Case 2: Multifactor Gaussian Model

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Question 1. Using the hints from slide 23, derive the discount bond price formula $P(t,m) = \exp\{A(t,m) - B(t,m)'Y_t\}$ for the model from slide 22 with monthly time-steps, $\delta = \frac{1}{12}$. Find the recursion equations for the function $A(t,m)$ and the (vector-valued) function $B(t,m)$.

Each period has a duration of δ, so the corresponding one-step discount factor is $e^{-\delta r_t}$. Hence, the discount bond price is given as:

$$P(t,m) = e^{-\delta r_t}\mathbb{E}_t^*[P(t+1,m)]$$

where $r_t = f(t) + g'Y_t$.
Furthermore, we also know the definition of the vector-AR(1) model:

$$Y_t - Y_{t-1} = -\delta K Y_{t-1} + \sqrt{\delta}\epsilon_t$$

where $\epsilon_t \sim N(0, I)$.

We can start off with the discount bond price:

$$P(t,m) = e^{-\delta r_t}\mathbb{E}_t^*[P(t+1,m)]$$

and substitute $r_t = f(t) + g'Y_t$ into the formula, which gives us:

$$P(t,m) = e^{-\delta(f(t)+g'Y_t)}\mathbb{E}_t^*[P(t+1,m)]$$

Furthermore, substituting the assumed bond price formula for $P(t+1,m)$ leaves us with:

$$P(t,m) = e^{-(f(t)+g'Y_t)\delta}\mathbb{E}_t^*\left[\exp\left\{A(t+1,m) - B(t+1,m)'Y_{t+1}\right\}\right]$$

Using the moment-generating function of the multivariate Gaussian from slide 23, we can calculate $\mathbb{E}_t^*\left[e^{-B(t+1,m)'Y_{t+1}}\right]$ and obtain:

$$\mathbb{E}_t^*\left[e^{-B(t+1,m)'Y_{t+1}}\right] = e^{-B(t+1,m)'(I-\delta K)Y_t + \frac{\delta}{2}B(t+1,m)'B(t+1,m)}$$

So for $P(t,m)$, we now have the following:

$$P(t,m) = e^{-(f(t)+g'Y_t)\delta}e^{A(t+1,m)}e^{-B(t+1,m)'(I-\delta K)Y_t + \frac{\delta}{2}B(t+1,m)'B(t+1,m)}$$

If we rearrange the terms, we can see that it ressembles $P(t,m) = \exp\{A(t,m) - B(t,m)'Y_t\}$:

$$\exp\{A(t,m) - B(t,m)'Y_t\} = \exp\Big\{ - f(t)\delta + A(t+1,m) + \frac{\delta}{2}B(t+1,m)'B(t+1,m) \\ - B(t+1,m)'(I-\delta K)Y_t - \delta g'Y_t\Big\}$$

Now that we derived the formula, we can start by matching the terms that are associated with Y_t to define $B(t,m)$:

$$-B(t,m)'Y_t = -\delta g'Y_t - B(t+1,m)'(I-\delta K)Y_t$$

$$-B(t,m)' = -\delta g' - B(t+1,m)'(I-\delta K)$$

$$\boxed{B(t,m) = \delta g + (I-\delta K)'B(t+1,m)}$$

where $\delta = \frac{1}{12}$.

Now we can match the remaining terms to define $A(t,m)$:

$$\boxed{A(t,m) = A(t+1,m) - \delta f(t) + \frac{\delta}{2}B(t+1,m)'B(t+1,m)}$$

where $\delta = \frac{1}{12}$.

Lastly, we know that at maturity the bond pays 1, hence:

$$P(m,m) = 1$$

$$P(m,m) = \exp\{A(m,m) - B(m,m)'Y_m\} = 1$$

which implies

$$A(m,m) = 0, \quad B(m,m) = 0.$$

Question 2. The model (3.38) in the book of W&M is a special case of the model from slide 22, but this is not immediately obvious. Note that the process Y_t is an unobserved auxiliary process, and we are allowed to "stretch and shift" this process to obtain an observationally equivalent model for r_t. We start with the W&M model (3.38) in vector notation:

$$\begin{cases} \tilde{Y}_t = b + (I-K)\tilde{Y}_{t-1} + G\epsilon_t^* \\ r_t = \iota'\tilde{Y}_t \end{cases}$$

where K and G are diagonal matrices and ι is a vector of ones. We now introduce the new process $Y_t = D\tilde{Y}_t + \nu$, where D is a diagonal matrix and ν is a vector. This is the "stretch and shift." Substitute $\tilde{Y}_t = D^{-1}(Y_t - \nu)$ into the equations above and find D and ν such that the W&M model expressed in terms of Y_t is in the same form as the model from slide 22.

Hint: for products of *diagonal* matrices like $D(I-K)D^{-1}$ or DG, we can simply take the product of the diagonals.

We are given the W&M model in vector notation:

$$\tilde{Y}_t = b + (I-K)\tilde{Y}_{t-1} + G\epsilon_t^*$$
$$r_t = \iota'\tilde{Y}_t$$

as well as the new process:

$$Y_t = D\tilde{Y}_t + v \quad \text{and} \quad \tilde{Y}_t = D^{-1}(Y_t - v)$$

We can start with the W&M model and substitute $\tilde{Y}_t = D^{-1}(Y_t - v)$, which gives us:

$$D^{-1}(Y_t - v) = b + (I-K)D^{-1}(Y_{t-1} - v) + G\epsilon_t^*$$

Multiplying both sides with D results in:

$$\begin{aligned} Y_t - v &= Db + D(I-K)D^{-1}(Y_{t-1} - v) + DG\epsilon_t^* \\ &= Db + (I-K)(Y_{t-1} - v) + DG\epsilon_t^* \\ &= Db + (I-K)Y_{t-1} - (I-K)v + DG\epsilon_t^* \end{aligned}$$

Using the equation above, we can now define Y_t as:

$$Y_t = Db + (I-K)Y_{t-1} - (I-K)v + DG\epsilon_t^* + v$$

By substracting $-Y_{t-1}$ on both sides we now get:

$$Y_t - Y_{t-1} = -Y_{t-1} + Db + (I-K)Y_{t-1} - (I-K)v + DG\epsilon_t^* + v$$

$$\begin{aligned} Y_t - Y_{t-1} &= (I-K-I)Y_{t-1} + v + Db - (I-K)v + DG\epsilon_t^* \\ &= -KY_{t-1} + v + Db - (I-K)v + DG\epsilon_t^* \end{aligned}$$

The equation above now resembles to $Y_t - Y_{t-1} = -\delta K Y_{t-1} + \sqrt{\delta}\epsilon_t^*$, however, for it to be equivalent the following is required:

1. The coefficients of Y_{t-1} must match:

$$-KY_{t-1} = -\delta K Y_{t-1}, \quad \text{(this condition already holds with } \delta = 1\text{)}$$

$$\Rightarrow \quad K = \delta K$$

2. The noise term must also match (= be scaled appropriately):

$$DG\epsilon_t^* = \sqrt{\delta}\epsilon_t^*$$

$$DG = \sqrt{\delta} \quad \Rightarrow \quad \boxed{D = \sqrt{\delta}G^{-1}}$$

3. v + Db - (I - K) v = 0, we can solve this to find the value of v

$$v + Db - (I - K)v = 0$$
$$v - (I - K)v = -Db$$

Factoring out v:

$$(I - I + K)v = -Db$$
$$Kv = -Db$$

We know that $K = \delta K$ from point 1, substitung this we get:

$$\delta K v = -Db$$

Now we can finally solve for v:

$$v = -\frac{Db}{\delta K}$$

Since $D = \sqrt{\delta} G^{-1}$, we obtain:

$$v = -\frac{\sqrt{\delta}}{\delta} K^{-1} G^{-1} b$$

$$\boxed{v = -\frac{1}{\sqrt{\delta}} K^{-1} G^{-1} b}$$