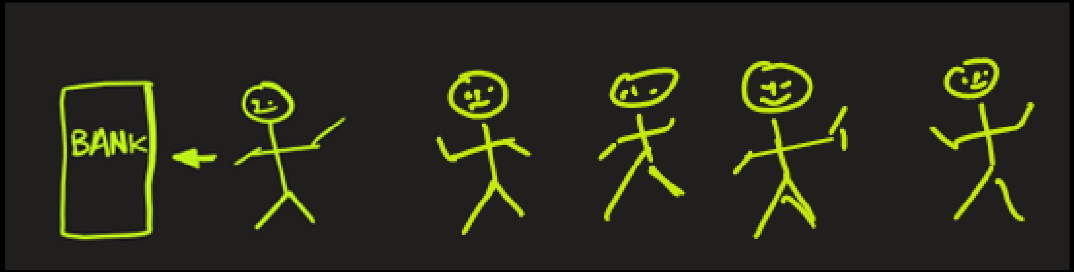
The probability of getting exactly **k** events in a fixed interval is:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

λ = average number of events (mean rate)
k= number of occurrences
e= 2.71828 (Euler's number)

It deals with types of situations where you have to determine (k)

- Number of customers arriving at a bank in the next 45 minutes.
- Number of phone calls received at call center in 2 hours
- Number of patients arriving at a hospital in next 2 minutes
- Number of typing errors per page
- Number of car accidents in a city per day
- Number of data packet arrivals at a server per second



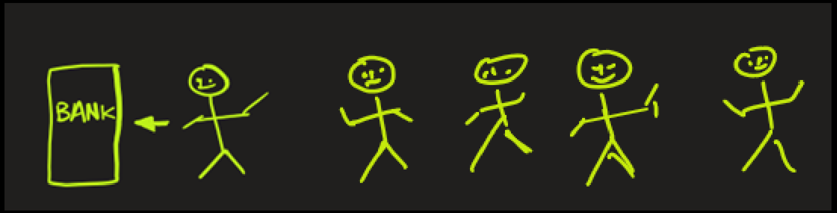
Poisson Distribution

$$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$$

Problem:

A bank receives an average of **5 customers per hour**. Assume that customers arrive at bank independent of each other and average is constant.

1) What is the probability that exactly **1 customers** will arrive in the next hour?
2) What is the probability that exactly **2 customers** will arrive in the next hour?
3) What is the probability that exactly **3 customers** will arrive in the next hour?
4) What is the probability that exactly **4 customers** will arrive in the next hour?



Ans:

λ = average number of customers per hour = 5 customers per hour

k = number of occurrences = number of customers arriving in next hour

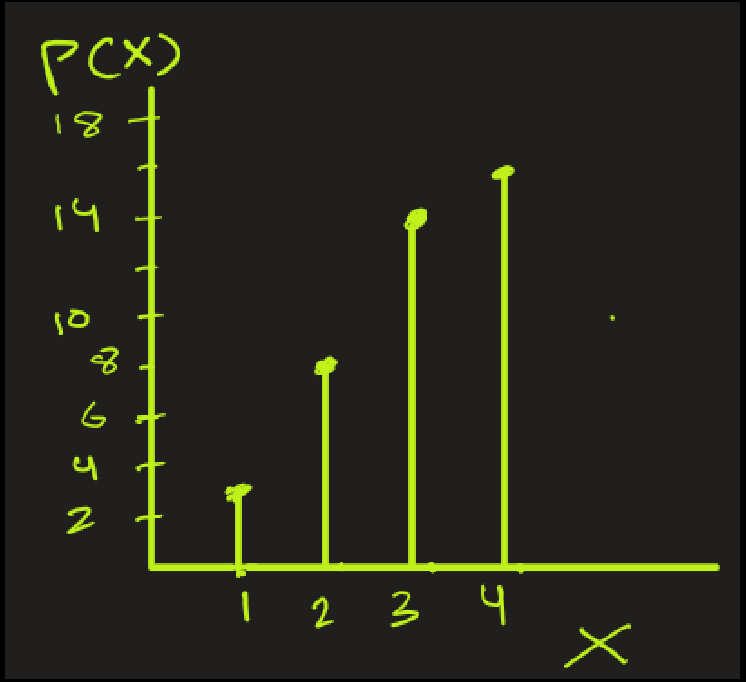
1) $P(X=1)=\frac{e^{-5}\times 5^1}{1!}$ = 0.0337 = 3.37 %

So, probability that exactly **1 customers** will arrive in the next hour is = 3.37 %

2) For 2 customers, $P(X=2)=\frac{e^{-5}\times 5^2}{2!}$ = 0.0842 = 8.42 %

3) For 3 customers, $P(X=3)=\frac{e^{-5}\times 5^3}{3!}$ = 0.1403 = 14.03 %

4) For 4 customers, answer is 0.1754 = 17.54 %



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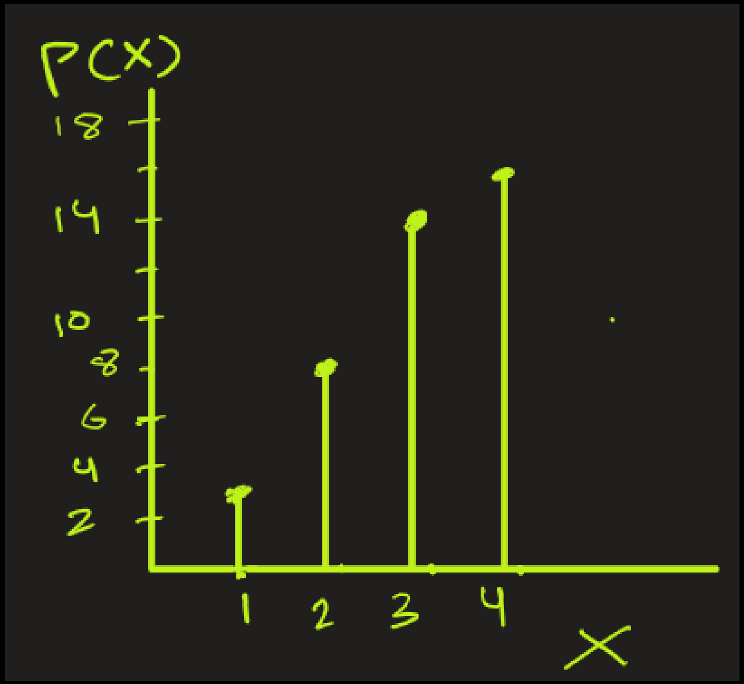
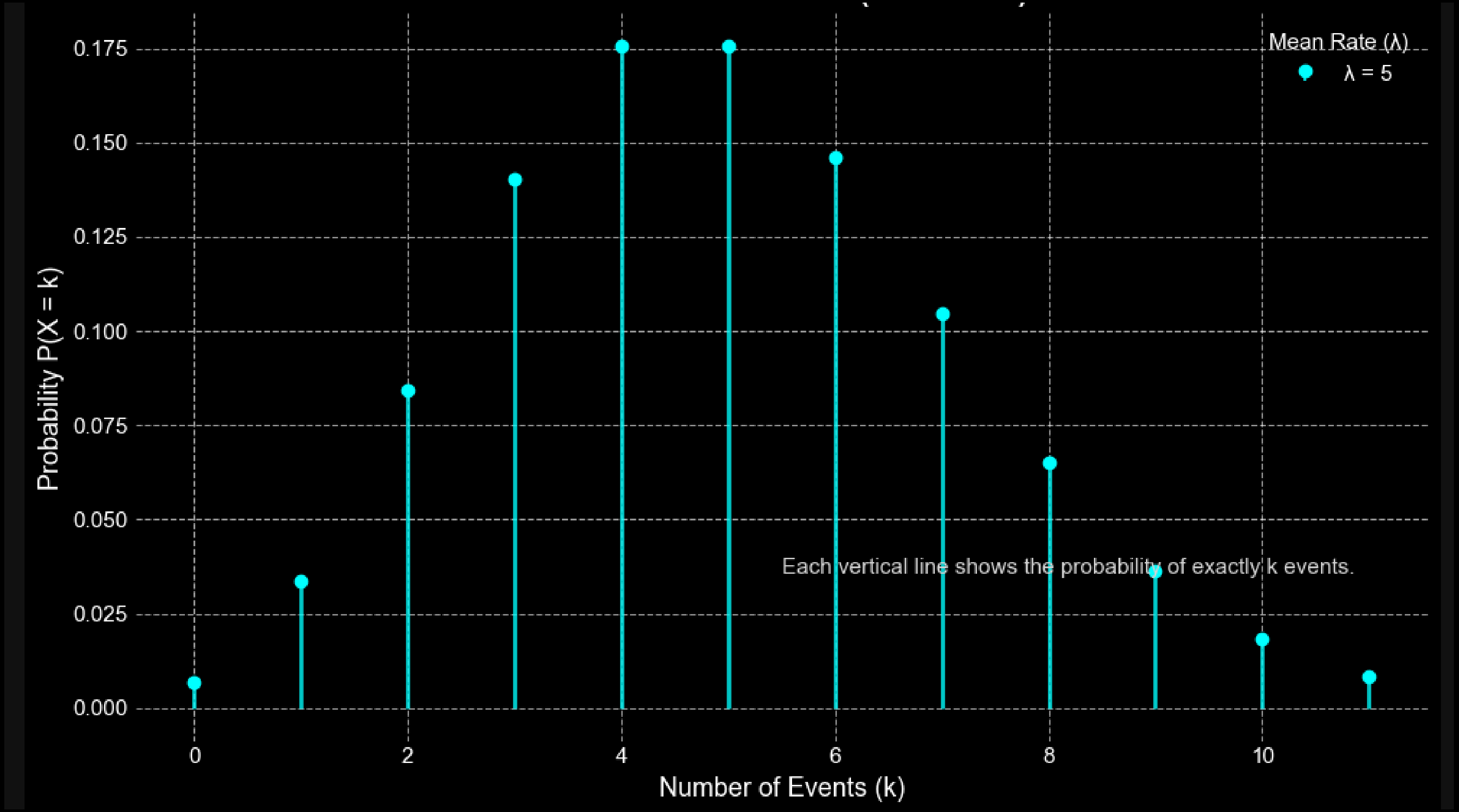
Poisson Distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

As we increase the value of k , the Poisson distribution curve will look as shown below.





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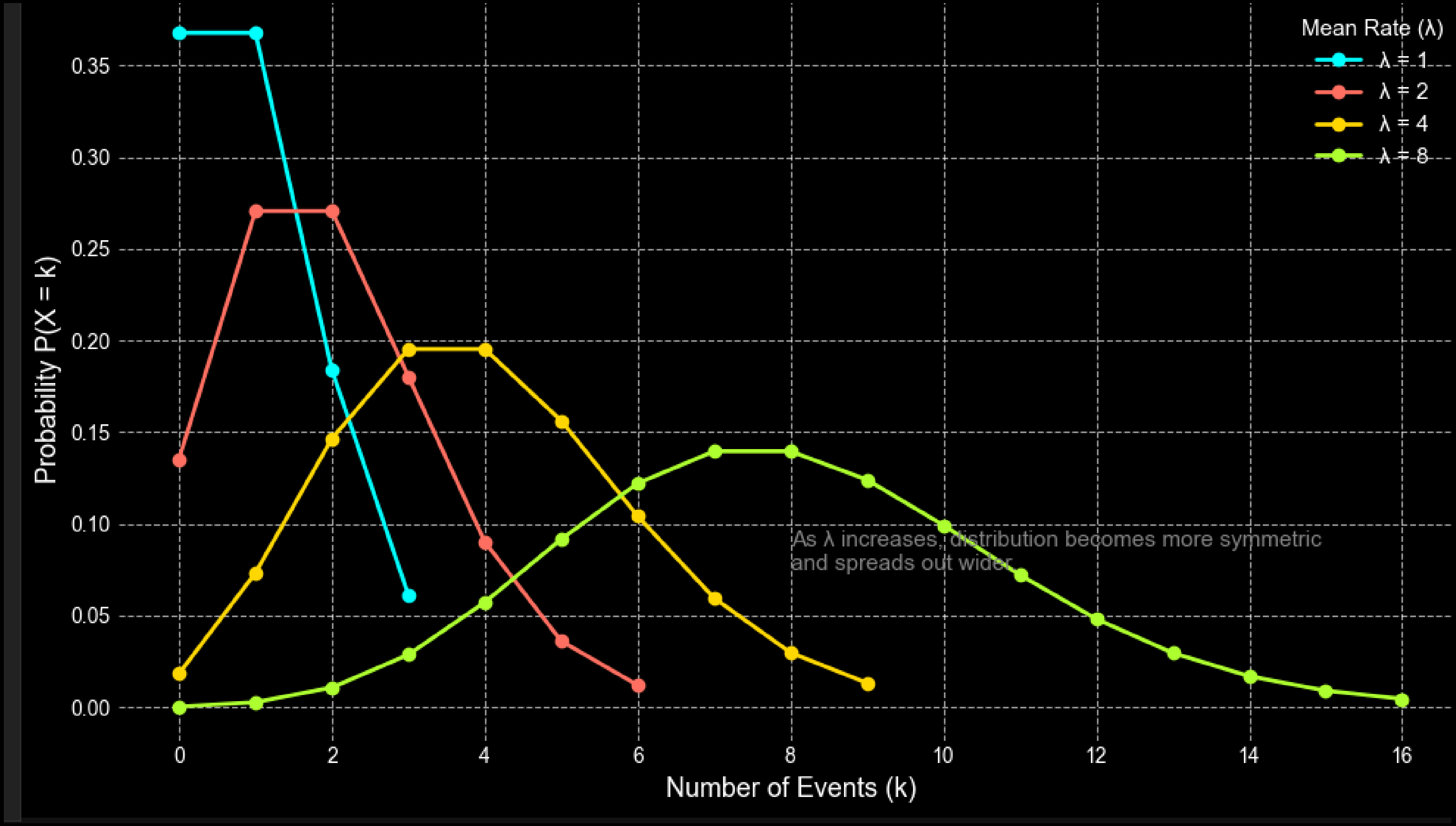
Poisson Distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

As a matter of fact, as we increase lambda the shape of the Poisson distribution curve becomes bell-shaped.



Poisson Distribution

$$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$$

Problem:

Scientists have observed that, on average, **2 meteorites** large enough to be detected by ground-based sensors strike Earth **per year**. Assume meteorite strikes:

- Occur **randomly and independently**,
- Happen at a **constant average rate (λ = 2 per year)**.

1) Find the probability that **2 or less meteorites** strike the Earth in a year.
2) Find the probability that **3 or more meteorites** strike the Earth in a year.

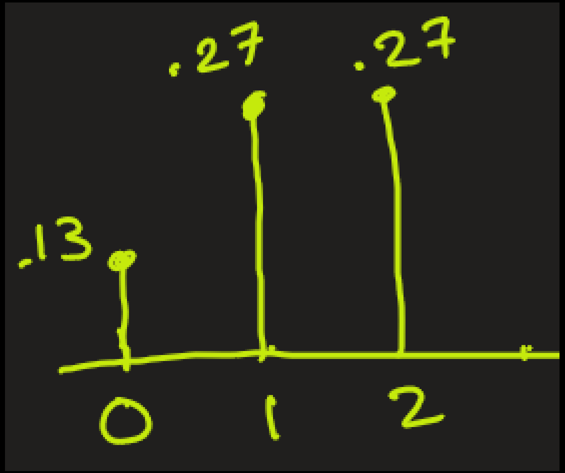
Ans:

1) Probability that **2 or less meteorites** strike the Earth in a year is
$P(X \leq 2)$

$= P(X=0) + P(X=1) + P(X=2)$ (Apply the formula) $$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$$

$= + 0.1353 + 0.2707 + 0.2707$

$= 0.6767 = 67.67\ \%$



Poisson Distribution

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

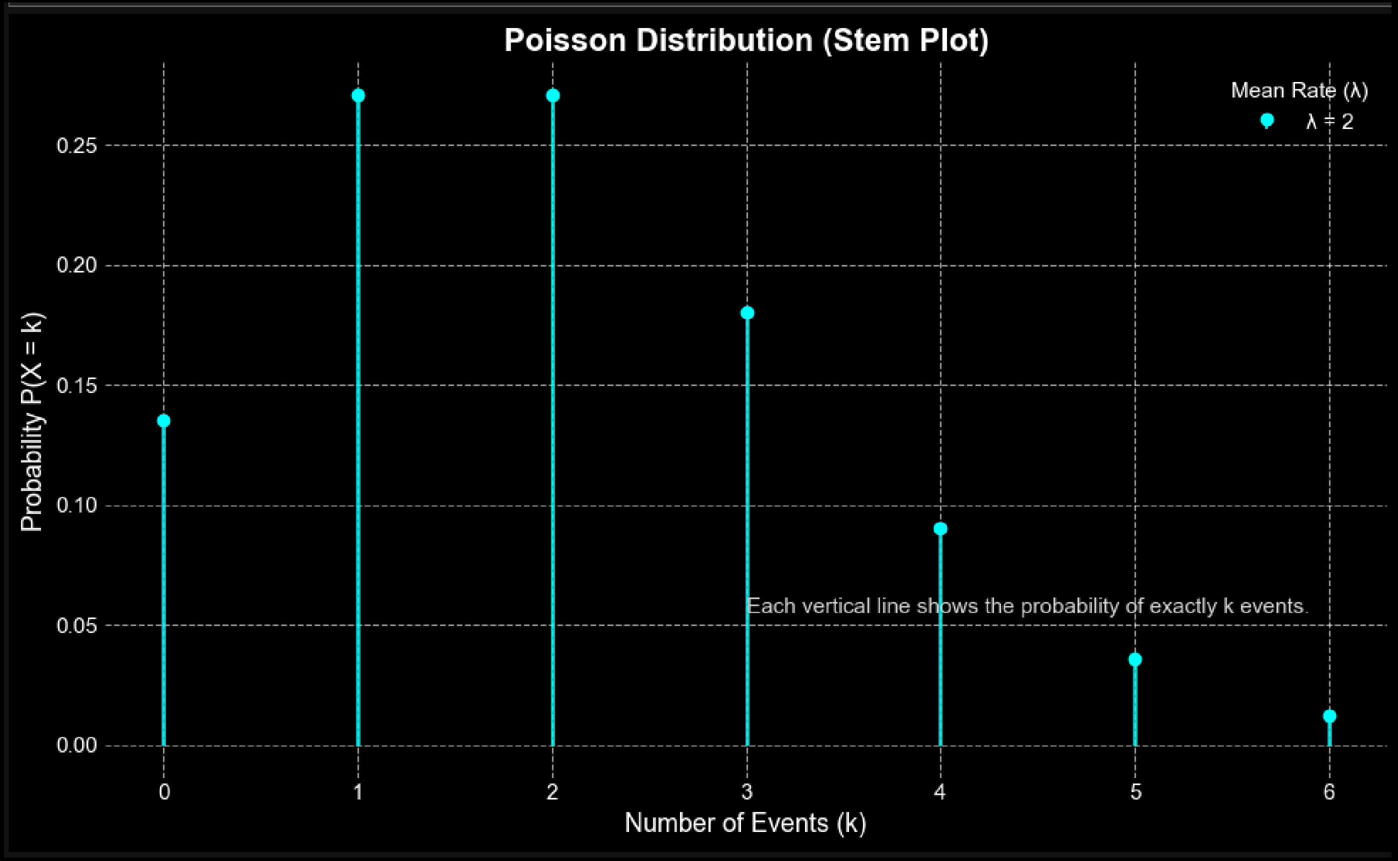
2) Probability that **3 or more meteorites** strike the Earth in a year is

P(X≥3)

= 1 − P(X≤2)

= 1 – 0.6767   (from previous result)

= 0.3233 = 32.33%



Side Note: Poisson is often used in astronomy and space science for modeling **meteorite impacts, Supernova occurrences, Gamma-ray bursts, Cosmic ray** detections because these are **rare, random, independent events** over space or time

STOP

Poisson Distribution

$$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$$

Prove that: Poisson distribution

$$P(x=k)=\frac{\lambda^k e^{-\lambda}}{k!}$$

satisfies $\sum_{k=0}^{\infty} P(X=k)=1$

Proof: $\sum_{k=0}^{\infty} P(x=k)$

$$=\sum \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}$$

$$= e^{-\lambda}\cdot e^{\lambda} = \underline{1}$$

Mean (Expected Value)

$$\mu = E[X] = \lambda$$

Interpretation: On average, there are λevents in the interval.

Variance

$$\sigma^2 = Var(X) = \lambda$$

Interesting fact: For Poisson, **mean = variance**.

Standard Deviation

$$\sigma = \sqrt{Var(X)} = \sqrt{\lambda}$$

Standard deviation measures how spread out the counts are around the mean.

Skewness

$$\text{Skewness} = \frac{1}{\sqrt{\lambda}}$$

Shows how asymmetric the distribution is.

Small λ → highly skewed; large λ → becomes more symmetric.

Kurtosis (Excess)

$$\text{Excess } Kurtosis = \frac{1}{\lambda}$$

Measures "peakedness" of the distribution.

Smaller λ → sharper peak, heavier tails; as λ → ∞, distribution approaches Normal.