7^{th} Grade Team Contest Solutions

IMSA *Mu Alpha Theta*

February 22, 2023

1. The price has been reduced by \$6, which is 6/16 of the original price. Converting this fraction to a percentage yields $\frac{6}{16} \times 100\% = \boxed{37.5\%}$.

2. Against the current, Trish's net speed is just 1 mph. It takes her 90 minutes (1.5 hours) to reach her destination, so the destination must be 1.5 miles away. On her way back, her net speed is 5 mph, and $\dfrac{1.5 \text{ miles}}{5 \text{ mph}} = \boxed{\dfrac{3}{10} \text{ hours}}$.

3. First, convert hours to days: 2024 hours $= 2024 \times \dfrac{1 \text{ day}}{24 \text{ hours}} = 84\frac{1}{3}$days. Now 84 days is $84/7 = 12$ weeks, leaving just a fraction of a day, so, to the nearest week, 2024 hours is $\boxed{12}$ weeks.

4. There is a $\frac{1}{6}$ chance of you choosing the cheap bulb, and a 12% chance that bulb is bad. Since both of those have to occur for your new bulb not to work, you multiply the probabilities. So there is a $\frac{1}{6} \times 12\% = 2\%$ chance your new bulb doesn't work. That means the probability that it will work is $100 - 2 = \boxed{98\%}$.

5. Since we only need the last two digits, we only ever look at the last two digits as we take power of 2024. But the last two digits of 2024^2 are 76. The last two digits of 2024^3 are then the same as the last two digits of 76×24 which are 24. That is, as you take powers of 2024, the last two digits switch back and forth between 24 and 76. Since the exponent, 2024, is even, the last two digits of 2024^{2024} are the same as the last two digits of 2024^2, so they are $\boxed{76}$.

6. From 2:00 p.m. to 4:00 p.m. is two hours, so 80% of the original 1000 mg of medication is used up, leaving just 200 mg. Then another 1000 mg are added, so the patient now has 1200 mg of medication in their body.. Two hours later, at 6:00 p.m. only 20

7. All of the sides of the discs are visible. The surface area of the side of a cylinder is given by the formula $2\pi rh$ where r is the radius and h is the height. So Max's cylinders have $2\pi 5 \cdot 5 + 2\pi 3 \cdot 3 + 2\pi 2 \cdot 2 = 76\pi$ square inches of area in total on their sides. Now as far as the visible part of the ends is concerned, if you simply looked down on the tower from above you would just see a circle of radius 5 inches, as the top of a higher-up disc is visible exactly where the base of that disc covers part of the disc below it. So the tops add another $\pi 5^2 = 25\pi$ square inches to the are, making a total of $\boxed{101\pi}$ square inches.

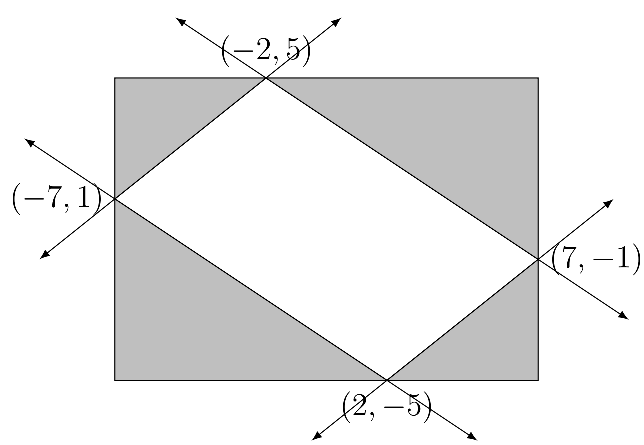
8. Notice that $x = (\sqrt{x})^2$. So inside the innermost cube root is $(\sqrt{x})^3$. That means the innermost cube root evaluates to $\sqrt{x}$. This process is repeated for each of the cube roots, so the left-hand side simplifies to $\sqrt{x}$. If $\sqrt{x} = 10$ then $x = \boxed{100}$.

9. If you were to write all the scores in increasing order, the last 85 you wrote down would finish out the first 40% of the scores. Then there are a lot of 90's, so the 50% mark is among these scores—the median score is 90. We can work out the mean score by assuming there were 100 people in the class and each 1% represents one person's score. Thus, 25 people scored 70, 10 people scored 80, 5 score 85, 50 scored 90, and rthe remaining 10 scored 100. That totals to $25 \cdot 70 + 10 \cdot 80 + 5 \cdot 85 + 50 \cdot 90 + 10 \cdot 100 = 8475$. Dividing by 100 to obtain the average gives a mean score of 84.75. Since the median score was 90, the difference is $\boxed{5.25}$.

10. Of the 20 work days in a month, Felix brings his lunch twice. If those two times are both on Clarence's forgetful days, then only one of them has to go out to lunch that day. The rest of Clarence's forgetful days, they both must go out to lunch. Since Clarence remembers his lunch half the time and forgets half the time, he forgets 10 of the 20 days. So there are at least $\boxed{8}$ days when both must go out to lunch.

11. The equation $|2x + 3y| = 11$ means that either $2x + 3y = 11$ or $2x + 3y = -11$. The graphs of these equations are parallel lines in the plane. Similarly, $|4x - 5y| = 33$ means that either $4x - 5y = 33$ or $4x - 5y = -33$, producing two more parallel lines. The quadrilateral is a parallelogram. The four vertices can be found by solving pairs of the equations. For example, the line graphed by $2x + 3y = 11$ and $4x - 5y = 33$ meet at $(7, -1)$, which can be found by doubling the first equation and subtracting from the second to obtain $-11y = 11$ so that $y = -1$ and this can be substituted back into one of the equations to find x. The other vertices may be found similarly at $(-7, 1)$, $(-2, 5)$ and $(2, -5)$.

    The easiest way to find the area of this parallelogram is by the method of *encasement*. We surround the parallelogram by a rectangle and then remove triangles to leave just the parallelogram:



    The top of the rectangle is at $y = 5$ and the bottom at $y = -5$ so the rectangle has a height of 10. The left- and right-hand edges of the rectangle are at $x = -7$ and $x = 7$

respectively, so the rectangle is 14 units wide. So its area is 140. The top-left and bottom-right triangles have height 4 and width 5, so each has an area of 10. The other two triangles have width 9 and height 6, so each has an area of 27. Removing these four triangles leaves an area of $140 - 2 \cdot 10 - 2 \cdot 27 = \boxed{66}$.

12. We call the side of the barn the length, ℓ of the pen and the distance that the rectangle sticks out from the wall the height h. Of course the area is $\ell h = 200$. Now the farmer must buy $\ell + 2h$ meters of fencing. Now a nice fact to know is that if you are trying to make $x + y$ as small as possible while making xy as big as possible, you should make x and y equal. This is because for a fixed value of $x + y$, you are fixing the perimeter of a rectangle and trying to make the area as large as possible, so you want a square. To apply this fact to our current problem, take the equation $\ell h = 200$ and multiply by 2: $\ell \cdot 2h = 400$. Now make $x = \ell$ and $y = 2h$. We want them to be equal, and since their product is 400 each should be $\sqrt{400 = 20}$ So $\ell = 20$, $h = 10$ and the amount of fencing that must be purchased is $\ell + 2h = \boxed{40}$ meters.

13. If Fievel picks the number n, his total will be $n+(n+1)+(n+2)+(n+3)+(n+4) = 5n+10$. So Fievel wants the biggest $n < 1000$ so that $5n+10$ is divisible by 13. It can be quickly checked the $n = \boxed{999}$ is the number Fievel is after.

14. Since 23 is small-ish, let's just start writing down the string until we get to the 23^{rd} digit: 1 4 9 16 25 36 49 64 81 100 121 144. . . , and the 23^{rd} digit is the first $\boxed{4}$ of 144.

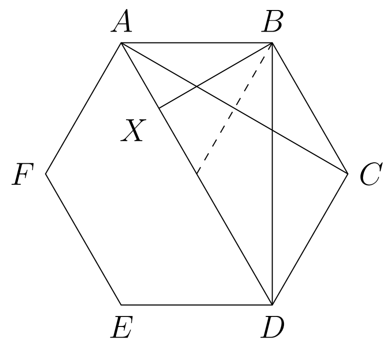
15. Arjun's first second always takes him from the center to one of the outside corners. There are six choices of which corner he could go to, so we will find the number of paths that he could take assuming his first step was to go directly to the right, and then multiply the result by six. In the second second, Arjun could:

    - Go up to the northeast vertex of the hexagon. From here there are two ways he could return to the center in the next two seconds: backtrack his last two moves, or go another step counterclockwise and then return to center. This is two possibiltiies.
    - Go down the the southeast vertex of the hexagon. As above, there are two ways for him to finish his trip.
    - Return to the center. Then in his next two steps he could travel to any of the six vertices of the hexagon and then back to the center again, leading to six more paths that work.

    So altogether there are 10 different paths Arjun could take that return him to the center in four seconds if his initial move is directly to the right. Multiplying by 6 for the different initial steps he could take gives a total of $\boxed{60}$ paths.

16. Let's see how much gets painted in one minute. Sarah, who takes 90 minutes to paint a room, would paint $\frac{1}{90}$ of a room in one minute. Working together, $\frac{1}{70}$ of the room is painted in one minute. Since Sarah paints $\frac{1}{90}$ of it, Sam must paint the remaining $\frac{1}{70} - \frac{1}{90} = \frac{1}{315}$ in that minute. Since Sam paints $\frac{1}{315}$ of a room in one minute, it must take him $\boxed{315}$ minutes to paint an entire room.

17. Draw perpendicular segment $\overline{BX}$ from point B to segment $\overline{AD}$:

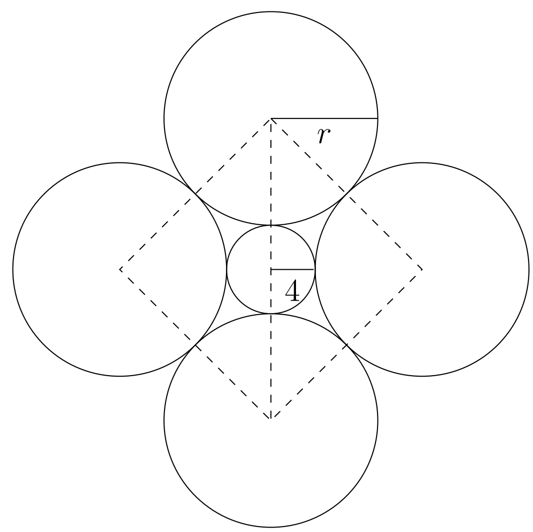


This segment is the altitude of both triangles $\triangle ABC$ (with base $\overline{BC}$) and $\triangle ABD$ (with base $\overline{AD}$. It is also the altitude of an equilateral triangle whose vertices are at A, B, and the center of the hexagon (as shown above with a dashed line as its third side). Since the equilateral triangle has sides of length 2, its altitude is $\sqrt{3}$ (use the Pythagorean theorem on right $\triangle AXB$ with hypotenuse $AB = 2$ and leg $AX = 1$). Thus the area of $\triangle ABC$ is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$ and the area of $\triangle ABD$ is $\frac{1}{2} \cdot 4 \cdot \sqrt{3} = 2\sqrt{3}$. The difference is $\boxed{\sqrt{3}}$.

18. The diagram is at right. Note that the four centers of the large circles form a square whose side has length $2r$, while the diagonal of this square has length $2r + 8$. But if a square has side s, its diagonal is $s\sqrt{2}$. So we obtain the relationship

$$2r\sqrt{2} = 2r + 8.$$

Then $r(2\sqrt{2} - 2) = 8$ or $r = \dfrac{8}{2\sqrt{2} - 2}$, which simplifies to $\boxed{4\sqrt{2} + 4}$.



19. There are four digits that Sally cannot use, she must use at least one 9, and there are five other digits. Let's do some casework depending on how many 9's there are.

- If there is just one 9, there are four choices of where that 9 could go ($9xxx$, $x9xx$, $xx9x$, and $xxx9$) and each x could be any of five digits. So there are $4 \cdot 5 \cdot 5 \cdot 5 = 500$ combinations with just one 9.
- If there are two 9's, there are six choices of where they could be ($99xx$, $9x9x$, $9xx9$, $x99x$, $x9x9$, and $xx99$). There are five choices for each x, so this combines for $6 \cdot 5 \cdot 5 = 150$ combinations.

- If there are three 9's, there are four choices of where the last digit could go, and 5 choices of what it could be, totaling $4 \cdot 5 = 20$ combinations with three 9's.
- If all four digits are 9, there is only one possibility!

Adding these up gives a total of $500 + 150 + 20 + 1 = \boxed{671}$ combinations.

20. You are likely to solve this by brute force, trying values of z until you find one that works. The smallest z that works is $\boxed{12}$. If you had a lot of time to play with this problem, you might make the following observations that allow a step-by-step solution.

Let r by any factor of z^2. Then the numbers $x = z + r$ and $y = z + \frac{z^2}{r}$ are a solution to the problem. Both are positive integers, and

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z+r} + \frac{r}{rz + z^2} = \frac{1}{z+r}\left(1 + \frac{r}{z}\right) = \frac{1}{z}.$$

So each factor of z^2 leads to a solution to the problem.

Conversely, a solution to the problem determines a factor of z^2. Let $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$. Multiply this equation by x, y, and z to obtain $zy + zx = yx$. Let $r = x - z$. Then write $r = x - z$. Then $ry = xy - zy = zx$ or $y = \frac{zx}{r} = \frac{z(z+r)}{r} = \frac{z^2}{r} + z$ and thus $y - z = \frac{z^2}{r}$ is also an integer, so r is a factor of z^2.

Thus, the solutions are in one-to-one correspondence with the factors of z^2. Let z be factored into primes as $z = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$. Then $z^2 = p_1^{2a_1} \cdot p_n^{2a_n}$ and the number of factors of z^2 is $(2a_1 + 1)(2a_2 + 1) \cdot (2a_n + 1)$. Since we need this to equal 15, which is $5 \cdot 3$ we have $a_1 = 2$ and $a_2 = 1$, and to make the number as small as possible we choose $p_1 = 2$ and $p_2 = 3$. Then $2^2 \cdot 3^1 = \boxed{12}$.