7^{th} Grade Individual Contest Solutions

IMSA *Mu Alpha Theta*

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1. Remember that percent means divide by 100. So 50% of 19.5 is 50/100 or 1/2 or 19.5, which is $\boxed{9.75}$

2. The catch in this problem is that some solvers will divide $11+4 = 15$ by $3 \cdot 7 = 21$. That is incorrect, because multiplication and division are performed in *left-to-right* order unless there are parentheses. So this problem should be computed as $4 + 11 = 15$, $15/3 = 5$, $5 \cdot 7 = 35$ and, finally, $35 - 5 = \boxed{30}$.

3. Factoring: $2024 = 2 \cdot 2 \cdot 2 \cdot 11 \cdot 23$. So $a = 3$, $b = 1$, and $c = 1$. Thus $2a - 11b + 23c = 6 - 11 + 23 = \boxed{18}$.

4. This is an exercise in dimensional analysis: $7 \text{ miles} \cdot \dfrac{1 \text{ hour}}{90 \text{ miles}} \cdot \dfrac{60 \text{ minutes}}{1 \text{ hour}} = \boxed{\dfrac{14}{3}}$ minutes.

5. Let Alice be A years old and Kate K years old. The problem states that $A = 8K$ and that $A + K = 63$. Then $8K + K = 63$ or $9K = 63$ so $K = 7$, and $A = 8 \cdot 7 = \boxed{56}$.

6. Banana's phone starts at \$80 and increases by 50%, which is \$40, to a total of \$120. Then it loses 50% of that, dropping to a price of \$60. Blueberry's phone also starts at \$80, but increases by 60%, which is \$48, to a total of \$128. Now it loses 60% of that, dropping to a price of \$51.20. The difference in price is now $\boxed{\$8.80}$.

7. One way to solve this is to use a system of equations. If A is the cost of a pen and B is the cost of a pencil, then the two purchases reveal that $2A+4B = 4$ and $7A+2B = 8$. A quick way to solve this is to double the second equation, $14A + 4B = 16$. Then subtract the first equation to yield $12A = 12$ so $A = 1$ and $B = 0.5$. Thus, one pen and one pencil would cost a total of $\boxed{\$1.50}$.

   Another way to solve this would be to start with twice Yolanda's purchase, so that 14 pens and 4 pencils cost \$16. Add 5 times Xavier's purchase, or that 10 pens and 20 pencils cost \$20. The total is that 24 pens and 24 pencils come to \$36, so dividing by 24 gives the result.

8. In the year 2000, the sum of the family members' ages are $0 + 24 + 25 = 49$. Every year, each person's age increases by one, so the total increases by 3. So after y years, the total is $49 + 3y$. This is supposed to equal 100, so $49 + 3y = 100$ or $3y = 51$. So $y = 17$, and the total was 100 in $\boxed{2017}$.

9. Call the length of the side of the square s. Then the perimeter is $4s$ and the area is s^2. Therefore $4s = 8s^2$. The solutions to this equation are $s = 0$ and $s = \boxed{1/2}$.

10.

11. Here is a table of the first 10 primes and their squares:

n	1	2	3	4	5	6	7	8	9	10
prime	2	3	5	7	11	13	17	19	23	29
square	4	9	25	49	121	169	289	361	529	841

Since no square is a prime and no prime is a square, if 529 comes out of the machine it must be that 23 was supposed to come out, which is the $\boxed{9^{\text{th}}}$ prime.

12. The easiest way to approach the question is to start at 1000 and go down by multiples of 4 until you hit a number the sum of whose digits is 22: 1000, 996, 992, 988, 984, 980, **976**, so the answer is $\boxed{976}$.

To be more sophisticated about the search, you can also note that numbers with the same sum of digits will differ from each other by multiplies of 9. The largest numbers less than 1000 whose sum of digits is 22 would be 994. So start counting down by 9's: 994, 985, **976**. If the numbers involved in this problem had been much more complicated, it would have been faster to combine these two facts to create a more efficient search.

13. Raising numbers to powers means multiplying them by themselves. But if 4 is multiplied by itself over and over, the units digits bounce back and forth between 4 and 6. If the exponent is even, the units digit will be 6 while an odd exponent leads to a unit digit of 4. Since the exponent is 4^{4^4} and is clearly even—it is a multiple of 4—the units digit of the expression will be $\boxed{6}$.

14. When the left-handed people enter the room, the number of handshakes that occur equals the number of left-handed people already in the room. So that will total $0+1+2+3+4 = 10$ since there are no left-handed people in the room when the first person enters, one left-handed person in the room when the second left-handed person enters, and so on. As for the right handed people, the handshakes correspond to the total number of people in the room. Since the right-handed people are the 2^{nd}, 4^{th}, 6^{th}, 8^{th}, and 10^{th} people to enter the room, there are 1, 3, 5, 7, and finally 9 people already there to shake hands with. $1 + 3 + 5 + 7 + 9 = 25$. So the total number of handshakes that take place is $10 + 25 = \boxed{35}$.

15. If we simply add the number of students who take English to the students who are taking math and those who are taking science, we find that the total enrollment in science, English, and math is $240+260+280 = 780$. But students who are in two classes are counted twice and students in all three subjects are counted three times in this total. Now add the number of students who take at least two subjects: $120 + 180 + 140 = 440$ students are in at least two subjects, but this counts the students who are enrolled in all three classes three times. Since there are 60 students enrolled in all three classes,

we subtract them out (three times!) to learn that $440 - 180 = 260$ students are taking exactly two of the three classes. In the original total of 780, these students were counted twice, and the students in all three classes were counted three times. So we subtract to find that $780 - 2 \cdot 260 - 3 \cdot 60 = \boxed{80}$ students are enrolled in only one class. The technique in this problem is called the method of *inclusion-exclusion* and is extremely important in counting complicated arrangements.

16. As you move from each number to the next, the factorial gets multiplied by the new number. For example, $4! = 3! \cdot 4$. Let's write out the prime numbers getting multiplied each time: $1! = 1$, $2! = 2$, $3! = 3 \cdot 2$, $4! = 2 \cdot 2 \cdot 3 \cdot 2$, $5! = 5 \cdot 2 \cdot 2 \cdot 3 \cdot 2$, and so forth. Notice that each time you get a 5, then since there are plenty of 2's in the product the 5 and a 2 can combine to make a 10, which puts another zero at the end of the number. So the number of zeros is the same as the number of multiples of 5 that are crossed between 1 and 100. *Except* that when you cross numbers like 25 that have *two* factors of 5 in them, you add *two* zeros. Thus you get one zero for 5, 10, 15, 20, 30, 35, 40, 45, 55, 60, 65, 70, 80, 85, 90, 95, and two zeroes for each of 25, 50, 75, and 100. So altogether you get $\boxed{24}$ zeros.

17. Using the factoring formula that $a^2 - b^2 = (a + b)(a - b)$ shows that $56x^4 - y^4 = (x^2 + y^2)(x^2 - y^2)$. Adding the second and third equations together gives $x^2 + 2xy + y^2 + x^2 - 2xy + y^2 = 8 + 6$ or $2x^2 + 2y^2 = 14$, so $x^2 + y^2 = 7$. Dividing by this produces $x^2 - y^2 = \boxed{8}$.

18. Notice that each time the laser beam bounces off of the $\overline{CD}$ wall and returns to the $\overline{AB}$ wall, its path together with the segment on the $\overline{AB}$ wall between those two places forms an equilateral triangle. Since all three sides of this triangle are the same length, we reason that the beam travels twice as far in total as it does horizontally. Since it must travel 17 meters horizontally, the total distance traveled is $\boxed{34}$ meters.

19. Since a total of 86% of the people surveyed preferred pie or cake, the remaining $100\% - 86\% = 14\%$ enjoy ice cream the most. Call the number of people in the survey p. Then $\frac{14}{100}p = 35$ so $p = 250$. Then cake is the favorite of 32% of 250, which is $\frac{32}{100} \cdot 250 = \boxed{80}$.

20. Let p be the probability that whoever rolls first wins the game. Now if the first player *doesn't* roll blue then we can pretend that we are just starting the game new with the roles of first and second player reversed! That is, if Alice rolls red, which happens 3/4 of the time, then Betty is the first player. So altogether Betty wins with probability $\frac{3}{4}p$. Since the total probability of either first or second player winning must be 1, we learn that $p + \frac{3}{4}p = 1$. Thus, the first player (Alice in this case) wins with probability $p = \boxed{4/7}$.