# Case 1: Martingale Pricing and Deflator

Asha de Meij - i6254733

February 6, 2025

**Question 1.** Prove that (irrespective of the choice of numéraire $N_t$), the state-price deflator $\phi_t$ is always the same. Hint: use the change of numéraire theorem from slide 29.

The following equation from slide 29 represents the **martingale pricing formula**, which holds for two different numéraires $N_t$ and $M_t$:

$$V_t = \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{N_t}{N_T}X_T\right] = \mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{N_t}{N_T}\frac{M_T}{M_t}\frac{M_t}{M_T}X_T\right] = M_t\mathbb{E}_t^{\mathbb{Q}^N}\left[\left(\frac{N_t}{N_T}\frac{M_T}{M_t}\right)\frac{X_T}{M_T}\right] = M_t\mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{X_T}{M_T}\right].$$

where $\frac{d\mathbb{Q}^M}{d\mathbb{Q}^N} = \frac{N_t}{N_T}\frac{M_T}{M_t}$ as stated in W&M, Ch 11.

However, the martingale pricing formula can be rewritten and we derive following:

$$Q_t = M_t\mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{X_T}{M_T}\right] = \mathbb{E}_t^{\mathbb{Q}^M}\left[\frac{M_t}{M_T}X_T\right] = \mathbb{E}_t^{\mathbb{P}}\left[\left(\frac{M_t}{M_T}\frac{d\mathbb{Q}^M}{d\mathbb{P}}\right)X_T\right] = \mathbb{E}_t^{\mathbb{P}}\left[\phi_T X_T\right]$$

as well as:

$$Q_t = N_t\mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{X_T}{N_T}\right]\mathbb{E}_t^{\mathbb{Q}^N}\left[\frac{N_t}{N_T}X_T\right] = \mathbb{E}_t^{\mathbb{P}}\left[\left(\frac{N_t}{N_T}\frac{d\mathbb{Q}^N}{d\mathbb{P}}\right)X_T\right] = \mathbb{E}_t^{\mathbb{P}}\left[\phi_T X_T\right]$$

Since both formulas (regardless of the chosen numéraire) lead to the same expectation, we can conclude the following:

$$\phi_T^{(M)} = \frac{M_t}{M_T}\frac{d\mathbb{Q}^M}{d\mathbb{P}} = \frac{N_t}{N_T}\frac{d\mathbb{Q}^N}{d\mathbb{P}} = \phi_T^{(N)}$$

This confirms that irrespective of the choice of numéraire the state-price deflator $\phi_t$ is always the same.

**Question 2.** Do the following calculations with the Black-Scholes deflator of slide 33. Show that $\frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T \cdot 1] = e^{-r(T-t)}$ (correct price for bond) and $\frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T S_T] = S_t$ (correct price for stock).

**2.1 Show:** $\frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T \cdot 1] = e^{-r(T-t)}$

The deflator in the Black-Scholes model is defined as: $\phi_t = \frac{1}{B_t}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = e^{-rt}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t-\left(\frac{\mu-r}{\sigma}\right)W_t^{\mathbb{P}}}$.

At maturity $T$, the deflator is given by: $\phi_T = e^{-rT}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T-\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}}$.

Thus, its reciprocal is following: $\frac{1}{\phi_T} = e^{rT}e^{\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T+\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}}$.

Now we compute $\mathbb{E}_t^{\mathbb{P}}[\phi_T.1]$:

$$\mathbb{E}_t^{\mathbb{P}}\left[\phi_T.1\right] = \mathbb{E}_t^{\mathbb{P}}\left[e^{-rT}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2-\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}}\right]$$

$$= e^{-rT}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}\mathbb{E}_t^{\mathbb{P}}\left[e^{-\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}}\right]$$

Since $W_T^{\mathbb{P}}$ follows the Brownian motion under measure $\mathbb{P}$, we can use the moment generating function for a normal variable, and get the following:

$$\mathbb{E}_t^{\mathbb{P}}\left[\phi_T\right] = e^{-rT}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}e^{-\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}+\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}$$

We can also express T as T = t + (T - 1), so we can rewrite the expectation as follows:

$$= e^{-r(T-t)}e^{-rt}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}e^{-\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}}e^{\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2(T-t)}$$

Since $\phi_t = e^{-rt}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t-\left(\frac{\mu-r}{\sigma}\right)W_t^{\mathbb{P}}}$, we can once again derive a new expression:

$$\mathbb{E}_t^{\mathbb{P}}\left[\phi_T\right] = e^{-r(T-t)}\phi_t$$

Now that $\mathbb{E}_t^{\mathbb{P}}[\phi_T 1]$ is computed, we can now compute $\frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T \cdot 1]$.

$$\frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T \cdot 1] = \frac{1}{\phi_t}e^{-r(T-t)}\phi_t = e^{-r(T-t)}$$

**2.2 Show:** $\frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T S_T] = S_t$

Again, the deflator in the Black-Scholes model is defined as: $\phi_t = e^{-rt}e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t-\left(\frac{\mu-r}{\sigma}\right)W_t^{\mathbb{P}}}$ and the reciprocal is $\frac{1}{\phi_T} = e^{rT}e^{\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T+\left(\frac{\mu-r}{\sigma}\right)W_T^{\mathbb{P}}}$.

Since $\frac{1}{\phi_T}$ is known, we can now compute $\mathbb{E}_t^{\mathbb{P}}[\phi_T S_T]$:

$$\mathbb{E}_t^{\mathbb{P}}[\phi_T S_T] = \mathbb{E}_t^{\mathbb{P}}\left[e^{-rT}\exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - \frac{\mu-r}{\sigma}W_T^{\mathbb{P}}\right)S_T\right]$$

$$= e^{-r(T-t)}e^{-rt}\mathbb{E}_t^{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - \frac{\mu-r}{\sigma}W_T^{\mathbb{P}}\right)S_T\right]$$

We can recognize that the term $\exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - \frac{\mu-r}{\sigma}W_T^{\mathbb{P}}\right)$ is the **Radon-Nikodym derivative** that transforms the probability measure from $\mathbb{P}$ (the real-world measure) to $\mathbb{Q}$ (the risk-neutral measure). The Radon-Nikodym derivative is defined as follows according to slide 33:

$$\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)_t = e^{-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t+\left(\frac{\mu-r}{\sigma}\right)W_t^{\mathbb{P}}}.$$

Using this, we can rewrite the expectation under $\mathbb{P}$ as an expectation under $\mathbb{Q}$, which results in:

$$\mathbb{E}_t^{\mathbb{P}}\left[\exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - \frac{\mu-r}{\sigma}W_T^{\mathbb{P}}\right)S_T\right] = \mathbb{E}_t^{\mathbb{Q}}[S_T].$$

This follows directly from the Girsanov's Theorem, now the the drift term changes from $\mu$ to $r$, making the process risk-neutral.
Under $\mathbb{Q}$ we have:

$$\mathbb{E}_t^{\mathbb{Q}}[S_T \mid \mathcal{F}_t] = S_t e^{r(T-t)}$$

Hence:

$$\mathbb{E}_t^{\mathbb{Q}}[S_T] = S_t e^{r(T-t)}$$

So we can conclude that:

$$\mathbb{E}_t^{\mathbb{P}}[\phi_T S_T] = e^{-r(T-t)}e^{-rt}S_t e^{r(T-t)} = S_t e^{-rt}$$

and

$$\mathbb{E}_t^{\mathbb{P}}\left[\phi_T S_T \mid \mathcal{F}_t\right]$$

$$= \mathbb{E}_t^{\mathbb{P}}\left[e^{-rT}\exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T - \frac{\mu-r}{\sigma}W_T^{\mathbb{P}}\right)S_T \mid \mathcal{F}_t\right] = \exp\left(-\frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t - \frac{\mu-r}{\sigma}W_t^{\mathbb{P}}\right)S_t e^{-rt}.$$

Now multiplying this by $\frac{1}{\phi_t}$ gives us $\quad \frac{1}{\phi_t}\mathbb{E}_t^{\mathbb{P}}[\phi_T S_T] = S_t$

**Question 3.** What is the economic interpretation of the state-price deflator $\phi_T$?

I think that the state-price deflator is a helpful tool that enables to express uncertain future cash flows in terms of the present value. It acts as a discounting factor that accounts for fluctuations in the general price level over time. The discount factor integrates both the risk-free rate and the additional risk tied to the uncertainty of future cash flows. By incorporating uncertainty, it plays a crucial role in asset pricing, ensuring arbitrage-free valuation. Additionally, it facilitates the transition from real-world probabilities to risk-neutral measures, making it fundamental in financial modeling.