Equations of motion of a 2R manipulator

R.B. Ashith Shyam

Figure 1 shows the schematic diagram of a 2R manipulator. The fixed or 0 co-ordinate system is denoted by {X, Y, Z}. There are two rotating co-ordinate systems λ_i, n_i, i = 1,2 which are respectively parallel and perpendicular to the respective links. The unit vectors and their derivatives

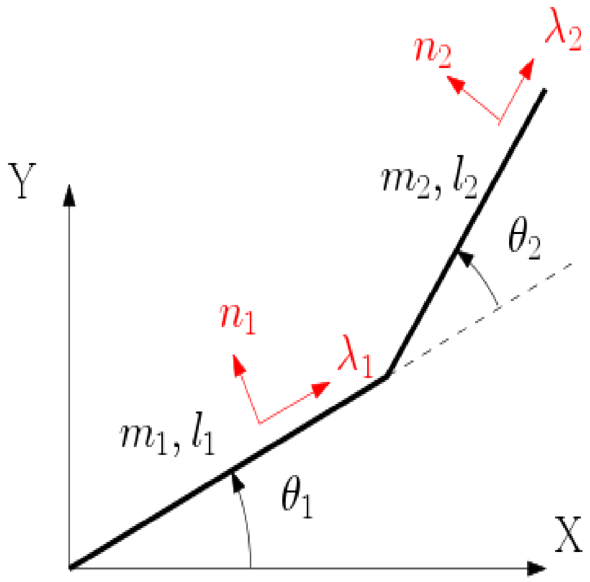

Figure 1: Schematic representation of a 2R manipulator

with respect to the fixed co-ordinate system can be written as

$$
\begin{aligned}
\lambda_1 &= \cos\theta_1 \hat{i} + \sin\theta_1 \hat{j} \\
n_1 &= -\sin\theta_1 \hat{i} + \cos\theta_1 \hat{j} \\
\lambda_2 &= \cos(\theta_1+\theta_2)\hat{i} + \sin(\theta_1+\theta_2)\hat{j} \\
n_2 &= -\sin(\theta_1+\theta_2)\hat{i} + \cos(\theta_1+\theta_2)\hat{j} \\
\dot{\lambda}_1 &= \dot{\theta}_1(-\sin\theta_1 \hat{i} + \cos\theta_1 \hat{j}) = \dot{\theta}_1 n_1 \\
\dot{n}_1 &= -\dot{\theta}_1(\cos\theta_1 \hat{i} + \sin\theta_1 \hat{j}) = -\dot{\theta}_1 \lambda_1 \\
\dot{\lambda}_2 &= (\dot{\theta}_1+\dot{\theta}_2)(-\sin(\theta_1+\theta_2)\hat{i} + \cos(\theta_1+\theta_2)\hat{j}) = (\dot{\theta}_1+\dot{\theta}_2)n_2 \\
\dot{n}_2 &= -(\dot{\theta}_1+\dot{\theta}_2)(\cos(\theta_1+\theta_2)\hat{i} + \sin(\theta_1+\theta_2)\hat{j}) = -(\dot{\theta}_1+\dot{\theta}_2)\lambda_2
\end{aligned}
$$

Lagranges Method

The position and velocity of a small element of the link dx at a distance x from the joint of links are given as

$$
\begin{aligned}
r_1 &= x\ \hat{\lambda}_1 \\
\dot{r}_1 &= x\ \dot{\hat{\lambda}}_1 = x\ (\dot{\theta}_1\ \hat{k} \times \hat{\lambda}_1) \\
&= x\ \dot{\theta}_1 \hat{n}_1 \\
r_2 &= l_1\ \hat{\lambda}_1 + x\ \hat{\lambda}_2 \\
\dot{r}_2 &= l_1\ \dot{\hat{\lambda}}_1 + x\ \dot{\hat{\lambda}}_2 \\
&= l_1\ \dot{\theta}_1 \hat{n}_1 + x\ (\dot{\theta}_1 + \dot{\theta}_2)\hat{n}_2
\end{aligned}
$$

The kinetic (T) and potential energies (V) are given by

$$
\begin{aligned}
T &= \frac{1}{2}\int_0^{l_1} \dot{r}_1\ .\ \dot{r}_1\ dm + \frac{1}{2}\int_0^{l_2} \dot{r}_2\ .\ \dot{r}_2\ dm \\
&= \frac{1}{2}\dot{\theta}_1^2 \int_0^{l_1} x^2\ (\frac{m_1}{l_1}dx) + \frac{1}{2}\int_0^{l_2} [l_1^2\dot{\theta}_1^2 + x^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2l_1\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_2)\ (\frac{m_2}{l_2}dx) \\
&= \frac{1}{6}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2 + \frac{1}{6}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_2) \\
V &= m_1 g(-\hat{j})\ .\ \frac{l_1}{2}\lambda_1 + m_2 g(-\hat{j})\ .\ (l_1\lambda_1 + \frac{l_2}{2}\lambda_2) \\
&= -m_1 g\frac{l_1}{2}\sin\theta_1 - m_2 g\{l_1 \sin\theta_1 + \frac{l_2}{2}\sin(\theta_1 + \theta_2)\}
\end{aligned}
$$

where the link is assumed to have uniform mass distribution over the its length. The Lagrangian, L is given by L = T - V

$$
\begin{aligned}
L = \frac{1}{6}m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 l_1^2\dot{\theta}_1^2 + \frac{1}{6}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_2) + \\
m_1 g\frac{l_1}{2}\sin\theta_1 + m_2 g\{l_1 \sin\theta_1 + \frac{l_2}{2}\sin(\theta_1 + \theta_2)\}
\end{aligned}
$$

The equations of motion then becomes

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= \frac{1}{3}m_1 l_1^2\dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + \frac{1}{3}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2}m_2 l_1 l_2\cos(\theta_2)(2\dot{\theta}_1 + \dot{\theta}_2) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= \frac{1}{3}m_1 l_1^2\ddot{\theta}_1 + m_2 l_1^2\ddot{\theta}_1 + \frac{1}{3}m_2 l_2^2\ddot{\theta}_1 + \frac{1}{3}m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\cos(\theta_2)(\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2) - \frac{1}{2}m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_2) \\
\frac{\partial L}{\partial \theta_1} &= m_1 g\frac{l_1}{2}\cos\theta_1 + m_2 g\{l_1\cos\theta_1 + \frac{l_2}{2}\cos(\theta_1 + \theta_2)\}
\end{aligned}
$$

Therefore the equation for θ_1 becomes

$$
\begin{aligned}
(\frac{1}{3}m_1 l_1^2 + m_2 l_1^2 + \frac{1}{3}m_2 l_2^2 + m_2 l_1 l_2\cos\theta_2)\ddot{\theta}_1 + (\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2\cos\theta_2)\ddot{\theta}_2 - \frac{1}{2}m_2 l_1 l_2\dot{\theta}_2^2\sin\theta_2 - \\
m_1 g\frac{l_1}{2}\cos\theta_1 - m_2 g\{l_1\cos\theta_1 + \frac{l_2}{2}\cos(\theta_1 + \theta_2) = 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= \frac{1}{3}m_2 l_2^2(\dot{\theta}_1 + \dot{\theta}_2) + \frac{1}{2}m_2 l_1 l_2\dot{\theta}_1\cos\theta_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= (\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2\cos\theta_2)\ddot{\theta}_1 + \frac{1}{3}m_2 l_2^2\ddot{\theta}_2 - \frac{1}{2}m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2 \\
\frac{\partial L}{\partial \theta_2} &= -\frac{1}{2}m_2 l_1 l_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 + m_2 g\frac{l_2}{2}\cos(\theta_1 + \theta_2)
\end{aligned}
$$

Therefore the equation for θ_2 becomes

$$(\frac{1}{3}m_2 l_2^2 + \frac{1}{2}m_2 l_1 l_2 \cos\theta_2)\ddot{\theta}_1 + \frac{1}{3}m_2 l_2^2 \ddot{\theta}_2 + \frac{1}{2}m_2 l_1 l_2 \dot{\theta}_1^2 \sin\theta_2 - m_2 g \frac{l_2}{2}\cos(\theta_1+\theta_2)$$

A slightly different formulation

The link is assumed to have a concentrated mass at a location $r_{i_{cm}}$ from the joints and is having inertia tensor about the centre of mass given by $I_{i_{cm}}$, i=1,2

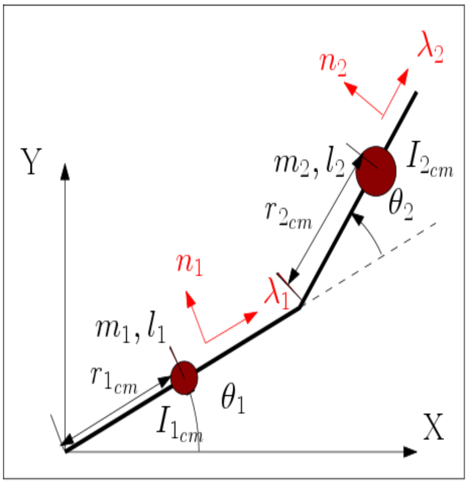

Figure 2: Schematic representation of a 2R manipulator

The equations of motions are

$$\begin{aligned}(I_1 + I_2 + m_2 l_1^2 + m_1 r_{1_{cm}}^2 + m_2 r_{2_{cm}}^2 + 2m_2 l_1 r_{2_{cm}}\cos\theta_2)\ddot{\theta}_1 + (I_2 + m_2 r_{2_{cm}}^2 + m_2 l_1 r_{2_{cm}}\cos\theta_2)\ddot{\theta}_2 - \\ m_2 l_1 r_{2_{cm}}\sin\theta_2(2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 + m_2 g(l_1\cos\theta_1 + r_{2_{cm}}\cos(\theta_1+\theta_2)) + m_2 g r_{2_{cm}}\cos\theta_2 = \tau_1 \\ (I_2 + m_2 r_{2_{cm}}^2 + m_2 l_1 r_{2_{cm}}\cos\theta_2)\ddot{\theta}_1 + (I_2 + m_2 r_{2_{cm}}^2)\ddot{\theta}_2 + m_2 l_1 r_{2_{cm}}\sin\theta_2 \dot{\theta}_1^2 + m_2 r_{2_{cm}} g\cos(\theta_1+\theta_2) = \tau_2\end{aligned} \tag{1}$$