7^{th} Grade Team Contest Solutions

IMSA *Mu Alpha Theta*

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1. Consecutive angles in a parallelogram are always supplementary, so add to 180°. Thus $(9x + 2) + (8x + 8) = 180$. Collecting like terms, $17x + 10 = 180$ so $17x = 170$ and $x = \boxed{10}$.

2. As always, perform the operation inside the innermost parentheses first. Since $3^2 = 9$ the formula becomes $2 \blacklozenge (1 \lozenge 9 \lozenge -10)$. Now there are two $\lozenge$ operations inside the parentheses, so the rules of order say they should be performed left-to-right. So first calculate $1 \lozenge 9 = 1 + 2 \cdot 9 + 3 = 22$. Then calculate $22 \lozenge -10 = 22 + 2 \cdot (-10) + 3 = 5$. Finally, calculate $2 \blacklozenge 5 = 2^5 = \boxed{32}$.

3. Since x is positive it is impossible for $x+1 = 0$, so multiplying both sides of the equation by $x + 1$ will not change the solutions. This yields $x(x + 1) = 5x$. Again, x is positive, so is not zero, so both sides can be divided by x to find $x + 1 = 5$ or $x = \boxed{4}$.

4. Since 10 of the tiles are red on the underside, that means that $8 \times 8 - 10 = 54$ tiles are black on the bottom. Thus, the probability desired is 54/64 which should be simplified to $\boxed{27/32}$.

5. If the original rectangle's sides have lengths a and b, then each of the four white triangles in the corners of the flag have area $\frac{1}{2}(\frac{1}{2}a)(\frac{1}{2}b)$, so the four of them together have an area of $\frac{1}{2}ab$. Since the area of the entire flag is ab, the blue and red areas together must cover the other $\frac{1}{2}ab$ of the flag's area. A similar argument demonstrates that the red and blue areas are equal, so each is $\frac{1}{4}ab$. Thus, dividing the white area by the blue area gives a final result of $\boxed{2}$.

   The same result would apply even if the original flag were not rectangular—it could be any quadrilateral. *Varignon's Theorem* says that if you connect the midpoints of the sides of a quadrilateral the result is a parallelogram whose area is half that of the original quadrailateral.

6. An octagon has 8 vertices. Each of them has two neighbors and 5 non-neighboring vertices. That means there are a total of $8 \cdot 5 = 40$ choices of a way to draw a line from one vertex to a non-neighboring vertex. But this counts each diagonal twice—once drawn one way and the second time drawn going in the opposite direction. So cutting 40 in half results in $\boxed{20}$ diagonals.

7. There are 50 multiples of 2, and 33 multiples of 3. But that double-counts multiples of 6, of which there are $100 \div 6 = 16$. Thus, there are a total of $50 + 33 - 16 = 67$ numbers which are multiples of either 2 or 3. In addition, 5, 25, 35, 55, 65, 85, and 95 are divisible by 5 but not by either 2 or 3 (multiples of 2 or 3 have already been counted!), so there are a total of $67 + 7 = 74$ numbers from 1 to 100 that are divisible by at least one of 2, 3, or 5. Since we are counting out of 100, that means $\boxed{74\%}$ of the numbers are divisible by 2, 3, or 5.

   In more general cases, where you would want to count such things with a much larger upper limit, you would want to rely on the *inclusion-exclusion* principle. This is an important counting trick—it is worth looking it up!

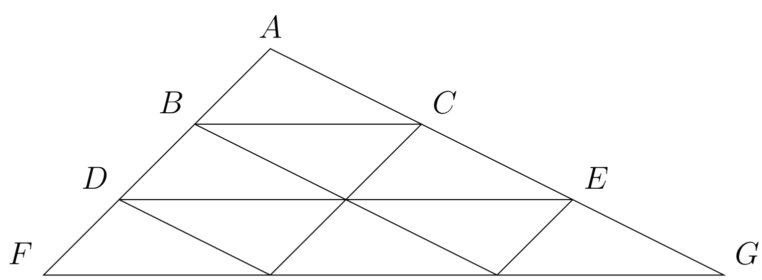
8. If every student scored as high as possible with the restrictions given in the problem, then 35 students scores 10 points, while the remaining 15 students each scored 6 points. The total of all scores is then $35 \cdot 10 + 15 \cdot 6 = 440$, and dividing by the number of students (50) to obtain the average gives $440/50 = \boxed{8.8}$.

9. If all four digits are used, the number would not be divisible by 3. The quick way to tell if a number is divisible by 3 is to add its digits, and the number is divisible by 3 if and only if the sum of its digits is. But $1 + 6 + 7 + 8 = 22$ is not divisible by 3. If we leave out the 1, though, the remaining digits add to $6 + 7 + 8 = 21$ which is divisible by 3. To make the number as large as possible, arrange the digits from largest to smallest: $\boxed{876}$.

10. Since xy is a multiple of 37, which is prime, at least one of the numbers is divisible by 37. So the numbers could be either $x = 37$ and $y = 63$, or vice versa, or else $x = 74$ and $y = 6$ or vice versa. The minimum possible difference is when x is as small as possible and y as large as possible, so $x = 26$ and $y = 74$, and $x - y = \boxed{-48}$.

11. Writing out the numbers in order gives 9 single-digit numbers, followed by 90 2-digit numbers, then 900 3-digit numbers. Altogether this is more than 2024 digits, to the 2024^{th} digit after the decimal point must occur somewhere among the 3-digit numbers. As pointed out, the first $9 + 90 \cdot 2 = 189$ digits are part of 1- or 2-digit numbers, so we are looking for the $2024 - 189 = \text{nth}1835$ digit that occurs among 3-digit numbers. Now the 100's, 200's, 300's, 400's, 500's, and 600's account for a total of $600 \cdot 3 = 1800$ digits. So we need the 35^{th} digit in the 700's. 700–710 account for $11 \cdot 3 = 33$ more digits, so we need the second digit of the next number (711), which is $\boxed{1}$.

12. A clever and quick way to get the answer is to draw a few more lines in the figure:



Because all the various lines are parallel and equally spaced, the 9 smaller triangles each have the same area. Quadrilateral $BDEC$ is made from three of these triangles, while $DFGE$ is made from five of them, so the ratio of the areas is $\boxed{3/5}$.

13. Compute $\sqrt{n} = \sqrt{2^4 3^6 5^6} = 2^2 3^3 5^3 = 13500$. Unfortunately, now it comes down to brute force counting. If there are no factors of 5 in the number, then since $2^4 3^6 = 11664 < 13500$, all the combinations of 2's and 3's without a 5 fit the description. There are $(4+1)(6+1) = 35$ of these because the choice of exponents of the 2 can be 0, 1, 2, 3, or 4 (5 choices) and there are 7 choices (0–6) for the exponent of the 3. Similarly:

   - If there is a single factor of 5, the 2's and 3's have to come out to less than $13500 \div 5 = 2700$. This is all choices of $2^a 3^b$ where b is at most 4 ($5 \times 5 = 25$ choices), plus $2^0 3^5$, $2^1 3^5$, $2^2 3^5$, $2^3 3^5$, $2^0 3^6$, and $2^1 3^6$, for a total of 31 possibilities with a single factor of 5.
   - With two factors of 5, the 2's and 3's have to multiply out to less than $13500/25 = 540$. This is all the combinations where the exponent on the 3 is at most 3 ($5 \times 4 = 20$ combinations), and $2^0 3^4$, $2^1 3^4$, $2^2 3^4$, $2^0 3^5$, and $2^1 3^5$—another 25 possibilities.
   - With three factors of 5, the 2's and 3's have to multiply out to less than 108. You should find 16 possibilities here.
   - With four factors of 5, the 2's and 3's have to multiply out to at most 20. There are 7 of these.
   - With five factors of 5, the 2's and 3's must come out to no more than 4, giving 2^0, 2^1, 2^2 and 3^1 as the only possibilities.
   - $5^6 > 13500$ so there are no possibilities in this case.

   The total number of all these factors is $35 + 31 + 25 + 16 + 7 + 4 = \boxed{118}$.

14. Sam first crawls from A to some other vertex. Then, since he never returns the way he came, his next vertex will be one he hasn't visited yet. For his next trip, he has a 50% chance to crawl back to A, and a 50% chance to crawl to the remaining unvisited vertex. If he crawls back to A on the third leg of his journey, after the 4^{th} move he is not at A. On the other hand, if he crawls to the last unvisited vertex in his third move, his next trip takes him back to A 50% of the time, and back to his second vertex the other 50% of the time. So Sam returns to vertex A after 4 moves 50% of 50% of the time—his probability is $\boxed{25\%}$.

15. If a card is showing a vowel, we must check its other side to see whether the statement is true; if the card is showing a non-vowel then it has no bearing on whether the statement is true or not. If a card shows a *non-prime* number, we must check its other side (because it the other side is a vowel, then the statement is false) but if the number is prime we don't have to check the other side, because we know the statement will be true if the other side is a vowel and not applicable anyway if it isn't a vowel. Thus, we have to flip over cards A, E, and 4—$\boxed{3}$ of them.

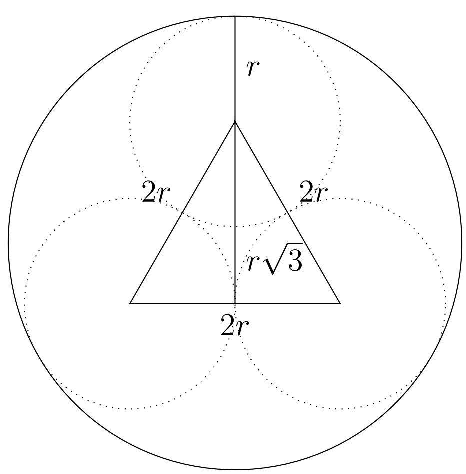
16. Rearrange the equation to read $2xy - 12x - 12y + 68 = 0$. This can be factored: $2(xy - 6x - 6y + 34) = 0$. Now this doesn't quite factor as is, but it is close! Rewrite it as $2(xy-6x-6y+36-2) = 2((x-6)(y-6)-2) = 0$. This tells us that $(x-6)(y-6) = 2$ and since $x < y$ we learn that $x-6 = 1$ while $y-6 = 2$ or else $x-6 = -2$ and $y-6 = -1$. In other words, $x = \boxed{7 \text{ or } 4}$.

17. Because both $\overline{AR}$ and $\overline{AS}$ are radii of the circle, they have the same length and triangle ARS is isosceles. Thus angles $\angle ARS$ and $\angle ASR$ are equal, so must both be $70°$ in order for triangle ARS to have angles adding to $180°$. Now in triangle ART, angle $\angle ART = \angle ARS = 70°$ and $\angle RTA = \angle RTB = 40°$, so again since all triangles have angles adding to $180°$ we learn that $\angle RAT$ measures $70°$. That means that supplementary angle $\angle RAB$ must be $\boxed{110°}$, which is also the measure of the desired arc.

18. The smallest number of marbles that leaves 1 when divide by 3 and 2 when divided by 4 is 10 marbles. After that, this pair of remainders repeats every 12 marbles (because $3 \times 4 = 12$). So counting by 12's starting at 10, the largest number we reach before going over 100 is $\boxed{94}$.

19. Since $16 = 2^4$, we learn that $\sqrt{49 - x^2} - \sqrt{25 - x^2} = 4$. Rearrange the equation:
$$\sqrt{49 - x^2} = 4 + \sqrt{25 - x^2}.$$
Square both sides:
$$49 - x^2 = 16 + 8\sqrt{25 - x^2} + 25 - x^2.$$
Subtract 41 and add x^2 to both sides:
$$8 = 8\sqrt{25 - x^2}.$$
So we learn that $\sqrt{25 - x^2} = 1$, and thus $\sqrt{49 - x^2} = 4 + \sqrt{25 - x^2} = 5$. So we get $2^5 2^1 = 32 \cdot 2 = \boxed{64}$.

20. Let the small circles have radius r, so that their total area is $3\pi r^2$. Connect their centers to form an equilateral triangle whose sides have length $2r$. Draw the altitude of this triangle. Since it is a leg of a right triangle whose hypotenuse is $2r$ and whose other leg is r, the Pythagorean theorem says that this height is $r\sqrt{3}$.



By drawing the three altitudes of an equilateral triangle and noting all the 30-60-90 triangles formed, you should be able to convince yourself that the center of an equilateral triangle is 2/3 of the height of the triangle from the vertex. Since this is also going to be the center of the large circle, putting all the pieces together shows that the large circle has radius equal to $(1+\frac{2}{3}\sqrt{3})r$. That makes its area $\pi\left(\frac{7+4\sqrt{3}}{3}\right)r^2$. Dividing this into the area of the smaller circles and doing a lot of algebra to simplify the resulting expression gives a final answer of $\boxed{63-36\sqrt{3}}$.